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# A MOSER/BERNSTEIN TYPE THEOREM IN A LIE GROUP WITH A LEFT INVARIANT METRIC UNDER A GRADIENT DECAY CONDITION

ARI AIOLFI, LEONARDO BONORINO, JAIME RIPOLL,

ABSTRACT. We say that a PDE in a Riemannian manifold  $M$  is *geometric* if, whenever  $u$  is a solution of the PDE on a domain  $\Omega$  of  $M$ , the composition  $u_\phi := u \circ \phi$  is also solution on  $\phi^{-1}(\Omega)$ , for any isometry  $\phi$  of  $M$ . We prove that if  $u \in C^1(\mathbb{H}^n)$  is a solution of a geometric PDE satisfying the comparison principle, where  $\mathbb{H}^n$  is the hyperbolic space of constant sectional curvature  $-1$ ,  $n \geq 2$ , and if

$$\limsup_{R \rightarrow \infty} \left( e^R \sup_{S_R} \|\nabla u\| \right) = 0,$$

where  $S_R$  is a geodesic sphere of  $\mathbb{H}^n$  centered at fixed point  $o \in \mathbb{H}^n$  with radius  $R$ , then  $u$  is constant. Moreover, given  $C > 0$ , there is a bounded non-constant harmonic function  $v \in C^\infty(\mathbb{H}^n)$  such that

$$\lim_{R \rightarrow \infty} \left( e^R \sup_{S_R} \|\nabla v\| \right) = C.$$

The first part of the above result is a consequence of a more general theorem proved in the paper which asserts that if  $G$  is a non compact Lie group with a left invariant metric,  $u \in C^1(G)$  a solution of a left invariant PDE (that is, if  $v$  is a solution of the PDE on a domain  $\Omega$  of  $G$ , the composition  $v_g := v \circ L_g$  of  $v$  with a left translation  $L_g : G \rightarrow G$ ,  $L_g(h) = gh$ , is also solution on  $L_g^{-1}(\Omega)$  for any  $g \in G$ ), the PDE satisfies the comparison principle and

$$\limsup_{R \rightarrow \infty} \left( \sup_{g \in B_R} \|\text{Ad}_g\| \sup_{S_R} \|\nabla u\| \right) = 0,$$

where  $\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$  is the adjoint map of  $G$  and  $\mathfrak{g}$  the Lie algebra of  $G$ , then  $u$  is constant.

## 1. INTRODUCTION

Finding a characterization of entire minimal graphs on  $\mathbb{R}^n$  is a long standing problem on Differential Geometry. The first well known result, due to Bernstein [2], asserts that the only entire solutions of the minimal surface equation (MSE) in  $\mathbb{R}^2$  are affine functions. J. Simons [12] extended Bernstein's result to  $\mathbb{R}^n$  for  $2 \leq n \leq 7$  and a celebrated work of Giorgi, Giusti and Bombieri [1] proved that Bernstein's theorem is false in  $\mathbb{R}^n$  for  $n \geq 8$ . A full characterization of entire minimal graphs of  $\mathbb{R}^n$  was given in terms of the gradient at infinity of the solutions: Indeed, J. Moser proved that an entire solution for the MSE in  $\mathbb{R}^n$ ,  $n \geq 2$ , is an affine function if

and only if the norm of the gradient of the solution is bounded (corollary of Theorem 6 of [7]).

Moser's result is not true in the hyperbolic space: There are entire solutions of the MSE in  $\mathbb{H}^n$ ,  $n \geq 2$ , which assume a prescribed continuous non-constant value at infinity and with bounded gradient (it is a consequence, for instance, of Theorem 3.14 of [11]). In this note we prove that if the norm of the gradient converges exponentially to zero, then the solution must be constant.

Our main result applies to the solutions of a quite broad class of partial differential equations, according to the following definitions. Although we don't have a proof, we believe that the exponential decay is also optimal for the MSE.

**Definition 1.** *We say that a partial differential equation (PDE) on a complete Riemannian manifold  $M$  satisfies the comparison principle if, given a bounded domain  $\Omega \subset M$ , if  $u$  and  $v$  are solutions of the PDE in  $\Omega$  and  $u \leq v$  in  $\partial\Omega$  that is*

$$\limsup_k (u(x_k) - v(x_k)) \leq 0$$

for any sequence  $x_k \in \Omega$  that leaves any compact subset of  $\Omega$ , then  $u \leq v$  in  $\Omega$ .

