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# Complex vs etale Abel Jacobi map for higher Chow groups and algebraicity of the zero locus of etale normal functions

Johann Bouali

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## Abstract

We prove, using  $p$ -adic Hodge theory for open algebraic varieties, that for a smooth projective variety over  $k \subset \mathbb{C}$  the complex abel jacobi map vanishes if the etale abel jacobi map vanishes. This implies that for a smooth projective morphism  $f : X \rightarrow S$  of smooth complex algebraic varieties over  $k \subset \mathbb{C}$  and  $Z \in \mathcal{Z}^d(X, n)^{f, \theta=0}$  an algebraic cycle flat over  $S$  whose cohomology class vanishes on fibers, the zero locus of the etale normal function associated to  $Z$  is contained in the zero locus of the complex normal function associated to  $Z$ . From the work of Saito or Charles, we deduce that the zero locus of the complex normal function associated to  $Z$  is defined over the algebraic closure  $\bar{k}$  of  $k$  if the zero locus of the etale normal function associated to  $Z$  is not empty. We also prove an algebraicity result for the zero locus of an etale normal function associated to an algebraic cycle.

## 1 Introduction

Let  $X$  be a connected smooth projective variety over  $\mathbb{C}$ . The Abel-Jacobi map associates to a cycle  $Z \in \mathcal{Z}^d(X)$  homologically equivalent to 0 of codimension  $d$ , a point  $AJ(X)(Z) \in J^d(X)$  in the intermediate Jacobian of  $X$ . In family, if  $f : X \rightarrow S$  is a smooth projective morphism of smooth connected complex varieties and  $Z \in \mathcal{Z}^d(X)$  is a relative cycle on  $S$  of codimension  $d$  homologically trivial on fibers, the Abel-Jacobi map provides a holomorphic and horizontal section of the relative intermediate Jacobian :  $\nu_Z : S \rightarrow J^d(X/S)$  called the associated normal function. By a Brosnan-Pearlstein theorem, the zero-locus  $V(\nu_Z) \subset S$  of  $\nu_Z$  is an algebraic subvariety ([3]). If  $X/S$  and  $Z$  are defined over a subfield  $k \subset \mathbb{C}$ , we conjecture in the spirit of the Bloch-Beilinson conjectures, that the zero-locus of the normal function is also defined over  $k$  (see [7]).

On the other hand, if  $k$  is of finite type over  $\mathbb{Q}$ ,  $X$  is a connected smooth projective variety over  $k$ , and  $p$  is a prime number, we can define via the continuous étale cohomology with  $\mathbb{Z}_p$  coefficients an etale Abel-Jacobi map which associate to a cycle  $Z \in \mathcal{Z}^d(X, n)$  homologically equivalent to 0 of codimension  $d$ , an element  $AJ^{et,p}(X)(Z) \in H^1(G, H^{2d-1-n}(X_{\bar{k}}, \mathbb{Z}_p))$  of the first degree Galois cohomology of the absolute galois group  $G := Gal(\bar{k}/k)$  with value in the etale cohomology of  $X_{\bar{k}}$  with  $\mathbb{Z}_p$  coefficients. In family, we get, for  $f : X \rightarrow S$  a smooth projective morphism of smooth connected varieties over  $k$  and  $Z \in \mathcal{Z}^d(X, n)$  a relative cycle on  $S$  of codimension  $d$  homologically trivial on fibers, a normal function  $\nu_Z^{et,p}$  associated to  $Z \in \mathcal{Z}^d(X, n)$  and thus a zero-locus  $V(\nu_Z^{et,p}) \subset S$  and more generally a torsion locus  $V(\nu_Z^{et,p}) \subset V_{tors}(\nu_Z^{et,p}) \subset S$  of  $\nu_Z^{et}$  which are subsets of closed points of  $S$ . For  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding, and  $V \subset S$  a subset, we denote  $V_{\mathbb{C}} := \pi_{k/\mathbb{C}}(S)^{-1}(V) \subset S_{\mathbb{C}}$  where  $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \rightarrow S$  is the projection.

Let  $k$  of finite type over  $\mathbb{Q}$  and  $f : X \rightarrow S$  a smooth projective morphism of smooth connected varieties over  $k$ . Let  $Z \in \mathcal{Z}^d(X)$  a relative cycle. F. Charles then proves that for any embedding  $\sigma : k \hookrightarrow \mathbb{C}$

- assuming that  $R^{2d-1}f_*^{an}(\mathbb{C})$  has no global sections then  $V(\nu_Z^{et,p})_{\mathbb{C}} = V(\nu_Z)$  ([7])
- if  $V(\nu_Z)(\bar{k})$  is non-empty then it is defined over  $k$  ([8]).

In this work, we show that, for  $k$  is of finite type over  $\mathbb{Q}$ ,  $X$  a connected smooth projective variety over  $k$ ,  $Z \in \mathcal{Z}^d(X, *)$  an (higher) algebraic cycle homologically equivalent to 0 of codimension  $d$ , and  $p$  a prime number, if  $AJ^{et,p}(X)(Z) = 0$  then  $AJ_\sigma(X)(Z) := AJ(X_{\mathbb{C}})(Z_{\mathbb{C}}) = 0$  for any embedding  $\sigma : k \hookrightarrow \mathbb{C}$  (c.f. theorem 5). This implies by definition that for  $f : X \rightarrow S$  a smooth projective morphism of smooth connected varieties over  $k$ , and  $Z \in \mathcal{Z}^d(X, *)$  a relative cycle on  $S$  of codimension  $d$  homologically trivial on fibers,

$$V(\nu_Z^{et,p})_{\mathbb{C}} \subset V(\nu_Z), V_{tors}(\nu_Z^{et,p})_{\mathbb{C}} \subset V_{tors}(\nu_Z)$$

for any embedding  $\sigma : k \hookrightarrow \mathbb{C}$  without any assumption (c.f. corollary 2(i)). We deduce that if  $V(\nu_Z^{et,p})$  is not empty,  $V(\nu_Z)$  is defined over  $k$  (c.f. corollary 2(ii)).

The proof of theorem 5 uses  $p$ -adic Hodge theory for open varieties to relate de Rham cohomology and its Hodge filtration to  $p$ -adic étale cohomology with its Galois action. Theorem 5 follows indeed from the fact that by proposition, 2 and proposition 1, we have for  $U \in \text{SmVar}(k)$ , and  $\sigma : k \hookrightarrow \mathbb{C}$  and  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  embeddings, for each  $j, l \in \mathbb{Z}$ , a canonical injective map

$$H^j \iota_{ev}^{G,d}(U) : H_{et}^j(U_{\bar{k}}, \mathbb{Z}_p)(l)^G \hookrightarrow F^l H^j(U_{\mathbb{C}}^{an}, 2i\pi\mathbb{Z}), \alpha \mapsto H^j \iota_{ev}^{G,d}(U)(\alpha) := ev(U)(w(\alpha)),$$

see section 3. More precisely, if the étale Abel-Jacobi image of a cycle  $Z$  is zero,

- there exist a Galois invariant class  $\alpha$  in the étale cohomology of  $((X \times \square^n) \setminus Z)_{\bar{k}}$  with non-zero boundary, where  $\bar{k}$  is an algebraic closure of  $k$ .
- Then, by  $p$  adic Hodge theory for  $(X \times \square^n) \setminus Z$ ,  $\alpha$  define a logarithmic de Rham class  $w(\alpha)$  laying inside the right degree of the Hodge filtration of  $((X \times \square^n) \setminus Z)_{\bar{k}}$  (c.f. proposition 2).
- Taking the image of this class by the complex period map with respect to the given embedding  $\sigma : k \hookrightarrow \mathbb{C}$ , we get a Betti cohomology class  $ev((X \times \square^n) \setminus Z)(w(\alpha))$  of  $((X \times \square^n) \setminus Z)_{\mathbb{C}}^{an}$  with  $2i\pi\mathbb{Z}$  coefficients (c.f. proposition 1).
- This last class  $(1/2i\pi)ev((X \times \square^n) \setminus Z)(w(\alpha))$  induce a splitting of the localization exact sequence of mixed Hodge structures :

$$0 \rightarrow H^{2d-1-n}(X_{\mathbb{C}}^{an}, \mathbb{Z}) \xrightarrow{j^*} H^{2d-1}(((X \times \square^n) \setminus Z)_{\mathbb{C}}^{an}, \mathbb{Z})^{[Z]} \xrightarrow{\partial} H_Z^{2d}((X \times \square^n)_{\mathbb{C}}^{an}, \mathbb{Z})^{[Z]} = \mathbb{Z}^{Hdg}(d) \rightarrow 0,$$

which means that the complex Abel-Jacobi image de  $Z$  is zero.

We also show that for  $U \in \text{SmVar}(k)$ ,

$$H^j \gamma(U) \circ H^j \iota_{ev}^{G,d}(U) : H_{et}^j(U_{\bar{k}}, \mathbb{Z}_p)(d)^G \xrightarrow{H^j \iota_{ev}^{G,d}(U)} H^j(U_{\mathbb{C}}^{an}, 2i\pi\mathbb{Z}) \simeq H^j(U_{\mathbb{C}}^{an}, \mathbb{Z}) \xrightarrow{H^j \gamma(U)} H_{et}^j(U_{\bar{k}}, \mathbb{Z}_p)$$

is the canonical embedding for each  $j \in \mathbb{Z}$ , where  $\gamma(U)$  is the canonical map given by reduction modulo  $p^*$  and from the comparison theorem between étale and Betti cohomology with torsion coefficient (see definition-proposition 1).

In section 6, we show that, for all expect finitely many prime numbers  $p \in \mathbb{N}$

$$V_{tors}(\nu_Z^{et,p}) = T \cap S_{(0)} \subset S,$$

where  $T \subset S$  is an constructible algebraic subset (c.f. definition 6 and theorem 6(i)). We also give a local version for  $V_{tors}(\nu_{Z, \sigma_p}^{et,p}) \subset S$  (c.f. theorem 6(ii)). The proof use De Yong alterations to get a stratification  $S = \sqcup_{\alpha \in \Lambda} S^\alpha$  by locally closed subsets and alterations  $\pi^\alpha : (X_{S^\alpha} \times \square^n)^\alpha \rightarrow X_{S^\alpha} \times \square^n$  such that

$$f \circ \pi^\alpha : ((X \times \square^n)_{\bar{k}_{\sigma_p}}^\alpha, \pi^{\alpha,-1}(|Z|)) \rightarrow S_{\bar{k}_{\sigma_p}}^\alpha$$

is a semi-stable morphism. We then use the  $p$  adic semi-stable comparison theorem for semi-stable morphisms  $f' : (X', D') \rightarrow S'$ , with  $S', X', D' \in \text{SmVar}(\bar{k}_{\sigma_p})$ , that is satisfying

- $f' : X' \rightarrow S$  is smooth projective,  $D' \subset X'$  is a normal crossing divisor,
- for all  $s \in S'$ ,  $D'_s \subset X'_s$  is a normal crossing divisor and  $(X'_s, D'_s)$  has integral model with semi-stable reduction,

(or more generally for log smooth morphism of schemes over  $(O_{\hat{k}_{\sigma_p}}, N_{\mathcal{O}})$ ), which gives, for each  $j \in \mathbb{Z}$  a canonical filtered isomorphism

$$H^j f'_* \alpha(U') : R^j f'_* \mathbb{Z}_{p, U'_{\mathbb{C}_p}} \otimes_{\mathbb{Z}_p} O_{\mathbb{B}_{st, S'}} \xrightarrow{\sim} R^j f'_{*Hdg}(OU', F_b) \otimes_{O_{S'}} O_{\mathbb{B}_{st, S'}}.$$

which is for each  $s \in S'$  compatible with the action of the Galois group  $Gal(\mathbb{C}_p/k(s))$ , the Frobenius and the monodromy: for all but finitely many primes  $p$ ,  $\hat{k}_{\sigma_p}(s)$  is unramified for all embeddings  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  and all  $s \in S'$ , so that we get a Frobenius action on  $R^j f'_{*Hdg}(OU', F_b)_s = H^j_{DR}(U'_s)$  for all  $s \in S'$ .

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## 2 Preliminaries and Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For  $\mathcal{S} \in \text{Cat}$  and  $X \in \mathcal{S}$ , we denote  $\mathcal{S}/X \in \text{Cat}$  the category whose objects are  $Y/X := (Y, f)$  with  $Y \in \mathcal{S}$  and  $f : Y \rightarrow X$  is a morphism in  $\mathcal{S}$ , and whose morphisms  $\text{Hom}((Y', f'), (Y, f))$  consists of  $g : Y' \rightarrow Y$  in  $\mathcal{S}$  such that  $f \circ g = f'$ .
- For  $\mathcal{S} \in \text{Cat}$  denote  $\text{Gr } \mathcal{S} := \text{Fun}(\mathbb{Z}, \mathcal{S})$  is the category of graded objects.
- Denote by Ab the category of abelian groups. For  $R$  a ring denote by  $\text{Mod}(R)$  the category of (left)  $R$  modules. We have then the forgetful functor  $o_R : \text{Mod}(R) \rightarrow \text{Ab}$ .
- Denote by AbCat the category of small abelian categories.
- For  $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$  a ringed topos, we denote by
  - $\text{PSh}(\mathcal{S})$  the category of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$  and  $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$  the category of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$ , whose objects are  $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes O_{\mathcal{S}} \rightarrow M\}$ , together with the forgetful functor  $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ ,
  - $C(\mathcal{S}) = C(\text{PSh}(\mathcal{S}))$  and  $C_{O_{\mathcal{S}}}(\mathcal{S}) = C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}))$  the big abelian category of complexes of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$ ,
  - $C_{O_{\mathcal{S}}(2)fil}(\mathcal{S}) := C_{(2)fil}(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})) \subset C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$ , the big abelian category of (bi)filtered complexes of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$  such that the filtration is biregular and  $\text{PSh}_{O_{\mathcal{S}}(2)fil}(\mathcal{S}) = (\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$ .
- For  $\mathcal{X} \in \text{Cat}$  a (pre)site and  $p$  a prime number, we consider the full subcategory

$$\text{PSh}_{\mathbb{Z}_p}(\mathcal{X}) \subset \text{PSh}(\mathbb{N} \times \mathcal{X}), \quad F = (F_n)_{n \in \mathbb{N}}, \quad p^n F_n = 0, \quad F_{n+1}/p^n \xrightarrow{\sim} F_n$$

$C_{\mathbb{Z}_p}(\mathcal{X}) := C(\text{PSh}_{\mathbb{Z}_p}(\mathcal{X})) \subset C(\mathbb{N} \times \mathcal{X})$  and

- $\mathbb{Z}_p := \mathbb{Z}_{p, \mathcal{X}} := ((\mathbb{Z}/p^* \mathbb{Z})_{\mathcal{X}}) \in \text{PSh}_{\mathbb{Z}_p}(\mathcal{X})$  the diagram of constant presheaves on  $\mathcal{X}$  indexed by  $\mathbb{N}$ ,
- $\underline{\mathbb{Z}}_p := \underline{\mathbb{Z}}_{p, \mathcal{X}} \in \text{PSh}(\mathcal{X})$  the constant presheaf on  $\mathcal{X}$ ,

–  $l_{p,\mathcal{X}} := (p(*)) : \underline{\mathbb{Z}}_{p,\mathcal{X}} \rightarrow \mathbb{Z}_{p,\mathcal{X}}$  the projection map in  $\text{PSh}(\mathbb{N} \times \mathcal{X})$

Recall that filtered colimits do NOT commute with infinite limits in general. In particular, for  $f : \mathcal{X}' \rightarrow \mathcal{X}$  a morphism of (pre)site and  $F = (F_n)_{n \in \mathbb{N}} \in \text{PSh}_{\mathbb{Z}_p}(\mathcal{X})$ ,  $\varprojlim_{n \in \mathbb{N}} f^* F_n$  is NOT isomorphic to  $f^* \varprojlim_{n \in \mathbb{N}} F_n$  in  $\text{PSh}(\mathcal{X}')$  in general.

- Denote by  $\text{Sch} \subset \text{RTop}$  the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). For  $X \in \text{Sch}$ , we denote by
  - $\text{Sch}^{ft}/X \subset \text{Sch}/X$  the full subcategory consisting of objects  $X'/X = (X', f) \in \text{Sch}/X$  such that  $f : X' \rightarrow X$  is an morphism of finite type
  - $X^{et} \subset \text{Sch}^{ft}/X$  the full subcategory consisting of objects  $U/X = (X, h) \in \text{Sch}/X$  such that  $h : U \rightarrow X$  is an etale morphism.
  - $X^{sm} \subset \text{Sch}^{ft}/X$  the full subcategory consisting of objects  $U/X = (X, h) \in \text{Sch}/X$  such that  $h : U \rightarrow X$  is a smooth morphism.

For a field  $k$ , we consider  $\text{Sch}/k := \text{Sch}/\text{Spec } k$  the category of schemes over  $\text{Spec } k$ . We then denote by

- $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$  the full subcategory consisting of algebraic varieties over  $k$ , i.e. schemes of finite type over  $k$ ,
- $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$  the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
- $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$  the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

For a morphism of field  $\sigma : k \hookrightarrow K$ , we have the extension of scalar functor

$$\otimes_k K : \text{Sch}/k \rightarrow \text{Sch}/K, X \mapsto X_K := X_{K,\sigma} := X \otimes_k K, (f : X' \rightarrow X) \mapsto (f_K := f \otimes I : X'_K \rightarrow X_K).$$

which is left adjoint to the restriction of scalar

$$\text{Res}_{k/K} : \text{Sch}/K \rightarrow \text{Sch}/k, X = (X, a_X) \mapsto X = (X, \sigma \circ a_X), (f : X' \rightarrow X) \mapsto (f : X' \rightarrow X)$$

The adjunction maps are

- for  $X \in \text{Sch}/k$ , the projection  $\pi_{k/K}(X) : X_K \rightarrow X$  in  $\text{Sch}/k$ , for  $X = \cup_i X_i$  an affine open cover with  $X_i = \text{Spec}(A_i)$  we have by definition  $\pi_{k/K}(X_i) = n_{k/K}(A_i)$ ,
- for  $X \in \text{Sch}/K$ ,  $I \times \Delta_K : X \hookrightarrow X_K = X \times_K K \otimes_k K$  in  $\text{Sch}/K$ , where  $\Delta_K : K \otimes_k K \rightarrow K$  is the diagonal which is given by for  $x, y \in K$ ,  $\Delta_K(x, y) = x - y$ .

