ON THE SEMI-CLASSICAL ANALYSIS OF SCHRÖDINGER OPERATORS WITH LINEAR ELECTRIC POTENTIALS ON A BOUNDED DOMAIN

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ABSTRACT. The aim of this paper is to establish the asymptotic expansion of the eigenvalues of the Stark Hamiltonian, with strong uniform electric field and Dirichlet boundary conditions on a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$. This work aims at generalizing the recent results of Cornean, Krejcirik, Pedersen, Raymond and Stockmeyer in dimension 2. More precisely, in dimension N, in the strong electric field limit, we derive, under certain local convexity conditions, a full asymptotic expansion of the low-lying eigenvalues. To establish our main result, we perform the construction of quasi-modes. The "optimality" of our constructions is then established thanks to a reduction to model operators and localization estimates. In addition, we apply our techniques to find a full asymptotic expansion of the low-lying eigenvalues of more general operator $-h^2\Delta + V(\mathbf{x})$ where h is a small parameter and V is a smooth real potential satisfies some additional conditions.

1. INTRODUCTION

1.1. **A Stark Hamiltonian.** Consider a system of one-particle moving in a uniform electric field $\mathbf{F}$. In a simplified model one may consider independently electrons or holes.The total energy Stark Hamiltonian (Schrödinger operator) for such a system takes the following form:

$$\frac{-\hbar^2}{2m}\Delta + q\mathbf{F}\cdot\mathbf{x} + V_{\text{conf}}, \tag{1.1}$$

where

- $\hbar$ is Plank constant divided by 2π,
- $m > 0$ is the effective mass of the particle (electron or hole),
- q is the charge of the particle (electron or hole),
- V_{conf} is some confining potential.

The Hamiltonian in (1.1) describes an important phenomenon in semiconductor physics which is the *quantum confined Stark effect*, where has been considered in many remarkable works [5, 4, 21, 23, 19, 13, 22, 16]. The recent paper [17] provides more details on the physical orientation of this phenomenon.

In this paper, we work in the simplest case where we model the confinement potential as infinite potential walls, i.e. we consider the first two terms of the Hamiltonian in (1.1) restricted to the confinement domain with Dirichlet boundary condition. We are interested in the study of the low-lying eigenvalues of the following N-dimensional Hamiltonian restricted to an open set Ω of $\mathbb{R}^N$, $N \geq 2$:

$$\mathscr{L}_h := -h^2\Delta + x_1 = -h^2(\partial_{x_1}^2 + \ldots + \partial_{x_N}^2) + x_1, \tag{1.2}$$

acting on a dense subspace of the square integral functions $L^2(\Omega)$ with Dirichlet boundary conditions.
This operator appears in the physics semi-conductors where h is given by

$$h = \frac{\hbar}{\sqrt{2mqF}}\,.$$

Notice that we have chosen coordinates such that $\mathbf{F}$ is parallel to the x_1-axis.

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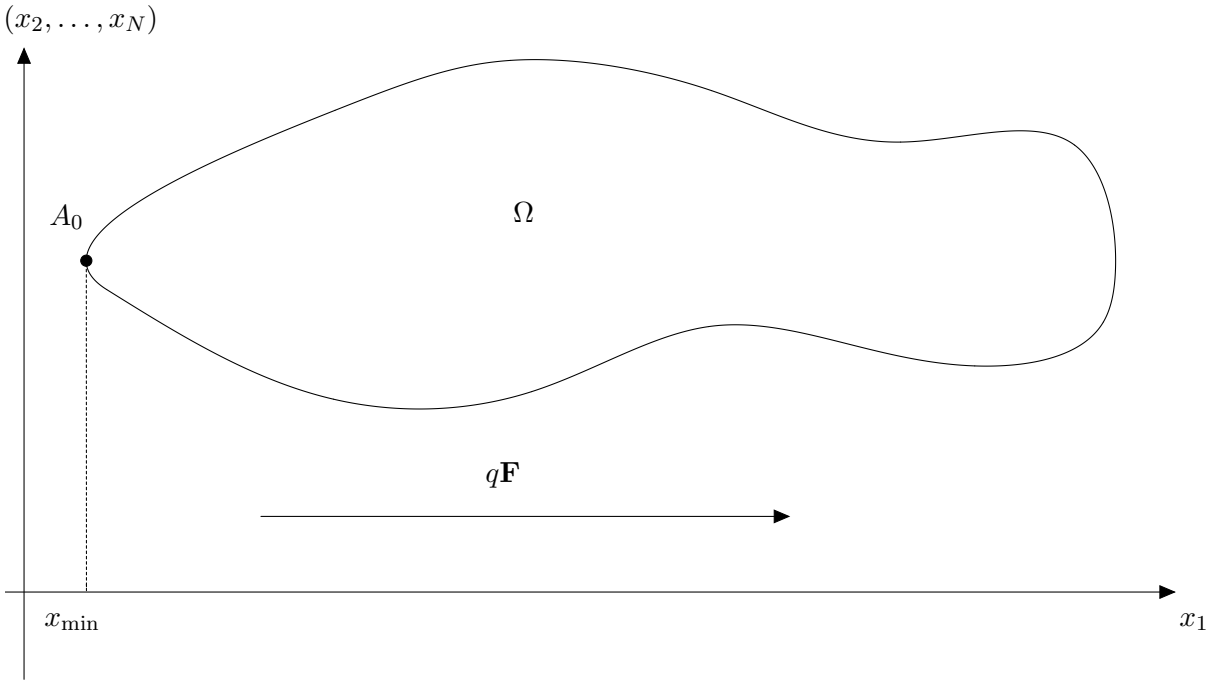
The low-lying eigenvalues of this operator have been studied, partially numerically, for different geometries in several papers such as in [18, 20] for squares, in [23] for rectangles, in [19, 15, 13] for disks, and in [9] for annul. Concerning the analysis in the case of imaginary electric fields, the reader might also want to consider [1, 2, 8].

For now, we will work with domains satisfying the following conditions.
Assumption: Let Ω is an open bounded and connected domain of $\mathbb{R}^N$ with a smooth boundary. We assume that there exists a unique point $A_0 \in \partial\Omega$ such that the first component of A_0 is given by:

$$x_{\min} = \inf_{(x_1,x_2,\ldots,x_N)\in\Omega} x_1 = \min_{\mathbf{x}\in\overline{\Omega}} x_1 \,.$$

We also assume that Ω strictly convex domain near A_0. We resume this assumption with this figure.



To be more precise, the operator $\mathscr{L}_h$ is defined via the Lax-Milgram theorem, from the closed and semi-bounded quadratic form

$$\mathscr{Q}_h(\varphi) = h^2 \int_\Omega |\nabla\varphi(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} + \int_\Omega x_1|\varphi(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}, \quad \forall\varphi \in H^1_0(\Omega)\,.$$

The domain of $\mathscr{L}_h$ is contained in $H^1_0(\Omega)$ and $\mathscr{L}_h$ acts as in (1.2). This is the Dirichlet realization of the Stark Hamiltonian. If $\mathscr{Q}_h(\cdot,\cdot)$ is the bilinear form associated with the quadratic form $\mathscr{Q}_h(\cdot)$, then

$$\mathrm{Dom}(\mathscr{L}_h) = \{\varphi \in H^1_0(\Omega) : \psi \longmapsto \mathscr{Q}_h(\varphi,\psi) \text{ is continuous for the } L^2(\Omega)-\text{topology}\}\,.$$

Since Ω is bounded and regular, then the embedding of $H^1_0(\Omega)\hookrightarrow L^2(\Omega)$ is compact, therefore the self-adjoint operator $\mathscr{L}_h$ has compact resolvent. Its spectrum is purely discrete formed by a non-decreasing sequence of eigenvalues denoted by $\big(\lambda_n(h)\big)_{n\geq 1}$, where each eigenvalue is repeated according to its multiplicity. The purpose of this paper is to understand the behavior as $h \to 0$ (in the limit of strong electric field) of the eigenvalues $\lambda_n(h)$.

1.2. **Main result.** In [5], by constructing suitable test functions, and applying the min-max principle, the authors found a three-term asymptotic expansion of the low-lying eigenvalues. It is proved that, in $\mathbb{R}^2$, for every fixed

$n \in \mathbb{N}^*$,

$$\lambda_n(h) = x_{\min} + z_1 h^{\frac{2}{3}} + h(2n-1)\sqrt{\frac{\kappa_0}{2}} + \mathscr{O}(h^{\frac{4}{3}}) \quad \text{as} \quad h \to 0\,. \tag{1.3}$$

where $z_1 \approx 2.338$ is the absolute value of the smallest zero of the Airy function and κ_0 is the positive curvature at A_0 .
The aim of this paper is to improve the asymptotic expansion in (1.3) and to give a asymptotic completeness of $\lambda_n(h)$. More precisely, we establish a full asymptotic expansion of the low-lying eigenvalues, by involving the eigenvalues of the Weingarten map of $\partial\Omega$ at A_0 defined by

$$W_{A_0} = \mathrm{d}_{A_0}\nu : T_{A_0}\partial\Omega \to T_{A_0}\partial\Omega \tag{1.4}$$

where, for all $p \in \partial\Omega$, $\nu(p)$ is the outward-pointing normal to $T_p\partial\Omega$.

Remark 1.1. From Section 7.2, the assumption on Ω induces that the matrix of W_{A_0} is positive definite. Their positives eigenvalues $(\kappa_i)_{i\geq 1}$ are related to the curvature of the boundary.

(1) The κ_i are called principal curvatures of $\partial\Omega$ at A_0 .
(2) $\det W_{A_0}$ is the Gaussian curvature from $\partial\Omega$ at A_0 .
(3) $\dfrac{\mathrm{Tr}\, W_{A_0}}{N-1}$ is the mean curvature from $\partial\Omega$ at A_0.

The main result in this paper is:

Theorem 1.1. *For any positive n, there exists a real sequence $\{\zeta_{j,n}\}_{j\geq 0}$ such that, for any positive $M \in \mathbb{N}$, we have*

$$\lambda_n(h) \underset{h\to 0}{=} x_{\min} + z_1 h^{\frac{2}{3}} + \beta_{1,n} h + h^{4/3} \sum_{j=0}^{M} h^{j/6}\zeta_{j,n} + \mathscr{O}\big(h^{\frac{4}{3}+\frac{M+1}{6}}\big)\,,$$

with $\beta_{1,n}$ is the n-th eigenvalue of the harmonic oscillator in dimension $N-1$

$$\sum_{i=1}^{N-1} -\partial_{u_i}^2 + \frac{\kappa_i}{2}\, u_i^2\,,$$

where κ_i are the eigenvalues of the Weingarten map of $\partial\Omega$ at A_0.

This result is essentially a generalization of that stated in (1.3), but the proof presented here, based on the construction of quasimodes. The "optimality" of our constructions is then established thanks to decay estimates and the reduction to model operators involves a 1-dimensional Airy operator in $\mathbb{R}_+$ and a harmonic oscillator in dimension $N-1$.

Remark 1.2. $\beta_{1,n}$ is the n-th element counted with multiplicity of

$$\Sigma = \left\{ \sum_{i=1}^{N-1} \sqrt{\frac{\kappa_i}{2}}\,(2n_i - 1)\,, \quad n \in {\mathbb{N}^*}^{N-1} \right\}.$$

Notice that for $N = 2$ one is reduced to

$$\Sigma = \left\{ \sum_{i=1}^{N-1} (2n-1)\sqrt{\frac{\kappa_0}{2}}, \quad n \in \mathbb{N}^* \right\},$$

and all the elements are of multiplicity one.

Remark 1.3. It is possible to prove asymptotic expansions of the low-lying eigenvalues of the Hamiltonian $\mathscr{L}_h$, if we replace the potential x_1 by $V(\mathbf{x})$, where V is a smooth real potential satisfies some additional conditions. One of this conditions is: V has a unique minimum $\mathbf{x}_0 \in \partial\Omega$, and $\nabla V(\mathbf{x}_0) \neq 0$ with a condition of the sign. Firstly, we can show that, the low energy eigenfunctions will localized around $\mathbf{x}_0$. We can quantify this thanks to classical Agmon estimates and reduce the investigation to an effective Hamiltonian on a tiny domain around

$\mathbf{x}_0$.
Secondly, we can introduce local boundary coordinates in the neighborhood of $\mathbf{x}_0 \in \partial\Omega$ in order to straighten a portion of the boundary. Let ϕ denote the embedding of $\partial\Omega$ in $\mathbb{R}^N$ and G the induced metrics on $\partial\Omega$. If $\mathrm{dist}(\mathbf{x}, \partial\Omega) \leq \delta$, for δ sufficiently small, we introduce the diffeomorphism Φ defined by the formula

$$\mathbf{x} = \Phi(s,t) = \phi(s) - t\nu(\phi(s))\,,$$

where $\nu(\mathbf{x})$ is the unit outward pointing normal vector at the point $\mathbf{x} \in \partial\Omega$, t is the distance to the boundary and $s = (s_1, \ldots, s_{N-1})$ is the tangential variable on $\partial\Omega$. We assume that $\mathbf{x}_0 = \Phi(0,0)$.
Thanks to the Agmon estimates in coordinates (s,t), the operator $\mathscr{L}_h$ is expressed formally in coordinates (s,t) as:

$$\mathscr{L}_h = V(\mathbf{x}_0) - h^2\partial_t^2 - \nabla V(\mathbf{x}_0)\cdot\nu(\mathbf{x}_0)t - h^2\Delta^{G(0)} + \frac{1}{2}\frac{d^2}{ds^2}V(\phi)|_{s=0}(s)(s) + \mathscr{O}(h^{\frac{7}{6}}),$$

where $\Delta^{G(0)}$ is the Laplace-Beltrami operator on $\partial\Omega$ in the neighborhood of $\mathbf{x}_0$.
To appears the Airy operator and the harmonic oscillator, it will assume its conditions:

(1) $\nabla V(\mathbf{x}_0)\cdot\nu(\mathbf{x}_0) < 0$.
(2) $\dfrac{d^2}{ds^2}V(\phi)|_{s=0} := H$ is definite positive matrix, this equivalent to $\mathbf{x}_0$ is a non degenerate minimum of V.

