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Aristotle on Common Axioms

Vincenzo De Risi

Aristotle claimed that each science is grounded in principles that are proper to it. He also recognized, however, the existence of principles that are common to many sciences, that he called axioms. The sources of Aristotle's theory of axioms seem to have been both philosophical and mathematical, and the outcomes of the theory are some very refined views on the analogy between different scientific domains, and possibly a theory of non-syllogistic inferences. I sketch here Aristotle's theory of axioms, its motivations, its developments, and its applications to the theory of science. Finally, I offer a tentative interpretation of axioms that may shed some light on several open interpretative questions in Aristotle's epistemology.

§1. Introduction: Common Axioms and Universal Science

In the *Posterior Analytics*, with the intention of laying the foundations to a general account of scientific demonstration, Aristotle presented a rich and important theory of the first principles (*ἀρχαί*) of science. Aristotle's main polemical target was Plato, who famously argued time and again that only philosophy (or dialectics) could provide the foundations for all other disciplines.¹ Aristotle, by contrast, advocated the autonomy of non-philosophical sciences and intended to secure *specific* foundations for each of them individually, on account of the fact that each science investigates a specific *genus* of things (*τὸ γένος ὑποκείμενον*), and that there is no genus common to them all. For Aristotle, the worst methodological mistake in science consisted in mixing up different genera (*μετάβασις εἰς ἄλλο γένος*), as in employing the results established in one science to draw conclusions in the domain of another.

Most of Aristotle's efforts in his epistemological works were accordingly devoted to discussing and expanding the *proper* principles of science, which he understood as the

I thank Francesco Ademollo, Orna Harari, Marko Malink, Mattia Mantovani, Marco Panza, and David Rabouin for their brilliant remarks and careful corrections of previous drafts of this essay. I have learned a lot from their comments, and the paper has substantially improved thanks to their help.

¹ See for instance PLATO, *Resp.* H, 533b-534e.

veritable foundations of each discipline. Aristotle called them *theses* (θέσεις), and further divided them into *definitions*, *hypotheses*, and possibly *postulates* and yet other principles. The *Posterior Analytics* are mostly devoted to explaining the epistemic role of these proper principles. What comes as striking is rather to find out that, in addition to the above-mentioned principles proper to each science, Aristotle also posited some principles *common* to a plurality of disciplines. He called these principles *axioms*, or *common axioms* (κοινὰ ἀξιιώματα).

Common axioms seem to have a twofold origin in Aristotle's system. On the one hand, they clearly stem from the Academic discussions on dialectics as the foundation of all sciences. Aristotle, to be sure, wanted to oppose these Platonic claims, but agreed that there was some truth in them, and himself developed a theory of first philosophy as a grounding discipline *of sorts*. Aristotle's first philosophy deals indeed with very general principles, such as the Principle of Contradiction or the Principle of Excluded Middle, which apply to every object and must be accepted in every science. These logical and ontological principles are arranged by Aristotle among the ἀξιιώματα.

On the other hand, the theory of common principles had a mathematical source. In Plato's time, mathematics was not considered to be a unitary discipline, and it was rather a name encompassing the different sciences of plane geometry, stereometry, arithmetic, and sometimes also optics, astronomy, harmonics, and others.² Similarly, Greek mathematicians considered solid bodies, plane figures, lines, points, and numbers, as objects of different sorts, with few or no connections to one another—objects belonging to different *genera*, as Aristotle would have said.³ This general picture of the mathematical sciences and their plural objects was however challenged during Aristotle's lifetime. Aristotle states, in fact, that some mathematicians had begun to prove theorems that could be universally applied to different kinds of mathematical objects, and in this way they had generalized previous theories that proved in different ways the same theorem for lines, plane figures and numbers. Aristotle's claim is confirmed by Proclus' testimony that Eudoxus had "increased the number of general theorems", thereby unifying mathematical sciences that had been, up to that point, separate disciplines.⁴ Aristotle's explicit example of these general theorems comes from the theory of proportions (ἀναλογίαι), and while it is uncertain which theory of proportions Aristotle had in mind, one can make the educated guess that this was indeed Eudoxus' theory.⁵ The same

² Plato still distinguished, for instance, plane geometry and stereometry as two different disciplines. See *Resp.* H, 528b-c.

³ See DE RISI 2021A, as an example of a multi-sorted (as opposed to set-theoretic) approach to Greek mathematics.

⁴ See PROCLUS, *In primum Euclidis* 67, which seems to be based on a book by Eudemus on the history of ancient mathematics. The fact that Plato did not recognize common axioms as principles of mathematics (they do not fit at all with his description of ὑποθέσεις in the important passages of *Meno*, 86e and *Resp.* Z, 510d; H, 533b-c), may confirm that they were first thematized in Eudoxus' time, shortly after the composition of the *Republic*. Some principles similar to Euclid's common notions are however (very obliquely) mentioned in *Parm.* 154b and *Theaet.* 155a. These passages may offer some evidence that these principles were already being discussed by the mathematical community.

⁵ In the early *Top.* Θ 3, 158^b29-35, Aristotle mentions an older (pre-Eudoxian) anthyphairtic theory of proportions; for a similar passage in the *Metaphysics*, see PRITCHARD 1997. On Aristotle and the theory of proportions, see

theory, or a close version of it, was the one later expounded in Euclid's Fifth Book of the *Elements*. Eudoxus' theory of proportions dealt with a novel kind of mathematical object, called *magnitude* (μέγεθος), which encompassed lines, plane figures, solid bodies, and possibly angles and other mathematical items. As a consequence, a Eudoxian "general" theorem on proportions may be employed to find the solution of an array of related problems in different mathematical sciences. This new theory was extremely powerful and represented an important turning point in the history of Greek mathematics.⁶

The formulation of general theorems reverberated in the theory of principles, since such results should be themselves grounded on some basic assumptions. Euclid's *Elements*, indeed, are prefaced by a set of principles labeled "common notions" (κοινὰ ἔννοιαι) that seem to refer to any mathematical objects whatsoever and are applied, in the course of the proofs, to lines, surfaces, solids, angles, magnitudes in general, as well as numbers. The third of these common notions, for instance, states that *equal things subtracted from equal things result in equal things*. This principle (henceforth CN₃) is mentioned by Aristotle several times as the main example of a "common axiom", and he explicitly says that this kind of general principles were called "axioms" (ἀξιιώματα) *by the mathematicians*.⁷ We may easily surmise, then, that some common axioms, or common notions, had been spelled out in Aristotle's time (possibly by Eudoxus himself) in order to ground general theorems. This new kind of principle was discussed by Aristotle and was later incorporated into Euclid's *Elements*.⁸

In an isolated passage, Aristotle also refers to a "universal mathematics", i.e. a καθόλου μαθηματική (later to be called *mathesis universalis*), that was *common* (κοινή) to several

MENDELL 2007 and RABOUIN 2016. On Eudoxus' theory of proportions, see the classic KNORR 1975. For a discussion of Eudoxus' dating and involvement with the Academy, see for instance ZHMUD 1998.

⁶ In *An. post.* A 5, 74^a17-25, Aristotle says that a theorem of the theory of proportions used to be proven by ancient mathematicians through different demonstrations, bearing on numbers, lines, solids and time intervals, respectively, but more recent mathematicians had unified all these demonstrations into one. The theorem that Aristotle has in mind may well be the one of the alternation of proportion (cf. also *An. post.* B 17, 99^a8-11, discussed later at note 36). In Euclid's *Elements*, such a theorem has *two* demonstrations: a first which is valid for lines, surfaces and solids (i.e. magnitudes in general), given in *Elements* V, 16; and a second, valid for numbers, given in *Elements* VII, 13. The *Elements*, therefore seem to represent a stage of generalization predating the proof hinted at by Aristotle. But also more in general, Book X of Euclid's *Elements*, which is often attributed to Theaetetus, still classifies ratios according to the dimensionality of the objects involved (lines vs. surfaces). This is not the case with the theory of proportions expounded in Book V of the *Elements* (commonly attributed to Eudoxus) which offers a unified treatment of ratios between "magnitudes", be they lines, surfaces, solid bodies, or other geometrical objects. On the other hand, *Elements* X, 5 and 6 mix together magnitudes and numbers. It is clear that all these theorems express different conceptions of the generality and scope of the proofs.

⁷ *Metaph.* Γ 3, 1005^a18-20. It is not entirely clear why Euclid called κοινὰ ἔννοιαι what Aristotle called κοινὰ ἀξιιώματα, but the term may have enjoyed some flexibility, and Aristotle himself calls these principles κοινὰ δόξαι in *Metaph.* B 2, 996^b28. It has been suggested that the term κοινή ἔννοια is Stoic, and that the label of these principles at the beginning of the *Elements* was first given, or modified, in post-Euclidean times. See TODD 1973.

⁸ In DE RISI 2021B, I have claimed that Euclid's first three common notions were originally devised in order to ground a theory of equivalence for plane figures, which is expounded in *Elements* I, 35-45. They were later generalized so as to encompass other kinds of mathematical objects. I have also claimed that at the times of Aristotle and Euclid it is likely that only the first three common notions of the *Elements* had been spelled out, while the others were later additions to the text.

genera of objects.⁹ We have no independent testimony attributing to Eudoxus or other mathematicians of the fourth century the idea of a universal mathematics, and it is debated among the interpreters whether the general theorems mentioned by Aristotle were ever put together into a “universal mathematics” as an independent science.¹⁰

Be this as it may, Aristotle clearly envisaged the possibility of conceiving of a unitary science of mathematics (encompassing both arithmetic and geometry) and discussed it in several passages of his work. Once again, Aristotle’s main targets seem to be various theories developed in the Academy. On the one hand, universal mathematics may have appeared to Aristotle as an analogue of Platonic dialectics for mathematics, that is to say a fabricated super-science intended to provide a foundation for mathematical disciplines. On the other hand, it seems that universal mathematics had suggested several ontologies of mathematical objects. Aristotle opposed some reductionists ideas aimed to derive geometrical magnitudes from numbers, for instance, that may have been advanced by Pythagorizing Platonists, and that were probably connected with a parallel reduction of geometry to arithmetic—considered itself as the *mathesis universalis*.¹¹ It seems, also, that some other philosophers suggested that universal mathematics was not to be identified with arithmetic or geometry, and was rather a different science having *objects of its own*, that were neither numbers nor magnitudes but higher-level entities. Aristotle strongly opposed this view as well, and denied that there might be a mathematical science dealing with

no number, no line, no surface, no solid, but something else.¹²

⁹ The main passages in which Aristotle mentions a universal mathematics are E 1, 1026^a25-27, and K 7, 1064^b7-9, the second of which may be spurious and depending on the first. A possible further reference may be *Metaph.* Γ 2, 1004^a2-9, in which Aristotle mentions a “first part of mathematics”. On the development of this notion from Aristotle to the early modern age, see RABOUIN 2009.