The extension of scalar functor restrict to a functor

$$\otimes_k K : \text{Var}(k) \rightarrow \text{Var}(K), X \mapsto X_K := X_{K,\sigma} := X \otimes_k K, (f : X' \rightarrow X) \mapsto (f_K := f \otimes I : X'_K \rightarrow X_K).$$

and for  $X \in \text{Var}(k)$  we have  $\pi_{k/K}(X) : X_K \rightarrow X$  the projection in  $\text{Sch}/k$ . An algebraic variety  $X \in \text{Var}(K)$  is said to be defined over  $k$  if there exists  $X_0 \in \text{Var}(k)$  such that  $X \simeq X_0 \otimes_k K$  in  $\text{Var}(K)$ . For  $X = (X, a_X) \in \text{Var}(k)$ , we have  $\text{Sch}^{ft}/X = \text{Var}(k)/X$  since for  $f : X' \rightarrow X$  a morphism of schemes of finite type,  $(X', a_X \circ f) \in \text{Var}(k)$  is the unique structure of variety over  $k$  of  $X' \in \text{Sch}$  such that  $f$  becomes a morphism of algebraic varieties over  $k$ , in particular we have

- $X^{et} \subset \text{Sch}^{ft}/X = \text{Var}(k)/X$ ,
- $X^{sm} \subset \text{Sch}^{ft}/X = \text{Var}(k)/X$ .

- For  $X \in \text{Sch}$  and  $s \in \mathbb{N}$ , we denote by  $X_{(s)} \subset X$  its points of dimension  $s$ , in particular  $X_{(0)} \subset X$  are the closed points of  $X$ .
- For  $X \in \text{Sch}$  and  $k$  a field we denote by  $X(k) := \text{Hom}_{\text{Sch}}(\text{Spec } k, X)$  the  $k$  points of  $X$ . We get  $X(k)_{in} \subset X$  the image of the  $k$ -points of  $X$ . For  $k \subset k'$  a subfield,  $\mathbb{A}_{k'}^N(k)_{in} = k^N \subset k'^N \subset \mathbb{A}_k^N$  and  $\mathbb{A}_k^N(k')_{in} = \pi_{k/k'}(\mathbb{A}_k^N)(k'^N) \subset \mathbb{A}_k^N$ .
- For  $X \in \text{Sch}$ , we denote  $X^{pet} \subset \text{Sch}/X$  the pro etale site (see [2]) which is the full subcategory of  $\text{Sch}/X$  whose object consists of weakly etale maps  $U \rightarrow X$  (that is flat maps  $U \rightarrow X$  such that  $\Delta_U : U \rightarrow U \times_X U$  is also flat) and whose topology is generated by fpqc covers. We then have the canonical morphism of site

$$\nu_X : X^{pet} \rightarrow X, (U \rightarrow X) \mapsto (U \rightarrow X)$$

For  $F \in C(X^{et})$ ,

$$\text{ad}(\nu_X^*, R\nu_{X*})(F) : F \rightarrow R\nu_{X*}\nu_X^*F$$

is an isomorphism in  $D(X^{et})$ , in particular, for each  $n \in \mathbb{Z}$

$$H^n \Gamma(X, k) : \mathbb{H}_{et}^n(X, F) \xrightarrow{\sim} \mathbb{H}_{pet}^n(X, \nu_X^*F)$$

are isomorphisms, where

$$k := k \circ \text{ad}(\nu_X^*, \nu_{X*})(E_{et}(F)) : E_{et}(F) \rightarrow \nu_{X*}E_{pet}(\nu_X^*F)$$

is the canonical map in  $C(X^{et})$  which is a quasi-isomorphism.

- Let  $k$  a field of characteristic zero and  $k_0 \subset k$  a subfield. We say that  $k$  is of finite type over  $k_0$  if  $k$  is generated as a field by  $k_0$  and a finite set  $\{\alpha_1, \dots, \alpha_r\} \subset k$  of elements of  $k$ , that is  $k = k_0(\alpha_1, \dots, \alpha_r)$ . If  $k$  is of finite type over  $k_0$  then it is of finite transcendence degree  $d \in \mathbb{N}$  over  $k_0$  and  $k = k_0(\alpha_1, \dots, \alpha_d)(\alpha_{d+1})$  with  $\{\alpha_1, \dots, \alpha_{d+1}\} \subset k$  such that  $k_0(\alpha_1, \dots, \alpha_d) = k_0(x_1, \dots, x_d)$  and  $\alpha_{d+1}$  is an algebraic element of  $k$  over  $k_0(\alpha_1, \dots, \alpha_d)$ . Note that if  $k$  is of finite type over  $k_0$  then it is NOT algebraically closed. We denote  $\bar{k}$  the algebraic closure of  $k$ . Then  $\bar{k}$  is also transcendence degree  $d$  over  $k$ .
- Let  $C$  a field of characteristic zero. Let  $X \in \text{Var}(C)$ . Then there exist a subfield  $k \subset C$  of finite type over  $\mathbb{Q}$  such that  $X$  is defined over  $k$  that is  $X \simeq X_0 \otimes_k \mathbb{C}$  with  $X_0 \in \text{Var}(k)$ .
- Let  $X = (X, O_X) \in \text{RCat}$  a ringed topos, we have in  $C(X)$  the subcomplex of presheaves of abelian group

$$OL_X : \Omega_{X, \log}^\bullet \hookrightarrow \Omega_X^\bullet, \text{ s.t. for } X^o \in X \text{ and } p \in \mathbb{N}, \\ \Omega_{X, \log}^p(X^o) := \langle df_{\alpha_1}/f_{\alpha_1} \wedge \dots \wedge df_{\alpha_p}/f_{\alpha_p}, f_\alpha \in \Gamma(X^o, O_X)^* \rangle \subset \Omega_X^p(X^o),$$

where  $\Omega_X^\bullet := DR(X)(O_X) \in C(X)$  is the De Rham complex and  $\Gamma(X^o, O_X)^* \subset \Gamma(X^o, O_X)$  is the subring consisting of invertible elements for the multiplication.

- Let  $X \in \text{Var}(k)$ . Considering its De Rham complex  $\Omega_X^\bullet := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ . If  $X \in \text{SmVar}(k)$ , then  $H_{DR}^j(X) = \mathbb{H}_{et}^j(X, \Omega_X^\bullet)$  since  $\Omega^\bullet \in C(\text{SmVar}(k))$  is  $\mathbb{A}^1$  local and admits transfert (see [5]).
- Let  $X \in \text{Var}(k)$ . Let  $X = \cup_{i=1}^s X_i$  an open affine cover. For  $I \subset [1, \dots, s]$ , we denote  $X_I := \cap_{i \in I} X_i$ . We get  $X_\bullet \in \text{Fun}(P([1, \dots, s]), \text{Var}(k))$ . Since quasi-coherent sheaves on affine noetherian schemes are acyclic, we have for each  $j \in \mathbb{Z}$ ,  $H_{DR}^j(X) = \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet)$ .

- Denote by  $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$  the full subcategory of analytic spaces over  $\mathbb{C}$ , and by  $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$  the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). For  $X \in \text{AnSp}(\mathbb{C})$ , we denote by  $X^{et} \subset \text{AnSp}(\mathbb{C})/X$  the full subcategory consisting of objects  $U/X = (X, h) \in \text{AnSp}(\mathbb{C})/X$  such that  $h : U \rightarrow X$  is an etale morphism. By the Weirstrass preparation theorem (or the implicit function theorem if  $U$  and  $X$  are smooth), a morphism  $r : U \rightarrow X$  with  $U, X \in \text{AnSp}(\mathbb{C})$  is etale if and only if it is an isomorphism local. Hence for  $X \in \text{AnSp}(\mathbb{C})$ , the morphism of site  $\pi_X : X^{et} \rightarrow X$  is an isomorphism of site.
- Denote by  $\text{CW} \subset \text{Top}$  the full subcategory of  $\text{CW}$  complexes. Denote by  $\text{Diff}(\mathbb{R}) \subset \text{RTop}$  the full subcategory of differentiable (real) manifold.
- Let  $k \subset \mathbb{C}$  a subfield. For  $X \in \text{Var}(k)$ , we denote by

$$\text{an}_X : X_{\mathbb{C}}^{an} \xrightarrow{\sim} X_{\mathbb{C}}^{an, et} \xrightarrow{\text{an}_{X_{\mathbb{C}}^{et}}} X_{\mathbb{C}}^{et} \xrightarrow{\pi_{k/\mathbb{C}}(X^{et})} X^{et},$$

the morphism of site given by the analytical functor.

- Let  $k \subset \mathbb{C}$  a subfield. For  $X \in \text{Var}(k)$ , we denote by

$$\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^{\bullet}$$

the canonical embedding in  $C(X_{\mathbb{C}}^{an})$ . It induces the embedding in  $C(X_{\mathbb{C}}^{an})$

$$\beta(X) : 2i\pi\mathbb{Z}_{X_{\mathbb{C}}^{an}} \xrightarrow{\iota_{2i\pi\mathbb{Z}/\mathbb{C}, X_{\mathbb{C}}^{an}}} \mathbb{C}_{X_{\mathbb{C}}^{an}} \xrightarrow{\alpha(X)} \Omega_{X_{\mathbb{C}}^{an}}^{\bullet}$$

For  $X \in \text{SmVar}(k)$ ,  $\alpha(X)$  is a quasi-isomorphism by the holomorphic Poincare lemma.

- Let  $k \subset \mathbb{C}_p$  a subfield. For  $X \in \text{Var}(k)$ , we denote by

$$\beta_{pet}(X_{\mathbb{C}_p}) : \underline{\mathbb{Z}}_{X_{\mathbb{C}_p}^{pet}} \xrightarrow{\iota_{X_{\mathbb{C}_p}^{pet}}} \Omega_{X_{\mathbb{C}_p}^{pet}, \log}^{\bullet} \xrightarrow{OL_{X_{\mathbb{C}_p}^{et}}} \Omega_{X_{\mathbb{C}_p}^{pet}}^{\bullet}$$

the canonical embedding in  $C(X_{\mathbb{C}_p}^{pet})$ .

- For  $X \in \text{Sch}$  noetherian irreducible and  $d \in \mathbb{N}$ , we denote by  $\mathcal{Z}^d(X)$  the group of algebraic cycles of codimension  $d$ , which is the free abelian group generated by irreducible closed subsets of codimension  $d$ .
  - For  $X \in \text{Sch}$  noetherian irreducible and  $d \in \mathbb{N}$ , we denote by  $\mathcal{Z}^d(X, \bullet) \subset \mathcal{Z}^d(X \times \square^{\bullet})$  the Bloch cycle complex which is the subcomplex which consists of algebraic cycles which intersect  $\square^*$  properly.
  - For  $X \in \text{Var}(k)$  irreducible and  $d, n \in \mathbb{N}$ , we denote by  $\mathcal{Z}^d(X, n)_{hom}^{\partial=0} \subset \mathcal{Z}^d(X, \bullet)^{\partial=0}$  the the subabelian group consisting of algebraic cycles which are homologically trivial.
  - For  $f : X \rightarrow S$  a dominant morphism with  $S, X \in \text{Sch}$  noetherian irreducible and  $d \in \mathbb{N}$ , we denote by  $\mathcal{Z}^d(X, \bullet)^f \subset \mathcal{Z}^d(X, \bullet)$  the subcomplex consisting of algebraic cycles which are flat over  $S$ .
  - For  $f : X \rightarrow S$  a dominant morphism with  $S, X \in \text{Var}(k)$  irreducible and  $d, n \in \mathbb{N}$  we denote by  $\mathcal{Z}^d(X, n)_{f, hom}^{f, \partial=0} \subset \mathcal{Z}^d(X, n)^{f, \partial=0}$  the subabelian group consisting of algebraic cycles which are flat over  $S$  and homological trivial on fibers.
- We denote  $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$  (with boundary). For  $X \in \text{Top}$  and  $R$  a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^{\vee}) \otimes R$$

and for  $l \in \mathbb{Z}$  its singular cohomology  $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$ . In particular, we get by functoriality the complex

$$C_{X,R,\text{sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We will consider the canonical embedding

$$C_{\text{sing}}^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X) : C_{\text{sing}}^*(X, 2i\pi\mathbb{Z}) \hookrightarrow C_{\text{sing}}^*(X, \mathbb{C}), \alpha \mapsto \alpha \otimes 1$$

whose image consists of cochains  $\alpha \in C_{\text{sing}}^j(X, \mathbb{C})$  such that  $\alpha(\gamma) \in 2i\pi\mathbb{Z}$  for all  $\gamma \in \mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)$ . We get by functoriality the embedding in  $C(X)$

$$\begin{aligned} C_{\text{sing}}^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}, X} : C_{X, 2i\pi\mathbb{Z}, \text{sing}}^* &\hookrightarrow C_{X, \mathbb{C}, \text{sing}}^* \\ (U \subset X) \mapsto (C_{\text{sing}}^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(U) : C_{\text{sing}}^*(U, 2i\pi\mathbb{Z}) &\hookrightarrow C_{\text{sing}}^*(U, \mathbb{C})) \end{aligned}$$

We recall we have

- For  $X \in \text{Top}$  locally contractile, e.g.  $X \in \text{CW}$ , and  $R$  a ring, the inclusion in  $C_R(X)$   $c_X : R_X \rightarrow C_{X,R,\text{sing}}^*$  is by definition an equivalence top local and that we get by the small chain theorem, for all  $l \in \mathbb{Z}$ , an isomorphism  $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$ .
- For  $X \in \text{Diff}(\mathbb{R})$ , the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- Let  $X \in \text{AnSm}(\mathbb{C})$ . Let  $X = \cup_{i=1}^r \mathbb{D}_i$  an open cover with  $\mathbb{D}_i \simeq D(0, 1)^d$ . Since a convex open subset of  $\mathbb{C}^d$  is biholomorphic to an open ball we have  $\mathbb{D}_I := \cap_{i \in I} \mathbb{D}_i \simeq D(0, 1)^d$  (where  $d$  is the dimension of a connected component of  $X$ ). We get  $\mathbb{D}_\bullet \in \text{Fun}(P([1, \dots, r]), \text{AnSm}(\mathbb{C}))$ .
- For  $k$  a field, we denote by  $\text{Vect}(k)$  the category of vector spaces and  $\text{Vect}_{\text{fil}}(k)$  the category of filtered vector spaces. Let  $k \subset K$  a field extension of field of characteristic zero.
  - For  $(V, F) \in \text{Vect}_{\text{fil}}(k)$ , we get a filtered  $K$  vector space  $(V \otimes_k K, F) \in \text{Vect}_{\text{fil}}(K)$  by  $F^j(V \otimes_k K) := (F^j V) \otimes_k K$ . In this case, we say that the filtration  $F$  on  $V \otimes_k K$  is defined over  $k$ .
  - For  $(V', F) \in \text{Vect}_{\text{fil}}(K)$  and  $h : V \otimes_k K \xrightarrow{\sim} V'$  and isomorphism of  $K$  vector space, we get  $(V, F_h) \in \text{Vect}_{\text{fil}}(k)$  by  $F_h^j V := h^{-1}(F^j V') \cap V$  (considering the canonical embedding  $n : V \hookrightarrow V \otimes_k K$ ,  $n(v) := v \otimes 1$ ).
  - For  $(V, F) \in \text{Vect}_{\text{fil}}(k)$ , we have  $F^j(V \otimes_k K) \cap V = F^j V$ .
  - For  $(V', F) \in \text{Vect}_{\text{fil}}(K)$  and  $h : V \otimes_k K \xrightarrow{\sim} V'$  and isomorphism of  $K$  vector space, we have  $h((F_h^j V) \otimes_k K) \subset F^j V'$ . Of course this inclusion is NOT an equality in general. The filtration  $F$  on  $V'$  is NOT defined over  $k$  in general.
- We also consider
  - $\text{Top}_2$  the category whose objects are couples  $(X, Y)$  with  $X \in \text{Top}$  and  $Y \subset X$  a subset and whose set of morphisms  $\text{Hom}((X', Y'), (X, Y))$  consists of  $f : X' \rightarrow X$  continuous such that  $Y' \subset f^{-1}(Y)$  (i.e.  $f(Y') \subset Y$ ),
  - $\text{RTop}_2$  the category whose objects are couples  $(X, Y)$  with  $X = (X, O_X) \in \text{RTop}$  and  $Y \subset X$  a subset and whose set of morphisms  $\text{Hom}((X', Y'), (X, Y))$  consists of  $f : X' \rightarrow X$  of ringed spaces such that  $Y' \subset f^{-1}(Y)$ ,
  - $\text{Top}^2$  the category whose objects are couples  $(X, Z)$  with  $X \in \text{Top}$  and  $Z \subset X$  a closed subset and whose set of morphisms  $\text{Hom}((X', Z'), (X, Z))$  consists of  $f : X' \rightarrow X$  continuous such that  $f^{-1}(Z) \subset Z'$  (i.e.  $f(X' \setminus Z') \subset X \setminus Z$ ), in particular we have the canonical functor  $\text{Top}^2 \rightarrow \text{Top}_2$ ,  $(X, Z) \mapsto (X, X \setminus Z)$ ,



- $\text{RTop}^2$  the category whose objects are couples  $(X, Z)$  with  $X = (X, O_X) \in \text{RTop}$  and  $Z \subset X$  a closed subset and whose set of morphisms  $\text{Hom}((X', Z'), (X, Z))$  consists of  $f : X' \rightarrow X$  of ringed spaces such that  $f^{-1}(Z) \subset Z'$ , in particular we have the canonical functor  $\text{RTop}^2 \rightarrow \text{RTop}_2, (X, Z) \mapsto (X, X \setminus Z)$ .

A (generalized) cohomology theory is in particular a functor  $H^* : \text{Top}_2 \rightarrow \text{GrAb}$ , e.g singular cohomology

$$H_{\text{sing}}^* : \text{Top}^2 \rightarrow \text{Gr Ab}, (X, Y) \mapsto H_{\text{sing}}^*(X, Y, R).$$

where  $R$  is a commutative ring. It restrict to a functor  $H^* : \text{Top}^2 \rightarrow \text{GrAb}, (X, Z) \mapsto H_Z^*(X) := H^*(X, X \setminus Z)$ .

