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# Entropic Lower Bound of Cardinality for Sparse Optimization

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## Abstract

We introduce a family of cardinality's lower bounds, defined as ratios of norms. We prove that the tightest bound of the family is obtained as a limit case, and involves a Shannon entropy. We then use this entropic lower bound in sparse optimization problems to approximate cardinality requirements. This provides a nonlinear nonconvex relaxed problem, which can be efficiently solved by off-the-shelf nonlinear solvers. In the numerical study, we focus on the case where the optimization is performed on the simplex, and where the classical  $\ell_1$  penalization does not yield sparse solution. The Finance Index Tracking problem is taken as an example and illustrates the efficiency of the proposed approach.

**Keywords:** Sparse Optimisation, Cardinality,  $\ell_0$ -pseudo-norm, Shannon entropy, Index Tracking.

## 1 Introduction

In numerous fields such as finance, energy or machine learning, decision makers aim to control the cardinality of the solution vector, i.e., the number of representative features (assets in portfolio optimization (Bertsimas and Shioda, 2007), shutdowns/start-ups in thermal power plants scheduling (Bialecki et al., 2014a), Support Vectors in machine learning (Bi et al., 2003), ...).

In optimization terminology, the cardinality of a solution encoded by a vector  $x \in \mathbb{R}^n$  is the number of non-zero elements, i.e.,  $|\{i \in [n] : x_i \neq 0\}|$ , and is often written  $\text{card}(x)$ . Both correspond to the so-called  $\ell_0$ -pseudo-norm, denoted  $\|x\|_0$ . Note that  $\ell_0$  does not satisfy the absolute homogeneity condition: for all  $x \in \mathbb{R}^n$  and  $\alpha \neq 0$ ,  $\|\alpha x\|_0 = \|x\|_0$ . The optimization under cardinality requirements is called *sparse optimization*.

Sparsity has mainly emerged from the signal processing and machine learning communities, under names such as compressed sensing (Donoho, 2006) and sparse learning (Bi et al., 2003). In machine learning, the Sparse Support Vector Machine aims at finding a minimal cardinality linear classifier which can separate two classes of labeled data. The sparsity of the solution helps for better interpretability of the solution which is crucial in automated analysis of large text corpora. One major case of using sparsity is the feature selection which refers to the necessity of selecting representative

variables from datasets containing a large number of features, many of them being irrelevant or redundant. For example, in finance, feature selection is used to restrict asset allocation to a limited number of assets in the portfolio. Sparsity allows reducing a priori the dimension of a large-scale problem when performing a sparse regression that may be more efficient than the classical one, by selecting a small set of predictors in a least-squares sense.

Sparsity is also very useful in energy management where many problems involve cardinality constraints. A first example is the intra-day problem, consisting in updating day-ahead generation schedule by modifying a limited number of power units schedules (Bialecki et al., 2014a, Bialecki et al., 2014b). Two other examples concern operation of power plants. During start-up, some components of thermal power plants go from 20°C to 1300 – 1900°C in a few seconds leading, over the long term, to damages, reducing their lifespan. Saving durability of these plants consists in limiting the number of shutdowns/start-ups. Finally, when operating nuclear power plants, it is necessary to limit the number of “deep” drops in power because a nuclear reaction at low power for a long time generates unwanted isotopes that “poison the heart” and to limit the daily number of production variations (modulations) so as not to over-consume the boron (neutron absorber) because a reduction in the boron available in the core makes the plant more difficult to operate and leads to its premature shutdown for refueling.

Optimization problems involving the  $\ell_0$ -norm of the decision vector belong to the class of sparse optimization problems and take one of the two following general form:

(i)  $\|x\|_0$  in the objective function:

$$\min_{x \in X} \{f(x) + \lambda \|x\|_0 \mid g(x) \leq 0\} \quad (P_\lambda)$$

(ii)  $\|x\|_0$  in constraints:

$$\min_{x \in X} \{f(x) \mid g(x) \leq 0, \|x\|_0 \leq k (< n)\} \quad (P_k)$$

In both formulations,  $X \subseteq \mathbb{R}^n$  is the set defining the constraints. The objective function  $f$  corresponds to a given criterion and is often considered as convex in machine learning applications (such as *least-squares problems* (LSQ) while it may be nonconvex in energy applications. The parameter  $\lambda \geq 0$  is viewed as a regularization parameter used to manage the trade-off between the criterion  $f(x)$  and the sparsity of  $x$ . In the sequel, we suppose that the following assumption holds:

**Assumption 1.1.** *We suppose that the decision variables are nonnegative quantities, i.e.,  $X \subseteq \mathbb{R}_+^n$ .*

**Remark 1.1.** *Assumption 1.1 is not restrictive since, for cardinality requirements, the sign does not affect the value:  $\text{card}(x) = \text{card}(|x|)$ .*

In selection problems, a stronger constraint on the decision vector arises:  $x$  must belong to the probability simplex, defined as  $\Delta_n := \{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i = 1\}$ . In this specific case, the  $\ell_1$ -norm is constrained to be one.