¹⁰ In particular, the important RABOUIN 2009 sheds doubts on the hasty conclusion that Aristotle’s discussion of general theorems was connected with his references to a *mathesis universalis*, as well as on its identification with any extant mathematical theory. The Eudoxian theory of proportions expounded in Book V of the *Elements*, insofar as it deals with any kind of magnitudes (lines, surfaces, solid bodies) has been pointed out as a specimen of Aristotle’s *mathesis universalis*. Yet, it does not apply to numbers, and in this respect it appears more as a fragment of a “universal geometry” of sorts than as a universal mathematics in Aristotle’s sense—and in fact, in the passage in which Aristotle mentions universal mathematics, he explicitly *contrasts* this latter with geometry.

¹¹ A large part of the discussion of *Metaph.* M 9, in fact, revolves around Speusippus’ views, and in general on whether the number may be considered a principle for lines, surfaces and solids (and so, possibly, arithmetic being the first mathematical science), or whether these three geometrical genera may be reducible to one another. A passage from DIOG. LAERT. IV, 2 (frag. 70 in TARÁN 1981) states that Speusippus was the first who considered what is *common* (κοινόν) to several μαθήματα, which might be a reference to a *mathesis universalis* (but also a tamer reference to μαθήματα as sciences in general).

¹² See *An. post.* A 24, 85^a37-85^b1. A similar statement is read in *Metaph.* M 2, 1077^a9-12, in which Aristotle says that the mathematicians formulate some universal axioms or theorems (the Greek only has ἕναι καθόλου), that should apply to substances that are “neither number, nor point, nor magnitude, nor time” (but, Aristotle adds, such substances do not exist). Cf. also *Metaph.* M 3, 1077^b17-22: “Just as universal propositions in mathematics are not about separate objects over and above magnitudes and numbers, but are about these...”.

These criticisms notwithstanding, Aristotle clearly praised the mathematical power of Eudoxus' theory of proportions, and explicitly thematized the notion of μέγεθος in his philosophy of mathematics.¹³ He also praised general theorems applying transversally to several kinds of objects, and intended to elaborate a theory of science that could accommodate these important results.

Aristotle needed a theory of mathematics that could oppose the Platonic “univocation” of this science and its ontology, and at the same time account for the new discoveries pushing towards a *mathesis universalis*. It is clear, therefore, that Aristotle perceived similar philosophical issues in general mathematics and first philosophy. It comes therefore as no surprise that he conceived of a theory of the “common” principles of science (axioms) that could account for both these hyper-generic disciplines.

This meant that common axioms such as CN3, that apply to numbers, lines, surfaces, and solids, do not apply to “something else” over and above the previous items. It also meant that such axioms may not ground a science of their own, the καθόλου μαθηματική, as if they were the *proper principles* of a higher genus. These common principles, therefore, had to be sharply distinguished from the *theses*. Similarly, the logical and ontological principles of dialectics, such as the Principle of Contradiction, cannot be said to apply to a definite genus of things nor to ground a single science, and are therefore axioms as well. Aristotle's *Posterior Analytics* attempted to provide a unified theory of these κοινὰ ἀξιώματα in mathematics, first philosophy and science in general. This theory had the aim, on the one hand, of explaining the universal applicability of axioms; and, on the other, of reassessing the claim that each science is nonetheless grounded on its own proper foundations.

In what follows, I will sketch Aristotle's theory of axioms and the way in which it may accomplish his epistemological goals. We will first look at the main texts in which the theory of common axioms is presented (§2). Then we will explore Aristotle's original solution to the problem of reconciling the existence of common principles with the idea that each science should be grounded on proper principles (§3). We will further look at the role of common axioms in Aristotle's theory of demonstration, and offer a standard reading of it: the “schematic” interpretation of axioms (§4). This latter reading, however, presents some serious exegetical issues, and I will defend an alternative interpretation of Aristotle's theory (the “inferential” interpretation of common axioms), that may solve these problems (§5). Finally, I expound some more general, but also more conjectural consequences of the inferential interpretation for Aristotle's theory of demonstration (§6).¹⁴

¹³ An important question is whether, according to Aristotle, the notion of a *magnitude* constitutes a veritable genus or a weaker form of unity among the objects of geometry, and therefore whether geometry should be considered to be a unitary discipline. In several passages, such as the one in *Metaph.* Δ 13, Aristotle seems to incline towards the idea that lines, surfaces and solid bodies represent three different *summa genera* which are irreducible to one another, even though they may all be labeled as *magnitudes* (μέγεθη). This is the opinion, for instance, of APOSTLE 1952, pp. 105-106. A contrary stance seems however to be taken in the array of objections in *Metaph.* M 9, 1085^a9-23, mentioned above. I will not discuss this difficult issue here, since it does not affect the interpretation of Aristotle's common axioms.

¹⁴ The most extensive treatments of common axioms in recent literature are to be found in MUELLER 1991 and MCKIRAHAN 1992. My own interpretation of axioms is completely at odds with McKirahan's—who distinguishes between logical and mathematical axioms; accepts a schematic reading of them; denies their primary inferential

§2. Logical and Mathematical Axioms

Aristotle's main discussion of axioms takes place in two passages in the *Posterior Analytics* and one passage in the *Metaphysics*.¹⁵

In *An. post.* A 2, Aristotle offers a first classification of the principles of science, and distinguishes between theses and axioms (ἀξιιώματα), by saying that both kinds of principles are unprovable, but axioms have the further feature that they must be grasped by anyone who is going to learn anything whatever. He does not provide any examples in this connection, but a few lines above he explicitly spells out the Principle of Excluded Middle.¹⁶ The characterization of axioms as principles needed to learn anything at all fits well with the status of the highest logical principles such as the Excluded Middle or the Principle of Contradiction. In this passage, axioms are not qualified as “common”, nor are theses ascribed to a single genus. The whole context seems much more philosophical than mathematical.¹⁷

Aristotle offers a classification of principles again in *An. post.* A 10. Here the context is more explicitly mathematical, and the divide between theses and *common* axioms (κοινὰ ἀξιιώματα) falls on their being either genus-specific or common to several sciences. The discussion of theses is expanded into an array of cases that had not been considered in the previous chapter, and new terms as “postulates” (αἰτήματα) or specific “hypotheses” (ὑποθέσεις, in a different sense from in the previous chapter) are introduced. The philosophical “definitions” (ὁρισμοί) mentioned in A 2 are here substituted by a reference to the “terms” (ὄροι) employed in a demonstration, and ὄροι is indeed the word commonly found at the beginning of the lists of definitions in Greek *mathematical* texts. The examples are

role; takes categories to be genera; and thinks that Aristotle embraced a fully-fledged *mathesis universalis*. Mueller's essay has the great advantage of providing some of the mathematical background of Aristotle's theory of axioms, but still rejects the possibility of an inferential interpretation of axioms. I will deal with all these points in the course of the present essay.

¹⁵ The term ἀξιιώματα only rarely appears in Aristotle's works, the main occurrences being limited to the technical notion employed in the *Analytics* and the *Metaphysics*. It is sometimes also employed, however, in the general, non-technical sense of a premise in *Top.* Θ 1, 156^a23, or *Soph. El.* 24, 179^b14. In *An. pr.* B 11, 62^a12-13 an ἀξιιώματα is an ἐνδοξον, this term possibly betraying the dialectical origin of these principles. It seems, therefore, that Aristotle gave a more technical meaning to the term only in his later works. This may draw on the constitution of a mathematical theory of common axioms (possibly devised by Eudoxus) during Aristotle's lifetime. On Aristotle's mathematical terminology, see EINARSON 1936.

¹⁶ In *An. post.* A 1, 71^a11.

¹⁷ “An immediate deductive principle I call a thesis if it cannot be proven but need not be grasped by anyone who is to learn anything. If it must be grasped by anyone who is going to learn anything whatever, I call it an axiom (there are principles of this kind); for it is of this sort of principle in particular that we normally use this name. A thesis which assumes either of the parts of a contradictory pair – what I mean is that something is or that something is not – I call a hypothesis. A thesis which does not I call a definition” (*An. post.* A 2, 72^a14-24; transl. Barnes modified). Note, however, that the Greek uses the indefinite ὅστις where Barnes (and many others) translate “anything whatever”. The meaning might also be “some (unspecified) thing”, referred to a particular science, and in the latter interpretation a principle would be an axiom if there is a science for which it is necessary to learn it. Given Aristotle's example in this connection (the Excluded Middle), as well as the parallel passages in the *Metaphysics* referring to the Principle of Contradiction, I endorse Barnes' stronger translation.

mathematical throughout, and as an instance of a *common axiom* we do not find the Excluded Middle but rather the statement on the subtraction of equals, CN₃.¹⁸

The reason for this double presentation in the *Posterior Analytics* is unclear. It is possible that Aristotle was first introducing the general ideas of his theory of science (A 2), and then returning in greater detail to the same topic (A 10), but there are too many small discrepancies between the two chapters to suggest that we are in fact confronted with two different versions of the same discussion, one philosophical and one mathematical, that were put together into a single treatise. Indeed in the two chapters we seem to witness the twofold origin of Aristotle's interest on the topic.

This does not mean, however, that the *Posterior Analytics* offer two different theories of axioms, such that the logical “axioms” needed to be grasped in order to learn anything whatever are different principles from the “common axioms” employed in the mathematical sciences. On the contrary, the double discussion of logical and mathematical axioms is intertwined in the whole text of the *Posterior Analytics*, showing that we are confronted with different aspects of the same kind of principles rather than with different sorts of axioms. The two examples of CN₃ and the Excluded Middle are very recurrent in Aristotle's works, and they represent the standard instances of axioms in several contexts.¹⁹ A passage in *An. post.* A 11 explicitly mentions both the Excluded Middle and CN₃ as falling under the same label:

All the sciences associate with one another in respect of the common principles [κοινά] (I call common the principles which they use as the basis from which to demonstrate—not those about which they prove nor what they prove); and dialectic associates with them all, and so would any science which attempted to give universal proofs of the common principles (e.g. that everything is asserted or denied, or that equals from equals leave equals, or any principles of this sort).²⁰

The passage is also relevant insofar as it explicitly connects both *logical and mathematical* common axioms to “all the sciences” (πᾶσαι αἱ ἐπιστήμαι), and therefore seems to envisage

¹⁸ Here is the main passage on common axioms: “Of the (principles) used in the demonstrative sciences some are proper to each science and others common—but common by analogy, since something is only useful insofar as it is in the genus under the science. Proper: e.g. that a line is such-and-such, and straight so-and-so. Common: e.g. that if equals are removed from equals, the remainders are equal” (*An. post.* A 10, 76^a36-41; transl. Barnes modified). The term “common axioms” does not appear as such in this passage, and Aristotle simply adds a few lines below that the “common things” from which all demonstrations begin are called axioms (τὰ κοινὰ λεγόμενα ἀξιιώματα).

¹⁹ The Principle of the Excluded Middle appears in *An. post.* A 1, 71^a11; A 11, 77^a22 and 77^a30; A 32, 88^a36-^b1. The passage in *An. Post.* A 11, 77^a22 has a clear correspondence in *An. Pr.* B 11, 62^a13, which is the only reference to axioms in the latter work. The common axiom CN₃ is mentioned in *An. post.* A 10, 76^b21 and 76^a41-42; A 11, 77^a26-31; *An. pr.* A 24, 41^b22-23; *Metaph.* K 4, 1061^b20.

