ON THE STRONG HYBRID SOLUTION OF AN N-PERSON GAME

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Abstract

We propose a new notion of coalitional equilibrium, the strong hybrid solution, which is a refinement of Zhao's hybrid solution. It is well suited to study situations where people co-operate within coalitions but where coalitions compete with one another. In the strong hybrid solution, as opposed to the hybrid solution, the strategy profile assigned to each coalition is strongly Pareto optimal. We show that there exists a strong hybrid solution whenever preferences are partially quasi-transferable.

***Keywords**: N-Person Game, coalition structure, hybrid solution, strong hybrid solution*

1 INTRODUCTION

To study cooperation in strategic non-transferable utility games, one can rely on the concepts of coalitional equilibria. Those concepts differ in the way outsiders are assumed to react to the formation of a deviating coalition. The three traditional concepts of coalitional equilibria, all introduced in Aumann [1959], include the α-core, the β-core (Aumann [1961, 1959]) and the strong Nash equilibrium (Aumann [1959]). An action profile is in the α-core if no coalition can guarantee a higher payoff for each of its members by choosing another action, independently of the actions

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taken by the other players. On the other hand, an action profile is in the β-core if no deviating coalition can guarantee a higher payoff for each of its members, because coalition outsiders have a dominant blocking strategy. In a strong Nash equilibrium, no coalition can guarantee a higher payoff for each of its members assuming that coalition outsiders stick to their equilibrium strategies.

Scarf [1971] gave existence results of α-core in games with nontransferable utilities, which were later extended for local α and β-cores by Yano [1990]. Kajii [1992] showed the non-emptiness of the α-core without assuming that agents' preference relations are transitive or complete. Zhao [1999a, 1999b] obtained general existence results for α and β-cores in games with transferable utility. More recently, Uyanik [2015] addressed the existence of α-core in discontinuous games. In addition, Yang [2017a, 2018] generalized Kajii's theorem to games with infinitely many players, Yang [2017b] considered the essential stability of α-core and Yang and Meng [2017] tackled the Hadamard well-posedness of the α-core.[1] Martins-da-Rocha and Yannelis [2011] established the existence of the α-core in games where agents' preferences are non-ordered and with infinite dimensional strategy spaces. Askoura et al. [2013] analyzed the ex-ante α-core in games with uncertainty while Noguchi [2018] focused on the α-cores in games with asymmetric information.

Yet, the coalitional equilibrium concepts initially defined by Aumann are probably too demanding. They require that *each* conceivable coalition should not break away from a prealably agreed strategy profile. As a consequence, the class of games for which the α-core and the β-core exist is not large. In that connection, it is worthy to note that in many "real-world" situations not all the coalitions are relevant (for instance, it is doubtful that certain countries can ever cooperate, so different are their views on the issue at hand, be it for international trade, environmental protection and so on). To put it another way, some coalition can form more easily than others.

Zhao [1992] is a major attempt to study coalition games with restricted formation of coalitions. Specifically, he introduces the *hybrid solution*, a solution concept pertaining to the case where there is cooperation inside coalitions, but competition between coalitions. The main issue here is the internal stability of the coalitions belonging to a *given* coalition structure.

Consider for instance Catalogna in Spain, Corsica in France, or Scotland in Great Britain. In each of these cases, a region (that is, a subcoalition of a given coalition) contemplates breaking away from an existing coalition (a Member Sate of the European Union). But to the best of

[1] Roughly said, a strategy profile in the α-core of a game is essential if whenever we slightly modify the game, the α-core of the modified game has elements which are close to the strategy profile. A game is Hadamard well-posed if its α-core has a unique element and if this element changes continuously with the conditions defining the game.

our knowledge, none of these regions endeavors to join, or merge, with *another* member of the European Union (that is, another coalition).

In relation to the remark above, the hybrid solution is a strategy profile that is stable against deviations by any subcoalition of a coalition belonging to a given partition of the players. More precisely, any subcoalition of a given coalition S can be blocked by *some* decisions taken by the outsiders in the coalition, assuming that all the other players (in the other coalitions) stick to their strategies. Here, blocking means that a subcoalition cannot choose a strategy such that the payoff obtained by any of its members is strictly greater than what he obtains with the hybrid solution, when the outsiders in the coalition minimize his gain.

Zhao (*ibid*) showed the existence of the hybrid solution and Zhao [1999a, 1999b] extended the notions of TU α and β cores to the hybrid solution.[2] Existence of a version of the hybrid solution for games with non-ordered preferences, infinite dimensional action spaces (namely Hausdorff topological vector spaces) and infinitely many players was studied in Yang and Yuan [2019]. In their approach, however, the definition of blocking is slightly different from Zhao's. It is actually similar to the definition used in the α-core.[3]

This paper introduces a new coalitional equilibrium concept, namely the strong hybrid solution, that applies for any given coalition structure $\mathfrak{C}$. This definition furthers the approach of Yang and Yuan [2019] in the sense that it introduces yet another definition of blocking. Here, we propose to say that an alternative strategy chosen by a subcoalition is blocked by the outsiders in the coalition if it cannot guarantee a payoff at least as great as the reference one for each of its members, and strictly higher than the reference one for at least one of its members, independently of the actions taken by all the other players (like in the hybrid solution, we assume that the players outside the coalition stick to their strategies). To wit, because coalition members are assumed to *cooperate*, there is no reason why they would not collectively choose an alternative strategy profile that does no good nor bad for all of them, and strictly benefits to at least one member. With this definition, one can show that the set of strong hybrid solution is a subset of the Yang and Yuan's hybrid solutions, which are themselves a subset of Zhao's hybrid solutions. Hence the assertion that the notion of strong hybrid solution is a refinement of the hybrid solution. Our main contribution is to provide an existence result for the strong hybrid solution. This existence result is obtained under

[2] Zhao [1996] also applies the hybrid solution concept to an exchange economy with externalities. Zhao [2018] reviews different solution concepts for TU games in Industrial Organization.

[3] This definition implies that there is no subcoalition which can choose a strategy such that the payoff obtained by any of its members is no lower (as opposed to not strictly greater as in Zhao [1992]) than what he obtains with the hybrid solution, when the outsiders in the coalition minimize his gain.

the assumption that preferences are partially quasi-transferable.[4]

The remainder of the paper is organized as follows. Section 2 sets out formal definitions, establishes some properties for the strong hybrid solution and compare it with alternative ones. Section 3 establishes our existence result for the strong hybrid solution. Section 4 studies the condition of partially quasi-transferable preferences in two economic models. Section 5 provides some concluding remarks.