Since the  $\|\cdot\|_0$  is lower semicontinuous on  $\mathbb{R}^n$  and is discontinuous at the origin, optimization problems involving the  $\ell_0$ -norm are nonconvex and hence very challenging. They are inherently of combinatorial nature and hence, not solvable in polynomial-time in general. Huge research effort has been made in sparse optimization and several approaches have been proposed. Let us cite :

- **The convex approximation.** A typical example is the famous Least Absolute Shrinkage and Selection Operator (LASSO) penalty technique. It consists in replacing the nonconvex term  $\|x\|_0$

by the convex approximation  $\|x\|_1$ . This approach has first been proposed for linear regression in (Tibshirani, 1996). Since then, the  $\ell_1$ -regularization technique has been extensively studied and improved (Gribonval and Nielsen, 2003, Zou, 2006, Knight and W. Fu, 2000, ...) This leads to very efficient and scalable algorithms in many cases. For example, the main approaches to sparse learning replace the (hard) cardinality requirements with some simpler (convex) functions such as the  $\ell_1$ -norm. However, in several applications of great interest, in energy for instance, even the optimization problems obtained with this convex approximation are tractable, solutions obtained are far from the expected one. It is worth pointing out that replacing cardinality by the convex approximation based on  $\ell_1$ -norm is not applicable to the optimization over the probability simplex (selection problems) i.e. when the variables are discrete probability distributions. The now-standard approaches fail and a number of heuristics have been proposed for this case, for example ignoring the last (sum) constraint and scaling the result, which are suboptimal.

- **The nonconvex approximation.** This approach consists in approximating  $\|x\|_0$  by a continuous nonconvex function. Various functions have been proposed to approximate the  $\ell_0$  term (Bradley and Mangasarian, 1998, W.J. Fu, 1998, Weston et al., 2003) and several types of algorithms have been designed to solve related optimization problems, including algorithms based on the Difference of Convex functions (DC) (Chen et al., 2010, Gasso et al., 2009, Guan and Gray, 2013, Ong and Le Thi, 2013, Thiao et al., 2008, Pham Dinh and Le Thi, 2014) or based on Successive or Local Linear Approximation (Bradley and Mangasarian, 1998, Zou and Li, 2008). Nonconvex approximations can be better than convex relaxations by guarantying a higher sparsity level, but the related nonconvex optimization problems are more difficult to solve.
- **Heuristic approach.** In addition to the mathematical programming based approaches, heuristic methods have also been applied, especially greedy algorithms, designed to directly tackle cardinality minimization problem. Two noteworthy examples are the matching pursuit (Mallat and Zhang, 1993) and the orthogonal matching pursuit (Pati et al., 1993).

Table 1 gives some additional entries in the litterature.

	Problem	Optimality	Resolution
(Bertsimas and Shioda, 2007)	Sparse LSQ Portfolio selection	Global	Branch & Bound
(Nadistic et al., 2020)	Sparse LSQ	Global	Branch & Bound
(Ben Mhenni et al., 2021)	Sparse LSQ	Global	Branch & Bound
(Tibshirani, 1996)	Sparse LSQ	Relaxation	Convex penalization (LASSO)
(Soubies et al., 2015)	Sparse LSQ	Relaxation	Continuous nonsmooth penalty
(Haddou and T., 2019)	Sparsity	Relaxation	Nonconvex penalization
(Atamturk and Gomez, 2019)	Sparse regression	Lower Bound	SDP (convex)
(Chancelier and De Lara, 2019)	GSO	Lower Bound	Caprac conjugacy
(Soussen et al., 2011)	Sparse LSQ	Heuristic	Penalization + Greedy

Table 1: Literature overview on sparse optimization.

LSQ: Least squares problem  
GSO : General Sparse Optimization

In this context, we propose an approach based on constructing a set of lower bounds of  $\ell_0$ -pseudo-norm expressed as ratios of norms (Theorem 2.2). In particular, we prove that the best lower bound we obtained is expressed as a function of Shannon entropy (Shannon, 1948) and  $\ell_1$ -norm. In (Sakai and Iwata, 2016), the authors bring to light sharp extreme relations between Shannon entropy and

$\ell_\alpha$ -norm ( $\alpha > 0$ ). Here, we obtain a relation for  $\alpha = 0$ . Then, we insert this new bound in sparse optimization problems, and show that the relaxed problem is a smooth nonlinear problem (yet non convex), see Proposition 2.3, and that a local solution can be obtained by using a nonlinear solvers like IPOPT (Wächter and Biegler, 2006). Some numerical experiments on the Finance Index Tracking problem illustrate the efficiency of the proposed approach (Section 3).

