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ON STABLE AND FIXED POLYNOMIALS

J. NOVACOSKI AND M. SPIVAKOVSKY

ABSTRACT. Let ν be a rank one valuation on $K[x]$ and Ψ_n the set of key polynomials for ν of degree $n \in \mathbb{N}$. We discuss the concepts of being Ψ_n -stable and (Ψ_n, Q) -fixed. We discuss when these two concepts coincide. We use this discussion to present a simple proof of Proposition 8.2 of [3] and Theorem 1.2 of [5].

1. INTRODUCTION

Let ν be a rank one valuation on $K[x]$. For $n \in \mathbb{N}$ set

$$\Psi_n = \{Q \in K[x] \mid Q \text{ is a key polynomial for } \nu \text{ and } \deg(Q) = n\}.$$

Suppose that Ψ_n is non-empty and bounded (i.e., there exists $a \in K[x]$ such that $\nu(a) > \nu(Q)$ for every $Q \in \Psi_n$) and that $\nu(\Psi_n)$ does not have a maximum. Set $K[x]_n = \{a \in K[x] \mid \deg(a) < n\}$. For each $f \in K[x]$ and $Q \in \Psi_n$ the Q -**expansion** of f is the expression

$$f = a_0 + a_1Q + \dots + a_rQ^r = l(Q)$$

where $l(X) \in K[x]_n[X]$. We denote the value r (which does not depend on the choice of $Q \in \Psi_n$) in the previous expression by $\deg_X(f)$. The truncation of ν on Q is given by

$$\nu_Q(f) := \min_{0 \leq i \leq r} \{\nu(a_iQ^i)\}.$$

Let

$$S_n = \{f \in K[x] \mid \nu_Q(f) < \nu(f) \text{ for every } Q \in \Psi_n\}.$$

A polynomial f is said to be Ψ_n -**stable** if it does not belong to S_n . A monic polynomial $F \in K[x]$ is called a **limit key polynomial** for Ψ_n if it belongs to S_n and has the smallest degree among polynomials in S_n .

In [4], Kaplansky introduces the concept of *pseudo-convergent sequences*. These objects are strongly related to the set Ψ_1 . For a given such sequence \underline{a} we can define what it means for a polynomial $f \in K[x]$ to be fixed by \underline{a} . Here, we generalize this concept for any of the sets Ψ_n . We say that $f = l(Q)$, $l(X) \in K[x]_n[X]$, is (Ψ_n, Q) -**fixed** if there exists $Q' \in \Psi_n$, $\nu(Q) < \nu(Q')$ such that $\nu(f) = \nu(l(Q' - Q))$.

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Our first main result (Proposition 3.2) is that we can choose a suitable $Q \in \Psi_n$ such that for any $f \in K[x]$ with $\deg_X(f) \leq \deg_X(F)$, we obtain that f is Ψ_n -stable if and only if it is (Ψ_n, Q) -fixed.

We will fix a suitable $Q \in K[x]$ (see (2) and (3)). Then we fix a limit ordinal λ and any cofinal well-ordered (with respect to ν) subset

$$\{Q_\rho\}_{\rho < \lambda} \subseteq \{Q' \in \Psi_n \mid \epsilon(Q) < \epsilon(Q')\}.$$

This means that if $\rho < \sigma < \lambda$, then $\nu(Q_\rho) < \nu(Q_\sigma)$ and that for every $Q' \in \Psi_n$, there exists $\rho < \lambda$ such that $\nu(Q') < \nu(Q_\rho)$. For each $\rho < \lambda$ set $h_\rho := Q - Q_\rho \in K[x]_n$ and $\gamma_\rho = \nu(Q_\rho)$. It follows from the definition that $\{h_\rho\}_{\rho < \lambda}$ is a *pseudo-convergent sequence* for ν . For simplicity, we will denote ν_{Q_ρ} by ν_ρ .

Let $p = \text{char}(K\nu)$ and I be the set of all non-negative powers of p . As an application of Proposition 3.2 we can prove the following.