**Definition 2.** *We say that a PDE in a Riemannian manifold  $M$  is a geometric PDE if, whenever  $u$  is a solution of the PDE on a domain  $\Omega$  of  $M$ , the composition  $u_\phi := u \circ \phi$  is also solution on  $\phi^{-1}(\Omega)$  for any isometry  $\phi$  of  $M$ .*

Any PDE depending only on  $u$ ,  $\nabla u$ ,  $\nabla^2 u$  and other geometric operators as  $\Delta$  and  $\text{div}$  are invariant by isometries and hence are geometric PDE's. Those of the form

$$\text{div} \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) + C = 0, \quad C \text{ constant},$$

where  $a \in C^1([0, \infty))$ ,  $a' > 0$ ,  $a(0) = 0$  also satisfy the comparison principle (Proposition 3.1 of [11]). This class includes the minimal surface equation with

$$a(s) = \frac{s}{\sqrt{1+s^2}}$$

and the  $p$ -Laplace PDE where

$$a(s) = s^{p-1}, \quad p > 1.$$

Also the solutions of PDE's of the form  $F(\nabla^2 u) = 0$  studied in [3] satisfy the comparison principle (Lemma B of [3]). Comparison principles for more general geometric PDE's of the form  $F(u, \nabla u, \nabla^2 u) = 0$  are studied in [9]. On another hand, a PDE depending on given function on  $M$  may not be geometric. For example  $\Delta u = f$  is geometric if and only if  $f$  is invariant by the isometries of  $M$ . Hence, if  $M$  is a homogeneous manifold then  $\Delta u = f$  is geometric if and only if  $f$  is constant. In general, PDE's depending only

on the differentiable structure of  $M$  will not be geometric as  $X(X(u)) = 0$ , where  $X$  is a vector field on  $M$ . We prove:

**Theorem 3.** *Let  $\mathbb{H}^n$  be the hyperbolic space of constant sectional curvature  $-1$ ,  $n \geq 2$ . Let  $u \in C^1(\mathbb{H}^n)$  be a solution of a geometric PDE satisfying the comparison principle. If*

$$\limsup_{R \rightarrow \infty} \left( e^{2R} \sup_{S_R} \|\nabla u\| \right) = 0,$$

where  $S_R$  is a geodesic sphere of  $\mathbb{H}^n$  centered at fixed point  $o \in \mathbb{H}^n$  with radius  $R$ , then  $u$  is constant. Moreover, given  $C > 0$ , there is a bounded non constant harmonic function  $v \in C^\infty(\mathbb{H}^n)$  such that

$$\lim_{R \rightarrow \infty} \left( e^R \sup_{S_R} \|\nabla v\| \right) = C.$$

It remains opened the problem of weather the condition on the decay of the gradient can be improved or if there are examples of harmonic functions having faster gradient decay which are not constant.

The first part of Theorem 3 is actually a consequence of a more general result that applies to a Lie group with a left invariant metric and to the solutions of a broader class of partial differential equations which we call *left invariant* PDE's. To state it we first recall and introduce some notations and definitions.

Let  $G$  be a Lie group,  $e$  its neutral element and  $\mathfrak{g} = T_e G$  its Lie algebra. Given  $g \in G$  denote by  $R_g$  and  $L_g$  the right and left translations of  $G$ ,  $R_g(x) = xg$  and  $L_g(x) = gx$ ,  $x \in G$ , and by  $C_g = L_g \circ R_g^{-1}$  the conjugation. Let  $\text{Ad}_g = d(C_g)_e : \mathfrak{g} \rightarrow \mathfrak{g}$  be the adjoint map of  $G$ . Given an inner product  $\langle \cdot, \cdot \rangle_e$  in  $\mathfrak{g}$ , let  $\langle \cdot, \cdot \rangle$  be the left invariant metric of  $G$  determined by  $\langle \cdot, \cdot \rangle_e$ , namely

$$\langle u, v \rangle_g := \langle d(L_g^{-1})_g u, d(L_g^{-1})_g v \rangle_e, \quad g \in G, \quad u, v \in T_g G.$$

**Definition 4.** *We say that a PDE in  $G$  is a left invariant PDE if, whenever  $u$  is a solution of the PDE on a domain  $\Omega$  of  $G$ , the composition  $u_g := u \circ L_g$  is also solution on  $L_g^{-1}(\Omega)$  for any  $g \in G$ .*

Clearly geometric PDE's on  $G$  are left invariant. However, the converse is not true. For example, considering  $\mathbb{R}^n$  as a commutative Lie group, the PDE  $\Delta u = X(u)$ , where  $X$  is a vector field of  $\mathbb{R}^n$  is left invariant but not geometric in general.

Given a bounded subset  $S$  of  $G$  we set

$$S^{-1} = \{g^{-1} \mid g \in S\},$$

and denote by  $B(S)$  the smallest closed geodesic ball of  $G$  containing  $S \cup S^{-1}$ .