## 2 Entropic Lower Bound of $\|x\|_0$ and use in Sparse Optimization

### 2.1 A family of lower bounds

We define the  $\ell_p$ -norm of a vector  $x \in \mathbb{R}^n$ ,  $p \geq 1$ , as:

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We remind the known lower bounds of  $\|x\|_0$  as ratios of norms ( $\forall x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ ):

$$B_1(x) = \frac{\|x\|_1}{\|x\|_\infty} \leq \|x\|_0 \quad (1)$$

$$B_2(x) = \left( \frac{\|x\|_1}{\|x\|_2} \right)^2 \leq \|x\|_0. \quad (2)$$

These lower bounds may far from  $\|x\|_0$  in practice. Before introducing a family of new lower bounds, we first recall the following useful property:

**Lemma 2.1** (Hardy et al., 1952). : *If  $\mu$  is a probability measure, and if  $M_p(f, \mu)$  denotes the  $p$ -norm of a function  $f$  with respect to the measure  $\mu$ , then, the function  $p \mapsto M_p(f, \mu)$  defined for  $p > 0$  is non-increasing.*

Theorem 2.2 provides a set of more efficient lower bounds  $(B_q)_{q \in [1, +\infty[}$  based on ratio of norms. We prove that the quality of the bound improves with the parameter  $q$ . The limit case constitutes the tighter bound of the family, and involves a Shannon entropy and the  $\ell_1$  norm :

**Theorem 2.2.** 1.  $\forall x \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ , and for  $q \in [1, +\infty]$ ,  $B_q(x) := \left( \frac{\|x\|_1}{\|x\|_p} \right)^q$ , where  $1/p + 1/q = 1$ , is a non-increasing function of  $q$  constructing a set of ordered lower bounds of  $\|x\|_0$  :  $1 \leq q \leq q' \implies B_q(x) \leq B_{q'}(x) \leq \|x\|_0$ .

$$B_1(x) = \frac{\|x\|_1}{\|x\|_\infty} \leq B_2(x) = \left( \frac{\|x\|_1}{\|x\|_2} \right)^2 \leq \dots \leq B_\infty(x) \leq \|x\|_0 \quad (3)$$

2. The lower bound  $B_\infty(x) := \lim_{q \rightarrow \infty} B_q(x)$  is expressed as a function of Shannon entropy and  $\ell_1$  norm :

$$B_\infty(x) = \frac{\|x\|_1}{\prod_{i \in [n]} |x_i|^{x_i / \|x\|_1}} = \|x\|_1 \exp \left( -\frac{1}{\|x\|_1} \sum_{i \in [n]} x_i \log x_i \right) \quad (4)$$

*Proof.* 1. By Hölder's inequality, we have :

$$\|x\|_1 = \sum_{i \in n} |x_i \times 1| \leq \|x\|_p \left( \sum_{i \in S} 1^q \right)^{1/q} = \|x\|_p (\|x\|_0)^{1/q}$$

from which we deduce that  $B_q(x) = \left(\frac{\|x\|_1}{\|x\|_p}\right)^q \leq \|x\|_0$  for all  $q \geq 1$ .

Besides, using Lemma 2.1, we have  $B_q(x) = \left(\sum_{i \in [n]} x_i^{p-1} \frac{x_i}{\|x\|_1}\right)^{\frac{1}{p-1}} = M_{p-1}\left(x, \frac{x}{\|x\|_1}\right)$ , where in  $M_{p-1}\left(x, \frac{x}{\|x\|_1}\right)$ , the first argument is thought of as a function on the discrete set  $[n]$ , whereas the second one is thought of as a probability measure on the same set. As  $p$  is a decreasing function of  $q$ , we immediately deduce that  $q \mapsto B_q(x)$  is non-decreasing.

2. For all  $q \in [1, \infty]$ , the map  $x \mapsto B_q(x)$  is positively homogeneous of degree 0. Then  $B_q(x) = B_q(x/\|x\|_1)$ . We may replace  $x$  by  $x/\|x\|_1$ . Then, as  $q \rightarrow \infty$  (or equivalently, as  $p \rightarrow 1$ ), we have :

$$\begin{aligned} -\log B_q(x) &= \frac{q}{p} \log \left( \sum_{i \in [n]} x_i^p \right) = (q-1) \log \left( \sum_{i \in [n]} x_i x_i^{p-1} \right) \\ &= (q-1) \log \left( \sum_{i \in [n]} x_i (1 + (p-1) \log x_i + o(p-1)) \right) \\ &= (q-1) \log \left( 1 + (p-1) \left( \sum_{i \in [n]} x_i \log x_i \right) + o(p-1) \right) \\ &\xrightarrow{q \rightarrow \infty} \sum_{i \in [n]} x_i \log x_i + o(p-1) \end{aligned}$$

and so,  $B_q(x) \rightarrow \left(\prod_{i \in [n]} x_i^{x_i}\right)^{-1} = B_\infty(x)$ . □

The entropic lower bound  $B_\infty(x)$  can be viewed as a perspective function of the exponential, composed with the Shannon's entropy of  $x$  :  $H(x) := -\sum_{i \in [n]} x_i \log(x_i)$ . In the case  $\|x\|_1 = 1$ , it simplifies to the exponential of the Shannon entropy. We refer to Figure 1 for a numerical example of the bound  $B_\infty$ . This illustrates, in particular, the concavity of this nonlinear bound.