²⁰ *An. post.* A 11, 77^a26-31. I use and modify Barnes' translation, who renders “κοινά” as “common items” rather than as “common principles” or “common axioms”. Note that the passage is arguing about Platonists, and it may be read as stating that *they* should think that logical and mathematical axioms fall under the same label. I do not think that this is the case, and rather believe that Aristotle himself endorsed such a view. But the above dialectical reading may account for some different interpretations of Aristotle's theory (such as Theophrastus').

an unrestricted notion of their “commonality” (which is probably in line with *An. post.* A 2, and its claim on the need of axioms in order to learn anything whatever).

The convergence of the two characterizations of axioms is further developed in the third important passage dealing with this kind of principles, in *Metaph.* Γ 3. Here the axioms at stake are the logical and ontological principles, such as the Excluded Middle and the Principle of Contradiction. Aristotle famously says that the Principle of Contradiction is the best known principle, the one on which no one can err, the principle that must be known in order to learn anything else, and is a perfectly unhypothetical principle.²¹ These (and other) features of this axiom dovetail with the characterization of axioms in *An. post.* A 2 (first principles needed in order to learn anything whatever). Nonetheless, a large part of this chapter of the *Metaphysics* is in fact devoted to belaboring the point that such logical principles are investigated by the philosopher insofar as they pertain to being *qua* being and, therefore, hold for all genera whatsoever:

It is obvious that the investigation of these axioms pertains to one science, namely the science of the philosopher, for they apply to all existing things, and not to a particular class separate and distinct from the rest. Moreover all thinkers employ them – because they are axioms of being *qua* being, and every genus possesses being – but employ them only in so far as their purpose require; i.e., so far as the genus extends about which they are carrying their proofs. Hence these axioms apply to all things *qua* being (for this is what is common [κοινόν] to them), it is the function of him who studies being *qua* being to investigate them as well.²²

The main argument in the passage is therefore precisely that such logical axioms are *common* and hyper-generic, just as was the case with *An. post.* A 10.

In *Metaph.* Γ, mathematical principles such as CN₃ are not mentioned, but Aristotle claims that *equality* is a notion common to several genera, and therefore it is investigated by first philosophy. In this respect CN₃, articulating a fundamental property of equality, seems to be regarded by Aristotle as being on the same footing of the other logical-ontological principles.²³

Further on, the long discussion about the peculiar “demonstration” by ἐλεγχος of the Principle of Contradiction carried out in *Metaph.* Γ 3 is applied (with important modifications) to the Principle of the Excluded Middle as well (in *Metaph.* Γ 7), and we have seen above that in *An. post.* A 11 Aristotle explicitly mentioned the issue of the demonstrability of the Excluded Middle together with that of CN₃, and indeed of all common axioms

²¹ *Metaph.* Γ 3, 1005^b12-19.

²² *Metaph.* Γ 3, 1005^a21-29 (transl. Tredennik). In a similar vein, Aristotle begins the *Rhetoric* (A 1, 1354^a1-4) by stating that there are principles for the particular sciences and principles that are common (κοινά) to all. Just as *dialectic* deals with the latter (here the reference seems to be the discussion of common axioms in the *Metaphysics*), rhetoric itself, conceived of as the “countermelody” (ἀντίστροφος) of dialectic, is the art of expressing any science and any argument, irrespective of their particular genus. Aristotle does not mention explicitly axioms in this passage, and the κοινά may refer to either notions or principles.

²³ *Metaph.* Γ 2, 1004^a18.

(δεικνύναι τὰ κοινά). Whatever we may think of these dialectical arguments on principles, the very fact that they seem to concern both logical and mathematical axioms shows that Aristotle envisaged a uniform treatment of ἀξιιώματα in general.

Based on these considerations, it seems reasonable to claim that the *Posterior Analytics* and the *Metaphysics* expound the same theory of axioms, just as it seems safe to conclude that Aristotle regarded logical axioms, such as the Excluded Middle, and mathematical axioms, such as CN₃, as instances of the same kind of principles.²⁴

At first glance, this claim might come as a surprise, since the two kinds of axioms seem to possess very different epistemic features. The Principles of Contradiction and the Excluded Middle seem to be *much more general* than an axiom on the subtraction of equals, that only applies to mathematical objects. More than this: in the passage in *Metaph. Γ*, Aristotle can be seen to be arguing that a principle needed to know anything whatever (i.e. an axiom according to *An. post. A 2*) must be a principle common to all sciences (i.e. an axiom according to *An. post. A 10*). The inverse implication however is far more problematic to justify, and it is hard to see how CN₃ could possibly be a principle, without which one could not learn anything whatever. As a matter of fact, in the same *Metaph. Γ* Aristotle states that the Principle of Contradiction is *the principle of all other axioms* (ἀρχὴ τῶν ἄλλων ἀξιωματῶν πάντων), therefore explicitly stressing its preeminence over all common principles such as CN₃.²⁵

Given the apparent heterogeneity of the logical and mathematical principles, and their double origins in dialectical discussions and Eudoxian theories, both ancient and modern interpreters have detected a tension in Aristotle's treatment of axioms and attempted to separate once more the pieces of a theory that Aristotle had attempted to join together. Since the generation after Aristotle, a difference has been established between the two kinds of common axioms, and we have a reference by Themistius on the fact that Aristotle's student Theophrastus had himself already distinguished between common axioms that are truly universal, such as the logical-ontological principles, and common axioms that apply to mathematics only.²⁶ Theophrastus' reading was generally accepted by ancient, medieval and early modern commentators, and it is still endorsed by many modern interpreters.²⁷

Modern scholars have reinforced Theophrastus' interpretation by noting that in various contexts Aristotle employed the word "common" (κοινόν), in the double sense of "common to

²⁴ I may add that in a (probably spurious) passage in *Metaph. K 4, 1061^b20*, the theory of logical axioms expounded in *Metaph. Γ* is explicitly connected with CN₃, this being the only reference to this axiom in the whole *Metaphysics*.

²⁵ *Metaph. Γ*, 1005^b34-35.

²⁶ The passage is in THEM. *In an. post. 10*: "Theophrastus defines an axiom as follows. An axiom is a statement either concerning things of the same genus, such as 'equals subtracted from equals', or concerning absolutely everything, such as 'either the affirmation or the negation'".

²⁷ The only Aristotelian passage that may help such an interpretation, as far as I know, is in *An. post. A 32, 88^a36-88^b3*, where Aristotle distinguishes common principles such as the Principle of Contradiction, and principles pertaining to a genus, such as "quantity". The principles pertaining to a genus are, I would say, hypotheses and definitions (theses), and Aristotle is using "quantity" in an equivocal sense to mean "either numbers or magnitudes". But the text is open to other readings, such as understanding that both kinds of principles are common axioms, the first being logical common axioms, and the second being mathematical common axioms (for further interpretations see MIGNUCCI 1975, pp. 627-29, and BARNES 1993, p. 196).

all” and “common to some”, without bothering too much to distinguish between the two meanings. This happens, for instance, in Aristotle’s famous distinction between proper sensibles and common sensibles. The latter are sometimes regarded as common to *all* senses, and sometimes simply as sensibles that can be perceived by *more than one sense*.²⁸ It has been surmised, therefore, that in the case of common axioms mentioned in *Analytics* a similar meaning of commonality is at play, and that Aristotle did not distinguish between principles that are common to all sciences (the logical principles) and principles common to more than one science (the mathematical principles). He may have intended *κοινόν* in the weaker sense of “common to several items, and possibly to all of them”.

It must be stressed, in any case, that the (alleged) difference between the two kinds of common axioms rests entirely on conceptual interpretation, and it is not grounded on any explicit Aristotelian claim. On the contrary, it seems to go directly against Aristotle’s express intention to equate the epistemic role of logical and mathematical axioms. Moreover, we have seen that in *An. post.* A 11 Aristotle also explicitly refers to CN3 as a mathematical example of a common axiom applying to *all* sciences (*πᾶσαι αἱ ἐπιστημῆμαι*). In the final sections of this paper, therefore, I propose a unitary interpretation of the role of *κοινὰ ἀξιώματα* in Aristotle’s epistemology, that takes “commonality” in the stronger meaning of (formal) generality.

Before offering such a reading, however, we must look at Aristotle’s arguments for accepting common axioms among principles in the first place.

§3. *Generic Unity and Unity by Analogy*

Aristotle’s theory of common axioms was devised in order to account for universal logical and mathematical principles in the context of a theory of science with the aim of grounding individual (i.e. genus-specific) disciplines in proper principles. To this effect, Aristotle developed some important epistemological strategies, which were especially deployed in the case of the *mathematical* common axioms.

Mathematics, for Aristotle, deals with quantity (*ποσόν*), and a principle such as CN3 does indeed express a general claim about equality, sums and differences, which are the main notions that Aristotle connects with quantity as such.²⁹ In several passages, Aristotle explains that everything *having parts* may be considered to be a quantity, and further distinguishes between *continuous* and *discrete* parts of a whole. A quantity, the parts of which are continuous, is called a *magnitude* (*μέγεθος*); a quantity with discrete parts is called a *number* (*ἀριθμός*).³⁰ This important distinction between quantities demarcates two different mathematical sciences: geometry, dealing with magnitudes, and arithmetic, dealing with

²⁸ Aristotle’s examples of common sensibles are often sensibles perceived by some senses (e.g. *De sensu* 4, 442^b7, where he says that common sensibles are common “at least to touch and sight”). He explicitly characterizes common sensibles as sensibles perceived by all senses in *De an.* B 6, 418^a19 (cf. also *De an.* Γ 1, 425^a15). The neat comparison between this use of *κοινόν* in psychology and the one concerning axioms is in BARNES 1993, p. 100.

²⁹ See for instance *Cat.* 6, 6^a26-27: “Most distinctive of a quantity is its being called both equal and unequal” (transl. Ackrill).

³⁰ The main passages on the classification of quantities are *Cat.* 6 and *Metaph.* Δ 13.

numbers.³¹ A common axiom such as CN₃ should apply to the whole mathematical domain of geometry and arithmetic, and to the mixed, subordinate sciences such as astronomy or music.

In spelling out this classification of the mathematical disciplines, Aristotle deals with quantity *as if* it were a genus, with specific differences (continuity and discreteness) and subordinate species (magnitudes and numbers). This would open the perspective that a single science (universal mathematics) could investigate the single genus of quantity, and principles such as CN₃ would be *theses* (i.e. *proper* principles) of a *mathesis universalis* rather than common axioms. Geometry and arithmetic would be downplayed to the role of subordinate sciences of the καθόλου μαθηματική.

Aristotle, however, resisted this idea. He claimed that quantity is a *category* rather than a genus, and therefore it does not produce the kind of unity that a genus imposes on its subordinate species. According to Aristotle, since there is no *genus* of quantity, there is not, and cannot be, a science of mathematics in general. Instead, magnitudes and numbers are themselves *summa genera*, that do *not* communicate with one another thanks to superordinate kinds of objects (alleged universal quantities, which would be “no number, no line, no surface, no solid, but something else”). It can therefore be concluded that, for Aristotle, mathematical sciences (in the plural) deal with numbers and magnitudes (and other items); but there is no *one* science (mathematics) that deals with a single genus of objects (quantities).