- Denote  $\text{Sch}^2 \subset \text{RTop}^2$  the subcategory whose objects are couples  $(X, Z)$  with  $X = (X, O_X) \in \text{Sch}$  and  $Z \subset X$  a closed subset and whose set of morphisms  $\text{Hom}((X', Z'), (X, Z))$  consists of  $f : X' \rightarrow X$  of locally ringed spaces such that  $f^{-1}(Z) \subset Z'$ .
- Let  $k$  a field of characteristic zero. Denote  $\text{SmVar}^2(k) \subset \text{Var}^2(k) \subset \text{Sch}^2/k$  the full subcategories whose objects are  $(X, Z)$  with  $X \in \text{Var}(k)$ , resp.  $X \in \text{SmVar}(k)$ , and  $Z \subset X$  is a closed subset, and whose morphisms  $\text{Hom}((X', Z') \rightarrow (X, Z))$  consists of  $f : X' \rightarrow X$  of schemes over  $k$  such that  $f^{-1}(Z) \subset Z'$ .
- Let  $k$  a field of characteristic zero. Let

$$H^* : \text{SmVar}^2(k) \rightarrow \text{Gr AbCat}, (X, Z) \mapsto H_Z^*(X)$$

a mixed Weil cohomology theory in sense of [9] (e.g. (filtered) De Rham, etale or Betti cohomology, Hodge or  $p$  adic realization). For  $X \in \text{SmVar}(k)$  and  $Z \subset X$  a closed subset, we denote

$$H_Z^*(X)^0 := \ker(H_Z^*(X) \rightarrow H^*(X)).$$

For  $X \in \text{SmVar}(k)$  and  $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0}$ , we consider the subobject  $H^{2d-1}(U)^{[Z]} \subset H^{2d-1}(U)$  where  $j : U := (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$  is the complementary open subset, given by the pullback by  $H_Z^{2d}(X \times \square^n)^{[Z]} := [Z] \subset H_Z^{2d}(X \times \square^n)^0$

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{2d-1}(X \times \square^n) = H^{2d-n-1}(\dot{X}^*) & \longrightarrow & H^{2d-1}(U) & \xrightarrow{\partial} & H_Z^{2d}(X \times \square^n)^0 \longrightarrow 0 \\ & & \uparrow = & & \uparrow \subset & & \uparrow \subset \\ 0 & \longrightarrow & H^{2d-1}(X \times \square^n) = H^{2d-n-1}(\dot{X}^*) & \longrightarrow & H^{2d-1}(U)^{[Z]} & \xrightarrow{\partial} & H_Z^{2d}(X \times \square^n)^{[Z]} := [Z] \longrightarrow 0 \end{array}$$

of the first row exact sequence. In particular the second row is also an exact sequence.

- We denote by  $\text{logSch}$  the category of log schemes whose objects are couples  $(X, M) := (X, M, \alpha)$  where  $X = (X, O_X) \in \text{Sch}$ ,  $M \in \text{Shv}(X)$  is a sheaf of monoid and  $\alpha : M \rightarrow O_X$  is a morphism of sheaves of monoid. In particular we have a canonical functor

$$\text{Sch}^2 \rightarrow \text{logSch}, (X, Z) \mapsto (X, M_Z), M_Z := (f \in O_X \text{ s.t. } f|_{X \setminus Z} \in O_{X \setminus Z}^*) \subset O_X$$

Let  $k$  a field of characteristic zero. We denote by  $\text{log Var}(k) \subset \text{log Sch}/k$  the full subcategory of log varieties.

- Let  $p$  be a prime number. For  $K$  a  $p$  adic field (i.e. a finite extension of  $\mathbb{Q}_p$ ), we consider the canonical functor of Huber (see [5])

$$\mathcal{R} : \text{Var}(K) \rightarrow \text{Sch}/O_K, X \mapsto \mathcal{R}(X) := (X, O_X^o) = X^O, O_X^o := (f \in O_X, \text{ s.t. } |f(x)|_p \leq 1 \forall x \in X(O_K))$$

where  $O_K \subset K$  is the ring of integers of  $K$ .

We have the followings facts :

- Let  $k$  a field of characteristic zero. Denote  $G := Gal(\bar{k}/k)$  its absolute Galois group. Then the functor

$$\Gamma(\bar{k})(-) : \text{PSh}(k^{et}) \rightarrow \text{Mod}(\bar{k}, G), F \mapsto \Gamma(\bar{k}, F)$$

is an equivalence of category whose inverse is

$$G(-) : \text{Mod}(\bar{k}, G) \rightarrow \text{PSh}(k^{et}), V \mapsto G(V) := V := ((k'/k) \mapsto V^{Aut(k'/k)}).$$

In particular, for each  $V \in \text{Mod}(\bar{k}, G)$  and  $j \in \mathbb{Z}$ , we get an isomorphism

$$H^j G(V) : H^j(G, V) \xrightarrow{\sim} \text{Ext}_G^j(\bar{k}, V).$$

- For  $X \in \text{Sch}$ , we have

- $\nu_X^* \Omega_{X^{et}}^\bullet = \Omega_{X^{pet}}^\bullet$  and hence isomorphisms

$$H^* \Gamma(X, k) : \mathbb{H}_{et}^*(X, \Omega_{X^{et}}^\bullet) \xrightarrow{\sim} \mathbb{H}_{pet}^*(X, \Omega_{X^{pet}}^\bullet),$$

where

$$k := k \circ \text{ad}(\nu_X^*, \nu_{X^*})(-) : E_{et}(\Omega_{X^{et}}^\bullet) \rightarrow \nu_{X^*} E_{pet}(\Omega_{X^{pet}}^\bullet)$$

is the canonical map in  $C(X^{et})$  (which is a quasi-isomorphism),

- $\nu_X^* \Omega_{X^{et}, \log}^\bullet = \Omega_{X^{pet}, \log}^\bullet$  and hence isomorphisms

$$H^* \Gamma(X, k) : \mathbb{H}_{et}^*(X, \Omega_{X^{et}, \log}^\bullet) \xrightarrow{\sim} \mathbb{H}_{pet}^*(X, \Omega_{X^{pet}, \log}^\bullet)$$

where

$$k := k \circ \text{ad}(\nu_X^*, \nu_{X^*})(-) : E_{et}(\Omega_{X^{et}, \log}^\bullet) \rightarrow \nu_{X^*} E_{pet}(\Omega_{X^{pet}, \log}^\bullet)$$

is the canonical map in  $C(X^{et})$  (which is a quasi-isomorphism).

- Let  $k$  a field of characteristic zero. The complex of presheaves

$$\Omega_{\log}^\bullet \in C(\text{SmVar}(k)), X \mapsto \Omega_{X, \log}^\bullet(X)$$

is  $\mathbb{A}^1$  local and admits transfers. Hence for  $X \in \text{SmVar}(k)$

$$H^* \Gamma(X, k) : \mathbb{H}^*(X, \Omega_{X, \log}^\bullet) \xrightarrow{\sim} \mathbb{H}_{et}^*(X, \Omega_{X^{et}, \log}^\bullet)$$

are isomorphisms.

- Let  $k \subset K$  a field extension.

- Let  $X \in \text{Var}(k)$ . We have then the canonical isomorphism in  $C_{Aut(K/k), \text{fil}}(X_K)$

$$w(k/K) : (\Omega_X^\bullet \otimes_k K, F_b) \xrightarrow{\sim} (\Omega_{X_K}^\bullet, F_b)$$

given by the universal property of derivation of a ring.

- Let  $X \in \text{SmVar}(k)$ . Let  $\bar{X} \in \text{PSmVar}(k)$  a smooth compactification of  $X$  with  $D := \bar{X} \setminus X$  a normal crossing divisor. We have then the canonical isomorphism in  $C_{Aut(K/k), \text{fil}}(\bar{X}_K)$

$$w(k/K) : (\Omega_{\bar{X}}^\bullet(\log D) \otimes_k K, F_b) \xrightarrow{\sim} (\Omega_{\bar{X}_K}^\bullet(\log D), F_b)$$

given by the preceeding point. In particular, we get for all  $j, l \in \mathbb{Z}$ ,

$$* F^l H^j w(k/K) : F^l H_{DR}^j(X) \otimes_k K \xrightarrow{\sim} F^l H_{DR}^j(X_K),$$

$$* H^j w(k/K) : H_{DR}^j(X) \xrightarrow{\sim} H_{DR}^j(X_K)^G.$$

Let  $\sigma : k \hookrightarrow \mathbb{C}$  a subfield of finite type over  $\mathbb{Q}$ . Consider  $k \subset \bar{k} \subset \mathbb{C}$  the algebraic closure of  $k$ . Let  $X \in \text{Var}(k)$ . Let  $p$  a prime number. Let  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  an embedding. We have then for  $k'/k$  a field extension and  $\sigma' : k' \hookrightarrow \mathbb{C}$  and  $\sigma'_p : k' \hookrightarrow \mathbb{C}_p$  field embedding such that  $\sigma'|_k = \sigma$  and  $\sigma'_{p|k} = \sigma_p$ , the following diagram in  $C(\mathbb{N} \times X^{et})$

$$\begin{array}{ccccc}
& & \text{an}_{X^*} \mathbb{Z}_{p, X_{\mathbb{C}}^{an}} \xleftarrow{l_{p, X_{\mathbb{C}}^{an}} = (/p^*)} \text{an}_{X^*} \mathbb{Z}_{X_{\mathbb{C}}^{an}} \xrightarrow{\text{an}_{X^*} \beta(X)} \text{an}_{X^*} \Omega_{X_{\mathbb{C}}^{an}}^{\bullet} & & \\
& \nearrow \text{ad}(\text{an}_{X^*}, \text{an}_{X^*})(\mathbb{Z}_{p, X_{\bar{k}}^{et}}) & & & \uparrow \Omega(\text{an}_X) \\
& \mathbb{Z}_{p, X_{\bar{k}}^{et}} \xrightarrow{\text{ad}(\nu_X, \nu_{X^*})(\mathbb{Z}_{p, X_{\bar{k}}^{et}})} \mathbb{V}_{X_{\bar{k}}^*} \mathbb{Z}_{p, X_{\bar{k}}^{et}} \xleftarrow{l_{p, X_{\bar{k}}}} \mathbb{V}_{X_{\bar{k}}^*} \mathbb{Z}_{p, X_{\bar{k}}^{pet}} & & & \\
& \text{ad}(\pi_{\bar{k}/\mathbb{C}_p}(X_{\bar{k}}^{pet})^*, \pi_{\bar{k}/\mathbb{C}_p}(X_{\bar{k}}^{pet})^*)(\mathbb{Z}_{p, X_{\bar{k}}^{pet}}) & & & \\
& \searrow \Omega(\pi_{k'/\mathbb{C}_p}(X_{k'})) & \Omega_{X_{k'}^{et}, \log} & \xrightarrow{OL_{X_{k'}^{et}}} & \Omega_{X_{k'}^{et}} \\
& \mathbb{Z}_{p, X_{\mathbb{C}_p}^{pet}} \xrightarrow{l_{X_{\mathbb{C}_p}^{pet}}} \Omega_{X_{\mathbb{C}_p}^{pet}, \log} & \xrightarrow{OL_{X_{\mathbb{C}_p}^{pet}}} & \Omega_{X_{\mathbb{C}_p}^{pet}} & \downarrow \Omega(\pi_{k'/\mathbb{C}_p}(X_{k'}))
\end{array}$$

with as above

$$\text{an}_X : X_{\mathbb{C}}^{an} \simeq X_{\mathbb{C}}^{an, et} \xrightarrow{\text{an}_X} X_{\mathbb{C}}^{et} \xrightarrow{\pi_{\bar{k}/\mathbb{C}}(X_{\bar{k}}^{et})} X_{\bar{k}}^{et}, \quad \text{an}_X : X_{\mathbb{C}}^{an} \simeq X_{\mathbb{C}}^{an, et} \xrightarrow{\text{an}_X} X_{\mathbb{C}}^{et} \xrightarrow{\pi_{k'/\mathbb{C}}(X_{k'}^{et})} X_{k'}^{et}$$

We also have for  $X \in \text{Var}(k)$ , the canonical map

$$\begin{aligned}
H^j \gamma(X) : H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{Z}) &\xrightarrow{H^j(/p^*)} H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{Z}_p) \xrightarrow{\sim} H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}_p) \\
&\xrightarrow{H^j(\text{an}_X^*)^{-1}} H_{et}^j(X_{\mathbb{C}}, \mathbb{Z}_p) \xrightarrow{H^j(\pi_{\bar{k}/\mathbb{C}}(X)^*)^{-1}} H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)
\end{aligned}$$

for each  $j \in \mathbb{Z}$ , where  $H^j(\text{an}_X^*)$  is an isomorphism by the comparison theorem between etale cohomology and Betti cohomology with torsion coefficients (see SGA4).

We will use the following easy lemma

**Lemma 1.** *Let  $l$  a prime number. Let  $k \subset \mathbb{C}_l$  a subfield a finite type over  $\mathbb{Q}$ . Denote  $G := \text{Gal}(\bar{k}/k)$  its absolute Galois group. Consider  $\pi_{k/\mathbb{C}_l}(X_{\bar{k}}) : X_{\mathbb{C}_l} \rightarrow X_{\bar{k}}$  the projection. Let  $X \in \text{Var}(k)$ . Let  $p$  a prime number. Let  $j \in \mathbb{Z}$  and  $\alpha \in H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)$ . Then  $\alpha \in H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)^G$  if and only if*

$$\pi_{k/\mathbb{C}_l}(X_{\bar{k}})^* \alpha \in H_{et}^j(X_{\mathbb{C}_l}, \mathbb{Z}_p)^{\text{Aut}(\mathbb{C}_l/k)},$$

where  $\text{Aut}(\mathbb{C}_l/k)$  denote the group of (algebraic) automorphism of  $\mathbb{C}_l$  which fix  $k$ .

*Proof.* Obvious. □

### 3 Integral complex and $p$ -adic periods of a smooth algebraic variety over a field $k$ of finite type over $\mathbb{Q}$

#### 3.1 Complex integral periods

Let  $k$  a field of finite type over  $\mathbb{Q}$ .

Let  $X \in \text{SmVar}(k)$  a smooth variety. Let  $X = \cup_{i=1}^s X_i$  an open affine cover. We have for  $\sigma : k \hookrightarrow \mathbb{C}$  the evaluation period embedding map which is the morphism of bi-complexes

$$\begin{aligned} ev(X) \bullet \bullet : \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) &\rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}, \\ w_I^l \in \Gamma(X_I, \Omega_{X_I}^l) &\mapsto (ev(X)_I^l(w_I^l) : \phi_I^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C}, I}^{an})^\vee \otimes \mathbb{C} \mapsto ev_I^l(w_I^l)(\phi_I^l) := \int_{\mathbb{I}^l} \phi_I^{l*} w_I^l) \end{aligned}$$

given by integration. By taking all the affine open cover  $(j_i : X_i \hookrightarrow X)$  of  $X$ , we get for  $\sigma : k \hookrightarrow \mathbb{C}$ , the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X) \bullet \bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for  $j \in \mathbb{Z}$ , the evaluation period map

$$H^j ev(X) = H^j ev(X) \bullet \bullet : H_{DR}^j(X) = H^j \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

**Remark 1.** We also have for  $\sigma : k \hookrightarrow \mathbb{C}$  the composition

$$\bar{ev}(X) \bullet \bullet : \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \xrightarrow{ev(X) \bullet \bullet} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C} \xrightarrow{\mathbb{Z}(X)(i) \circ \text{an}^{-, -}} \text{Hom}_{\text{Fun}(\Delta^\bullet, \text{Var}(k))}(\mathbb{D}_{k, et}^\bullet, X_\bullet)^\vee \otimes \mathbb{C}$$

where  $i : I^\bullet \mathbb{D}_{k, et}^\bullet$  is the embedding, which is given by integration : for  $w_I^l \in \Gamma(X_I, \Omega_{X_I}^l)$  and  $\phi_I^l \in \text{Hom}_{\text{Fun}(\Delta^\bullet, \text{Var}(k))}(\mathbb{D}_{k, et}^l, X_I)$ ,

$$\bar{ev}_I^l(w_I^l)(\phi_I^l) = \int_{\mathbb{I}^l} \phi_I^{l, an*} w_I^l.$$

Let  $X \in \text{SmVar}(k)$ . Note that

$$H^* ev(X_{\mathbb{C}}) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{H^* R\Gamma(X_{\mathbb{C}}^{an}, E_{\text{Zar}}(\Omega(\text{an} X)))} H_{DR}^*(X_{\mathbb{C}}^{an}) \xrightarrow{H^* R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X))} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and the quasi-isomorphism  $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^\bullet$  in  $C(X_{\mathbb{C}}^{an})$ . Hence,

$$H^* ev(X) = H_{DR}^*(X) \xrightarrow{\Omega(\pi_{k/\mathbb{C}}(X))} H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{H^* ev(X_{\mathbb{C}})} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is injective. The elements of the image  $H^* ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$  are the periods of  $X$ .

Let  $X \in \text{SmVar}(k)$  a smooth variety. Let  $X = \cup_{i=1}^s X_i$  an open affine cover with  $X_i := X \setminus D_i$  with  $D_i \subset X$  smooth divisors with normal crossing. Let  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding and  $X_{\mathbb{C}}^{an} = \cup_{i=1}^r \mathbb{D}_i$  an open cover with  $\mathbb{D}_i \simeq D(0, 1)^d$ . Since a convex open subset of  $\mathbb{C}^d$  is biholomorphic to an open ball we have  $\mathbb{D}_I := \cap_{i \in I} \mathbb{D}_i \simeq D(0, 1)^d$  (where  $d$  is the dimension of a connected component of  $X$ ). Denote by  $j_\bullet : X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet \rightarrow X_{\bullet, \mathbb{C}}^{an}$  is the open embeddings. We then have the period morphism of tri-complexes

$$\begin{aligned} ev(X_{\mathbb{C}}^{an}) \bullet \bullet \bullet : \Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet) &\rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}, \\ w_{I, J}^l \in \Gamma(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \text{an}_X^* \Omega_{X_I}^l) &\mapsto \\ (ev_{I, J}^l(w_{I, J}^l) : \phi_{I, J}^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C}, I}^{an} \cap \mathbb{D}_J)^\vee \otimes \mathbb{C} &\mapsto ev_{I, J}^l(w_{I, J}^l)(\phi_{I, J}^l) := \int_{\mathbb{I}^l} \phi_{I, J}^{l*} w_{I, J}^l) \end{aligned}$$

given by integration. We have then the factorization

$$\begin{aligned} H^j ev(X) : H^j_{DR}(X) &:= \mathbb{H}^j(X, \Omega_X^\bullet) = \mathbb{H}^j_{et}(X, \Omega_{X^{et}}^\bullet) \xrightarrow{H^j \Omega(\pi_{k/C}(X))} \\ H^j_{DR}(X_{\mathbb{C}}) &:= \mathbb{H}^j(X_{\mathbb{C}}, \Omega_X^\bullet) = \mathbb{H}^j_{et}(X_{\mathbb{C}}, \Omega_{X^{et}}^\bullet) \xrightarrow{j_\bullet^* \circ \text{an}_{X_\bullet}^*} H^j \Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet) \\ &\xrightarrow{H^j ev(X_{\mathbb{C}}^{an})} H^j_{\text{sing}}(X_{\mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an} \cap \mathbb{D}_\bullet)^\vee \otimes \mathbb{C}). \end{aligned}$$

where for the left hand side, the first equality follows from the fact that  $\Omega^\bullet \in C(\text{SmVar}(k))$  is  $\mathbb{A}^1$  local and admits transfers, and the equality of the right hand side follows from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces.