2 Definitions and Properties

In this section, we first present our setting. Then, we state some definitions and properties for different notions of coalitional equilibria (including ours).

2.1 Preliminaries

Consider an n-person game $\mathrm{G} = \langle I, X_i, u_i \rangle$ where $I = \{1, ..., n\}$ is the set of players, X_i (a nonempty set in $\mathbb{R}^{l(i)}$, $l(i) \geq 1$) is player i's strategy set and $u_i : X \to \mathbb{R}$ is player i's payoff function, where $X = \prod\limits_{i \in I} X_i$.

For each nonempty coalition S of players, and for any subset K of S, define $-S = \{i \in I : i \notin S\}$ and $S - K = \{j \in S : j \notin K\}$. If S is a singleton $\{i\}$, we denote by $-i$ all the players but i. We also denote by $X_S = \prod\limits_{i \in S} X_i$ the set of strategies of coalition S's players. A coalition structure $\mathfrak{C} = \{P_1, ..., P_m\}$ is a partition of I. That is, $\bigcup\limits_{i=1,...,m} P_i = I$ and $P_i \cap P_j = \emptyset$, for each $i \neq j$. A coalition structure $\mathfrak{C}$ being given, we can express any strategy profile $x = (x_1, ..., x_n) \in X$ as $x = (x_{P_1}, x_{P_2}, .., x_{P_m})$, where $x_{P_i} \in X_{P_i}$.

We say that a game G is compact, convex, and continuous if for all $i \in I$, X_i is compact, convex, and u_i is continuous on X.

For each C in the coalition structure $\mathfrak{C}$, we denote by Δ_C the $|C| - 1$ simplex of $\mathbb{R}^{|C|}$, that is

$$\Delta_C = \left\{ \lambda \in \mathbb{R}_+^{|C|} : \sum_{i=1}^{|C|} \lambda_i = 1 \right\}.$$

Let E be a vector space and Z be a convex set in E. A function $f : Z \to \mathbb{R}$ is concave on Z if, for any z_1, z_2 in Z and for any $\theta \in [0, 1]$, $\theta f(z_1) + (1 - \theta) f(z_2) \leq f(\theta z_1 + (1 - \theta) z_2)$. It is convex

[4] Following Rader [1972], preferences are said to be quasi-transferable if whenever a strategy profile x is at least as good as another one x' for a group of players, and strictly better for at leat one agent, then there is a third profile x'' which strictly better than x' for all the agents. Our notion of partially quasi-transferable preferences builds on Rader's definition.

on Z if $-f$ is concave on Z. A function $f : Z \to \mathbb{R}$ is quasiconcave on Z if, for any z_1, z_2 in Z and for any $\theta \in [0,1]$, $\min\{f(z_1), f(z_2)\} \leq f(\theta z_1 + (1-\theta)z_2)$. It is quasiconvex on Z if $-f$ is quasiconcave on Z. A function $f : Z \to \mathbb{R}$ is strictly-quasiconcave on Z if, for any z_1, z_2 in Z with $z_1 \neq z_2$ and for any $\theta \in (0,1)$, $\min\{f(z_1), f(z_2)\} < f(\theta z_1 + (1-\theta)z_2)$. A correspondence $C : Z \twoheadrightarrow E$ is said to be closed, or has closed graph, if its graph

$$\mathrm{Gr}(C) = \{(x,y) \in Z \times E : \ y \in C(x)\}$$

is a closed subset in $Z \times E$. (For more details see Aliprantis and Border [2006]).

For any two vectors $a, b \in \mathbb{R}^n$, we write $a \succeq b$ if $a_i \geq b_i$ for all $i = 1, ..., n$, $a \gg b$ if $a_i > b_i$ for all $i = 1, ..., n$ and $a \succ b$ if $a \succeq b$ and $a \neq b$.

2.2 Coalitional Equilibria

We first recall the definitions of the α and β-cores of an n-person game introduced in Aumann [1961, 1959]. A strategy profile $\overline{x}$ is in the α-core if for every coalition K and for each deviation x_K of K, players in $-K$ can choose a strategy profile x_{-K} such that the payoff obtained by any member of K at (x_K, x_{-K}) cannot be higher than its value when all the players stick to $\overline{x}$. A strategy $\overline{x}$ is in the β-core if for every coalition K, players in $-K$ can choose a strategy $\widetilde{x}_{-K}$ such that for each deviation x_K of K, the payoff obtained by any member of K at $(x_K, \widetilde{x}_{-K})$ cannot be higher than its value when all the players stick to $\overline{x}_K$.

We next recall the formal definition of the *hybrid solution*.

Definition 2.1 (Zhao [1992]) *Given a coalition structure $\mathfrak{C}$, a feasible strategy $\overline{x} \in X$ is a hybrid-solution if, for every coalition S in $\mathfrak{C}$ and for any subcoalition K of S there is no strategy profile x_K in X_K such that $\underline{u}_K(x_K, \overline{x}_{-S}) \gg u_K(\overline{x})$, where $\underline{u}_K(x_K, \overline{x}_{-S}) = \{\inf_{x_{S-K}} u_i(x_K, x_{S-K}, \overline{x}_{-S}),\ i \in K\}$.*

A slightly different definition of the hybrid solution appears as follows in Yang and Yuan [2019].

Definition 2.2 (Yang and Yuan [2019]) *Given a coalition structure $\mathfrak{C}$, a feasible strategy $\overline{x} \in X$ is a hybrid solution if, for every coalition S in $\mathfrak{C}$ and for any subcoalition K of S there is no x_K in X_K such that $u_K(x_K, x_{S-K}, \overline{x}_{-S}) \gg u_K(\overline{x})$, for all x_{S-K} in X_{S-K}.*

The next property states a link between the two definitions above of the hybrid solution.

Property 2.1 *Any hybrid solution in the sense of Yang and Yuan [2019] is a hybrid solution in the sense of Zhao [1992].*

This result stems from the fact that for the hybrid solution in the sense of Yang and Yuan [2019], there is *no* x_K such that $\underline{u}_K(x_K, \overline{x}_{-S}) \succeq u_K(\overline{x})$.

We next introduce our notion of *strong hybrid solution*.

