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Hypertemperature effects in heterogeneous media and thermal flux at small-length scales

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30 homogenization of second-order elliptic equations is Fourier's law of heat conduction. In its
 31 simplest form, Fourier's law relates the heat flux \mathbf{q} as a linear function of the temperature
 32 gradient, that is:

$$33 \quad \mathbf{q} = -\kappa \nabla u, \quad (1.1)$$

34 where u is the absolute temperature and $\kappa > 0$ the thermal conductivity that depends
 35 on properties of the material. In general, the coefficient κ may depend on temperature,
 36 space, and/or time, but often varies so little in cases of interest that it is reasonable to
 37 neglect this variation.

38 The theory of periodic homogenization led to a deeper exploration of Fourier's heat con-
 39 duction law as it pertains to heterogeneous periodic material with different conductivities.
 40 Effective heat fluxes were derived taking into account microstructure morphology and vol-
 41 ume fraction. The classical problem in the periodic homogenization for stationary heat
 42 conduction states:

$$-\operatorname{div} \left(\mathbf{K} \left(\frac{\mathbf{x}}{\varepsilon} \right) \nabla u_\varepsilon \right) = f \text{ in } \Omega, \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \quad (1.2)$$

43 where $\mathbf{K}(\mathbf{y}) \in L^\infty(Y, \mathbb{R}^{3 \times 3})$ is uniformly elliptic, symmetric, and Y -periodic with $Y =$
 44 $[0, 1]^3$. If $\Omega \subset \mathbb{R}^3$ is uniformly Lipschitz open set, then there exists a unique solution u_ε to
 45 (1.2) such that it converges weakly to a function u in $H_0^1(\Omega)$, where the function $u \in H_0^1(\Omega)$
 46 is the unique solution to,

$$-\operatorname{div} \left(\mathbf{K}^{\text{eff}} \nabla u \right) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \quad (1.3)$$

47 with $\mathbf{K}_{ij}^{\text{eff}} := \sum_{l=1}^3 \int_Y \mathbf{K}_{il}(\mathbf{y}) (\delta_{jl} - \partial_{y_l} w_j) d\mathbf{y}$ and $w_j \in \{H_{\text{per}}^1(Y) \mid \int_Y w_j d\mathbf{y} = 0\}$ solution to,

$$-\operatorname{div}_y \left(\mathbf{K}(\mathbf{y}) (\nabla_y w_j - \mathbf{e}_j) \right) = 0 \text{ in } Y, \\ w_j \text{ is } Y\text{-periodic}. \quad (1.4)$$

48 The convergence of u_ε to u in $H_0^1(\Omega)$ is only weak. If one seeks to improve the convergence
 49 then, usually, a corrector type term is introduced,

$$50 \quad u_\varepsilon - u - \varepsilon \hat{u} \left(\cdot, \frac{\cdot}{\varepsilon} \right) \rightarrow 0 \text{ in } H^1(\Omega), \quad (1.5)$$

51 where $\hat{u}(\mathbf{x}, \mathbf{y}) := -\sum_{k=1}^N w_k(\mathbf{y}) \frac{\partial u}{\partial x_k}(\mathbf{x})$ is the correction term. In order for the expression
 52 to belong in $H^1(\Omega)$ it is required that $w_k \in W^{1,\infty}(Y)$ and $u \in H^2(\Omega)$ (see [4, Sect. 5,
 53 pg. 33]). With the introduction of two-scale convergence [28] (see also [1] and [24]) a
 54 rigorous justification was provided for the multiple-scale method and the corrector result
 55 (in N -dimensional space) in (1.5) was made rigorous through the following theorem:

56 **Proposition 1.1.** [1, Thm 2.6], [8, Prop. 9.12], [24, Thm. 22] Let \hat{u} be given by,

$$57 \quad \hat{u}(\mathbf{x}, \mathbf{y}) = - \sum_{k=1}^N w_k(\mathbf{y}) \frac{\partial u}{\partial x_k}(\mathbf{x}) + \tilde{u}_1(\mathbf{x}), \quad (1.6)$$

58 and suppose that $\nabla_{\mathbf{y}} w_k \in L^r(Y, \mathbb{R}^N)$, $k = 1, \dots, N$ and $\nabla u \in L^s(\Omega, \mathbb{R}^N)$ with $1 < r, s < \infty$
59 and such that,

$$60 \quad \frac{1}{r} + \frac{1}{s} = \frac{1}{2}. \quad (1.7)$$

61 Then,

$$62 \quad \nabla u_\varepsilon - \nabla u - \nabla_{\mathbf{y}} \hat{u}(\cdot, \frac{\cdot}{\varepsilon}) \rightarrow 0 \text{ in } L^2(\Omega, \mathbb{R}^N). \quad (1.8)$$

63 As one can immediately observe, the main unease with the above proposition is the higher
64 integrability required on both local and homogenized solutions. However, with the intro-
65 duction of periodic unfolding operators [9, 10], one can obtain a more general corrector
66 result without requiring any regularity assumption on the cell function w_k , stating

$$67 \quad \nabla u_\varepsilon - \nabla u - \sum_{k=1}^N \mathcal{Q}_\varepsilon\left(\frac{\partial u}{\partial x_k}\right) \nabla_{\mathbf{y}} w_k \rightarrow 0 \text{ in } L^2(\Omega, \mathbb{R}^N), \quad (1.9)$$

68 where \mathcal{Q}_ε is the scale-splitting operator defined in [9]; see the elegant proof in [11]. More-
69 over, based on the scale-splitting operator, upper bound estimates on the convergence rate
70 in terms of ε were obtained in [23] (with some additional regularity assumptions on the
71 homogenized solution). Furthermore, the upper bound estimates on the convergence rate
72 can be made tighter by using boundary layer correctors (see e.g. [27], [2], [32]). Therefore,
73 it seems that in order to lift the (restrictive) regularity conditions in Proposition 1.1 one
74 must have knowledge of the operator \mathcal{Q}_ε .

75 In this work, we commence with a higher-gradient heat equation model so as when passing
76 to the limit the homogenized problem coincides with (1.3), however, the local solutions w_k
77 satisfy a higher-gradient local problem and, hence, possess better regularity properties than
78 classical local solutions. This aforementioned higher regularity of the local solution com-
79 pensated by the mild assumption that the data are of the form $f := \operatorname{div} \mathbf{F}$, $\mathbf{F} \in L^3(\Omega, \mathbb{R}^3)$
80 allows us to prove Proposition 1.1 with weaker than usual assumptions and without using
81 the scale-splitting operator. The approach marshalled in this article is motivated from
82 generalized continuum mechanic theories (see e.g. [17]). It is well understood that the
83 effective properties of heterogeneous materials can depend not only on the volume frac-
84 tion of the phases or their geometrical distribution but also on the absolute size of the
85 constituents (see e.g. [19]). By introducing physically relevant scaling arguments related
86 to the absolute size of the constituents, we introduce a new length scale parameter that
87 models scale-dependent thermal effects (see e.g. [16, 15]). Hence, by arguing as in the
88 work of [30], the solution of the enriched microscopic problem can be seen as a vanishing-
89 viscosity solution that coincides with the classical homogenized solution of (1.3). The

90 higher regularity that the enriched microscopic model possesses is transferred to the local
 91 solution with the minimal assumptions of a uniformly Lipschitz open set Ω and non-smooth
 92 coefficients.

93 We have organized the paper as follows: In Section 2 we explain in detail the scaling
 94 argument we employ, present the enriched microscopic model, provide some motivation
 95 for its use, and prove some general qualitative results as they pertain to existence and
 96 uniqueness of solution as well as the variational nature of the problem. In section Section 3
 97 we state the main results, we discuss their consequences, and demonstrate symmetry
 98 relations for the higher-gradient effective coefficients as well as explore their variational
 99 structure. Section 4 is dedicated to proving the main results in Section 3. Finally, we
 100 reserve Section 5 for some discussion and remarks.