**Remark 2.** Let  $X \in \text{SmVar}(k)$  a smooth variety. Let  $X = \cup_{i=1}^s X_i$  an open affine cover with  $X_i := X \setminus D_i$  with  $D_i \subset X$  smooth divisors with normal crossing. Let  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding and  $X_{\mathbb{C}}^{an} = \cup_{i=1}^r \mathbb{D}_i$  an open cover with  $\mathbb{D}_i \simeq D(0, 1)^d$ . Since a convex open subset of  $\mathbb{C}^d$  is biholomorphic to an open ball we have  $\mathbb{D}_I := \cap_{i \in I} \mathbb{D}_i \simeq D(0, 1)^d$  (where  $d$  is the dimension of a connected component of  $X$ ). Denote by  $j_\bullet : X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet \rightarrow X_{\mathbb{C}, \bullet}^{an}$  the open embeddings. Then,

$$j_\bullet^* \circ \text{an}_{X_\bullet}^* := \Omega(j_\bullet \circ \text{an}_{X_\bullet}) : \Gamma(X_{\bullet, \mathbb{C}}, \Omega_{X^{et}}^\bullet) \rightarrow \Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet)$$

is a quasi-isomorphism by the Grothendieck comparison theorem for De Rham cohomology and the acyclicity of quasi-coherent sheaves on noetherian affine schemes.

**Lemma 2.** Let  $X \in \text{SmVar}(k)$  a smooth variety. Let  $X = \cup_{i=1}^s X_i$  an open affine cover with  $X_i := X \setminus D_i$  with  $D_i \subset X$  smooth divisors with normal crossing. Let  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding and  $X_{\mathbb{C}}^{an} = \cup_{i=1}^r \mathbb{D}_i$  an open cover with  $\mathbb{D}_i \simeq D(0, 1)^d$ . Let

$$w = [ \sum_{I, J, l, \text{card} I + \text{card} J + l = j} w_{I, J}^l ] = \sum_{I, J, l, \text{card} I + \text{card} J + l = j} [w_{I, J}^l] \in H^j \Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet).$$

Then the following assertions are equivalent :

- (i)  $H^j ev(X)(w) \in H^j_{\text{sing}}(X_{\mathbb{C}}^{an}, \mathbb{Z}(2i\pi))$ ,
- (ii) for all  $I, J, l$  such that  $\text{card} I + \text{card} J + l = j$ , there exist a lift

$$[\tilde{w}_{I, J}^l] \in H^l \Gamma(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \Omega_{X_{\mathbb{C}}^{an}}^\bullet)$$

of  $[w_{I, J}^l]$  with respect to the spectral sequence associated to the filtration on the total complex associated to the bi-complex structure such that

$$H^l ev(X_{\mathbb{C}}^{an})_{I, J}^\bullet([\tilde{w}_{I, J}^l]) \in H^l_{\text{sing}}(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \mathbb{Z}(2i\pi)).$$

*Proof.* Follows immediately from the fact that  $ev(X_{\mathbb{C}}^{an})_{\bullet, \bullet}^\bullet$  define by definition a morphism of spectral sequence for the filtration given by the bi-complex structure.  $\square$

The main proposition of this section is the following :

**Proposition 1.** Let  $X \in \text{SmVar}(k)$ . Let  $X = \cup_{i=1}^s X_i$  an open affine cover with  $X_i := X \setminus D_i$  with  $D_i \subset X$  smooth divisors with normal crossing. Let  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding. Let

$$w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega_X^\bullet) = \mathbb{H}^j_{et}(X, \Omega_X^\bullet).$$

Then if  $w \in H^j OL_{X^{et}}(\mathbb{H}^j_{et}(X, \Omega_{X^{et}, \log}^\bullet))$  then  $H^j ev(X)(w) \in H^j_{\text{sing}}(X_{\mathbb{C}}^{an}, \mathbb{Z}(2i\pi))$ , where we recall (see section 2) the inclusion in  $C(X^{et})$

$$OL_{X^{et}} : \Omega_{X^{et}, \log}^\bullet \hookrightarrow \Omega_{X^{et}}^\bullet$$

is the sub-complex of logarithmic forms.

*Proof.* Let

$$w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet) = H^j\Gamma(X_\bullet, \Omega_X^\bullet).$$

where  $X = \cup_{i=1}^s X_i$  is an open affine cover with  $X_i := X \setminus D_i$  with  $D_i \subset X$  smooth divisors with normal crossing. Let  $X_{\mathbb{C}}^{an} = \cup_{i=1}^s \mathbb{D}_i$  an open cover with  $\mathbb{D}_i \simeq D(0, 1)^d$ . Then by definition  $H^j ev(X)(w) = H^j ev(X_{\mathbb{C}}^{an})(j_\bullet^* \circ \text{an}_{X_\bullet}^* w)$  with

$$j_\bullet^* \circ \text{an}_{X_\bullet}^* w \in H^j\Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet)$$

and we have the following commutative diagram

$$\begin{array}{ccc} \Gamma(X_\bullet, \Omega_X^\bullet) & \xrightarrow{j_\bullet^* \circ \text{an}_{X_\bullet}^*} & \Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet) \\ \Gamma(-, OL_{X^{et}}) \uparrow & \nearrow j_\bullet^* \circ \text{an}_{X_\bullet}^* & \\ \Gamma(X_\bullet, \Omega_{X^{et}, \log}^\bullet) & & \end{array}$$

Now, if  $w = H^j OL_{X^{et}}(w_L)$  with

$$w_L \in \mathbb{H}_{et}^j(X, \Omega_{X^{et}, \log}^\bullet) = \Gamma(X_\bullet, \Omega_{X^{et}, \log}^\bullet),$$

we have

$$j_\bullet^* \circ \text{an}_{X_\bullet}^* w = j_\bullet^* \circ \text{an}_{X_\bullet}^* w_L = \left[ \sum_{I, J, l, \text{card} I + \text{card} J + l = j} w_{L, I, J}^l \right] = \sum_{I, J, l, \text{card} I + \text{card} J + l = j} [w_{L, I, J}^l] \in H^j\Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet).$$

Let for each  $(I, J, l)$  take a lift

$$[\tilde{w}_{L, I, J}^l] \in H^l\Gamma(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \Omega_{X_{\mathbb{C}}^{an}}^\bullet)$$

of  $[w_{L, I, J}^l]$  with respect to the spectral sequence associated to the filtration on the total complex associated to the bi-complex structure. We have by a standard computation, for each  $(I, J)$

- $H_{\text{sing}}^*(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \mathbb{Z}) = \langle \gamma_1, \dots, \gamma_{\text{card} I} \rangle$ .
- $w_{L, I, J}^l = \sum_{\mu' \in P([1, \dots, s]), \text{card} \mu' = l} n_{IJ}^{\mu'} dz_{\mu'_1} \wedge \dots \wedge dz_{\mu'_l} + d\alpha \in \Gamma(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \Omega_{X_{\mathbb{C}}^{an}}^\bullet)$ .

Then for  $\mu \in P([1, \dots, s])$  with  $\text{card} \mu = l$ , we get

$$H^l ev(X_{\mathbb{C}}^{an})_{I, J}([\tilde{w}_{L, I, J}^l])(\gamma_\mu) = \sum_k n_{IJ}^\mu \delta_{\mu', \mu} \in 2i\pi\mathbb{Z}.$$

where  $\gamma_\mu := \gamma_{\mu_1} \cdots \gamma_{\mu_l}$ . We conclude by lemma 2.  $\square$

Let  $X \in \text{SmVar}(k)$ . Let  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding. By proposition 1, we have a commutative diagram of graded algebras

$$\begin{array}{ccc} H_{DR}^*(X) & \xrightarrow{H^* ev(X)} & H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C}) \\ \uparrow H^* OL_{X^{et}} & & \uparrow H^* C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X_{\mathbb{C}}^{an}) \\ \mathbb{H}_{et}^*(X, \Omega_{X^{et}, \log}^\bullet) & \xrightarrow{H^* ev(X) \circ H^* OL_{X^{et}}} & H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Z}) \end{array}$$

where

$$C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X_{\mathbb{C}}^{an}) : C_{\text{sing}}^\bullet(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Z}) \hookrightarrow C_{\text{sing}}^\bullet(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the subcomplex consisting of  $\alpha \in C_{\text{sing}}^j(X_{\mathbb{C}}^{\text{an}}, \mathbb{C})$  such that  $\alpha(\gamma) \in 2i\pi\mathbb{Z}$  for all  $\gamma \in C_j^{\text{sing}}(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z})$ . Recall that

$$H^*ev(X_{\mathbb{C}}) = H^*R\Gamma(X_{\mathbb{C}}^{\text{an}}, \alpha(X)) \circ \Gamma(X_{\mathbb{C}}^{\text{an}}, E_{\text{zar}}(\Omega(\text{an}_X))) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{\sim} H_{\text{sing}}^*(X_{\mathbb{C}}^{\text{an}}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and  $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{\text{an}}}^{\bullet}$ , which gives the periods elements  $H^*ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{\text{an}}, \mathbb{C})$ . On the other side the induced map

$$H^*ev(X_{\mathbb{C}}) : H^*OL_{X^{\text{et}}}(\mathbb{H}_{\text{et}}^*(X_{\mathbb{C}}, \Omega_{X^{\text{et}}, \log})) \hookrightarrow H^*\iota_{2i\pi\mathbb{Z}/\mathbb{C}}H_{\text{sing}}^*(X_{\mathbb{C}}^{\text{an}}, 2i\pi\mathbb{Z})$$

is NOT surjective in general since the left hand side is invariant by the action of Galois group  $Gal(\mathbb{C}/k)$  whereas the right hand side is not.

### 3.2 $p$ adic integral periods

Let  $k$  a field of finite type over  $\mathbb{Q}$ . Denote  $\bar{k}$  the algebraic closure of  $k$  and  $G = Gal(\bar{k}/k)$  the absolute Galois group of  $k$ . Let  $X \in \text{SmVar}(k)$  a smooth variety. Take a compactification  $\bar{X} \in \text{PSmVar}(k)$  of  $X$  such that  $D := \bar{X} \setminus X \subset \bar{X}$  is a normal crossing divisor, and denote  $j : X \hookrightarrow \bar{X}$  the open embedding. Let  $p \in \mathbb{N}$  a prime number. Consider an embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ . Then  $k \subset \bar{k} \subset \mathbb{C}_p$  and  $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ , where  $\hat{k}_{\sigma_p}$  is the  $p$ -adic field which is the completion of  $k$  with respect the  $p$  adic norm given by  $\sigma_p$ . Denote  $\hat{G}_{\sigma_p} = Gal(\mathbb{C}_p/\hat{k}_{\sigma_p}) = Gal(\bar{\mathbb{Q}}_p/\hat{k}_{\sigma_p})$  the Galois group of  $\hat{k}_{\sigma_p}$ . We have then the commutative diagram in  $C_{\mathbb{B}_{\text{dr}}, \text{fil}, \hat{G}_{\sigma_p}}(\bar{X}_{\mathbb{C}_p}^{\text{an}, \text{pet}})$

$$\begin{array}{ccc} j_*E_{\text{pet}}(\mathbb{B}_{\text{dr}, X_{\mathbb{C}_p}}, F) & \xrightarrow{j_*E_{\text{pet}}(\alpha(X))} & j_*E_{\text{pet}}((\Omega_{X_{\mathbb{C}_p}}^{\bullet}, F_b) \otimes_{O_{X_{\mathbb{C}_p}}} (O_{\mathbb{B}_{\text{dr}, X_{\mathbb{C}_p}}}, F)) \\ \uparrow \subset & & \uparrow \subset \\ E_{\text{pet}}(\mathbb{B}_{\text{dr}, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F) & \xrightarrow{\alpha(X)} & E_{\text{pet}}((\Omega_{\bar{X}_{\mathbb{C}_p}}^{\bullet}(\log D_{\mathbb{C}_p}), F_b) \otimes_{O_{\bar{X}_{\mathbb{C}_p}}} (O_{\mathbb{B}_{\text{dr}, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}}, F)) \\ E_{\text{pet}}(j_*\iota'_{X_{\mathbb{C}_p}})j := E_{\text{pet}}(l \rightarrow l.1)j \uparrow & & \uparrow E_{\text{pet}}(j_*(n \circ OL_{X_{\mathbb{C}_p}^{\text{pet}}}))j := E_{\text{pet}}(w \mapsto (w \otimes 1))j \\ j_*E_{\text{pet}}(\underline{\mathbb{Z}}_p_{X_{\mathbb{C}_p}}) & \xrightarrow{j_*E_{\text{pet}}(\iota_{X_{\mathbb{C}_p}^{\text{pet}}}) := j_*E_{\text{pet}}(l \rightarrow l.1)} & j_*E_{\text{pet}}(\Omega_{X_{\mathbb{C}_p}, \log}^{\bullet}, F_b) \end{array}$$

where for  $j' : U' \hookrightarrow X'$  an open embedding with  $X' \in \text{RTop}$  and  $\tau$  a topology on  $\text{RTop}$  we denote for  $m : j_*Q \rightarrow Q'$  with  $Q \in \text{PSh}_O(U')$ ,  $Q' \in \text{PSh}_O(X')$  the canonical map in  $C_O(X')$

$$E_{\tau}^0(m)_j : j_*E_{\tau}^0(Q) \rightarrow E_{\tau}^0(j_*Q) \xrightarrow{E_{\tau}^0(m)} E_{\tau}^0(Q'),$$

giving by induction the canonical map  $E_{\tau}(m)_j : j_*E_{\tau}(Q) \rightarrow E_{\tau}(Q')$  in  $C_O(X')$ . The main results of [15] state that

- the map in  $C_{\mathbb{B}_{\text{dr}}, \text{fil}, \hat{G}_{\sigma_p}}(\bar{X}_{\hat{k}_{\sigma_p}}^{\text{an}, \text{pet}})$

$$\alpha(X) : (\mathbb{B}_{\text{dr}, \bar{X}_{\hat{k}_{\sigma_p}}, \log D_{\hat{k}_{\sigma_p}}}, F) \hookrightarrow (\Omega_{\bar{X}_{\hat{k}_{\sigma_p}}}^{\bullet}(\log D_{\hat{k}_{\sigma_p}}), F_b) \otimes_{O_{\bar{X}_{\hat{k}_{\sigma_p}}}} (O_{\mathbb{B}_{\text{dr}, \bar{X}_{\hat{k}_{\sigma_p}}, \log D_{\hat{k}_{\sigma_p}}}}, F)$$

is a filtered quasi-isomorphism, that is, the induced map in  $C_{\mathbb{B}_{\text{dr}}, \text{fil}, \hat{G}_{\sigma_p}}(\bar{X}_{\mathbb{C}_p}^{\text{an}, \text{pet}})$

$$\alpha(X) := \alpha(X)_{\mathbb{C}_p} : (\mathbb{B}_{\text{dr}, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F) \hookrightarrow (\Omega_{\bar{X}_{\mathbb{C}_p}}^{\bullet}(\log D_{\mathbb{C}_p}), F_b) \otimes_{O_{\bar{X}_{\mathbb{C}_p}}} (O_{\mathbb{B}_{\text{dr}, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}}, F)$$

is thus a filtered quasi-isomorphism,

- the map in  $D_{\mathbb{Z}_p, fil}$

$$T(a_X, a_X, \otimes)(Rj_*\mathbb{Z}_p, X^{et}) : R\Gamma(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes (\mathbb{B}_{dr, \mathbb{C}_p}, F) \rightarrow R\Gamma(\bar{X}_{\mathbb{C}_p}, (\mathbb{B}_{dr, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F))$$

is an isomorphism.

Hence, we get the isomorphism in  $D_{fil}(\mathbb{B}_{dr}, \hat{G}_{\sigma_p})$

$$\begin{aligned} R\alpha(X) &:= R\Gamma(\bar{X}_{\mathbb{C}_p}, \alpha(X)) \circ T(a_X, a_X, \otimes)(Rj_*\mathbb{Z}_p, X^{et}) : \\ R\Gamma(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes (\mathbb{B}_{dr, \mathbb{C}_p}, F) &\xrightarrow{\sim} R\Gamma(\bar{X}_{\mathbb{C}_p}, (\Omega_{\bar{X}_{\mathbb{C}_p}^{et}}(\log D_{\mathbb{C}_p}), F_b) \otimes_{O_{\bar{X}_{\mathbb{C}_p}}} (O_{\mathbb{B}_{dr, \bar{X}_{\mathbb{C}_p}, \log D_{\mathbb{C}_p}}, F)) \\ &\xrightarrow{\cong} R\Gamma(\bar{X}_{\mathbb{C}_p}, (\Omega_{\bar{X}_{\mathbb{C}_p}^{et}}(\log D_{\mathbb{C}_p}), F_b) \otimes_{O_{\bar{X}_{\mathbb{C}_p}}} j_*Hdg(O_{X_{\mathbb{C}_p}}, F_b) \otimes_{O_{X_{\mathbb{C}_p}}} (O_{\mathbb{B}_{dr, X_{\mathbb{C}_p}}, F})) \end{aligned}$$

which gives for each  $n \in \mathbb{Z}$  a filtered isomorphism of  $\hat{G}_{\sigma_p}$ -modules

$$H^n R\alpha(X) : H_{et}^n(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes \mathbb{B}_{dr, \mathbb{C}_p} \xrightarrow{\sim} H_{DR}^n(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{dr, \mathbb{C}_p}$$

so that we can recover the Hodge filtration on  $H_{DR}^*(X)$  by the action of  $\hat{G}_{\sigma_p}$ .