Under this conditions, we can write:

$$\lambda_n(h) \underset{h\to 0}{=} V(\mathbf{x}_0) + z_1(\nabla V(\mathbf{x}_0)\cdot\nu(\mathbf{x}_0))^{\frac{2}{3}}h^{\frac{2}{3}} + \beta_{1,n}h + \mathscr{O}(h^{\frac{7}{6}})\,,$$

where $\beta_{1,n}$ is the n-th eigenvalue of the harmonic oscillator in dimension $N-1$

$$\sum_{i=1}^{N-1} -\partial_{u_i}^2 + \frac{\Lambda_i}{2}u_i^2\,,$$

where Λ_i are the eigenvalues of the matrix $G(0)^{-1}H$. The full asymptotic expansion can be computed, by constructing a sequence of trial states as Section 4.

1.3. **Organization of the paper.** This paper is organized as follows: In Section 2.1, we show that low energy eigenfunctions and their derivatives are exponentially localized around the potential minimum $x_{\min}$. This will allow us to work in a thin tubular neighborhood of A_0. As a consequence, we find an approximated operator $\mathscr{L}_{h,\delta}$, whose low energy eigenvalues are those of $\mathscr{L}_h$ up to an exponentially small error, see Section 2.2. In Section 3, we locally straighten the boundary by introducing a system of local coordinates: " tubular coordinates". The operator $\mathscr{L}_{h,\delta}$ is expressed in this coordinates and acts on a tiny domain around the origin with Dirichlet boundary conditions. In Section 4 and 5, we construct quasimodes, thanks to reduction to model operators and localization estimates, we provide the full asymptotic expansion of the low-lying eigenvalues stated in Theorem 1.1, see Section 6. Section 7 is dedicated to the analysis of simplified model which shall be used as local approximations for operator $\mathscr{L}_{h,\delta}$.

2. Preliminaries

2.1. **Concentration of bound states near the potential minimum.** The following proposition states that the eigenfunctions associated with the low-lying eigenvalues are localized in x_1 near $x_{\min}$ at the scale $h^{2/3}$. For more details on the proofs of the proposition and corollary of this section, see [5].

Proposition 2.1. *Let $M > 0$. There exist $\varepsilon, C, h_0 > 0$ such that, for all $h \in (0, h_0)$, for all eigenevalues λ such that $\lambda \leqslant x_{\min} + Mh^{\frac{2}{3}}$, and all corresponding eigenfunctions ψ,*

$$\int_\Omega e^{\varepsilon|x_1 - x_{\min}|^{\frac{3}{2}/h}}|\psi|^2\,\mathrm{d}\mathbf{x} \leqslant C\|\psi\|^2\,, \tag{2.1}$$

and

$$\int_\Omega e^{\varepsilon|x_1 - x_{\min}|^{\frac{3}{2}/h}}|h\nabla\psi|^2\,\mathrm{d}\mathbf{x} \leqslant Ch^{2/3}\|\psi\|^2\,. \tag{2.2}$$

Let $\eta > 0$, we denote by $\complement B\left(A_0, \eta\right)$ the complement of the open ball $B\left(A_0, \eta\right)$.

Corollary 2.1. *Let $M, \eta > 0$. There exist $c_\eta, C_\eta, h_0 > 0$ such that, if $h \in (0, h_0)$, then any eigenfunction ψ corresponding to an eigenvalue $\lambda \leqslant x_{\min} + Mh^{\frac{2}{3}}$ satisfies the estimates*

$$\int_{\Omega \cap \complement B(A_0,\eta)} |\psi|^2 \, \mathrm{d}\mathbf{x} \leqslant C_\eta e^{-c_\eta/h} \|\psi\|^2 \,,$$

and

$$\int_{\Omega \cap \complement B(A_0,\eta))} |\nabla\psi|^2 \, \mathrm{d}\mathbf{x} \leqslant C_\eta e^{-c_\eta/h} \, \|\psi\|^2 \,.$$

Proof. Since Ω is bounded then the set $\bar{\Omega} \cap \complement B\left(A_0, \eta\right)$ is compact. The map $\bar{\Omega} \cap \complement B\left(A_0, \eta\right) \ni (x_1, \ldots, x_N) \mapsto x_1 - x_{\min}$ is continuous and positive. By compactness it has a positive minimum. The conclusion follows from Proposition 2.1. □

As a consequence, for small h, the ground states of the operator $\mathscr{L}_h$ are concentrated around $x_{\min}$ (cf. Corollary 3.1). We can now reduce our investigation to a neighborhood of A_0 .

2.2. **Reduction near the boundary.** Let $\delta > 0$ small enough, we introduce the δ-neighborhood of the boundary

$$\Omega_\delta = \{\, \mathbf{x} = (x_1, \ldots, x_N) \in \Omega \,:\, \mathrm{dist}(\mathbf{x}, A_0) < \delta \,\} \,.$$

Consider the quadratic form, defined on the variational space

$$W_\delta = \{u \in H^1_0(\Omega_\delta) \,:\, u(\mathbf{x}) = 0 \,,\, \forall\, \mathbf{x} \in \Omega \text{ such that } \mathrm{dist}(\mathbf{x}, A_0) = \delta\} \,,$$

by the formula

$$\mathscr{D}_{h,\delta}(\varphi) = h^2 \int_{\Omega_\delta} |\nabla\varphi(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} + \int_{\Omega_\delta} x_1 |\varphi(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} \,. \tag{2.3}$$

Let the self-adjoint operator associated with the quadratic form $\mathscr{D}_{h,\delta}$ be defined on $L^2(\Omega_\delta)$ and with the Dirichlet condition on the boundary of Ω_δ

$$\mathscr{L}_{h,\delta} = -h^2\Delta + x_1 \,,$$

Let $(\lambda_n(h,\delta))_{n\in\mathbb{N}^*}$ be the sequence of min-max values of the operator $\mathscr{L}_{h,\delta}$. It is standard to deduce from the min-max principle and using the decay estimates of Proposition 2.1 the following proposition (see [10]). It is an important result which compares the eigenvalues and gives an approximation of $\lambda_n(h)$.

Proposition 2.2. *Let $n \geq 1$ and $\delta > 0$ small enough. There exist constants $C_\delta, k_\delta > 0, h_0 \in (0,1)$ such that, for all $h \in (0, h_0)$ and $\lambda_n(h) \leqslant x_{\min} + Mh^{\frac{2}{3}}$,*

$$\lambda_n(h,\delta) \leq \lambda_n(h) + C_\delta \exp(-k_\delta/h \,) \,.$$

Moreover, we have, for all $n \geq 1$, $h > 0$

$$\lambda_n(h) \leq \lambda_n(h,\delta) \,.$$

3. Local boundary coordinates

In this section, we introduce local coordinates in the neighborhood of $\mathbf{x}_0 \in \partial\Omega$, in order to straighten a portion of the boundary. Throughout this section, we mainly refer to [7, App. F] and [11, 14], although these coordinates were also used in [3] in a two-dimensional framework.

3.1. **The coordinates and the metric.** We fix $\epsilon > 0$ such that the distance function

$$t(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \partial\Omega)$$

is smooth on $\Omega_\epsilon = \{\, \mathbf{x} \in \Omega \,:\, \mathrm{dist}(\mathbf{x}, \partial\Omega) < \epsilon \,\}$.
Let $\mathbf{x}_0 \in \partial\Omega$ be fixed and choose a chart $\phi = (\phi_1, \dots, \phi_N) : \mathcal{V}_{\mathbf{x}_0} \to \phi(\mathcal{V}_{\mathbf{x}_0})$ such that $\mathbf{x}_0 \in \phi(\mathcal{V}_{\mathbf{x}_0})$ and $\mathcal{V}_{\mathbf{x}_0}$ is an open set of $\mathbb{R}^{N-1}$. We set

$$s_0 = \phi^{-1}(\mathbf{x}_0)\,.$$

Then, in these coordinates, the metric on the surface $\phi(\mathcal{V}_{\mathbf{x}_0})$ induced by the euclidian metric g_0 of $\mathbb{R}^N$ writes

$$G = \sum_{1 \le i,j \le N-1} G_{ij} ds_i \otimes ds_j\,,$$

where

$$\forall i, j = 1, \dots, N-1, \quad G_{ij}(s_1, \dots, s_{N-1}) = \partial_{s_i}\phi(s_1, \dots, s_{N-1}) \cdot \partial_{s_j}\phi(s_1, \dots, s_{N-1})\,.$$

Now, we define local coordinates in a neighborhood of $\mathbf{x}_0$ in Ω. For $\epsilon > 0$ small enough, we consider

$$\mathcal{V}_{\mathbf{x}_0} \times]0, \epsilon[\ni (s_1, \dots, s_{N-1}, t) = (s, t) \mapsto \Phi(s,t) = (\Phi_1(s,t), \dots, \Phi_N(s,t)) = \mathbf{x} = \phi(s) - \nu(\phi(s))\, t\,, \tag{3.1}$$

where $\nu(\mathbf{x})$ is the unit outward pointing normal vector at the point $\mathbf{x} \in \partial\Omega$.
This defines a diffeomorphism of $\mathcal{V}_{\mathbf{x}_0} \times]0, \epsilon[$ onto $\mathcal{V}_{\mathbf{x}_0}$ in Ω_ϵ and its inverse gives local coordinates on $\mathcal{V}_{\mathbf{x}_0}$. We set

$$U_0 = \Phi(\mathcal{V}_{\mathbf{x}_0} \times]0, \epsilon[)\,.$$

Note that t denotes the normal variable in the sense that for a point $\mathbf{x} \in U_0$ such that

$$t(\mathbf{x}) = \mathrm{dist}\,(\mathbf{x}, \phi(s_1, \dots, s_{N-1})) = \mathrm{dist}(\mathbf{x}, \partial\Omega)\,.$$

In particular, $t = 0$ is the equation of the surface $U_0 \cap \partial\Omega$.
The matrix of the metric g_0 in these new coordinates (s,t) is

$$\mathrm{d}\Phi^T \mathrm{d}\Phi = \left(\begin{array}{c|c} \big(G_{ij}(s) - 2K_{ij}(s)t + t^2 L_{ij}(s)\big)_{i,j=1,\dots,N-1} & 0_{\mathbb{R}^{N-1}} \\ \hline 0^T_{\mathbb{R}^{N-1}} & 1 \end{array}\right),$$

where $K_{ij} = \partial_{s_i}\phi \cdot \partial_{s_j}\nu(\phi)$ and $L_{ij} = \partial_{s_i}\nu(\phi) \cdot \partial_{s_j}\nu(\phi)$ are the coefficients of the second and third fundamental form of $\partial\Omega$.
We conclude that

$$\begin{aligned} g_0 &= dt \otimes dt + \sum_{1 \le i,j \le N-1} \big[G_{ij}(s) - 2tK_{ij}(s) + t^2 L_{ij}(s)\big] ds_i \otimes ds_j \\ &= dt \otimes dt + G - 2tK + t^2 L\,, \end{aligned}$$

we denote by K and L the second and the third fundamental forms on $\partial\Omega$. In the coordinates s and with respect to the canonical basis, their matrices are given by

$$K = \sum_{1 \le i,j \le N-1} K_{ij} ds_i \otimes ds_j \quad \text{and} \quad L = \sum_{1 \le i,j \le N-1} L_{ij} ds_i \otimes ds_j\,.$$

We denote by $g = (g_{ij})_{i,j}$ the matrix of the metric g_0 in the (s,t) coordinates and the determinant of g is denoted by $|g|$. We have

$$|g|^{1/2} = |\mathrm{d}\Phi^T \mathrm{d}\Phi| = \big[\det\,(G - 2tK + t^2 L)\big]^{1/2} = |G|^{1/2} \big[\det\,(I - 2tG^{-1}K + t^2 G^{-1}L)\big]^{1/2}\,.$$

By the differential calculus of the determinant and since t is close to zero, we get

$$|g|^{1/2} = \big(1 - t\,\mathrm{tr}\,(G^{-1}K) + t^2 p(s,t)\big)\,|G|^{1/2}\,, \tag{3.2}$$

where p is a bounded function in the neighborhood $\mathcal{V}_{\mathbf{x}_0} \times]0, \epsilon[$.

3.2. **The operator in boundary coordinates.** For $\varphi \in H_0^1(\Phi^{-1}(\Omega_\delta))$, the quadratic form is written in the new coordinates (s,t) as follows

$$\widetilde{\mathscr{Q}}_{h,\delta}(\varphi) = h^2 \int_{\mathcal{V}_{A_0}\times]0,\delta[} |\partial_t \tilde{\varphi}|^2 |g|^{1/2} \,\mathrm{d}s\mathrm{d}t + h^2 \sum_{1\leq i,j\leq N-1} \int_{\mathcal{V}_{A_0}\times]0,\delta[} g^{ij}\partial_{s_i}\tilde{\varphi}\,\overline{\partial_{s_j}\tilde{\varphi}}\,|g|^{1/2}\,\mathrm{d}s\mathrm{d}t$$
$$+ \int_{\mathcal{V}_{A_0}\times]0,\delta[} \Phi_1(s,t)|\tilde{\varphi}|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t\,,$$

where (g^{ij}) are the coefficients of the inverse matrix of (g_{ij}) and $\tilde{\varphi} = \varphi\circ\Phi$.
The associated operator is $\widetilde{\mathscr{L}}_{h,\delta}$

$$\widetilde{\mathscr{L}}_{h,\delta} = -h^2|g|^{-1/2}\partial_t(|g|^{1/2}\partial_t) - h^2 \sum_{1\leq i,j\leq N-1} |g|^{-1/2}\partial_{s_i}(g^{ij}|g|^{1/2}\partial_{s_j}) + \Phi_1(s,t)\,,$$

acting on $L^2(\mathcal{V}_{A_0}\times]0,\delta[;|g|^{1/2}\,\mathrm{d}s\mathrm{d}t)$.
The operator $\mathscr{L}_{h,\delta}$ is unitary equivalent to the Dirichlet realization of $\widetilde{\mathscr{L}}_{h,\delta}$. From Proposition 2.2, we can focus on the spectral analysis of $\widetilde{\mathscr{L}}_{h,\delta}$.