**Theorem 5.** *Let  $\Omega$  be a bounded  $C^1$  domain on a Lie group  $G$  with a left invariant metric and let  $u \in C^1(\overline{\Omega})$  be a solution of a left invariant PDE satisfying the comparison principle on  $\Omega$ . Then the following estimates for the gradient hold:*

$$(1) \quad \sup_{\Omega} \|\nabla u\| \leq \sup_{g \in \Omega \cup \Omega^{-1}} \|\text{Ad}_g\|^2 \sup_{\partial\Omega} \|\nabla u\|$$

and

$$(2) \quad \sup_{\Omega} \|\nabla u\| \leq \sup_{g \in B(\Omega)} \|\text{Ad}_g\| \sup_{\partial\Omega} \|\nabla u\|,$$

Clearly  $\|\text{Ad}_g\| = 1$  for any  $g \in G$  if the metric of  $G$  is bi-invariant. In this case, then, the solution satisfies the maximum principle for the gradient that is, it holds the equality in (1).

**Corollary 6.** *Let  $u \in C^1(G)$  be a solution of a left invariant PDE satisfying the comparison principle on a non compact Lie group  $G$  with a left invariant metric. If*

$$\limsup_{R \rightarrow \infty} \left( \sup_{g \in B_{2R}} \|\text{Ad}_g\| \sup_{S_R} \|\nabla u\| \right) = 0,$$

where  $B_R$  is the geodesic ball of  $G$  centered at  $e$  with radius  $R$  and  $S_R = \partial B_R$ , then  $u$  is constant. Moreover, if  $G$  has negative sectional curvature then one can replace  $\sup_{g \in B_{2R}} \|\text{Ad}_g\|$  by  $\sup_{g \in \partial B_{2R}} \|\text{Ad}_g\|$

The last assertion of the Corollary 6 is a consequence of the following general fact, namely, if  $G$  has negative sectional curvature then the norm of the adjoint map satisfies the strong maximum principle (Proposition 11).

A well known paper of J. Milnor [6] studies Lie groups with a left invariant metric. More recently, 3-dimensional Lie groups with a left invariant metric have been classified and an explicit description is given in [8]. Finally we mention that it follows by the Iwasawa decomposition that any symmetric space of noncompact type is isometric to a Lie group with a left invariant metric [4].

## 2. PRELIMINARY FACTS, PROOFS OF THEOREM 5 AND COROLLARY 6

We introduce some notation and prove some preliminary results to be used in Theorem 5 and Corollary 6. Given  $h$  and  $g$  in  $G$ , we denote

$$R(g)_h = \|(R_g)_{\star, h}\|$$

where  $(R_g)_{\star, h}$  is the derivative of  $R_g$  at the point  $h$ .

**Lemma 7.** *The quantity  $R(g)_h$  does not depend on  $h$ .*

So, from now on we denote it

$$R(g) = \|(R_g)_{\star, e}\|$$

and introduce, for a subset  $S$  of  $G$

$$R_S = \max_{g \in S} R(g).$$

*Proof.* We first notice that the left and right translations commute, i.e., if  $g_1, g_2, h \in G$ ,

$$L_{g_1} \circ R_{g_2}(h) = R_{g_2} \circ L_{g_1}(h) = g_1 h g_2.$$

Let  $h, g \in G$ . Then

$$\begin{aligned} R(g)_h^2 &= \sup_{X \in T_h G, \|X\|=1} \langle (R_g)_* X, (R_g)_* X \rangle_{hg} \\ &= \sup_{X \in T_h G, \|X\|=1} \langle (L_{(hg)^{-1}})_* (R_g)_* X, (L_{(hg)^{-1}})_* (R_g)_* X \rangle_e \\ &= \sup_{X \in T_h G, \|X\|=1} \langle (L_{g^{-1}})_* (L_{h^{-1}})_* (R_g)_* X, (L_{g^{-1}})_* (L_{h^{-1}})_* (R_g)_* X \rangle_e \\ (3) \quad &= \sup_{X \in T_h G, \|X\|=1} \langle (L_{g^{-1}})_* (R_g)_* (L_{h^{-1}})_* X, (L_{g^{-1}})_* (R_g)_* (L_{h^{-1}})_* X \rangle_e. \end{aligned}$$

For  $X \in T_h G$ ,  $Z = (L_{h^{-1}})_* X$  belongs to  $T_e G$ ; moreover

$$\|X\|_h = \|(L_{h^{-1}})_* X\|_1$$

and we can rewrite (3) as

$$\begin{aligned} (3) &= \sup_{Z \in T_e G, \|Z\|=1} \langle (L_{g^{-1}})_* (R_g)_* Z, (L_{g^{-1}})_* (R_g)_* Z \rangle_e \\ &= \sup_{Z \in T_e G, \|Z\|=1} \langle (R_g)_* Z, (R_g)_* Z \rangle_g = (R_g)_* e. \end{aligned}$$