101 2 Problem set-up

102 2.1 Generalized Fourier's law

103 Fourier's law of heat conduction is widely regarded as a limiting approximation of some
 104 general (potentially nonlinear) constitutive law for the heat flux that may depend on
 105 higher-gradients [12] [7], [33]. For instance, the temperature of a rarefied gas at the slip
 106 regime, namely when $0.001 < \text{Kn} < 0.3$ where Kn is the non-dimensional Knudsen number,
 107 deviates from Fourier's law of heat conduction [33]. Moreover, in the same article a
 108 generalized heat conduction model, from a phenomenological point of view, was postulated
 109 under the assumption that the gas is isotropic. The authors' speculation was that the heat
 110 flux in a rarefied gas in the slip regime, depends linearly on on the temperature gradient
 111 but also on higher-order temperature derivatives,

$$112 \quad \mathbf{q} = \mathbf{K}\nabla u + \mathbf{L}:\nabla\nabla u + \mathbf{M}:\nabla\nabla\nabla u, \quad (2.1)$$

113 where \mathbf{K} is a second order tensor, \mathbf{L} is a third order tensor, and \mathbf{M} is a fourth order tensor
 114 with $:$ and $\dot{}$ denoting second and third order contraction, respectively. In the case where
 115 we have isotropy and the medium is centro-symmetric, the higher-gradient heat flux can
 116 reduce further,

$$117 \quad \mathbf{q} = \mathbf{K}\nabla u + \mathbf{M}:\nabla\nabla\nabla u, \quad (2.2)$$

118 where $\mathbf{K}_{ij} := \eta\delta_{ij}$ and $\mathbf{M}_{ijkl} := \ell_1\delta_{ik}\delta_{jl} + \ell_2\delta_{il}\delta_{jk} + \ell_3\delta_{ij}\delta_{kl}$ for scalars η and $\ell_i, i = 1, 2, 3$ that
 119 will be assumed to be constant or piece-wise constant. The heat conduction inequality
 120 asserts that,

$$121 \quad \mathbf{q} \cdot \nabla u \leq 0, \quad (2.3)$$

122 for all temperature fields. Furthermore, the tensor \mathbf{M} has a physical meaning and is often
 123 referred to as the *spatial retardation* (see [7]).

124 In addition to the above mentioned works, emerging primarily from the physics commu-
 125 nity, non-classical laws of Fourier's heat conduction have, for many years now, attracted
 126 considerable attention from the theoretical mechanics community as well (see e.g. [18],
 127 [5]). In recent years, the motivation for deriving non-classical heat conduction models in
 128 the mechanics field stemmed from trying to understand the presence of thermal fluctua-
 129 tion fields in heterogeneous materials with a microstructure. Specifically, the authors in
 130 [18] postulate the existence of a free energy function that has an added dependence on
 131 the gradient of the entropy density variable. Based on this enhanced free energy, an en-
 132 hanced heat equation was derived containing a term with a characteristic length related to
 133 material parameters that *can account for scale-size thermal effects in microheterogeneous*
 134 *bodies*.

135 Finally, all of the above theoretical or computational non-classical approaches seem to
 136 have found some validation in recent experimental work where evidence of size-dependent
 137 thermal effects were reported in heterogeneous materials (see e.g. [16, 15]).

138 2.2 Scale-dependent thermal effects

139 In this paragraph, we postulate the modified heat flux in (2.2) and derive an additional
 140 length scale parameter that encapsulates the size-dependent thermal effects in the context
 141 of an idealized periodic microstructure. We assume that our working domain Ω is periodic
 142 with period ℓ and with overall characteristic length L . Define the dimensionless coordinates
 143 and temperature,

$$144 \quad \mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad u^*(\mathbf{x}^*) = \frac{u(\mathbf{x})}{L}. \quad (2.4)$$

145 Moreover, we can define the following non-dimensional tensors:

$$\mathcal{K}\mathcal{K}^* = \mathcal{K}, \quad \mathcal{M}\mathcal{M}^* = \mathcal{M}, \quad (2.5)$$

146 where $\mathcal{K} := \max_{\mathbf{z} \in Y_\ell} |\mathbf{K}(\mathbf{z})|$, $\mathcal{M} := \max_{\mathbf{z} \in Y_\ell} |\mathbf{M}(\mathbf{z})|$ with $Y_\ell := (-\ell/2, \ell/2]^3$ the periodic cell
 147 characterizing the body Ω . We can now introduce an additional length scale relation
 148 between \mathcal{K} and \mathcal{M} as follows:

$$149 \quad \mathcal{M} = \ell_{\text{TE}}^2 \mathcal{K}. \quad (2.6)$$

150 Thus, the non-dimensional heat flux becomes,

$$151 \quad q_i = \sum_{j=1}^3 \left(\mathcal{K} \mathcal{K}_{ij}^* \frac{\partial u^*}{\partial x_j^*} + \sum_{k,l=1}^3 \mathcal{K} \left(\frac{\ell_{\text{TE}}}{L} \right)^2 \frac{\partial}{\partial x_j^*} \left(\mathcal{M}_{ijkl}^* \frac{\partial^2 u^*}{\partial x_k^* \partial x_l^*} \right) \right). \quad (2.7)$$

152 If we use the notation, $\mathbf{q}^* := \mathcal{K}^{-1} \mathbf{q}$ then we have a non-dimensional form of the heat flux,

153
$$\mathbf{q}^* = \mathbf{K}^* \nabla u^* + \left(\frac{\ell_{\text{TE}}}{L} \right)^2 \text{div}^* (\mathbf{M}^* : \nabla^* \nabla^* u^*). \quad (2.8)$$

154 We remark that since the coefficients \mathbf{K} and \mathbf{M} are Y_ℓ periodic the corresponding normalized
 155 coefficients \mathbf{K}^* and \mathbf{M}^* are Y^* periodic where $Y^* := \frac{\ell}{L} Y$ with $Y := (-1/2, 1/2]^3$. Finally,
 156 henceforth, if no confusion arises we will drop the $*$ notation in order to expedite our
 157 presentation.

158 2.3 The microscopic problem

159 We consider a material with a periodic microstructure of period $\varepsilon := \ell/L \ll 1$ occupying
 160 a region $\Omega \subset \mathbb{R}^3$. The region Ω that the heterogeneous material occupies is assumed
 161 to be a uniformly Lipschitz open set (see [14, Definition 2.65]). The exterior boundary
 162 component will be denoted by $\Sigma := \partial\Omega$ while the vector \mathbf{n} will denote the unit normal on Σ
 163 pointing in the outward direction. The ε periodic problem, generated by defining the non-
 164 dimensional number ε as the ratio of ℓ/L , will permit us to obtain an effective equation
 165 when $\varepsilon \rightarrow 0$. However, unlike in classical homogenization problems, different cases ought
 166 to be considered depending on how the intrinsic length scale ℓ_{TE} scales with ℓ (or L).
 167 Here, since we are interested in recovering Fourier's classical law of heat conduction as an
 168 effective limit, we will only consider the scaling,

169
$$\ell_{\text{TE}}/\ell \sim 1. \quad (2.9)$$

170 The physical meaning of the above scaling, is that the intrinsic length ℓ_{TE} is comparable
 171 with the length of the heterogeneities. Naturally, other type of scalings are possible,
 172 however, we will not address other cases here. We refer the reader to [30] for different
 173 type of scalings in the context of generalized continuum mechanics.