An insightful connection may be made with Aristotle’s extended discussion, in the *Posterior Analytics*, on the epistemic need to find the *primary subject* of a property. The property of having an interior angle sum of two right angles, for instance, belongs to triangles as its primary subjects. Aristotle insists that *isosceles* triangles cannot be said to have this property *per se*, insofar as they have this certain angle sum *qua* triangles and not *qua* isosceles. In this case, the genus (triangle) is the primary subject of the property, even though, of course, its species (such as the isosceles triangle) also share this property.³² Should one ask, now, which are the primary subjects of CN₃, it would be clear that they cannot be numbers, nor lines, nor surfaces, nor solids, since none of them may encompass the others. The common axiom applies indeed to all these objects, which are not, however, species of any common genus. But since the common genus is missing (the alleged “something else” mentioned by Aristotle) the κοινὰ ἀξιώματα (contrary to proper principles) *do not have any primary subject*.

Still, according to Aristotle, the generalization of magnitudes and numbers into “quantities” is not *entirely* nominal. Rather, magnitudes and numbers *do have* a sort of intrinsic commonality, which depends on the unity conferred to them by their belonging to the same category. The term “quantity”, in short, is not an *equivocal* name, arbitrarily (νόμῳ) encompassing two altogether different kinds of objects. It is not univocal either, however, as if it expressed a veritable genus. In Aristotle’s view, the category of quantity indeed offers a unity, but a *unity by analogy* (that is to say, *by proportion*) of numbers and magnitudes:

³¹ See for instance *An. Post.* A 7, 75^a38-9 on the separation between arithmetic and geometry, and the mistake of μετάβασις εἰς ἄλλο γένος which is made by proving a geometrical theorem by arithmetical means.

³² See for instance *An. Post.* A 4, 73^a34-73^b3, but the example of the isosceles triangle is recurrent in this and other works.

Of the (principles) used in the demonstrative sciences some are proper to each science and others common—but common by analogy [κατ'ἀναλογίαν], since something is only useful insofar as it is in the genus under the science.³³

This is Aristotle's breakthrough in the epistemology of common axioms. Geometry and arithmetic are different sciences, and they cannot be fused together into a universal mathematics. As such, they are actually grounded on *their own* specific principles (*theses*) that are *not* common to one another (e.g. the definition of a line, or the definition of a number). Nonetheless, there are relations among numbers and relations among magnitudes that are similar to one another. These relations can be captured by some principles expressing their common behavior. Such principles are common axioms, showing the structural similarities between different domains of objects, by analogy.³⁴

Thus, for instance, “equal” means different things when referred to numbers or geometrical magnitudes. Numbers are generally conceived of, in Greek mathematics, as collections of units: and to be equal, in numbers, may therefore mean that two collections of units may be put in biunivocal correspondence. Geometrical magnitudes, on the other hand, have quite different conditions of equality: for instance, they may be dissected into equal collections of superimposable parts.³⁵ Yet, no matter how different the notions of equality among numbers and magnitudes may be, it is still true that equal parts, subtracted from equal wholes, make the remainders equal. This is a structural similarity between numbers and magnitudes concerning mereology and equality. As such, it may be expressed by a “common axiom” capturing the “analogy” among these two different domains. Other principles may be stated as well, and the collection of these principles suggests a kind of “unity” of the domain of numbers and magnitudes, that may be formulated as the unity κατ'ἀναλογίαν of a category—the category of quantity. Aristotle himself makes the further example of the rule of alternation in proportions (his basic example of a theorem of universal mathematics), which is, and yet is not, the same for the two different genera of numbers and geometrical lines:

E.g. why do proportionals alternate? The explanation in the cases of lines and of numbers is different—and also the same: as lines it is different, as having such-and-such a ratio it is the same. And so in all cases.³⁶

³³ *An. post.* A 10, 76^a37-40 (transl. Barnes modified). The Greek word ἀναλογία is generally translated as “analogy” in philosophical contexts and as “proportion” in mathematical ones. It is the same notion, however, employed by Eudoxus and Aristotle for similar aims. Aristotle explicitly says that an analogical unity does not require a genus-specific unity in *Metaph.* Δ 6, 1017^a1-3.

³⁴ See for instance *Metaph.* N 6, 1093^b16-21: “For all things have connections and have unity by analogy. There is something analogous, indeed, in all categories of being. As the straight is to the line, so the plane to the surface, and, possibly, the odd to the number and the white to the color”.

³⁵ *Cat.* 6, 6^a27-30. On numbers, see for instance the passage on the equal number of sheep and dogs in *Phys.* Δ 14, 224^a2-3. See again my DE RISI 2021B for a fuller treatment of the matter in Euclid's geometry.

³⁶ *An. post.* B 17, 99^a8-11 (transl. Barnes). The example on alternation (ἐναλλάξ) is the same as Aristotle had mentioned in his description of the mathematical breakthrough of the invention of universal theorems in *An. post.* A 5, 74^a17-25 (see above, note 6). The passage in *An. post.* B 17 continues by remarking that not every case in which we use an identical term to express a relation (such as “proportional”) is *analogous* to any other. There are more

This being and not-being the same is expressed in the formula $\kappa\alpha\tau' \acute{\alpha}\nu\alpha\lambda\omicron\gamma\acute{\iota}\alpha\nu$.

The treatment of *logical* axioms, such as the Principle of Contradiction and the Excluded Middle, is also based on the same idea. While, according to Aristotle's *Metaphysics*, there is no science common to being *qua* being, there are universal relations among beings that are the same in all genera. The fact that it is impossible for the same thing to hold good and not to hold good simultaneously for the same thing and in the same respect (i.e. the Principle of Contradiction in Aristotle's famous formulation)³⁷ expresses a relation among things that is structurally similar in all genera of objects.

We should be careful, of course, not to over-modernize Aristotle: the notions of structural identity and isomorphism (or the notion of a structure *tout court*) do not belong as such to Aristotle's epistemology. Nonetheless, the idea of a similarity of relations, which is the starting point of the modern notion of isomorphism and structure, is certainly to be found in Eudoxus' mathematical theory of proportions ($\acute{\alpha}\nu\alpha\lambda\omicron\gamma\acute{\iota}\alpha\iota$) as it is expounded in Euclid's *Elements*.³⁸ In such a theory, indeed, ratios are conceived of as very abstract relations between magnitudes, and "analogies" as identities among these relations. Aristotle's philosophical theory of analogy is likely to depend on, and generalize over, the basic ideas of this new theory of proportions. It is remarkable that Eudoxus' very refined theory of mathematical analogies may have brought this mathematician to develop the idea of a universal mathematics as a general science of magnitudes. Aristotle's "meta-mathematical" generalization of the notion of analogy transferred the idea of a structural identity from mathematics to epistemology and applied it to a plurality of "analogous" scientific theories. In this way, Aristotle employed Eudoxus' conceptual breakthrough against the very idea of a universal mathematics as a unified superordinate science, and showed that a general theory of magnitudes and ratios still allows for a plurality of structurally similar, and yet different, mathematical sciences.

Thus, even if Aristotle cannot be credited for a fully-fledged theory of isomorphism between theories, he saw the possibility of spelling out *structurally identical principles* ruling different theories. In this sense, Eudoxus may have first envisaged the idea of "common

frequent cases of mere homonymy. Thus, when we say that two geometrical figures are "similar" and that two colors are "similar", here similarity is not predicated $\kappa\alpha\tau' \acute{\alpha}\nu\alpha\lambda\omicron\gamma\acute{\iota}\alpha\nu$ in the two cases, for there is not any identical structure shared in the two cases. Indeed, colors and figures do not belong to the same category. Here is the passage: "The explanation of a color's being similar to a color and a figure to a figure is different for the different cases. Here similarity is homonymous: in the latter case it is presumably having proportional sides and equal angles; in the case of colors it is the fact that perception of them is one and the same (or something else of this sort)". A more general discussion on the analogy of principles ($\acute{\alpha}\rho\chi\alpha\acute{\iota}$), this time conceived of as physical and metaphysical causes, is to be found in *Metaph.* Λ 4. On the notion of analogy in Aristotle, see at least COURTINE 2005 and AUBENQUE 2009.

³⁷ *Metaph.* Γ 3, 1005^b19-20.

³⁸ An important breakthrough towards modern structuralism took place in Leibniz' theory of relations and theory of "expression" (isomorphism), and it is likely that Dedekind's (and others') discussions on structuralism in the 19th century drew in fact on Leibniz' ideas. On the other hand, Leibniz' route towards his own proto-structuralism and theory of relations is likely to have drawn precisely on Eudoxus' theory of proportions (as hinted at, for instance, in §47 of Leibniz' *Fifth Paper to Clarke*, in GP VII, pp. 400-402). On Leibniz' theory of relations and expression, and its sources, see MUGNAI 1992 and MUGNAI 2012. On the birth of modern structuralism, see RECK, SCHIEMER 2020.

notions” of the kind we find in Euclid, and Aristotle developed this view in his epistemology by thematizing the need of κοινὰ ἀξιιώματα alongside the *proper* principles of each science. Thanks to this idea, Aristotle attempted to solve the mathematical and dialectical puzzles concerning the universality of science.

According to Aristotle, then, each science is based on some principles that are proper to it (definitions, hypotheses, and possibly other kinds of “theses”). Nonetheless, these principles are complemented by others that are not proper to the science in question but are in common with other *analogous* disciplines.

We must now look at how these common principles may be applied to the individual sciences.

§4. *Specialized Common Axioms: The Schematic Interpretation*

A κοινὸν ἀξιῶμα states some very general relations between all *beings* whatsoever. The Principle of Contradiction or CN₃ are indeed formulated so as to apply to things or beings in general (the Greek makes use of the neuter pronouns expressing the maximum possible generality). Aristotle, however, still wants every science to be *only* grounded on principles referring to the genus that is investigated by it. *All* principles of arithmetic, say, must be about *numbers*.

Aristotle is explicit in stating that axioms, which are hyper-generic principles, are concretely *applied* in science in some specialized version. They are only employed in a demonstration to the extent in which they deal with the genus under consideration. We have already mentioned the most important passages to this effect:

Of the principles used in the demonstrative sciences some are proper to each science and others common—but common by analogy, since they are only useful in so far as they bear on the genus under the science ... It is sufficient to assume each of these in so far as it bears on the genus; for it will produce the same results even if it is assumed as holding not of everything but only for magnitudes (or, for arithmeticians, for numbers).

...

It is obvious that the investigation of these axioms pertains to one science, namely the science of the philosopher, for they apply to all existing things, and not to a particular class separate and distinct from the rest. Moreover all thinkers employ them – because they are axioms of being *qua* being, and every genus possesses being – but employ them only in so far as their purpose require; i.e., so far as the genus extends about which they are carrying their proofs.³⁹

Ancient commentators offered the first and most influential interpretation of these passages. It was first expounded (in the texts we have) by Alexander of Aphrodisias, and later almost unfailingly repeated by other Greek, Latin and modern interpreters. I will call it the *schematic interpretation* of Aristotle’s common axioms.