□

**Lemma 8.** *If  $g \in G$ ,*

$$R(g) = \|\text{Ad}_g\|.$$

*Proof.* If  $X \in T_e G$ , by definition of the metric

$$\begin{aligned} \langle (R_g)_* X, (R_g)_* X \rangle_g &= \langle (L_{g^{-1}})_* (R_g)_* X, (L_{g^{-1}})_* (R_g)_* X \rangle_e \\ &= \langle \text{Ad}_g(X), \text{Ad}_g(X) \rangle_e. \end{aligned}$$

□

Denote by  $d$  the Riemannian distance in  $G$ . The next lemma is elementary and its proof is therefore omitted:

**Lemma 9.** *Let  $\Omega$  be a  $C^1$  open subset of  $G$ . Let  $u \in C^1(\overline{\Omega})$  and  $x_0 \in \overline{\Omega}$  be given and assume that  $\|\nabla u\|(x_0) \neq 0$ . Let  $\gamma : [0, \varepsilon) \rightarrow \Omega$  be an arc length geodesic such that  $\gamma(0) = x_0$  and*

$$\gamma'(0) = \frac{\nabla u(x_0)}{\|\nabla u(x_0)\|} \text{ or } \gamma'(0) = -\frac{\nabla u(x_0)}{\|\nabla u(x_0)\|}.$$

*Then*

$$\|\nabla u\|(x_0) = \lim_{t \rightarrow 0} \frac{|u(\gamma(t)) - u(x_0)|}{d(\gamma(t), x_0)}.$$

**Lemma 10.** *Let  $a, b, g$  be elements in  $G$ . Denoting by  $d$  the distance on  $G$ , we have*

$$\begin{aligned} d(a.g, b.g) &\leq R(g)d(a, b) \\ d(a, b) &\leq R(g^{-1})d(a.g, b.g). \end{aligned}$$

*Proof.* If  $\gamma$  is a minimizing geodesic between  $a$  and  $b$  then  $d(a, b) = \text{length}(\gamma)$ . Also, since  $R_g \circ \gamma$  is a path between  $a.g$  and  $b.g$ ,

$$d(a.g, b.g) \leq \text{length}(R_g \circ \gamma) \leq R(g) \text{length}(\gamma) = R(g)d(a, b).$$

We have thus proved the first inequality of the lemma. We derive

$$d(a, b) = d((a.g)g^{-1}, (b.g)g^{-1}) \leq R(g^{-1})d(a.g, b.g)$$

which proves the second inequality.  $\square$

*Proof of Theorem 5.* The proof is an extension of Lemma 12.7 of [5] (see also [13]). Let  $\Omega$  be a bounded  $C^1$  domain on  $G$  with and let  $u \in C^1(\overline{\Omega})$  be a solution of a left invariant PDE satisfying the comparison principle. Take

$$k = \sup \left\{ \frac{|u(x) - u(z)|}{d(x, z)}, x \in \Omega, z \in \partial\Omega \right\}.$$

From Lemma 9, for the estimate (1) it is then enough to prove that

$$(4) \quad |u(x) - u(y)| \leq k \sup_{z \in \Omega \cup \Omega^{-1}} \|\text{Ad}_z\|^2 d(x, y)$$

for all  $x, y \in \Omega$ . Given  $x_1, x_2 \in \Omega$ , set  $z = x_1 \cdot x_2^{-1}$ ,

$$\Omega_z = \{z^{-1} \cdot x \in G \mid x \in \Omega\} = \{x \in G \mid z \cdot x \in \Omega\}$$

and define  $u_z \in C^1(\overline{\Omega}_z)$  by  $u_z(x) = u(zx)$ . We have  $\Omega \cap \Omega_z \neq \emptyset$  since  $x_2$  belongs to both  $\Omega$  and  $\Omega_z$ . By the comparison principle

$$\sup \{u(x) - u_z(x) \mid x \in \Omega \cap \Omega_z\} = \sup \{u(x) - u_z(x) \mid x \in \partial(\Omega \cap \Omega_z)\}.$$

Then, in particular

$$u(x_2) - u(x_1) = u(x_2) - u_z(x_2) \leq \sup \{u(x) - u(zx) \mid x \in \partial(\Omega \cap \Omega_z)\}.$$

Let  $x_0 \in \partial(\Omega \cap \Omega_z)$  be such that

$$u(x_0) - u(zx_0) = \sup \{u(x) - u(zx) \mid x \in \partial(\Omega \cap \Omega_z)\}.$$

If  $x_0 \in \partial(\Omega \cap \Omega_z)$  then either  $x_0 \in \partial\Omega$  or  $zx_0 \in \partial\Omega$ . It then follows from the hypothesis that

$$u(x_2) - u(x_1) \leq kd(x_0, zx_0).$$

Now, using Lemma 10 twice, we have

$$\begin{aligned} d(x_0, (x_1x_2^{-1})x_0) &\leq R(x_0)d(1, x_1x_2^{-1}) \leq R(x_0)R(x_2^{-1})d(x_2, (x_1x_2^{-1})x_2) \\ &= R(x_0)R(x_2^{-1})d(x_2, x_1) \\ &\leq \sup_{z \in \Omega \cup \Omega^{-1}} \|\text{Ad}_z\|^2 d(x_2, x_1) \end{aligned}$$

proving (1).