DEFINITION 2.3 *Given a coalition structure $\mathfrak{C}$, a strategy profile $\overline{x}$ in X is a* **strong hybrid solution** *of game G if for every coalition S in $\mathfrak{C}$ and for any subcoalition K of S, there does not exist x_K in X_K such that for each x_{S-K} in X_{S-K} $u_K(x_K, x_{S-K}, \overline{x}_{-S}) \succ u_K(\overline{x})$, with $u_K(.) = (u_i(.),\ i \in K)$.*

That is, $\overline{x} \in X$ is a strong hybrid solution if, given any coalition S in $\mathfrak{C}$ and any subcoalition K of S, for each deviation x_K of K, players in $S - K$ have a strategy x_{S-K} such that at $(x_K, x_{S-K}, \overline{x}_{-S})$, the payoffs of players in K cannot be higher than what they would obtain by playing $\overline{x}_K$, when the players of $-K$ play $\overline{x}_{-K}$. This notion of coalitional equilibrium is stable against the deviations of each subcoalition K of each coalition S in $\mathfrak{C}$.

The hybrid solution and the strong hybrid solution rely on different definitions of a blocking coalition. In a strong hybrid solution an alternative strategy of a subcoalition is blocked by the outsiders in the coalition if it cannot guarantee a payoff at least as great as the reference one for each of its members, and strictly higher than the reference one for at least one of its members, independently of the actions taken by outsiders (the players outside the coalition sticking to their strategies). By contrast, with the hybrid solution an alternative strategy of a subcoalition is blocked by the outsiders in the coalition if it cannot guarantee a payoff strictly higher than the reference one for *all* its members, independently of the actions taken by outsiders (the players outside the coalition sticking to their strategies).

In our view, the strong hybrid solution is more consistent with the assumption that coalition members *cooperate* whereas coalitions compete with one another. To wit, because coalition members are assumed to *cooperate*, there is no reason why they would not collectively choose an alternative strategy profile that does no good nor bad for all of them, and strictly benefits to at least one member.
From definition 2.3, it is immediate to obtain the two following properties.

Property 2.2 *The strong hybrid solution is individually rational, i.e., for each deviation x_i of every player i in coalition S, there exists a punishment strategy x_{S-i} of players in $S - i$ such that player i cannot be better off when players in $-S$ play $\overline{x}_{-S}$.*

Property 2.3 *The strong hybrid solution is such that for every coalition S in $\mathfrak{C}$, the strategy profile $\overline{x}_S$ is a solution to the following multiobjective optimization problem $\langle X_S, \{u_i(., \overline{x}_{-S}), \}_{i\in S}\rangle$.*[5]

In particular, this implies that $\overline{x}_S$ is Pareto-efficient in X_S.

The next property states that the strong hybrid solution is a refinement of the hybrid solutions of Zhao [1992] and Yang and Yuan [2019]).

Property 2.4 *Any strong hybrid solution is also a hybrid solution (both in the senses of Zhao [1992] and Yang and Yuan [2019]).*

That is because, every strong hybrid solution is a hybrid solution in the sense of Yang and Yuan [2019], and then from property 2.1, is also a hybrid solution in the sense of Zhao [1992].

The next example illustrates the differences between the notions of coalitional equilibria defined above.

EXAMPLE 2.1 *Consider a three-player game G such that $I = \{1, 2, 3\}$, $X_i = [0, 1]$, $i = 1, 2, 3$ and*

$$\begin{cases} u_1(x) = 1 - (x_1 - x_2)^2 + x_3 + \begin{cases} \frac{3}{2}x_1 & \text{if } x_1 \geq \frac{1}{2}, \\ -\frac{3}{2}x_1 + \frac{3}{2} & \text{if } x_1 \leq \frac{1}{2} \end{cases} \\ u_2(x) = x_2 + x_3, \ u_3(x) = x_3. \end{cases}$$

Let $\mathfrak{C} = \{\{1, 2\}, \{3\}\}$. The only strong hybrid solution is given by $(\overline{x}_1, \overline{x}_2, \overline{x}_3) = (1, 1, 1)$. The set of hybrid solutions (both in the senses of Zhao [1992] and Yang and Yuan [2019]) is

$$\left\{ \overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) \in X : \overline{x}_1 \in [\frac{7 - 2\sqrt{6}}{4}, 1], \overline{x}_2 = \overline{x}_3 = 1 \right\}.$$

Observe that the unique strong hybrid solution Pareto-dominates all the hybrid solutions (except, of course, when the later coincides with the former). Since by assumption players 1 and 2 cooperate, it seems relevant that they will grasp any opportunity to increase their payoffs, even if only one player's payoff actually increases.

We now relate the strong hybrid solution to other solutions concepts. Recall that a strategy profile $\overline{x} \in X$ is a ***Nash equilibrium*** of the game G if,

$$u_i(y_i, \overline{x}_{-i}) \leq u_i(\overline{x}), \text{ for all } i \in I \text{ and for all } y_i \in X_i.$$

[5] Due to its connection with mathematical programming, multiobjective programming is more appealing than alternative terms such as vector maximization or multicriteria decision making. More importantly, Borm et al. [1988] first used the term multiobjective games (MOG). However, several later authors had used different terms such as Multicriteria Games, which were unnecessary because it not only caused confusion in the field but also showed a bit disrespect for the original contributors.

The next result property is easily obtained.

Property 2.5 *If the coalition structure $\mathfrak{C}$ is such that $\mathfrak{C} = \{\{1\}, \{2\}, ..., \{n\}\}$, then the set of strong hybrid solutions and Nash equilibria coincide.*

We next consider a refinement of the Nash equilibria.

DEFINITION 2.4 (Aumann [1959]) A strategy profile $\overline{x}$ in X is a ***strong Nash equilibrium*** of game G if, for every coalition S, there is no y_S in X_S such that $u_S(y_S, \overline{x}_{-S}) \succ u_S(\overline{x})$.

That is, a strategy profile is a strong equilibrium if no coalition can profitably deviate from this profile. A straightforward implication is that any strong equilibrium is both Pareto efficient and stable with regard to the deviation of any coalition. We then have:

Property 2.6 *Any strong equilibrium is also a strong hybrid solution.*

We consider yet another coalition equilibrium notion, which adopts the concept of absolute optimal solution in multiobjective programming (Zhao [1983]) and in games (Nessah and Tazdait [2013]).

DEFINITION 2.5 *(Nessah and Tazdait [2013]) Given a coalition structure $\mathfrak{C}$, a feasible strategy $\overline{x} \in X$ is an* **absolute optimal solution relatively to** $\mathfrak{C}$ *(or $\mathfrak{C}$-absolute optimal solution) of game G if, for every coalition S in $\mathfrak{C}$, we have $u_j(y_S, \overline{x}_{-S}) \leq u_j(\overline{x})$, for each j in S and y_S in X_S.*

Hence, a strategy profile $\overline{x}$ is an absolute optimal solution relatively to $\mathfrak{C}$ if no player in any coalition S in $\mathfrak{C}$, can be better off when players in S deviate from their $\mathfrak{C}$-absolute optimal strategy profile $\overline{x}_S$. This equilibrium is stable against deviations from any player in S.