174 Therefore, under the scaling in (2.9), the (generalized) heat flux becomes,

175
$$\mathbf{q}^\varepsilon = \mathbf{K} \left(\frac{\mathbf{x}}{\varepsilon} \right) \nabla u_\varepsilon + \varepsilon^2 \text{div} \left(\mathbf{M} \left(\frac{\mathbf{x}}{\varepsilon} \right) : \nabla \nabla u_\varepsilon \right). \quad (2.10)$$

176 The microscopic problem is then characterized by the following equation and boundary
 177 conditions,

$$\begin{aligned} -\text{div} \left(\mathbf{K} \left(\frac{\mathbf{x}}{\varepsilon} \right) \nabla u_\varepsilon - \varepsilon^2 \text{div} \left(\mathbf{M} \left(\frac{\mathbf{x}}{\varepsilon} \right) : \nabla \nabla u_\varepsilon \right) \right) &= f && \text{in } \Omega, \\ \varepsilon^2 \mathbf{M} \left(\frac{\mathbf{x}}{\varepsilon} \right) \nabla \nabla u_\varepsilon : \mathbf{n} \otimes \mathbf{n} &= 0 && \text{on } \Sigma, \\ u_\varepsilon &= 0 && \text{on } \Sigma, \end{aligned} \quad (2.11)$$

178 where f is some given source that belongs in $L^2(\Omega)$. We remark, that prescribing a
 179 homogeneous Dirichlet boundary condition, as is usually the case, is no longer sufficient.
 180 We require, additionally, to prescribe a zero heat flux for what we refer to as a *normal*
 181 *double heat flux* that is directly related to the *spatial retardation* coefficient \mathbf{M} (see next
 182 section on the weak formulation of the problem). In direct analogy with second-gradient
 183 elasticity, such a term is referred to as normal double traction (see [26, 20, 21]).

184 **2.3.1 Notation and assumptions**

- 185 - We employ the Einstein notation of repeated indices unless otherwise stated.
 186 - Throughout the work we assume that the uniform strong ellipticity condition holds,
 187 i.e., there exist positive (generic) constants c_1 and c_2 such that:

$$\begin{aligned} c_1|\mathbf{w}|^2 &\leq \mathbf{w}\cdot\mathbf{K}(\mathbf{x})\cdot\mathbf{w} \leq c_2|\mathbf{w}|^2, \\ c_1|\mathbf{w}|^2|\mathbf{q}|^2 &\leq \mathbf{w} \otimes \mathbf{q}:\mathbf{M}(\mathbf{x}):\mathbf{w} \otimes \mathbf{q} \leq c_2|\mathbf{w}|^2|\mathbf{q}|^2, \end{aligned} \quad (2.12)$$

188 for all $\mathbf{w}, \mathbf{q} \in \mathbb{R}^3 - \{\mathbf{0}\}$.

189 **2.3.2 Auxiliary formulas**

190 For the readers' convenience and for the expediency of the our results, we introduce certain
 191 formulas that we will make use of in obtaining the variational formulation of (2.11). These
 192 formulas, among others, can also be found in [20, Appendix].

193 For any smooth enough scalar function ξ defined on Σ or on a neighborhood of Σ , the
 194 tangential and normal components of $\nabla\xi$ are,

$$(\nabla\xi)_\tau = -\mathbf{n} \times (\mathbf{n} \times \nabla\xi) = \nabla\xi - (\nabla\xi)_n\mathbf{n}, \quad (\nabla\xi)_n := \nabla\xi \cdot \mathbf{n}. \quad (2.13)$$

196 Moreover, we introduce the surface gradient of ξ using the projection operator $\Pi := \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$,

$$\nabla_s \xi = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\nabla\xi = \Pi\nabla\xi. \quad (2.14)$$

198 Thus, we can write down a useful integration by parts formula on surfaces,

$$\int_\Sigma \nabla_s \xi \, ds = \int_\Sigma \xi(\operatorname{div} \mathbf{n})\mathbf{n} \, ds + \int_{\partial\Sigma} \llbracket \xi \boldsymbol{\nu} \rrbracket \, dl, \quad (2.15)$$

200 where

$$\nu_i = \epsilon_{ijk} t_j n_k, \quad (2.16)$$

202 is the unit normal vector on $\partial\Sigma$ and tangent to Σ , \mathbf{t} is the unit tangent vector to $\partial\Sigma$, and
 203 ϵ is the Levi-Civita tensor. Lastly, we remark, the jump term on (2.15) is on a ridge, i.e.
 204 the line on Σ where the tangent plane of Σ is discontinuous.

205 **Remark 2.1.** *The above formulas are used with a high degree of frequency in emulsions*
 206 *and capillary fluids (see e.g. [31]). We refer the reader to the appendix of reference [20]*
 207 *or [21] for an excellent exposition of the above formulae and related topics.*

208 **2.4 Weak formulation**

209 The primary setting for for the variational formulation of (2.11) is the space $H^2(\Omega) \cap H_0^1(\Omega)$
 210 where the Sobolev space $H^2(\Omega)$ is a Hilbert space with norm,

211
$$\|u\|_{H^2(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega, \mathbb{R}^3)}^2 + \|\nabla \nabla u\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})}^2 \right)^{1/2}. \quad (2.17)$$

212 Since the varational formulation of (2.11) is not a standard one, we write down the details
 213 for the readers convenience using the notation introduced in Section 2.3.2. Hence, if we
 214 multiply (2.11) by a test function $v \in \{\phi \in C^\infty(\bar{\Omega}) \mid \phi = 0 \text{ on } \Sigma\}$ and integrate by parts
 215 several times (including integration by parts on surfaces using formula (2.15)) we obtain,

$$\begin{aligned} & - \int_{\Sigma} \left(\mathbf{K}_{ij} \left(\frac{\mathbf{x}}{\varepsilon} \right) \partial_{x_j} u_\varepsilon - \varepsilon^2 \partial_{x_k} \left(\mathbf{M}_{ikpq} \left(\frac{\mathbf{x}}{\varepsilon} \right) \partial_{x_p x_q}^2 u_\varepsilon \right) \right) n_i v \, ds \\ & + \int_{\Omega} \mathbf{K}_{ij} \left(\frac{\mathbf{x}}{\varepsilon} \right) \partial_{x_j} u_\varepsilon \partial_{x_i} v \, d\mathbf{x} - \varepsilon^2 \left\{ \int_{\Sigma} \mathbf{M}_{ikpq} \left(\frac{\mathbf{x}}{\varepsilon} \right) \partial_{x_p x_q}^2 u_\varepsilon n_k n_m n_i \partial_{x_m} v \, d\sigma \right. \\ & + \int_{\partial \Sigma} \left[\mathbf{M}_{ikpq} \left(\frac{\mathbf{x}}{\varepsilon} \right) \partial_{x_p x_q}^2 u_\varepsilon n_k \Pi_{i\ell} \nu_\ell v \right] \, dl - \int_{\Sigma} \Pi_{\ell m} \partial_{x_m} \left(\mathbf{M}_{ikpq} \left(\frac{\mathbf{x}}{\varepsilon} \right) \partial_{x_p x_q} u_\varepsilon n_k \Pi_{i\ell} \right) v \, d\sigma \left. \right\} \\ & + \varepsilon^2 \int_{\Omega} \mathbf{M}_{ikpq} \left(\frac{\mathbf{x}}{\varepsilon} \right) \partial_{x_p x_q} u_\varepsilon \partial_{x_i x_k} v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x}. \end{aligned} \quad (2.18)$$

216 Using the fact that we have imposed a homogeneous Dirichlet boundary condition and
 217 a zero normal double heat flux for the spatial retardation on Σ , we can see that the
 218 variational formulation (in vectorial form) reduces to the following: Find $u_\varepsilon \in H^2(\Omega) \cap$
 219 $H_0^1(\Omega)$ such that,

220
$$\int_{\Omega} \mathbf{K} \left(\frac{\mathbf{x}}{\varepsilon} \right) \nabla u_\varepsilon \cdot \nabla v \, d\mathbf{x} + \varepsilon^2 \int_{\Omega} \mathbf{M} \left(\frac{\mathbf{x}}{\varepsilon} \right) \nabla \nabla u_\varepsilon : \nabla \nabla v \, d\mathbf{x} = \int_{\Omega} f v \, d\mathbf{x}, \quad (2.19)$$

221 for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$.

