



**HAL**  
open science

# Finite volume approximation and well-posedness of conservation laws with moving interfaces under abstract coupling conditions

Boris Andreianov, Abraham Sylla

## ► To cite this version:

Boris Andreianov, Abraham Sylla. Finite volume approximation and well-posedness of conservation laws with moving interfaces under abstract coupling conditions. 2023. hal-03873169v2

**HAL Id: hal-03873169**

**<https://hal.science/hal-03873169v2>**

Preprint submitted on 21 Jan 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Finite volume approximation and well-posedness of conservation laws with moving interfaces under abstract coupling conditions

Boris Andreianov<sup>1,2\*</sup> and Abraham Sylla<sup>3\*</sup>

<sup>1\*</sup>Institut Denis Poisson CNRS UMR 7013, Université de Tours, Université d'Orléans, Parc Grandmont, 37200, Tours, France.

<sup>2</sup>Peoples' Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation.

<sup>3\*</sup>Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, , Via R. Cozzi 53, 20125, Milano, Italy.

\*Corresponding author(s). E-mail(s):

[boris.andreianov@univ-tours.fr](mailto:boris.andreianov@univ-tours.fr); [abraham.sylla@unimib.it](mailto:abraham.sylla@unimib.it);

## Abstract

Scalar conservation law  $\partial_t \rho(t, \mathbf{x}) + \partial_x(\mathbf{f}(t, \mathbf{x}, \rho)) = \mathbf{0}$  with a flux  $\mathbf{C}^1$  in the state variable  $\rho$ , piecewise  $\mathbf{C}^1$  in the  $(t, \mathbf{x})$ -plane admits infinitely many consistent notions of solution which differ by the choice of interface coupling. Only the case of the so-called vanishing viscosity solutions received full attention, while different choice of coupling is relevant in modeling situations that appear, *e.g.*, in road traffic and in porous medium applications. In this paper, existence of solutions for a wide set of coupling conditions is established under some restrictions on  $\mathbf{f}$ , *via* a finite volume approximation strategy adapted to slanted interfaces and to the presence of interface crossings. The notion of solution, restated under the form of an adapted entropy formulation which is consistently approximated by the numerical scheme, implies uniqueness and stability of solutions. Numerical simulations are presented to illustrate the reliability of the scheme.

**Keywords:** Conservation laws, Discontinuous flux, Moving Interface, Interface coupling conditions, Finite Volume Scheme, Godunov Flux, Existence of Solutions, Well-posedness

**MSC Classification:** 35L65 , 65M08 , 65M12 , 35A01 , 35A02

# 1 Introduction

In the domain  $\Omega = ]0, +\infty[ \times \mathbb{R}$ , consider the formal Cauchy problem

$$\begin{cases} \partial_t \rho(t, x) + \partial_x (\mathbf{f}(t, x, \rho(t, x))) = 0 & (t, x) \in \Omega \\ \rho(0, x) = \rho_o(x) & x \in \mathbb{R}, \end{cases} \quad (1)$$

where  $\rho_o \in \mathbf{L}^\infty(\mathbb{R})$  and  $\mathbf{f} : \Omega \times \mathbb{R} \mapsto \mathbb{R}$  is a Carathéodory function, as specified below. The physical range of values of the state variable may be restricted to a compact subset  $[a, b] \subset \mathbb{R}$  in many applications.

## 1.1 A state of the art for well-posedness of (1)

We start by exposing the important PDE aspects of the problem and highlight the absence, in the literature, of general *existence* results in the situations where the issues of solution admissibility and of uniqueness are well understood.

In the case where  $\mathbf{f}$  extends to a locally Lipschitz continuous on  $\bar{\Omega} \times \mathbb{R}$  function, and under some restrictions on the growth of  $\mathbf{f}$  in  $\rho$  to ensure global in time existence (see in particular [1]), the definition of entropy solutions in the sense of Kruzhkov [2] is the cornerstone of the well-posedness theory. Beyond this essentially classical setting, discontinuous-flux problems and related problems of conservation laws with embedded interfaces appear in applications such as sedimentation, porous media, road traffic, *etc.* By "discontinuous-flux" we mean only problems where the flux is continuous (and even locally Lipschitz or even  $\mathbf{C}^1$ ) in the state variable  $\rho$  while it may present discontinuities in the  $(t, x)$ -plane. Sense can be given to (1) (in particular, a weak formulation can be written) in the general situation where  $\mathbf{f}$  is a Carathéodory function (measurable in  $(t, x)$ , continuous in  $\rho$ ); to the best of the authors' knowledge, no consistent theory has been constructed yet in this very general setting. The most general setting where partial existence and uniqueness results were established concerns fluxes  $\mathbf{f}$  that are either regulated in  $(t, x)$  (like in [3]) or at least  $\mathbf{BV}_{\text{loc}}$  in  $(t, x)$ , for all  $\rho \in \mathbb{R}$ , see in particular [4, 5]. To the authors' knowledge, the existence results under the  $\mathbf{BV}$  kind assumption on the flux, as well as for regulated flux in [3], concern exclusively solutions constructed by the classical vanishing viscosity strategy going back to Rayleigh, Hopf and Kruzhkov. The early concepts of admissibility such as the minimal jump condition ([6]) or the  $\Gamma$ -condition ([7], see also [8]) at interfaces select the vanishing viscosity solution, *cf.* [9]; they were elaborated in the context of sedimentation applications. However, vanishing viscosity limits are not always appropriate from the modeling perspective. Optimal entropy solutions (*i.e.* solutions which maximize the flow across interfaces) were a second class of solutions identified as relevant in applications in porous media, see [10–12]; they also appear in the context of road traffic with varying road conditions (see [13] and references therein). It has been realized that the optimal solutions may be different from the vanishing viscosity limits (see, in particular, [4, Sect. 7]). More generally, co-existence of infinitely many different kinds of solutions, equally consistent from the purely mathematical standpoint, was explicitly pointed out in [11]. Later

on, the different solution notions ( $(A, B)$ -connections) put forward in [11] were linked to different vanishing capillarity limits in the porous medium context [14, 15]. Moreover, fully analogous situation arose from modeling of road traffic by a classical, continuous-flux Lighthill-Whitham-Richards equation with point constraints on the flux ([16, 17]) where different solvers stem from different levels of constraint. Thus, different notions of solution correspond to different modeling assumptions at the interface, quite analogously to what happens when one prescribes different boundary conditions to a given PDE (we refer to [18] for the viewpoint of "Interface Coupling Conditions" and highlighted analogies with nonlinear boundary conditions for scalar conservation laws [19]). For this reason, writing (1) is formal, even having in mind a Kruzhkov-like entropy formulation: indeed, one needs to specify the expected (expected, given the underlying modeling context) behavior of solutions at interfaces which are the jump sets of  $(t, x) \mapsto \mathbf{f}(t, x, \rho)$ . From this perspective, these jump sets should be common for all values of  $\rho$ . A rather general set of assumptions on  $\mathbf{f}$  that allows to interpret (1) as a discontinuous-flux conservation law is elaborated in [20]. Coupling at interfaces and the associated uniqueness analysis for the reference setting of [20] (see also [4, 21] for simpler but still rather complex variants) are mimicked from the model case, which we now discuss.

The model case for the general problem (1) features the flux function  $\mathbf{f}$  discontinuous with respect to the space variable across the interface  $\{x = 0\}$ . The expression for such flux reads as follows:

$$\mathbf{f}(t, x, \rho) = \begin{cases} g(\rho) & \text{if } x < 0 \\ f(\rho) & \text{if } x > 0. \end{cases} \quad (2)$$

Problem (1)-(2) has been the main playground for understanding the issue of admissibility of solutions and of their uniqueness (see [22, 23] for a different line of research on this topic which applies to particular flux configurations, and where discontinuities need not to arrange along interfaces). General structure of interface coupling leading to  $\mathbf{L}^1$ -contractive solution semigroup for (1), (2) has been described in [9] in terms of " $\mathbf{L}^1 D$  germs" recalled below. It gave a common framework to a number of uniqueness arguments developed in the literature ([4, 6, 7, 11, 24-26]). It has been exploited for the sake of uniqueness and stability analysis of road traffic models with point constraints [17] and of a non-conservative fluid-particle interaction model [27, 56, 57] involving moving interfaces. The very general uniqueness result of [20] highlights the fact that abstract "germ formulations" readily lead to uniqueness of the associated solutions, far beyond the model problem case (in this respect, let us underline that [20] deals with the multi-dimensional analogue of (1)). For the sake of completeness, let us point out a very different uniqueness result of [3], which requires much weaker than  $\mathbf{BV}$  regularity assumptions on the  $(t, x)$ -dependence of  $\mathbf{f}$  but is only applicable to vanishing viscosity solutions.

In what concerns existence for (1) - for a given choice of interface coupling conditions, as highlighted above - the situation is far less explored. We refer in particular to [20, Remark 2.10]. Typically, to prove existence for

(1) having in mind a specific interface coupling, it is required to construct sequences of approximate solutions, pass to the limit using the appropriate compactness structures, and obtain at the limit entropy formulations encoding, in particular, the coupling expected at the interfaces. To the best of the authors' knowledge only two situations were explored systematically beyond the model flux (2) case. A huge majority of works on the subject is concerned with the vanishing viscosity interface coupling. The existence results of, e.g., [3, 4, 21, 24, 28–32] are based either on a vanishing viscosity approximation (which can be a very tricky one, see [21]) or on a numerical finite volume approximation which enforces, at the numerical level, the continuity of the state variable  $\rho$  at interfaces. Actually, the notion of vanishing viscosity solution corresponds to an implicit assumption of continuity - up to an interface layer, like for the case of Bardos-LeRoux-Nédélec boundary-value problems - of solution  $\rho$  at interfaces ([18, 21, 33]), and such continuity is explicitly or implicitly incorporated into the above mentioned approximation schemes. Another natural construction procedure is the smoothing of the discontinuities of  $\mathbf{f}$  (see, e.g., [34] in the model case, see also [35] for a more elaborate situation coming from vanishing viscosity approximation of a triangular system of conservation laws). It may produce relevant solutions in particular situations, but it cannot be used to produce solutions for any kind of coupling. The adapted viscosity procedure of [9] was a rather artificial attempt to produce solutions with more general interface coupling, but it has not been extended beyond the model case. Note in passing that the above viscosity, smoothing and discretization methods are applicable for multi-dimensional generalization of (1), and some of the above references deal with multiple space dimensions. In one space dimension, wave-front tracking approximations were used for constructing solutions ([36] for the vanishing viscosity case, [37] for the general setting fully comparable to [9]) but beyond the model situations, their use for problems of the kind (1) was mainly restricted to road traffic models in concrete situations. Road traffic with point flux limitations at interfaces (which is a specific choice of coupling, relevant in traffic modeling) is the second context where solutions to (1) were constructed for slanted, curved and possibly crossing interfaces either through wave-front tracking or through finite volume approximation. The latter is addressed in detail in the recent work [38]; we refer to its introduction for a set of references to related works.

From the viewpoint of the PDE theory adopted in this paragraph, the present contribution addresses - via numerical analysis tools - the problem of existence, beyond the known situations. We extend the approximation strategy elaborated in [38] for handling very general interface couplings; to do so, we find it useful to modify (in a way equivalent to the previously known formulations) the concept of admissible solution to (1).

## 1.2 The general objectives, the key contribution and the simplifying assumption

Firstly, our goal is to provide a convergent finite volume approach to the construction of solutions to (1) with piecewise  $\mathbf{C}^1$  dependence on  $(t, x)$  and with general interface coupling not covered by known existence results. To this end, we exploit the constructions put forward in [38] (the general structure

of the scheme, the treatment of interfaces and of interface crossings, and the local compactness arguments) which were developed for a particular interface coupling relevant to traffic models. The coupling formalism of [38] does not extend to the general situation we have in mind. To achieve our general result, we complement the elements devised and detailed in [38] with an original treatment of terms supported at the interfaces. To make the idea apparent and to circumvent heavy technicalities of a straightforward analysis, we first develop the formalism and the convergence analysis in a simple model case. Then we operate a series of reductions to show that the analysis in the general situation can be reduced, in a non-technical way, to the simple model situation.