We have the following key proposition :

**Proposition 2.** *Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $X \in \text{SmVar}(k)$  a smooth variety. Take a compactification  $\bar{X} \in \text{PSmVar}(k)$  of  $X$  such that  $D := \bar{X} \setminus X \subset X$  is a normal crossing divisor, and denote  $j : X \hookrightarrow \bar{X}$  the open embedding. Let  $p \in \mathbb{N}$  a prime number. Let  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  an embedding. Consider its completion  $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$  with respect to the  $p$  adic norm induced by  $\sigma_p$ . Then  $\hat{G}_{\sigma_p} := \text{Gal}(\mathbb{C}_p/\hat{k}_{\sigma_p}) \subset G := \text{Gal}(\bar{k}/k)$ .*

(i) Let  $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)$ . Consider then

$$\nu_X^* \alpha := H^j \Gamma(X_{\mathbb{C}_p}, k)(\alpha) \in H_{pet}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p) = H_{pet}^j(X_{\mathbb{C}_p}, \underline{\mathbb{Z}_p})$$

where  $k := k \circ \text{ad}(\nu_X^*, \nu_{X^*})(\mathbb{Z}_p) : E_{et}(\mathbb{Z}_p) \rightarrow \nu_{X^*} E_{pet}(\mathbb{Z}_p)$  is the canonical map in  $C(\mathbb{N}, X^{et})$  which is a quasi-isomorphism, and its associated De Rham log class

$$w_L(\alpha) := H^j \iota_{X_{\mathbb{C}_p}^{pet}}(\nu_X^* \alpha) \in \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{pet}, \log}^\bullet) = \mathbb{H}_{et}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{et}, \log}^\bullet)$$

which gives

$$\begin{aligned} w(\alpha) &:= H^j OL_{X_{\mathbb{C}_p}^{et}}(w_L(\alpha)) = H^j \beta(X_{\mathbb{C}_p}^{pet})(\alpha) \\ &\in \mathbb{H}_{pet}^j(\bar{X}_{\mathbb{C}_p}, \Omega_{\bar{X}_{\mathbb{C}_p}^{pet}}^\bullet(\log D_{\mathbb{C}_p})) = \mathbb{H}_{et}^j(\bar{X}_{\mathbb{C}_p}, \Omega_{\bar{X}_{\mathbb{C}_p}^{et}}^\bullet(\log D_{\mathbb{C}_p})) = H_{DR}^j(X_{\mathbb{C}_p}) \end{aligned}$$

If  $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}}$  then

$$H^j R\Gamma(X_{\mathbb{C}_p}, \alpha(X))(\alpha \otimes 1) = (w(\alpha) \otimes 1) \in F^l H_{DR}^j(X_{\hat{k}}) \otimes_{\hat{k}} \mathbb{B}_{dr, \mathbb{C}_p}$$

so that  $w(\alpha) \in F^l H_{DR}^j(X_{\mathbb{C}_p})$ .

(ii) Let  $\alpha \in H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(l)$ . Referring to (i), we consider

- $w_L(\alpha) := w_L(\pi_{\bar{k}/\mathbb{C}_p}(X_{\bar{k}})^* \alpha) \in \mathbb{H}_{et}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{et}, \log}^\bullet)$ ,
- $w(\alpha) := H^j OL_{X_{\mathbb{C}_p}^{et}}(w_L(\alpha)) \in H_{DR}^j(X_{\mathbb{C}_p})$ .

where  $\pi_{\bar{k}/\mathbb{C}_p}(X_{\bar{k}}) : X_{\mathbb{C}_p} \rightarrow X_{\bar{k}}$  is the projection. If  $\alpha \in H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(l)^G$  then  $w_L(\alpha) \in \mathbb{H}_{et}^j(X, \Omega_{X^{et}, \log}^\bullet)$  and  $w(\alpha) = H^j OL_{X^{et}}(w_L(\alpha)) \in F^l H_{DR}^j(X)$ .



*Proof.* (i): It follows immediately from the fact that  $H^j R\alpha(X)$  is a filtered quasi-isomorphism compatible with the Galois action of  $\hat{G}_{\sigma_p}$  by [15] and from the commutative diagram

$$\begin{array}{ccccc}
H_{pet}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p) & \xrightarrow{H^j \Gamma(X_{\mathbb{C}_p}, \iota_X^{et})} & \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{et}, \log}^\bullet) & \xrightarrow{H^j \Gamma(X_{\mathbb{C}_p}, (w \mapsto (w \otimes 1)))} & \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{et}}^\bullet) \\
\downarrow \text{koad}(\text{an}_{\bar{X}_{\mathbb{C}_p}}^*, \text{an}_{X_{\mathbb{C}_p}^*})(-) & & \downarrow \text{koad}(\text{an}_{\bar{X}_{\mathbb{C}_p}}^*, \text{an}_{X_{\mathbb{C}_p}^*})(-) & & \downarrow \text{koad}(\text{an}_{\bar{X}_{\mathbb{C}_p}}^*, \text{an}_{X_{\mathbb{C}_p}^*})(-) \\
H_{pet}^j(X_{\mathbb{C}_p}^{an}, \mathbb{Z}_p) & \xrightarrow{H^j \Gamma(X_{\mathbb{C}_p}^{an}, \iota_X^{pet})} & \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}^{an}, \Omega_{X_{\mathbb{C}_p}^{pet}, \log}^\bullet) & \xrightarrow{H^j \Gamma(X_{\mathbb{C}_p}^{an}, (w \mapsto (w \otimes 1)))} & \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}^{an}, \Omega_{X_{\mathbb{C}_p}^{pet}}^\bullet)
\end{array}$$

where the left and right column are isomorphisms by GAGA (note that the middle column is NOT an isomorphism in general).

(ii):

- By lemma 1 and the fact that

$$\iota_{X_{\mathbb{C}_p}^{pet}} : \mathbb{Z}_{p, X} \rightarrow \Omega_{X_{\mathbb{C}_p}^{pet}, \log}^\bullet, \quad OL_{X_{\mathbb{C}_p}^{pet}} : \Omega_{X_{\mathbb{C}_p}^{pet}, \log}^\bullet \rightarrow \Omega_{X_{\mathbb{C}_p}^{pet}}^\bullet, \quad \beta(X_{\mathbb{C}_p}) : \mathbb{Z}_{p, X} \rightarrow \Omega_{X_{\mathbb{C}_p}^{pet}}^\bullet$$

are  $Aut(\mathbb{C}_p/k)$  invariant, where we recall  $Aut(\mathbb{C}_p/k)$  is the group of field automorphisms of  $\mathbb{C}_p$  which fix  $k$ ,  $w_L(\alpha) \in \mathbb{H}_{et}^j(X, \Omega_{X^{et}, \log}^\bullet)$  and  $w(\alpha) = H^j OL_{X^{et}}(w_L(\alpha)) \in H_{DR}^j(X)$ . Note that  $\mathbb{C}_p^{Aut(\mathbb{C}_p/k)} = k$  since  $k$  is of finite transcendence degree over  $\mathbb{Q}$ .

- On the other hand, by (i)  $w(\alpha) \in F^l H_{DR}^j(X_{\mathbb{C}_p})$ .

Hence  $w(\alpha) \in F^l H_{DR}^j(X) = F^l H_{DR}^j(X_{\mathbb{C}_p}) \cap H_{DR}^j(X)$  and  $w(\alpha) \in H^j OL_{X^{et}}(\mathbb{H}_{et}^j(X, \Omega_{X^{et}, \log}^\bullet))$ . The equality  $F^l H_{DR}^j(X) = F^l H_{DR}^j(X_{\mathbb{C}_p}) \cap H_{DR}^j(X)$  is given by the filtered isomorphism in  $C_{fil}(\bar{X}_{\mathbb{C}_p})$

$$w(k/\mathbb{C}_p) : (\Omega_{\bar{X}}^\bullet(\log D), F_b) \otimes_k \mathbb{C}_p \xrightarrow{\sim} (\Omega_{\bar{X}_{\mathbb{C}_p}}^\bullet(\log D_{\mathbb{C}_p}), F_b)$$

which say that the Hodge filtration is defined over  $k$ : see section 2.  $\square$

**Definition-Proposition 1.** *Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $X \in \text{SmVar}(k)$ . By proposition 2(ii) and proposition 1, we have for  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  and  $\sigma : k \hookrightarrow \mathbb{C}$  embeddings, and each  $j, l \in \mathbb{Z}$ , a canonical injective map*

$$H^j \iota_{ev}^{G, d}(X) : H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(l)^G \hookrightarrow F^l H^j(X_{\mathbb{C}}^{an}, 2\pi i \mathbb{Z}), \quad \alpha \mapsto H^j \iota_{ev}^{G, d}(X)(\alpha) := ev(X)(w(\alpha)).$$

Moreover,

$$H^j \gamma(X) \circ H^j \iota_{ev}^{G, d}(X) : H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)(d)^G \xrightarrow{H^j \iota_{ev}^{G, d}(X)} H^j(X_{\mathbb{C}}^{an}, 2i\pi \mathbb{Z}) \simeq H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}) \xrightarrow{H^j \gamma(X)} H_{et}^j(X_{\bar{k}}, \mathbb{Z}_p)$$

is the canonical embedding for each  $j \in \mathbb{Z}$ , where  $\gamma(X)$  is the canonical map (see section 2) given by reduction modulo  $p^*$  and from the comparison theorem between etale and Betti cohomology with torsion coefficient.

*Proof.* Follows from the following commutative diagram in  $C(X^{et} \times \mathbb{N})$

$$\begin{array}{ccccccc}
E_{et}(\mathbb{Z}_p, X_{\bar{k}}^{et}) & \longrightarrow & E_{usu}(\mathbb{Z}_p, X_{\mathbb{C}}^{an}) & \longleftarrow & E_{usu}(\mathbb{Z}_{X_{\mathbb{C}}^{an}}) & & \\
\uparrow & & & & \searrow & & \\
E_{pet}(\mathbb{Z}_p, X_{\mathbb{C}_p}^{pet})^G & \longrightarrow & E_{pet}(\Omega_{X_{\mathbb{C}_p}^{pet}, \log}^\bullet)^G & \longleftarrow & E_{zar}(\Omega_{X, \log}^\bullet) & \longrightarrow & E_{usu}(\Omega_{X_{\mathbb{C}}^{an}}^\bullet)
\end{array}$$

$\square$

## 4 The complex and etale Abel Jacobi maps and normal function

### 4.1 The complex Abel Jacobi map for higher Chow group and complex normal functions

Let  $k$  a field of finite type over  $\mathbb{Q}$ . Consider an embedding  $\sigma : k \hookrightarrow \mathbb{C}$ . Then  $k \subset \bar{k} \subset \mathbb{C}$ , where  $\bar{k}$  is the algebraic closure of  $k$ . We have then the quasi-isomorphism  $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^{\bullet}$  in  $C(X_{\mathbb{C}}^{an})$ .

- For  $X \in \text{SmVar}(k)$ , we consider

$$((H_{DR}^j(X), F), H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}), H^j \alpha(X)) \in MHM_{k, gm}(k) \subset \text{Vect}_{fil}(k) \times_I \text{Ab}$$

where  $F$  is the Hodge filtration on  $H_{DR}^j(X) \otimes_k \mathbb{C}$  and

$$H^j R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X)) : H^j(X_{\mathbb{C}}^{an}, \mathbb{C}) \xrightarrow{\sim} H_{DR}^j(X) \otimes_k \mathbb{C}$$

Recall the geometric mixed Hodge structures (see [5]) are mixed Hodge structure by the Hodge decomposition theorem on smooth proper complex varieties.

- For  $f : X \rightarrow S$  a morphism with  $S, X \in \text{SmVar}(k)$ , we consider

$$(H^j Rf_{*Hdg}(O_X, F_b), H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{an}}, H^j Rf_* \alpha(X)) \in MHM_{k, gm}(S) \subset \text{PSh}_{\mathcal{D}(1,0)fil}(S) \times_I P_{fil, k}(S_{\mathbb{C}}^{an})$$

where

$$H^j Rf_* \alpha(X) : H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{an}} \xrightarrow{\sim} DR(S)(o_{fil} H^j Rf_{*Hdg}(O_X, F_b) \otimes_k \mathbb{C}) = H^j \int_f O_X \otimes_k \mathbb{C}$$

Recall the geometric mixed Hodge modules (see [5]) are mixed Hodge modules by a theorem of Saito for proper morphisms of smooth complex varieties.

Let  $X \in \text{SmVar}(k)$ . We have for  $j, d \in \mathbb{N}$ , the generalized Jacobian

$$J_{\sigma}^{j,d}(X) := H^j(X_{\mathbb{C}}^{an}, \mathbb{C}) / (F^d H^j(X_{\mathbb{C}}^{an}, \mathbb{C}) \oplus H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}))$$

where  $F$  is given the Hodge filtration on  $H_{DR}^j(X) \otimes_k \mathbb{C}$  and

$$H^j R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X)) : H^j(X_{\mathbb{C}}^{an}, \mathbb{C}) \xrightarrow{\sim} H_{DR}^j(X) \otimes_k \mathbb{C}.$$

If  $X \in \text{PSmVar}(k)$  and  $2d \geq n$ ,  $J_{\sigma}^{j,d}(X)$  is a complex torus since

$$((H_{DR}^j(X), F), H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}), H^j R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X))) \in HM_{k, gm}(k) \subset MHM_{k, gm}(k)$$

is a pure Hodge structure. For  $X \in \text{PSmVar}(k)$ , we have a canonical isomorphism of abelian groups

$$I_{\sigma}^{j,d}(X) : J_{\sigma}^{j,d}(X) \xrightarrow{\sim} \text{Ext}_{MHM_{k, gm}(k)}^1(\mathbb{Z}_{\text{pt}}^{Hdg}(d), (H_{DR}^j(X), H^j(X_{\mathbb{C}}^{an}, \mathbb{Z}), H^j R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X)))).$$

**Definition 1.** Let  $X \in \text{SmVar}(k)$  irreducible. Let  $\bar{X} \in \text{PSmVar}(k)$  a compactification of  $X$  with  $D := \bar{X} \setminus X \subset \bar{X}$  a normal crossing divisor. The map of complexes of abelian groups (see [4])

$$\mathcal{R}_X : \mathcal{Z}^d(X, \bullet) \rightarrow C_{\bullet}^{\mathcal{D}}(\bar{X}_{\mathbb{C}}^{an}, D_{\mathbb{C}}^{an}), Z \mapsto \mathcal{R}_X := (T_Z, \Omega_Z, R_Z)$$

where  $C_{\bullet}^{\mathcal{D}}(\bar{X}_{\mathbb{C}}^{an}, D_{\mathbb{C}}^{an})$  is the Deligne homology complex induces the complex Abel Jacobi map for higher Chow groups

$$AJ_{\sigma}(X) : \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0} \rightarrow J_{\sigma}^{2d-1-n, d}(X), Z \mapsto AJ_{\sigma}(X)(Z) := D^{-1}(R'_Z), \\ R'_Z = R_Z - \Omega'_Z + T'_Z, ; \text{ with } \partial T'_Z = T_Z, \partial \Omega'_Z = \Omega_Z$$

where

$$D : C_{\bullet}^{\mathcal{D}}(X_{\mathbb{C}}^{an}) \rightarrow C_{\bullet}^{\mathcal{D}}(\bar{X}_{\mathbb{C}}^{an}, D_{\mathbb{C}}^{an})$$

is the Poincare dual for Deligne homology.

**Theorem 1.** *Let  $k \subset \mathbb{C}$  a subfield. Let  $X \in \text{PSmVar}(k)$ .*

(i) *For  $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0}$ , we have*

$$\begin{aligned} AJ_\sigma(X)(Z) &= I_\sigma^{j,d}(X)^{-1}(0 \rightarrow (H_{DR}^{2d-1-n}(X), H_{\text{sing}}^{2d-1-n}(X_{\mathbb{C}}^{\text{an}}, \mathbb{Z}), H^{2d-1-n}R\Gamma(X_{\mathbb{C}}^{\text{an}}, \alpha(X))) \\ &\quad \xrightarrow{(j^*, j^*, 0)} \\ & (H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|), H_{\text{sing}}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{\text{an}}, \mathbb{Z}), H^{2d-1-n}R\Gamma((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{\text{an}}, \alpha(X \times \square^n))^{[Z]} \\ &\quad \xrightarrow{(\partial, \partial, 0)} \\ & (H_{DR, |Z|}^{2d}(X \times \square^n), H_{\text{sing}, |Z|}^{2d}((X \times \square^n)_{\mathbb{C}}^{\text{an}}, \mathbb{Z}), R\Gamma_{|Z|}((X \times \square^n)_{\mathbb{C}}^{\text{an}}, \alpha(X \times \square^n)))^{[Z]} = \mathbb{Z}_{\text{pt}}^{\text{Hdg}}(n-d) \rightarrow 0) \end{aligned}$$

where  $j : (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$  is the open embedding and

$$H_{\text{Hdg}, |Z|}^{2d}(X \times \square^n)^{[Z]} \subset H_{\text{Hdg}, |Z|}^{2d}(X \times \square^n), \quad H_{\text{Hdg}}^{2d-1}((X \times \square^n) \setminus |Z|)^{[Z]} \subset H_{\text{Hdg}}^{2d-1}((X \times \square^n) \setminus |Z|).$$

are the subobjects given by the pullback of the class of  $Z$  (see section 2).

(ii) *Let  $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0}$ . Then  $AJ_\sigma(X)(Z) = 0$  if and only if there exist  $w \in H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|)^{[Z]}$  such that*

- $w \in F^d H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|)$ ,
- $ev((X \times \square^n) \setminus |Z|)(w) \in H_{\text{sing}}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{\text{an}}, \mathbb{Z}(2i\pi))$
- $\partial w \neq 0$ .

*Proof.* (i): See [6].