222 **Remark 2.2.** *The weak form in (2.18) also provides a way for recovering the strong form*
 223 *of problem (2.11) in the sense of distributions. For more details, the interested reader can*
 224 *consult [30].*

225 **Remark 2.3.** *Existence and uniqueness of a solution in (2.19) that belongs in $H^2(\Omega) \cap$*
 226 *$H_0^1(\Omega)$ is a matter of applying the Lax-Milgram lemma together with Poincaré's inequality*
 227 *in $H_0^1(\Omega)$ and the assumptions regarding the ellipticity of the tensors in Section 2.3.1.*
 228 *Hence, immediately, we can derive the following estimate from (2.19),*

229
$$\left(\|u_\varepsilon\|_{H^1(\Omega)}^2 + \varepsilon^2 \|\nabla \nabla u_\varepsilon\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})}^2 \right)^{1/2} \leq c(\Omega) \|f\|_{L^2(\Omega)}. \quad (2.20)$$

230 **2.5 Variational formulation**

231 The weak solution to (2.19) can be classified as the unique minimum of the functional
 232 $\mathcal{J}_\varepsilon(\theta)$,

233
$$u_\varepsilon = \arg \min_{\theta \in H^2(\Omega) \cap H_0^1(\Omega)} \mathcal{J}_\varepsilon(\theta), \quad (2.21)$$

234 where

235
$$\mathcal{J}_\varepsilon(\theta) := \frac{1}{2} \int_{\Omega} \kappa\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla \theta \cdot \nabla \theta \, d\mathbf{x} + \frac{1}{2} \int_{\Omega} \varepsilon^2 M\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla \nabla \theta : \nabla \nabla \theta \, d\mathbf{x} - \int_{\Omega} f \theta \, d\mathbf{x}. \quad (2.22)$$

236 A standard computation of the variational derivative of \mathcal{J}_ε will recover (2.19) and the
 237 Euler-Lagrange equations in succession.

238 **3 Main results**

239 **3.1 Homogenization via unfolding Γ -convergence**

240 We define the following domain decompositions (see [9, 13, 10, 11]):

$$K_\varepsilon^- := \left\{ \ell \in \mathbb{Z}^d \mid \varepsilon(\ell + Y) \subset \bar{\Omega} \right\}, \quad \Omega_\varepsilon^- := \text{int} \left(\bigcup_{\ell \in K_\varepsilon^-} \varepsilon(\ell + Y) \right), \quad \Lambda_\varepsilon^- := \Omega \setminus \Omega_\varepsilon^-. \quad (3.1)$$

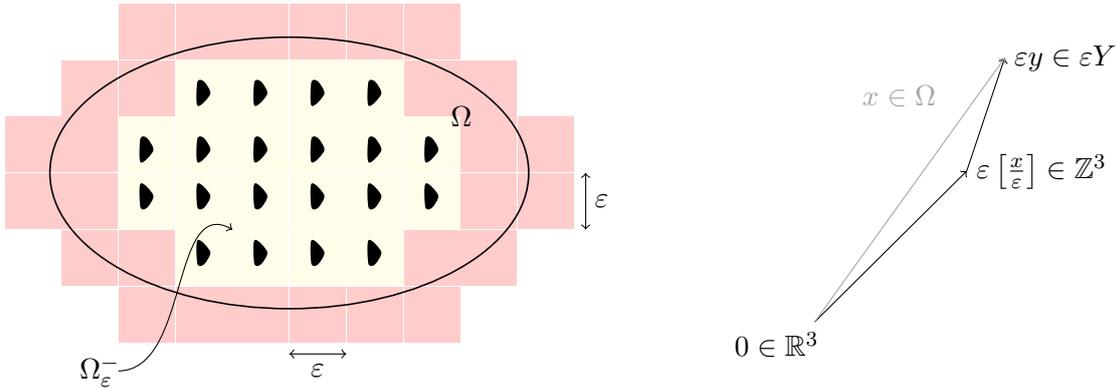


Figure 1: Unfolding operator on a periodic grid

241 Let $[z]_Y = ([z_1], [z_2], [z_3])$ denote the integer part of $z \in \mathbb{R}^3$ and denote by $\{z\}_Y$ the
 242 difference $z - [z]_Y$ which belongs to Y . Regarding our multiscale problem that depends on a
 243 small length parameter $\varepsilon > 0$, we can decompose any $x \in \mathbb{R}^3$ using the maps $[\cdot]_Y : \mathbb{R}^3 \mapsto \mathbb{Z}^3$
 244 and $\{\cdot\}_Y : \mathbb{R}^3 \mapsto Y$ the following way (see Fig. 3.1 (right)),

245
$$\mathbf{x} = \varepsilon \left(\left[\frac{\mathbf{x}}{\varepsilon} \right]_Y + \left\{ \frac{\mathbf{x}}{\varepsilon} \right\}_Y \right). \quad (3.2)$$

246 **Definition 3.1.** ([10, Def. 2.1, pg. 1588]) For any Lebesgue measurable function φ on Ω
 247 we define the periodic unfolding operator by,

$$248 \quad \mathcal{T}_\varepsilon(\varphi)(\mathbf{x}, \mathbf{y}) = \begin{cases} \varphi\left(\varepsilon \left[\frac{\mathbf{x}}{\varepsilon}\right]_Y + \varepsilon \mathbf{y}\right) & \text{for a.e. } (\mathbf{x}, \mathbf{y}) \in \Omega_\varepsilon^- \times Y \\ 0 & \text{for a.e. } (\mathbf{x}, \mathbf{y}) \in \Lambda_\varepsilon^- \times Y. \end{cases} \quad (3.3)$$

249 Regarding properties of the unfolding operator, the reader can consult [9, 13, 10, 11].

250 **Definition 3.2.** ([6, Def. 12 and Prop. 14, pg. 458]) Let $F_\varepsilon : L^p(\Omega) \rightarrow \overline{\mathbb{R}}$ be a sequence
 251 of functionals and $F : L^p(\Omega \times Y) \rightarrow \overline{\mathbb{R}}$ ($p > 1$). We say that F_ε unfolding Γ -converges to
 252 F if for all $u \in L^p(\Omega \times Y)$:

253 1. For every sequence $u_\varepsilon \in L^p(\Omega)$ such that $\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u$ in $L^p(\Omega \times Y)$ one has,

$$254 \quad F(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon); \quad (3.4)$$

255 2. there exists a sequence $u_\varepsilon \in L^p(\Omega)$ such that $\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u$ in $L^p(\Omega \times Y)$ and

$$256 \quad F(u) = \lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon). \quad (3.5)$$

257 **Theorem 3.1.** The sequence of functionals \mathcal{J}_ε unfolding Γ -converge in the weak topology
 258 of $H^2(\Omega) \cap H_0^1(\Omega)$ to \mathcal{J}_{eff} as $\varepsilon \rightarrow 0$,

$$259 \quad \mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}_{\text{eff}}, \quad (3.6)$$

260 where

$$261 \quad \mathcal{J}_{\text{eff}}(\theta) := \frac{1}{2} \int_{\Omega} \mathbf{K}^{\text{eff}} \nabla \theta \cdot \nabla \theta \, d\mathbf{x} - \int_{\Omega} f \theta \, d\mathbf{x}, \quad (3.7)$$

$$262 \quad \mathbf{K}_{ik}^{\text{eff}} := \int_Y \mathbf{K}(\mathbf{y}) (\nabla_y w_k - \mathbf{e}_k) \cdot (\nabla_y w_i - \mathbf{e}_i) + \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y w_k : \nabla_y \nabla_y w_i \, d\mathbf{y}, \quad (3.8)$$

263 with w_k the Y -periodic local solution to,

$$264 \quad -\text{div}_y (\mathbf{K}(\mathbf{y}) (\nabla_y w_k - \mathbf{e}_k)) - \text{div}_y (\mathbf{M}(\mathbf{y}) \nabla_y \nabla_y w_k) = 0 \text{ in } Y. \quad (3.9)$$

265 **Remark 3.1.** The first part of (3.8) is what one would expect as a result from the homog-
 266 enization of second order linear elliptic equations while the second part is new, specific to
 267 the inclusion of the second-gradient thermal effects.