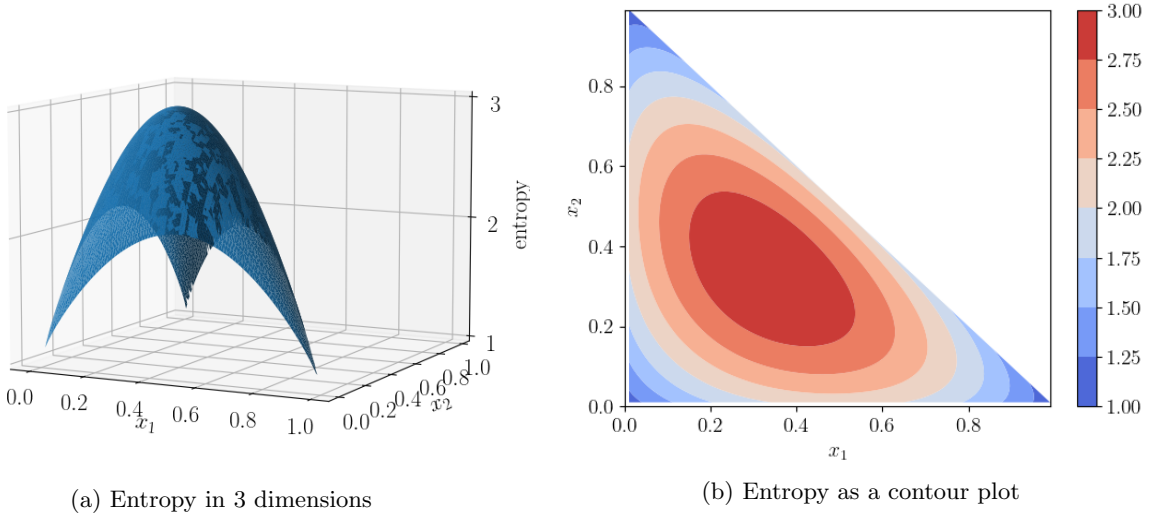


Figure 1: Shannon entropy  $H(x)$  for  $x \in \Delta_3$ . The two first dimensions  $x_1$  and  $x_2$  are displayed, and the third one is implicitly defined as  $x_3 = 1 - x_1 - x_2$ .

## 2.2 Sparse optimization

We now focus on the integration of the previously defined entropic bound (4) in a sparse optimization problem: let us assume a generic problem of the form  $(P_k)$ . The corresponding relaxation is then

$$\begin{aligned} \min_{x \in X} \quad & f(x) \\ \text{s. t.} \quad & g(x) \leq 0 \\ & B_\infty(x) \leq k \end{aligned} \tag{\tilde{P}_k}$$

**Proposition 2.3.** *The problem  $(\tilde{P}_k)$  can be equivalently reformulated as*

$$\begin{aligned} \min_{x \in X} \quad & f(x) \\ \text{s. t.} \quad & g(x) \leq 0 \\ & \Gamma(x, \|x\|_1) \leq 0 \end{aligned} \tag{5}$$

where

(i)  $\Gamma : (x_1, \dots, x_n, z) \in \mathbb{R}_+^{n+1} \mapsto z \log(z) - \sum_{i \in [n]} x_i \log(kx_i)$ ,

(ii) the Jacobian of  $\Gamma$  is defined as  $\frac{\partial \Gamma}{\partial x_i}(x, z) = -1 - \log(kx_i)$ ,  $\frac{\partial \Gamma}{\partial z}(x, z) = 1 + \log(z)$ ,

(iii) and the Hessian of  $\Gamma$  is  $H_\Gamma := \text{diag}(-1/x_1, \dots, -1/x_n, 1/z)$  for  $(x, z) \in \mathbb{R}_{>0}^{n+1}$ .

*Proof.*

$$\begin{aligned} \log B_\infty(x) \leq \log k &\iff \log \|x\|_1 - \frac{\sum_{i \in [n]} x_i \log x_i}{\|x\|_1} \leq \log k \\ &\iff \Gamma(x, \|x\|_1) \leq 0 . \end{aligned}$$

□

The relaxation problem that we obtain is not convex (the function  $\Gamma$  is concave), and there is no guarantee in finding the global optimum of this relaxation. Nonetheless, this problem numerically leads to solutions which are both sparse and with satisfactory objective value, see Section 3.

## 3 Numerical Experiments on the Finance Index Tracking Problem

### 3.1 Index Tracking

A financial index is a number representing the value of the set of assets (stocks or bonds) which reflects the value of a specific market or a segment of it. Insofar as an index is not a financial instrument that we can directly trade, a stock or a bond market index is effectively equivalent to a hypothetical portfolio of assets. In order to gain access to an index, it is necessary to use financial instruments such as options, futures and exchange-traded funds, or to create a portfolio of assets that closely tracks a given index. For a given index, fund managers have the choice between two basic investment strategies. The active strategy assumes that the markets are not perfectly efficient so that fund managers, thanks to their know-how, makes specific investments and hope to add value by choosing high performing assets outperforming an investment benchmark index. On the contrary, the passive strategy assumes that the market cannot be beaten in the long run, so that fund managers expect a return that closely replicates the investment weighting and returns of a benchmark index.