³⁹ *An. post.* A 10, 76^a37-76^b2; *Metaph.* Γ 3, 1005^a21-29.

According to it, common axioms may be conceived of, in modern parlance, as “axiom schemata” of sorts. Rather than encompassing a number of related principles of one theory, though, these schemata would be inter-theoretical: each “analogous” theory would be grounded on one instance of the axiom schema. A statement such as CN₃ (“*If equal things be subtracted from equal things, the remainders are equal things*”), thus, would be a general principle about quantity that should be specialized as a proper principle of arithmetic or geometry in order to be actually employed: “*If equal numbers be subtracted from equal numbers, the remainders are equal numbers*”, “*If equal magnitudes be subtracted from equal magnitudes, the remainders are equal magnitudes*”. The same happens with equal lines, equal surfaces, equal times or motions, and so forth. The structural similarity between numbers and other magnitudes (i.e. their categorial unity $\kappa\alpha\tau'\acute{\alpha}\nu\alpha\lambda\omicron\gamma\acute{\iota}\alpha\nu$) would allow for stating the above principles as the common schema CN₃, dealing with all “things” in general.⁴⁰

According to the schematic interpretation, the specialized principles on numbers, lines, etc., are *assertions* that may be assumed as premises of syllogistic reasoning, and therefore may play the role of principles and starting points in a demonstration. For instance, if we want to formalize in Aristotle’s logic a geometrical demonstration concerning angles, we should construct a “syllogism” having as a premise a specialized version of CN₃, such as “*If equal angles be subtracted from equal angles, the remainders are equal angles*”, and as another premise that “angle *a* is equal to angle *b*, and angle *c* is equal to angle *d*”, in order to conclude that “angle (*a* – *c*) is equal to angle (*b* – *d*)”. This is not a syllogism according to any standard definition, of course, and we see here at work the main problem of adapting Aristotle’s logic to mathematical reasoning. The general idea should however be clear: a common axiom is not itself a premise of a reasoning since it is not even an assertion, but its specialized instances may work as premises and therefore enter into the Aristotelian model of a deductive science.

It is important to stress that this interpretation is entirely conjectural. Nowhere in his writings does Aristotle offer an explicit example of a specialized principle, or the way in which an axiom is actually employed in a proof, and the above references to the genus-specific instances of CN₃ (on numbers and magnitudes, or on angles) were made up and suggested by Alexander and the other commentators.⁴¹

⁴⁰ See ALEX. *In metaph.* 265: “For each of the genera with which the sciences are concerned is a kind of being; this is what these people cut off from what is common and on it they use the axioms, so far as it is also useful for them to do so: the geometer applies the axioms to magnitudes, and uses them to that extent; the arithmetician applies them to numbers. They do so because the axioms belong to all things and are not proper to any one kind of thing. For example, the geometer assumes as an axiom that ‘things equal to the same thing are equal to one another’ and substitutes ‘magnitudes’, for his treatise is about magnitudes. The arithmetician, in his turn, transfers the axioms to numbers” (transl. Madigan). Among ancient commentators, this view is restated in THEM. *In an. post.* 29-30, and PHILOP. *In an. post.* 123 and 141-42.

⁴¹ The passage mentioning CN₃ in *Metaph.* K 4, 1061^b17-27 (see above, note 19) is the most liable to be interpreted along the lines of Alexander’s reading. The Eleventh Book of the *Metaphysics*, however, is most certainly spurious, and probably written in the early Peripatetic school. Here is the passage: “Since even the mathematician uses the common axioms only in a particular application, it will be the province of first philosophy to study the principles of these as well. That when equals are taken from equals the remainders are equal is an axiom common to all quantities; but mathematics isolates a particular part of its proper subject matter and studies it separately; e.g. lines or angles or numbers or some other kind of quantity, but not *qua* being, but only in so far as each of them is continuous in one, two or three dimensions” (transl. Tredennick, modified). It may be noted that this passage

This schematic interpretation agrees with Theophrastus' general idea that the axioms of quantity and the axioms of logic are two different kinds of principles. The schematic interpretation, indeed, takes very seriously the idea that categories are *genera of sorts*, which are further divided into *species of sorts*. Thus, since quantity would be a kind of “quasi-genus” divided into the “quasi-species” of numbers and magnitudes, a general statement about quantity (such as CN3) may be specialized into particular statements about numbers or about magnitudes.

This seems to be the most important but also the most problematic feature of the schematic interpretation. A statement such as “*If equal angles be subtracted from equal angles, the remainders are equal angles*” would be, according to Aristotle's classification of principles, a *thesis* (being a claim about one particular genus). In the schematic interpretation, therefore, a common axiom would be nothing but a general expression for a collection of theses sharing a similar structure; it would be a *schema of theses*. As a matter of fact, this is explicitly stated by ancient commentators, who regarded common axioms as collections of proper principles.⁴² But nothing similar is *ever* suggested by Aristotle, who always stresses the *irreducibility* of these different kinds of principles: common axioms are considered by him to be totally different from theses, to the point that their difference demarcates the first division among ἀρχαί.⁴³

One may attempt to defend the schematic interpretation and speculate that Aristotle would have not considered a specialized axiom (such as “*If equal angles be subtracted from equal angles, the remainders are equal angles*”) to be a thesis after all. Insofar as this assertion expresses a special case of a general principle, it would just be “a common axiom limited in scope” or “the common axiom itself, *qua* specialized” rather than a proper principle in the Aristotelian sense. This latter reading would at least account for Aristotle's distinction between axioms and theses. The problem remains that a principle such as the one on “equal angles” above would be labeled as a specialized version of the common axiom CN3 (rather than a thesis) *only insofar as it happens* that another genus of things (say, numbers) also obeys an analogous principle. The state of affairs that both angles and numbers fall under the scope of a common principle is to be taken at face-value as a brute fact, since no common genus of them may be found (as an alleged *primary subject* of predication) that might explain their

does not mention *magnitudes* (μεγέθη) but only specific instances of them (such as lines). This is connected with the problem of understanding whether magnitudes are for Aristotle a veritable genus or not (see above, *note* 13), and therefore whether, according to the schematic interpretation, CN3 should be specialized into a statement on magnitudes or (e.g.) on lines, in order to be employed in a demonstration. Ancient commentators, in any case, showed no doubts that μέγεθη constitute a proper genus.

⁴² PHILOP. *In an. post.* 141-42, even stated this view about the Principle of Contradiction: “So the principle of contradiction without qualification is *common* to every science, but it becomes *proper* insofar as it concerns magnitudes, numbers, or the attributes of these things” (transl. McKirahan).

⁴³ I may add that in *Metaph.* Γ 3, 1005^b14, Aristotle says that the Principle of Contradiction (a common axiom) is *not hypothetical* (ἀνυπόθετον). While the notion of hypothesis hinted at in this passage may be that of an assumption that is provable (i.e. following one of the looser definitions of ὑπόθεσις given in *An. post.* A 10, rather than the metaphysical one offered in *An. post.* A 2; such seems to be Ross' suggestion), I would still find it odd if Aristotle could call ἀνυπόθετον a principle that he had conceived of as a collection of ὑποθέσεις. On the relation between Aristotelian hypotheses and the unhypothetical first principle, see UPTON 1985.

similar structure. Being common, according to this reading, is not an intrinsic feature of axioms as such – which distinguishes them from theses – but is derived from their scope of application. In a way, according to the schematic reading, mathematical axioms are theses (accidentally) gone wider.

In conclusion, the schematic interpretation succeeds in accounting for Aristotle's claim that each science only proceeds by and through principles that are referring to its subject-matter, accepting at the same time the existence of some common axioms. I still see three main, related problems stemming from such an interpretation. (#1): The idea of separating mathematical and logical axioms introduces a division among axioms where Aristotle appears to have seen none. The schematic interpretation fails to provide a unitary account of the role of axioms. It rather assumes that Aristotle was not able to reconcile the different sources from which his theory originated, namely, the mathematical and dialectical claims on the univocation of being. (#2): The schematic interpretation reduces axioms to collections of proper principles, and is incapable of tracing a real divide between axioms and theses. In this sense, it offers a weak explanation of Aristotle's main epistemological claim that principles are first and foremost distinguished into proper and common ἀρχαί. (#3): The schematic interpretation also has the additional issue (which is a standard problem of many readings of Aristotle's theory of demonstration) that common axioms are not expressed in a subject-predicate form that would make them eligible to be assumed as syllogistic premises in a demonstration. Nonetheless, according to the schematic interpretation, specialized common axioms (on angles, on numbers, etc.) should be assumed as premises of syllogistic reasoning. The three claims above are to some extent independent from one another (claim #2 being the core of the schematic interpretation), but they are usually found together in ancient and modern interpreters, and indeed all together they make a very consistent interpretation of Aristotle's theory of axioms.

In the next sections of this essay, I advance a different reading of axioms, with the aim of solving these exegetical problems.

§5. *The Inferential Interpretation of Common Axioms*

In his influential commentary on the *Analytics*, David Ross briefly suggested an interpretation of axioms which is completely at odds with the standard reading. Logical principles such as the Principle of Contradiction or the Excluded Middle, Ross claimed, cannot be thought of as syllogistic premises of a demonstration. Rather, they shape and constrain the logical form of the demonstration itself. Axioms, therefore, are not to be interpreted as schemata of assertions, but rather as *formal* principles of sorts.⁴⁴

⁴⁴ ROSS 1949, pp. 531-32: “even if we insert the law of contradiction as a premiss, we shall still have to use it as a principle in order to justify our advance from that and any other premiss to a conclusion”. In the course of the years, several other interpreters have endorsed weaker forms of this “inferential” interpretations. Something to this effect, for instance, may be found in GRANGER 1976, pp. 81-82, where axioms are called “outils de raisonnement” (but later on also “schémas de règles”, p. 93).