Moreover, the numerical strategy we devise here keeps in mind a more elaborate scenario. In Section 5 we give two numerical illustrations which are extracted from complex PDE-ODE coupled problems known in the literature. Let us highlight the fact that, although the interfaces in the setting of our present contribution are given beforehand, such more complex problems where the interface evolves along with the solution itself are tractable within the same numerical approach. We refer to [39] for a simple example and to [40] for further work in this direction, based on the strategy of the present contribution.

Secondly, the paper contributes to a better theoretical understanding of the PDE problem at hand. Focusing on a finite volume approach leads us to reformulate the underlying solution notion in terms most suitable for numerical analysis of the problem. Indeed, the numerical scheme consistent with the desired interface coupling should use specific fluxes at the locations of the interfaces; for the sake of maximal generality, we use the Godunov flux associated with the underlying *germ* (cf. [9] and Section 2.2). Since the germ depends on the inclination  $\dot{y}(t)$  of the interface  $x = y(t)$  and the interfaces are approximated, we require a mild restriction on the family of germs prescribing the interface coupling. Identification of the proper Carathéodory structure for families of germs is a useful consequence of our approach. The essential tool for convergence analysis is the adapted entropy formulation incorporating remainder terms (cf. [9]). The cornerstone of our contribution is the identification of a novel form of the remainder term. It is crucial for our purposes because it is directly linked to the Godunov numerical approximation at interfaces.

Accordingly, the secondary goal achieved (partially) in this paper is to complement the general theory of discontinuous-flux conservation laws with an existence result in a wide setting that was never considered in the literature. Note that uniqueness in this setting is not a novelty (see, e.g., [18]): it is inherent to the chosen germ formalism for interface coupling. While we wish to achieve wide generality in what concerns the choice of interface coupling and the geometry of the interfaces (under the assumption of a finite number of interfaces), we restrict our attention to the simplest representative case in what concerns the assumptions on the flux in the regions between the interfaces. This is done in order not to face straightforward but cumbersome numerical technicalities available in the literature on finite volume approximation of conservation laws with smooth  $(t, x)$ -dependent flux (e.g., [51]). Therefore we make the simplifying assumption of taking a piecewise constant in  $(t, x)$  flux, which highlights the original elements of the analysis conducted in the present

paper, but which limits the outreach of the existence result. Nonetheless, the existence strategy of the present contribution is extendable to piecewise regular in  $(t, x)$  fluxes.

Indeed, let us point out that developments necessary in order to cover the general piecewise  $\mathbf{C}^1$  flux case are essentially of technical nature, as well as the more original developments that would allow to carry the construction and the resulting existence claims to the multi-dimensional situation (see in particular Remark 5). Such useful but heavy developments are postponed to future work.

### 1.3 The outline

The paper is organized as follows. In Section 2 we make precise the assumptions on  $\mathbf{f}$  in (1) and on the coupling enforced on each of the interfaces associated with the  $(t, x)$ -discontinuity of  $\mathbf{f}$ . We state the definition of solution for which uniqueness follows from [20] (but we also rely upon the analysis of [38] for a technically simpler proof, since our geometrical assumptions on the structure of interfaces are much stronger than those of [20]), and reformulate the notion of solution in terms of adapted entropy inequalities. At this point, a specific Carathéodory structure of the family of interface couplings at hand comes into play. We illustrate the resulting setting with an example of *transmission maps* covering many situations known in the literature (another example being explored in detail in [38]). To conclude Section 2, we state the main result of existence *via* convergence of a Finite Volume scheme that is described in the subsequent Sections 3 and 4. We also provide comments about the compactness assumptions we take and on the convergence of other approximation procedures. In Section 3, we coin the key tools of our study in the model case (2) with  $t$ -dependent coupling at the interface  $\{(t, 0) : t \in ]0, +\infty[ \}$ . We link our adapted entropy formulation to the Godunov numerical approach at the interface, formulate the numerical scheme and sketch the compactness and convergence analysis, treating in detail the interface terms. In Section 4 we briefly develop the adaptations needed in the scheme to take into account slanted interfaces and interface crossings, and perform the reduction of the general case with multiple, possibly crossing interfaces to the case on an isolated interface; we then call upon the convergence analysis of the model case, showing that it also applies to the general setting. Finally, in Section 5 we illustrate the construction by two examples inspired by concrete applications in pedestrian modeling and in fluid-particle interaction.

## 2 Flux and interface coupling structure. Notion of solution, uniqueness and existence result

Our interest in this contribution goes to interface coupling and its approximation, therefore we assume that the space-time heterogeneity of the flux is reduced to the presence of sharp interfaces; in this way we avoid technical details related to  $t$  and  $x$  dependence of  $\mathbf{f}$  in regions between interfaces. In simple words, we assume that  $\mathbf{f}$  is piecewise constant with respect to  $(t, x)$ . The approximation scheme, the convergence analysis and the resulting new existence results we obtain can be extended to the more general

piecewise  $\mathbf{C}^1$  fluxes, but we do not pursue this objective in the present contribution. Note that the multi-dimensional extension is also feasible. We refer to Remark 2, Remarks 4, 5 for a brief discussion of feasible generalizations to flux heterogeneous between interfaces.

## 2.1 Piecewise constant flux

We assume we are given a finite family of interfaces  $(y_i)_{i \in \llbracket 1; J \rrbracket}$  ( $J \in \mathbb{N}^*$ ) defined on  $] \tau_i, T_i[$  ( $0 \leq \tau_i < T_i \leq +\infty$ ). Introduce the notations:

$$\forall i \in \llbracket 1; J \rrbracket, \quad \Gamma_i = \{(t, y_i(t)) \in \overline{\Omega} \mid t \in ] \tau_i, T_i[ \}. \quad (3)$$

Note that the extension to locally finite number of interfaces in straightforward. We suppose that for all  $i \in \llbracket 1; J \rrbracket$ ,  $y_i \in \mathbf{W}^{1,\infty}(] \tau_i, T_i[)$ . This notation means that what can be seen as crossing points between interfaces will be considered as endpoints of the interfaces. We denote by  $(\mathcal{C}_m)_{1 \leq m \leq M}$  the set of all endpoints of the interfaces  $\Gamma_i$ ,  $i \in \llbracket 1; J \rrbracket$ . As suggested hereabove, we assume that in each of the regions of the  $(t, x)$ -plane delimited by the interfaces,  $\mathbf{f}$  depends on  $\rho$  only (*i.e.* the flux is homogeneous in each such region).

## 2.2 Germs and Godunov fluxes for interface coupling

Further, to each interface we attach a two-parameter family of subsets of  $\mathbb{R}^2$  denoted by  $\mathcal{G}_s^i(t)$ ,  $t \in ] \tau_i, T_i[$ ,  $s \in \mathbb{R}$ . In the terminology of [9, 18] recalled in Section 3 below,  $\mathcal{G}_s^i(t)$  is assumed to be a complete  $\mathbf{L}^1 D$  germ for the couple of fluxes

$$g_s^i(\rho) = \mathbf{f}(t, y_i(t)^-, \rho) - s\rho, \quad f_s^i(\rho) = \mathbf{f}(t, y_i(t)^+, \rho) - s\rho \quad (\forall t \in ] \tau_i, T_i[); \quad (4)$$

note that due to the piecewise constant assumption on  $\mathbf{f}$ , the expressions of  $g_s^i, f_s^i$  are actually  $t$ -independent. We will need  $g_{\dot{y}(t)}^i, f_{\dot{y}(t)}^i$  and  $\mathcal{G}_{\dot{y}(t)}^i(t)$  to define the interface coupling conditions associated with the interfaces in the formal problem (1), see Definition 2 below; but we exploit  $\mathcal{G}_s^i(t)$ , for  $s$  in a vicinity of  $\dot{y}(t)$ , in order to construct approximate solutions *via* a numerical scheme. The assumption we impose on the family  $\{\mathcal{G}_s^i(t)\}_{t \in ] \tau_i, T_i[, s \in \mathbb{R}}$  is a Carathéodory-kind assumption: the family should be continuous in  $s$  and measurable in  $t$ . To this end, we need to define a topology on the set of  $\mathbf{L}^1 D$  germs; the one we take is inferred from our numerical approach and it is described in terms of the Godunov flux associated to the Riemann solver determined by the germ at hand. Note that definitions of a neighbourhood of a germ and associated measurability properties were elaborated and discussed in [9, 18], but we elaborate a different and more practical viewpoint here. For given  $s \in \mathbb{R}$  and a fixed  $t_0 \in ] \tau_i, T_i[$ , consider the flux (2) with  $g = g_s^i, f = f_s^i$ , with interface coupling prescribed (in the sense of [9], see also Section 3 below) by the maximal  $\mathbf{L}^1 D$  germ  $\mathcal{G}_s^i(t_0)$  and with Riemann initial data

$$\rho_o(x) = \begin{cases} \kappa_L & \text{if } x < 0 \\ \kappa_R & \text{if } x > 0. \end{cases}$$



This problem admits a unique solution, which we denote  $\mathbf{RS}_s^i(\kappa_L, \kappa_R, t_0)$ . Since a maximal  $\mathbf{L}^1 D$  germ is also complete (see [9, 18]), this solution is self-similar, therefore the Godunov flux

$$g_s^i(\mathbf{RS}_s^i(\kappa_L, \kappa_R, t_0))|_{x=0^-} \equiv f_s^i(\mathbf{RS}_s^i(\kappa_L, \kappa_R, t_0))|_{x=0^+} \quad (5)$$

is well defined. We call  $\mathbf{F}_s^{i,int}(\kappa_L, \kappa_R, t_0)$  this common value which is seen as a real-valued function, constant in  $t$ . Note that an alternative would be to use the formalism of [37], which directly prescribes the interface coupling in terms of a Riemann solver defined at the interface.

We are now in a position to define the Carathéodory structure on families of germs used in this paper.

**Definition 1** Given  $i \in \llbracket 1; J \rrbracket$  and a family  $\{\mathcal{G}_s^i(t)\}_{t \in ]\tau_i, T_i[, s \in \mathbb{R}}$  of maximal  $\mathbf{L}^1 D$  germs associated with fluxes  $g_s^i, f_s^i$ , we say that the family is Carathéodory (measurable in  $t \in ]\tau_i, T_i[$ , continuous in  $s \in \mathbb{R}$ ) if for every  $(\kappa_L, \kappa_R) \in \mathbb{R}^2$  the associated function

$$\mathbf{F}^{i,int}(\kappa_L, \kappa_R) : \left. \begin{array}{l} \mathbb{R} \times ]\tau_i, T_i[ \longrightarrow \mathbb{R} \\ (s, t) \longmapsto \mathbf{F}_s^{i,int}(\kappa_L, \kappa_R, t) \end{array} \right|$$

is a Carathéodory function.

Recall that  $s$  will stand for  $\dot{y}(t)$  (the slope of the interface) or for its approximations, while  $t$  replaces the fixed value  $t_0$  in the definition of the Godunov fluxes associated with the family of germs.

## 2.3 Notion of solution and uniqueness

We are now in a position to define solutions; the definition readily leads to uniqueness. For  $\kappa \in \mathbb{R}$ , denote by  $\Phi(t, x, \rho, \kappa)$  the Kruzhkov entropy flux corresponding to  $\mathbf{f}(t, x, \rho)$ , *i.e.*

$$\Phi(t, x, \rho, \kappa) = \text{sgn}(\rho - \kappa)(\mathbf{f}(t, x, \rho) - \mathbf{f}(t, x, \kappa)).$$

**Definition 2** Consider a piecewise constant flux  $\mathbf{f} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  with a set of interfaces of the form (3). Assume that for each  $i \in \llbracket 1; J \rrbracket$  we are given a family  $\{\mathcal{G}_s^i(t)\}_{t \in ]\tau_i, T_i[, s \in \mathbb{R}}$  of maximal  $\mathbf{L}^1 D$  germs associated with fluxes  $g_s^i, f_s^i$  in (4); assume that this family is Carathéodory, in the sense of Definition 1.