Currently, passive strategies seem to attract more interest from investors. Index tracking, also known as index replication, is one of the most popular passive portfolio management strategy to use the market index to determine the portfolio weights by reproducing the performance of a market index, i.e., to match the performance of a theoretical portfolio as closely as possible. Index tracking allows to get the desired returns from the overall market growth with the lower variability and the lower expense ratio for the investment. The smaller the number of assets needed to mimic is, the smaller the incurred transaction costs will be. Nevertheless, the tracking error is likely to be higher when a small number of assets is used.

To create a tracking portfolio, the simplest technique, called full replication, is to buy appropriate amounts of all the assets that make up the index. Provided that the true index construction weights are available, it allows a perfect tracking. However, it has several disadvantages, one related to the fact that a portfolio can consist of thousands of stocks and the other to the fact that there can be many small or illiquid stocks. These last types of shares increase the risk associated with their sale, which is more difficult, and generate an arbitrage cost that is all the more significant as it is frequent. One of the ways to overcome these drawbacks is to construct a sparse index tracking portfolio (Beasley et al., 2003, Jansen and Van Dijk, 2002) by limiting the number of assets to approximately replicate an index. It corresponds to tracking a signal using a sparse mixture of a given set of time series, see e.g. (Benidis et al., 2018). A sparse portfolio simplifies the execution of the portfolio and tends to avoid illiquid stocks that usually correspond to the assets with small weights in an index, since in a sparse setting most of these assets are discarded. Furthermore, since only a small number of assets is used, the transaction costs are reduced significantly due to the reduction of the fixed (minimum) costs in the commission fees. For more details, see (Benidis et al., 2018, Calafiore and El Ghaoui, 2014).

### 3.2 Formulation as a Sparse Regression Problem

Following (Calafiore and El Ghaoui, 2014), we give the main steps leading to formulate the Index Tracking problem as Sparse Regression Problem. Let a single financial asset  $j$  on which we invest a sum  $S_j$  at the beginning of a period. If the rate of return (or return) of this single asset is denoted  $r_j$ , we will earn  $S_{j,end} = (1 + r_j)S_j$  at the end of the period with  $r_j = \frac{S_{j,end} - S_j}{S_j}$ . For  $n$  assets, we define a vector  $r \in \mathbb{R}^n$  where the  $j$ -th component is the rate of return of the  $j$ -th asset.  $r(k) \in \mathbb{R}^n$  represents the vector of simple returns of the components assets during the  $k$ -th period of time  $[(k - 1)\Delta, k\Delta]$ , where  $\Delta$  is a fixed duration.

Let the entries of  $x \in \mathbb{R}^n$  are the fractions of an investor's total wealth invested in each of  $n$  different assets. Investing at the beginning of the period a total sum  $S$  over all assets is made by allocating a fraction  $x_j$ ,  $j = 1, \dots, n$  of  $S$  in the  $j$ -th asset. The non-negative vector  $x \in \mathbb{R}_+^n$  represents the portfolio "mix", and its components sum to one. At the end of the period, the total value of the portfolio is  $S_{end} = \sum_{j=1}^n (1 + r_j)x_j S$ . The rate of return of the portfolio is the relative increase in wealth  $\frac{S_{end} - S}{S} = \sum_{j=1}^n (1 + r_j)x_j - 1 = \sum_{j=1}^n x_j - 1 + \sum_{j=1}^n r_j x_j = r^T x$ ; i.e., the standard inner product between the vector  $r$  of individual returns  $r_j$ ,  $j = 1, \dots, n$  and the vector of the portfolio allocation weights  $x$ . The  $m \times n$  matrix  $R$  gives the (close price) data of the component assets. The component  $y_k$  of the vector  $y \in \mathbb{R}^T$  represents the return of some target financial index over the  $j$ -th period, for  $j = 1, \dots, n$ . Vector  $y$  is the close price of the target index. Then, the so-called *index tracking* problem is to construct a portfolio  $x$  so as to track as close as possible the "benchmark" index returns



$y$ . Since the vector of portfolio returns over the considered time horizon is :

$$z = Rx, \quad R \in \mathbb{R}^{m \times n} .$$

We may seek for the portfolio  $x$  with minimum Least Squares tracking error, by minimizing  $\|Rx - y\|_2^2$ . However, we need to take into account the fact that the elements of  $x$  represent relative *weights*, that is they are non-negative and they sum up to one. In addition, a cardinality constraint is added for constructing a sparse index tracking portfolio. For given  $R \in \mathbb{R}^{m \times n}$  and  $y \in \mathbb{R}^m$ , this leads to the following sparse regression problem :

$$(P_k) \quad \min_{x \in \mathbb{R}_+^n} \left\{ \|y - Rx\|^2 \mid \sum_{i=1}^n x_i = 1, \text{card}(x) \leq k \right\} .$$

*Remark:*  $(P_n)$  is the problem *without sparsity requirement*.

The constraint  $x \geq 0, 1^T x = 1$  makes the use of LASSO penalty impossible.