For proving (2) we assume, wlg, that the smallest geodesic ball  $B$  of radius  $R$  containing  $\Omega \cup \Omega^{-1}$  is centered at the neutral element  $e$  of  $G$ . Then

$$d(x_0, (x_1 x_2^{-1})x_0) = d(x_2 x_2^{-1} x_0, (x_1 x_2^{-1})x_0) \leq R(x_2^{-1} x_0) d(x_2, x_1).$$

Observe that

$$d(x_2^{-1} x_0, e) = d(x_0, x_2) \leq 2R$$

since

$$x_0, x_2 \in \bar{\Omega} \subset B$$

and then

$$R(x_2^{-1} x_0) \leq \sup_{z \in B} \|\text{Ad}_z\|,$$

concluding with the proof of Theorem 5.  $\square$

**Proof of Corollary 6.** Given  $R > 0$ , if  $g \in S_R$  then

$$R = d(e, g) = d(g^{-1}, e) = d(e, g^{-1})$$

so that  $S_R^{-1} = S_R$ . It follows that  $B_R^{-1} = B_R$ , where  $B_R$  is the geodesic ball centered at  $e$  with radius  $R$ . Then, if  $u \in C^1(G)$  is an entire solution satisfying the hypothesis of Corollary 6, it follows from Theorem 5 that

$$\max_{B_R} \|\nabla u\| \leq \max_{g \in B_R} \|\text{Ad}_g\| \max_{S_R} \|\nabla u\|,$$

from which the proof follows.  $\square$

We close this section by proving a strong maximum principle for the adjoint map:

**Proposition 11.** *Let  $G$  be a Lie group with negative sectional curvature and let  $\Lambda$  be any open subset of  $G$ . Then*

$$\|\text{Ad}_h\| < \sup_{g \in \partial\Lambda} \|\text{Ad}_g\|$$

for all  $h \in \Lambda$ . In particular

$$\sup_{g \in \Lambda} \|\text{Ad}_g\| = \sup_{g \in \partial\Lambda} \|\text{Ad}_g\|.$$

*Proof.* By contradiction, assume that

$$\|\text{Ad}_h\| = \sup_{g \in \Lambda} \|\text{Ad}_g\|$$

for some  $h \in \Lambda$ . There exists  $x \in T_e G$ ,  $\|x\| = 1$ , such that  $\|\text{Ad}_h\| = \|\text{Ad}_h(x)\|$ . Let  $X$  be the right invariant vector field of  $G$  such that  $X(e) = x$ . Choose  $u \in T_h G$ ,  $\|u\| = 1$ , such that  $u \neq X(h)$  and let  $\gamma(t)$ ,  $t \geq 0$ , be the



geodesic parametrized by arc length in  $G$  such that  $\gamma(0) = h$  and  $\gamma'(0) = u$ . Then

$$(5) \quad \left. \frac{d}{dt} \|\text{Ad}_{\gamma(t)}(x)\|^2 \right|_{t=0} = 0$$

$$(6) \quad \left. \frac{d^2}{dt^2} \|\text{Ad}_{\gamma(t)}(x)\|^2 \right|_{t=0} \leq 0.$$

We have

$$\begin{aligned} \|\text{Ad}_{\gamma(t)}(x)\|^2 &= \langle \text{Ad}_{\gamma(t)}(x), \text{Ad}_{\gamma(t)}(x) \rangle \\ &= \left\langle d\left(L_{\gamma(t)}^{-1}\right)_{\gamma(t)} \left( d\left(R_{\gamma(t)}\right)_e(x), d\left(L_{\gamma(t)}^{-1}\right)_{\gamma(t)} \left( d\left(R_{\gamma(t)}\right)_e(x) \right) \right\rangle \\ &= \left\langle d\left(R_{\gamma(t)}\right)_e(x), d\left(R_{\gamma(t)}\right)_e(x) \right\rangle, \quad t \geq 0. \end{aligned}$$

Then

$$\|\text{Ad}_{\gamma(t)}(x)\|^2 = \langle X(\gamma(t)), X(\gamma(t)) \rangle = \|X(\gamma(t))\|^2, \quad t \geq 0.$$