Consider a function  $\rho \in \mathbf{L}^\infty(\Omega)$  such that  $\rho$  has strong one-sided traces in the sense of [41, 42] on interfaces  $\Gamma_i$ ; we denote them  $\gamma_L^i \rho, \gamma_R^i \rho : ]\tau_i, T_i[ \rightarrow \mathbb{R}$ .

We say that such a function  $\rho$  is a  $\mathcal{G}_s^i(t)$ -entropy solution to (1) with initial data  $\rho_o \in \mathbf{L}^\infty(\mathbb{R})$  if for all test functions  $\varphi \in \mathbf{C}_c^\infty(\bar{\Omega} \setminus \cup_{i=1}^J \Gamma_i)$ ,  $\varphi \geq 0$  and for all  $\kappa \in \mathbb{R}$ , the classical Kruzhkov entropy inequalities are satisfied:

$$\int_0^{+\infty} \int_{\mathbb{R}} \left( |\rho - \kappa| \partial_t \varphi + \Phi(t, x, \rho, \kappa) \partial_x \varphi \right) dx dt + \int_{\mathbb{R}} |\rho_o(x) - \kappa| \varphi(0, x) dx \geq 0 \quad (6)$$

and moreover, for all  $i \in \llbracket 1; J \rrbracket$ , for a.e.  $t \in ]\tau_i, T_i[$  there holds:

$$\left( (\gamma_L^i \rho)(t), (\gamma_R^i \rho)(t) \right) \in \mathcal{G}_{\dot{y}_i(t)}^i(t). \quad (7)$$

Note that it is easy to assess that a solution  $\rho$  in the above sense actually belongs to  $\mathbf{C}(\mathbb{R}^+, \mathbf{L}_{\text{loc}}^1(\mathbb{R}))$  in the sense that it is time-continuous taking values in the  $\mathbf{L}_{\text{loc}}^1$  space of functions of the space variable; see, for instance [43, 44].

*Remark 1* Regarding the assumption of existence of strong one-sided traces for  $\rho$  on interfaces  $\Gamma_i$ , we have two important comments. First, our existence result will require uniform convexity or uniform concavity in  $\rho$  of the flux  $\mathbf{f}$  on each of the regions of the  $(t, x)$ -plane delimited by the interfaces. Under this assumption which implies the genuine nonlinearity of the flux, existence of strong traces is well known since [41]. Second, using the machinery of [42], one can circumvent the genuine nonlinearity assumption with the help of a kind of singular mappings which allow to define the relevant traces and with the help of “reduced germs” to replace (7) (see [9, Remark 2.2, Definition 3.5]). For these two reasons, we consider that the trace assumption is non-restrictive in practice.

*Remark 2* For the sake of conciseness, in this contribution we have chosen to formulate Definition 2 in the context of piecewise constant  $\mathbf{f}$ . Extension of this notion of solution to piecewise regular fluxes, heterogeneous but  $\mathbf{C}^1$  (or even merely Lipschitz continuous) in  $(t, x)$  in regions delimited by the interfaces, is straightforward. Note that also the existence of strong one-sided interface traces extends to this framework under mild assumptions, see in particular [20, 45, 46].

The uniqueness proof under the assumption of a finite number of interfaces is standard: see, in particular, [9, 21]; see also [38, Theorem 1.11]. Moreover, even in the much more general situation of **SBV** fluxes uniqueness is proved in [20]. We state the corresponding claim for the sake of completeness, along with the  $\mathbf{L}^1$  contraction result.

**Theorem 1** *In the situation of Definition 2, there exists at most one  $\mathcal{G}_s^i(t)$ -entropy solution for every initial datum  $\rho_o \in \mathbf{L}^\infty(\mathbb{R})$ , moreover, if  $\hat{\rho}_o \in \mathbf{L}^\infty(\mathbb{R})$  with  $\hat{\rho}_o - \rho_o \in \mathbf{L}^1(\mathbb{R})$  and  $\hat{\rho}$  is the associated  $\mathcal{G}_s^i(t)$ -entropy solution, then for all  $t > 0$  there holds*

$$\|\hat{\rho}(t, \cdot) - \rho(t, \cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq \|\hat{\rho}_o - \rho_o\|_{\mathbf{L}^1(\mathbb{R})}.$$

Note also that the continuous dependence on interface coupling conditions can be obtained along the same line of argumentation, see [9, Proposition 3.21] for a prototype statement.

Finally, note that the Carathéodory assumption on the family  $\{\mathcal{G}_s^i(t)\}_{t \in ]\tau_i, T_i[, s \in \mathbb{R}}$  plays no role in the uniqueness proof (Definition 2 makes sense and the uniqueness argument applies under a weaker assumption of measurability of the family  $\{\mathcal{G}_{y_i(t)}^i(t)\}_{t \in ]\tau_i, T_i[}$ , cf. [18, Appendix], is enough); actually, we will exploit the Carathéodory assumption on the family of germs for the sake of proving existence of solutions. Moreover, we expect that the Carathéodory assumption is important for stability of solutions under perturbation of interface locations. To sum up, we believe that this assumption is an important one on the way to a consistent theory of problem (1) and therefore, we have included it into Definition 2.

## 2.4 Adapted entropy inequalities and existence result

Definition 2 is particularly well suited for uniqueness proof, but it cannot be used directly to establish existence of solutions. It became standard in the literature to use different kinds of “adapted entropy inequalities” in order to describe the interface coupling, in the place of (7). A typical well-posedness proof (*cf.* [9]) then consists in proving existence for the adequate adapted entropy formulation, and proving uniqueness by deriving explicit trace conditions that can be cast under the form (7). In the case of traffic models with flux limitation, following [16] one uses entropy inequalities of Kruzhkov (with a constant value  $\kappa$ ) with a remainder term  $\mathcal{R}(\kappa)$  supported by the interface; see [16, Definition 3.2], [39, Definition 2.1] or [38, Definition 2.1]. The choice of a constant  $\kappa$  can be done for the vanishing viscosity interface coupling, see [4, 28]. The corresponding remainder term  $\mathcal{R}(\kappa)$  appears quite naturally even in a very general context, see [5], but this natural formulation leads to uniqueness only if the so-called crossing condition is fulfilled (see [9]). The case without the crossing condition is significantly more delicate but it can be handled as well by introducing a singular form of remainder term  $\mathcal{R}(\kappa)$ , see [21, 47] (see also [31]).

However, the presence of remainder terms supported on the interfaces is not the main adaptation of the classical Kruzhkov entropy conditions. Adapted entropy inequalities with  $\kappa$  that may jump across interface offer more flexibility; *e.g.*, in many situations including the road traffic with limited flux, only one entropy inequality (with a special choice of the piecewise constant function  $\kappa$ ) is needed, see [15, 17, 26] and [9, Sections 4.8, 4.9]. However, beyond the homogeneous situation with constant in time choice of coupling across interfaces, adapted entropy inequalities should be written for arbitrary choice of the piecewise constant  $\kappa$ , which jumps across the interfaces of  $\mathbf{f}$ , see in particular [21, 30]; these inequalities incorporate a remainder term that depends both upon the piecewise constant  $\kappa$  and on the prescribed coupling (*i.e.* on the given family of germs along interfaces).

We start by reformulating the definition of  $\mathcal{G}_s^i(t)$ -entropy solution under the form of adapted entropy inequalities which incorporate a remainder term  $\mathcal{R}_{y_i(t)}^i(\kappa_L, \kappa_R, t)$  different from those previously proposed in [9, 18] but satisfying the key structural properties that eventually lead to the equivalence with the formulation of Definition 2. More precisely, we first derive the trace property (7) from the adapted entropy inequalities (9) stated in Proposition 2. As a consequence of the uniqueness of  $\mathcal{G}_s^i(t)$ -entropy solution (Theorem 1), the reciprocal implication of Proposition 2 can be proved as soon one can ensure the existence of solutions in the sense of inequality (9), see Remark 9.

**Proposition 2** *Consider a piecewise constant flux  $\mathbf{f} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  with a set of interfaces of the form (3). Assume that for each  $i \in \llbracket 1; J \rrbracket$ , we are given a family  $\{\mathcal{G}_s^i(t)\}_{t \in ]\tau_i, T_i[, s \in \mathbb{R}}$  of maximal  $\mathbf{L}^1 D$  germs associated with fluxes  $g_s^i, f_s^i$  in (4); assume that this family is Carathéodory, in the sense of Definition 1. Denote by  $\mathcal{K}$  the set of all the functions, piecewise on  $\Omega$ , that share the same interfaces as  $\mathbf{f}$ . For a function  $\kappa \in \mathcal{K}$  and  $i \in \llbracket 1; J \rrbracket$ , we denote by  $\kappa_L^i, \kappa_R^i$  the one-sided limits of  $\kappa$  on  $\Gamma_i$  (being  $\kappa_L^i = \kappa(t, y_i(t)^-)$  and  $\kappa_R^i = \kappa(t, y_i(t)^+)$  for all  $t \in ]\tau_i, T_i[$ ).*

Define for all  $i \in \llbracket 1; J \rrbracket$ ,  $t \in ]\tau_i, T_i[$ ,  $s \in \mathbb{R}$  and  $(\kappa_L, \kappa_R) \in \mathbb{R}^2$  the "remainder term"

$$\mathcal{R}_s^i(\kappa_L, \kappa_R, t) := \left| g(\kappa_L) - \mathbf{F}_s^{i,int}(\kappa_L, \kappa_R, t) \right| + \left| \mathbf{F}_s^{i,int}(\kappa_L, \kappa_R, t) - f(\kappa_R) \right|, \quad (8)$$

where  $\mathbf{F}_s^{i,int}(\cdot, \cdot, t)$  is the Godunov flux defined in (5) associated with the germ family  $\{\mathcal{G}_s^i(t)\}_{t \in ]\tau_i, T_i[, s \in \mathbb{R}}$ .

Assume that  $\rho \in \mathbf{L}^\infty(\Omega)$  satisfies, for all test functions  $\varphi \in \mathbf{C}_c^\infty(\bar{\Omega})$ ,  $\varphi \geq 0$  and for all  $\kappa \in \mathcal{K}$ , the following adapted entropy inequalities:

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}} \left( |\rho - \kappa| \partial_t \varphi + \Phi(t, x, \rho, \kappa) \partial_x \varphi \right) dx dt + \int_{\mathbb{R}} |\rho_o(x) - \kappa| \varphi(0, x) dx \\ & + \sum_{i=1}^J \int_{\tau_i}^{T_i} \mathcal{R}_{y_i(t)}^i(\kappa_L, \kappa_R, t) \varphi(t, y_i(t)) dt \geq 0. \end{aligned} \quad (9)$$

Then  $\rho$  is the  $\mathcal{G}_s^i(t)$ -entropy solution to (1) with initial data  $\rho_o \in \mathbf{L}^\infty(\mathbb{R})$ .

*Remark 3* As demonstrated in [38], in the context of formulations of the kind (9) we can replace the set  $\mathbf{C}_c^\infty(\bar{\Omega})$  for the test functions by  $\mathbf{C}_c^\infty(\bar{\Omega} \setminus \cup_{m=1}^M \mathcal{C}_m)$ . This permits to relax, in a small vicinity of the cross-points  $\mathcal{C}_m$ , the consistency constraints on the approximation procedure used to construct solutions, see Section 4.2.

With this reformulation of the notion of solution we are concerned with, we are in a position to state the main result of the paper.

**Theorem 3** Consider a piecewise constant flux  $\mathbf{f} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  with a set of interfaces of the form (3). Assume that for each  $i \in \llbracket 1; J \rrbracket$  we are given a family  $\{\mathcal{G}_s^i(t)\}_{t \in ]\tau_i, T_i[, s \in \mathbb{R}}$  of maximal  $\mathbf{L}^1 D$  germs associated with fluxes  $g_s^i, f_s^i$  in (4); assume that this family is Carathéodory, in the sense of Definition 1.