**Numerical experiments.** We conducted two experiments with data from (Calafiore, 2021). The results have been obtained on a laptop i7-1065G7 CPU@1.30GHz.

In the first experiment, we consider the following sparse techniques with a limited index tracking data set index with  $n = 50$  assets over a period of  $m = 229$  time steps ( the limited number of assets being the limiting dimension that the SDP method can accept) :

(i) *Greedy heuristic:* solve  $(P_n)$ , take the  $k$  greatest value of  $x$  and renormalize

(ii) *Reversed greedy heuristic:* solve  $(P_n)$ . While  $\text{card}(x) > k$ ,

(a) constrain to 0 the lowest nonzero entry

(b) solve  $(P_n)$  with the additional constraint

(iii) *SDP approach:* computation of method `sdp2` of (Atamturk and Gomez, 2019)

(iv) *Mixed-integer programming:* exact solving using CPLEX

(v) *Entropy lower bound:* solve the problem  $\min_{x \in \mathbb{R}_+^n} \left\{ \|y - Rx\|^2 \mid \sum_{i=1}^n x_i = 1, B_\infty(x) \leq k \right\}$

**Remark 3.1.** *The plain method based on  $\|x\|_\infty \geq 1/k$  (Pilanci et al., 2012) have been tested, but does not produce solutions with significant sparsity for this specific problem.*

We aim at finding a vector  $x \in \mathbb{R}^n$  with sparsity  $k = 10$ . Figures 2 to 4 illustrate the obtained results. For the different methods tested, we carried out various simulations by varying the desired cardinality along the  $x$ -axis in an interval ranging from 5 (a high degree of sparsity is required with only 5 non-zero values out of the 50) to 45 (the desired vector is practically dense). The quality of the solution to the problem can be assessed according to two criteria: the value of the objective function at the optimum and the respect of the cardinality constraint. The main comments that can be made from these results are listed below:

(i) Compliance with the cardinality constraint is all the more difficult to satisfy when the desired cardinality is low (see e.g. the time of the exact solver). Beyond a certain degree of sparsity (here, about 30), the problem becomes easy to solve for all the methods tested.

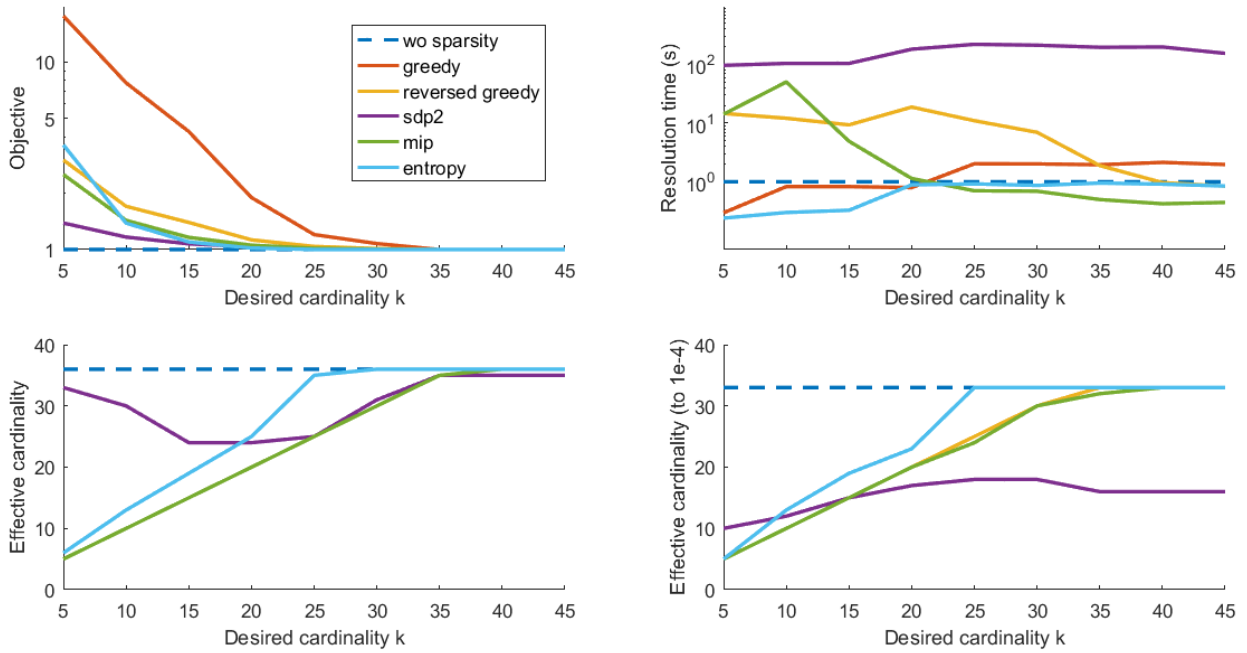


Figure 2: Index tracking for  $n = 50$ .

At the bottom left, the cardinality constraint is satisfied below the diagonal  $y = x$ .

At the bottom right, we display the cardinality for the solution filtered with a threshold of  $10^{-4}$ .