After a dilation and a translation of the s coordinates of the section 3.1, we may assume that $A_0 = \phi(0)$ and $\nu(A_0) = (-1,0,\dots,0)$. By Taylor expansion near $(0,0)$, for $(s,t)\in\mathcal{V}_{A_0}\times]0,\delta[$, we have

$$\Phi_1(s,t) = x_{\min} + \frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1(s)(s) + t + \mathscr{O}(|s|^3 + t|s|^2). \tag{3.3}$$

3.3. **Agmon Estimates in tubular variables.** The following proposition is a slight adaptation in coordinates (s,t) of Proposition 2.1. They states that the eigenfunctions associated with the low-lying eigenvalues are localized near A_0 at a scale $h^{1/2}$ in the s_i direction for $i\in\{1,\dots,N-1\}$, and at a scale $h^{2/3}$ in the t direction.

Proposition 3.1. *Let $M>0$. There exist $\epsilon, C, h_0 > 0$ such that, for all $h\in(0,h_0)$, and for all eigenfunctions ψ of $\widetilde{\mathscr{L}}_{h,\delta}$ corresponding to eigenvalues λ with $\lambda \leq x_{\min} + Mh^{2/3}$, we have:*

$$\int_{\mathcal{V}_{A_0}\times]0,\delta[} e^{\epsilon t^{\frac{3}{2}}/h}|\psi|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t \leq C\|\psi\|^2\,,$$

$$\int_{\mathcal{V}_{A_0}\times]0,\delta[} e^{\epsilon t^{\frac{3}{2}}/h}|h\,\nabla_{s,t}\psi|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t \leq Ch^{2/3}\|\psi\|^2\,.$$

where $\nabla_{s,t}$ is the differential in the tubular coordinates (s,t).

Therefore, the operator $\widetilde{\mathscr{L}}_{h,\delta}$ can be replaced by

$$\widehat{\mathscr{L}}_{h,\delta} = -h^2|g|^{-1/2}\partial_t(|g|^{1/2}\partial_t) - h^2 \sum_{1\leq i,j\leq N-1} |g|^{-1/2}\partial_{s_i}(g^{ij}|g|^{1/2}\partial_{s_j}) + \Phi_1(s,t)\,,$$

with Dirichlet boundary conditions, acting on $L^2(\mathcal{V}_{A_0}\times]0,h^{\frac{2}{3}-\eta}[;|g|^{1/2}\,\mathrm{d}s\mathrm{d}t)$, for some $\eta\in(0,1/3)$.

Let $\mu_n(h)$ be the associated (ordered) eigenvalues of $\widehat{\mathscr{L}}_{h,\delta}$. The decay estimates of Proposition 3.1 are still satisfied by the eigenfunctions of $\widehat{\mathscr{L}}_{h,\delta}$ with eigenvalues $\lambda \leq x_{\min} + Mh^{\frac{2}{3}}$. By using this exponential decay, there exist $C, h_0 > 0$ such that, for all $h\in(0,h_0)$.

$$\mu_n(h) - \frac{1}{C}e^{-Ch^{-3\eta/2}} \leqslant \lambda_n(h,\delta) \leqslant \mu_n(h)\,. \tag{3.4}$$

Thus, modulo an exponentially small error, the asymptotic analysis of $\lambda_n(h)$ is reduced to that of $\mu_n(h)$.

By shrinking the spectral window, we can even get a localization with respect to the s variable, as stated in the next proposition.

Proposition 3.2. *Let $M > 0$ and $\eta \in (0, 1/3)$. There exist $\varepsilon, C, h_0 > 0$ such that, for all $h \in (0, h_0)$, and for all eigenfunctions ψ of $\widehat{\mathscr{L}_{h,\delta}}$ corresponding to eigenvalues λ with $\lambda \leq x_{\min} + h^{2/3} z_1 + Mh$, we have:*

$$\int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} e^{\varepsilon |s|^2/h} \, |\psi|^2 |G|^{1/2} \, \mathrm{d}s\mathrm{d}t \leq C \|\psi\|^2 \,,$$

and

$$\int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} e^{\varepsilon |s|^2/h} |h \nabla_s \psi|^2 \, |G|^{1/2} \, \mathrm{d}s \, \mathrm{d}t \leq Ch \|\psi\|^2 \,.$$

Proof. We let $\varphi(s) = \varepsilon |s|^2/2$ and write the Agmon formula:

$$\left\langle \widehat{\mathscr{L}_{h,\delta}} \psi, e^{2\varphi/h} \psi \right\rangle = \left\langle \widehat{\mathscr{L}_{h,\delta}} e^{\varphi/h} \psi, e^{\varphi/h} \psi \right\rangle - \| e^{\varphi/h} \nabla_{s,t} \varphi \psi \|^2 \,. \tag{3.5}$$

Since ψ is the eigenfunction of $\widehat{\mathscr{L}_{h,\delta}}$ corresponding to an eigenvalue λ, we get

$$\int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} \Big[|h \nabla_{s,t} e^{\varphi/h} \psi|^2 + \Phi_1(s,t) |e^{\varphi/h} \psi|^2 - (\lambda + |\nabla_{s,t} \varphi|^2) |e^{\varphi/h} \psi|^2 \Big] |g|^{1/2} \, \mathrm{d}s\mathrm{d}t = 0 \,,$$

with ∇^g is the differential in the tubular coordinates (s,t).
First, we drop the tangential derivative:

$$\int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} \Big[|h \partial_t e^{\varphi/h} \psi|^2 + \Phi_1(s,t) |e^{\varphi/h} \psi|^2 - (\lambda + |\nabla_{s,t} \varphi|^2) |e^{\varphi/h} \psi|^2 \Big] |g|^{1/2} \, \mathrm{d}s\mathrm{d}t \leq 0 \,.$$

We observe from (3.3), as $s = 0$ is a non degenerate minimum, then $\mathrm{Hess}_{(0)} \phi_1$ is a positive definite matrix, then there exist $C^* > 0$, such that

$$\mathrm{Hess}_{(0)} \phi_1 \, (s) \, (s) \geq C^* |s|^2,$$

from (3.3), we obtain

$$\Phi_1(s,t) \geq x_{\min} + t + C^* |s|^2 \,.$$

Introducing the last inequality in the above integral, we get:

$$\begin{aligned} &\int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} \Big[|h \partial_t e^{\varphi/h} \psi|^2 + t |e^{\varphi/h} \psi|^2 \Big] |g|^{1/2} \, \mathrm{d}s\mathrm{d}t \\ &\qquad + \int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} \Big[(-\lambda + x_{\min} + C^* |s|^2 - |\nabla_{s,t} \varphi|^2) |e^{\varphi/h} \psi|^2 \Big] |g|^{1/2} \, \mathrm{d}s\mathrm{d}t \leq 0 \,. \end{aligned}$$

From (3.2), on $\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[$, for sufficiently small h, there exists $C > 0$ such that

$$|g|^{1/2} \geq (1 - Ch^{\frac{2}{3}-\eta}) \, |G|^{1/2} \,.$$

By using this in the above integrals, we have that

$$\begin{aligned} &(1 - Ch^{\frac{2}{3}-\eta}) \int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} \Big[|h \partial_t e^{\varphi/h} \psi|^2 + t |e^{\varphi/h} \psi|^2 - h^{\frac{2}{3}} z_1 |e^{\varphi/h} \psi|^2 \Big] |G|^{1/2} \, \mathrm{d}s\mathrm{d}t \\ &\quad + \int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} (-\lambda + x_{\min} + C^* |s|^2 + h^{\frac{2}{3}} z_1 - |\nabla_{s,t} \varphi|^2 - Ch^{\frac{4}{3}-\eta}) |e^{\varphi/h} \psi|^2 |g|^{1/2} \, \mathrm{d}s\mathrm{d}t \leq 0 \,. \end{aligned}$$

By using the min-max principle, we notice that

$$\int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} \Big[|h \partial_t e^{\varphi/h} \psi|^2 + t |e^{\varphi/h} \psi|^2 \Big] |G|^{1/2} \, \mathrm{d}s\mathrm{d}t \geq h^{\frac{2}{3}} z_1 \int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} |e^{\varphi/h} \psi|^2 |G|^{1/2} \, \mathrm{d}s\mathrm{d}t \,.$$

Therefore,

$$\int_{\mathcal{V}_{A_0} \times]0, h^{\frac{2}{3}-\eta}[} (-\lambda + x_{\min} + C^* |s|^2 + h^{\frac{2}{3}} z_1 - |\nabla_{s,t} \varphi|^2 - Ch^{\frac{4}{3}-\eta}) |e^{\varphi/h} \psi|^2 |g|^{1/2} \, \mathrm{d}s\mathrm{d}t \leq 0 \,.$$

So that, using the assumption of the location of λ and for h small enough, then:

$$\int_{\mathcal{V}_{A_0}\times]0,h^{\frac{2}{3}-\eta}[}(-2Mh+C^*|s|^2-|\nabla_{s,t}\varphi|^2)|e^{\varphi/h}\psi|^2|G|^{1/2}\,\mathrm{d}s\mathrm{d}t\leq 0\,.$$

We have

$$|\nabla_{s,t}\varphi|^2=|\partial_t\varphi|^2+\sum_{1\leq i,j\leq N-1}g^{ij}\partial_{s_i}\varphi\,\overline{\partial_{s_j}\varphi}=\varepsilon^2\sum_{1\leq i,j\leq N-1}g^{ij}s_is_j\,,$$

where (g^{ij}) is the inverse matrix of (g_{ij}). We can choose $\varepsilon>0$ such that

$$\varepsilon^2\sum_{1\leq i,j\leq N-1}g^{ij}s_is_j\leq\varepsilon^2|s|^2\,.$$

It follows:

$$\int_{\mathcal{V}_{A_0}\times]0,h^{\frac{2}{3}-\eta}[}\big(-2Mh+(C^*-\varepsilon^2)|s|^2\big)|e^{\varphi/h}\psi|^2|G|^{1/2}\,\mathrm{d}s\mathrm{d}t\leq 0\,.$$

For ε small enough, the conclusion follows as in the proof of Proposition 2.1 in [5]. □

4. Construction of quasimodes

This section aims to explain how the following proposition follows. For the proof, we will follow the same strategy as in [12].

Proposition 4.1. *For any positive n, there exists a sequence $\{\zeta_{j,n}\}_{j\geq 0}$ such that, for any positive $M\in\mathbb{N}$, as $h\to 0$, we have*

$$\|\widehat{\mathscr{L}_{h,\delta}}\Phi_{n,M}-h^{2/3}\lambda_{n,M}\Phi_{n,M}\|\leq Ch^{\frac{4}{3}+\frac{M+1}{6}}\|\Phi_{n,M}\|\,,$$

where

$$h^{2/3}\lambda_{n,M}=x_{\min}+h^{2/3}z_1+h\beta_{1,n}+h^{4/3}\sum_{j=0}^{M}h^{j/6}\zeta_{j,n}\,,$$

with $z_1,\beta_{1,n}$ are defined in Theorem 1.1 and $\Phi_{n,M}$ a family of quasimodes introduced in (4.23).

4.1. Rescaled operator.

We recall that

$$\widehat{\mathscr{L}_{h,\delta}}=-h^2|g|^{-1/2}\partial_t(|g|^{1/2}\partial_t)-h^2\sum_{1\leq i,j\leq N-1}|g|^{-1/2}\partial_{s_i}(g^{ij}|g|^{1/2}\partial_{s_j})+\Phi_1(s,t)\,,$$

with Dirichlet boundary conditions, acting on $L^2(\mathcal{V}_{A_0}\times]0,h^{\frac{2}{3}-\eta}[;|g|^{1/2}\,\mathrm{d}s\mathrm{d}t)$, also we recall that

$$\Phi_1(s,t)=x_{\min}+\frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)+t+\mathscr{O}(|s|^3+t|s|^2)\,.$$

We know that, the eigenfunctions are localized near the boundary at the scale $h^{\frac{1}{2}}$ in the s_i direction and $h^{2/3}$ in the t direction (cf. Proposition 3.1 and 3.2). This suggests to use the rescaling:

$$(s,t)=(h^{\frac{1}{2}}\sigma,h^{\frac{2}{3}}\tau)\,.$$

Then, we have

$$\Phi_1(h^{\frac{1}{2}}\sigma,h^{\frac{2}{3}}\tau)=x_{\min}+\frac{h}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(\sigma)\,(\sigma)+h^{2/3}\tau+\mathscr{O}(h^{3/2}|\sigma|^3+h^{5/3}\tau|\sigma|^2)\,.$$

This change of variables transforms the above expression of $\widehat{\mathscr{L}_{h,\delta}}$ into

$$\begin{aligned}\widehat{\mathscr{H}_{h,\delta}}=&-h^{2/3}\big[|g|^{-1/2}\partial_\tau(|g|^{1/2}\partial_\tau)-\tau\big]\\&+h\bigg[-\sum_{1\leq i,j\leq N-1}|g|^{-1/2}\partial_{\sigma_i}(g^{ij}|g|^{1/2}\partial_{\sigma_j})+\frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(\sigma)\,(\sigma)\bigg]\\&+x_{\min}+h^{4/3}h^{1/6}q_{1,h}(\sigma,\tau)\,,\end{aligned}$$

where the function $q_{1,h}$ satisfies

$$|q_{1,h}(\sigma,\tau)| \leq C(|\sigma|^3 + h^{1/6}\tau|\sigma|^2)\,.$$

4.2. **Formal expansion.** We have the formal expansion of the operator $\widehat{\mathscr{H}_{h,\delta}}$ (for the details see Lemma 4.1)

$$h^{-2/3}\widehat{\mathscr{H}_{h,\delta}} = x_{\min}h^{-2/3} + \mathscr{L}_0 + h^{1/3}\mathscr{L}_1 + h^{2/3}Q_h\,. \tag{4.1}$$

where

$$\begin{aligned}
&\mathscr{L}_0 = -\partial_\tau^2 + \tau,\\
&\mathscr{L}_1 = -\Delta^{G_0} + \frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(\sigma)\,(\sigma)\,, \quad G_0 = G(0),\\
&Q_h = h^{1/6}q_{1,h}(\sigma,\tau) + q_{2,h}(\sigma,\tau)\partial_\tau + h^{1/6}\sum_{1\leq j\leq N-1} q_{3,h}(\sigma,\tau)\partial_{\sigma_j}\,,
\end{aligned} \tag{4.2}$$

where the Laplace-Beltrami operator on $\partial\Omega$ in the neighborhood of A_0 is

$$\Delta^{G_0} = \sum_{1\leq i,j\leq N-1} |G_0|^{-1/2}\partial_{\sigma_i}(G_0^{ij}|G_0|^{1/2}\partial_{\sigma_j})\,. \tag{4.3}$$

where the functions $q_{j,h},$ for $j \in \{2,3\}$ satisfy

$$|q_{2,h}(\sigma,\tau)| \leq C(\tau + |\sigma|) \quad \text{and} \quad |q_{3,h}(\sigma,\tau)| \leq Ch^{1/6}\tau\,.$$