Assume moreover that the following confinement assumption holds:

$$\exists a, b \in \mathbb{R} : a < b \text{ and for a.e. } (t, x) \in \Omega \quad \mathbf{f}(t, x, a) = a, \quad \mathbf{f}(t, x, b) = b. \quad (10)$$

Assume further that  $\mathbf{f}$  is  $\mathbf{C}^2$  in the state variable  $\rho \in [a, b]$  for a.e.  $(t, x) \in \Omega$  and that the uniform convexity/concavity assumption holds:

$$\exists c > 0, \forall \rho \in [a, b], \quad |\partial_\rho^2 \mathbf{f}(t, x, \rho)| \geq c. \quad (11)$$

Finally, suppose that for each  $i \in \llbracket 1; J \rrbracket$ ,

$$(\kappa_L, \kappa_R, s, t) \mapsto \mathbf{F}_s^{i,int}(\kappa_L, \kappa_R, t) \quad (12)$$

is Lipschitz continuous on  $[a, b]^2$  uniformly in  $t \in ]\tau_i, T_i[$  and locally uniformly in  $s \in \mathbb{R}$ .

Then for any initial datum  $\rho_o \in \mathbf{L}^\infty(\mathbb{R})$  taking values in  $[a, b]$ , there exists a  $\mathcal{G}_s^i(t)$ -entropy solution (which is unique, due to Theorem 1) of the discontinuous-flux conservation law (1). Moreover, it can be obtained as the limit of a sequence of approximate solutions generated by a finite volume numerical scheme described in Section 4.

Let us stress that, due to the piecewise constant in  $(t, x)$  structure of  $\mathbf{f}$ , assumption (11) is a concise way to state that, in each of the regions delimited by the interfaces  $\Gamma_i$ ,  $\mathbf{f}$  is subject either to the uniform convexity or to the uniform concavity assumption. Observe that the change of convexity across the interface is relevant, *e.g.*, in the Hughes model of pedestrian evacuation [48], *cf.* Section 5. Further, Assumption (10) is a standard way to ensure uniform  $L^\infty$  bounds on the approximate solutions (see, *e.g.*, [4]); for instance in traffic and porous media models, it is satisfied with  $[a, b] = [0, 1]$ , the physical range of the state variable.

Next, we guess that Assumption (12) is verified in all practical cases. In particular, for the interface coupling based on transmission maps (see Section 2.5 below) the Lipschitz property is checked in [33], while for the flux-limitation coupling considered in [38], this property is obvious from the definition of the Godunov flux (see in particular [49] for the definition of the Godunov flux in this case).

We stress that the convergence and existence result of Theorem 3 requires strong compactness of the sequence of approximate solutions. Robust local compactness arguments that do not depend on the choice of the interface coupling are most appropriate in our setting. The standard and popular approach of this kind is based upon compensated compactness (or, in more involved variants, on parametrized  $H$ -measures [5, 50]). Because our contribution exploits extensively the work [38], we borrow to [38, Appendix A] (adapted from the work [32]) a different local compactness argument, based upon one-sided Lipschitz (OSL) regularization property of local entropy solutions and some of their approximations. This argument requires the assumption of uniform convexity or concavity of the fluxes  $f_s^i$ . Three remarks are in order.

*Remark 4* The OSL compactness technique we use in this paper does extend to  $t$ -dependent flux in regions between interfaces, see [32, 39], but it is not clear that it can be extended to  $x$ -dependent flux in regions between the interfaces.

*Remark 5* Compensated compactness tools would permit to justify the existence and convergence result of Theorem 3 under weaker assumptions on the genuine non-linearity of  $\mathbf{f}$  with respect to  $\rho$  in each of the subdomains separated by interfaces, moreover, smooth  $(t, x)$ -dependence can be handled in this framework at the price of lengthy but standard technicalities. We refer to [4] for such arguments, in the setting of a Lax-Friedrichs scheme with a meshing different from the one we consider in the present work. Furthermore, the compensated compactness approach can be applied to multi-dimensional scalar conservation laws. Multi-dimensional variants of the localization arguments, reducing intricate interface geometries to the basic case of plane interfaces analogous to what we do in Section 4, should be detailed in this case. These technical developments are left for future work.

*Remark 6* As soon as the existence of a solution is established for a dense set of initial data, the weak convergence methods (see, *e.g.*, the "entropy process" framework put forward in [51]) can be applied to justify convergence of other numerical schemes that are consistent with the weak formulation and the adapted entropy inequalities. We refer to [9, Theorem 3.28, Theorem 6.5] and [17] for this line of argumentation.

## 2.5 The example of transmission map coupling

In [38], the example of flux limitation interface coupling was treated; here, let us observe that the associated family of germs is indeed Carathéodory, due to the continuity in  $s$  of the fluxes  $g_s^i, f_s^i$  and to the measurability of the constraints  $t \mapsto q_i(t)$  (with the notation of [38]). Moreover, the remainder term denoted by  $\mathcal{R}(\kappa, q_i(t))$  used in [17, 38, 39], for the choice of constant  $\kappa$ , is precisely  $\mathcal{R}_s^i(\kappa, \kappa, t)$  in our framework.

In this section, let us provide another important example of interface coupling fulfilling the Carathéodory assumption, and give the explicit expression of the associated Godunov fluxes. This example is based upon [33], see also [18] for a summary. For the sake of conciseness, let us consider the case of a single interface defined for  $t \in ]0, +\infty[$ , *i.e.*  $J = 1$ ; in this case, we drop the interface label  $i$  in the notation. Given fluxes  $g, f$  on  $[a, b] \subset \mathbb{R}$  such that  $g(a) = f(a)$  and  $g(b) = f(b)$ , set

$$g_s(\rho) = g(\rho) - s\rho, \quad f_s(\rho) = f(\rho) - s\rho$$

and denote by  $\mathbf{God}g_s, \mathbf{God}f_s : [a, b]^2 \rightarrow \mathbb{R}$  the Godunov fluxes associated to  $g_s, f_s$ , respectively. The classical explicit formula for the Godunov fluxes shows that the dependence of  $\mathbf{God}g_s(\kappa_L, \kappa_R), \mathbf{God}f_s(\kappa_L, \kappa_R)$  on  $s$  is continuous.

To define the interface coupling, given  $\{\beta(t)\}_{t \in ]0, +\infty[}$  a family of maximal monotone graphs in  $\mathbb{R} \times \mathbb{R}$ , we can define for all  $s \in \mathbb{R}$  and  $t \in ]\tau_i, T_i[$  the maximal  $\mathbf{L}^1D$  germ

$$\mathcal{G}_s(t) = \left\{ (\rho_L, \rho_R) \in [a, b]^2, \exists (k_-, k_+) \in \beta(t) \text{ such that} \right. \\ \left. g_s(\rho_L) = (\mathbf{God}g_s)(\rho_L, k_-) = (\mathbf{God}f_s)(k_+, \rho_R) = f_s(\rho_R) \right\}$$

and the associated interface Godunov flux

$$\mathbf{F}_s^{int}(\kappa_L, \kappa_R, t) = (\mathbf{God}g_s)(\kappa_L, k_-) = (\mathbf{God}f_s)(k_+, \kappa_R), \quad (13)$$

where  $(k_-, k_+) \in \beta(t)$ , being understood that the equality

$$(\mathbf{God}g_s)(\kappa_L, k_-) = (\mathbf{God}f_s)(k_+, \kappa_R)$$

may not define uniquely the couple  $(k_-, k_+) \in \beta(t)$  but their common value is defined uniquely. We refer to [33] for the justification of the above claims. Note that the case  $\beta(t) = \text{Id}$  corresponds to the fundamental case of vanishing viscosity interface coupling.

We point out that the family  $\{\mathcal{G}_s^i(t)\}_{t \in ]0, +\infty[, s \in \mathbb{R}}$  is Carathéodory, in the sense of Definition 1, provided the family  $\{\beta(t)\}_{t \in ]0, +\infty[}$  is measurable in a natural sense. To this end, let us represent any maximal monotone graph  $\beta$  in  $\mathbb{R} \times \mathbb{R}$  as

$$\beta = \{(\pi_L(p), \pi_R(p)), p \in \mathbb{R}\},$$

where  $\pi_{L,R} : \mathbb{R} \rightarrow \mathbb{R}$  are 1-Lipschitz non-decreasing functions verifying  $\pi_L(p) + \pi_R(p) = p$  for all  $p \in \mathbb{R}$ . In other words, we parametrize  $\beta$  by the sum  $p =$

$\kappa_L + \kappa_R$  of the two components of a point  $(\kappa_L, \kappa_R) \in \beta$ . Then the natural way to impose measurability of a family  $\{\beta(t)\}_t$  is to consider the corresponding parametrizations  $\pi_{L,R}(t, \cdot)$  and require that these functions be Carathéodory. Then, in view of the construction (13), it is clear that, given  $(\kappa_L, \kappa_R)$ ,  $(s, t) \mapsto \mathbf{F}_s^{int}(\kappa_L, \kappa_R, t)$  is continuous with respect to  $s$  because the Godunov fluxes are continuous with respect to  $s$ ; and measurable with respect to  $t$  because they are obtained solving an equation of the form  $F_s(t, p) = 0$  for a measurable in  $t$ , monotone and continuous in  $p$  function  $F_s$ .

### 3 The basic ingredients in the model case with variable interface coupling

Let us now provide the key tools to our study for the model case (1)-(2) with a time-dependent coupling at the interface  $\{x = 0\}$ . Therefore, in this section, we consider a flux  $\mathbf{f} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  given by (2) with  $f, g \in \mathbf{C}^2([a, b])$  verifying the confinement assumption (10) and the uniform convexity/concavity assumption (11). Throughout the section, we denote by  $\Phi_f$  (resp.  $\Phi_g$ ) the classical Kruzhkov entropy flux associated with  $f$  (resp.  $g$ ) so that:

$$\forall x \in \mathbb{R}^*, \forall \rho, \kappa \in [a, b], \quad \Phi(x, \rho, \kappa) = \begin{cases} \Phi_g(\rho, \kappa) & \text{if } x < 0 \\ \Phi_f(\rho, \kappa) & \text{if } x > 0. \end{cases}$$

Suppose also that we are given a family  $\{\mathcal{G}(t)\}_{t>0}$  of maximal  $\mathbf{L}^1D$  germs associated with fluxes  $f$  and  $g$ . We suppose that this family is Carathéodory in the sense of Definition 1 which, in this context, means that for all  $\kappa_L, \kappa_R \in [a, b]$ , the associated function

$$t \in \mathbb{R} \mapsto \mathbf{F}^{int}(\kappa_L, \kappa_R, t)$$

is a Carathéodory function.

#### 3.1 Stability and uniqueness

For the sake of completeness, we recall the abstract definition of  $\mathbf{L}^1D$  germ, see [9, Definition 3.1] or [18, Definition 1.1]. We restrict ourselves to subsets of  $[a, b]^2$  in view of the confinement assumption verified by  $f$  and  $g$ .

**Definition 3** A family  $\{\mathcal{G}(t)\}_{t>0}$  of subsets of  $[a, b]^2$  is called  $\mathbf{L}^1D$  germ associated with fluxes  $f, g$  if for all  $t > 0$ :

- (i) for all  $(\kappa_L, \kappa_R) \in \mathcal{G}(t)$ ,  $g(\kappa_L) = f(\kappa_R)$ ;
- (ii) for all  $(\kappa_L, \kappa_R), (c_L, c_R) \in \mathcal{G}(t)$ ,

$$\Phi_g(\kappa_L, c_L) - \Phi_f(\kappa_R, c_R) \geq 0. \quad (14)$$

We say that the family is maximal if for all  $t > 0$ ,  $\mathcal{G}(t)$  is not a strict subset of some other  $\mathbf{L}^1D$  germ.