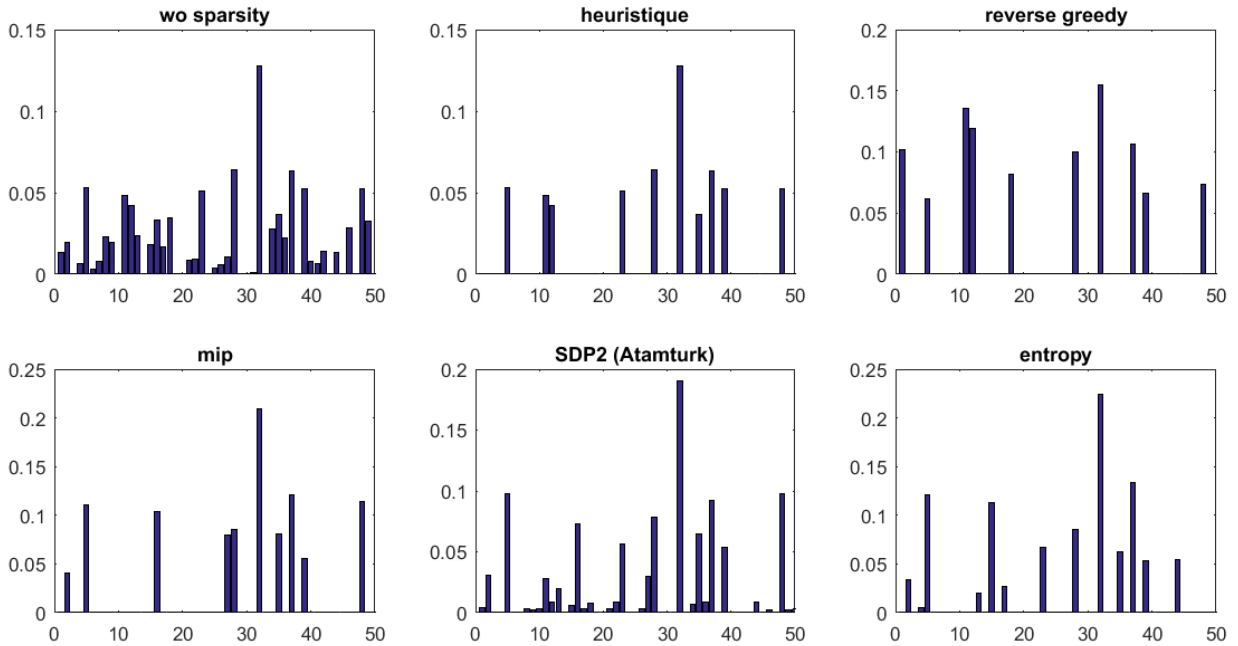


Figure 3: Index tracking for  $n = 50$  and  $k = 10$

- (ii) Concerning the value of the objective function, we note that the “greedy” method is the least efficient of all, while the “reversed greedy” method is competitive. From a desired cardinality of 10, the results of the entropic method are very close to the MIP method (exact resolution of the problem).
- (iii) Regarding the respect of the cardinality constraint, we observe that, for strong sparsity requirements, the SDP2 technique absolutely does not respect the desired cardinality unlike the entropic method. The fact of filtering at  $10^{-4}$  the values of cardinalities obtained does not change the