Lemma 4.1. *For $0 < h < \frac{1}{2}$, $0 \leq \tau \leq h^{-\eta}$ and $\sigma_i = \mathcal{O}\left(h^{-1/2}\right)$, we have the following identities:*

$$|g|^{-1/2}\partial_\tau(|g|^{1/2}\partial_\tau) = \partial_\tau^2 + (\partial_\tau|g|^{1/2})|g|^{-1/2}\partial_\tau = \partial_\tau^2 + h^{2/3}q_{2,h}(\sigma,\tau)\partial_\tau\,, \tag{4.4}$$

and

$$|g|^{-1/2}\partial_{\sigma_i}(g^{ij}|g|^{1/2}\partial_{\sigma_j}) = |G_0|^{-1/2}\partial_{\sigma_i}(G_0^{ij}|G_0|^{1/2}\partial_{\sigma_j}) + h^{1/2}q_{3,h}(\sigma,\tau)\partial_{\sigma_j}\,, \tag{4.5}$$

where the functions $q_{j,h}$, for $j \in \{2,3\}$ satisfy

$$|q_{2,h}(\sigma,\tau)| \leq C(\tau + |\sigma|) \quad \textit{and} \quad |q_{3,h}(\sigma,\tau)| \leq Ch^{1/6}\tau\,.$$

Proof. We have the following asymptotic expansions:

$$\begin{aligned}
M(\sigma,\tau) &:= |g|^{1/2}(\sigma,\tau)\\
&= \left(1 - h^{\frac{2}{3}}\tau\mathrm{tr}(G^{-1}K)(h^{\frac{1}{2}}\sigma) + h^{\frac{4}{3}}\tau^2 p(h^{\frac{1}{2}}\sigma, h^{\frac{2}{3}}\tau)\right)|G|^{1/2}(h^{\frac{1}{2}}\sigma)\\
&= M(0,0) + \tau\frac{\partial M}{\partial\tau}(0,0) + \sum_{i=1}^{N-1}\sigma_i\frac{\partial M}{\partial\sigma_i}(0,0) + hF_1(\sigma,\tau)\,,
\end{aligned}$$

where

- $M(0,0) = |G|^{1/2}(0)$,
- $\dfrac{\partial M}{\partial\tau}(0,0) = -h^{2/3}\mathrm{tr}(G^{-1}K)(0)|G|^{1/2}(0)$,
- $\dfrac{\partial M}{\partial\sigma_i}(0,0) = h^{1/2}\dfrac{\partial|G|^{1/2}}{\partial\sigma_i}(0)$.

Then, we have

$$|g|^{1/2}(\sigma,\tau) = |G|^{1/2}(0) - h^{2/3}\tau\mathrm{tr}(G^{-1}K)(0)|G|^{1/2}(0) + h^{1/2}\sum_{1\leq i\leq N-1}\sigma_i\frac{\partial|G|^{1/2}}{\partial\sigma_i}(0) + hF_1(\sigma,\tau)\,.$$

From the previous inequality, we have

$$|g|^{-1/2}(\sigma,\tau) = |G|^{-1/2}(0) + h^{1/2}F_2(\sigma,\tau)\,. \tag{4.6}$$

This gives us the following identities:

$$(\partial_\tau |g|^{1/2})|g|^{-1/2}(\sigma,\tau) = -h^{2/3}F_3(\sigma,\tau)\,. \tag{4.7}$$

From Section 3, we have

$$g_{ij}(\sigma,\tau) = G_{ij}(h^{\frac{1}{2}}\sigma) - 2h^{2/3}\tau K_{ij}(h^{\frac{1}{2}}\sigma) + h^{4/3}\tau^2 L_{ij}(h^{\frac{1}{2}}\sigma) = G_{ij}(0) + h^{1/2}F_4(\sigma,\tau)\,, \tag{4.8}$$

and thus,

$$g^{ij}(\sigma,\tau) = G^{ij}(0) + h^{1/2}F_5(\sigma,\tau)\,, \tag{4.9}$$

where, for $0 < h < \frac{1}{2}, 0 \le \tau \le h^{-\eta}$ and $\sigma_i = \mathcal{O}\left(h^{-1/2}\right)$, the functions F_j, $j = 1, \cdots, 5$, satisfy

$$F_1(\sigma,\tau) \le C(h^{1/3}\tau^2 + |\sigma|^2)\,, \quad F_3(\sigma,\tau) \le C(\tau + |\sigma|) \quad \text{and}$$
$$F_j(\sigma,\tau) \le C(h^{1/6}\tau + |\sigma|), \forall j = 2,4,5\,.$$

This gives us the following identities:

$$|g|^{-1/2}\partial_\tau(|g|^{1/2}\partial_\tau) = \partial_\tau^2 + (\partial_\tau|g|^{1/2})|g|^{-1/2}\partial_\tau = \partial_\tau^2 + h^{2/3}q_{2,h}(\sigma,\tau)\partial_\tau\,. \tag{4.10}$$

and

$$|g|^{-1/2}\partial_{\sigma_i}(g^{ij}|g|^{1/2}\partial_{\sigma_j}) = |G|^{-1/2}(0)\partial_{\sigma_i}(G^{ij}(0)|G|^{1/2}(0)\partial_{\sigma_j}) + h^{1/2}\partial_{\sigma_i}q_{3,h}(\sigma,\tau)\partial_{\sigma_j}\,. \tag{4.11}$$

where the functions $q_{i,h}$, for $i \in \{2,3\}$ satisfy for $0 < h < \frac{1}{2}, 0 \le \tau \le h^{-\eta}$ and $|\sigma| = \mathcal{O}\left(h^{-1/2}\right)$

$$|q_{2,h}(\sigma,\tau)| \le C(\tau + |\sigma|) \quad \text{and} \quad |q_{3,h}(\sigma,\tau)| \le Ch^{1/6}\tau\,.$$

□

4.3. **Construction of trial states.** In this section, we construct trial states that will give with the lower bounds, an accurate expansion of the eigenvalues of the operator $\widehat{\mathscr{L}_{h,\delta}}$ valid to any order. The key to the construction is to expand the operator Q_h in powers of $h^{1/6}$.

The functions $q_{1,h}$, $q_{2,h}$ and $q_{3,h}$ in (4.2) admit Taylor expansions that can be rearranged in the form:

$$\begin{aligned} h^{1/6}q_{1,h}(\sigma,\tau) &= \sum_{k=0}^{\infty} q_{1,k}(\sigma,\tau)h^{k/6}\,,\\ q_{2,h}(\sigma,\tau) &= \sum_{k=0}^{\infty} q_{2,k}(\sigma,\tau)h^{k/6}\,,\\ h^{1/6}q_{3,h}(\sigma,\tau) &= \sum_{k=0}^{\infty} q_{3,k}(\sigma,\tau)h^{k/6}\,, \end{aligned} \tag{4.12}$$

with the functions $q_{1,k}(\sigma,\tau)$, $q_{2,k}(\sigma,\tau)$ and $q_{3,k}(\sigma,\tau)$ being polynomial functions of (σ,τ). The operator Q_h in (4.2) admits the formal expansion

$$Q_h = \sum_{k=0}^{\infty} Q_k h^{k/6}\,, \tag{4.13}$$

where

$$Q_k = q_{1,k}(\sigma,\tau) + q_{2,k}(\sigma,\tau)\partial_\tau + \sum_{1\le j\le N-1} q_{3,k}(\sigma,\tau)\partial_{\sigma_j}\,. \tag{4.14}$$

Furthermore, for every $k \ge 0$ and $f \in \mathcal{S}\left(\mathbb{R}^{N-1}\times\overline{\mathbb{R}_+}\right)$, $Q_k f \in \mathcal{S}\left(\mathbb{R}^{N-1}\times\overline{\mathbb{R}_+}\right)$.
For all $M \ge 0$, we introduce two operators

$$\widehat{\mathscr{H}_{h,M}} = x_{\min}h^{-2/3} + \mathscr{L}_0 + h^{1/3}\mathscr{L}_1 + h^{2/3}F_M \quad \text{and} \quad F_M = \sum_{k=0}^{M} Q_k h^{k/6}\,. \tag{4.15}$$

Thanks to (4.1) and (4.2), for every $f \in \mathcal{S}(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+})$, there exists a constant $C > 0$ such that, for all $h \in (0,1)$,

$$\|h^{-2/3}\widehat{\mathscr{H}_{h,\delta}}f - \widehat{\mathscr{H}_{h,M}}f\|_{L^2(\mathbb{R}^{N-1}\times\mathbb{R}_+)} \leq Ch^{\frac{2}{3}+\frac{M+1}{6}}. \tag{4.16}$$

Let $n \in \mathbb{N}$. We will construct a sequence of real numbers $(\zeta_{n,j})_{j=0}^{\infty}$ and two sequences of real-valued Schwartz functions $(v_{n,j})_{j=1}^{\infty} \subset \mathcal{S}(\mathbb{R}^{N-1})$, $(g_{n,j})_{j=0}^{\infty} \subset \mathcal{S}\left(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+}\right)$ such that

$$\forall j, \quad g_{n,j}|_{\tau=0} = 0\,,$$

and for all $M \in \mathbb{N}$, the function

$$\Psi_{n,M}(\sigma,\tau) = u_0(\tau)f_n(\sigma) + \sum_{j=1}^{M+1} h^{\frac{j+1}{6}} u_0(\tau)v_{n,j}(\sigma) + h^{2/3}\sum_{j=0}^{M} h^{j/6} g_{n,j}(\sigma,\tau)\,,$$

in the Schwartz space $\mathcal{S}\left(\mathbb{R}^{N-1} \times \overline{\mathbb{R}_+}\right)$ and satisfies (for all $h \in (0, h_{n,M})$)

$$\|\widehat{\mathscr{H}_{h,M}}\Psi_{n,M} - \lambda_{n,M}\Psi_{n,M}\|_{L^2(\mathbb{R}^{N-1}\times\mathbb{R}_+)} \leq C_{n,M}h^{\frac{2}{3}+\frac{M+1}{6}}\,, \tag{4.17}$$

where

$$\lambda_{n,M} = x_{\min}h^{-2/3} + \beta_{0,n} + h^{1/3}\beta_{1,n} + h^{2/3}\sum_{j=0}^{M} h^{j/6}\zeta_{j,n}\,, \tag{4.18}$$

$C_{n,M} > 0$ and $h_{n,M}$ are two constants determined by the values of n and M solely, and $\beta_{0,n}, \beta_{1,n}, u_0$ and f_n to be determined.

For $M = 0$: We want to find a real numbers $\zeta_{0,n}, \beta_{0,n}, \beta_{1,n}$ and two functions $v_{n,1}(\sigma), g_{n,0}(\sigma,\tau), u_0(\tau), f_n(\sigma)$ such that

$$\Psi_{n,0}(\sigma,\tau) = u_0(\tau)f_n(\sigma) + h^{1/3}u_0(\tau)v_{n,1}(\sigma) + h^{2/3}g_{n,0}(\sigma,\tau)$$

and

$$\lambda_{n,0} = x_{\min}h^{-2/3} + \beta_{0,n} + h^{1/3}\beta_{1,n} + h^{2/3}\zeta_{0,n}\,.$$

In order that (4.17) is satisfied for $M = 0$, it is sufficient to select $\zeta_{0,n}$, $g_{n,0}(\sigma,\tau)$, $u_0(\tau)$, $f_n(\sigma)$ and $v_{n,1}(\sigma)$ as follows:

$$\begin{cases} (\mathscr{L}_0 - \beta_{0,n})u_0(\tau)f_n(\sigma) = 0\,, \\ (\mathscr{L}_0 - \beta_{0,n})u_0(\tau)v_{n,1}(\sigma) + (\mathscr{L}_1 - \beta_{1,n})u_0(\tau)f_n(\sigma) = 0\,, \\ (\mathscr{L}_0 - \beta_{0,n})g_{n,0}(\sigma,\tau) + (\mathscr{L}_1 - \beta_{1,n})u_0(\tau)v_{n,1}(\sigma) + (F_0 - \zeta_{0,n})u_0(\tau)f_n(\sigma) = 0\,. \end{cases}$$

• For the first equation, the natural choice is then to choose u_0 the eigenfunction of $\mathscr{L}_0$, with $\beta_{0,n}$ is the corresponding eigenvalue, therefore

$$u_0(\tau) = Ai(\tau - z_1) \quad \text{and} \quad \beta_{0,n} = z_1\,, \tag{4.19}$$

where Ai is the $L^2(\mathbb{R}^+)$-normalized Airy function and $-z_1$ is its first negative zero.

• The second equation is equivalent to

$$(\mathscr{L}_1 - \beta_{1,n})u_0(\tau)f_n(\sigma) = 0\,,$$

then, we choose $\beta_{1,n}$ is the n-th eigenvalue on $L^2(\mathbb{R}^{N-1}, |G_0|^{1/2}\mathrm{d}\sigma)$ of $\mathscr{L}_1$ determined in Section 7.1 associated with eigenfunction f_n, .