*Remark 7* With this convention, it is readily seen that any maximal  $\mathbf{L}^1D$  germ  $\mathcal{G}(t)$  associated with fluxes  $f, g$  verifying the confinement condition (10) contains the couples  $(a, a)$  and  $(b, b)$ .

Let us give the arguments of the proof of Theorem 1 for this model case. Following the proof of [38, Lemma 2.7], we derive the Kato inequality:

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}} \left( |\rho - \hat{\rho}| \partial_t \varphi + \Phi(x, \rho, \hat{\rho}) \partial_x \varphi \right) dx dt + \int_{\mathbb{R}} |\rho_o(x) - \hat{\rho}_o(x)| \varphi(0, x) dx \\ & + \int_0^{+\infty} \left( \Phi_f(\gamma_R \rho(t), \gamma_R \hat{\rho}(t)) - \Phi_g(\gamma_L \rho(t), \gamma_L \hat{\rho}(t)) \right) \varphi(t, 0) dt \geq 0. \end{aligned}$$

The  $\mathbf{L}^1D$  dissipativity assumption on the family of germs  $\{\mathcal{G}(t)\}_{t>0}$  (14) and property (7) of solutions ensure that for a.e.  $t > 0$ ,

$$\Phi_f(\gamma_R \rho(t), \gamma_R \hat{\rho}(t)) - \Phi_g(\gamma_L \rho(t), \gamma_L \hat{\rho}(t)) \leq 0.$$

Upon a suitable choice of test function, see the proof of [38, Theorem 2.8], the stability estimate of Theorem 1 follows. This leads to uniqueness.

### 3.2 Adapted entropy inequalities with the Godunov remainder terms

We now turn to the proof of Proposition 2 where the remainder term becomes:

$$\mathcal{R}(\kappa_L, \kappa_R, t) = |g(\kappa_L) - \mathbf{F}^{int}(\kappa_L, \kappa_R, t)| + |\mathbf{F}^{int}(\kappa_L, \kappa_R, t) - f(\kappa_R)|.$$

Recall that for  $t > 0$  and  $\kappa_L, \kappa_R \in [a, b]$ ,  $\mathbf{F}^{int}(\kappa_L, \kappa_R, t)$  denotes the Godunov flux associated with the family  $\{\mathcal{G}(t)\}_{t>0}$ . We start with the following lemma.

**Lemma 4** *Let  $(\kappa_L, \kappa_R) \in [a, b]$ .*

$$(\kappa_L, \kappa_R) \in \mathcal{G}(t) \implies \forall t > 0, \mathcal{R}(\kappa_L, \kappa_R, t) = 0.$$

*Proof* Indeed, saying that  $(\kappa_L, \kappa_R) \in \mathcal{G}(t)$  means that the piecewise constant function

$$\kappa(x) = \begin{cases} \kappa_L & \text{if } x < 0 \\ \kappa_R & \text{if } x > 0 \end{cases}$$

is the unique solution in the sense of Definition 2 to the problem (1) with the flux (2) and with initial data  $\rho_o = \kappa$ . Consequently, for all  $t > 0$ ,

$$g(\kappa_L) = \mathbf{F}^{int}(\kappa_L, \kappa_R, t) = f(\kappa_R) \quad \text{and} \quad \forall t > 0, \mathcal{R}(\kappa_L, \kappa_R, t) = 0,$$

concluding the proof.  $\square$

We now turn to the proof of Proposition 2.



Suppose that  $\rho$  verifies (9). Clearly,  $\rho$  satisfies (6) if  $\varphi \in \mathbf{C}_c^\infty(\overline{\Omega})$ ,  $\varphi \geq 0$  vanishes along the interface  $\{x = 0\}$ . Following the proof of [38, Proposition 2.6], we obtain:

$$\int_0^{+\infty} \left( \Phi_g(\gamma_L \rho(t), \kappa_L) - \Phi_f(\gamma_R \rho(t), \kappa_R) + \mathcal{R}(\kappa_L, \kappa_R, t) \right) \varphi(t, 0) dt \geq 0. \quad (15)$$

Since  $t \mapsto \mathcal{R}(\kappa_L, \kappa_R, t)$  is measurable and bounded, the function  $t \mapsto \mathcal{R}(\kappa_L, \kappa_R, t)$  belongs to  $\mathbf{L}_{\text{loc}}^1(]0, +\infty[)$ . Consequently, almost every  $t > 0$  is a Lebesgue point of this function. From (15), we deduce that for a.e.  $t > 0$ ,

$$\Phi_g(\gamma_L \rho(t), \kappa_L) - \Phi_f(\gamma_R \rho(t), \kappa_R) + \mathcal{R}(\kappa_L, \kappa_R, t) \geq 0.$$

In particular, using Lemma 4,

$$\forall (\kappa_L, \kappa_R) \in \{\mathcal{G}(t)\}_{t>0}, \text{ for a.e. } t > 0, \quad \Phi_f(\gamma_R \rho(t), \kappa_R) \leq \Phi_g(\gamma_L \rho(t), \kappa_L). \quad (16)$$

This last inequality implies that

$$\text{for a.e. } t > 0, \quad \left( (\gamma_L \rho)(t), (\gamma_R \rho)(t) \right) \in \mathcal{G}(t)$$

by maximality of the family of germs. We proved that  $\rho$  is a  $\mathcal{G}_s^i(t)$ -entropy solution to (1) with initial data  $\rho_o \in \mathbf{L}^\infty(\mathbb{R}, [a, b])$ .

### 3.3 Finite volume scheme, compactness and convergence

We turn to the proof of the existence statement (Theorem 3) with the construction of finite volume scheme for which we prove the convergence. Let us recall some standard notations.

For a fixed spatial mesh size  $\Delta x > 0$  and time mesh size  $\Delta t > 0$ , let  $x_j = j\Delta x$  ( $j \in \mathbb{Z}$ ) and  $t^n = n\Delta t$  ( $n \in \mathbb{N}$ ). Set  $\lambda = \Delta t/\Delta x$ . We define the cell grids:

$$\overline{\Omega} = \bigcup_{n \in \mathbb{N}} \bigcup_{j \in \mathbb{Z}} \mathcal{P}_{j+1/2}^n, \quad \mathcal{P}_{j+1/2}^n = [t^n, t^{n+1}[ \times ]x_j, x_{j+1}[.$$

For  $n \in \mathbb{N}$  and  $j \in \mathbb{Z}$ , let us denote  $\rho_{j+1/2}^n$  an approximation of the solution  $\rho$  on  $\mathcal{P}_{j+1/2}^n$ .

Let  $\rho_o \in \mathbf{L}^\infty(\mathbb{R}, [a, b])$ , where  $a, b \in \mathbb{R}$  are the numbers defined in the confinement assumption 10. We denote by  $\mathbf{F}_f = \mathbf{F}_f(u, v)$  (resp.  $\mathbf{F}_g = \mathbf{F}_g(u, v)$ ) any monotone numerical flux associated with  $f$  (resp.  $g$ ). Following the assumptions of Theorem 3, we assume that  $(\kappa_L, \kappa_R) \mapsto \mathbf{F}^{int}(\kappa_L, \kappa_R, \cdot)$  is Lipschitz continuous on  $[a, b]^2$ , uniformly on  $t > 0$ . This can be expressed as:

$$\begin{aligned} & \exists \mathbf{C} > 0, \quad \forall t > 0, \quad \forall x, y, u, v \in [a, b], \\ & |\mathbf{F}^{int}(x, y, t) - \mathbf{F}^{int}(u, v, t)| \leq \mathbf{C} (|x - u| + |y - v|). \end{aligned} \quad (17)$$

In the analysis below, the following properties of the Godunov flux  $(\kappa_L, \kappa_R, t) \mapsto \mathbf{F}^{int}(\kappa_L, \kappa_R, t)$  are used in addition to (17).

**Lemma 5** *The Godunov flux defined in Section 2.2 has the following properties:*

(i) for all  $t > 0$ ,  $\mathbf{F}^{int}(\cdot, \cdot, t)$  is nondecreasing with respect to its first argument and nonincreasing with respect to its second argument;

(ii) for all  $t > 0$ ,

$$g(a) = \mathbf{F}^{int}(a, a, t) = f(a), \quad g(b) = \mathbf{F}^{int}(b, b, t) = f(b). \quad (18)$$

*Proof* Point (i) via the Crandall-Tartar Lemma follows from the order-preservation property encoded in  $\mathbf{L}^1D$  germs (cf. [52]). Point (ii) comes from the fact that the couples  $(a, a)$  and  $(b, b)$  belong to  $\{\mathcal{G}(t)\}_{t>0}$  (see Remark 7). Identities (18) follow as in the proof of Lemma 4.  $\square$

We now proceed to the definition of the scheme. Fix  $n \in \mathbb{N}$  and  $j \in \mathbb{Z}$ . Away from the interface, *i.e.* for  $j \notin \{-1, 0\}$ , our scheme reduces to a classical three-point finite volume scheme on a uniform grid. More precisely,

$$\rho_{j+1/2}^{n+1} = \rho_{j+1/2}^n - \lambda(\mathbf{F}_g(\rho_{j+1/2}^n, \rho_{j+3/2}^n) - \mathbf{F}_g(\rho_{j-1/2}^n, \rho_{j+1/2}^n)), \quad j \leq -2 \quad (19)$$

and

$$\rho_{j+1/2}^{n+1} = \rho_{j+1/2}^n - \lambda(\mathbf{F}_f(\rho_{j+1/2}^n, \rho_{j+3/2}^n) - \mathbf{F}_f(\rho_{j-1/2}^n, \rho_{j+1/2}^n)), \quad j \geq 1. \quad (20)$$

To handle the coupling at the interface, introduce the mean numerical flux

$$\mathbf{God}_{int}^n(\rho_{-1/2}^n, \rho_{1/2}^n) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{F}^{int}(\rho_{-1/2}^n, \rho_{1/2}^n, t) dt \quad (21)$$

and use it to define the numerical solution in the remaining cells:

$$\rho_{-1/2}^{n+1} = \rho_{-1/2}^n - \lambda(\mathbf{God}_{int}^n(\rho_{-1/2}^n, \rho_{1/2}^n) - \mathbf{F}_g(\rho_{-3/2}^n, \rho_{-1/2}^n)). \quad (22)$$

and

$$\rho_{1/2}^{n+1} = \rho_{1/2}^n - \lambda(\mathbf{F}_f(\rho_{1/2}^n, \rho_{3/2}^n) - \mathbf{God}_{int}^n(\rho_{-1/2}^n, \rho_{1/2}^n)). \quad (23)$$

For the sake of simplicity, we choose  $\mathbf{F}_f$  and  $\mathbf{F}_g$  equal to one of the standard numerical fluxes: Rusanov, Lax-Friedrichs, Godunov or Engquist-Osher. This way, the CFL condition reduces to

$$2 \underbrace{\max \{\|f'\|_{\mathbf{L}^\infty}, \|g'\|_{\mathbf{L}^\infty}, \mathbf{C}\}}_{:=\mathbf{L}} \lambda \leq 1, \quad (24)$$

with  $\mathbf{C}$  defined in (17).