Theorem 1.1. *Let F be a limit key polynomial for Ψ_n and write $F = L(Q)$ for some $L(X) \in K[x]_n[X]$. Then we have the following.*

(i): *There exists $\sigma < \lambda$ such that for every $\theta > \sigma$ the polynomial*

$$F_p = L(h_\theta) + \sum_{i \in I} \partial_i L(h_\theta) Q_\theta^i$$

is a limit key polynomial for Ψ_n . Here $\partial_i L$ denotes the Hasse derivative of $L(X)$ (as a polynomial in $K(x)[X]$) of order i .

(ii): *For each $i \in I \cup \{0\}$ there exists $a_i \in K[x]_n$ such that*

$$\overline{F}_p = \sum_{i \in I \cup \{0\}} a_i Q_\theta^i$$

is a limit key polynomial for Ψ_n .

Kaplansky proved the above result in the case $n = 1$. In that case, (ii) follows trivially from (i). Our proof of Theorem 1.1 follows Kaplansky's proof.

If $n > 1$, then (ii) was proven in [3] (Proposition 8.2). An alternative proof of it was presented in [5] (Theorem 1.2). The advantage of our proof is that it is much simpler and presents as algorithm on how to construct the limit key polynomial of this form (from a given limit key polynomial). Also, our proof does not require that are in the equicharacteristic case.

2. PRELIMINARIES

Throughout this paper ν will denote a rank one valuation on $K[x]$. For $f \in K[x]$ we denote

$$(1) \quad \epsilon(f) = \max_{1 \leq b \leq \deg(f)} \left\{ \frac{\nu(f) - \nu(\partial_b f)}{b} \right\},$$

where $\partial_b f$ denotes the Hasse derivative of f of order b . We denote

$$I(f) = \left\{ b, 1 \leq b \leq \deg(f) \mid \epsilon(f) = \frac{\nu(f) - \nu(\partial_b f)}{b} \right\}.$$

A monic polynomial $Q \in K[x]$ is said to be a **key polynomial for ν** if for every $f \in K[x]$, if $\epsilon(f) \geq \epsilon(Q)$, then $\deg(f) \geq \deg(Q)$. For a polynomial $f \in K[x]$ let

$$f = f_0 + f_1Q + \dots + f_rQ^r$$

be the Q -expansion of f . We set

$$S_Q(f) = \{i, 0 \leq i \leq r \mid \nu_Q(f) = \nu(f_iQ^i)\} \text{ and } \delta_Q(f) = \max S_Q(f).$$

Throughout this paper we will fix a limit key polynomial F for Ψ_n and denote $d := \deg_X(F)$. Set

$$B = \lim_{Q \in \Psi_n} \nu(Q) \text{ and } \overline{B} = \lim_{Q \in \Psi_n} \nu_Q(F).$$

Take $Q_0 \in \Psi_n$ and choose $Q \in \Psi_n$ such that

$$(2) \quad \epsilon(Q) - \epsilon(Q_0) > d(B - \nu(Q))$$

and

$$(3) \quad \epsilon(Q) - \epsilon(Q_0) > \overline{B} - \nu_Q(F).$$

Write $F = L(Q)$ for $L(X) \in K[x]_n[X]$.

The next result is well-known. We will reprove it here because we need this slightly stronger statement.

Lemma 2.1. *Let Q be a key polynomial and take f such that $f = qQ + r$ with $\gamma = \max\{\epsilon(f), \epsilon(r)\} < \epsilon(Q)$. Then we have*

$$\nu_Q(qQ) - (\epsilon(Q) - \gamma) \geq \nu(f) = \nu(r).$$

Proof. Take $b \in I(qQ)$. Since $\epsilon(qQ) = \max\{\epsilon(Q), \epsilon(q)\} \geq \epsilon(Q)$ (Corollary 4.4 of [1]) we have

$$\nu(qQ) - b\epsilon(Q) \geq \nu(\partial_b(qQ)) \geq \min\{\nu(\partial_b(f)), \nu(\partial_b(r))\} \geq \min\{\nu(f), \nu(r)\} - b\gamma.$$

Consequently, $\nu(f) = \nu(r)$ and

$$\nu(qQ) - \nu(f) \geq b(\epsilon(Q) - \gamma) \geq \epsilon(Q) - \gamma.$$

Applying the above discussion to ν_Q instead of ν we obtain the result. \square

Take $f \in K[x]$ and $Q' \in \Psi_n$ with $\epsilon(Q) \leq \epsilon(Q')$ with $\epsilon(f) < \epsilon(Q_0)$ and write

$$f = qQ' + r \text{ with } \deg(r) < \deg(Q') = \deg(Q_0).$$

By (2), (3), Lemma 2.1 and the fact that $\epsilon(r) < \epsilon(Q_0)$ we have

$$(4) \quad \nu_{Q'}(qQ') > \nu(f) + d(B - \nu(Q))$$

and

$$(5) \quad \nu_{Q'}(qQ') > \nu(f) + \overline{B} - \nu_Q(F).$$

The next result is a well-known result about key polynomials.