Property 2.7 *Any absolute optimal solution relatively to $\mathfrak{C}$ is also a strong hybrid solution.*

3 EXISTENCE RESULT

This section establishes a general existence result for a strong hybrid solution. This result relies on the specific assumption that preferences are partially quasi-transferable as well as on the arguments used in the proof of Theorem 3 in Zhao [1992].

To explain this specific assumption we shall need some notations and definitions. Firstly, without loss of generality, we assume that $\min_{x \in X} u_i(x) > 0$ for each $i \in I$. Secondly, recall that for each $x \in X$, $S \subseteq C$, $\underline{u}_S(x_S, \overline{x}_{-C})$ is defined as follows

$$\underline{u}_S(x_S, \overline{x}_{-C}) = \left\{ \inf_{y_{C-S} \in X_{C-S}} u_i(x_S, y_{C-S}, \overline{x}_{-C}),\ i \in S \right\}.$$

Definition 3.1 ***(Partially quasi-transferable preferences)*** *Given a coalition structure $\mathfrak{C}$, a game G satisfies the assumption of partially quasi-transferable preferences if, for every $x \in X$, $C \in \mathfrak{C}$, $\overline{x}_C \in X_C$, $\gamma \in \Delta_C$, every $S \subseteq C$, $t > 0$ such that*

$$\underline{u}_S(x_S, x_{-C}) \succ u_S(\overline{x}_C, x_{-C}) \succeq t\gamma_S,$$

there exist a sub-coalition $\tilde{S} \subseteq C$ and a strategy $\tilde{x}_{\tilde{S}} \in X_{\tilde{S}}$ such that

$$\underline{u}_{\tilde{S}}(\tilde{x}_{\tilde{S}}, x_{-C}) \gg t\gamma_{\tilde{S}}.$$

This condition actually states that if the payoff of each member i of coalition S is such that $\underline{u}_i(x_S, x_{-C}) \geq u_i(\overline{x}_C, x_{-C}) \geq t\gamma_i$ (with a strict inequality for at least one i in the first inequation), it is possible to find a subcoalition $\tilde{S}$ whose members are able to secure a payoff satisfying the condition: $\underline{u}_i(\tilde{x}_{\tilde{S}}, x_{-C}) > t\gamma_i$.

Definition 3.1 is connected with the notion of quasi-transferability of preferences that we recall next.

Definition 3.2 *We say that a game satisfies the quasi-transferable preferences property if for each coalition S, whenever $x_S \succeq_i y_S$ for all $i \in S$, and there is j in S such that $x_S \succ_j y_S$, then there is $\bar{x}_S$ in X_s such that $\bar{x}_S \succ_i y_S$ for all i in S.*

This definition is adapted from a Rader [1972] (who introduced it in a general equilibrium context) and was also used in, *e.g*, Diamantaras and Wilkie [1996]. In relation to the above definition, the assumption of partially quasi-transferable preferences refers to a transformation of the initial preferences, and only concerns a subset of agents (hence the term partially) of the initial coalition (notice that the agents in the subcoalition are assumed to do better despite the fact that their former teammates in $S \setminus \tilde{S}$ joins people in $C \setminus S$ do lower their payoffs).

We shall illustrate definition 3.1 in the following example.

Example 3.1 Let us consider the following four players contribution game in which each player can either contribute to a collective action or retract his contribution. Let $I = \{1, 2, 3, 4\}$ be the set of players, $X_i = [0, 1]$ be the strategy space of player's i, and player i's payoff function be given by

$$u_i(x) = -ix_i + \sum_{j \neq i} x_j,\ i = 1, 2, 3, 4.$$

Let the coalition structure $\mathfrak{C} = \{\{1, 2\}, \{3, 4\}\}$. This game satisfies the assumption of partially quasi-transferable preferences. Indeed, let $x \in X$, $C = \{i, j\} \in \mathfrak{C}$, $\overline{x}_C \in X_C$, $\gamma \in \Delta_C$, $t > 0$, $S \subseteq C$ such that

$$\underline{u}_S(x_S, x_{-C}) \succ u_S(\overline{x}_C, x_{-C}) \succeq t\gamma_S.$$

We distinguish the following cases:

1) If $S = \{h\}$ with $h = i, j$. Then obviously we have

$$\underline{u}_h(x_h, x_{-C}) > u_h(\overline{x}_C, x_{-C}) \succeq t\gamma_h.$$

2) If $S = C$, then

$$\begin{cases} \underline{u}_i(x_i, x_j, x_{-C}) > u_i(\overline{x}_i, \overline{x}_j, x_{-C}) \succeq t\gamma_i, \\ \underline{u}_j(x_i, x_j, x_{-C}) \geq u_j(\overline{x}_i, \overline{x}_j, x_{-C}) \succeq t\gamma_j. \end{cases}$$

This implies that

$$\begin{cases} -ix_i + x_j > -i\overline{x}_i + \overline{x}_j, \\ x_i - jx_j \geq \overline{x}_i - j\overline{x}_j. \end{cases}$$

We can show that both $x_i < \overline{x}_i$ and $x_j < \overline{x}_j$. Since $-ix_i + x_j > -i\overline{x}_i + \overline{x}_j$, then $x_i < \overline{x}_i + \frac{x_j - \overline{x}_j}{i}$. Let us choose $\epsilon > 0$ such that

$$x_i + \epsilon < \overline{x}_i + \frac{x_j - \overline{x}_j}{i}.$$

Therefore there exists a strategy $\tilde{x}_{\tilde{S}} \in X_{\tilde{S}}$ (defined by $\tilde{x}_i = x_i + \epsilon$ and $\tilde{x}_j = x_j$) such that

$$\underline{u}_S(\tilde{x}_S, x_{-C}) \gg t\gamma_S.$$

Our existence result is as follows.