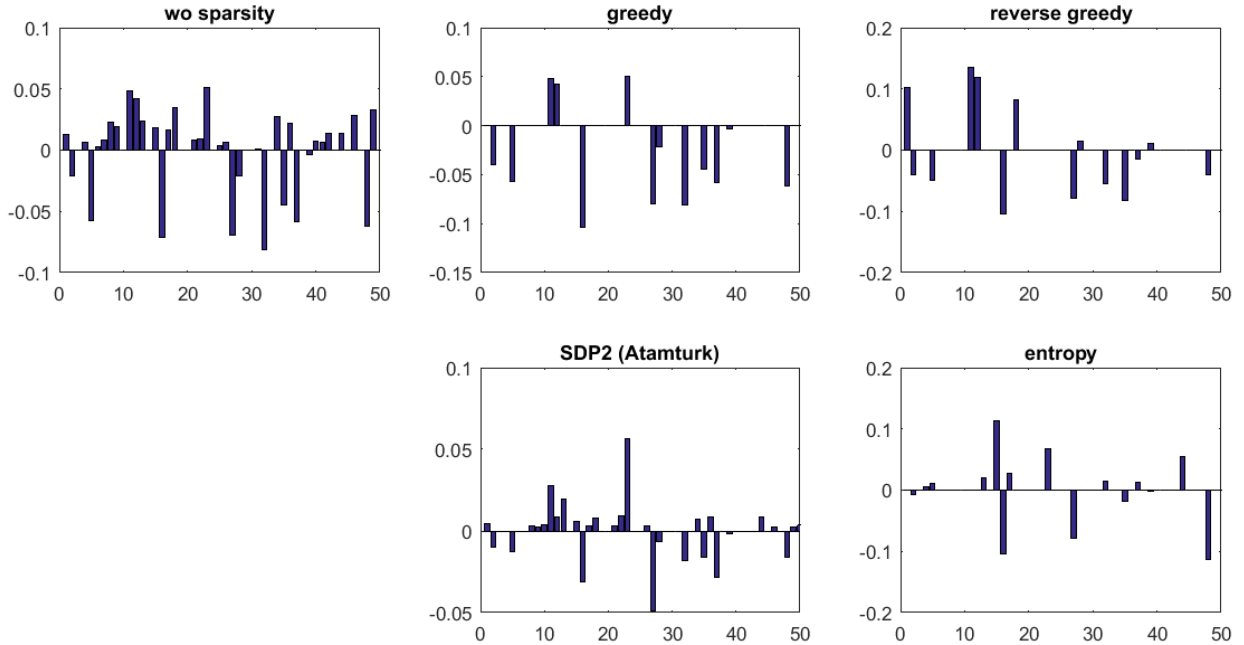


Figure 4: Index tracking for  $n = 50$  and  $k = 10$ .  
The bars represent the difference with exact solution.

fact that the SDP2 method cannot calculate a solution which respects the desired cardinality. The reversed greedy method provides solutions that respect the cardinality constraint.

- (iv) Concerning computation times, unsurprisingly the SDP2 method is the most expensive by far, even for very low sparsity requirements. The exact MIP method is also expensive but the computation time becomes logically lower as the sparsity requirement weakens. The “reserved greedy” method requires a computation time that remains quite high, regardless of the level of cardinality requirement. The entropic method, on the other hand, makes it possible to calculate solutions in short times, even when the cardinality requirement is very strong. This is an important point in practice, especially for large problem instances, thus further disqualifying the SDP approach and even the exact approach.

In a second experiment, we illustrate the possibility to compute a sparse solution via entropic bound even in high dimension ( $n = 430$  assets) and hard cardinality requirement ( $k = 6$ ). Figure 5 compares the solution obtained without cardinality constraint (left subfigure) with the relaxed problem (right subfigure). Our technique is highly scalable since its computational time is low (around 1 second for our technique against around 3 seconds for the problem without sparsity requirement). Moreover, the sparsity requirement is almost fully satisfied, as the effective cardinality of the solution is 7 (the target was  $k = 6$ ).

## 4 Conclusions and Perspectives

By using ratios of norms, we proposed a new lower bound of cardinality, based on Shannon entropy. Despite its non-convexity, the use of this entropic bound in a sparse optimization problem is easy, and a local solution can be found very rapidly by using nonlinear solvers. Early results obtained on Index Tracking Finance problem are good regarding other approaches (heuristics, SDP,... ) and the proposed approach seems promising.

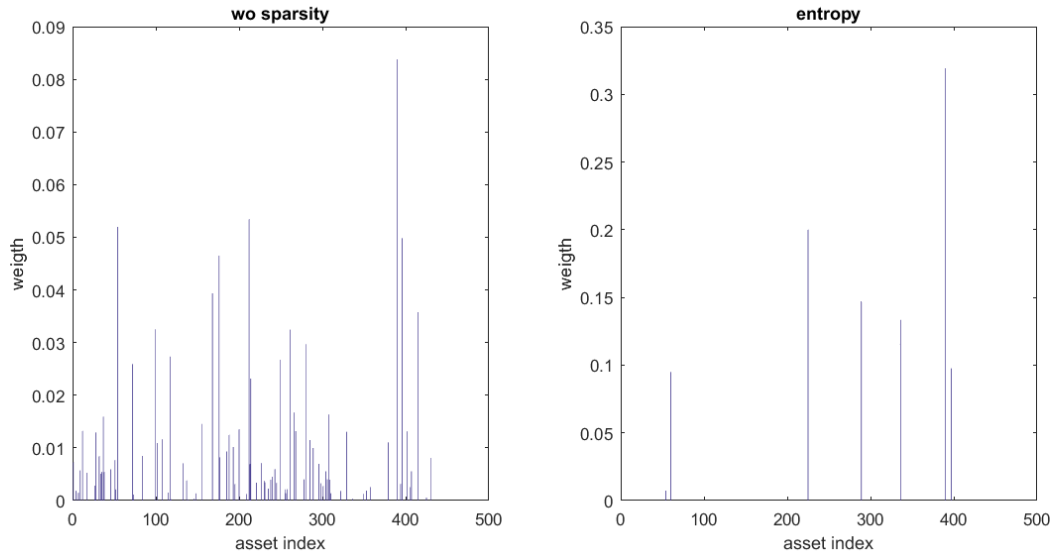


Figure 5: Index Tracking for  $n = 430$  and  $k = 6$ .

Among the various perspectives opened to future investigation, we can mention the search for efficient bounds and estimates of cardinality (results on estimates can be found in Bialecki et al., 2015). Extensive simulations on various applications, including Machine Learning, in order to evaluate the efficiency of our approach would be worth considering. Finally, a close look on the relations between Shannon entropy and  $\ell_0$ -pseudonorm should also be done to possibly get approximation guarantees in the sparse optimization problem.

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