Lemma 2.2. *Take $Q, Q' \in \Psi_n$ be such that $\nu(Q) < \nu(Q')$. For $f \in K[x]$ let $f = \sum_{i=0}^r f_i Q^i$ be the Q' -expansion of f . Then*

$$\nu_Q(f) = \min_{0 \leq i \leq r} \{\nu_Q(f_i Q^i)\}.$$

Proof. Since Q' is monic and has the smallest degree among all polynomials f such that $\nu_Q(f) < \nu(f)$, it is a (Mac Lane-Vaquié) key polynomial for ν_Q (Theorem 31 of [2]). In particular, Q' is ν_Q -minimal and the result follows from Proposition 2.3 of [6]. \square

Lemma 2.3. *Let $Q' \in \Psi_n$ such that $\epsilon(Q) < \epsilon(Q')$. For any $f \in K[x]$ let*

$$f = a_0 + a_1 Q + \dots + a_r Q^r \text{ and } f = b_0 + b_1 Q' + \dots + b_r Q'^r$$

be the Q and Q' -expansions of f , respectively. For $l = \delta_Q(f)$ we have

$$\nu(a_l - b_l) > \nu(a_l).$$

In particular, $\nu(a_l) = \nu(b_l)$.

Proof. Let $h := Q - Q'$ so that $Q = Q' + h$. Then

$$a_i Q^i = \sum_{j=0}^i \binom{i}{j} a_i h^j Q'^{i-j}.$$

For each i , $0 \leq i \leq r$, and j , $0 \leq j \leq i$, let

$$(6) \quad \binom{i}{j} a_i h^j = a_{ij0} + a_{ij1} Q' + \dots + a_{ijn} Q'^n$$

be the Q' expansion of $a_i h^j$. Then

$$b_l = \sum_{i-j+k=l} a_{ijk}.$$

For i , $0 \leq i \leq r$, and j , $0 \leq j \leq i$, if $k := l + j - i > 0$, then by (4), we have

$$\begin{aligned} \nu(a_{ijk}) + k\nu(Q') &> \nu(a_i h^j) + d(B - \nu(Q)) \\ &\geq \nu(a_i Q^i) + (j - i)\nu(Q) + k(B - \nu(Q)) \\ &\geq \nu(a_l Q^l) + (j - i)\nu(Q) + k(B - \nu(Q)) \\ &\geq \nu(a_l) + (l + j - i)\nu(Q) + k(B - \nu(Q)) = \nu(a_l) + kB. \end{aligned}$$

Since $B > \nu(Q')$ we have $\nu(a_{ijk}) > \nu(a_l)$.

Suppose now that $k := l + j - i = 0$ (i.e., that $i = l + j$). If $j = 0$, then by definition $a_{ijk} = a_l$. If $j > 0$, then $i > l$. Since $l = \delta_Q(f)$ we have $\nu(a_i Q^i) > \nu(a_l Q^l)$. Then by Lemma 2.2, applied to (6), we have

$$\nu_Q(a_{ijk} Q'^k) \geq \nu_Q(a_i h^j) = \nu(a_i Q^i) + (j - i)\nu(Q) > \nu(a_l) + (l + j - i)\nu(Q).$$

Since $\nu_Q(Q') = \nu(Q)$ we obtain that $\nu(a_{ijk}) > \nu(a_l)$ and the result follows. \square

For each $\rho < \lambda$ and $f \in K[x]$, let

$$f = a_{\rho 0}(f) + a_{\rho 1}(f)Q_\rho + \dots + a_{\rho r}(f)Q_\rho^r$$

be the Q_ρ -expansion of f . The value of $a_{\rho 0}(f)$ will be very important in what follows.