THEOREM 3.1 *A coalition structure $\mathfrak{C}$ being given, assume that the game G is continuous, compact and convex and the functions u_i are quasiconcave in x_C, for each $C \in \mathfrak{C}$, $i \in C$. Then if the game satisfies the partially quasi-transferable preferences assumption, it has a strong hybrid solution.*

The proof of this theorem uses the lemmata 3.1 and 3.2 established in Zhao [1992]. Let us introduce some notations used in the statements of these results. Let $C \in \mathfrak{C}$ and S any subcoalition in C ($S \subseteq C$). Let $t_S : \Delta_C \times X_{-C} \to \mathbb{R}$ be the function defined by

$$t_S(\alpha, y) = \max_{t>0,\, x_S \in X_S} t$$
$$\begin{cases} \underline{u}_S(x_S, y) - t\alpha_S \succeq 0, \\ ce_{C-S} - t\alpha_{C-S} \succeq 0, \end{cases} \qquad \text{(Problem 1)}$$

where $\alpha_C = (\alpha_S, \alpha_{C-S}) \in \Delta_C$, $e_C = (1, \ldots, 1) \in \mathbb{R}^{|C|}$ and $c \in \mathbb{R}_+$ is such that $u_i(x) < c$ for all $i \in N$ and $x \in X$. Finally, let $\overline{t} : \Delta_C \times X_{-C} \to \mathbb{R}$ be defined by

$$\overline{t}_C(\alpha, y) = \max_{S \subseteq C} t_S(\alpha, y).$$

The first of the two lemmas drawn from Zhao [1992] is presented next.

Lemma 3.1 *Zhao [1992] When the game G is convex, continuous, and the function u_i are quasiconcave in x_C, for each $C \in \mathfrak{C}$, for all i, the following two properties hold.*

1) *The function $f_C : \Delta_C \times X_{-C} \to \mathbb{R}^n$ defined by*

$$f_C(\alpha, y) = \bar{t}_C(\alpha, y)\alpha$$

is continuous in (α, y).

2) *The correspondence $\theta_C : \Delta_C \times X_{-C} \twoheadrightarrow C$ defined by*

$$\theta_C(\alpha, y) = \left\{S \subseteq C : \ t_S(\alpha, y) = \bar{t}_C(\alpha, y)\right\}$$

is nonempty and has a closed graph.

Now for each $S \subseteq C$ in $\mathfrak{C}$, let $h_S : C \to \Delta_C$ be the function defined by $h_S(i) = \frac{1}{|S|}$ if $i \in S$ and 0 otherwise. Define also the function $g_C : \Delta_C \times \Delta_C \to \Delta_C$ by

$$g_C(\alpha, \beta) = \left(\frac{\alpha_i + \max\left\{0, \beta_i - \frac{1}{n}\right\}}{1 + \sum_{j=1}^{n} \max\left\{0, \beta_j - \frac{1}{n}\right\}}, \ i = 1, \ldots, |C| \right)$$

and the correspondence $\sigma_C : \Delta_C \times X_{-C} \twoheadrightarrow \Delta_C$ by

$$\sigma_C(\alpha, y) = \mathrm{co}\left\{h_S : \ S \in \theta(\alpha, y)\right\}.$$

Furthermore. We say that a collection $B = \{S_1, \ldots, S_{|B|}\}$ of subsets of C is balanced if there exists $y_{S_j} \in \mathbb{R}_{++}$ for each $S_j \in B$, such that for all x_C in X_C, $\sum_{j, x_C \in S_j} y_{S_j} = 1$.

Lemma 3.2 *(Zhao [1992]) When the game G is convex, continuous, and the function u_i are quasiconcave in x_S for each $S \in C$ and $C \in \mathfrak{C}$, for all i, then the correspondence $\pi_C : \Delta_C \times \Delta_C \times X_{-C} \twoheadrightarrow \Delta_C \times \Delta_C$ defined by*

$$\pi_C(\alpha, \beta, y) = \{g_C(\alpha, \beta)\} \times \sigma_C(\alpha, y)$$

satisfies the two following properties.

1) *It has closed bounded nonempty convex values and a closed graph.*

2) *For each fixed $y \in X_{-C}$, the correspondence $\pi'_C(\alpha, \beta) = \pi_C(\alpha, \beta, y)$ has a fixed point (α^*, β^*) and at this fixed point $\theta_C(\alpha^*, y)$ is a balanced set.*

In addition to the two above lemmata, our proof of Theorem 3.1 will make use of a third lemma which we shall prove next. This lemma refers to the notion of strong core solution which we state beforehand:

DEFINITION 3.3 Let a coalition C in $\mathfrak{C}$ be given. A strategy profile $\overline{x}_C$ is said to be a **strong core solution** of the subgame $\tilde{G}(\overline{x}_{-C}) = \big(S, X_S, \underline{u}_S(x_S, \overline{x}_{-C})\big)$ if, for any subcoalition S of C, there does not exist x_S in X_S such that $\underline{u}_S(x_S, \overline{x}_{-C}) \succ u_S(\overline{x})$.

We shall also say that the game $\tilde{G}(\overline{x}_{-C}) = \big(S, X_S, \underline{u}_S(x_S, \overline{x}_{-C})\big)$ is balanced if for all y in $\mathbb{R}^{|C|}$ and any balanced collection of coalitions θ such that for each S in θ, there exists $x_S \in X_S$ satisfying $\underline{u}_S(x_S, \overline{x}_{-C}) \geq y_S$, then there exists $x_C \in X_C$ such that $u_C(x_C, \overline{x}_{-C}) \geq y_C$.

LEMMA 3.3 *Assume that the game G is continous, compact and convex. Assume also that the functions u_i are quasiconcave in x_S, for each $S \in C$ and $C \in \mathfrak{C}$, and such that $\min_X u_i(x) > 0$ for all i. Suppose further that preferences are partially quasi-transferable. Then the correspondence $\delta_C : \Delta_C \times X_{-C} \twoheadrightarrow X_C$ defined by*

$$\delta_C(\alpha, y) = \{x_C \in X_C :\ u_C(x_C, y) - t_C(\alpha, y)\alpha \succeq 0\}$$

satisfies the next two properties.

1) *It has closed bounded nonempty convex values and a closed graph.*

2) *If $\theta_C(\alpha, y)$ is a balanced family and $\tilde{G}(y)$ is a balanced game for each y, then each $x_C \in \delta_C(\alpha, y)$ is a strong core solution of $\tilde{G}(y)$.*

PROOF. The first assertion is then a direct consequence of the continuity of the functions u_i, $t_C(\alpha, y)\alpha$ and the quasiconcavity of u_i in x_C.

Let us prove the second assertion. As $\theta_C(\alpha, y)$ is a balanced family and $\tilde{G}(y)$ is a balanced game for each y, we can show as in the proof of lemma 3 in Zhao that $t_C(\alpha, y) = \bar{t}_C(\alpha, y)$.