Since right invariant vector fields are Killing fields and Killing fields restricted to geodesics are Jacobi vector fields,

$$J(t) := X(\gamma(t))$$

is a Jacobi field along  $\gamma$ ,  $t \geq 0$ . Thus,  $\|\text{Ad}_{\gamma(t)}(x)\|^2$  is equal to the square of the norm of the Jacobi field  $J(t)$  along  $\gamma$  satisfying the initial conditions

$$\begin{aligned} J(0) &= X(h) \\ J'(0) &= \nabla_{\gamma'(0)} X = \nabla_u X. \end{aligned}$$

We have, from (5)

$$\left. \frac{d}{dt} \|J(t)\|^2 \right|_{t=0} = 2 \langle J'(0), J(0) \rangle = 0.$$

And, using the Jacobi equation,

$$\begin{aligned} \frac{d^2}{dt^2} \|J(t)\|^2 &= 2 \langle J''(t), J(t) \rangle + 2 \langle J'(t), J'(t) \rangle \\ &= -2 \langle R(\gamma', J)\gamma', J(t) \rangle + 2 \|J'(t)\|^2. \end{aligned}$$

Since  $u \neq X(h)$  the vector fields  $J(t) = X(\gamma(t))$  and  $\gamma'(t)$  are linearly independent along  $\gamma$  and we then have

$$\frac{d^2}{dt^2} \|J(t)\|^2 = -2K(\gamma', J) \|\gamma' \wedge J\|^2 + 2 \|J'\|^2.$$

Since  $K < 0$  it follows that

$$\left. \frac{d^2}{dt^2} \|J(t)\|^2 \right|_{t=0} > 0$$

contradicting (6). This proves the proposition.  $\square$

## 3. THE HYPERBOLIC SPACE. PROOF OF THEOREM 3.

We begin by calculating the norm of the adjoint map in the hyperbolic space to apply Corollary 6. We present an explicit construction of the Lie group structure of the hyperbolic space, that comes from the Iwasawa decomposition [4].

In the half-space model

$$\mathbb{H}^n = \{(x_1, \dots, x_n), | x_n > 0\}, ds^2 = \frac{\delta_{ij}}{x_n^2},$$

of the hyperbolic  $n$ -dimensional space,  $n \geq 2$ , given  $s > 0$  and  $t := (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$ , define

$$\begin{aligned} a_s, n_t &: \mathbb{H}^n \rightarrow \mathbb{H}^n, \\ a_s(x_1, \dots, x_n) &= s(x_1, \dots, x_n) \\ n_t(x_1, \dots, x_n) &= (x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n). \end{aligned}$$

Set

$$G := AN = \{a_s \circ n_t \mid s > 0, t \in \mathbb{R}^{n-1}\}$$

where

$$\begin{aligned} A &= \{a_s, s > 0\} \\ N &= \{n_t, t \in \mathbb{R}^{n-1}\}. \end{aligned}$$

Given  $p \in \mathbb{H}^n$  there is one and only one  $g_p \in G \subset \text{Iso}(\mathbb{H}^n)$  such that  $p = g_p((0, \dots, 0, 1))$ . Indeed: If  $p = (x_1, \dots, x_n) \in \mathbb{H}^n$  define  $n := n_{(x_1, \dots, x_{n-1}, 0)}$ ,  $a := a_{x_n} \in \text{Iso}(\mathbb{H}^n)$ , that is

$$\begin{aligned} n(z_1, \dots, z_n) &= (z_1 + x_1, \dots, z_{n-1} + x_{n-1}, z_n) \\ a &= x_n(z_1, \dots, z_n), \quad (z_1, \dots, z_n) \in \mathbb{H}^n. \end{aligned}$$

Then, taking

$$(7) \quad g_p = n \circ a$$

we have

$$g_p(0, \dots, 0, 1) = n(a(0, \dots, 0, 1)) = n(0, \dots, 0, x_n) = (x_1, \dots, x_n) = p.$$

Note that  $n$  and  $a$  are not uniquely determined by  $p$ , but  $g_p$  is.

One may see that with the operation

$$p \cdot q := (g_p \circ g_q)((0, \dots, 0, 1))$$

$\mathbb{H}^n$  is a Lie group (a solvable Lie group indeed. This is a general fact that holds for symmetric spaces of non compact type ([4], Chapter VI)). Moreover, given  $p \in \mathbb{H}^n$  we have, for any  $q \in \mathbb{H}^n$ ,

$$L_p(q) = p \cdot q = (g_p \circ g_q)((0, \dots, 0, 1)) = g_p(g_q((0, \dots, 0, 1))) = g_p(q)$$

that is  $L_p = g_p$ . Since  $g_q \in \text{Iso}(\mathbb{H}^n)$  it follows that the left translation  $L_p$  is an isometry of  $\mathbb{H}^n$  with respect to the hyperbolic metric that is, the

hyperbolic metric is left invariant with respect to the Lie group structure of  $\mathbb{H}^n$ .