(ii): Follows from (i) □

Let  $f : X \rightarrow S$  a morphism with  $S, X \in \text{SmVar}(k)$ . We have for  $j, d \in \mathbb{N}$  such that  $H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}}$  is a local system and  $F^d H^j Rf_* \text{Hdg}(O_X, F_b) \subset H^j \int_f(O_X)$  are locally free sub  $O_S$  modules, the generalized relative intermediate Jacobian

$$J_\sigma^{j,d}(X/S) := (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \otimes O_{S_{\mathbb{C}}^{\text{an}}}) / (F^d (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \otimes O_{S_{\mathbb{C}}^{\text{an}}}) \oplus H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{\text{an}}})$$

where  $F$  is given by the Hodge filtration on  $H^j \int_f(O_X)$  and

$$H^j Rf_* \alpha(X) : H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \xrightarrow{\sim} DR(S)(H^j \int_f(O_X)).$$

A generalized normal function is then a section  $\nu \in \Gamma(S_{\mathbb{C}}^{\text{an}}, J_\sigma^{j,d}(X/S))$  which is horizontal (i.e.  $\nabla \nu = 0$ ). For  $s \in S$ , we get immediately that  $i_s^{* \text{mod}} J_\sigma^{j,d}(X/S) = J_\sigma^{j,d}(X_s)$ . In particular we get for  $f : X \rightarrow S$  a smooth projective morphism with  $S, X \in \text{SmVar}(k)$  and  $j, d \in \mathbb{N}$ , the relative intermediate Jacobian

$$J_\sigma^{j,d}(X/S) := (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \otimes O_{S_{\mathbb{C}}^{\text{an}}}) / (F^d (H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}} \otimes O_{S_{\mathbb{C}}^{\text{an}}}) \oplus H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{\text{an}}})$$

where  $F$  is given by the Hodge filtration on  $H^j \int_f(O_X) = H^j Rf_* \Omega_{X/S}^\bullet$  and  $H^j Rf_* \alpha(X)$ . A normal function is then a section  $\nu \in \Gamma(S_{\mathbb{C}}^{\text{an}}, J_\sigma^{j,d}(X/S))$  which is horizontal. For  $f : X \rightarrow S$  a smooth projective morphism with  $S, X \in \text{SmVar}(k)$ , we have a canonical isomorphism of abelian groups

$$I_\sigma^{j,d}(X/S) : J_\sigma^{j,d}(X/S) \xrightarrow{\sim} \text{Ext}_{\text{MHM}(S)}^1(\mathbb{Z}_S^{\text{Hdg}}(d), (H^j Rf_* \text{Hdg}(O_X, F_b), H^j Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{\text{an}}}, H^j Rf_* \alpha(X))).$$

**Definition-Proposition 2.** *Let  $f : X \rightarrow S$  a morphism with  $S, X \in \text{SmVar}(k)$ . Let  $j : S^o \hookrightarrow S$  an open subset such that for all  $j, d \in \mathbb{Z}$ ,  $j^* H^j Rf_* \mathbb{C}_{X_{\mathbb{C}}^{\text{an}}}$  is a local system and  $j^* F^d H^j Rf_* \text{Hdg}(O_X, F_b) \subset$*

$j^*H^j \int_f(O_X)$  is a locally free sub  $O_S$  module. Let  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding. Let  $d, n \in \mathbb{N}$ . We have then, denoting  $X^o := X \times_S S^o$  and using definition 1, the map

$$AJ_\sigma(X^o/S^o) : \mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0} \rightarrow \Gamma(S_{\mathbb{C}}^{o, an}, J_\sigma^{2d-n-1, d}(X^o/S^o)) \subset \Gamma(S_{\mathbb{C}}^{o, an}, \bigoplus_{s \in S_{\mathbb{C}}^{o, an}} i_{s*} J_{\sigma'}^{2d-n-1, d}(X_s)),$$

$$Z \mapsto AJ_\sigma(X^o/S^o)(Z) := \nu_Z := ((s \in S_{\mathbb{C}}^o) \mapsto (AJ_{\sigma'}(X_s)(Z_s) \in J_{\sigma'}^{2d-n-1, d}(X_s)))$$

where  $\mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0} \subset \mathcal{Z}^d(X, n)^{f, \partial=0}$  denote the sub-abelian group consisting of algebraic cycles  $Z$  with  $Z_s := i_s^* Z \in \mathcal{Z}^d(X_s, n)_{hom}^{\partial=0}$ , and  $\sigma' : k(s) \hookrightarrow \mathbb{C}$  is the embedding given by  $s$  extending  $\sigma : k \hookrightarrow \mathbb{C}$ , denoting again  $s := \pi k/\mathbb{C}(S)(s) \in S$ ,  $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \rightarrow S$  being the projection.

*Proof.* Standard : to show that

$$\nu_Z := (s \in S_{\mathbb{C}}^o) \mapsto AJ_\sigma(X_s)(Z_s) \in \Gamma(S_{\mathbb{C}}^{o, an}, \bigoplus_{s \in S_{\mathbb{C}}^{o, an}} i_{s*} J_\sigma^{2d-n-1, d}(X_s))$$

is holomorphic and horizontal we consider a compactification  $\bar{f} : \bar{X} \rightarrow S$  of  $f$  with  $\bar{X} \in \text{SmVar}(k)$  and use trivializations of  $f : (\bar{X}_{\mathbb{C}}^{o, an}, (\bar{X} \setminus X)_{\mathbb{C}}^{o, an}) \rightarrow S_{\mathbb{C}}^{o, an}$  which gives trivialization of the local system  $j^*Rf_*\mathbb{Z}_{S_{\mathbb{C}}^{o, an}}$  (see [4] for example).  $\square$

**Corollary 1.** *Let  $f : X \rightarrow S$  a smooth projective morphism with  $S, X \in \text{SmVar}(k)$ . Let  $d, n \in \mathbb{N}$ . For  $Z \in \mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0}$ , we have*

$$AJ_\sigma(X/S)(Z) = I_\sigma^{j, d}(X/S)^{-1}(0 \rightarrow (H^{2d-1-n}Rf_{*Hdg}(O_X, F_b), H^{2d-1-n}Rf_*\mathbb{Z}_{X_{\mathbb{C}}^{an}}, H^{2d-1-n}Rf_*\alpha(X))$$

$$\xrightarrow{(j^*, j^*, 0)} (H^{2d-1}R(f \circ j)_{*Hdg}(O_{(X \times \square^n) \setminus |Z|}, F_b), H^{2d-1}R(f \circ j)_{*\mathbb{Z}_{((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{an}}},$$

$$H^{2d-1}R(f \circ j)_{*\alpha}((X \times \square^n) \setminus |Z|)^{[Z]} \xrightarrow{(\partial, \partial, 0)}$$

$$(H^{2d}Rf_{*Hdg}R\Gamma_{|Z|}^{Hdg}(O_{X \times \square^n}, F_b), (H^{2d}Rf_*R\Gamma_{|Z|}\mathbb{Z}_{(X \times \square^n)_{\mathbb{C}}^{an}}), Rf_*\Gamma_{|Z|}\alpha(X \times \square^n)^{[Z]} = \mathbb{Z}_S^{Hdg}(n-d) \rightarrow 0)$$

where  $j : (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$  is the open embedding and

$$(H^{2d}Rf_{*Hdg}R\Gamma_{|Z|}^{Hdg}\mathbb{Z}_{X \times \square^n}^{Hdg})^{[Z]} \subset H^{2d}Rf_{*Hdg}R\Gamma_{|Z|}^{Hdg}\mathbb{Z}_{X \times \square^n}^{Hdg},$$

$$(H^{2d}R(f \circ j)_{*Hdg}\mathbb{Z}_{(X \times \square^n) \setminus |Z|}^{Hdg})^{[Z]} \subset H^{2d}R(f \circ j)_{*Hdg}\mathbb{Z}_{(X \times \square^n) \setminus |Z|}^{Hdg}.$$

are the subobjects given by the pullback of the class of  $Z$ .

*Proof.* Follows from theorem 1 by definition of the Abel Jacobi map and by the base change for mixed hodge modules.  $\square$

We have the following main result of [3] :

**Theorem 2.** *Let  $f : X \rightarrow S$  a smooth projective morphism with  $S, X \in \text{SmVar}(k)$ . Let  $d, n \in \mathbb{N}$ . Let  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding. For  $Z \in \mathcal{Z}^d(X, n)_{f_{hom}}^{f, \partial=0}$ , the zero locus  $V(\nu_Z) \subset S_{\mathbb{C}}$  of*

$$\mu_Z := AJ_\sigma(X/S)(Z) \in \Gamma(S_{\mathbb{C}}^{an}, J_\sigma^{2d-1-n, d}(X/S))$$

is an algebraic subvariety.

*Proof.* See [3]: if  $\bar{S} \in \text{PSmVar}(k)$  is a compactification of  $S$  with  $\bar{S} \setminus S = \cup_i D_i \subset \bar{S}$  a normal crossing divisor, there exist an analytic subset  $\Sigma(\nu_Z) \subset \bar{S}_{\mathbb{C}}$  such that  $V(\nu_Z) = \Sigma(\nu_Z) \cap S_{\mathbb{C}}$ . By GAGA  $\Sigma(\nu_Z) \subset \bar{S}_{\mathbb{C}}$  is algebraic subvariety. Hence  $V(\nu_Z) \subset S_{\mathbb{C}}$  is an algebraic subvariety.  $\square$

## 4.2 The etale Abel Jacobi map for higher Chow group and etale normal functions

Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $\bar{k}$  the algebraic closure of  $k$  and denote by  $G = Gal(\bar{k}/k)$  its galois group. Let  $p \in \mathbb{N}$  a prime integer.

**Definition 2.** Let  $X \in \text{SmVar}(k)$  irreducible. Let  $\bar{X} \in \text{PSmVar}(k)$  a compactification of  $X$  with  $D := \bar{X} \setminus X \subset \bar{X}$  a normal crossing divisor. Denote  $G = Gal(\bar{k}/k)$  the absolute galois group. The cycle class map

$$\mathcal{R}_X^{et,p} : \mathcal{Z}^d(X, n)^{\partial=0} \rightarrow H_{p_X(|Z|), et}^{2d-n}(X, D, \hat{\mathbb{Z}}_p) \rightarrow H_{et}^{2d-n}(X, D, \hat{\mathbb{Z}}_p),$$

to continuous etale cohomology induces the etale Abel Jacobi map for higher Chow groups

$$AJ_{et,p}(X) : \mathcal{Z}^d(X, n)_{hom}^{\partial=0} \rightarrow \text{Ext}_G^1(\bar{k}, H_{et}^{2d-1-n}(X_{\bar{k}}, D_{\bar{k}}, \mathbb{Z}_p)),$$

$$Z \mapsto AJ_{et,p}(X)(Z) := L^1\mathcal{R}_X^{et,p}(Z)/L^2\mathcal{R}_X^{et,p}(Z),$$

where  $L$  is the filtration given by the Leray spectral sequence of the map of sites  $a_X :: X^{et} \rightarrow \text{Spec}(k)^{et}$ .

**Theorem 3.** Let  $X \in \text{PSmVar}(k)$ . Denote  $G = Gal(\bar{k}/k)$  the absolute galois group.

(i) For  $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$ , we have

$$AJ_{et,p}(X)(Z) = (0 \rightarrow H_{et}^{2d-1-n}(X_{\bar{k}}, \mathbb{Z}_p) \xrightarrow{j^*} H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} \xrightarrow{\partial} H_{|Z|, et}^{2d}((X \times \square^n)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} = \bar{k}(n-d) \rightarrow 0)$$

with  $j : (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$  the open embedding, and

$$H_{|Z|, et}^{2d}((X \times \square^n)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} \subset H_{|Z|, et}^{2d}((X \times \square^n)_{\bar{k}}, \mathbb{Z}_p),$$

$$H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)^{[Z]} \subset H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)$$

are the subobjects given by the pullback by the class of  $Z$  (see section 2).

(ii) Let  $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$ . Then  $AJ_{et,p}(X)(Z) = 0$  if and only if there exist

$$\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)^{[Z]}$$

such that  $\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p)^G$  and  $\partial\alpha \neq 0$ .

*Proof.* (i): See [14].

(ii): Follows from (i). □

**Definition 3.** Let  $f : X \rightarrow S$  a morphism with  $S, X \in \text{SmVar}(k)$ . Let  $j : S^\circ \hookrightarrow S$  an open subset such that for all  $j \in \mathbb{Z}$ ,  $j^*H^j Rf_*\mathbb{Z}_{p, X_{\bar{k}}}$  is a local system. Let  $d, n \in \mathbb{N}$ . We have then, denoting  $X^\circ := X \times_S S^\circ$  and using definition 2, the map

$$AJ_{et,p}(X^\circ/S^\circ) : \mathcal{Z}^d(X, n)_{f, hom}^{f, \partial=0} \rightarrow \Gamma(S^\circ, \bigoplus_{s \in S^\circ_{(0)}} i_{s*} \text{Ext}_{Gal(\bar{k}/k(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s, \bar{k}}, \mathbb{Z}_p))),$$

$$Z \mapsto AJ_{et,p}(X^\circ/S^\circ)(Z) := \nu_Z^{et,p} :=$$

$$((s \in S^\circ_{(0)}) \mapsto (AJ_{et,p}(X_s)(Z_s) \in \text{Ext}_{Gal(\bar{k}/k(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s, \bar{k}}, \mathbb{Z}_p))))$$

where  $\mathcal{Z}^d(X, n)_{f, hom}^{f, \partial=0} \subset \mathcal{Z}^d(X, n)^{f, \partial=0}$  denote the subabelian group consisting of algebraic cycles  $Z$  with  $Z_s := i_s^*Z \in \mathcal{Z}^d(X_s, n)_{hom}$ . Recall that  $i_s : \{s\} \hookrightarrow S_{(0)} \subset S$  is a closed Zariski point of  $S$ .

We now localize, for each prime number  $l$  and each embedding  $\sigma_l : k \hookrightarrow \mathbb{C}_l$  the definition given above.

**Definition 4.** Let  $X \in \text{SmVar}(k)$  irreducible. Let  $\bar{X} \in \text{PSmVar}(k)$  a compactification of  $X$  with  $D := \bar{X} \setminus X \subset \bar{X}$  a normal crossing divisor. Let  $\sigma_l : k \hookrightarrow \mathbb{C}_l$  an embedding. Then  $k \subset \bar{k} \subset \mathbb{C}_l$ , where  $\bar{k}$  is the algebraic closure of  $k$  and  $k \subset \hat{k}_{\sigma_l} \subset \mathbb{C}_l$  where  $\hat{k}_{\sigma_l}$  is the completion of  $k$  with respect to  $\sigma_l$ . Denote  $\hat{G}_{\sigma_l} := \text{Gal}(\mathbb{C}_l \hat{k}_{\sigma_l})$ . The cycle class map

$$\mathcal{R}_{X, \sigma_l}^{et, p} : \mathcal{Z}^d(X, n)^{\partial=0} \rightarrow H_{p_X(|Z|), et}^{2d-n}(X_{\hat{k}_{\sigma_l}}, D_{\hat{k}_{\sigma_l}}, \hat{\mathbb{Z}}_p) \rightarrow H_{et}^{2d-n}(X_{\hat{k}_{\sigma_l}}, D_{\hat{k}_{\sigma_l}}, \hat{\mathbb{Z}}_p),$$

to continuous etale cohomology induces the etale Abel Jacobi map for higher Chow groups

$$\begin{aligned} AJ_{et, p, \sigma_l}(X) : \mathcal{Z}^d(X, n)_{hom}^{\partial=0} &\rightarrow \text{Ext}_{\hat{G}_{\sigma_l}}^1(\mathbb{C}_l, H_{et}^{2d-1-n}(X_{\mathbb{C}_l}, D_{\mathbb{C}_l}, \mathbb{Z}_p)), \\ Z &\mapsto AJ_{et, p, \sigma_l}(X)(Z) := L^1 \mathcal{R}_{X, \sigma_l}^{et, p}(Z) / L^2 \mathcal{R}_{X, \sigma_l}^{et, p}(Z), \end{aligned}$$

where  $L$  is the filtration given by the Leray spectral sequence of the map of sites  $a_X :: X_{\hat{k}_{\sigma_l}}^{et} \rightarrow \text{Spec}(\hat{k}_{\sigma_l})^{et}$ .

We have then the commutative diagram

$$\begin{array}{ccc} & & \text{Ext}_G^1(\bar{k}, H_{et}^{2d-1-n}(X_{\bar{k}}, D_{\bar{k}}, \mathbb{Z}_p)) \\ & \nearrow^{AJ_{et, p}(X)} & \downarrow \text{Ext}^1(\text{ad}(\pi_{k/\hat{k}_{\sigma_l}}^*, \pi_{k/\hat{k}_{\sigma_l}^*})(-), \text{ad}(\pi_{k/\hat{k}_{\sigma_l}}^*, \pi_{k/\hat{k}_{\sigma_l}^*})(-)) \\ \mathcal{Z}^d(X, n)_{hom}^{\partial=0} & & \\ & \searrow_{AJ_{et, p, \sigma_l}(X)} & \\ & & \text{Ext}_{\hat{G}_{\sigma_l}}^1(\mathbb{C}_l, H_{et}^{2d-1-n}(X_{\mathbb{C}_l}, D_{\mathbb{C}_l}, \mathbb{Z}_p)) = \text{Ext}_{\hat{G}_{\sigma_l}}^1(\bar{k}, H_{et}^{2d-1-n}(X_{\bar{k}}, D_{\bar{k}}, \mathbb{Z}_p)) \end{array}$$

where the right column arrow is given by the restriction  $\pi_{k/\hat{k}_{\sigma_l}} : \hat{G}_{\sigma_l} \hookrightarrow G$ .

**Theorem 4.** Let  $X \in \text{PSmVar}(k)$ . Let  $\sigma_l : k \hookrightarrow \mathbb{C}_l$  an embedding and  $k \subset \hat{k}_{\sigma_l} \subset \mathbb{C}_l$  the completion of  $k$  with respect to  $\sigma_l$ .

(i) For  $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$ , we have

$$\begin{aligned} AJ_{et, p, \sigma_l}(X)(Z) &= (0 \rightarrow H_{et}^{2d-1-n}(X_{\mathbb{C}_l}, \mathbb{Z}_p) \xrightarrow{j^*} H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]} \\ &\xrightarrow{\partial} H_{|Z|, et}^{2d}((X \times \square^n)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]} = \mathbb{C}_l(n-d) \rightarrow 0) \end{aligned}$$

with  $j : X \times \square^n \setminus |Z| \hookrightarrow X$  the open embedding and

$$\begin{aligned} H_{|Z|, et}^{2d}((X \times \square^n)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]} &\subset H_{|Z|, et}^{2d}((X \times \square^n)_{\mathbb{C}_l}, \mathbb{Z}_p), \\ H_{|Z|, et}^{2d}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]} &\subset H_{|Z|, et}^{2d}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p), \end{aligned}$$

are the subobjects given by the pullback by the class of  $Z$  (see section 2).

(ii) Let  $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$ . Then  $AJ_{et, p, \sigma_l}(X)(Z) = 0$  if and only if there exist

$$\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p)^{[Z]}$$

such that  $\partial\alpha \neq 0$  and  $\alpha \in H_{et}^{2d-1}(((X \times \square^n) \setminus |Z|)_{\mathbb{C}_l}, \mathbb{Z}_p)^{\hat{G}_{\sigma_l}}$ .