Now let $\overline{x}_C \in \delta_C(\alpha, y)$. If $\overline{x}_C$ is not a strong core solution of $\tilde{G}(y)$, then there is a coalition $S \subseteq C$ and $x_S \in X_S$ such that

$$\underline{u}_S(x_S, y) \succ u_S(\overline{x}_C, y). \tag{3.1}$$

Since $\overline{x}_C \in \delta_C(\alpha, y)$ and $t_C(\alpha, y) = \bar{t}_C(\alpha, y)$, then (3.1) becomes

$$\underline{u}_S(x_S, y) \succ \bar{t}_C(\alpha, y)\alpha_S.$$

But by the choice of the constant c and since $\overline{x}_C \in \delta_C(\alpha, y)$, then

$$ce_{C-S} \gg \bar{t}_C(\alpha, y)\alpha_{C-S}.$$

As preference are partially quasi-transferable, there exist $\tilde{S} \subseteq C$, $\tilde{x}_{\tilde{S}} \in X_{\tilde{S}}$ and $\tilde{t} > \overline{t}_C(\alpha, y)$ such that

$$\begin{cases} \underline{u}_{\tilde{S}}(\tilde{x}_{\tilde{S}}, y) \succeq \tilde{t}\alpha_{\tilde{S}} \\ ce_{C-\tilde{S}} \succeq \tilde{t}\alpha_{C-\tilde{S}}. \end{cases}$$

Hence, $\tilde{t}$ and $\tilde{x}_{\tilde{S}}$ is feasible for the Problem 1. Then $\tilde{t} \leq \overline{t}_C(\alpha, y)$ which contradicts the relation $\tilde{t} > \overline{t}_C(\alpha, y)$. ■

Proof of Theorem 3.1.

Let Ω be the set defined by

$$\Omega = \left(\prod_{i=1,\ldots,r} (\Delta_{C_i} \times \Delta_{C_i}) \right) \times X$$

where r is the cardinal of $\mathfrak{C}$.

Let $\Psi : \Omega \to 2^{\Omega}$ be the correspondence defined by

$$\Psi\left(\left((\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\right), x\right) = \left(\prod_{i=1,\ldots,r} \pi_{C_i}(\alpha_i, \beta_i, x_{-C_i}) \right) \times \left(\prod_{i=1,\ldots,r} \delta_{C_i}(\alpha_i, x_{-C_i}) \right).$$

By Lemmas 3.1-3.3, the correspondence Ψ is defined on a compact convex set and has nonempty closed convex values and a closed graph. Then by Kakutani fixed point Theorem, Ψ has a fixed point $(\overline{\alpha}_i, \overline{\beta}_i)$, for each $i = 1, \ldots, r$ and there is a vector $\overline{x} = (\overline{x}_{C_1}, \ldots, \overline{x}_{C_r})$ such that $(\overline{\alpha}_i, \overline{\beta}_i) \in \pi_{C_i}(\overline{\alpha}_i, \overline{\beta}_i, \overline{x}_{-C_i})$ and $\overline{x}_{C_i} \in \delta_{C_i}(\overline{\alpha}_i, \overline{x}_{-C_i})$. Since u_i is continuous on X, quasiconcave in x_S and the game $\tilde{G}(x_{-C_i})$ is balanced (Scarf, 1971), therefore by Lemma 3.3, $\overline{x}$ is a strong hybrid solution. ■

4 Examples

In this section we study the condition of partially quasi-transferable preferences in two examples of games having a strong hybrid solution. The first example includes three versions of the garbage game introduced by Shapley and Shubik [1969]. The second example is a coalitional game with environmental externalities.

4.1 Some versions of the garbage game

• A garbage game with side-payments

Consider the version of Shapley and Shubik [1969] garbage game studied in Laffont [1982], in which four neighbors decide on where to dump their garbage. Assume that each neighbor has an endowment of a transferable good equal to one and a quantity of garbage also equal to one. Let $z^{ji} \geq 0$ be the quantity of good transfered by neighbor j to neighbor i, and $x^{ji} \geq 0$, the quantity of garbage that neighbor j locates in the garden of neighbor i. For each i, $0 \leq \sum_{j=1}^{4} z^{ij} \leq 1$, $0 \leq \sum_{j=1}^{4} x^{ij} \leq 1$, $0 \leq z^{ij}$.

Suppose that all agents i have the same payoff function

$$u_i(x^{ji}, z^j i) = \sum_{j=1}^{4} z^{ji} - \sum_{i=1}^{4} x^{ji}, \tag{4.1}$$

and assume that there are only two coalitions. The first coalition includes neighbors 1, 2, 3 while the second only includes neighbor 4.

Then, one can check that the strategy profile $z^{ii} = 1$ for all i, $x^{i4} = 1$, for all $i = 1, 2, 3$, $x^{41} = 1/3$, $x^{44} = 0$ is both a hybrid and a strong hybrid solution. Notice that this game is continous, compact and convex and the functions u_i are quasi-concave. Let us check that indivudals' preferences are partially quasi-transferable. To do this we shall study the two coalitions in turn.

Case $C = \{4\}$.
The property is clearly satisfied for this coalition.

Case $C = \{1, 2, 3\}$.

Let us examine all the subcoalitions S included in $\{1, 2, 3\}$ (as well as the coalition $\{1, 2, 3\}$ itself).

• $S = \{1, 2, 3\}$.

Suppose that the following inequalities hold

$$\sum_{j=1}^{4} z^{ji} - \sum_{j=1}^{4} x^{ji} \geq \sum_{j=1}^{3} \bar{z}^{ji} + z^{4i} - \sum_{j=1}^{3} \bar{x}^{ji} - x^{4i} \geq \gamma_i t, \; i = 1, 2, 3. \tag{4.2}$$

and that at at least one of the left-hand sides inequalities hold strictly. One can check that it is always possible to rearrange the transfers among individuals in such a way that all the left-hand sides inequalities become strict.

• $S = \{1, 2\}$.

Assume that the two inequalities hold (and at least one holds strictly).

$$\sum_{j=1,j\neq 3}^{4} z^{j1} - \sum_{j=1,j\neq 3}^{4} x^{j1} - 1 \geq \sum_{j=1}^{3} \bar{z}^{j1} + z^{41} - \sum_{j=1}^{3} \bar{x}^{j1} - x^{41} \geq \gamma_1 t \tag{4.3}$$

$$\sum_{j=1,j\neq 3}^{4} z^{j2} - \sum_{j=1,j\neq 3}^{4} x^{j2} - 1 \geq \sum_{j=1}^{3} \bar{z}^{j2} + z^{42} - \sum_{j=1}^{3} \bar{x}^{j2} - x^{42} \geq \gamma_2 t. \tag{4.4}$$

Again, we can always design some transfers of good ensuring that the left-hand sides inequalities are both strict.