### 3.3.1 Stability and discrete entropy inequalities

**Lemma 6** ( $L^\infty$  stability) *The numerical scheme (19)-(23) is monotone and stable:*

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad \rho_{j+1/2}^n \in [a, b]. \quad (25)$$

*Proof* The monotonicity of the scheme follows from the same arguments used in the proof of [39, Proposition 2] for instance. Note in particular that the Godunov interface fluxes  $\mathbf{F}^{int}$  are monotone (which implies the monotonicity of  $\mathbf{God}_{int}^n$  due to (21)). The stability estimate is proved by induction with the introduction, for all  $n \in \mathbb{N}$  and  $j \in \mathbb{Z}$ , of the function  $\mathbf{H}_j^n = \mathbf{H}_j^n(x, y, z)$  used to express  $\rho_{j+1/2}^{n+1}$  from  $\rho_{j-1/2}^n, \rho_{j+1/2}^n$  and  $\rho_{j+3/2}^n$ . More precisely,

$$\forall j \in \mathbb{Z} (j \leq -2), \forall u, v, w \in [a, b], \quad \mathbf{H}_j^n(u, v, w) = v - \lambda(\mathbf{F}_g(v, w) - \mathbf{F}_g(u, v))$$

or

$$\forall u, v, w \in [a, b], \quad \mathbf{H}_0^n(u, v, w) = v - \lambda(\mathbf{God}_{int}^n(v, w) - \mathbf{F}_g(u, v)).$$

The key point of the proof is the fact that  $a$  and  $b$  are stationary states of the scheme. Indeed, as a consequence of Lemma 5 (iii), we have:

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad \mathbf{H}_j^n(a, a, a) = a \quad \text{and} \quad \mathbf{H}_j^n(b, b, b) = b.$$

We refer to the proof of [39, Proposition 2] for details.  $\square$

**Corollary 7** (Discrete entropy inequalities) *Fix  $\kappa_L, \kappa_R \in [a, b]$  and define*

$$\forall j \in \mathbb{Z}, \quad \kappa_{j+1/2} = \kappa_L \mathbf{1}_{\{j \leq -1\}} + \kappa_R \mathbf{1}_{\{j \geq 0\}}.$$

*Then the numerical scheme (19)-(23) fulfills the following discrete entropy inequalities for all  $n \in \mathbb{N}$  and  $j \in \mathbb{Z}$ :*

$$\begin{aligned} & \left( |\rho_{j+1/2}^{n+1} - \kappa_{j+1/2}| - |\rho_{j+1/2}^n - \kappa_{j+1/2}| \right) \Delta x \\ & \leq \begin{cases} -(\Phi_{j+1}^n - \Phi_j^n) \Delta t & \text{if } j \notin \{-1, 0\} \\ -(\Phi_{int}^n - \Phi_{-1}^n) \Delta t + |g(\kappa_L) - \mathbf{God}_{int}^n(\kappa_L, \kappa_R)| \Delta t & \text{if } j = -1 \\ -(\Phi_1^n - \Phi_{int}^n) \Delta t + |\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - f(\kappa_R)| \Delta t & \text{if } j = 0, \end{cases} \quad (26) \end{aligned}$$

where  $\Phi_j^n$  and  $\Phi_{int}^n$  are the numerical entropy fluxes:

$$\Phi_j^n = \begin{cases} \mathbf{F}_g(\rho_{j-1/2}^n \vee \kappa_L, \rho_{j+1/2}^n \vee \kappa_L) - \mathbf{F}_g(\rho_{j-1/2}^n \wedge \kappa_L, \rho_{j+1/2}^n \wedge \kappa_L) & \text{if } j \leq -1 \\ \mathbf{F}_f(\rho_{j-1/2}^n \vee \kappa_R, \rho_{j+1/2}^n \vee \kappa_R) - \mathbf{F}_f(\rho_{j-1/2}^n \wedge \kappa_R, \rho_{j+1/2}^n \wedge \kappa_R) & \text{if } j \geq 1 \end{cases}$$

$$\Phi_{int}^n = \mathbf{God}_{int}^n(\rho_{-1/2}^n \vee \kappa_L, \rho_{1/2}^n \vee \kappa_R) - \mathbf{God}_{int}^n(\rho_{-1/2}^n \wedge \kappa_L, \rho_{1/2}^n \wedge \kappa_R).$$

*Proof* This is mostly a consequence of the scheme monotonicity. Fix  $n \in \mathbb{N}$  and  $j \in \mathbb{Z}$ . Suppose first that  $j \notin \{-1, 0\}$ . In this case, all the constant states  $\kappa_L, \kappa_R \in [a, b]$  are

stationary solutions of the scheme. Consequently,

$$\begin{aligned}
 |\rho_{j+1/2}^{n+1} - \kappa_{j+1/2}| &= \rho_{j+1/2}^{n+1} \vee \kappa_{j+1/2} - \rho_{j+1/2}^{n+1} \wedge \kappa_{j+1/2} \\
 &= \mathbf{H}_j^n \left( \rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n \right) \vee \mathbf{H}_j^n (\kappa_{j+1/2}, \kappa_{j+1/2}, \kappa_{j+1/2}) \\
 &\quad - \mathbf{H}_j^n \left( \rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n \right) \wedge \mathbf{H}_j^n (\kappa_{j+1/2}, \kappa_{j+1/2}, \kappa_{j+1/2}) \\
 &\leq \mathbf{H}_j^n \left( \rho_{j-1/2}^n \vee \kappa_{j+1/2}, \rho_{j+1/2}^n \vee \kappa_{j+1/2}, \rho_{j+3/2}^n \vee \kappa_{j+1/2} \right) \\
 &\quad - \mathbf{H}_j^n \left( \rho_{j-1/2}^n \wedge \kappa_{j+1/2}, \rho_{j+1/2}^n \wedge \kappa_{j+1/2}, \rho_{j+3/2}^n \wedge \kappa_{j+1/2} \right) \\
 &= |\rho_{j+1/2}^{n+1} - \kappa_{j+1/2}| - \lambda (\Phi_{j+1}^n - \Phi_j^n),
 \end{aligned}$$

which is exactly (27) in the case  $j \notin \{-1, 0\}$ .

Suppose now that  $j = -1$  for instance. The previous observation regarding the constants  $\kappa_L, \kappa_R \in [a, b]$  is not longer valid in this case since

$$\mathbf{H}_{-1}^n(\kappa_L, \kappa_L, \kappa_R) = \kappa_L - \lambda (\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - g(\kappa_L)).$$

Instead, we deduce:

$$\begin{aligned}
 \mathbf{H}_{-1}^n \left( \rho_{-3/2}^n \wedge \kappa_L, \rho_{-1/2}^n \wedge \kappa_L, \rho_{3/2}^n \wedge \kappa_R \right) - \lambda (\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - g(\kappa_L))^- \\
 \leq \kappa_L \leq \mathbf{H}_{-1}^n \left( \rho_{-3/2}^n \vee \kappa_L, \rho_{-1/2}^n \vee \kappa_L, \rho_{3/2}^n \vee \kappa_R \right) + \lambda (\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - g(\kappa_L))^+,
 \end{aligned}$$

where we denoted by  $z^+$  (resp.  $z^-$ ) the positive part (resp. negative part) of the real number  $z$ . We deduce:

$$\begin{aligned}
 |\rho_{-1/2}^{n+1} - \kappa_L| &= \rho_{-3/2}^{n+1} \vee \kappa_L - \rho_{-3/2}^{n+1} \wedge \kappa_L \\
 &\leq \mathbf{H}_{-1}^n \left( \rho_{-3/2}^n \vee \kappa_L, \rho_{-1/2}^n \vee \kappa_L, \rho_{1/2}^n \vee \kappa_R \right) + \lambda (\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - g(\kappa_L))^+ \\
 &\quad - \mathbf{H}_{-1}^n \left( \rho_{-3/2}^n \wedge \kappa_L, \rho_{-1/2}^n \wedge \kappa_L, \rho_{1/2}^n \wedge \kappa_R \right) + \lambda (\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - g(\kappa_L))^- \\
 &= |\rho_{-1/2}^n - \kappa_L| - \lambda (\Phi_{int}^n - \Phi_{-1}^n) + \lambda |\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - g(\kappa_L)|,
 \end{aligned}$$

which is exactly (26) in the case  $j = -1$ . The obtaining of (26) in the case  $j = 0$  is similar so we omit the details of the proof for this case.  $\square$

We now derive continuous entropy inequalities similar to (9) verified by  $\rho_\Delta$ . Let us define the approximate entropy flux:

$$\Phi_\Delta(x, \rho_\Delta, \kappa) = \sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{Z}^*} \Phi_j^n \mathbb{1}_{[t^n, t^{n+1}[x]x_j, x_{j+1}[$$

and the approximate remainder term: if  $t \in [t^n, t^{n+1}[$ ,

$$\mathcal{R}_\Delta(\kappa_L, \kappa_R, t) = |g(\kappa_L) - \mathbf{God}_{int}^n(\kappa_L, \kappa_R)| + |\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - f(\kappa_R)|.$$

**Proposition 8** (Approximate entropy inequalities) Fix  $\kappa_L, \kappa_R \in [a, b]$ . Define

$$\kappa(x) = \kappa_L \mathbb{1}_{\{x < 0\}} + \kappa_R \mathbb{1}_{\{x > 0\}}.$$

Let  $\varphi \in \mathbf{C}_c^\infty(\bar{\Omega})$ ,  $\varphi \geq 0$ , let  $n \in \mathbb{N}$ . Then as  $\Delta \rightarrow 0$ , we have

$$\begin{aligned} & \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \left( |\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta(x, \rho_\Delta, \kappa) \partial_x \varphi \right) dx dt + \int_{\mathbb{R}} |\rho_\Delta(t^n, x) - \kappa| \varphi(t^n, x) dx \\ & - \int_{\mathbb{R}} |\rho_\Delta(t^{n+1}, x) - \kappa| \varphi(t^{n+1}, x) dx + \int_{t^n}^{t^{n+1}} \mathcal{R}_\Delta(\kappa_L, \kappa_R, t) \varphi(t, 0) dt \\ & \geq \mathcal{O}(\Delta x \Delta t) + \mathcal{O}(\Delta t^2). \end{aligned} \tag{27}$$

*Proof* For the sake of clarity, let us also introduce the following notations:

$$\mathcal{R}_L^n = |g(\kappa_L) - \mathbf{God}_{int}^n(\kappa_L, \kappa_R)|; \quad \mathcal{R}_R^n = |\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - f(\kappa_R)|.$$

Define  $(k_{j+1/2})_{j \in \mathbb{Z}}$  as in Corollary 7. Let us multiply the discrete entropy inequalities (26) by  $\varphi_{j+1/2}^{n+1} \Delta x$  and sum over  $j \in \mathbb{Z}$ :

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} |\rho_{j+1/2}^{n+1} - \kappa_{j+1/2}| \varphi_{j+1/2}^{n+1} \Delta x \\ & \leq \sum_{\substack{j \in \mathbb{Z} \\ j \notin \{-1, 0\}}} \left( |\rho_{j+1/2}^n - \kappa_{j+1/2}| \Delta x - (\Phi_{j+1}^n - \Phi_j^n) \Delta t \right) \varphi_{j+1/2}^{n+1} \\ & + |\rho_{-1/2}^n - \kappa_L| \varphi_{-1/2}^{n+1} \Delta x - (\Phi_{int}^n - \Phi_{-1}^n) \varphi_{-1/2}^{n+1} \Delta t + \mathcal{R}_L^n \varphi_{-1/2}^{n+1} \Delta t \\ & + |\rho_{1/2}^n - \kappa_R| \varphi_{1/2}^{n+1} \Delta x - (\Phi_1^n - \Phi_{int}^n) \varphi_{1/2}^{n+1} \Delta t + \mathcal{R}_R^n \varphi_{1/2}^{n+1} \Delta t. \end{aligned}$$

We now proceed to the Abel's transformation as well as adding some quantities and their opposites to obtain:

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} |\rho_{j+1/2}^{n+1} - \kappa_{j+1/2}| \varphi_{j+1/2}^{n+1} \Delta x - \sum_{j \in \mathbb{Z}} |\rho_{j+1/2}^n - \kappa_{j+1/2}| \varphi_{j+1/2}^n \Delta x \\ & - \sum_{j \in \mathbb{Z}} |\rho_{j+1/2}^n - \kappa_{j+1/2}| \left( \varphi_{j+1/2}^{n+1} - \varphi_{j+1/2}^n \right) \Delta x \leq \underbrace{\sum_{j \in \mathbb{Z}} \Phi_j^n \left( \varphi_{j+1/2}^{n+1} - \varphi_{j-1/2}^{n+1} \right) \Delta t}_A \\ & + \underbrace{\left( \mathcal{R}_L^n \varphi_{-1/2}^{n+1} + \mathcal{R}_R^n \varphi_{1/2}^{n+1} \right) \Delta t}_B + \underbrace{(\Phi_{int}^n - \Phi_0^n) (\varphi_{1/2}^{n+1} - \varphi_{-1/2}^{n+1}) \Delta t}_C. \end{aligned}$$

The left-hand side of this inequality is equal to

$$\begin{aligned} & \int_{\mathbb{R}} |\rho_\Delta(t^{n+1}, x) - \kappa(x)| \varphi(t^{n+1}, x) dx - \int_{\mathbb{R}} |\rho_\Delta(t^n, x) - \kappa(x)| \varphi(t^n, x) dx \\ & - \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} |\rho_\Delta(t, x) - \kappa(x)| \partial_t \varphi dx dt. \end{aligned}$$

We now estimate the members of the right-hand side.