*Proof.* Similar to the proof of theorem 3. □

**Definition 5.** Let  $f : X \rightarrow S$  a morphism with  $S, X \in \text{SmVar}(k)$ . Let  $j : S^\circ \hookrightarrow S$  an open subset such that for all  $j \in \mathbb{Z}$ ,  $j^* H^j Rf_* \mathbb{Z}_{p, X_{\bar{k}}}$  is a local system. Let  $d, n \in \mathbb{N}$ . Let  $\sigma_l : k \hookrightarrow \mathbb{C}_l$  an embedding and  $k \subset \hat{k}_{\sigma_l} \subset \mathbb{C}_l$  the completion of  $k$  with respect to  $\sigma_l$ . Denoting  $X^\circ := X \times_S S^\circ$ , we consider

$$\begin{aligned} AJ_{et,p,\sigma_l}(X^\circ/S^\circ) : \mathcal{Z}^d(X, n)_{f_{hom}}^{f,\partial=0} &\rightarrow \Gamma(S^\circ, \bigoplus_{s \in S^\circ_{(0)}} i_{s*} \text{Ext}^1(\text{Gal}(\mathbb{C}_l/\hat{k}_{\sigma_l}(s)), H_{et}^{2d-n-1}(X_{s,\bar{k}}, \mathbb{Z}_p))), \\ Z &\mapsto AJ_{et,p,\sigma_l}(X^\circ/S^\circ)(Z) := \nu_{Z,\sigma_l}^{et,p} := \\ &((s \in S^\circ_{(0)}) \mapsto (AJ_{et,p,\sigma_l}(X_s)(Z_s) \in \text{Ext}_{\text{Gal}(\mathbb{C}_l/\hat{k}_{\sigma_l}(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s,\bar{k}}, \mathbb{Z}_p)))) \end{aligned}$$

where  $\mathcal{Z}^d(X, n)_{f_{hom}}^{f,\partial=0} \subset \mathcal{Z}^d(X, n)^{f,\partial=0}$  denote the subabelian group consisting of algebraic cycles  $Z$  with  $Z_s := i_s^* Z \in \mathcal{Z}^d(X_s, n)_{hom}$ . Recall that  $i_s : \{s\} \hookrightarrow S_{(0)} \subset S$  is a closed Zariski point of  $S$ .

## 5 The vanishing of the etale Abel Jacobi map implies the vanishing of the complex Abel Jacobi map

The  $p$  adic Hodge theory for open varieties implies the following main theorem :

**Theorem 5.** Let  $k$  a field of finite type over  $\mathbb{Q}$ . Denote  $\bar{k}$  the algebraic closure of  $k$  and  $G = \text{Gal}(\bar{k}/k)$  its absolute Galois group. Let  $X \in \text{PSmVar}(k)$ . Consider an embedding  $\sigma : k \hookrightarrow \mathbb{C}$ . Let  $p \in \mathbb{N}$  a prime number. Let  $Z \in \mathcal{Z}^d(X, n)_{hom}^{\partial=0}$ . Consider the exact sequences

- $0 \rightarrow H_{et}^{2d-1-n}(X_{\bar{k}}, \mathbb{Z}_p) \xrightarrow{j^*} H_{et}^{2d-1}((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p \xrightarrow{\partial} H_{et,|Z|}^{2d}((X \times \square^n)_{\bar{k}}, \mathbb{Z}_p)^0 \rightarrow 0$
- $0 \rightarrow H_{DR}^{2d-1-n}(X) \xrightarrow{j^*} H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|) \xrightarrow{\partial} H_{DR,|Z|}^{2d}(X \times \square^n)^0 \rightarrow 0,$

where  $j : (X \times \square^n) \setminus |Z| \hookrightarrow X \times \square^n$  is the open embedding. Consider the following assertions :

- (i)  $AJ_{et,p}(X)(Z) = 0 \in \text{Ext}_1^G(\bar{k}, H_{et}^{2d-1-n}(X_{\bar{k}}, \mathbb{Z}_p)(d-n)),$
- (i)' there exist  $\alpha \in H_{et}^{2d-1}((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p(d)^{[Z]}$  such that  $\alpha \in H_{et}^{2d-1}((X \times \square^n) \setminus |Z|)_{\bar{k}}, \mathbb{Z}_p(d)^G$  and  $\partial \alpha \neq 0,$
- (ii) there exist  $w \in H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|)^{[Z]}$  such that  $w \in F^d H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|),$   
 $w \in H^{2d-1} OL_{X^{et}}(\mathbb{H}_{et}^{2d-1}((X \times \square^n) \setminus |Z|, \Omega_{X^{et}, \log}^\bullet)),$   
and  $\partial w \neq 0,$
- (iii) there exist  $w \in H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|)^{[Z]}$  such that  $w \in F^d H_{DR}^{2d-1}((X \times \square^n) \setminus |Z|),$   
 $H^{2d-1} ev((X \times \square^n) \setminus |Z|)(w) \in H_{\text{sing}}^{2d-1}((X \times \square^n) \setminus |Z|)_{\mathbb{C}}^{an}, 2i\pi\mathbb{Z},$   
and  $\partial w \neq 0,$
- (iii)'  $AJ_\sigma(X)(Z) = 0 \in J_\sigma^{2d-1-n,d}(X),$

where the inclusion  $OL_{X^{et}} : \Omega_{X^{et}, \log}^\bullet \hookrightarrow \Omega_{X^{et}}^\bullet$  of  $C(X^{et})$  is the subcomplex of logarithmic forms and  $e_X : X^{et} \rightarrow X$  is the canonical morphism of site (see section 2). Then (i) is equivalent to (i)', (i)' implies (ii), (ii) implies (iii), (iii) is equivalent to (iii)'. Hence (i) implies (iii)'.

*Proof.* (i) is equivalent to (i)': see theorem 3(ii),

(i)' implies (ii): follows from proposition 2(ii): we take an embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  and  $w = w(\alpha),$   
 $w_L = w_L(\alpha),$

(ii) implies (iii): follows from proposition 1,

(iii) is equivalent to (iii)': see theorem 1(ii). □

It implies the following :

**Corollary 2.** (i) Let  $k$  a field of finite type over  $\mathbb{Q}$ . Denote  $\bar{k}$  the algebraic closure of  $k$ . Let  $f : X \rightarrow S$  a smooth projective morphism with  $S, X \in \text{SmVar}(k)$ . Consider an embedding  $\sigma : k \hookrightarrow \mathbb{C}$ . Let  $p \in \mathbb{N}$  a prime number. Then for  $Z \in \mathcal{Z}^d(X, n)_{f_{\text{hom}}}^{f, \partial=0}$ , we have

$$V(\nu_Z^{et,p})_{\mathbb{C}} \subset V(\nu_Z) \subset S_{\mathbb{C}}, V_{\text{tors}}(\nu_Z^{et,p})_{\mathbb{C}} \subset V_{\text{tors}}(\nu_Z) \subset S_{\mathbb{C}}$$

where

–  $V(\nu_Z) \subset V_{\text{tors}}(\nu_Z) \subset S_{\mathbb{C}}$  is the zero locus, resp. torsion locus, of the complex normal function

$$\nu_Z =: AJ_{\sigma}(X/S)(Z) \in \Gamma(S_{\mathbb{C}}^{an}, J_{\sigma}^{2d-1-n,d}(X/S))$$

associated to  $Z$  (see proposition-definition 2),

–  $V(\nu_Z^{et,p}) \subset V_{\text{tors}}(\nu_Z^{et,p}) \subset S$  is the zero locus, resp. torsion locus of the etale normal function

$$\nu_Z^{et,p} \in \Gamma(S, \bigoplus_{s \in S_{(0)}} i_{s*} \text{Ext}_{\text{Gal}(\bar{k}/k(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s,\bar{k}}, \mathbb{Z}_p))(d-n))$$

associated to  $Z$  (see definition 3) and

$$V(\nu_Z^{et,p})_{\mathbb{C}} := \pi_{k/\mathbb{C}}(S)^{-1}(V(\nu_Z^{et,p})), V_{\text{tors}}(\nu_Z^{et,p})_{\mathbb{C}} := \pi_{k/\mathbb{C}}(S)^{-1}(V_{\text{tors}}(\nu_Z^{et,p}))$$

where we recall  $\pi_{k/\mathbb{C}}(S) : S_{\mathbb{C}} \rightarrow S$  is the projection.

(ii) Let  $\sigma : k \hookrightarrow \mathbb{C}$  a subfield of finite type over  $\mathbb{Q}$ . Let  $f : X \rightarrow S$  a smooth projective morphism with  $S, X \in \text{SmVar}(k)$ . Then for  $Z \in \mathcal{Z}^d(X, n)_{f_{\text{hom}}}^{f, \partial=0}$ , the zero locus  $V(\nu_Z) \subset S_{\mathbb{C}}$  of the complex normal function

$$\nu_Z =: AJ_{\sigma}(X/S)(Z) \in \Gamma(S_{\mathbb{C}}^{an}, J_{\sigma}^{2d-1-n,d}(X/S))$$

associated to  $Z$  is defined over  $\bar{k}$  if  $V(\nu_Z^{et,p}) \neq \emptyset$ .

*Proof.* (i): Follows immediately from theorem 5 since for  $s \in S_{(0)}$ ,  $k(s)$  is of finite type over  $\mathbb{Q}$  and for  $s' \in \pi_{k/\mathbb{C}}(S)^{-1}(s)$  denoting  $\sigma' : k(s) \hookrightarrow \mathbb{C}$  the embedding given by  $s'$ , we have by definition

- $\nu_Z(s') =: AJ_{\sigma'}(X/S)(Z)(s') := AJ_{\sigma'}(X_s)(Z_s) \in J_{\sigma'}^{2d-1-n,d}(X_s)$ .
- $\nu_Z^{et,p}(s) =: AJ_{et,p}(X/S)(Z)(s) := AJ_{et,p}(X_s)(Z_s) \in \text{Ext}_{\text{Gal}(\bar{k}/k(s))}^1(\bar{k}, H_{et}^{2d-n-1}(X_{s,\bar{k}}, \mathbb{Z}_p))(d-n)$ .

(ii): Since  $V(\nu_Z^{et,p}) \subset S$  contain a  $\bar{k}$  point,  $V(\nu_Z) \subset S_{\mathbb{C}}$  contain a  $\bar{k}$  point by (i). Hence by the work of [17] or [8],  $V(\nu_Z) \subset S_{\mathbb{C}}$  is defined over  $\bar{k}$ .  $\square$

## 6 Algebraicity of the zero locus of etale normal functions

Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $f : X \rightarrow S$  a smooth proper morphism with  $S, X \in \text{SmVar}(k)$  connected. Let  $p$  a prime number. Let  $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0, f}$ . By the definition, we have

$$V(\nu_Z^{et,p}) \subset \bigcap_{l \in \mathbb{N}, l \text{ prime}} \bigcap_{\sigma_l : k \hookrightarrow \mathbb{C}_l} V(\nu_{Z, \sigma_l}^{et,p}) \subset S_{(0)}$$

and

$$V_{\text{tors}}(\nu_Z^{et,p}) \subset \bigcap_{l \in \mathbb{N}, l \text{ prime}} \bigcap_{\sigma_l : k \hookrightarrow \mathbb{C}_l} V_{\text{tors}}(\nu_{Z, \sigma_l}^{et,p}) \subset S_{(0)}.$$

In this section, we investigate the algebraicity of  $V_{\text{tors}}(\nu_Z^{et,p}) \subset S$  and of  $V_{\text{tors}}(\nu_{Z, \sigma_l}^{et,p}) \subset S$ ,  $\sigma_l : k \hookrightarrow \mathbb{C}_l$ .



**Remark 3.** Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $f : X \rightarrow S$  a smooth proper morphism with  $S, X \in \text{SmVar}(k)$  connected. Let  $p$  a prime number. Let  $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0, f}$ . We can show, using [13], that we have in fact

$$V_{\text{tors}}(\nu_Z^{\text{et}, p}) = \bigcap_{l \in \mathbb{N}, l \text{ prime}} \bigcap_{\sigma_l: k \hookrightarrow \mathbb{C}_l} V_{\text{tors}}(\nu_{Z, \sigma_l}^{\text{et}, p}) \subset S_{(0)}.$$

We don't need this result so we don't give the details.

Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $p$  be a prime number. Let  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  be an embedding. We have then  $\hat{k}_{\sigma_p}$  the completion of  $k$  with respect to  $\sigma_p$  and we denote  $O_{\hat{k}_{\sigma_p}} \subset \hat{k}_{\sigma_p}$  its ring of integers. We then consider the canonical functor of Huber (see section 2)

$$\mathcal{R} : \text{Var}(\hat{k}_{\sigma_p}) \rightarrow \text{HubSp}(\hat{k}_{\sigma_p}, O_{\hat{k}_{\sigma_p}}) \rightarrow \text{Sch}/O_{\hat{k}_{\sigma_p}}, X \mapsto \mathcal{R}(X) = X^{\mathcal{O}}$$

which associated to a variety over a  $p$  adic field its canonical integral model. Let  $f : X \rightarrow S$  a smooth projective morphism with  $S, X \in \text{SmVar}(k)$ . Let  $Z \subset X$  a closed subset and  $j : U = X \setminus Z \hookrightarrow X$  the open complementary subset. We have then  $f := f_{\hat{k}_{\sigma_p}} : (X, Z)_{\hat{k}_{\sigma_p}} \rightarrow S_{\hat{k}_{\sigma_p}}$  the morphism in  $\text{SmVar}^2(\hat{k}_{\sigma_p})$  induced by the scalar extension functor and

$$f^{\mathcal{O}} := \mathcal{R}(f_{\hat{k}_{\sigma_p}}) : (X, Z)_{\hat{k}_{\sigma_p}}^{\mathcal{O}} \rightarrow S_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$$

its canonical integral model in  $\text{Sch}^2/O_{\hat{k}_{\sigma_p}}$  to which we denote

$$f := f^{\mathcal{O}} : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}}) := (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, (M_Z, N_{\mathcal{O}})) \rightarrow (S_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{\mathcal{O}})$$

the corresponding morphism in  $\text{logSch}$ , where for  $K$  a  $p$  adic field and  $Y \in \text{Sch}/O_K$ ,  $(Y, N_{\mathcal{O}}) := (Y, M_{Y_k})$  with  $k = O_K/(\pi)$  the residual field. We have then the morphisms of sites

$$v_{X, N} : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})^{\text{Falt}} \rightarrow (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})^{\text{ket}}, u_{X, N} : (X_{\mathbb{C}_p}, M_{Z_{\mathbb{C}_p}})^{\text{ket}} \rightarrow (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})^{\text{Falt}}$$

where  $(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})^{\text{Falt}}$  denote the Falting site, and for  $(Y, N) \in \text{logSch}$ ,  $(Y, N)^{\text{ket}} \subset \text{logSch}/(Y, N)$  is the small Kummer etale site. If  $(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})$  is log smooth, we consider an hypercover

$$a_{\bullet} : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}, \bullet}, N_{U, \mathcal{O}}) \rightarrow (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})$$

in  $\text{Fun}(\Delta, \text{logSch})$  by small log schemes in sense of [1]. The main result of [1] say that

- if  $(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})$  is log smooth, the embedding in  $C((X_{\hat{k}_{\sigma_p}}^{\mathcal{O}})^{\text{Falt}})$

$$\alpha(U) : (\mathbb{B}_{\text{st}, X_{\hat{k}_{\sigma_p}}}, N_{U, \mathcal{O}}) \hookrightarrow a_{\bullet*} DR(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}, \bullet}/S_{\hat{k}_{\sigma_p}}^{\mathcal{O}})(O_{\mathbb{B}_{\text{st}, X_{\hat{k}_{\sigma_p}}}}, N_{U, \mathcal{O}})$$

is a filtered quasi-isomorphism compatible with the action of  $\text{Gal}(\mathbb{C}_p, \hat{k}_{\sigma_p})$ , the Frobenius  $\phi_p$  and the monodromy  $N$ ,

- if  $f : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}}) \rightarrow (S_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{\mathcal{O}})$  is log smooth, the morphism in  $D_{\text{fil}}((S_{\hat{k}_{\sigma_p}}^{\mathcal{O}})^{\text{Falt}})$

$$\begin{aligned} T(f, \mathbb{B}_{\text{st}}) : Rf_* (\mathbb{B}_{\text{st}, X_{\hat{k}_{\sigma_p}}}, N_{U, \mathcal{O}}) &\xrightarrow{T(f, f, \otimes)(-) \circ \text{ad}(u_{X, N}^*, Ru_{X, N, *})(-)} Rf_* \mathbb{Z}_{p, (X_{\mathbb{C}_p}, M_{Z_{\mathbb{C}_p}})^{\text{ket}}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}} \\ &\xrightarrow{\text{ad}(j^*, Rj_*)(\mathbb{Z}_{p, (X_{\mathbb{C}_p}, M_{Z_{\mathbb{C}_p}})^{\text{et}}})} R(f \circ j)_* \mathbb{Z}_{p, U_{\mathbb{C}_p}^{\text{et}}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}} \end{aligned}$$

is an isomorphism, where the last map is an isomorphism by [12] theorem 7.4.

This gives if  $(X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}})$  is log smooth, for each  $j \in \mathbb{Z}$ , a filtered isomorphism of filtered abelian groups

$$H^j R\alpha(U) : H_{\text{et}}^j(U_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{st}, \hat{k}_{\sigma_p}} \xrightarrow{H^j T(a_X, \mathbb{B}_{\text{st}})^{-1}} H_{\text{et}}^j((X, N)^{\text{Falt}})(\mathbb{B}_{\text{st}, X_{\hat{k}_{\sigma_p}}}, N_{U, \mathcal{O}}) \\ \xrightarrow{H^j \text{RG}((X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}}), \alpha(U))} H_{\text{DR}}^j(U_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{\text{st}, \hat{k}_{\sigma_p}}$$

compatible with the action of  $\text{Gal}(\mathbb{C}_p/\hat{k}_{\sigma_p})$ , of the Frobenius  $\phi_p$  and the monodromy  $N$ . More generally, this gives if  $\hat{k}_{\sigma_p}(s)$  is unramified for all  $s \in S_{\hat{k}_{\sigma_p}}$  and if  $f : (X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{U, \mathcal{O}}) \rightarrow (S_{\hat{k}_{\sigma_p}}^{\mathcal{O}}, N_{\mathcal{O}})$  is log smooth, an isomorphism in  $\text{Shv}_{\text{fil}, G, \phi_p, N}(S_{\hat{k}_{\sigma_p}})$

$$H^j f_* \alpha(U) : R^j f_* \mathbb{Z}_{p, U_{\mathbb{C}_p}^{\text{et}}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}} \xrightarrow{H^j T(f, \mathbb{B}_{\text{st}})^{-1}} Rf_*(\mathbb{B}_{\text{st}, X_{\hat{k}_{\sigma_p}}}, N_{U, \mathcal{O}}) \\ \xrightarrow{H^j Rf_* \alpha(U)} H^j \int_f j_* \text{Hdg}(O_{U_{\hat{k}_{\sigma_p}}}, F_b) \otimes_{O_{S_{\hat{k}_{\sigma_p}}}} O_{\mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}}}$$

that is a filtered isomorphism compatible with the action of  $G = \text{Gal}(\mathbb{C}_p/\hat{k}_{\sigma_p})$ , of the Frobenius  $\phi_p$  and the monodromy  $N$ , writing for short again  $f = f \circ j$ .