268 **Theorem 3.2.** Let u_ε and u be the unique minimizers of (2.22) and (3.7), respectively.
 269 Moreover, let $f := \operatorname{div} \mathbf{F}$, $\mathbf{F} \in L^3(\Omega, \mathbb{R}^3)$ then,

$$270 \quad \nabla u_\varepsilon - \nabla u - \nabla_y \hat{u}(\cdot, \frac{\cdot}{\varepsilon}) \rightarrow 0 \text{ in } L^2(\Omega, \mathbb{R}^3), \quad (3.10)$$

271 as $\varepsilon \rightarrow 0$ with

$$272 \quad \hat{u}(\mathbf{x}, \mathbf{y}) = -w_k(\mathbf{y}) \frac{\partial u}{\partial x_k}(\mathbf{x}) + \tilde{u}_1(\mathbf{x}). \quad (3.11)$$

273 **Remark 3.2.** We note in the above theorem, the assumptions that $\nabla_y w_k \in L^r$ and $\nabla u \in$
 274 L^s with $1/r + 1/s = 1/2$ are no longer needed. Furthermore, if we compare (3.11) with G .
 275 Griso's result in (1.9), we see that (3.11) is more accessible to computations, hence, more
 276 practical.

277 3.2 Remarks on the homogenized coefficients

278 3.2.1 Symmetry

279 Since we work within a variational framework, the homogenized coefficients inherit the
 280 symmetry that is imposed on them from the framework. If one were to obtain the effective
 281 tensor through a multiple scale expansion then the tensor would have the following, non-
 282 symmetric, form:

$$283 \quad \mathbf{K}_{ik}^{\text{eff}} := \int_Y \mathbf{K}_{ij}(\mathbf{y}) (\delta_{jk} - \frac{\partial w_k}{\partial y_j}) d\mathbf{y}. \quad (3.12)$$

284 Naturally, the two forms, (3.8) and (3.12), are equivalent (since the tensors \mathbf{K} and \mathbf{M} are
 285 assumed to be isotropic). The proof is virtually identical to the classical case (see [34]),
 286 however, we feel it should be included here since it is a higher-gradient generalization of
 287 the classical case:

288 Multiply with a test function (3.9) and integrate by parts to obtain,

$$289 \quad 0 = \int_Y \mathbf{M}_{mjpq} \frac{\partial^2 w_k}{\partial y_p \partial y_q} \frac{\partial^2 v}{\partial y_m \partial y_j} d\mathbf{y} + \int_Y \mathbf{K}_{mj} \frac{\partial(w_k - y_k)}{\partial y_j} \frac{\partial v}{\partial y_m} d\mathbf{y}. \quad (3.13)$$

290 Selecting $v = w_i$ we have,

$$291 \quad 0 = \int_Y \mathbf{M}_{mjpq} \frac{\partial^2 w_k}{\partial y_p \partial y_q} \frac{\partial^2 w_i}{\partial y_m \partial y_j} d\mathbf{y} + \int_Y \mathbf{K}_{mj} \frac{\partial(w_k - y_k)}{\partial y_j} \frac{\partial w_i}{\partial y_m} d\mathbf{y}. \quad (3.14)$$

292 If we add and subtract coordinate y_i on the w_i term of the second integral we obtain,

$$\begin{aligned} & \int_Y \mathbf{K}_{ij}(\mathbf{y}) (\delta_{jk} - \frac{\partial w_k}{\partial y_j}) d\mathbf{y} \\ &= \int_Y \mathbf{M}_{mjpq} \frac{\partial^2 w_k}{\partial y_p \partial y_q} \frac{\partial^2 w_i}{\partial y_m \partial y_j} d\mathbf{y} + \int_Y \mathbf{K}_{mj} \frac{\partial(w_k - y_k)}{\partial y_j} \frac{\partial(w_i - y_i)}{\partial y_m} d\mathbf{y}. \end{aligned} \quad (3.15)$$

293 3.2.2 Variational characterization of the higher-order effective tensor

294 By (3.8), we can see that \mathbf{K}^{eff} is symmetric as well, and is determined in its entirety
 295 from the knowledge of the quadratic form $\mathbf{K}^{\text{eff}}\boldsymbol{\xi} \cdot \boldsymbol{\xi}$ for any constant vector $\boldsymbol{\xi} \in \mathbb{R}^3$. Using
 296 definition (3.8) one can check that,

$$297 \quad \mathbf{K}^{\text{eff}}\boldsymbol{\xi} \cdot \boldsymbol{\xi} = \int_Y \mathbf{K}(\mathbf{y})(-\boldsymbol{\xi} + \nabla_y w_\xi) \cdot (-\boldsymbol{\xi} + \nabla_y w_\xi) d\mathbf{y} + \int_Y \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y w_\xi : \nabla_y \nabla_y w_\xi d\mathbf{y}, \quad (3.16)$$

298 where w_ξ is the solution of the following cell problem,

$$\begin{cases} -\operatorname{div}_y (\mathbf{K}(\nabla_y w_\xi - \boldsymbol{\xi}) - \mathbf{M} : \nabla_y \nabla_y w_\xi) = 0 \text{ in } Y, \\ \mathbf{y} \mapsto w_\xi(\mathbf{y}) \text{ is } Y\text{-periodic.} \end{cases} \quad (3.17)$$

299 Using completely standard techniques stemming from the calculus of variations we can
 300 write for our case,

$$301 \quad \mathbf{K}^{\text{eff}}\boldsymbol{\xi} \cdot \boldsymbol{\xi} = \inf_{v \in \mathbf{H}_{\text{per}}^2(Y)} \left(\int_Y \mathbf{K}(\mathbf{y})(\nabla_y v - \boldsymbol{\xi}) \cdot (\nabla_y v - \boldsymbol{\xi}) d\mathbf{y} + \int_Y \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y v : \nabla_y \nabla_y v d\mathbf{y} \right), \quad (3.18)$$

302 with $\mathbf{H}_{\text{per}}^2(Y) := \{v \in \mathbf{H}^2(Y) \mid v \text{ is } Y\text{-periodic}\}$. It follows immediately that the effective
 303 coefficients are elliptic (satisfy Legendre condition), i.e. $\mathbf{K}^{\text{eff}}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \mathcal{K}|\boldsymbol{\xi}|^2$ where $\mathcal{K} =$
 304 $\int_Y K(\mathbf{y}) d\mathbf{y}$.