**Proposition 12.** *Consider  $\mathbb{H}^n$  as a Lie group and let  $e$  be its neutral element. If  $B_R$  is the closed geodesic ball of  $\mathbb{H}^n$  centered at  $e$  with radius  $R$  then*

$$(8) \quad \max_{g \in B_R} \|\text{Ad}_g\| = \cosh R + \sinh R.$$

*Proof.* We claim that it is enough to consider the 2-dimensional case. Indeed: Let  $g \in B_R$  and  $x \in T_e \mathbb{H}^n$  be such that

$$\|\text{Ad}_g(x)\| = \max_{h \in B_R} \|\text{Ad}_h\|.$$

Let  $y \in T_e \mathbb{H}^n$  be a nonzero vector tangent to the geodesic from  $e$  to  $g$ . There exists a totally geodesic hyperbolic plane  $\mathbb{H}^2$  of  $\mathbb{H}^n$  such that  $e \in \mathbb{H}^2$  and  $x, y \in T_e \mathbb{H}^2$ . Since  $\mathbb{H}^2$  is a Lie subgroup of  $\mathbb{H}^n$  we have

$$\|\text{Ad}_g(x)\| = \max_{h \in D_R} \|\text{Ad}_h\|$$

where  $D_R = B_R \cap \mathbb{H}^2$  is the closed geodesic disk centered at  $e$  with radius  $R$ . It is clear that the maximum of the norm of the adjoint map does not depend on hyperbolic plane containing  $e$ . This proves our claim.

Considering the half-plane model for  $\mathbb{H}^2$  with  $e = (0, 1)$ , given  $p \in \mathbb{H}^2$ , the conjugation  $C_p : \mathbb{H}^2 \rightarrow \mathbb{H}^2$  is given by

$$(9) \quad C_p(q) = (g_p \circ g_q \circ g_p^{-1})(0, 1) = g_p(g_q(g_p^{-1}(0, 1))).$$

Expanding (9) we arrive to

$$C_{(x,y)}(z, w) = (-xw + yz + x, w)$$

from which we obtain, at a given  $X := (a, b) \in T_{(0,1)} \mathbb{H}^2$ ,

$$\text{Ad}_{(x,y)}(a, b) = d(C_{(x,y)})_e(a, b) = (-xb + ya, b).$$

Since at  $(0, 1)$  the hyperbolic metric coincides with the Euclidean metric and since  $\text{Ad}_{(x,y)}(X) \in T_{(0,1)} \mathbb{H}^2$  we obtain

$$(10) \quad \|\text{Ad}_{(x,y)}(X)\| = \sqrt{(-xb + ya)^2 + b^2}.$$

Since the identity of  $\mathbb{R}_+^2$  is a conformal map between the Euclidean and hyperbolic geometries, the hyperbolic geodesic circle in the half plane model of  $\mathbb{H}^2$  centered at  $(0, 1)$  with hyperbolic radius  $R$  is also an Euclidean circle. Moreover, because an Euclidean symmetry with respect to a vertical line is also a hyperbolic isometry, the Euclidean center of the circle is at some point  $(0, y_0)$  of the vertical line  $x = 0$ . The Euclidean circle is parametrized by

$$(11) \quad (0, y_0) + r(\cos \theta, \sin \theta), \quad \theta \in [0, 2\pi),$$

where  $r$  is the Euclidean radius. Since  $\gamma(t) = (0, e^t)$  is an arc length hyperbolic geodesic such that  $\gamma(0) = (0, 1)$  we have  $t = d_{\mathbb{H}^2}((0, 1), (0, e^{\pm t}))$ . In particular

$$R = d_{\mathbb{H}^2}((0, 1), (0, e^R)) = d_{\mathbb{H}^2}((0, 1), (0, e^{-R})).$$

It follows that the points  $(0, e^R)$  and  $(0, e^{-R})$  are both in the hyperbolic circle centered at  $(0, 1)$  and with hyperbolic radius  $R$ . Then the Euclidean circle must contain both points  $(0, e^R)$  and  $(0, e^{-R})$ . Since these points are in the same vertical straight line, the Euclidean center of Euclidean circle is

$$\frac{(0, e^R) + (0, e^{-R})}{2} = (0, \cosh R).$$

And the Euclidean radius  $r$  of this hyperbolic circle is just half of the Euclidean distance between the points  $(0, e^R)$  and  $(0, e^{-R})$  that is,

$$r = \frac{1}{2}d_{\mathbb{R}^2}((0, e^R), (0, e^{-R})) = \frac{e^R - e^{-R}}{2} = \sinh R.$$