The argument is similar for the other two-person subcoalitions. Moreover, one can check that all one-person S coalitions satisfy the partially quasi-transferable preferences property as well. Therefore preferences are partially quasi-transferable.

• A garbage game with bounded payoff for one player

We now pay focus on a variant of the garbage game considered above. Assume that the payoff function of all neighbors but neighbor 2 is given by

$$u_i(x^{ji}, z^{ji}) = 3 \times \left(\sum_{j=1}^{4} z^{ji} \right) - \sum_{j=1}^{4} x^{ji}, i = 1, 2, 4. \tag{4.5}$$

Furthermore, assume that the payoff function of neighbor 2 reads

$$u_i(x^{ji}, z^{ji}) = \min \left\{ \frac{1}{2}, \sum_{j=1}^{4} z^{j2} \right\}. \tag{4.6}$$

That is, neighbor 2's payoff is bounded above and does not depend on the quantity of garbage located in his garden. Thus, preferences are not partially quasi-transferable (since that property is not satisfied for the subcoalition $\{2\}$). Yet the game is continous, compact and convex and the functions u_i are quasi-concave.

Consider the strategy profile: $z^{ii} = 1$ (for $i = 1, 3, 4$), $z^{21} = 1/2$, $x^{i4} = 1$, $x^{4i} = 1/3$, $x^{44} = 0$. One can check that profile is a strong hybrid solution. Therefore, the assumption that preferences are partially quasi-transferable is *not* necessary for the existence of a strong hybrid solution.

Also notice that some hybrid solutions are not strong. To see this, let us pay attention to the strategy profile: $z^{ii} = 1$, $x^{i4} = 1$, $x^{4i} = 1/3$, $x^{44} = 0$ and concentrate on the coalition $\{1, 2, 3\}$. Consider the subcoalition $\{1, 2\}$. One can see that it is impossible to increase the payoff of neighbor 2 since it is equal to its maximum value. The same remark applies for coalition $\{2, 3\}$. Now consider coalition $\{1, 3\}$. Both neighbor cannot jointly increase their payoffs since if they deviate, player 2 can split his garbage equally between them. Hence the strategy profile is a hybrid solution. Yet, it is not a strong hybrid solution. For instance, let us concentrate on coalition $\{1, 2\}$. Suppose that neighbor 2 gives a quantity of good $1/2$ to neighbor 1. Then his payoff remains equal to 1/2. Assume that to punish the deviation by neighbors 1 and 2, neighbor 3 transfers all his garbage in the garbage of neighbor 1. The latter's payoff is then equal to 25/6, whereas the payoff associated with the candidate equilibrium strategy profile equals 8/3. Thus in the deviation considered above, neighbor 2's payoff is unchanged, but neighbor 1's increases.

• A garbage game with altruism

There are four players and two coalitions as before. In this version of the garbage games, we assume that players 1, 3 and 4 have the same preferences, which are given by the payoff function $3 \times \left(\sum_{j=1}^{4} z^{ji}\right) - \sum_{i=1}^{4} x^{ji}, i = 1, 3, 4$. Preferences of player 2 are given by: $\sum_{j=1}^{4} z^{j2} + z^{21}$. Thus, player 2 is altruistic: his payoff increases when he transfers some good to player 1.

Let us check that preferences are partially quasi-transferable. We consider the two coalitions in turn.

Case $C = \{4\}$.
The property is clearly satisfied for this coalition.

Case $C = \{1, 2, 3\}$.

Let us consider all the subcoalitions S included in $\{1, 2, 3\}$ (and the coalition $\{1, 2, 3\}$ itself).

• $S = \{1, 2, 3\}$.

Suppose that the following inequalities hold

$$3\left(\sum_{j=1}^{4} z^{j1}\right) - \sum_{j=1}^{4} x^{j1} \geq 3\left(\sum_{j=1}^{3} \bar{z}^{j1} + z^{41}\right) - \sum_{j=1}^{3} \bar{x}^{j1} - x^{41} \geq \gamma_1 t \tag{4.7}$$

$$\sum_{j=1}^{4} z^{j2} + z^{21} \geq \sum_{j=1}^{3} \bar{z}^{j2} + z^{42} + \bar{z}^{21} \geq \gamma_2 t \tag{4.8}$$

$$3\left(\sum_{j=1}^{4} z^{j3}\right) - \sum_{j=1}^{4} x^{j3} \geq 3\left(\sum_{j=1}^{3} \bar{z}^{j3} + z^{43}\right) - \sum_{j=1}^{3} \bar{x}^{j3} - x^{43} \geq \gamma_3 t, \tag{4.9}$$

and that at least one of the left-hand sides inequalities hold strictly. Let us focus one, say player i, for whom the inequality is strict. Clearly, is always possible to rearrange the transfers among individuals in such a way that all the left-hand sides inequalities become strict.

• $S = \{1, 2\}$

Suppose that we have the following inequalities

$$3\left(\sum_{j=1,j\neq 3}^{4} z^{j1}\right)-\sum_{j=1,j\neq 3}^{4} x^{j1}-1\geq 3\left(\sum_{j=1}^{3}\bar{z}^{j1}+z^{41}\right)-\sum_{j=1}^{3}\bar{x}^{j1}-x^{41}\geq \gamma_1 t \quad (4.10)$$

$$\sum_{j=1,j\neq 3}^{4} z^{j2}+z^{21}\geq \sum_{j=1}^{3}\bar{z}^{j2}+z^{42}+\bar{z}^{21}\geq \gamma_2 t. \quad (4.11)$$

We can reason as before. When one of the left-hand side inequality is strict, one can make a transfer that striclty benefits the other player of the deviating coalition.

• $S=\{2,3\}$

Suppose that

$$3\left(\sum_{j\neq 1}^{4} z^{j3}\right)-\sum_{j\neq 1}^{4} x^{j3}-1\geq 3\left(\sum_{j=1}^{3}\bar{z}^{j3}+z^{43}\right)-\sum_{j=1}^{3}\bar{x}^{j3}-x^{43}\geq \gamma_3 t \quad (4.12)$$

$$\sum_{j\neq 1}^{4} z^{j2}+z^{21}\geq \sum_{j=1}^{3}\bar{z}^{j2}+z^{42}+\bar{z}^{21}\geq \gamma_2 t. \quad (4.13)$$

The same reasoning used above applies.