305 4 Proof of the main results

306 4.1 Proof of Theorem 3.1

307 **Theorem 3.1.** *The sequence of functionals \mathcal{J}_ε unfolding Γ -converge in the weak topology*
 308 *of $\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ to \mathcal{J}_{eff} as $\varepsilon \rightarrow 0$,*

$$309 \quad \mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}_{\text{eff}}, \quad (3.6)$$

310 where

$$311 \quad \mathcal{J}_{\text{eff}}(\theta) := \frac{1}{2} \int_\Omega \mathbf{K}^{\text{eff}} \nabla \theta \cdot \nabla \theta d\mathbf{x} - \int_\Omega f \theta d\mathbf{x}, \quad (3.7)$$

$$312 \quad \mathbf{K}_{ik}^{\text{eff}} := \int_Y \mathbf{K}(\mathbf{y})(\nabla_y w_k - \mathbf{e}_k) \cdot (\nabla_y w_i - \mathbf{e}_i) + \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y w_k : \nabla_y \nabla_y w_i d\mathbf{y}, \quad (3.8)$$

313 with w_k the Y -periodic local solution to,

$$314 \quad -\operatorname{div}_y (\mathbf{K}(\mathbf{y})(\nabla_y w_k - \mathbf{e}_k) - \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y w_k) = 0 \text{ in } Y. \quad (3.9)$$

315 *Proof.* Define,

$$316 \quad \mathcal{E}_\varepsilon(\theta) := \begin{cases} \frac{1}{2}a(\theta, \theta) & \text{if } \theta \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega) \\ +\infty & \text{if } \theta \in \mathbf{L}^2(\Omega) \setminus \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \end{cases} \quad (4.1)$$

317 with

$$318 \quad a(\theta, \theta) := \int_{\Omega} \mathbf{K}\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla \theta \cdot \nabla \theta \, d\mathbf{x} + \int_{\Omega} \varepsilon^2 \mathbf{M}\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla \nabla \theta : \nabla \nabla \theta \, d\mathbf{x}, \quad (4.2)$$

319 and

$$320 \quad \mathcal{E}_{\text{eff}}(\theta) := \begin{cases} \frac{1}{2} \int_{\Omega} \mathbf{K}^{\text{eff}} \nabla \theta \cdot \nabla \theta \, d\mathbf{x} & \text{if } \theta \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega) \\ +\infty & \text{if } \theta \in \mathbf{L}^2(\Omega) \setminus \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega). \end{cases} \quad (4.3)$$

321 It suffices to show that $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_{\text{eff}}$ where the energy \mathcal{E}_ε is finite.

322 Γ -limit inferior inequality.

323 Let $u_\varepsilon \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ be a solution to (2.11) then based on the estimates in (2.20) and
324 properties of the periodic unfolding operator \mathcal{T}_ε (see [10] for a definition) we have,

- 325 • $\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u$ in $\mathbf{L}^2(\Omega, \mathbf{H}^2(\mathbf{Y}))$
- 326 • $\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla_x u + \nabla_y \hat{u}$ in $\mathbf{L}^2(\Omega, \mathbf{H}^1(\mathbf{Y}, \mathbb{R}^3))$
- 327 • $\mathcal{T}_\varepsilon(\varepsilon \nabla \nabla u_\varepsilon) \rightharpoonup \nabla_y \nabla_y \hat{u}$ in $\mathbf{L}^2(\Omega \times \mathbf{Y}, \mathbb{R}^{3 \times 3})$.

328 Then, for all $\varphi \in \mathbf{L}^2(\Omega, \mathbf{C}_{\text{per}}^1(\mathbf{Y}, \mathbb{R}^3))$ we have,

$$0 \leq \frac{1}{2} \int_{\Omega} \mathbf{K}\left(\frac{\mathbf{x}}{\varepsilon}\right) (\nabla u_\varepsilon - \varphi) \cdot (\nabla u_\varepsilon - \varphi) \, d\mathbf{x} + \frac{1}{2} \int_{\Omega} \varepsilon^2 \mathbf{M}\left(\frac{\mathbf{x}}{\varepsilon}\right) (\nabla \nabla u_\varepsilon - \nabla \varphi) : (\nabla \nabla u_\varepsilon - \nabla \varphi) \, d\mathbf{x}. \quad (4.4)$$

329 Opening up the parentheses in the above expression we obtain,

$$\begin{aligned} \mathcal{E}_\varepsilon(u_\varepsilon) &\geq \int_{\Omega} \mathbf{K}\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla u_\varepsilon \cdot \varphi \, d\mathbf{x} - \frac{1}{2} \int_{\Omega} \mathbf{K}\left(\frac{\mathbf{x}}{\varepsilon}\right) \varphi \cdot \varphi \, d\mathbf{x} \\ &\quad + \int_{\Omega} \varepsilon^2 \mathbf{M}\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla \nabla u_\varepsilon : \nabla \varphi \, d\mathbf{x} - \frac{1}{2} \int_{\Omega} \varepsilon^2 \mathbf{M}\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla \varphi : \nabla \varphi \, d\mathbf{x} \\ &= \int_{\Omega \times \mathbf{Y}} \mathbf{K}(\mathbf{y}) \mathcal{T}_\varepsilon(\nabla u_\varepsilon) \cdot \mathcal{T}_\varepsilon(\varphi) \, d\mathbf{y} d\mathbf{x} - \frac{1}{2} \int_{\Omega \times \mathbf{Y}} \mathbf{K}(\mathbf{y}) \mathcal{T}_\varepsilon(\varphi) \cdot \mathcal{T}_\varepsilon(\varphi) \, d\mathbf{y} d\mathbf{x} \\ &\quad + \int_{\Omega \times \mathbf{Y}} \varepsilon^2 \mathbf{M}(\mathbf{y}) \mathcal{T}_\varepsilon(\nabla \nabla u_\varepsilon) : \mathcal{T}_\varepsilon(\nabla \varphi) \, d\mathbf{y} d\mathbf{x} - \frac{1}{2} \int_{\Omega \times \mathbf{Y}} \varepsilon^2 \mathbf{M}(\mathbf{y}) \mathcal{T}_\varepsilon(\nabla \varphi) : \mathcal{T}_\varepsilon(\nabla \varphi) \, d\mathbf{y} d\mathbf{x} \\ &\stackrel{\varepsilon \rightarrow 0}{\rightarrow} \int_{\Omega \times \mathbf{Y}} \mathbf{K}(\mathbf{y}) (\nabla_x u + \nabla_y \hat{u}) \cdot \varphi \, d\mathbf{y} d\mathbf{x} - \frac{1}{2} \int_{\Omega \times \mathbf{Y}} \mathbf{K}(\mathbf{y}) \varphi \cdot \varphi \, d\mathbf{y} d\mathbf{x} \\ &\quad + \int_{\Omega \times \mathbf{Y}} \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y \hat{u} : \nabla_y \varphi \, d\mathbf{y} d\mathbf{x} - \frac{1}{2} \int_{\Omega \times \mathbf{Y}} \mathbf{M}(\mathbf{y}) \nabla_y \varphi : \nabla_y \varphi \, d\mathbf{y} d\mathbf{x} \end{aligned} \quad (4.5)$$

330 Since the space $L^2(\Omega, C_{\text{per}}^1(Y, \mathbb{R}^3))$ is dense in $L^2(\Omega \times Y, \mathbb{R}^3)$, the above inequality holds
 331 for a sequence of regular functions of the form $\varphi_\varepsilon(\mathbf{x}, \mathbf{y})$ Y -periodic in \mathbf{y} with the following
 332 convergence properties,

- 333 • $\mathcal{T}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \nabla_x u + \nabla_y \hat{u}$ in $L^2(\Omega \times Y, \mathbb{R}^3)$
- 334 • $\mathcal{T}_\varepsilon(\varepsilon \nabla \varphi_\varepsilon) \rightharpoonup \nabla_y \nabla_y \hat{u}$ in $L^2(\Omega \times Y, \mathbb{R}^{3 \times 3})$.