Ross also pointed to a couple of passages in the *Posterior Analytics* that should substantiate this reading. In them, Aristotle states, indeed, that a demonstration proves its conclusion *through* ($\delta\iota\acute{\alpha}$) the axioms, rather than *from* ($\epsilon\acute{\chi}$) them as premises.⁴⁵ In yet another passage, Aristotle explicitly claims that, in general, demonstrations do not assume the Principle of Contradiction as a premise, even though he seems to leave open the possibility that particular demonstrations may do so.⁴⁶

Recent interpreters have generally disregarded the textual evidence presented by Ross as inconclusive, and rejected his general suggestion that axioms may be formal principles of sorts. I think we have to agree with them on the poverty of the textual evidence substantiating Ross' claim. I have already remarked, however, that Theophrastus' and Alexander's readings, as well as the schematic interpretation as a whole, do not score any better. The schematic interpretation is generally accepted thanks to a certain intrinsic conceptual plausibility, joined with an ancient lineage, but it is scarcely based on Aristotle's actual statements. In sum, any interpretation of Aristotle's conception of axioms must complement textual evidence with some conceptual guidance, and in this respect Ross' general remarks on the role of the Principle of Contradiction in logical deduction seem to advance a sound point about the role of logical principles. I may add that Ross' interpretation is not unprecedented. In the Middle Ages, an inferential interpretation of common axioms (or some of them) was held by a minority of authors, such as Albert the Great and Giles of Rome. They claimed that axioms are principles *secundum virtutem* (i.e. formally) rather than *secundum substantiam* (i.e. semantically), and that they are only causes of the conclusion through the mediation of the proper principles assumed as premises.⁴⁷

I do believe, in fact, that there are good reasons to follow Ross' suggestion and develop a fully-fledged *inferential interpretation* of Aristotle's axioms. This should be extended well beyond Ross' initial idea. Ross, indeed, confined his interpretative suggestion to the *logical* axioms themselves, and claimed that *mathematical* common axioms were regarded by Aristotle as assertions and syllogistic premises.⁴⁸ Ross was thus preserving the most important (and, I think, critical) aspect of Theophrastus' reading, namely, the distinction between two

⁴⁵ The passages mentioned by Ross, in which Aristotle employs the term $\delta\iota\acute{\alpha}$, are *An. post.* A 10, 76^b10 and *An. post.* A 32, 88^a36-88^b3. Ross' argument has been rebutted by MIGNUCCI 1975, pp. 141-43, who showed that Aristotle uses of $\delta\iota\acute{\alpha}$ and $\epsilon\acute{\chi}$ to mean the starting points from which, and the means through which, a proof is performed, are quite inconsistent. A few lines below one of the passages just quoted, in *An. post.* A 10, 76^b15 and again 22, for instance, Aristotle mentions common axioms as principles $\epsilon\acute{\chi}$ $\tilde{\omega}\nu$ a demonstration is carried out. Also in the other passage quoted by Ross, Aristotle employs the expression $\epsilon\acute{\chi}$ $\tilde{\omega}\nu$ (*An. post.* A 32, 88^a37). The same is stated in *An. post.* A 7, 75^a42 and in *Metaph.* B 2, 997^a8-9.

⁴⁶ *An. post.* A 11, 77^a5-25. The passage is, however, quite obscure, and has been the subject of divergent readings. For a recent interpretation, see HARARI 2004, pp. 51-55.

⁴⁷ For a few passages in Albert, see his commentary on the *Posterior Analytics* I, 5, 4 and 8 (BORNET 1890, pp. 138 and 148-49) and the *Ethics*, II, VI, 5, q. 5 (KÜBEL 1987, p. 426). I take the references from the discussion in CORBINI 2006, pp. 72-74, who also mentions Giles (offering further references) in pp. 83-85.

⁴⁸ The same was the case for the above-mentioned medieval authors. It is remarkable that the main passage used by Ross to foster his own inferential interpretation of logical axioms is *An. post.* A 10, 76^b2-12, which gives the example of a mathematical demonstration and is probably referring to *mathematical* common axioms.

sorts of axioms. By contrast, I advance the hypothesis that *all* axioms (logical and mathematical alike) were primarily conceived of by Aristotle as formal principles.

According to this interpretation, a mathematical inference on angles employing CN₃ would be construed as: “angle *a* is equal to angle *b*”, and “angle *c* is equal to angle *d*”; therefore: “angle (*a* – *c*) is equal to angle (*b* – *d*)”. In this inference CN₃ is employed as a kind of *rule* allowing the conclusion to be drawn from the two assertions used as premises.

In this interpretation, no application is made of the specialized principle: “*If equal angles be subtracted from equal angles, the remainders are equal angles*”. The latter does not appear in the proof either as a premise or a rule. It is true that CN₃, according to this reading, only applies to angles in the present context; and its scope when it applies to other objects (such as numbers) is strictly speaking different, and only *analogous* to the scope that we use in the deduction of angles. Nonetheless, CN₃ is applied to angles (or numbers, or magnitudes, etc.) as a formal principle; and its use is restricted not by itself (i.e. being specialized), but by the subject matter to which it applies. Common axiom CN₃ *per se*, in fact, applies to the *category* of quantity as such, and not to a plurality of genera. In this sense, it would be misleading to lay down a specialized form of CN₃ such as the one on angles that we have mentioned above (which is a thesis, in Aristotle’s sense). In fact, we have seen that Aristotle did *not* make any reference to specialized forms of CN₃. I would surmise that this missing reference is not to be accounted for as an accident of the text of the *Posterior Analytics*, that was later corrected and integrated in the ancient commentaries mentioning such specialized principles. Specialized principles simply *do not exist* in Aristotle’s epistemology.⁴⁹

I should stress that in order to endorse the inferential interpretation, we do not need to claim that axioms *cannot be* syllogistic premises and assertions, or that they are indeed devoid of any semantic content. We may indeed modify Ross’ view that *some* common axioms are used inferentially (the logical ones), whereas *some other* common axioms are used as premises (the mathematical ones). Aristotle’s texts may support the more refined claim that *all* axioms are employed as formal principles, and *all* of them may *also* be formulated in assertoric form and be assumed to be syllogistic premises. The inferential interpretation only maintains that axioms are employed in demonstrations as formal principles, since no deduction can happen without them; but it allows axioms to be also assertions, and to be employed as premises in special cases.⁵⁰ This seems to be important for a historically sound attempt at reconstructing Aristotle’s theory of principles. We cannot credit Aristotle with a neat distinction between our notions of a semantic axiom vs. a rule of deduction. According to him, axioms (like any other principles) are *about something* insofar as they are principles, and indeed are *true* statements about their subject matter. They *are*, therefore, assertions; and Aristotle often presents them

⁴⁹ Aristotle applies CN₃ to angles in his example of *An. pr.* A 24. The issue is further complicated by the fact that Aristotle seems to deny that angles are *quantities* at all, and instead categorizes them among *qualities* (see *De caelo* B 14, 296^b20; B 14, 297^b19; Δ 4, 311^b34, where *similar* (ὅμοιοι) angles are mentioned; cf. SIMPLICIUS, *In de caelo* 538, for a similar observation). I think that the schematic interpretation cannot really account for the fact that CN₃, dealing with quantities in general, may be specialized into a statement on angles. On the contrary, the inferential interpretation, by applying CN₃ in its full generality to angles, seems to fit with Aristotle’s usage. According to such an interpretation, angles are simply *employed as quantities* in the demonstration. On some notions of angles in Greek geometry, see my DE RISI, forthcoming.

⁵⁰ This seems to be a viable interpretation of the above-mentioned passage in *An. post.* A 11, 77^a5-25.

in this form. An axiom such as CN₃ is a true statement about quantity, and the Principle of Contradiction expresses a fundamental truth about being *qua* being. Nonetheless, quantity and being are not genera, and much less the *primary subjects* of these statements. Therefore, I would claim, these axioms *cannot be employed as assertions* and syllogistic premises in any standard scientific demonstration—since scientific demonstrations only make use of genus-specific premises. The only way in which axioms may enter a scientific demonstration is by shaping the deduction itself, and the inferential interpretation claims that, according to Aristotle, they may indeed play this role.⁵¹

I would like to venture here a further conjecture. The dual character of axioms, counting both as assertions and formal principles, may perhaps be better explained if we conceive of axioms as assertions *grounding* and validating inferential rules. We know, after all, that axioms such as the Principle of Contradiction and the Excluded Middle (or the *dictum de omni*) are used by Aristotle in order to ground the validity of the moods of syllogistic figures.⁵² We can surmise, therefore, that a principle such as CN₃ should not be employed *itself* as a rule of inference (as I did in the above-mentioned example on angle equality), but should rather be regarded as a statement grounding (in some way to be better explained) a valid deductive rule. According to this latter interpretation, we would have a uniform reading of common axioms as assertions, while at the same time their role in demonstration would be totally different from that of proper principles insofar as axioms would still only shape the *form* of valid reasoning. In the absence of any textual evidence to this effect, I would not belabor this idea any further and in the rest of the paper I will not distinguish between formal rules and assertions grounding such rules. I can add, however, that Galen may have suggested something along these lines when advancing his theory of “relational syllogisms”.⁵³

⁵¹ Indeed, in *An. post.* A 2, 72^a7-8 Aristotle introduces for the first time the notion of a principle in science, by stating that *any* principle whatsoever (therefore also an axiom) is an assertion and a premise. Some interpreters have employed this passage to discredit Ross’ idea that an axiom may be a formal principle (a rule of sorts); see an argument to this effect in MUELLER 1991, p. 68, fn. 11. I think, on the contrary, that the passage is consistent with the fact that for Aristotle the Principle of Contradiction is an assertion, but it is nonetheless employed in demonstration as a rule of sorts.

⁵² The debate on the principles grounding Aristotle’s syllogistics is ample and difficult, and it is not entirely clear the role that the Principle of Contradiction and the Excluded Middle play in this connection. I will not enter this topic: for a recent assessment, see the important MALINK 2013.

⁵³ Galen’s treatment of relational syllogisms is in *Inst. log.* XVI-XVIII, where he also mentions several Euclidean common notions, and CN₃ in particular (in XVI, 8), but also non-mathematical axioms (such as “Every man is the son of his father”, XVI, 11) that validate other relational inferences. The details of Galen’s theory are notoriously opaque. At times, he clearly took axioms to be premises of inferences; at other times, he seems to have regarded them as formal rules of sorts that do not enter among the premises of an inference. MORISON 2008 aptly signals all passages in which Galen takes axioms to be syllogistic premises (*Inst. log.* I, 3; XVI, 12; XVI, 13) and in which he denies that they are premises (XVI, 1; XVI, 2; XVI, 3; XVI, 4; XVIII, 4; XVIII, 5). Among the various explanations of this confusing behavior, it has been suggested that Galen may have considered mathematical axioms, such as CN₃, precisely as assertions validating rules, and therefore as *sui generis* premises (say, “formal premises”) of syllogistic inferences. For a solution along these lines, see BARNES 2007, pp. 437-38, where he calls “Galen’s metatheorem” the idea that syllogisms and relational syllogisms are validated by common axioms. On the legacy of Galen’s theory of demonstration in the works of the Aristotelian commentators, see CHIARADONNA 2009.

The extensive discussion on first philosophy carried out in *Metaph. Γ* seems to take axioms as assertions and true statements on a wide domain of objects (being *qua* being in the case of the Principle of Contradiction, for instance). Nonetheless, I would say that the very significance of the whole discussion in *Metaph. Γ* shows that axioms cannot possibly be considered to be on a par with proper principles, nor collections of theses, and that their epistemic role was conceived of by Aristotle to be completely different from those. Aristotle was here launching his strongest attack against Plato's conception of dialectics as a hyper-science of sorts, dealing with all genera of things. Aristotle's conception of first philosophy is modeled in opposition to this idea, and he did not see philosophy as a discipline dealing with a hyper-genus or a quasi-genus (being *qua* being), but rather as a *formal* science with different aims and goals with respect to the genus-specific sciences. First philosophy is, so to speak, orthogonal to all other sciences, rather than above them. Consequently, it cannot prove (contrary to Plato's opinion) the proper principles of each particular science (the theses), and cannot be regarded as their foundation. First philosophy, nonetheless, still deals with the common principles, the ἀξιώματα. These axioms, like the science to which they belong, are not hyper-generic or quasi-generic principles (i.e. principles modeled after the theses, and sharing a similar, but wider, demonstrative role), but *formal principles* shaping the structure of being *qua* being. Thus, notwithstanding the fact that the Principle of Contradiction and the Excluded Middle are generally presented in *Metaph. Γ* as true assertions (for they indeed *are* true assertions), I would say that the theory expounded throughout the Book clearly shows that their primary function and meaning must be itself orthogonal to that of the proper principles: formal, rather than semantic.