Proposition 2.4. *For $f \in K[x]$ with $\deg_X(f) \leq d$, write $f = l(Q)$ for $l(X) \in K[x]_n[X]$. For every $\rho < \lambda$ we have*

$$(7) \quad \nu_\rho(l(h_\rho)) \geq \nu_\rho(f).$$

Moreover, the equality holds in (7) if and only if

$$\nu(a_{\rho 0}(f)) = \nu_\rho(f) = \nu(l(h_\rho)).$$

Proof. By definition

$$f = l(Q) = l(Q_\rho + h_\rho) = Q_\rho p(x) + l(h_\rho) \text{ for some } p(x) \in K[x].$$

Hence $a_{\rho 0}(f) = a_{\rho 0}(l(h_\rho))$. Let

$$(8) \quad l(h_\rho) = a_{\rho 0}(f) + b_1 Q_\rho + \dots + b_l Q_\rho^l$$

be the Q_ρ -expansion of $l(h_\rho)$. We will show that $\nu(b_i Q_\rho^i) > \nu_\rho(f)$ for every i , $1 \leq i \leq l$, and this will imply our result.

Let $f = a_0 + a_1 Q + \dots + a_r Q^r$ be the Q -expansion of f , so that

$$(9) \quad l(h_\rho) = a_0 + a_1 h_\rho + \dots + a_r h_\rho^r.$$

For each j , $1 \leq j \leq r$, consider the Q_ρ -expansion

$$(10) \quad a_j h_\rho^j = a_{\rho 0j} + a_{\rho 1j} Q_\rho + \dots + a_{\rho lj} Q_\rho^l$$

of $a_j h_\rho^j$. Comparing (8), (9) and (10), it is enough to show that

$$\nu(a_{\rho ij} Q_\rho^i) > \nu(a_{\rho 0}(f)) \text{ for every } i, j, 1 \leq i \leq l \text{ and } 1 \leq j \leq r.$$

For a fixed j , $1 \leq j \leq r$, by (4) applied to (10) we have

$$(11) \quad \nu(a_{\rho ij} Q_\rho^i) > \nu(a_j Q^j) + s(B - \nu(Q)).$$

Since $\nu(Q) = \nu(h_\rho) = \nu_\rho(Q)$, if $\nu(a_0) < \nu(a_i Q^i)$ for every i , $1 \leq i \leq r$, then

$$\nu_\rho(f) = \nu(a_0) = \nu(l(h_\rho))$$

and we are done. Suppose not and take $l = \delta_Q(f) > 0$. By (11) and the fact that $\nu(a_{\rho l}(f)) = \nu(a_l)$ (Lemma 2.3), we have

$$\begin{aligned} \nu(a_{\rho lj} Q_\rho^l) &> \nu(a_j Q^j) + l(B - \nu(Q)) \geq \nu(a_l) + l\nu(Q) + l(B - \nu(Q)) \\ &= \nu(a_l) + lB > \nu(a_l) + l\nu(Q_\rho) \geq \nu_\rho(f). \end{aligned}$$

This completes the proof. \square

Corollary 2.5. *If $\deg(f) < \deg(F)$, then there exists ρ such that*

$$\nu(l(h_\sigma)) = \nu(f) = \nu_\sigma(f) = \nu(a_{\sigma 0}(f))$$

for every σ , $\rho < \sigma < \lambda$.

Proof. It is well-known that if f is Ψ_n -stable, then there exists $Q' \in \Psi_n$ such that $0 \in S_{Q'}(f)$. The result follows immediately. \square

3. THE TAYLOR EXPANSION OF A POLYNOMIAL

We will consider the ring $K(x)[X]$ where X is an indeterminate and let ∂_i denote the i -th Hasse derivative with respect to X . Then, for every $l(X) \in K(x)[X]$ and $a, b \in K[x]$ we have the Taylor expansion

$$l(b) = l(a) + \sum_{i=1}^{\deg_X l} \partial_i l(a)(b-a)^i.$$

Lemma 3.1 (Lemma 4 of [4]). *Let Γ be an ordered abelian group, $\beta_1, \dots, \beta_n \in \Gamma$ and $\{\gamma_\rho\}_{\rho < \lambda}$ an increasing sequence in Γ , without a last element. If t_1, \dots, t_n are distinct positive integers, then there exist b , $1 \leq b \leq n$, and $\rho < \lambda$, such that*

$$\beta_i + t_i \gamma_\sigma > \beta_b + t_b \gamma_\sigma \text{ for every } i, 1 \leq i \leq n, i \neq b \text{ and } \sigma > \rho.$$