It follows from (11) that the geodesic disk  $D_R$  of  $\mathbb{H}^2$  centered at  $(0, 1)$  with radius  $R$  is given by

$$D_R = \{(\sinh r \cos \theta, \cosh r + \sinh r \sin \theta) \mid \theta \in \mathbb{R}, 0 \leq r \leq R\}.$$

From (10),

$$\|\text{Ad}_{(x,y)}(X)\|^2 = (\cosh r + \sinh r \sin \theta)^2 \left( \frac{-b \sinh r \cos \theta}{\cosh r + \sinh r \sin \theta} + a \right)^2 + b^2$$

and we may see that the biggest value of right hand side of the last inequality occurs at  $\theta = \pi/2$  and  $a = 1, b = 0$ . From Proposition 11 (which, in the case of the hyperbolic space, can also be directly confirmed from the expression above) we obtain (8).  $\square$

*Proof of Theorem 3.* The first part of the proof is a direct consequence of Corollary 6 and Proposition 12. For the second part consider a polar coordinate system  $(r, \theta, \varphi_1, \dots, \varphi_{n-2})$  of  $\mathbb{H}^n$  centered at a point  $o \in \mathbb{H}^n$  that is,  $\Theta = (\theta, \varphi_1, \dots, \varphi_{n-2})$  are spherical coordinates of the unit sphere  $\mathbb{S}^{n-1}$  centered at the origin of  $T_o\mathbb{H}^n$  and  $p \in \mathbb{H}^n \setminus \{o\}$  is parametrized by

$$p = \exp_o(r\Theta), \quad r > 0, \quad \Theta \in \mathbb{S}^{n-1}.$$

Let  $v$  be the function that depends only on  $r$  and  $\theta$  given by

$$v(r, \theta) = \frac{C(n-1)}{2} \cdot \frac{\int_0^r (\sinh s)^{n-1} ds}{(\sinh r)^{n-1}} \cos \theta.$$

Remind that in this coordinate system, the metric has the form

$$\begin{aligned} ds^2 = & dr^2 + \sinh^2 r d\theta + \sinh^2 r \sin^2 \theta d\varphi_1 + \sinh^2 r \sin^2 \theta \sin^2 \varphi_1 d\varphi_2 \\ & + \dots + \sinh^2 r \sin^2 \theta \sin^2 \varphi_1 \dots \sin^2 \varphi_{n-3} d\varphi_{n-2} \end{aligned}$$

and, since  $v = v(r, \theta)$ , the Laplacian of  $v$  can be expressed by

$$\Delta v = \frac{\partial^2 v}{\partial r^2} + (n-1) \coth r \frac{\partial v}{\partial r} + (n-2) \frac{\cot \theta}{\sinh^2 r} \cdot \frac{\partial v}{\partial \theta} + \frac{1}{\sinh^2 r} \cdot \frac{\partial^2 v}{\partial \theta^2}.$$

Hence, by a straight calculation, we have that  $\Delta v = 0$ . Moreover, since  $v$  depends only on  $r$  and  $\theta$  and the fields  $E_1 := \frac{\partial}{\partial r}$  and  $E_2 := \frac{\partial}{\partial \theta}$  are orthogonal, we have

$$\begin{aligned} \|\nabla v(r, \theta)\|^2 &= \frac{|E_1(v)|^2}{\|E_1\|^2} + \frac{|E_2(v)|^2}{\|E_2\|^2} = \left( \frac{|\partial_r v|^2}{1} + \frac{|\partial_\theta v|^2}{\sinh^2 r} \right) \\ &= \left( \frac{C(n-1)}{2} \right)^2 \left( 1 - (n-1) \cosh r (\sinh r)^{-n} \int_0^r (\sinh s)^{n-1} ds \right)^2 \cos^2 \theta \\ &\quad + \left( \frac{C(n-1)}{2} \right)^2 \left( (\sinh r)^{-n} \int_0^r (\sinh s)^{n-1} ds \right)^2 \sin^2 \theta. \end{aligned}$$

Then,

$$\|\nabla v(r, \theta)\| = \frac{C}{2(1 + \cosh r)} \quad \text{for any } r \geq 0 \text{ if } n = 2$$

and

$$\|\nabla v(r, \theta)\| \approx C e^{-r} \sin \theta \quad \text{as } r \rightarrow +\infty \text{ if } n > 2.$$

In both cases

$$\sup_{S_R} \|\nabla v(R, \theta)\| \approx C e^{-R} \quad \text{as } R \rightarrow +\infty.$$

Therefore,  $v$  is a bounded non constant harmonic function such that

$$\lim_{R \rightarrow \infty} \left( e^R \sup_{S_R} \|\nabla v\| \right) = C,$$

completing the proof.  $\square$

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