**Estimating A.** We write:

$$\begin{aligned}
A &= \Delta t \int_{\mathbb{R}} \Phi_{\Delta}(x, \rho_{\Delta}, \kappa(x)) \partial_x \varphi(t^{n+1}, x) dx \\
&\quad + \underbrace{\lambda \sum_{j \in \mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x-\Delta x}^x \int_x^y \Phi_j^n \partial_{xx}^2 \varphi(t^n, z) dz dy dx}_{A_1} \\
&= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \Phi_{\Delta}(x, \rho_{\Delta}, \kappa(x)) \partial_x \varphi dx dt + A_1 \\
&\quad + \underbrace{\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \int_t^{t^{n+1}} \Phi_{\Delta}(x, \rho_{\Delta}, \kappa(x)) \partial_{tx}^2 \varphi(\tau, x) d\tau dx dt}_{A_2},
\end{aligned}$$

and we have the estimations:

$$|A_1| \leq 4\mathbf{L} \sup_{t \geq 0} \|\partial_{xx}^2 \varphi(t, \cdot)\|_{\mathbf{L}^1} \Delta x \Delta t, \quad |A_2| \leq \mathbf{L} \sup_{t \geq 0} \|\partial_{tx}^2 \varphi(t, \cdot)\|_{\mathbf{L}^1} \Delta t^2.$$

**Estimating B.**

$$\begin{aligned}
B &= \mathcal{R}_L^n \left( \varphi(t^{n+1}, 0) + \underbrace{\frac{1}{\Delta x} \int_{-\Delta x}^0 (\varphi(t^{n+1}, x) - \varphi(t^{n+1}, 0)) dx}_{\leq \|\partial_x \varphi\|_{\mathbf{L}^\infty} \Delta x} \right) \\
&\quad + \mathcal{R}_R^n \left( \varphi(t^{n+1}, 0) + \underbrace{\frac{1}{\Delta x} \int_0^{\Delta x} (\varphi(t^{n+1}, x) - \varphi(t^{n+1}, 0)) dx}_{\leq \|\partial_x \varphi\|_{\mathbf{L}^\infty} \Delta x} \right) \\
&= \mathcal{R}_{\Delta}(\kappa_L, \kappa_R, t^n) \varphi(t^{n+1}, 0) \Delta t + \mathcal{O}(\Delta x \Delta t) \\
&= \int_{t^n}^{t^{n+1}} \mathcal{R}_{\Delta}(\kappa_L, \kappa_R, t) \varphi(t, 0) dt \\
&\quad + \underbrace{\int_{t^n}^{t^{n+1}} \mathcal{R}_{\Delta}(\kappa_L, \kappa_R, t) (\varphi(t^{n+1}, 0) - \varphi(t, 0)) dt}_{\leq (\|g\|_{\mathbf{L}^\infty} + \|f\|_{\mathbf{L}^\infty}) \|\partial_t \varphi\|_{\mathbf{L}^\infty} \Delta t^2} + \mathcal{O}(\Delta x \Delta t) \\
&= \int_{t^n}^{t^{n+1}} \mathcal{R}_{\Delta}(\kappa_L, \kappa_R, t) \varphi(t, 0) dt + \mathcal{O}(\Delta x \Delta t) + \mathcal{O}(\Delta t^2).
\end{aligned}$$

**Estimating C.** Finally,

$$|C| \leq (2\mathbf{L}) \Delta x \Delta t,$$

concluding the proof of the statement.  $\square$

*Remark 8* If  $\varphi$  is supported in time in some  $[0, T]$  ( $T > 0$ ), with  $T \in [t^N, t^{N+1}[$ , then by summing (27) over  $n \in \llbracket 0; N+1 \rrbracket$ , we obtain:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left( |\rho_{\Delta} - \kappa| \partial_t \varphi + \Phi_{\Delta}(x, \rho_{\Delta}, \kappa) \partial_x \varphi \right) dx dt + \int_{\mathbb{R}} |\rho_{\Delta}^0 - \kappa| \varphi(0, x) dx \\
&\quad + \int_0^T \mathcal{R}(\kappa_L, \kappa_R, t) \varphi(t, 0) dt \geq \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t).
\end{aligned} \tag{28}$$

### 3.3.2 Compactness and convergence

We now prove existence of solutions in the sense of inequality (9).

The compactness of the sequence  $(\rho_\Delta)_\Delta$  is obtained by deriving local **BV** bounds. Since  $f$  and  $g$  are assumed to have uniform convexity/concavity, we can use the OSL technique put forward by [32] and described in detail in [38, Appendix]. This technique "does not see" the interface coupling. This provides the existence of  $\rho \in \mathbf{L}^\infty(\Omega, [a, b])$  such that a subsequence of  $(\rho_\Delta)_\Delta$  converges to  $\rho$  a.e. on  $\Omega$ . To pass to the limit in (28) and conclude the analysis, we need to ensure that

$$\int_0^T \mathcal{R}_\Delta(\kappa_L, \kappa_R, t) \varphi(t, 0) dt \xrightarrow{\Delta \rightarrow 0} \int_0^T \mathcal{R}(\kappa_L, \kappa_R, t) \varphi(t, 0) dt.$$

This convergence claim comes from the measurability and the boundedness of  $t \mapsto \mathcal{R}(\kappa_L, \kappa_R, t)$ . Since  $t \mapsto \mathcal{R}_\Delta(\kappa_L, \kappa_R, t)$  is obtained by taking the mean values of  $t \mapsto \mathcal{R}(\kappa_L, \kappa_R, t)$  on a uniform grid, we are ensured that for all Lebesgue points  $t > 0$  of  $\mathcal{R}(\kappa_L, \kappa_R, \cdot)$  (which are a.e.  $t > 0$ ),  $(\mathcal{R}_\Delta(\kappa_L, \kappa_R, t))_\Delta$  converges to  $\mathcal{R}(\kappa_L, \kappa_R, t)$ . This last ingredient implies that by letting  $\Delta \rightarrow 0$  in (28), we obtain that  $\rho$  satisfies (9).

*Remark 9* As mentioned after Proposition 2, now that we proved the existence of solutions in the sense of inequality (9), we can prove, under the assumptions of Theorem 3, the uniqueness for the adapted entropy formulation (9) and the reciprocal statement of Proposition 2.

Indeed, first, let us observe that combining the last step of the convergence proof reasoning and Section 3.2, we proved a well-posedness result for solutions in the sense of inequality (9). Existence is obtained by the scheme. Moreover, two solutions in the sense of inequality (9) are also  $\mathcal{G}(t)$ -entropy solutions by Section 3.2; for those solutions we have uniqueness, see Theorem 1, leading to uniqueness for the adapted entropy formulation of Proposition 2.

Now fix  $\rho_o \in \mathbf{L}^\infty(\mathbb{R}, [a, b])$  and suppose that  $\rho$  is a  $\mathcal{G}(t)$ -entropy solution to (1) with initial data  $\rho_o$ . Suppose that  $\rho$  is not the solution in the sense of inequality (9). Let us then denote by  $\sigma \neq \rho$  the solution in the sense of inequality (9). As mentioned before,  $\sigma$  is also a  $\mathcal{G}(t)$ -entropy solution, and therefore equal to  $\rho$  by uniqueness. This contradicts the fact that  $\rho$  is not the solution in the sense of inequality (9). We conclude that  $\rho$  verifies (9).

## 4 Multi-interface problem with general interface coupling

This section is very brief because it borrows almost all of its contents to the corresponding [38, Section 4]. Indeed, the only changes are that we use Godunov interface flux in a more general situation leading to a more abstract form of the remainder term; and that we do not use any more the constraint inequalities required for sharp characterization of solutions for the adapted entropy formulation of [38]. We proceed in two steps. First the case of a single interface is dealt with calling upon the analysis of the previous section in what concerns

the remainder term  $\mathcal{R}_s$ . Second, the accurate use of partitions of unity along with the choice of test functions vanishing near the cross-points permits to reduce the general case described in Section 2.1 to the case of a single slanted interface.

#### 4.1 The case of a single slanted interface

This section builds on the work done in [38, Sections 1-3]. If  $y$  denotes the trajectory of the interface, the remainder term has now the form:

$$\mathcal{R}_{\dot{y}(t)}(\kappa_L, \kappa_R, t) = \left| g_{\dot{y}(t)}(\kappa_L) - \mathbf{F}_{\dot{y}(t)}^{int}(\kappa_L, \kappa_R, t) \right| + \left| \mathbf{F}_{\dot{y}(t)}^{int}(\kappa_L, \kappa_R, t) - f_{\dot{y}(t)}(\kappa_R) \right|.$$

Using Definition 2, the proof of uniqueness follows the ones of [38, Lemma 2.7, Theorem 2.8].

Regarding the existence, the only difference with the model case is that here we need to discretize the interface and to adapt the mesh in a neighbourhood of the discrete interface. We define the sequence of approximate slopes:

$$\forall n \in \mathbb{N}, s^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \dot{y}(t) dt; \quad \forall t \geq 0, s_\Delta(t) = \sum_{n \in \mathbb{N}} s^n \mathbb{1}_{[t^n, t^{n+1}[}(t),$$

which converges to  $\dot{y}$  in  $\mathbf{L}_{\text{loc}}^1([0, +\infty[)$ . The mesh is produced in the same way as in [38, Section 4], see in particular Figure 1 below.

One only needs to make precise the numerical flux used at the slanted mesh boundaries. At time step  $t^n$ , the approximate coupling Godunov flux is defined by:

$$\mathbf{God}_{int}^{s^n}(\rho_{-1/2}^n, \rho_{1/2}^n) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{F}_{s^n}^{int}(\rho_{-1/2}^n, \rho_{1/2}^n, t) dt. \quad (29)$$

Note that by assumption, the family of germs  $\{\mathcal{G}_s(t)\}_{t>0, s \in \mathbb{R}}$  is Carathéodory which implies that for all  $(\kappa_L, \kappa_R) \in [a, b]^2$ ,  $s \mapsto \mathbf{God}_{int}^{s^n}(\kappa_L, \kappa_R)$  is continuous since for all  $t > 0$ ,  $s \mapsto \mathbf{F}_s^{int}(\kappa_L, \kappa_R, t)$  is continuous.