Proposition 3.2. *Take $f \in K[x]$ such that $\deg_X(f) \leq d$ and write $f = l(Q)$ for $l(X) \in K[x]_n[X]$. Then there exists $\rho < \lambda$ such that*

$$\nu_\sigma(f) = \nu(l(h_\sigma)) \text{ for every } \sigma, \rho < \sigma < \lambda.$$

In particular, f is Ψ_n -stable if and only if it is (Ψ_n, Q) -fixed.

Proof. Since for every j , $1 \leq j \leq \deg_X(l)$, we have $\deg(\partial_j l(Q)) < \deg(F)$ we can use Corollary 2.5 to obtain $\rho < \lambda$ such that

$$(12) \quad \beta_j := \nu_\sigma(\partial_j l(h_\sigma)) = \nu(\partial_j l(h_\sigma)) \text{ for every } j, 1 \leq j \leq \deg_X(l) \text{ and } \rho < \sigma.$$

By Lemma 3.1, there exist b , $1 \leq b \leq \deg_X(l)$, and $\rho < \lambda$ such that for every λ , $\rho < \sigma < \lambda$ and $i \neq b$ we have

$$(13) \quad \beta_b + b\gamma_\rho < \beta_i + i\gamma_\rho \text{ and (12) is satisfied.}$$

For σ , $\rho < \sigma < \lambda$, since

$$f - l(h_\sigma) = \sum_{i=1}^{\deg_X(f)} \partial_i l(h_\sigma) Q_\sigma^i$$

we have

$$(14) \quad \nu_\sigma(f - l(h_\sigma)) = \beta_b + b\gamma_\sigma$$

and

$$(15) \quad \nu(f - l(h_\sigma)) = \beta_b + b\gamma_\sigma.$$

By Proposition 2.4 we have $\nu_\sigma(l(h_\sigma)) \geq \nu_\sigma(f)$. This and (14) imply that

$$(16) \quad \nu(l(h_\sigma)) \geq \nu_\sigma(l(h_\sigma)) \geq \beta_b + b\gamma_\sigma.$$

If $\nu_\sigma(f) = \nu(f)$, then $\nu(l(h_\sigma)) \geq \nu(f)$. Suppose, aiming for a contradiction, that $\nu(l(h_\sigma)) > \nu(f)$. Then by (15) we have

$$\nu(f) = \beta_b + b\gamma_\sigma.$$

For any $\sigma' > \sigma$, by (16) we would obtain

$$\nu(l(h_{\sigma'})) \geq \beta_b + b\gamma_{\sigma'} > \beta_b + b\gamma_\sigma = \nu(f)$$

and this contradicts (15) (with σ replaced by σ'). Hence, $\nu(l(h_\sigma)) = \nu(f)$.

Suppose now that $\nu_\sigma(f) < \nu(f)$. If $\nu_\sigma(l(h_\sigma)) > \nu_\sigma(f)$, then by (14) we have $\nu_\sigma(f) = \beta_b + b\gamma_\sigma$. Consequently,

$$\nu(f - l(h_\sigma)) \geq \min\{\nu(f), \nu(l(h_\sigma))\} > \nu_\sigma(f) = \beta_b + b\gamma_\sigma,$$

what is a contradiction to (15). Hence,

$$\nu_\sigma(l(h_\sigma)) = \nu_\sigma(f).$$

By the second part of Proposition 2.4 we obtain that $\nu_\sigma(f) = \nu(l(h_\sigma))$. This, (14)–(16) and the fact that $\nu_\sigma(f) < \nu(f)$ imply that

$$\nu_\sigma(f) = \nu(l(h_\sigma)) = \beta_b + b\gamma_\sigma.$$

□

Remark 3.3. In the proof of Theorem 1.1 we will use the explicit calculation of $\nu(l(h_\sigma))$ obtained in the previous proposition.