• For the third equation, we can select $\zeta_{0,n}$ such that

$$(F_0 - \zeta_{0,n})\, u_0(\tau)f_n(\sigma) \perp u_0(\tau)f_n(\sigma) \quad \text{in } L^2\left(\mathbb{R}^{N-1} \times \mathbb{R}_+\right)$$

This is given by the formula

$$\zeta_{0,n} = \langle F_0\,(u_0 \otimes f_n)\,, u_0 \otimes f_n\rangle_{L^2(\mathbb{R}^{N-1}\times\mathbb{R}_+)}\,. \tag{4.20}$$

Consequently,

$$h_1(\sigma) := -\int_0^\infty u_0(\tau)\,(F_0-\zeta_{0,n})\,u_0(\tau)f_n(\sigma)d\tau \perp f_n(\sigma) \quad \text{in } L^2(\mathbb{R}^{N-1})\,.$$

Since $\mathscr{L}_1-\beta_{1,n}$ can be inverted on the orthogonal complement of the n-th eigenfunction $f_n(\sigma)$, we may select $v_{n,1}(\sigma)$ to be

$$v_{n,1}(\sigma) = (\mathscr{L}_1-\beta_{1,n})^{-1}\,h_1(\sigma)\,.$$

As a consequence of the choice of $\zeta_{0,n}$ and $v_{n,1}(\sigma)$, we get that, for all σ,

$$\begin{aligned} w_1(\sigma,\cdot) &:= -u_0(\cdot)\,(\mathscr{L}_1-\beta_{1,n})\,v_{n,1}(\sigma) - (F_0-\zeta_{0,n})\,u_0(\cdot)f_n(\sigma) \\ &= -u_0(\cdot)h_1(\sigma) - (F_0-\zeta_{0,n})\,u_0(\cdot)f_n(\sigma) \perp u_0(.) \quad \text{in} \quad L^2\left(\mathbb{R}_+\right)\,. \end{aligned}$$

The operator $\mathscr{L}_0-\beta_{0,n}$ can be inverted on the orthogonal complement of $\{u_0(\tau)\}$, and the inverse is an operator in $L^2\left(\mathbb{R}_+\right)$ which sends $\mathcal{S}\left(\overline{\mathbb{R}_+}\right)$ into itself. The proof of this is standard and follows the same argument as in [6, Lemma A.5]. In that way, we may select $g_{0,n}(\sigma,\tau)$ to be

$$g_{n,0}\left(\sigma,\cdot\right) = \left(\mathscr{L}_0-\beta_{0,n}\right)^{-1}\left(w_1(\sigma,\cdot)\right)\,.$$

The iteration process. Here we suppose that we have selected $(\zeta_{0,n},\dots,\zeta_{M,n};\Psi_{n,0},\dots,\Psi_{n,M})$ such that the following induction hypothesis is satisfied

$$\left(\widehat{\mathscr{H}_{h,M}}-\lambda_{n,M}\right)\Psi_{n,M} = h^{\frac{2}{3}+\frac{M+1}{6}}\sum_{j=0}^{N_M} h^{j/6}r_{M,j}(\sigma,\tau)\,, \tag{4.21}$$

where $r_{M,j}$ are Schwartz functions. We have to selected $\zeta_{M+1,n}$ and two functions $v_{n,M+1}(\sigma)$, $g_{n,M+1}(\sigma,\tau)$ such that

$$\begin{aligned} \Psi_{n,M+1} &= \Psi_{n,M} + h^{\frac{M+3}{6}}u_0(\tau)v_{n,M+2}(\sigma) + h^{\frac{2}{3}+\frac{M+1}{6}}g_{n,M+1}(\sigma,\tau) \quad \text{and} \\ \lambda_{n,M+1} &= \lambda_{n,M} + h^{\frac{2}{3}+\frac{M+1}{6}}\zeta_{M+1,n} \end{aligned}$$

satisfy (4.17) and (4.21) for M replaced by $M+1$ (and some collection of Schwartz functions $r_{M+1,j(\sigma,\tau)}$). Notice that we have

$$\widehat{\mathscr{H}_{h,M+1}} = \widehat{\mathscr{H}_{h,M}} + h^{\frac{2}{3}+\frac{M+1}{6}}Q_{M+1}$$

and

$$\begin{aligned} &\left(\widehat{\mathscr{H}_{h,M+1}}-\lambda_{n,M+1}\right)\Psi_{n,M+1} \\ &= \left(\widehat{\mathscr{H}_{h,M}}-\lambda_{n,M}\right)\Psi_{n,M} \\ &+h^{\frac{2}{3}+\frac{M+1}{6}}\left[(\mathscr{L}_0-\beta_{0,n})g_{n,M+1}(\sigma,\tau) + (\mathscr{L}_1-\beta_{1,n})u_0(\tau)v_{n,M+2}(\sigma) + (Q_{M+1}-\zeta_{M+1,n})u_0(\tau)f_n(\sigma)\right] \\ &+h^{\frac{2}{3}+\frac{M+1}{6}}\left[(Q_{M+1}-\zeta_{M+1,n})\left(\Psi_{n,M}-u_0(\tau)f_n(\sigma)\right)\right] \\ &+h^{\frac{2}{3}+\frac{M+1}{6}}\left[\left(h^{1/3}\mathscr{L}_1 + h^{2/3}F_{M+1} - (\lambda_{n,M}-\beta_{0,n})\right)g_{n,M+1}\right] \\ &+h^{\frac{2}{3}+\frac{M+1}{6}}\left[h^{1/3}F_{M+1} - (\lambda_{n,M+1}-\beta_{0,n}-\beta_{1,n}h^{1/3})u_0(\tau)v_{n,M+2}(\sigma)\right]\,. \end{aligned} \tag{4.22}$$

It is sufficient to select $(\zeta_{M+1,n},v_{n,M+2},g_{n,M+1})$ such that

$$\begin{aligned} (\mathscr{L}_0-\beta_{0,n})g_{n,M+1}(\sigma,\tau) + (\mathscr{L}_1-\beta_{1,n})u_0(\tau)v_{n,M+2}(\sigma) \\ + (Q_{M+1}-\zeta_{M+1,n})u_0(\tau)f_n(\sigma) + r_{M,0}(\sigma,\tau) = 0\,. \end{aligned}$$

We select $\zeta_{M+1,n}$ such that

$$(Q_{M+1}-\zeta_{M+1,n})u_0(\tau)f_n(\sigma) + r_{M,0}(\sigma,\tau) \perp u_0(\tau)f_n(\sigma) \quad \text{in } L^2\left(\mathbb{R}^{N-1}\times\mathbb{R}_+\right)\,,$$

i.e.

$$\zeta_{M+1,n} = \langle Q_{M+1}(u_0\otimes f_n), u_0\otimes f_n\rangle_{L^2(\mathbb{R}^{N-1}\times\mathbb{R}_+)} + \langle r_{M,0}, u_0\otimes f_n\rangle_{L^2(\mathbb{R}^{N-1}\times\mathbb{R}_+)}\,.$$

Consequently,

$$h_{M+2}(\sigma) := -\int_0^\infty u_0(\tau)\big((Q_{M+1}-\zeta_{M+1,n})u_0(\tau)f_n(\sigma)+r_{M,0}(\sigma,\tau)\big)d\tau \perp f_n(\sigma)$$

in $L^2(\mathbb{R}^{N-1})$. We can select $v_{n,M+2}$ as follows:

$$v_{n,M+2}(\sigma) = (\mathscr{L}_1-\beta_{1,n})^{-1}\, h_{M+2}(\sigma)\,.$$

As a consequence of the choice of $\zeta_{M+1,n}$ and $v_{n,M+2}(\sigma)$, we get that, for all σ,

$$\begin{aligned} w_{M+2}(\sigma,\cdot) := &-u_0(\cdot)\,(\mathscr{L}_1-\beta_{1,n})\,v_{n,M+2}(\sigma)\\ &-(Q_{M+1}-\zeta_{M+1,n})u_0(\tau)f_n(\sigma)-r_{M,0}(\sigma,\tau)\perp u_0(\cdot)\quad\text{in}\quad L^2(\mathbb{R}_+)\,. \end{aligned}$$

Finally, we may select $g_{n,M+1}(\sigma,\tau)$ as follows:

$$g_{n,M+1}(\sigma,\cdot) = (\mathscr{L}_0-\beta_{0,n})^{-1}\,(w_{M+2}(\sigma,\cdot))\,.$$

4.4. **End of the proof.** Consider a smooth function χ_h with compact support equal to 1 near $(s,t)=(0,0)$ on a scale of order $h^{1/2-\theta}$ in the s direction, and on a scale of order $h^{2/3-\theta}$ in the t direction, for some $\theta\in(0,\eta)$. Let us introduce the following family of test functions

$$\Phi_{n,M}(s,t) = h^{\frac{-1}{3}\frac{-(N-1)}{4}}\underbrace{\chi\left(\frac{t}{h^{2/3-\theta}}\right)\chi\left(\frac{s}{h^{1/2-\theta}}\right)}_{\chi_h(s,t)}\Psi_{n,M}(h^{-1/2}s,h^{-2/3}t)\,, \tag{4.23}$$

We notice that the support of $\chi-1$ and the supports of the derivative of χ are located in a region where $h^{-1/2}|s|\geq Ch^{-\theta}$ and $h^{-2/3}t\geq Ch^{-\theta}$, thus all integrals with contain such derivatives will be of order h^∞ due to the exponential localization of $\Psi_{n,M}$.
Thanks to (4.16) and (4.17), we may write

$$\|\widehat{\mathscr{L}_{h,\delta}}\Phi_{n,M}-h^{2/3}\lambda_{n,M}\Phi_{n,M}\|\leq Ch^{\frac{4}{3}+\frac{M+1}{6}}\|\Phi_{n,M}\|\,.$$

5. Lower bound

The following proposition provides a lower bound of the eigenvalue $\mu_n(h)$, as $h\to 0$.

Proposition 5.1. *For any positive n, the eigenvalue $\mu_n(h)$ has, as $h\to 0$, the lower bound*

$$\mu_n(h)\geq x_{\min}+z_1h^{2/3}+\beta_{1,n}\,h+\mathscr{O}(h^{4/3})$$

where the $\beta_{1,n}$ is defined in Theorem 1.1.

Let $k\geq 1$, and consider a family of eigenfunctions $(\psi_{n,h})_{n=1,\dots,k}$ of $\widehat{\mathscr{L}_{h,\delta}}$ associated with the eigenvalues $(\mu_n(h))_{n=1,\dots,k}$. Let $\mathcal{E}_k(h)$ the vector subspace of $\mathrm{Dom}(\widehat{\mathscr{L}_{h,\delta}})$ spanned by the family $(\psi_{n,h})_{n=1,\dots,k}$. For all $\psi\in\mathcal{E}_k(h)$, ψ satisfies the same decay estimates as in Propositions 3.1 and 3.2.
Let $\psi\in\mathcal{E}_k(h)$, since $|g|^{1/2}\leq(1+C_*t)|G|^{1/2}$, using the decay estimates of Propositions 3.1, we have the important inequality ($\delta=h^{\frac{2}{3}-\eta}$)

$$\int_{\mathcal{V}_{A_0}\times]0,\delta[}|\psi|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t\leq\int_{\mathcal{V}_{A_0}\times]0,\delta[}|\psi|^2|G|^{1/2}\,\mathrm{d}s\mathrm{d}t+Ch^{2/3}\int_{\mathcal{V}_{A_0}\times]0,\delta[}|\psi|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t\,, \tag{5.1}$$

with C_*, C are constants independent of s and t (the value of C might change from one formula to another).
We also have

$$\begin{aligned}\langle\widehat{\mathscr{L}_{h,\delta}}\psi,\psi\rangle = &\int_{\mathcal{V}_{A_0}\times]0,\delta[}h^2|\partial_t\psi|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t+h^2\sum_{1\leq i,j\leq N-1}\int_{\mathcal{V}_{A_0}\times]0,\delta[}g^{ij}\partial_{s_i}\psi\,\overline{\partial_{s_j}\psi}\,|g|^{1/2}\,\mathrm{d}s\mathrm{d}t\\ &+\int_{\mathcal{V}_{A_0}\times]0,\delta[}\Phi_1(s,t)|\psi|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t\,.\end{aligned}$$

We recall that $|g|^{1/2} = \big(1 - t\,\mathrm{tr}(G^{-1}K) + t^2 p(s,t)\big)\,|G|^{1/2}$ so that, using the decay estimates of Propositions 3.1, there exists a $C > 0$ such that

$$\langle \widehat{\mathscr{L}_{h,\delta}}\psi, \psi\rangle \geq \int_{\mathcal{V}_{A_0}\times]0,\delta[} |h\partial_t\psi|^2 |G|^{1/2}\,\mathrm{d}s\mathrm{d}t + h^2 \sum_{1\leq i,j\leq N-1} \int_{\mathcal{V}_{A_0}\times]0,\delta[} g^{ij}\partial_{s_i}\psi\ \overline{\partial_{s_j}\psi}\,|G|^{1/2}\,\mathrm{d}s\mathrm{d}t$$
$$+ \int_{\mathcal{V}_{A_0}\times]0,\delta[} \Phi_1(s,t)|\psi|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t - Ch^{4/3}\|\psi\|^2\,.$$

From (3.3), we have

$$\Phi_1(s,t) \geq x_{\min} + \frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(s)\,(s) + t - \tilde{C}(|s|^3 + t|s|^2)\,.$$

We can deduce that

$$\langle \widehat{\mathscr{L}_{h,\delta}}\psi, \psi\rangle \geq \int_{\mathcal{V}_{A_0}\times]0,\delta[} |h\partial_t\psi|^2 |G|^{1/2}\,\mathrm{d}s\mathrm{d}t \quad + h^2 \sum_{1\leq i,j\leq N-1} \int_{\mathcal{V}_{A_0}\times]0,\delta[} g^{ij}\partial_{s_i}\psi\ \overline{\partial_{s_j}\psi}\,|G|^{1/2}\,\mathrm{d}s\mathrm{d}t$$
$$+ \int_{\mathcal{V}_{A_0}\times]0,\delta[} \left(x_{\min} + \frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(s)\,(s) + t\right)|\psi|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t$$
$$- Ch^{4/3}\|\psi\|^2 - \tilde{C}\int_{\mathcal{V}_{A_0}\times]0,\delta[} (|s|^3 + t|s|^2)|\psi|^2|g|^{1/2}\,\mathrm{d}s\mathrm{d}t\,.$$