335 Upon extracting a (a non-re-labeled) sub-sequence, we obtain in the limit inferior of (4.5),

$$\begin{aligned}
 \liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) &\geq \frac{1}{2} \int_{\Omega \times Y} \left(\mathbf{K}(\mathbf{y})(\nabla_x u + \nabla_y \hat{u}) \cdot (\nabla_x u + \nabla_y \hat{u}) \right. \\
 &\quad \left. + \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y \hat{u} : \nabla_y \nabla_y \hat{u} \right) d\mathbf{y} d\mathbf{x} \\
 &\geq \frac{1}{2} \int_{\Omega} \inf_{v \in H_{\text{per}}^2(Y)} \int_Y \left(\mathbf{K}(\mathbf{y})(\nabla_x u + \nabla_y v) \cdot (\nabla_x u + \nabla_y v) \right. \\
 &\quad \left. + \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y v : \nabla_y \nabla_y v \right) d\mathbf{y} d\mathbf{x} \\
 &= \frac{1}{2} \int_{\Omega} \mathbf{K}^{\text{eff}} \nabla_x u \cdot \nabla_x u d\mathbf{x}
 \end{aligned}$$

336 Γ -limit superior inequality.

337 We will construct the recovery sequence for smooth functions initially and then use a
 338 diagonalization argument to complete the proof.

339 Let $\mathbf{u} \in H^2(\Omega) \cap H_0^1(\Omega)$. Using the density of $C_0^\infty(\Omega)$ in $H^2(\Omega) \cap H_0^1(\Omega)$ we can suppose
 340 without loss of generality that $u \in C_0^\infty(\Omega)$. Furthermore, let $\hat{u} \in C_{\text{per}}^2(Y)$ be a minimizer
 341 of,

$$342 \quad \inf_{v \in H_{\text{per}}^2(Y)} \left\{ \int_Y \mathbf{K}(\mathbf{y})(\nabla_x u + \nabla_y v) \cdot (\nabla_x u + \nabla_y v) d\mathbf{y} + \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y v : \nabla_y \nabla_y v \right\}. \quad (4.6)$$

343 Existence of such a minimizer is shown using classical arguments of coercivity and lower
 344 semi-continuity of,

$$345 \quad v \mapsto \inf_{v \in H_{\text{per}}^2(Y)} \left(\int_Y \mathbf{K}(\mathbf{y})(\nabla_y v - \boldsymbol{\xi}) \cdot (\nabla_y v - \boldsymbol{\xi}) d\mathbf{y} + \int_Y \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y v : \nabla_y \nabla_y v d\mathbf{y} \right). \quad (4.7)$$

346 Define the sequence,

$$347 \quad u_\varepsilon(\mathbf{x}) = u(\mathbf{x}) + \varepsilon \hat{u}\left(\frac{\mathbf{x}}{\varepsilon}\right). \quad (4.8)$$

348 Then,

- 349 • $\mathcal{T}_\varepsilon(u_\varepsilon) \rightarrow u$ in $L^2(\Omega, H^2(Y))$

350 • $\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightarrow \nabla_x u + \nabla_y \hat{u}$ in $L^2(\Omega, H^1(Y, \mathbb{R}^3))$

351 • $\mathcal{T}_\varepsilon(\varepsilon \nabla \nabla u_\varepsilon) \rightarrow \nabla_y \nabla_y \hat{u}$ in $L^2(\Omega \times Y)$.

352 Thus,

$$\begin{aligned} \mathcal{E}_\varepsilon(u_\varepsilon) &= \frac{1}{2} \int_{\Omega} \left(\mathbf{K}\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla u_\varepsilon + \varepsilon^2 \mathbf{M}\left(\frac{\mathbf{x}}{\varepsilon}\right) \nabla \nabla u_\varepsilon : \nabla \nabla u_\varepsilon \right) d\mathbf{y} d\mathbf{x} \\ &= \frac{1}{2} \int_{\Omega \times Y} \left(\mathbf{K}(\mathbf{y}) \mathcal{T}(\nabla u_\varepsilon) \cdot \mathcal{T}(\nabla u_\varepsilon) + \varepsilon^2 \mathbf{M}(\mathbf{y}) \mathcal{T}(\nabla \nabla u_\varepsilon) : \mathcal{T}(\nabla \nabla u_\varepsilon) \right) d\mathbf{y} d\mathbf{x}. \end{aligned} \quad (4.9)$$

353 Hence, passing to the limit as $\varepsilon \rightarrow 0$ in the expression above we obtain,

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \\ &= \int_{\Omega \times Y} \left(\mathbf{K}(\mathbf{y}) (\nabla_x u + \nabla_y \hat{u}) \cdot (\nabla_x u + \nabla_y \hat{u}) + \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y \hat{u} : \nabla_y \nabla_y \hat{u} \right) d\mathbf{y} d\mathbf{x} \\ &= \frac{1}{2} \int_{\Omega} \inf_{v \in H_{per}^2(Y)} \int_Y \left(\mathbf{K}(\mathbf{y}) (\nabla_x u + \nabla_y v) \cdot (\nabla_x u + \nabla_y v) + \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y v : \nabla_y \nabla_y v \right) d\mathbf{y} d\mathbf{x} \\ &= \frac{1}{2} \int_{\Omega} \mathbf{K}^{\text{eff}} \nabla_x u \cdot \nabla_x u d\mathbf{x} \end{aligned} \quad (4.10)$$

354 We can conclude the proof using a density and diagonalization argument. \square

355 **Remark 4.1.** *The Euler-Lagrange equation of (3.7) is,*

$$\int_{\Omega} \mathbf{K}^{\text{eff}} \nabla u \cdot \nabla \phi d\mathbf{x} = \int_{\Omega} f \phi d\mathbf{x} \quad (4.11)$$

356 for all $\phi \in H_0^1(\Omega)$. In the sense of distributions we can recover,

$$\begin{aligned} -\operatorname{div} \left(\mathbf{K}^{\text{eff}} \nabla u \right) &= f \text{ in } \mathcal{D}(\Omega), \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \quad (4.12)$$

357 with \mathbf{K}^{eff} given in (3.8).

358 *The above equation is precisely the limit problem that has been obtained countless times*
 359 *in the homogenization literature. The difference in this work is that the local solution*
 360 *used to construct the effective tensor \mathbf{K}^{eff} belong satisfy a higher-gradient problem with*
 361 *$w_k \in H_{per}^2(Y)$ (and not only in $H^1(Y)$). This newly found local solution regularity is*
 362 *combined with the theorem below to prove the classical corrector convergence result.*

363 The next theorem is included only for completion purposes and is a consequence of G.
 364 Stampacchia's interpolation theorem [35]. Practically, it allows one to control the L^p
 365 norm of the gradient of the weak solution of a second order elliptic system with constant
 366 coefficients, for $p \in [2, \infty)$, by controlling the integrability of the data. We refer the
 367 interested reader to [3, Thm. 3.28, pg. 76] (and references therein) for a modern treatment
 368 on the subject. Our aim here is to apply the theorem below to the second order elliptic
 369 system with constant coefficients in (4.12) to obtain the necessary integrability for the
 370 homogenized solution to compensate for the remaining regularity that is needed in the
 371 classical corrector convergence theorem.