4. PROOF OF THEOREM 1.1

We will adapt the proof by Kaplansky in [4]. For each i , $1 \leq i \leq d$, the polynomial $\partial_i L(Q)$ has degree smaller than $\deg(F)$, hence by Corollary 2.5 there exists $\rho_0 < \lambda$ such that

$$(17) \quad \beta_i := \nu(\partial_i L(Q)) = \nu(\partial_i L(h_\rho))$$

for every ρ , $\rho_0 < \rho < \lambda$.

Lemma 4.1. *If $i = p^t$ and $j = p^t r$ with $r > 1$ and $p \nmid r$, then there exists $\rho < \lambda$ such that*

$$\beta_i + i\gamma_\sigma < \beta_j + j\gamma_\sigma \text{ for every } \sigma, \rho < \sigma < \lambda.$$

Moreover, if C in the value group of ν is such that $C > \gamma_\rho$ for every $\rho < \lambda$, then

$$\beta_i + iC < \beta_j + jC.$$

Proof. From the Taylor formula (applied to $\partial_i L$) we have

$$\begin{aligned}\partial_i L(h_\sigma) - \partial_i L(h_\rho) &= \sum_{k=1}^{n-i} \partial_k \partial_i L(h_\rho) (h_\sigma - h_\rho)^k \\ &= \sum_{k=1}^{n-i} \binom{i+k}{i} \partial_{i+k} L(h_\rho) (h_\sigma - h_\rho)^k.\end{aligned}$$

By Lemma 3.1, for $\rho < \sigma$ large enough

$$(18) \quad \nu(\partial_i L(h_\sigma) - \partial_i L(h_\rho)) = \min_{1 \leq k \leq n-i} \left\{ \nu \left(\binom{i+k}{i} \partial_{i+k} L(h_\rho) (h_\sigma - h_\rho)^k \right) \right\}.$$

In particular, taking $k = j - i$, this gives

$$(19) \quad \nu(\partial_i L(h_\sigma) - \partial_i L(h_\rho)) \leq \nu \left(\binom{j}{i} \partial_j L(h_\rho) (h_\sigma - h_\rho)^{j-i} \right)$$

By (17) and (19) we have

$$\begin{aligned}\beta_i &\leq \nu(\partial_i L(h_\sigma) - \partial_i L(h_\rho)) \\ &\leq \nu \left(\binom{j}{i} \partial_j L(h_\rho) (h_\sigma - h_\rho)^{j-i} \right) \\ &= \nu \left(\binom{j}{i} \right) + \beta_j + (j-i)\gamma_\rho.\end{aligned}$$

Since $p \nmid \binom{j}{i}$ and $\text{char}(K\nu) = p$ we have $\nu \left(\binom{j}{i} \right) = 0$. Consequently,

$$\beta_i \leq \beta_j + (j-i)\gamma_\rho.$$

This means that for every $\sigma, \rho < \sigma < \lambda$, we have

$$\beta_i + i\gamma_\sigma < \beta_j + j\gamma_\sigma.$$

Take $C > \gamma_\rho$ for every $\rho < \lambda$. If $\beta_i + iC \geq \beta_j + jC$, then

$$\beta_i - \beta_j \geq (j-i)C > (j-i)\gamma_\sigma \text{ for every } \rho < \lambda$$

and this contradicts the first part. \square

The proof of the next result is very similar to the proof of Proposition 2.4.

Lemma 4.2. *Fix $\theta < \lambda$ and for each $i, 0 \leq i \leq r$, set $a_{i0} := a_{\theta 0}(\partial_i L(h_\theta))$. Then*

$$\nu_\theta(\partial_i L(h_\theta) - a_{i0}) + i\nu(Q_\theta) > \overline{B}.$$

Proof. Since $F = a_0 + a_1 Q + \dots + a_r Q^r$ we have

$$\begin{aligned}\partial_0 L(h_\theta) &= a_0 + a_1 h_\theta + \dots + a_r h_\theta^r \\ \partial_1 L(h_\theta) &= a_1 + \dots + r a_r h_\theta^{r-1} \\ &\vdots \\ \partial_r L(h_\theta) &= a_r.\end{aligned}$$