The last integral is of order $h^{4/3}$ (by using the exponential decay in the s and t variables; Proposition 3.1 and 3.2). Then there exists $C > 0$ such that

$$\langle \widehat{\mathscr{L}_{h,\delta}}\psi, \psi\rangle \geq \int_{\mathcal{V}_{A_0}\times]0,\delta[} \big[|h\partial_t\psi|^2 + t|\psi|^2\big]|G|^{1/2}\,\mathrm{d}s\mathrm{d}t + h^2\sum_{i,j}\int_{\mathcal{V}_{A_0}\times]0,\delta[} g^{ij}\partial_{s_i}\psi\ \overline{\partial_{s_j}\psi}|G|^{1/2}\,\mathrm{d}s\mathrm{d}t$$
$$+ \int_{\mathcal{V}_{A_0}\times]0,\delta[} \frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\,|G|^{1/2}\,\mathrm{d}s\mathrm{d}t + x_{\min}\|\psi\|^2 - Ch^{4/3}\|\psi\|^2\,. \tag{5.2}$$

Thus, by the min max principle, we have

$$\int_{\mathcal{V}_{A_0}\times]0,\delta[} \big[|h\partial_t\psi|^2 + t|\psi|^2\big]|G|^{1/2}\,\mathrm{d}s\mathrm{d}t \geq z_1 h^{2/3}\int_{\mathcal{V}_{A_0}\times]0,\delta[} |\psi|^2|G|^{1/2}\,\mathrm{d}s\mathrm{d}t\,,$$

using the inequalities in (5.1), we have

$$\int_{\mathcal{V}_{A_0}\times]0,\delta[} \big[|h\partial_t\psi|^2 + t|\psi|^2\big]|G|^{1/2}\,\mathrm{d}s\mathrm{d}t \geq z_1 h^{2/3}\|\psi\|^2 - Ch^{4/3}\|\psi\|^2\,. \tag{5.3}$$

By Taylor expansion near $s = 0$, we get

$$\Big||G(s)|^{1/2} - |G_0|^{1/2}\Big| \leq C\,|s|, \quad G_0 = G(0)\,. \tag{5.4}$$

From (5.4), there exists constants $C > 0$ such that

$$\left|\int_{\mathcal{V}_{A_0}\times]0,\delta[} \mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\,|G|^{1/2}\,\mathrm{d}s\mathrm{d}t - \int_{\mathcal{V}_{A_0}\times]0,\delta[} \mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\,|G(0)|^{1/2}\,\mathrm{d}s\mathrm{d}t\right|$$
$$\leq \left|\int_{\mathcal{V}_{A_0}\times]0,\delta[} \mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\,\big[|G(s)|^{1/2} - |G(0)|^{1/2}\big]\,\mathrm{d}s\mathrm{d}t\right|$$
$$\leq C\int_{\mathcal{V}_{A_0}\times]0,\delta[} \mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|s||\psi|^2\ \mathrm{d}s\mathrm{d}t$$
$$\leq C\int_{\mathcal{V}_{A_0}\times]0,\delta[} |s|^3|\psi|^2\ \mathrm{d}s\mathrm{d}t \leq C\int_{\mathcal{V}_{A_0}\times]0,\delta[} |s|^3|\psi|^2|G|^{1/2}\ \mathrm{d}s\mathrm{d}t.$$

From Proposition 3.2, we get

$$\begin{aligned}(5.5)\qquad &\int_{\mathcal{V}_{A_0}\times]0,\delta[} \mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\,|G|^{1/2}\,\mathrm{d}s\mathrm{d}t\\ &\qquad\geq \int_{\mathcal{V}_{A_0}\times]0,\delta[} \mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\,|G(0)|^{1/2}\,\mathrm{d}s\mathrm{d}t - Ch^{3/2}\|\psi\|^2\,.\end{aligned}$$

The same idea, proves that

$$\begin{aligned}(5.6)\qquad &\sum_{1\leq i,j\leq N-1}\int_{\mathcal{V}_{A_0}\times]0,\delta[} g^{ij}h\partial_{s_i}\psi\;\overline{h\partial_{s_j}\psi}|G|^{1/2}\,\mathrm{d}s\mathrm{d}t\\ &\qquad\geq \sum_{1\leq i,j\leq N-1}\int_{\mathcal{V}_{A_0}\times]0,\delta[} g^{ij}h\partial_{s_i}\psi\;\overline{h\partial_{s_j}\psi}|G(0)|^{1/2}\,\mathrm{d}s\mathrm{d}t - Ch^{3/2}\|\psi\|^2\,.\end{aligned}$$

Observing that,

$$(5.7)\qquad \left|g^{ij}(s,t)-G_0^{ij}\right|\leq C\left(|s|+t\right),\quad g^{ij}(0,0)=G_0^{ij}\,.$$

From Proposition 3.1 and 3.2, we get

$$\begin{aligned}(5.8)\qquad &\sum_{1\leq i,j\leq N-1}\int_{\mathcal{V}_{A_0}\times]0,\delta[} g^{ij}h\partial_{s_i}\psi\;\overline{h\partial_{s_j}\psi}|G(0)|^{1/2}\,\mathrm{d}s\mathrm{d}t\\ &\qquad\geq \sum_{1\leq i,j\leq N-1}\int_{\mathcal{V}_{A_0}\times]0,\delta[} G_0^{ij}h\partial_{s_i}\psi\;\overline{h\partial_{s_j}\psi}|G(0)|^{1/2}\,\mathrm{d}s\mathrm{d}t - Ch^{4/3}\|\psi\|^2\,.\end{aligned}$$

From (5.5), (5.6) and (5.8), we can easily see that

$$\begin{aligned}(5.9)\qquad &\int_{\mathcal{V}_{A_0}\times]0,\delta[}\left[\sum_{1\leq i,j\leq N-1} h^2g^{ij}\partial_{s_i}\psi\;\overline{\partial_{s_j}\psi}+\frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\right]|G|^{1/2}\,\mathrm{d}s\mathrm{d}t\\ &\geq\int_{\mathcal{V}_{A_0}\times]0,\delta[}\left[\sum_{1\leq i,j\leq N-1} h^2G_0^{ij}\partial_{s_i}\psi\;\overline{\partial_{s_j}\psi}+\frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\right]|G_0|^{1/2}\,\mathrm{d}s\mathrm{d}t - Ch^{4/3}\|\psi\|^2.\end{aligned}$$

Now, using the inequalities in (5.2), (5.3) and (5.9), and using the min-max principle, we have

$$\begin{aligned}(5.10)\qquad \mu_k(h)\|\psi\|^2 &\geq \langle\widehat{\mathscr{L}_{h,\delta}}\psi,\psi\rangle\\ &\geq\int_{\mathcal{V}_{A_0}\times]0,\delta[}\left[\sum_{1\leq i,j\leq N-1} h^2G_0^{ij}\partial_{s_i}\psi\;\overline{\partial_{s_j}\psi}+\frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\right]|G_0|^{1/2}\,\mathrm{d}s\mathrm{d}t\\ &\quad + x_{\min}\|\psi\|^2+z_1h^{2/3}\|\psi\|^2-Ch^{4/3}\|\psi\|^2\,.\end{aligned}$$

Therefore, we have

$$\begin{aligned}(5.11)\qquad &\int_{\mathcal{V}_{A_0}\times]0,\delta[}\left[\sum_{1\leq i,j\leq N-1} h^2G_0^{ij}\partial_{s_i}\psi\;\overline{\partial_{s_j}\psi}+\frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)\,|\psi|^2\right]|G_0|^{1/2}\,\mathrm{d}s\mathrm{d}t\\ &\leq(\mu_k(h)-x_{\min}-z_1h^{2/3}+Ch^{4/3})\|\psi\|^2\,.\end{aligned}$$

On the left hand side of the above identity, we recognize the quadratic form associated to the operator

$$\mathbb{1}\otimes\mathscr{H}_h,$$

where $\mathscr{H}_h=-h^2\Delta^{G_0}+\frac{1}{2}\,\mathrm{Hess}_{(0)}\phi_1\,(s)\,(s)$ on $L^2(\mathbb{R}^{N-1},|G_0|^{1/2}\mathrm{d}s)$.
From Section 7.1 and 7.2, we show that, for h small enough, $\mathscr{H}_h$ is unitarily equivalent to

$$\mathscr{M}_h=\sum_{i=1}^{N-1}-h^2\partial_{u_i}^2+\frac{1}{2}\lambda_iu_i^2$$

on $L^2(\mathbb{R}^{N-1}, \mathrm{d}u)$, where λ_i are the eigenvalues of the Weingarten map of $\partial\Omega$ at A_0. Their spectrum is given by

$$sp(\mathscr{H}_h) = sp(\mathscr{M}_h) = \{h\beta_{1,n}, n \geq 1\}\,,$$

where $\beta_{1,n}$ is the n-th eigenvalue of $\mathscr{N}_h = \sum_{i=1}^{N-1} -\partial_{v_i}^2 + \frac{1}{2}\lambda_i v_i^2$ on $L^2(\mathbb{R}^{N-1}, \mathrm{d}v)$.

The decay estimates, given that dim($\mathcal{E}_k(h)$)=k on $L^2(\mathbb{R}^{N-1}, |G_0|^{1/2}\mathrm{d}s)$. Notice that the set $\mathcal{E}_k(h)$ is contained in the form domain of $\mathscr{H}_h$. Thus, by the min-max principle, we have

$$\begin{aligned} h\beta_{1,k} &\leq \sup_{\psi\in\mathcal{E}_k(h)} \frac{\langle(\mathbb{1}\otimes\mathscr{H}_h)\psi,\psi\rangle_{L^2(\mathcal{V}_{A_0}\times]0,\delta[,|G_0|^{1/2}\mathrm{d}s\mathrm{d}t)}}{\|\psi\|^2_{L^2(\mathcal{V}_{A_0}\times]0,\delta[,|G_0|^{1/2}\mathrm{d}s\mathrm{d}t)}} \\ &\leq (\mu_k(h) - x_{\min} - z_1 h^{2/3} + Ch^{4/3}) \frac{\|\psi\|^2}{\|\psi\|^2_{L^2(|G_0|^{1/2}\mathrm{d}s\mathrm{d}t)}} \end{aligned} \tag{5.12}$$

From (5.1), (5.4) and Proposition 3.2, we obtain

$$\begin{aligned} \int_{\mathcal{V}_{A_0}\times]0,\delta[} |\psi|^2 |G_0|^{1/2}\,\mathrm{d}s\mathrm{d}t &\geq \int_{\mathcal{V}_{A_0}\times]0,\delta[} |\psi|^2 |G|^{1/2}\,\mathrm{d}s\mathrm{d}t - C\int_{\mathcal{V}_{A_0}\times]0,\delta[} |s||\psi|^2\,\mathrm{d}s\mathrm{d}t \\ &\geq (1 - Ch^{1/2})\|\psi\|^2. \end{aligned} \tag{5.13}$$

Using the last inequality in (5.12), we get

$$\mu_k(h) - x_{\min} - z_1 h^{2/3} + Ch^{4/3} \geq h\beta_{1,k} \frac{\|\psi\|^2_{L^2(|G_0|^{1/2}\mathrm{d}s\mathrm{d}t)}}{\|\psi\|^2} \geq h\beta_{1,k} - Ch^{3/2}\,.$$

This implies the desired lower bound and concludes the proof of Proposition 5.1.

6. Full asymptotic expansion of the eigenvalues

Proof of theorem 1.1: From Proposition 5.1, we have the following lower bound of $\mu_n(h)$:

$$\mu_n(h) \geq x_{\min} + z_1 h^{2/3} + \beta_{1,1}\, h + \mathscr{O}(h^{4/3})\,.$$

Thanks to Proposition 4.1, we know that, from spectral theorem, there exists an eigenvalue $\widetilde{\mu}_1(h)$ of the operator $\widehat{\mathscr{L}_{h,\delta}}$ satisfying, as $h \to 0_+$,

$$\widetilde{\mu}_1(h) = x_{\min} + h^{2/3} z_1 + \beta_{1,1}\, h + h^{4/3} \sum_{j=0}^{M} h^{j/6} \zeta_{j,1} + \mathscr{O}\big(h^{\frac{4}{3}+\frac{M+1}{6}}\big)\,.$$

Since $\beta_{1,1}$ is a simple eigenvalue of the harmonic oscillator ($\beta_{1,1} < \beta_{1,2}$, see Section 7.1), we deduce that

$$\mu_1(h) = x_{\min} + h^{2/3} z_1 + \beta_{1,1}\, h + h^{4/3} \sum_{j=0}^{M} h^{j/6} \zeta_{j,1} + \mathscr{O}\big(h^{\frac{4}{3}+\frac{M+1}{6}}\big)\,.$$

Let $n \geq 2$ and $N \in \mathbb{N}$. Thanks to Proposition 4.1, we know that, from spectral theorem, there exists a sequence of quasi eigenvalues $(\widetilde{\mu}_n(h))_{n\geq 2}$ of the operator $\widehat{\mathscr{L}_{h,\delta}}$ satisfying, as $h \to 0_+$,

$$\widetilde{\mu}_n(h) = x_{\min} + h^{2/3} z_1 + \beta_{1,n}\, h + h^{4/3} \sum_{j=0}^{N} h^{j/6} \zeta_{j,n} + \mathscr{O}\big(h^{\frac{4}{3}+\frac{M+1}{6}}\big)\,.$$

Now, we will arrange the sequence of quasi eigenvalues $\widetilde{\mu}_n(h)$ in increasing order modulo $\mathscr{O}\big(h^{\frac{4}{3}+\frac{M+1}{6}}\big)$ and we note them $\overline{\mu}_n(h)$. We have

$$\overline{\mu}_n(h) = x_{\min} + h^{2/3} z_1 + \beta_{1,n}\, h + h^{4/3} \sum_{j=0}^{M} h^{j/6} \alpha_{j,n} + \mathscr{O}\big(h^{\frac{4}{3}+\frac{M+1}{6}}\big)\,,$$

where $\alpha_{j,n}$ are a sequence determined from $\zeta_{j,n}$.