In the *Posterior Analytics*, the formal role of common axioms is more explicitly stated (and Ross, indeed, took his lead from this work). In particular, the inferential interpretation of axioms may explain some debated passages of the *Posterior Analytics* in which Aristotle mentions axioms as the *only* principles employed in a demonstration.⁵⁴ Hypotheses and definitions (i.e. proper principles) do not appear at all in these passages, and they are only indirectly hinted at through the remark that the demonstration is *about* something (the γένος ὑποκείμενον, which is probably represented in the proof by genus-specific definitions and hypotheses). We have already seen a passage from *An. post. A 11* to this effect, to which another from *An. post. A 7* may be added:

There are three things involved in demonstrations: first, what is being demonstrated, or the conclusion (this is what holds of some genus in itself); second, the axioms (*axioms are the principles from which the demonstrations proceed*); third, the underlying genus whose attributes—i.e. the accidents *per se* that the demonstrations make plain.

...

⁵⁴ This is especially important since in the passages quoted in the *note 45* above Aristotle states that common axioms are the principles ἐξ ὧν the proof is carried out, and in *An. post. A 32*, 88^b28-29, he says, conversely, that the principles ἐξ ὧν are common axioms. There is thus an identity between principles ἐξ ὧν and axioms. It is ironic that Ross considered these passages as hindrances for his own interpretation, insofar as they do not mention axioms as principles *through* (διὰ) which something is proven.

I call common the principles which they use as the basis from which to demonstrate—not those about which they prove nor what they prove.⁵⁵

If we take common axioms to be *premises* of syllogisms, this seems to imply that definitions and hypotheses do not appear as premises at all, since the whole demonstration is carried out from (ἐκ) and through (διὰ) common axioms. This is a quite implausible claim to ascribe to Aristotle, who elsewhere stresses the importance of *theses* in demonstrations and indeed has the aim of grounding science on proper principles. On the contrary, if we accept the idea that common axioms make the formal part of a demonstration, Aristotle's statements become more palatable: definitions and hypotheses, in this reading, may well be premises of syllogistic chains, but the proof is still said to be carried out through and from common axioms in the sense that these are the actual *inferential means* that allow a conclusion to be drawn from some premises. More generally, in the first book of the *Posterior Analytics*, Aristotle seems to attribute great weight to common axioms in the making of science, and his statements to this effect conflict with the many other passages (especially in other works) in which he stresses the role of definitions and hypotheses as principles of knowledge. This apparent oddity can be smoothed out if we think of axioms as formal principles, in which case in different passages of different works Aristotle would be highlighting the different, and yet necessary, roles of the formal and the semantic principles of demonstration.

Finally, it is important to acknowledge that there is a passage in Aristotle's works that appears to militate against an inferential interpretation of common axioms. In *An. pr.* A 24, Aristotle argues that at least one of the premises of a syllogism must be universal. He explains several issues that may arise in science from syllogisms with only particular premises and gives some non-mathematical examples. Then, he says that “this is even better seen in geometrical demonstrations” and offers a rather obscure example of a pre-Euclidean geometrical theorem that Aristotle seems to articulate in three “syllogisms” needing universal premises. The third of these inferences, Aristotle says, cannot employ a statement on the equality of differences between angles but instead needs CN₃ in its full strength in order to be universal and therefore properly deductive.⁵⁶ Ancient commentators such as Alexander and Philoponus read this passage as a confirmation of their schematic interpretation, according to which axioms are

⁵⁵ The first quotation is from *An. post.* A 7, 75^a39-75^b2 (cf. an analogous passage in A 10, 76^b11-16); the second is the already quoted *An. post.* A 11, 77^a27-28 (transl. Barnes modified). MIGNUCCI 1975, pp. 141-44, attempts to solve the problem raised by the first passage by claiming that there Aristotle was using a broader notion of “axiom” as to encompass proper principles as well. This strikes me as highly implausible. On the contrary, MUELLER 1991 straightforwardly accepts the conclusion that (1) common axioms are *premises* of syllogistic reasoning, and (2) Aristotle took them to be the only premises of a mathematical demonstration. Mueller attempts to explain this latter preposterous claim by pointing to the fact that Aristotle could mean by “demonstration” a specific part of the Euclidean proof, i.e. the part that Proclus later called precisely the ἀπόδειξις of the proof (PROCLUS, *In Euclidis* 203-207). Yet, there are no hints that Proclus' classification of the parts of a proof predated Proclus' late commentary on the *Elements* (see NETZ 1999), and in any case Aristotle's references to mathematical proofs often take into consideration other parts of the demonstration as well (see MENDELL 1998). Mueller seems to have been aware of some problems in his own interpretation, but still did not accept an inferential interpretation of axioms (see above, note 51).

⁵⁶ *An. pr.* A 24, 41^b13-28. The mathematical example has been the subject of much debate among interpreters, and we possess several reconstructions of it. See at least MCKIRAHAN 1992, MENDELL 1998, and MALINK 2015.

assumed to be universal premises of syllogisms (rather than rules of inference). These commentators even exploited this passage to claim that, for Aristotle, mathematics is entirely reducible to chains of syllogisms.⁵⁷ Modern interpreters generally also agree that in this example CN₃ is taken as an assertion and a syllogistic premise, thus blocking an inferential reading of axioms.⁵⁸ I note, however, that the schematic interpretation does not fare any better. On the one hand, interpreting that CN₃ is here used as a syllogistic premise would mean to assume a common axiom in its full generality rather than in a specialized form (this is Aristotle's main contention in this passage). But we have seen that Aristotle is very explicit elsewhere that this should not be done. The need of restricting the axioms to a genus, as soon as they are employed as syllogistic premises, is in fact the core of the schematic interpretation. Such an interpretation is therefore more damaged than supported by this passage. On the other hand, there is no reasonable way to frame CN₃ as a syllogistic premise, since it is not a judgment in subject-predicate form. It is highly unlikely, I think, that Aristotle would have considered this statement as a viable syllogistic premise for a mathematical theorem in *barbara*.⁵⁹ Finally, it should be noted that the passage in *An. pr.* A 24 seems to contradict Aristotle's explicit claim in the above-mentioned *An. post.* A 11, according to which a common axiom (the Principle of Contradiction) is not generally employed as a syllogistic premise. I can suggest two solutions in order to deal with the delicate passage in the *Prior Analytics*. It is simply possible that Aristotle had not yet developed a theory of common axioms at the time in which he wrote the *Prior Analytics*. In this work, after all, the word ἀξίωμα rarely appears, and it has a non-technical meaning. In this hypothesis, the passage would hinder neither the schematic nor the inferential interpretation of axioms, since the theory of axioms would only appear later in the *Posterior Analytics*.⁶⁰ Alternatively, it is also possible that Aristotle, contrary to the obvious interpretation of the passage ("this is even better seen in geometrical demonstrations"), was giving the mathematical example to make a general point: even geometrical demonstrations, that appear to deal with individual angles and lines (Aristotle

⁵⁷ ALEX. *In an. pr.* 268-69; PHILOP. *In an. pr.* 253-54. From these ancient commentaries, a long and important tradition of attempts at *barbarizing* mathematics (put it into syllogisms in *barbara*) took the lead, with significant developments in the middle ages and the early modern age.

⁵⁸ Note that this specific problem did not arise in Ross' interpretation, since for him *mathematical* axioms are assertions and not rules.

⁵⁹ A syllogistic reconstruction of this proof, along the lines hinted at by the ancient commentators, is given in the above-mentioned MCKIRAHAN 1992, MENDELL 1998, and MALINK 2015. The main logical issue in any syllogistic reconstruction of the argument is that it has to assume premises of the form "A and B are equal", thus treating equality as a standard monadic predicate with plural subjects. But whereas "A and B are red" may be analyzed into "A is red" and "B is red", this cannot be done with "equal", which is a relation and therefore behaves in a completely different way. It is possible that Aristotle did not realize this fact, but I note that in *An. pr.* A 36, 49^a2-3, Aristotle explicitly takes "equal" to imply an *oblique* predication in the dative (to be equal *to* something), and therefore it is difficult to think that he may have conceived of a syllogistic premise with the predicate "...are equal" as ancient and modern commentators claim.

⁶⁰ We have already mentioned that in *An. pr.* B 11, 62^a12-13 the word ἀξίωμα means ἐνδοξον. On the meaning of πρότασις in the *Prior Analytics*, see CRIVELLI, CHARLES 2011. On the relative chronology of the *Prior* and *Posterior Analytics*, see some conjectures in BARNES 1981. I am not entirely fond of this solution, since in §6 below I would like to connect Aristotle's views on common axioms with the theory of oblique inferences expounded in *An. pr.* A 36.

mentions particular angles identified by letters) need universal assumptions in order to be deductions. *A fortiori*, syllogisms would also need universal premises. This reading would be compatible with an inferential interpretation of CN₃, employed in the passage as a general example of a universal statement (or rule) used in science rather than as a specific instance of a universal syllogistic premise.

In conclusion, I think that this isolated and rather obscure example cannot be taken as strong evidence against an inferential interpretation of axioms. On the contrary, the latter interpretation may solve some important exegetical issues that affect more standard readings. On the one hand (issue #1 in §4 above), such an interpretation clearly puts together logical and mathematical axioms, thus providing grounds for Aristotle's unitary understanding of all sorts of axioms as formal principles. On the other hand (issue #2), it also distinguishes in a very sharp way common axioms (as formal principles) from proper (semantic) principles, thus explaining Aristotle's main divide in the classification of ἀρχαί. In the next section, we discuss in which sense the inferential interpretation may also contribute to explain non-syllogistic inferences in Aristotle's theory and the fact that axioms are not presented in the subject-predicate form (issue #3, further reinforced by the above example).

§6. *Axioms, Logic and the Categories*

Further developments of the inferential theory of axioms below presuppose a general interpretation of the meaning of Aristotle's logic and theory of categories, that I am unable to fully provide in the present paper.⁶¹ These developments are also grounded on a thinner textual basis than the previous discussion. As such, they should be especially regarded as interpretative proposals and conjectures.

I surmise that the inferential interpretation of common axioms may explain why they are not stated in the usual subject-predicate form, but rather in the form of relations. According to this interpretation, axioms do not express a property of anything in particular, and not even the properties of several things in different genera (as in the schematic interpretation). On the contrary, they express formal relations among things and properties in general. The main difficulty is of course that the kind of inferences licensed by common axioms (such as CN₃) are not syllogistic, and it is not entirely clear whether Aristotle's theory of science may accommodate for "asyllogistic" inferences (i.e. inferences that are not ruled by the moods expounded in the *Prior Analytics*).