If we write

$$\partial_i L(h_\theta) = b_{i0} + b_{i1} h_\theta + \dots + b_{is} h_\theta^s,$$

then

$$\nu(b_{ij}h_\theta^j) + i\nu(Q) \geq \nu(a_{i+j}Q^{i+j}) \geq \nu_Q(F).$$

For each $j > 1$, write

$$b_{ij}h_\theta^j = a_{i0j} + a_{i1j}Q_\theta + \dots + a_{isj}Q_\theta^s.$$

For every i, j and $k > 0$, by (5) we have

$$\begin{aligned} \nu(a_{ikj}Q_\theta^k) + i\nu(Q_\theta) &> \nu(b_{ij}h_\theta^j) + \overline{B} - \nu_Q(F) + i\nu(Q_\theta) \\ &> \nu_Q(F) + \overline{B} - \nu_Q(F) = \overline{B}. \end{aligned}$$

For every i , we have

$$\partial_i L(h_\theta) - a_{i0} = \sum_{k,j>0} a_{ikj}Q_\theta^j.$$

The result follows. \square

We proceed now with the proof of Theorem 1.1.

Proof of Theorem 1.1. For each $i = p^s$ with $1 \leq i \leq \deg(F)$ and $j = p^s r$, with $p \nmid r$, by Lemma 4.1 we have

$$(20) \quad \beta_i + iB < \beta_j + jB \text{ for } \rho \text{ large enough.}$$

Then there exists ρ_{ij} such that for every $\sigma > \rho_{ij}$ we have

$$(21) \quad \beta_i + i\gamma_\rho < \beta_j + j\gamma_\sigma \text{ for every } \rho < \lambda.$$

Take σ such that (21) is satisfied for every $i = p^s$ and $j = p^s r$, with $p \nmid r$. Write

$$I = \{l \mid 1 \leq l \leq d \text{ such that } l = p^i \text{ for some } i \in \mathbb{N}\}$$

and

$$J := \{1, \dots, d\} \setminus I.$$

Then, for every $j \in J$ there exists $i \in I$ such that

$$\beta_i + i\gamma_\rho < \beta_j + j\gamma_\sigma \text{ for every } \rho < \lambda.$$

This means that for every $\rho > \sigma$ we have

$$(22) \quad \nu \left(\sum_{j \in J} \partial_j L(h_\sigma)(h_\rho - h_\sigma)^j \right) \geq \min_{j \in J} \{\beta_j + j\gamma_\sigma\} > \min_{i \in I} \{\beta_i + i\gamma_\rho\} = \beta_b + i\gamma_b.$$

In order to prove **(i)**, take $\theta > \sigma$ and consider the polynomial

$$F_p(x) = L(h_\theta) + \sum_{i \in I} \partial_i L(h_\theta)(Q - h_\theta)^i =: L_p(Q).$$

Then

$$L(h_\rho) - L_p(h_\rho) = \sum_{j \in J} \partial_j L(h_\theta)(h_\rho - h_\theta)^j.$$

Consequently,

$$\nu(L(h_\rho) - L_p(h_\rho)) > \beta_b + b\gamma_\rho = \nu(L(h_\rho)).$$

The last equality follows from the proof of Proposition 3.2 (as observed in Remark 3.3). Hence,

$$\nu(L_p(h_\rho)) = \beta_b + b\gamma_\rho.$$

If $r \notin I$, then $\deg_X(F_p) < d$ so we apply Lemma 3.2 to obtain that $F_p \in S_n$. This is a contradiction to the minimality of the degree of F in S_n . Hence, $\deg_X(F_p) = d$ and F_p is monic. Since $\deg_X(F_p) \leq d$, by Lemma 3.2 we obtain that $F_p \in S_n$ and so is a limit key polynomial for Ψ_n .

In order to prove (ii), for each $i \geq 0$ take

$$a_i = a_{\theta 0}(\partial_i L(h_\theta))$$

and

$$\overline{F}_p := \sum_{i \in I \cup \{0\}} a_i Q_\theta^i.$$

By Lemma 4.2 we have

$$\nu_\rho(F_p - \overline{F}_p) > \overline{B} > \nu_\rho(F) \text{ for every } \rho, \theta < \rho < \lambda.$$

As before, we conclude that \overline{F}_p is a limit key polynomial for Ψ_n and this completes the proof. \square

Remark 4.3. One can prove (Proposition 3.5 of [5]) that $a_r = 1$ and in particular $\deg(F) = n \deg_X(F)$.

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