Imagine that, there exist $C > 0$, $k \in \mathbb{N}$, such that

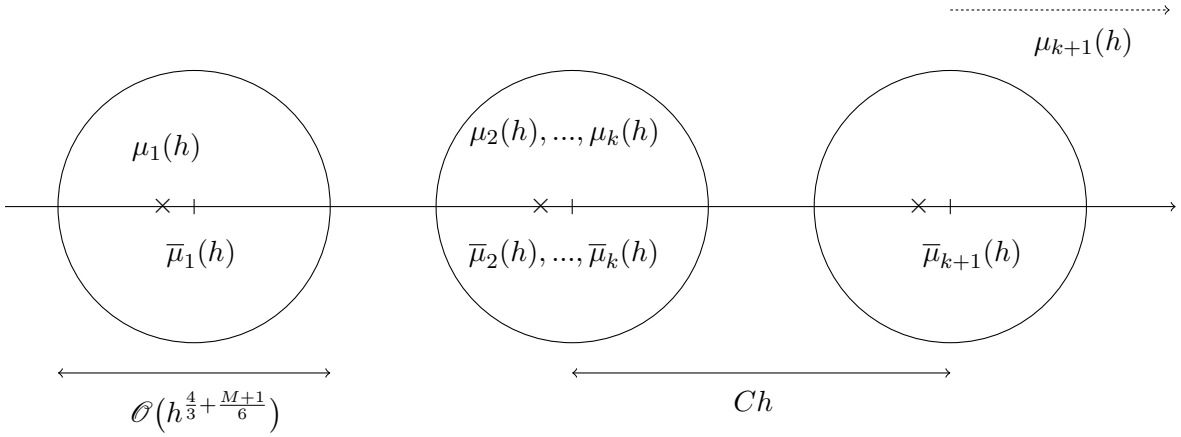
$$\begin{cases} \overline{\mu}_{i+1}(h) = \overline{\mu}_i(h) + o(h)\,, \forall i = 2, ..., k-1 \\ \overline{\mu}_{k+1}(h) - \overline{\mu}_k(h) \geq Ch \end{cases} \tag{6.1}$$

That is

$$\begin{cases} \beta_{1,2} = ... = \beta_{1,k} \\ \beta_{1,k+1} > \beta_{1,k}\,. \end{cases}$$

There are actually two possibilities:

1. $\overline{\mu}_2(h), ..., \overline{\mu}_k(h)$ coincide modulo $\mathscr{O}\big(h^{\frac{4}{3}+\frac{M+1}{6}}\big)$. From spectral theorem, the rank of the spectral projector near these values is at least $k-1$. Then, there exists an eigenvalues very close, modulo $\mathscr{O}\big(h^{\frac{4}{3}+\frac{M+1}{6}}\big)$ to $\overline{\mu}_i(h)$ for all $i \in \{2, ..., k\}$: it can only be $\mu_2(h), ..., \mu_k(h)$, since the other eigenvalues are elsewhere (see (6.1) and Proposition (5.1)).

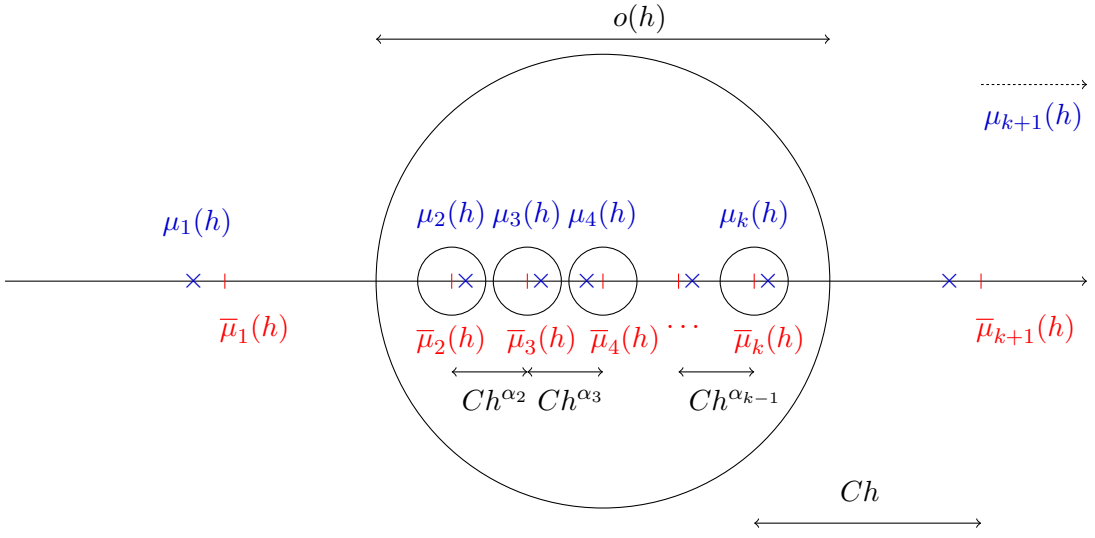


2. They differ modulo h^α with $\alpha < \frac{4}{3} + \frac{M+1}{6}$. There are actually three possibilities:

- There exist $C > 0$, $k \in \mathbb{N}$, $\alpha_i < \frac{4}{3} + \frac{M+1}{6}$, $\quad \forall i = 2, ..., k-1$, such that

$$\begin{cases} \overline{\mu}_{i+1}(h) - \overline{\mu}_i(h) \geq Ch^{\alpha_i}\,, \forall i = 2, ..., k-1 \\ \overline{\mu}_{k+1}(h) - \overline{\mu}_k(h) \geq Ch \end{cases} \tag{6.2}$$

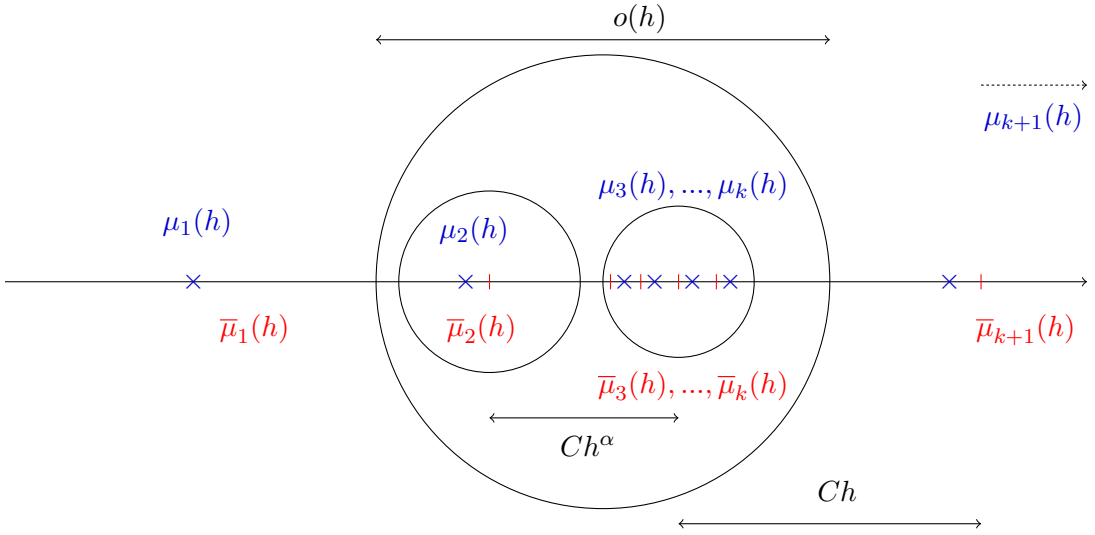
The construction of quasimode implies that there exists an eigenvalue very close, modulo $\mathscr{O}\big(h^{\frac{4}{3}+\frac{M+1}{6}}\big)$, to $\overline{\mu}_i(h)$ for all $i \in \{2, ..., k\}$ (it cannot be the same). Since the eigenvalues are increasing, then it can't be more than $\mu_2(h), ..., \mu_{k-1}(h)$ and $\mu_k(h)$ respectively, because the other eigenvalues are somewhere else(see (6.2) and Proposition 5.1).



- There exist $C > 0$, $k \in \mathbb{N}$, such that

$$\begin{cases} \overline{\mu}_3(h) - \overline{\mu}_2(h) \geq Ch^\alpha \\ \overline{\mu}_{i+1}(h) = \overline{\mu}_i(h) + \mathscr{O}\left(h^{\frac{4}{3}+\frac{M+1}{6}}\right), \forall i = 3, ..., k-1 \\ \overline{\mu}_{k+1}(h) - \overline{\mu}_k(h) \geq Ch \end{cases} \tag{6.3}$$

The rank of the spectral projector near $\{\overline{\mu}_i(h)\}_{i=3,...,k}$ is at least $k-2$. Than, there exists an eigenvalue very close, modulo $\mathscr{O}\left(h^{\frac{4}{3}+\frac{M+1}{6}}\right)$, to $\overline{\mu}_i(h)$ for all $i \in \{3, ..., k\}$. The construction of quasimode implies that there exists an eigenvalue very close, modulo $\mathscr{O}\left(h^{\frac{4}{3}+\frac{M+1}{6}}\right)$ to $\overline{\mu}_2(h)$ (and it cannot be the same). It can't be more than $\mu_2(h)$ near $\overline{\mu}_2(h)$ and $\mu_3(h), ..., \mu_k(h)$ near $\{\overline{\mu}_i(h)\}_{i=3,...,k}$, because the other eigenvalues are somewhere else(see (6.3) and Proposition 5.1).



- There exist $C > 0$, $k_0, k \in \mathbb{N}$ and $k_0 \leq k$ such that

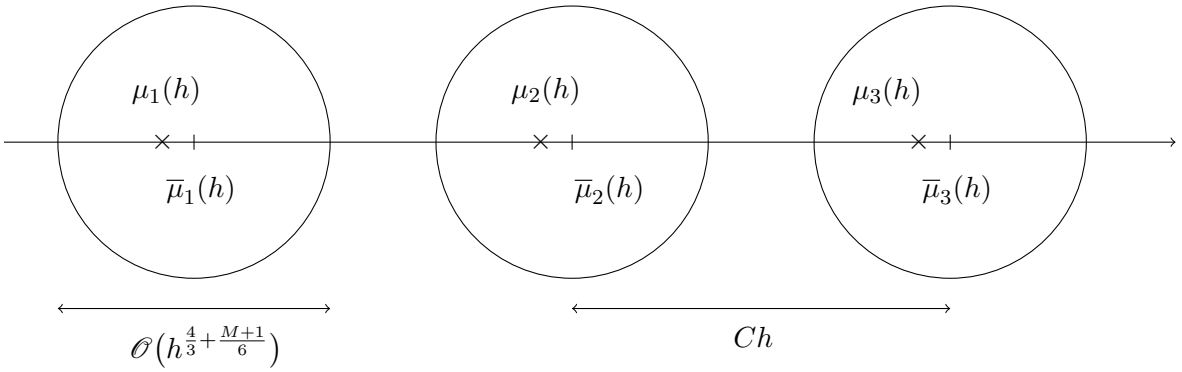
$$\begin{cases} \overline{\mu}_{i+1}(h) = \overline{\mu}_i(h) + \mathscr{O}\left(h^{\frac{4}{3}+\frac{M+1}{6}}\right), \forall i = 2, ..., k_0 - 1 \\ \overline{\mu}_{k_0+1}(h) - \overline{\mu}_{k_0}(h) \geq Ch^\alpha \\ \overline{\mu}_{i+1}(h) = \overline{\mu}_i(h) + \mathscr{O}\left(h^{\frac{4}{3}+\frac{M+1}{6}}\right), \forall i = k_0 + 1, ..., k - 1 \\ \overline{\mu}_{k+1}(h) - \overline{\mu}_k(h) \geq Ch \end{cases} \tag{6.4}$$

The rank of the spectral projector near $\{\overline{\mu}_i(h)\}_{i=2,...,k_0}$ is at least $k_0 - 1$ and near $\{\overline{\mu}_i(h)\}_{i=k_0+1,...,k}$ is at least $k - k_0$. Then, there exists an eigenvalues very close, modulo $\mathscr{O}\left(h^{\frac{4}{3}+\frac{M+1}{6}}\right)$, to $\overline{\mu}_i(h)$ for all $i \in \{2, ..., k_0\}$, but also another very close, modulo $\mathscr{O}\left(h^{\frac{4}{3}+\frac{M+1}{6}}\right)$ to $\overline{\mu}_i(h)$ for all $i \in \{k_0 + 1, ..., k\}$(and it cannot be the same). It can't be more than $\mu_2(h), ..., \mu_{k_0}(h)$ near $\{\overline{\mu}_i(h)\}_{i=2,...,k_0}$ and $\mu_{k_0+1}(h), ..., \mu_k(h)$ near $\{\overline{\mu}_i(h)\}_{i=k_0+1,...,k}$, because the other eigenvalues are somewhere else(see (6.3) and Proposition 5.1).

Now, if (6.1) is not satisfied:

$$\overline{\mu}_3(h) - \overline{\mu}_2(h) \geq Ch\,.$$

As a consequence of the spectral theorem, there exists an eigenvalue very close modulo $\mathscr{O}\left(h^{\frac{4}{3}+\frac{M+1}{6}}\right)$ to $\overline{\mu}_2(h)$, but also another very close to $\overline{\mu}_3(h)$ (and it cannot be the same). Since the eigenvalues are indeed separated, it can't be more than $\mu_2(h)$ and $\mu_3(h)$ respectively.



Thus, by (3.4) and Proposition 2.2, we have shown Theorem 1.1.