372 **Theorem 4.1.** *Let $u \in H_0^1(\Omega, \mathbb{R}^3)$ be a weak solution of the Dirichlet problem*

$$-\operatorname{div}(\mathbb{A}\nabla u) = -\operatorname{div} \mathbf{F}, \quad (4.13)$$

373 *where the constant coefficients $\mathbb{A}_{ij\alpha\beta}$ satisfy the Legendre-Hadamard condition, and $\mathbf{F} \in$
 374 $L^p(\Omega, \mathbb{R}^3)$ for some $2 \leq p < \infty$. Then, $\nabla u \in L^p(\Omega, \mathbb{R}^3)$ and*

$$\|\nabla u\|_{L^p(\Omega, \mathbb{R}^3)} \leq c \|\mathbf{F}\|_{L^p(\Omega, \mathbb{R}^3)} \quad (4.14)$$

376 A proof of this theorem can be found in [22][Thm. 7.1, pg. 138] and [3, Thm. 3.29, pg.
 377 79 and disc. on Sect. 3.5, pg. 78].

378 4.2 Proof of Theorem 3.2

379 **Theorem 3.2.** *Let u_ε and u be the unique minimizers of (2.22) and (3.7), respectively.
 380 Moreover, let $f := \operatorname{div} \mathbf{F}$, $\mathbf{F} \in L^3(\Omega, \mathbb{R}^3)$ then,*

$$\nabla u_\varepsilon - \nabla u - \nabla_y \hat{u}(\cdot, \frac{\cdot}{\varepsilon}) \rightarrow 0 \text{ in } L^2(\Omega, \mathbb{R}^3), \quad (3.10)$$

382 *as $\varepsilon \rightarrow 0$ with*

$$\hat{u}(\mathbf{x}, \mathbf{y}) = -w_k(\mathbf{y}) \frac{\partial u}{\partial x_k}(\mathbf{x}) + \tilde{u}_1(\mathbf{x}). \quad (3.11)$$

384 *Proof.* The proof of this theorem amounts to essentially a compensated regularity argu-
 385 ment, where the burden of regularity is split (un-equally) between the local solution
 386 and the homogenized solution. We argue as follows: by the assumption on the body
 387 force f we can apply Theorem 4.1 and obtain $\nabla u \in L^3(\Omega, \mathbb{R}^3)$. Moreover, since the lo-
 388 cal solution $w_k \in H_{\text{per}}^2(Y)$ we extend it by periodicity to the entire space. By applying
 389 standard Sobolev embedding theory we can obtain $\nabla_y w_k \in L^6(\mathbb{R}^3, \mathbb{R}^3)$ (see e.g. [14, The-
 390 orem 2.31]). Therefore, the above compensated regularity argument makes the expression
 391 $\nabla u_\varepsilon - \nabla u - \nabla_y \hat{u}(\cdot, \frac{\cdot}{\varepsilon})$ well defined in the L^2 norm and hence, we can prove Proposition
 392 9.12 in [8, pg. 185] directly. In what follows, we provide the actual steps of the argument

393 that is given in [8, Proposition 9.12, pg. 185] with the appropriate adjustments that the
 394 second-gradient thermal effects introduce.

395 Starting from the ellipticity of the second order tensor \mathbf{K} we have,

$$\begin{aligned}
 & c_1 \left\| \nabla u_\varepsilon(\mathbf{x}) - \nabla u(\mathbf{x}) - \nabla_y \hat{u}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right\|_{L^2(\Omega, \mathbb{R}^3)}^2 \\
 & \leq \int_{\Omega} \mathbf{K}(\frac{\mathbf{x}}{\varepsilon}) (\nabla u_\varepsilon(\mathbf{x}) - \nabla u(\mathbf{x}) - \nabla_y \hat{u}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})) \cdot (\nabla u_\varepsilon(\mathbf{x}) - \nabla u(\mathbf{x}) - \nabla_y \hat{u}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})) \, d\mathbf{x} \\
 & \leq \int_{\Omega} \mathbf{K}(\frac{\mathbf{x}}{\varepsilon}) (\nabla u_\varepsilon(\mathbf{x}) - \nabla u(\mathbf{x}) - \nabla_y \hat{u}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})) \cdot (\nabla u_\varepsilon(\mathbf{x}) - \nabla u(\mathbf{x}) - \nabla_y \hat{u}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon})) \, d\mathbf{x} \\
 & \quad + \int_{\Omega} \mathbf{M}(\frac{\mathbf{x}}{\varepsilon}) (\varepsilon \nabla \nabla u_\varepsilon - \nabla_y \nabla_y \hat{u}) : (\varepsilon \nabla \nabla u_\varepsilon - \nabla_y \nabla_y \hat{u}) \, d\mathbf{x}
 \end{aligned} \tag{4.15}$$

396 where the last inequality is due to the ellipticity condition of the tensor \mathbf{M} . Hence, opening
 397 up the expression above, using the structure of equation (2.11), and the symmetry of the
 398 tensors \mathbf{K} and \mathbf{M} we have,

$$\begin{aligned}
 & c_1 \left\| \nabla u_\varepsilon(\mathbf{x}) - \nabla u(\mathbf{x}) - \nabla_y \hat{u}(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}) \right\|_{L^2(\Omega, \mathbb{R}^3)}^2 \\
 & \leq \int_{\Omega} f u_\varepsilon \, d\mathbf{x} - 2 \int_{\Omega} \mathbf{K}(\frac{\mathbf{x}}{\varepsilon}) \nabla u_\varepsilon \cdot (\nabla u + \nabla_y \hat{u}) \, d\mathbf{x} + \int_{\Omega} \mathbf{K}(\frac{\mathbf{x}}{\varepsilon}) (\nabla u + \nabla_y \hat{u}) \cdot (\nabla u + \nabla_y \hat{u}) \, d\mathbf{x} \\
 & \quad - 2 \int_{\Omega} \varepsilon \mathbf{M}(\frac{\mathbf{x}}{\varepsilon}) \nabla \nabla u_\varepsilon : \nabla_y \nabla_y \hat{u} \, d\mathbf{x} + \int_{\Omega} \mathbf{M}(\frac{\mathbf{x}}{\varepsilon}) \nabla_y \nabla_y \hat{u} : \nabla_y \nabla_y \hat{u} \, d\mathbf{x}.
 \end{aligned} \tag{4.16}$$

399 By unfolding, we can see that we are able to pass to the limit in each expression due to
 400 estimate (2.20). Thus,

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} c_1 \left\| \nabla u_\varepsilon - \nabla u - \nabla_y \hat{u} \right\|_{L^2(\Omega, \mathbb{R}^3)}^2 & \leq \int_{\Omega \times Y} f u \, d\mathbf{y} \, d\mathbf{x} \\
 & \quad - \int_{\Omega \times Y} \mathbf{K}(\mathbf{y}) (\nabla u + \nabla_y \hat{u}) \cdot (\nabla u + \nabla_y \hat{u}) \, d\mathbf{y} \, d\mathbf{x} \\
 & \quad - \int_{\Omega \times Y} \mathbf{M}(\mathbf{y}) \nabla_y \nabla_y \hat{u} : \nabla_y \nabla_y \hat{u} \, d\mathbf{y} \, d\mathbf{x} = 0.
 \end{aligned} \tag{4.17}$$

401 On the last step we made use of equation (4.10). □

402 5 Discussion

403 Lastly, remark that it has been observed in [29] that the above generalized heat conduction
 404 problem can be formulated as a classical gradient system with an induced gradient flow
 405 equation,

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$$\partial_t u = +\nabla_{\mathcal{G}} D\mathcal{L}(u) \text{ in } \Omega, \quad (5.1)$$

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with \mathcal{G} the metric tensor and $\mathcal{L}(u) := \int_{\Omega} \log u \, d\mathbf{x}$ the entropy. Naturally, assumptions have to be made on the coefficients so that model is thermodynamically correct. Although, we will address the time-dependent case in future work, we would like to point out that this added gradient structure could be used to pass to the limit in the time dependent case through evolutionary Γ -convergence (see e.g. [25]).

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