**Definition 6.** Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $f : X \rightarrow S$  a smooth proper morphism with  $S, X \in \text{SmVar}(k)$ . Let  $p$  a prime number. Let  $Z \in \mathcal{Z}(X, n)_{f, \text{hom}}^{f, \partial=0}$ . Denote  $U := (X \times \square^n) \setminus |Z|$ . We have then the following exact sequence in  $\text{Vect}_{\mathcal{D}\text{-fil}}(S)$  (of filtered vector bundle with integrable connexion)

$$0 \rightarrow E_{\text{DR}}(X/S) := H^{2d-1-n} \int_f (O_X, F_b) \xrightarrow{j^*} E_{\text{DR}}(U/S)^{[Z]} := (H^{2d-1} \int_f j_* \text{Hdg}(O_U, F_b))^{[Z]} \\ \xrightarrow{\partial} E_{\text{DR}, Z}(X/S)^{[Z]} := (H^{2d} \int_f \Gamma_{|Z|}^{\text{Hdg}}(O_{X \times \square^n}, F_b))^{[Z]} \rightarrow 0.$$

Recall (see section 2) that  $(X \times \square^n, M_{|Z|}) \in \text{logVar}(k)$  denote log structure associated to  $(X \times \square^n, |Z|) \in \text{SmVar}^2(k)$ . There exist a finite set of prime numbers  $\delta(S)$  such that for all prime  $p \in \mathbb{N} \setminus \delta(S)$ , all embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  and all  $s \in S_{(0)}$ ,  $\hat{k}_{\sigma_p}(s)$  is unramified over  $\mathbb{Q}_p$ . Let  $p \in \mathbb{N} \setminus \delta(S)$  a prime number and  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  be an embedding.

- Take, using [10] theorem 8.2 (after considering an integral model  $(X, Z)^{\mathcal{R}} \in \text{Sch}^2/\mathcal{R}$  over  $\mathcal{R} \subset k$  of finite type over  $\mathbb{Z}$  with function field  $k$ ), for  $p \in \mathbb{N} \setminus \delta^0(X/S)$ , where  $\delta^0(X/S)$  is a finite set, an alteration  $\pi^0 : (X \times \square^n)^0 \rightarrow X \times \square^n$ , that is a generically finite morphism such that  $((X \times \square^n)^0, \pi^{0, -1}(|Z|))_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$  is semi-stable pair, that is  $\pi^{0, -1}(|Z|) \subset ((X \times \square^n)^0)$  is a normal crossing divisor and  $((X \times \square^n)^0, \pi^{0, -1}(|Z|))_{\hat{k}_{\sigma_p}}^{\mathcal{O}})_t$  has semi-stable reduction where  $t \in \text{Spec}(O_{\hat{k}_{\sigma_p}})$  is the closed point. Then there exists a closed subset  $\Delta \subset S$  such that for all  $s \in S^{\circ} := S \setminus \Delta$ ,  $((X \times \square^n)^0, \pi^{0, -1}(|Z|))_{\hat{k}_{\sigma_p}, s}^{\mathcal{O}}$  is a semi-stable pair.
- Take using [10] theorem 8.2, (after considering an integral model  $(X_{\Delta}, Z_{\Delta})^{\mathcal{R}} \in \text{Sch}^2/\mathcal{R}$  over  $\mathcal{R} \subset k$  of finite type over  $\mathbb{Z}$  with function field  $k$ ), for  $p \in \mathbb{N} \setminus \delta^1(X/S)$ , where  $\delta^1(X/S)$  is a finite set, an alteration  $\pi^1 : (X_{\Delta} \times \square^n)^1 \rightarrow X_{\Delta} \times \square^n$  such that  $((X_{\Delta} \times \square^n)^1, |Z_{\Delta}|)_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$  is a semi-stable pair. Then there exists a closed subset  $\Delta^2 \subset \Delta$  such that for all  $s \in S^1 := \Delta \setminus \Delta^2$ ,  $((X_{\Delta} \times \square^n)^1, |Z_{\Delta}|)_{\hat{k}_{\sigma_p}, s}^{\mathcal{O}}$  is a semi-stable pair.
- Go on by induction.

We obtain by the above finite induction, for  $p \in \mathbb{N} \setminus \delta(S, X/S)$ , with  $\delta(S, X/S) := \delta(S) \cup (\cup_{\alpha \in \Lambda} \delta^{\alpha}(X/S))$ , a stratification  $S = \sqcup_{\alpha \in \Lambda} S^{\alpha}$ ,  $\Lambda$  being a finite set, by locally closed subset  $S^{\alpha} \subset S$ , and alterations (i.e.

generically finite morphisms)  $\pi^\alpha : (X_{S^\alpha} \times \square^n)^\alpha \rightarrow X_{S^\alpha} \times \square^n$  such that

$$f \circ \pi^\alpha : (X_{S^\alpha} \times \square^n)_{\hat{k}_{\sigma_p}}^{\alpha, \mathcal{O}}, N_{U^\alpha, \mathcal{O}} \rightarrow (S_{\hat{k}_{\sigma_p}}^\alpha, N_{\mathcal{O}})$$

is log smooth, that is for all  $s \in S^\alpha$ ,  $((X_{S^\alpha} \times \square^n)^\alpha, \pi^{\alpha, -1}(|Z_{S^\alpha}|))_{\hat{k}_{\sigma_p}, s}^{\mathcal{O}}$  is a semi-stable pair. For each  $\alpha \in \Lambda$  we then set

$$T^\alpha := p_{S^\alpha}(\overline{((E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha) \otimes_{O_{S_{\hat{k}_{\sigma_p}}}} O_{\mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}}})^{\phi_p, N})^k} \\ \cap F^d E_{DR}(U^\alpha / S^\alpha) \cap \pi^{\alpha, -1}(E_{DR}(U_{S^\alpha} / S^\alpha)^{[Z_{S^\alpha}]}) \setminus E_{DR}((X \times \square^n)_{S^\alpha}^\alpha / S^\alpha)) \subset S^\alpha$$

and

$$\hat{T}_{\sigma_p}^\alpha := p_{S_{\hat{k}_{\sigma_p}}^\alpha}(\overline{((E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha) \otimes_{O_{S_{\hat{k}_{\sigma_p}}}} O_{\mathbb{B}_{st, S_{\hat{k}_{\sigma_p}}}})^{\phi_p, N}} \cap \\ F^d E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha) \cap \pi^{\alpha, -1}(E_{DR}(U_{S_{\hat{k}_{\sigma_p}}^\alpha} / S_{\hat{k}_{\sigma_p}}^\alpha)^{[Z_{S^\alpha}]}) \setminus E_{DR}((X \times \square^n)_{S_{\hat{k}_{\sigma_p}}^\alpha}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha)) \subset S_{\hat{k}_{\sigma_p}}^\alpha$$

where

- $U^\alpha := (X_{S^\alpha} \times \square^n)^\alpha \setminus \pi^{\alpha, -1}(|Z_{S^\alpha}|)$ ,
- $E_{DR}(U^\alpha / S^\alpha) := H^{2d-1} Rf_{*Hdg}(O_{U^\alpha}, F_b) \in \text{Vect}_{fil}(S^\alpha)$ ,  $E_{DR}(U_{S^\alpha} / S^\alpha)^{[Z_{S^\alpha}]} \subset E_{DR}(U_{S^\alpha} / S^\alpha)$  is the filtered subbundle given by the pullback of the class of  $[Z_{S^\alpha}]$ ,
- $\phi_p$  is the Frobenius operator,  $N$  is the monodromy operator, and  $(-)^k$  denote the  $k$ -Zariski closure inside  $E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha)$ , (note that for  $K'/k'$  a field extension,  $Y \in \text{Var}(k')$  and  $R \subset Y_{K'}$  a closed subset, we have  $\overline{R}^{k'}(k') = R(k')$ ),
- $p_{S^\alpha} : E_{DR}(U^\alpha / S^\alpha) \rightarrow S^\alpha$  and  $p_{S_{\hat{k}_{\sigma_p}}^\alpha} : E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha) \rightarrow S_{\hat{k}_{\sigma_p}}^\alpha$  are the projections.

By [11] theorem 1.8.4,

- $T^\alpha \subset S^\alpha$  is a constructible algebraic subset since it is the image of constructible algebraic subsets of the algebraic vector bundle  $E_{DR}(U^\alpha / S^\alpha)$  by the morphism  $p_{S^\alpha}$ ,
- $\hat{T}_{\sigma_p}^\alpha \subset S_{\hat{k}_{\sigma_p}}^\alpha$  is a constructible algebraic subset since it is the image of constructible algebraic subsets of the algebraic vector bundle  $E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha)$  by the morphism  $p_{S_{\hat{k}_{\sigma_p}}^\alpha}$ .

Hence,

$$T := \sqcup_{\alpha \in \Lambda} T^\alpha \subset S, \quad \hat{T}_{\sigma_p} := \sqcup_{\alpha \in \Lambda} \hat{T}_{\sigma_p}^\alpha \subset S_{\hat{k}_{\sigma_p}}$$

are constructible algebraic subsets.

**Lemma 3.** Let  $G$  be a group. Consider a commutative diagram of  $G$  modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & W & \longrightarrow & V & \xrightarrow{\partial} & K & \longrightarrow & 0 \\ & & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & & \\ 0 & \longrightarrow & W' & \longrightarrow & V' & \xrightarrow{\partial'} & K' & \longrightarrow & 0 \end{array}$$

whose rows are exact sequence and  $\pi^* : V \rightarrow V'$  is injective. Let  $\alpha \in V$ . Then  $\alpha \in V^G$  and  $\partial\alpha \neq 0$  if and only if  $\pi^*\alpha \in V'^G$  and  $\partial'\pi^*\alpha \neq 0$ .

*Proof.* Follows from the fact that  $\langle \alpha \rangle$  define a splitting  $W \oplus \langle \alpha \rangle \subset V$  of  $G$  modules.  $\square$

**Theorem 6.** *Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $f : X \rightarrow S$  a smooth proper morphism with  $S, X \in \text{SmVar}(k)$  connected. Let  $p \in \mathbb{N} \setminus \delta(S, X/S)$  a be prime number, where  $\delta(S, X/S)$  is the finite set given in definition 6. Let  $Z \in \mathcal{Z}^d(X, n)_{\text{hom}}^{\partial=0, f}$ .*

(i) *We have*

$$V_{\text{tors}}(\nu_Z^{\text{et}, p}) = T \cap S_{(0)} \subset S,$$

where  $T \subset S$  is given in definition 6.

(ii) *For each embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ , we have*

$$V_{\text{tors}}(\nu_{Z, \sigma_p}^{\text{et}, p})_{\hat{k}_{\sigma_p}} = \hat{T}_{\sigma_p} \cap S_{(0), \hat{k}_{\sigma_p}} \subset S_{\hat{k}_{\sigma_p}},$$

where  $\hat{T}_{\sigma_p} \subset S_{\hat{k}_{\sigma_p}}$  is given in definition 6, and for  $V \subset S$  a subset,  $V_{\hat{k}_{\sigma_p}} := \pi_{k/\hat{k}_{\sigma_p}}(S)^{-1}(V)$ , where  $\pi_{k/\hat{k}_{\sigma_p}}(S) : S_{\hat{k}_{\sigma_p}} \rightarrow S$  being the projection.

*Proof.* Let  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  be an embedding. For each  $\alpha \in \Lambda$ , by the semi-stable comparison theorem for

$$f^\alpha := f \circ \pi^\alpha : ((X_{S^\alpha} \times \square^n)^{\alpha, \mathcal{O}}, N_{U^\alpha}^{\mathcal{O}}) \rightarrow (S_{\hat{k}_{\sigma_p}}^{\alpha, \mathcal{O}}, N_{\mathcal{O}})$$

([1]) which is log smooth, we have the isomorphism in  $\text{Shv}_{\text{fil}, G, \phi_p, N}(S_{\hat{k}_{\sigma_p}}^{\alpha})$

$$H^j f_*^\alpha \alpha(U^\alpha) : R^j f_*^\alpha \mathbb{Z}_{p, U_{\mathbb{C}_p}^{\alpha, \text{et}}} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}^\alpha} \xrightarrow{\sim} H^j \int_{f^\alpha} j_* \text{Hdg}(O_{U_{\hat{k}_{\sigma_p}}^\alpha}, F_b) \otimes_{O_{S_{\hat{k}_{\sigma_p}}^\alpha}} O_{\mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}^\alpha}}, \quad (1)$$

recall that since  $p \in \mathbb{N} \setminus \delta(S)$ ,  $\hat{k}_{\sigma_p}(s)$  is unramified for all  $s \in S_{\hat{k}_{\sigma_p}}$ .

(i): We have

$$\begin{aligned} T^\alpha &:= pS^\alpha \left( \overline{((E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha) \otimes_{O_{S_{\hat{k}_{\sigma_p}}^\alpha}} O_{\mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}^\alpha})}^{\phi_p, N})} \right)^k \\ &\cap F^d E_{DR}(U^\alpha / S^\alpha) \cap \pi^{\alpha, -1}(E_{DR}(U_{S^\alpha} / S^\alpha)^{[Z_{S^\alpha}]}) \setminus E_{DR}((X \times \square^n)_{S^\alpha}^\alpha / S^\alpha) \\ &= pS^\alpha \left( \overline{((E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha) \otimes_{O_{S_{\hat{k}_{\sigma_p}}^\alpha}} O_{\mathbb{B}_{\text{st}, S_{\hat{k}_{\sigma_p}}^\alpha})}^{\phi_p, N})} \right)^k \\ &\cap F^d E_{DR}(U^\alpha / S^\alpha) \cap \pi^{\alpha, -1}(E_{DR}(U_{S^\alpha} / S^\alpha)^{[Z_{S^\alpha}]}) \setminus E_{DR}((X \times \square^n)_{S^\alpha}^\alpha / S^\alpha) \subset S^\alpha \end{aligned}$$

since for  $Y \subset E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha)$  we have  $\bar{Y}^k(k) = Y(k)$  and  $E_{DR}(U_{\hat{k}_{\sigma_p}}^\alpha / S_{\hat{k}_{\sigma_p}}^\alpha)(k) = E_{DR}(U^\alpha / S^\alpha)$ . Let  $s \in S_{(0)}^\alpha$ . Denote  $G := \text{Gal}(\bar{k}/k(s))$ .

- The map  $\pi_s^\alpha : (X_s \times \square^n)^\alpha \rightarrow X_s \times \square^n$  is generically finite since  $\pi^\alpha : (X \times \square^n)^\alpha \rightarrow X \times \square^n$  is generically finite and  $f$  is flat. Thus by lemma 3 and theorem 3,  $\nu_Z^{\text{et}, p}(s) := AJ^{\text{et}, p}(X_s)(Z_s) = 0$  if and only if there exists

$$\alpha \in H_{\text{et}}^{2d-1}(U_{S^\alpha, s}, \mathbb{Z}_p)^{[Z_s]}$$

such that  $\pi^{\alpha, -1}(\alpha) \in H_{\text{et}}^{2d-1}(U_s^\alpha, \mathbb{Z}_p)(d)^G$  and  $\partial \pi^{\alpha, -1}(\alpha) \neq 0$ .

- On the other hand by (1) and lemma 1,  $\alpha' \in H_{\text{et}}^{2d-1}(U_s^\alpha, \mathbb{Z}_p)(d)^G$  if and only if  $w(\alpha') \in F^d H_{DR}^{2d-1}(U_s^\alpha)$  and

$$w(\alpha') \in (H_{DR}^{2d-1}(U_{s, \hat{k}_{\sigma_p}}^\alpha) \otimes_{\hat{k}_{\sigma_p}(s)} \mathbb{B}_{\text{st}})^{\phi_p, N}.$$

Moreover  $w(\pi^{\alpha, -1}(\alpha)) = \pi^{\alpha, -1}(w(\alpha))$ .

(ii): Let  $s \in S_{(0)}^\alpha$  and  $s' \in \pi_{k/\hat{k}_{\sigma_p}}(S)^{-1}(s)$ .

- Since  $\pi_s^\alpha : (X_s \times \square^n)^\alpha \rightarrow X_s \times \square^n$  is generically finite (see the proof of (i)) we have by lemma 3 and theorem 4,  $\nu_{Z, \sigma_p}^{et, p}(s) := AJ_{\sigma_p}^{et, p}(X_s)(Z_s) = 0$  if and only if there exists

$$\alpha \in H_{et}^{2d-1}(U_{S^\alpha, s, \hat{k}_{\sigma_p}}, \mathbb{Z}_p)^{[Z_s]}$$

such that  $\pi^{\alpha, -1}(\alpha) \in H_{et}^{2d-1}(U_{s, \hat{k}_{\sigma_p}}^\alpha, \mathbb{Z}_p)(d)^{\hat{G}_{\sigma_p}}$  and  $\partial\alpha \neq 0$ .

- On the other hand by (1),  $\alpha' \in H_{et}^{2d-1}(U_{s, \hat{k}_{\sigma_p}}^\alpha, \mathbb{Z}_p)(d)^{\hat{G}_{\sigma_p}}$  if and only if  $w(\alpha') \in F^d H_{DR}^{2d-1}(U_{\hat{k}_{\sigma_p}, s}^\alpha)$  and

$$w(\alpha') \in (H_{DR}^{2d-1}(U_{s, \hat{k}_{\sigma_p}}^\alpha) \otimes_{\hat{k}_{\sigma_p}(s)} \mathbb{B}_{st})^{\phi_p, N} = (H_{DR}^{2d-1}(U_{s', \hat{k}_{\sigma_p}}^\alpha) \otimes_{\hat{k}_{\sigma_p}(s)} \mathbb{B}_{st})^{\phi_p, N}.$$

Moreover  $w(\pi^{\alpha, -1}(\alpha)) = \pi^{\alpha, -1}(w(\alpha))$ .

□

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