7. Analysis of the effective operator

7.1. **The model operator.** In this section, we want to study the spectrum of the operator defined by

$$\mathscr{L} = -\Delta^{G_0} + \frac{1}{2} H(\sigma)(\sigma)\,, \quad H = \mathrm{Hess}_{(0)}\phi_1\,, \quad G_0 = G(0).$$

where ϕ_1 is defined in Section 3.1 . The associate quadratic form on $H^1(\mathbb{R}^{N-1}; |G_0|^{1/2}\,\mathrm{d}\sigma)$ is

$$q_1(\psi) = \langle \mathscr{L}\psi, \psi\rangle_{L^2(|G_0|^{1/2}\,\mathrm{d}\sigma)} = \langle -\Delta^{G_0}\psi, \psi\rangle_{L^2(|G_0|^{1/2}\,\mathrm{d}\sigma)} + \frac{1}{2}\int \langle H\sigma, \sigma\rangle |\psi|^2 |G_0|^{1/2}\,\mathrm{d}\sigma\,.$$

By replacing Δ^{G_0} by its expression given by (4.3), we obtain

$$\langle -\Delta^{G_0}\psi, \psi\rangle_{L^2(|G_0|^{1/2}\,\mathrm{d}\sigma)} = -\sum_{1\le i,j\le N-1}\int \partial_{\sigma_i}(G_0^{ij}|G_0|^{1/2}\partial_{\sigma_j})\psi\,\overline{\psi}\,\mathrm{d}\sigma\,, \quad G_0^{ij} = G^{ij}(0)\,.$$

By integration by part and as ψ satisfies the Dirichlet conditions on the boundary, we have

$$\langle -\Delta^{G_0}\psi, \psi\rangle_{L^2(|G_0|^{1/2}\,\mathrm{d}\sigma)} = \sum_{1\le i,j\le N-1}\int G_0^{ij}\partial_{\sigma_j}\psi\,\overline{\partial_{\sigma_i}\psi}|G_0|^{1/2}\,\mathrm{d}\sigma = \int \langle G_0^{-1}\nabla_\sigma\psi, \nabla_\sigma\psi\rangle |G_0|^{1/2}\mathrm{d}\sigma\,.$$

Consequently, we have

$$q_1(\psi) = \int \langle G_0^{-1}\nabla_\sigma\psi, \nabla_\sigma\psi\rangle |G_0|^{1/2}\mathrm{d}\sigma + \frac{1}{2}\int \langle H\sigma, \sigma\rangle |\psi|^2 |G_0|^{1/2}\mathrm{d}\sigma\,. \tag{7.1}$$

We introduce the change of variable

$$\sigma = G_0^{-1/2} z\,.$$

Thus,

$$\nabla_\sigma = G_0^{1/2}\nabla_z \quad \text{and} \quad \mathrm{d}\sigma = |G_0|^{-1/2}\,\mathrm{d}z\,.$$

By replacing the previous equality in (7.1), and we use that G_0 is self-adjoint, we get

$$\begin{aligned} q_1(\psi) &= \int \langle G_0^{-1}G_0^{1/2}\nabla_z\bar{\psi}, G_0^{1/2}\nabla_z\bar{\psi}\rangle\,\mathrm{d}z + \frac{1}{2}\int \langle HG_0^{-1/2}z, G_0^{-1/2}z\rangle |\bar{\psi}|^2\,\mathrm{d}z \\ &= \int \langle \nabla_z\bar{\psi}, \nabla_z\bar{\psi}\rangle\,\mathrm{d}z + \frac{1}{2}\int \langle G_0^{-1/2}HG_0^{-1/2}z, z\rangle |\bar{\psi}|^2\,\mathrm{d}z\,, \end{aligned}$$

with $\bar{\psi}(z) = \psi(\sigma)$.
We define a quadratic form on $H^1(\mathbb{R}^{N-1})$ by

$$q_2(\psi) = \int \langle \nabla_z\psi, \nabla_z\psi\rangle\,\mathrm{d}z + \frac{1}{2}\int \langle G_0^{-1/2}HG_0^{-1/2}z, z\rangle |\psi|^2\,\mathrm{d}z\,.$$

The associated operator is unitarily equivalent to $\mathscr{L}$, and defined by

$$\mathscr{L}_2 = -\Delta_z + \frac{1}{2}\langle G_0^{-1/2} H G_0^{-1/2} z, z\rangle\,.$$

We set $S = G_0^{-1/2} H G_0^{-1/2}$. As H is a positive definite matrix, then S is also a positive definite matrix. Consequently, there exists an orthogonal matrix Q such that

$$Q^{-1} S Q = B\,,$$

where B is a diagonal matrix formed by the positive eigenvalues of S. Notice that

$$B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_{N-1} \end{pmatrix},$$

where λ_i are the eigenvalues of S.
Considering $z = Qu$, we get

$$\begin{aligned} q_2(\psi) &= \int |\nabla_u \psi|^2 \,\mathrm{d}u + \frac{1}{2}\int \langle Q^{-1} S Q u, u\rangle |\psi|^2 \,\mathrm{d}u \\ &= \int |\nabla_u \psi|^2 \,\mathrm{d}u + \frac{1}{2}\int \langle Bu, u\rangle |\psi|^2 \,\mathrm{d}u := q_3(\psi)\,. \end{aligned}$$

The operator associated with the quadratic form q_3 is unitarily equivalent to $\mathscr{L}$ and defined by

$$\mathscr{L}_3 = -\Delta_u + \frac{1}{2}\langle Bu, u\rangle = -\sum_{i=1}^{N-1} \partial_{u_i}^2 + \frac{1}{2}\sum_{i=1}^{N-1} \lambda_i u_i^2 = \sum_{i=1}^{N-1} \big(-\partial_{u_i}^2 + \frac{1}{2}\lambda_i u_i^2\big) = \sum_{i=1}^{N-1} \mathscr{H}_i\,,$$

where, $\mathscr{H}_i$ is the harmonic oscillator

$$\mathscr{H}_i = -\partial_{u_i}^2 + \frac{1}{2}\lambda_i u_i^2\,.$$

The n-th eigenvalues of $\mathscr{H}_i$ are given by

$$(2n-1)\sqrt{\frac{\lambda_i}{2}}\,.$$

Consequently, the n-th eigenvalues of $\mathscr{L}_3$ is the same of $\mathscr{L}_1$, it is the n-th element of the following set, by grouping the elements in increasing order

$$\left\{ \beta_{1,n} = \sum_{i=1}^{N-1} \sqrt{\frac{\lambda_i}{2}}\,(2n_i - 1)\,, \quad n \in {\mathbb{N}^*}^{N-1} \right\}. \tag{7.2}$$

Since the matrices S and $G_0^{-1} H$ are similar, then they have the same spectrum and therefore λ_i are the eigenvalues of $G_0^{-1} H$.

7.2. **Relation to the Weingarten map.** We recall from the Section 3.1, the coefficients of the first and second fundamental form, in the base $(\partial_1\phi, \ldots, \partial_{N-1}\phi)$, are given respectively by the formulas

$$\forall i, j = 1, \ldots, N-1, \quad G_{ij}(s) = \partial_i\phi(s) \cdot \partial_j\phi(s) \quad \text{and} \quad K_{ij} = \partial_i\phi \cdot \partial_j \nu(\phi)\,,$$

where ϕ is a parameterization of $\partial\Omega$ defined in Section 3.1, such that $\phi(0) = A_0$ and $\nu(A_0) = (-1, 0, \ldots, 0)$. We get

$$\begin{aligned} K_{ij}(s) = \partial_i\phi(s) \cdot \partial_j\nu(\phi)(s) = \partial_i\phi(s) \cdot \mathrm{d}_s\nu(\phi)(e_j) &= \partial_i\phi(s) \cdot \mathrm{d}_{\phi(s)}\nu[\partial_j\phi(s)] \\ &= \partial_i\phi(s) \cdot W_{\phi(s)}(\partial_j\phi(s))\,. \end{aligned}$$

Let $L(s) := (l_{ij}(\phi(s)))_{1\leq i,j\leq N-1}$ the matrix of $W_{\phi(s)}$ in the base $(\partial_1\phi(s), ..., \partial_{N-1}\phi(s))$. Then

$$W_{\phi(s)}(\partial_j\phi(s)) = \sum_{i=1}^{N-1} l_{ij}(\phi(s))\partial_i\phi(s)\,.$$

Therefore, we can show that

$$K_{ij}(s) = \partial_i\phi(s)\cdot W_{\phi(s)}(\partial_j\phi(s))] = \partial_i\phi(s)\cdot\sum_{k=1}^{N-1} l_{kj}(\phi(s))\partial_k\phi(s) = \sum_{k=1}^{N-1} l_{kj}(\phi(s)G_{ik}(s)\,.$$

We deduce that

$$K(s) = G(s)L(s)\,. \tag{7.3}$$

On the other hand

$$K_{ij}(s) = \partial_i\phi(s)\cdot\partial_j\nu(\phi)(s) = -\partial_i\partial_j\phi(s)\cdot\nu(\phi)(s)\,.$$

In $s = 0$, we get

$$K_{ij}(0) = -\partial_i\partial_j\phi(s)|_{s=0}\cdot\nu(A_0) = \partial_i\partial_j\phi_1(s)|_{s=0} = h_{ij}\,,$$

where $H = \text{Hess}_{(0)}\phi_1 = (h_{ij})_{1\leq i,j\leq N-1}$.
Thanks to (7.3), we obtain

$$H = G(0)L(0) = G_0L(0)\,.$$

Consequently,

$$L(0) = G_0^{-1}H\,.$$

From the previous section, λ_i are the eigenvalues of $G_0^{-1}H$ and by the last equality, we deduce that λ_i are the eigenvalues of the Weingarten map W_{A_0} of $\partial\Omega$ at A_0 .

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REFERENCES

[1] Y. Almog, D. S. Grebenkov, and B. Helffer. Spectral semi-classical analysis of a complex Schrödinger operator in exterior domains. *J. Math. Phys.*, 59(4):041501, 12, 2018. 2

[2] Y. Almog, D. S. Grebenkov, and B. Helffer. On a Schrödinger operator with a purely imaginary potential in the semiclassical limit. *Comm. Partial Differential Equations*, 44(12):1542–1604, 2019. 2

[3] Y. Almog and B. Helffer. Global stability of the normal state of superconductors in the presence of a strong electric current. *Comm. Math. Phys.*, 330(3):1021–1094, 2014. 5

[4] G. Bastard, E. E. Mendez, L. L. Chang, and L. Esaki. Variational calculations on a quantum well in an electric field. *Phys. Rev. B*, 28:3241–3245, Sep 1983. 1

[5] H. Cornean, D. Krejčiřík, T. G. Pedersen, N. Raymond, and E. Stockmeyer. On the two-dimensional quantum confined Stark effect in strong electric fields. *SIAM J. Math. Anal.*, 54(2):2114–2127, 2022. 1, 2, 4, 9

[6] S. Fournais and B. Helffer. Accurate eigenvalue asymptotics for the magnetic neumann laplacian. 56(1):1–67, 2006. 13

[7] S. Fournais and B. Helffer. *Spectral methods in surface superconductivity*, volume 77 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2010. 5

[8] D. S. Grebenkov and B. Helffer. On spectral properties of the Bloch-Torrey operator in two dimensions. *SIAM J. Math. Anal.*, 50(1):622–676, 2018. 2

[9] V.A. Harutyunyan. Cylindrical nanolayer in the strong uniform electrical field: The field localization of carriers and electrooptical transitions. *Physica E: Low-dimensional Systems and Nanostructures*, 41(4):695–700, 2009. 2

[10] B. Helffer. *Semi-classical analysis for the Schrödinger operator and applications*, volume 1336 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988. 5

[11] B. Helffer and A. Morame. Magnetic bottles for the Neumann problem: curvature effects in the case of dimension 3 (general case). *Ann. Sci. École Norm. Sup. (4)*, 37(1):105–170, 2004. 5

[12] Bernard Helffer and Ayman Kachmar. Eigenvalues for the robin laplacian in domains with variable curvature. *Transactions of the American Mathematical Society*, 369(5):3253–3287, 2017. 9

[13] J. Lee and H.N. Spector. Stark effect in the optical absorption in quantum wires. *Journal of Applied Physics*, 97(4):043511, 2005. 1, 2

[14] X. B. Pan. Surface superconductivity in 3 dimensions. *Trans. Amer. Math. Soc.*, 356(10):3899–3937, 2004. 5

[15] T. G. Pedersen. Stark effect in finite-barrier quantum wells, wires, and dots. *New Journal of Physics*, 19(4):043011, apr 2017. 2

[16] T. G. Pedersen. Stark effect in spherical quantum dots. *Phys. Rev. A*, 99:063410, Jun 2019. 1

[17] T G Pedersen, H D Cornean, D Krejčiřik, N Raymond, and E Stockmeyer. Stark-localization as a probe of nanostructure geometry. *New Journal of Physics*, 24(9):093005, sep 2022. 1

[18] B. Sukumar and K. Navaneethakrishnan. Polarizability of a carrier in an isolated well of a quantum-well wire. *Phys. Rev. B*, 41:12911–12914, Jun 1990. 2

[19] G. J. Vázquez, M. del Castillo-Mussot, and Harold N. Spector. Transverse stark effect of electrons in a semiconducting quantum wire. *physica status solidi (b)*, 240(3):561–564, 2003. 1, 2

[20] Sheng W., Yun K., and Chunjie H. Transverse stark effect in the optical absorption in a square semiconducting quantum wire. *Journal of Semiconductors*, 34(10):102001, oct 2013. 2

[21] S. Wang, Y. Kang, and C. Han. Transverse stark effect in the optical absorption in a square semiconducting quantum wire. *Journal of Semiconductors*, 34(10):102001, oct 2013. 1

[22] G. Wei, S. Wang, and G. Yi. Stark effect of electrons in semiconducting rectangular quantum boxes. *Microelectronics Journal*, 39(5):786–791, 2008. 1

[23] G. Yi, G. Wei, and H. Wu. Transverse stark effect in a rectangular semiconducting quantum wire. *physica status solidi (b)*, 244(12):4651–4659, 2007. 1, 2

LAREMA, FACULTÉ DES SCIENCES, 2 BOULEVARD LAVOISIER, UNIVERSITÉ D'ANGERS, 49045 ANGERS, FRANCE.
Email address: rayan.fahs@univ-angers.fr