• $S=\{1,3\}$.

Assume that the following two inequalities hold.

$$3\left(\sum_{j=1,j\neq 2}^{4} z^{j1}\right)-\sum_{j=1,j\neq 2}^{4} x^{j1}-1\geq 3\left(\sum_{j=1}^{3}\bar{z}^{j1}+z^{41}\right)-\sum_{j=1}^{3}\bar{x}^{j1}-x^{41}\geq \gamma_1 t \quad (4.14)$$

$$3\left(\sum_{j=1,j\neq 2}^{4} z^{j3}\right)-\sum_{j=1,j\neq 2}^{4} x^{j3}-1\geq 3\left(\sum_{j=1}^{3}\bar{z}^{j3}+z^{43}\right)-\sum_{j=1}^{3}\bar{x}^{j3}-x^{43}\geq \gamma_3 t, \quad (4.15)$$

and that at least one of them holds strictly. Again, we can always design some transfers of good ensuring the the left-hand sides inequalities are strict.

Finally, all the one-person S coalitions satisfy the partially quasi-transferable preferences property. Therefore preferences are partially quasi-transferable. The game satisfies all our assumptions. Therefore there exists a strong hybrid solution.

Now consider the following strategy profile: $z^{ii}=1$ for all i, $x^{i4}=1$ for all $i=1,2,3$, $z^{4i}=1/3$ for all $i=1,2,3$. This profile is a hybrid solution (it is impossible to (weakly) increase the welfare of any deviating coalition). Yet, it is not a strong hybrid solution. To see this, consider coalition $\{1,2\}$. Clearly if player 2 transfers a quantity $1/2$ to player 1, his welfare is unchanged while that of player 1 strictly increases (even when player 3 minimizes the deviants' payoffs).

Finally, consider the following strategy profile: $z^{ii} = 1$ for all $i \neq 2$, $z^{22} = 0$, $z^{21} = 1$, $x^{i4} = 1$ for all $i = 1, 2, 3$, $z^{4i} = 1/3$ for all $i = 1, 2, 3$. This profile is a strong hybrid solution (it is impossible to increase the welfare of any deviating coalition).

Notice that there is a set of hybrid solutions indexed by the transfer of good by player 2 to player 1. For instance, consider the solutions obtained as above but with $z^{22} = 1 - z^{21}$, $0 \leq z^{21} \leq 1$. Then there is a strong hybrid solution strictly included in the set of hybrid solutions (set $z^{21} = 1$).

The last two versions of the garbage game illustrate the fact that the condition of partially quasi-transferable preferences is not necessary for the existence of a strong hybrid solution, nor it is necessary for a strong hybrid solution to be strictly included in the set of hybrid solutions.

4.2 A coalitional game with environmental externalities

Consider a coalitional game with environmental externalities where there are n agents (*e.g.*, countries, firms). Assume that these agents produce and consume the same good and that production is a polluting activity. Let y_i denote the production of agent i and let x_i be the emission of pollutants resulting from this production. Also assume that $y_i = g_i(x_i)$ and that $u_i(x) = g_i(x_i) - v_i\big(\sum_{i \in I} x_i\big)$, where $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function such that $g_i(0) = 0$ and $v_i : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing on $\mathbb{R}_+$. Thus agent i's payoff increases with the consumption of the produced good ($g_i(x_i)$) and decreases with global pollution ($\sum_{i \in I} x_i$). Further, assume that $X_i = \{x_i : \, 0 \leq x_i \leq x_i^0\}$ (x_i^0 is agent i's maximum emission level), and that both g_i and v_i are continuous.

Now, suppose that there are $x \in X$, $x_S \in X_S$, $C \in \mathfrak{C}$, $\overline{x}_C \in X_C$, $\gamma \in \Delta_C$, $S \subseteq C$, $t > 0$ such that

$$\underline{u}_S(x_S, x_{-C}) \succ u_S(\overline{x}_C, x_{-C}) \succeq t\gamma_S.$$

This implies that there is an agent i in S such that

$$g_i(x_i) - v_i\left(\sum_{j \in S} x_j + \sum_{j \in C \setminus S} x_j^0 + \sum_{j \in -C} x_j\right) > t\gamma_i. \tag{4.16}$$

Here, the agents $j \in C \setminus S$ set their emission levels equal to x_j^0 in order to punish the deviation of coalition S. Necessarily, $x_i > 0$ (because $v(.)$ takes non-negative values, and $t\gamma_i$ is non-negative). Then by continuity of $g_i(.)$ there exists $\epsilon > 0$ such that $x_i - \epsilon > 0$, and such that

$$g_i(x_i - \epsilon) - v_i\left(x_i - \epsilon + \sum_{j \in S, j \neq i} x_j + \sum_{j \in C \setminus S} x_j^0 + \sum_{j \in -C} x_j\right) > t\gamma_i. \tag{4.17}$$

Now set $\tilde{S} = S$ and $\tilde{x}_k = x_k$, $k \neq i$, $\tilde{x}_i = x_i - \epsilon$. Since $v_k(.)$ is decreasing it holds that for every agent k in $\tilde{S}$:

$$g_k(x_k) - v_k\left(\sum_{j\in S} x_j - \epsilon + \sum_{j\in C\setminus S} x_j^0 + \sum_{j\in -C} x_j\right) > t\gamma_k. \tag{4.18}$$

This proves that preferences are partially quasi-transferable.

5 Conclusion

We have proposed a notion of strong hybrid solution, which is a refinement of Zhao's hybrid solution. Like this solution, the strong hybrid solution is well suited to study situations where people cooperate within coalitions but where coalitions compete with one another. In the strong hybrid solution, as opposed to Zhao's solution, the strategy profile assigned to each coalition is strongly-Pareto optimal. To put it another way, all possible gains of cooperation at the coalition level have been exhausted.

We have also given existence results for the strong hybrid solution in games with compact and convex strategy spaces.

We would like to point out that the concept of hybrid solutions, as well as that of strong hybrid solution introduced in this paper can be used to study the existence of the consensus equilibria introduced by Di et al. [2021] for the mining gap games related to the stability of Blockchain Ecosystems, which plays a key role for the development of Ecosystems. The strong hybrid solution may also be applied to study coalitional game with altruism (see, *e.g.*, the multiobjective games analyzed in Lejano and Ingram [2012]).

There are at least two issues that can be studied in further research. The first one is the existence of the strong hybrid solution in games with discontinuous payoffs. Another interesting issue is the existence of this solution with general strategy spaces, and with an arbitrarily large number of players.

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