As is well known, there are passages in which Aristotle seems to hint at a broader understanding of deduction, in which the notion of predication (the κατηγορεῖσθαι of a predicate to a subject) is extended to a more general notion of belonging (ὑπάρχειν) that may encompass at least some kinds of relations. Aristotle gives examples of deductions in which one premise, or both premises, allow for oblique predication, i.e. they are not such that in them a predicate "is said of" a subject: if wisdom is a science (standard direct predication) and wisdom is of the good (oblique ὑπάρχειν), the conclusion is that a science is of the good. These

⁶¹ In particular, I follow here a *logical* interpretation of the categories that is similar to that advanced by APELT 1891, and which has been later debated by several interpreters.

examples have been discussed at length by the interpreters, who in different degrees of generosity have attributed to Aristotle a fully-fledged theory of relations, or various confusions on the latter. It seems clear that Aristotle did not regard his extended logic as a treatment of relations in the strict sense, since he does not give any special relevance to the category of $\pi\rho\acute{o}\varsigma\ \tau\iota$ in this connection. He seems rather to have in mind that relations are treated as a special sort of oblique predication.⁶²

I would venture the hypothesis, then, that the principles ruling these kinds of oblique inferences may be *common axioms* interpreted along the lines of the inferential reading.

Aristotle's very scanty references to axioms and oblique $\acute{\upsilon}\pi\acute{\alpha}\rho\chi\epsilon\iota\nu$ make it difficult to provide textual evidence for this interpretative hypothesis. It may be noted, however, that as an example of oblique predication Aristotle mentions the "being equal to" or the "being double of". These relations are the main content of the mathematical *common notions* that we find in Euclid, and that were possibly formulated by Eudoxus.⁶³ The very plausible identification of Euclid's $\kappa\omicron\iota\iota\nu\alpha\iota\ \acute{\epsilon}\nu\nu\omicron\iota\alpha\iota$ with Aristotle's $\kappa\omicron\iota\iota\nu\delta\ \acute{\alpha}\xi\iota\acute{\omega}\mu\alpha\tau\alpha$ supports the idea that Aristotle's cases of oblique predication *in quantity* may have been ruled by common axioms such as CN₃.

This interpretation does not need to assume that a general logic of relations would be needed to implement these inferences. In Aristotle's theory of deduction no *general* rules for oblique and relational inferences are given beforehand. On the contrary, in the absence of a general logical framework that allows for a treatment of relations, there are a few cases of relations and oblique predications that have their own particular rules, which are encoded in common axioms. This might explain, at least, why Aristotle seems to have never envisaged or treated at any length a logic of relations, and rather conceived of these occurrences as particular cases. I think, therefore, that the inferential interpretation of common axioms may contribute to solving the difficult question of the existence of asylogistic inferences in Aristotle's works. If, on the contrary, we interpret common axioms as syllogistic premises, we would have to explain how they may be assumed in a syllogism, since they are not given in subject-predicate form. We would have either to admit that Aristotle simply could not fathom how to do this, or to imagine that Aristotle had conceived a logic of relations of sorts in order to draw conclusions from premises such as CN₃.⁶⁴ The lack of evidence for the latter seems to

⁶² The main text is *An. pr.* A 36, in which a broader notion of $\acute{\upsilon}\pi\acute{\alpha}\rho\chi\epsilon\iota\nu$ is introduced. The meaning of this whole chapter is very controversial. Aristotle offers a treatment of oblique inferences mainly based on some linguistic features of the Greek, and it is hard to evaluate its scope and significance. Many interpreters interested in Aristotle's philosophy of mathematics have employed this passage to argue that Aristotle had a conception of logic that was broad enough to accommodate at least some mathematical asylogistic inferences. The most significant attempt in this respect is the important paper by MENDELL 1998. See also, from the same year, KOUREMENOS 1998, who however has less bold claims about the reconstruction of Aristotle's theory of relational syllogisms. I follow their lead here, but the textual basis is thin.

⁶³ Examples of oblique inferences involving quantity are to be found in *An. pr.* A 36, 48^b40-49^a5. Aristotle makes examples of predications in which the copula must be interpreted as "to be equal to" also in other passages, such as in *An. pr.* B 25, 69^a30-34, or *An. post.* B 11, 94^a27-34.

⁶⁴ This seems to be the path taken by MENDELL 1998, who basically admits the possibility of inferring "A is equal to B" from "A and B are equal" (p. 206), thus resolving a theory of relations into a theory of inferences between monadic predications. I am aware that there are further difficulties in attributing a theory of asylogistic

me a further argument to conceive of common axioms as formal principles complementing standard syllogistic inferences.

In a further passage in the *Prior Analytics*, Aristotle states indeed that ὑπάρχειν is said according to all the categories.⁶⁵ This important statement seems to imply (should we further extend the present interpretation) that different common axioms rule different categorial contexts. The relations expressed by a common axiom such as CN₃ are not properties intrinsic (say) to triangles or lines, but relations that such objects acquire in so far as they are *predicated as quantities* (ὑπάρχειν “according to quantity”) in a given propositional context, such as a geometrical theorem. We can apply CN₃ to triangles not because triangles are quantities *per se* (they are rather *magnitudes*, and quantity is not a genus), but because sometimes, in doing geometry, we do employ triangles as quantities. A common axiom, in sum, does not express a collection of genus-specific properties of an object, but the *formal behavior of any thing whatsoever* in so far as it plays a certain role in a proposition and in an inference, and therefore it is predicated according to a given category.

In this respect, the domain of application of CN₃ is no more restricted than is the domain of the Principle of Contradiction: in so far as they are both formal (ontological) principles, they apply to all beings. We may admit, of course, degrees of *formal* generality, and state that the Principle of Contradiction, in so far as it applies to *all* categories (not just quantity) rules every kind of predication (every kind of ὑπάρχειν), and therefore it may be said to be the ἀρχὴ τῶν ἄλλων ἀξιωματῶν πάντων. But this greater generality does not concern the genera of things falling under the axiom, but the different ways of predication to which it applies. In this respect, we may fully account for Aristotle’s claim (in *An. Post.* A 11) that axioms are common to *all* sciences, without having to accept the suggestion of some interpreters that commonality would only mean “common to *some* things”. The meaning of “being common” (κοινόν) is rather to be equated with “being formal” or “being categorial”.

This interpretation also allows for a uniform reading of Aristotle’s statement in *An. Post.* A 2 to the effect that axioms are needed to learn *anything whatever*. We do not have to restrict the latter claim to the “logical” axioms only, such as the Principle of Contradiction: common axioms in general (also CN₃) are rather to be understood as formal principles, and therefore as principles employed in any reasoning whatsoever (as long as predication according to the relevant categories is employed).⁶⁶

inferences to Aristotle, and for instance a referee of the present essay rightly remarked that such a theory might become an insurmountable obstacle for Aristotle’s important claim (based on syllogistic inferences) that every demonstration is finite (*An. post.* A 19-22).

⁶⁵ *An. pr.* A 37, 49^a6-8. This chapter of the *Prior Analytics* is short and controversial, and seems not to connect smoothly with the previous discussion in *An. pr.* A 36. The significance of the term κατηγορίαι has been deflated by some interpreters as if Aristotle here meant just “predications” in a grammatical sense. I do not see much room for sharply distinguishing between grammatical and philosophical meanings of the term. For an interpretation of κατηγορίαι as categories in these passages, see ALEX. *In metaph.* 366; and very recently STRIKER 2009, p. 226.

⁶⁶ For the sentence on the Principle of Contradiction as the principle of all other axioms, see above *note* 25. For the passage in *An. Post.* A 11, see *note* 20. For the passage in *An. Post.* A 2, see *note* 16.

Even more importantly, an inferential reading of axioms may explain the analogical unity of things falling into different genera. The inferential interpretation agrees with the schematic interpretation on the fact that objects falling under different genera may have a unity *κατ'ἀναλογίαν*, which is captured by the axioms. This hyper-generic unity expressed by the axioms, however, is no longer regarded as an extrinsic, brute fact in the inferential interpretation. Numbers and geometrical magnitudes (two genera of things) do not simply *happen* to have relations and structures in common (as it is the case in the schematic interpretation). On the contrary, numbers and geometrical magnitudes (and times, motions, etc.) have analogous relations and structures *insofar as* they are predicated according to the same category in a propositional context. Their similar propositional roles make them fall under the same relations. The functional unity of the category explains the unity *κατ'ἀναλογίαν* expressed by *κοινὰ ἀξιώματα*.

This interpretation may therefore fully explain the sense in which the demonstration on the alternation of proportions “is different, and also the same” for lines and numbers.⁶⁷ The *deductive structure* of the two proofs is the same, and it employs the same common axioms to draw analogous conclusions. But the subject matter of the two proofs is different, as well as the definitions which explain the nature of the objects involved. This structural identity of several demonstrations dealing with different objects, which is a common feature of mathematical practice, is explained by Aristotle (I surmise) with a reference to the identical role (*κατ'ἀναλογίαν*) that different objects play in the proofs, insofar as they are employed as quantities.⁶⁸

We also see why Aristotle could, even if he consistently rejected the idea of a *mathesis universalis* over and above the individual mathematical disciplines, nonetheless refer to a universal mathematics as an established doctrine. This was certainly for historical reasons—Eudoxus and others had proven hyper-generic results in mathematics. But we see that there is a sense in which Aristotle could accept a *sui generis* science that did not revolve around any particular genus: this was the case of first philosophy, conceived of as a formal discipline. Universal mathematics, like first philosophy restricted to the *category* of quantity (as opposed to Platonic dialectic restricted to the *genera* of magnitudes and numbers) had the same status.⁶⁹

In sum, I have shown that an inferential interpretation of common axioms may solve several important exegetical issues. It puts logical and mathematical axioms on the same footing, as Aristotle himself seems to have done. It neatly separates axioms from proper principles by granting them a different role in demonstrations. It explains their not being formulated in a subject-predicate form, and gives some hints towards the solution of the difficult problem of the existence of non-syllogistic inferences in Aristotle's epistemology.

⁶⁷ Following the above-mentioned passage in *An. post.* B 17, 99^a8-11 (cf. note 36).

⁶⁸ See for instance the appendix to RABOUIN 2016, which offers parallel proofs of the same result about magnitudes and numbers in Euclid's *Elements* (X, 3 and VII, 2). I would say that Aristotle regarded these proofs as identical *κατ'ἀναλογίαν*.

⁶⁹ In this way, I endorse David Rabouin's idea of a *mathesis universalis* in Aristotle as a science of the “equal and unequal”, according to the category of quantity (and grounded on the common axioms on equality). See RABOUIN 2009, pp. 58-67.

Finally, it provides a better grounding for the whole theory of analogy by showing functional reasons for the commonality of axioms. In addition, Galen's reading of common axioms as rules of inferences (or grounds for these rules) shows that in antiquity such a theory was at least conceivable.⁷⁰

I am aware that in the lack of explicit textual evidence such an inferential interpretation of common axioms is destined to remain conjectural, but I think that this alternative reading is well worth entertaining.

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⁷⁰ See *Inst. log.* XVIII, 8. Galen even claimed that an interpretation of axioms as principles licensing inferences was older than him, and had already been endorsed, possibly by Posidonius.

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