The approximate remainder term that appears in the approximate adapted entropy inequalities is:

$$\mathcal{R}_{s_\Delta(t)}(\kappa_L, \kappa_R, t) = \left| g_{s^n}(\kappa_L) - \mathbf{God}_{int}^{s^n}(\kappa_L, \kappa_R) \right| + \left| \mathbf{God}_{int}^{s^n}(\kappa_L, \kappa_R) - f_{s^n}(\kappa_R) \right|$$

when  $t \in [t^n, t^{n+1}[$ . The convergence

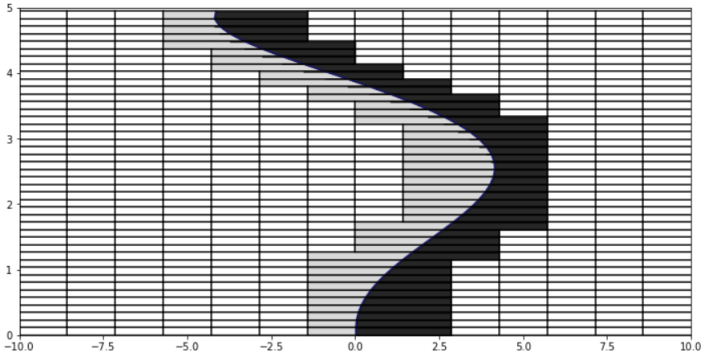
$$\int_0^{+\infty} \mathcal{R}_{s_\Delta(t)}(\kappa_L, \kappa_R, t) \varphi(t, y_\Delta(t)) dt \xrightarrow{\Delta \rightarrow 0} \int_0^{+\infty} \mathcal{R}_{\dot{y}(t)}(\kappa_L, \kappa_R, t) \varphi(t, y(t)) dt$$

comes from:

- the measurability of  $t \mapsto \mathcal{R}_{\dot{y}(t)}(\kappa_L, \kappa_R, t)$ ;



- the continuity of  $s \mapsto \mathcal{R}_s(\kappa_L, \kappa_R, t)$  ( $t > 0$ ) combined with the strong convergence of  $(s_\Delta)_\Delta$  to  $\dot{y}$ .



**Fig. 1** Illustration of the local modification to the mesh, in the  $(x, t)$  plane.

Existence of solutions in the sense of inequality (9) follows and like in Section 3.3.2, we can prove the equivalence between Definition 2 and inequality (9).

## 4.2 Isolating interfaces and neglecting cross-points

The construction of the mesh, of the finite volume scheme and the convergence analysis by reduction of the initial configuration of Section 2 to the case of a single slanted interface, tackled in Section 4.1 is identical to the reasoning of [38, Section 4]. Let us only highlight the fact that the choice of the precise coupling at the interfaces near the cross-points (see Figure 3 below) does not matter for the convergence of the scheme and its overall consistency. Indeed, the choice of the test functions in Remark 3 permits to disregard the discrete solution near cross-points. We only have to ensure that the discrete solutions take their values within  $[a, b]$ , for this sake we can use any numerical flux at the interfaces consistent with the stationary solutions  $a$  and  $b$ . In this way, we complete the proof of Theorem 3.

*Remark 10* In the practice of numerical approximation of problem (1) by the schemes we propose in this paper, any numerical flux which is monotone and whose Lipschitz constant is consistent with the CFL can be used on the interfaces in a small vicinity of the crossing points. This may lead to appearance of numerical artefacts (see [38, Section 4] for an example). However, the latter remain "under control" since, according to our analysis, they do not affect the convergence of the scheme.

## 5 Illustration with numerical simulations

In this section, we describe two simulations carried out in the framework of transmission maps set up on curved interfaces: the one-dimensional Hughes' model [48, 53–55] and the Burgers-particle interaction model [27, 56, 57]. Both

of them feature moving interfaces, actually they correspond to complex situations where the interface dynamics is coupled to the dynamics of the solution itself. Since the locations of the interfaces are not known *a priori*, the approach of this paper to construction of solutions is not directly applicable. However, if we consider the fixed-point strategy (see in particular [40] for the Hughes' model and [27] for the Burgers-particle model) then the step of resolution of the conservation law, given the interface locations, corresponds to the setting of our paper.

The Hughes' model makes appear solely one interface, while in the Burgers-particle(s) model multiple interfaces with crossing make sense (see, in particular, [57, 58]). In both situations, specific finite volume schemes were developed in the literature [59], [57, 58, 60, 61]. We will not apply these schemes; in order to stick to our main example detailed in Section 2.5, we will use interface fluxes based upon suitably identified transmission maps. Let us stress that in the Burgers-particle setting, the interface coupling is actually non-conservative but the adaptation of the transmission map approach to this situation is straightforward [18]; our theoretical result extends readily to such non-conservative interface couplings.

Note that the third example of simulations featuring slanted interfaces and interface crossings, in a setting significant in the road traffic context, can be found in [38]. The interface coupling corresponds, in this case, to flux limitation across interfaces, following [16, 39, 62]. The scheme of [38] is a particular case of the one considered in the present paper, with convergence analysis which exploited an adapted entropy formulation different from the one exhibited in the present paper. The analysis of the present paper applies to the situation considered in [38] (constant flux, featuring the same nonlinearity in the regions between the flux-limiting interfaces) and also to its generalization with piecewise constant flux.

## 5.1 One-dimensional Hughes model

The model of [53] for pedestrian evacuation through multiple exits, in the general multi-dimensional setting, features a conservation law with discontinuous field of directions given by the gradient of the solution to an eikonal equation. According to the analysis of [54] the one-dimensional situation in the corridor modeled by the interval  $[-1, 1]$  can be simplified, by explicitly solving the underlying eikonal equation, to

$$\partial_t \rho(t, x) + \partial_x (\text{sign}(x - \xi(t)) f(\rho(t, x))) = 0 \quad (30)$$

with  $f(\rho) = \rho v(\rho)$  a standard LWR (Lighthill-Whitham-Richards) flux, *e.g.* for  $v(\rho) = 1 - \rho$ , and  $\xi$  is a curve solving

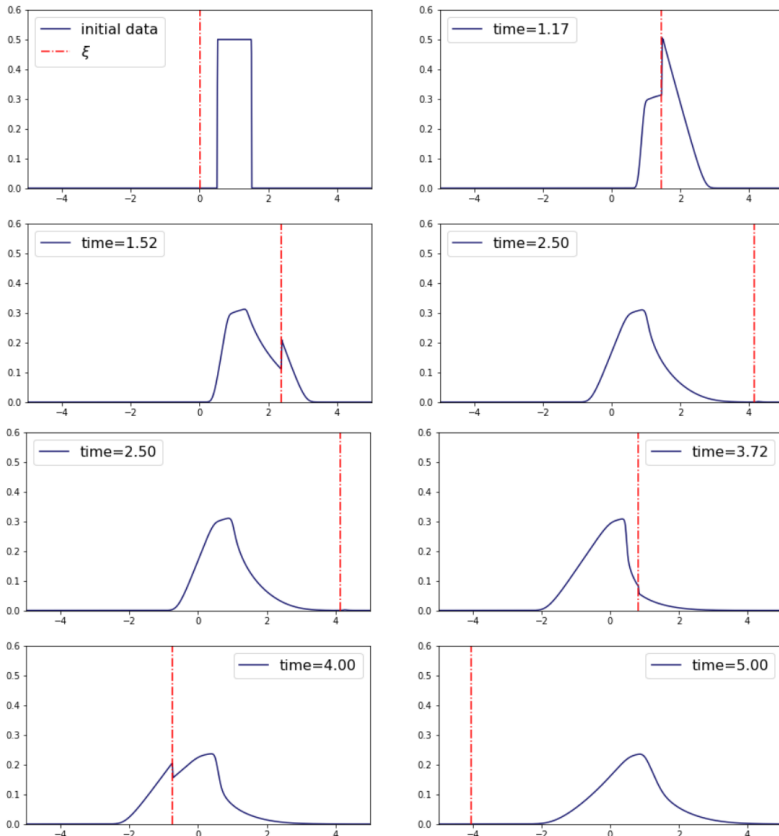
$$\int_{-1}^{\xi(t)} c(\rho(t, x)) \, dx = \int_{\xi(t)}^1 c(\rho(t, x)) \, dx$$

for some cost function  $c$ . In the situation considered in [40, 55],  $c(\rho) = 1 + \alpha\rho$ ,  $\alpha \in \mathbb{R}^+$ , is a linear cost and it is proved that the interface  $\xi$  is a Lipschitz

continuous function map of the time variable. The flux of (30) is equal to  $-f$  for  $x < \xi(t)$  and to  $+f$ , for  $x > \xi(t)$ . It can be proved (see [40]) that the coupling at the interface  $x = \xi(t)$  boils down to the mere continuity of the normal component of the flux, *i.e.*, to the Rankine-Hugoniot condition. Indeed, it can be checked that the set of all couples  $(\rho_L, \rho_R) \in [0, 1]^2$  fulfilling

$$-f(\rho_L) - \dot{\xi}(t)\rho_L = +f(\rho_R) - \dot{\xi}(t)\rho_R$$

constitutes an  $\mathbf{L}^1D$  germ. This means in particular that, whatever be the choice of a transmission map on the interface, the resulting solution corresponds to this unique possible germ (*cf.* [33]). For this reason, we choose the simplest transmission map  $\beta = \text{Id}$  to model the interface coupling in the Hughes' model (see [40] for details). Consider the Lipschitz curve  $\xi$  which graph is represented on Figure 1. We parametrize a corridor with the interval  $[-4, 4]$ . Below, we present the numerical solution to (30) with initial data  $\rho_o = 0.5 \times \mathbb{1}_{[0.5, 1.5]}$  and time horizon  $T = 5$ .



**Fig. 2** When  $\xi$  reaches the density, a "non classical" shock appears at the point where the directions of the propagation are opposite.

## 5.2 Fluid-particle interaction *via* a drag force

Fluid-particle interaction *via* a drag force was formalized in [56] as a one-dimensional model where the Burgers, equation with singular source carried by a curve,  $x = h(t)$

$$\partial_t \rho(t, x) + \partial_x (\rho(t, x)^2/2) = -\lambda(\rho(t, x) - h'(t))\delta_0(x - h(t)) \quad (31)$$

is coupled to an ODE with singular right-hand side

$$mh''(t) = \lambda(\rho(t, h(t)) - h'(t))$$

prescribing the evolution of  $h$ . While (31) features a non-conservative product in the source term, the interpretation of the problem in terms of a singular limit and of global conservation of the kinetic energy of the fluid-particle system gave rise to a definition in [56] of an adequate notion of solution. In [27, 57] the notion of solution for (31) was re-interpreted in terms of the “germ formalism” underlying the present paper, with a generalization allowing for a non-conservative germ. Further, in [18] this example was inserted into the framework of transmission maps, with the necessary non-conservative generalization of the latter. The transmission map in the context is constant in time, given by

$$\beta = \left\{ (k_-, k_+) \in \mathbb{R}^2, k_- - k_+ = \lambda \right\},$$

but one also introduces the associated “dissipation map”  $\psi$  taking the form

$$\psi_s(k_-, k_+) = \frac{\lambda}{2} (k_- + k_+ - 2s).$$

The fluxes in (31) are  $f_s(\rho) = g_s(\rho) = \frac{\rho^2}{2} - s\rho$  and the (conservative) Godunov transmission flux (13) should be replaced by two one-sided fluxes

$$\mathbf{F}_s^{int}(\kappa_L, \kappa_R)^- = (\mathbf{God}g_s)(\kappa_L, k_-), \quad \mathbf{F}_s^{int}(\kappa_L, \kappa_R)^+ = (\mathbf{God}f_s)(k_+, \kappa_R)$$

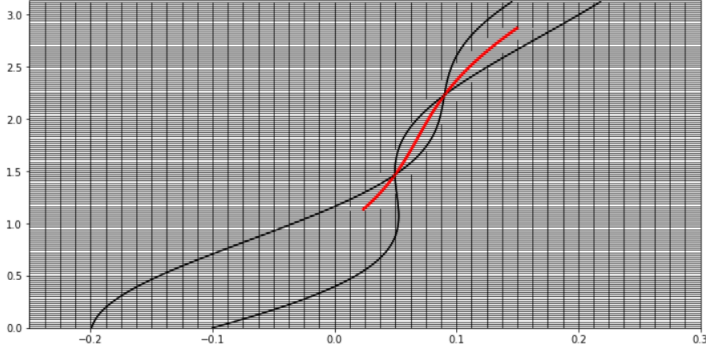
with  $(k_-, k_+) \in \beta$  found from the relation

$$(\mathbf{God}f_s)(k_+, \kappa_R) - (\mathbf{God}g_s)(\kappa_L, k_-) + \psi_s(k_-, k_+) = 0. \quad (32)$$

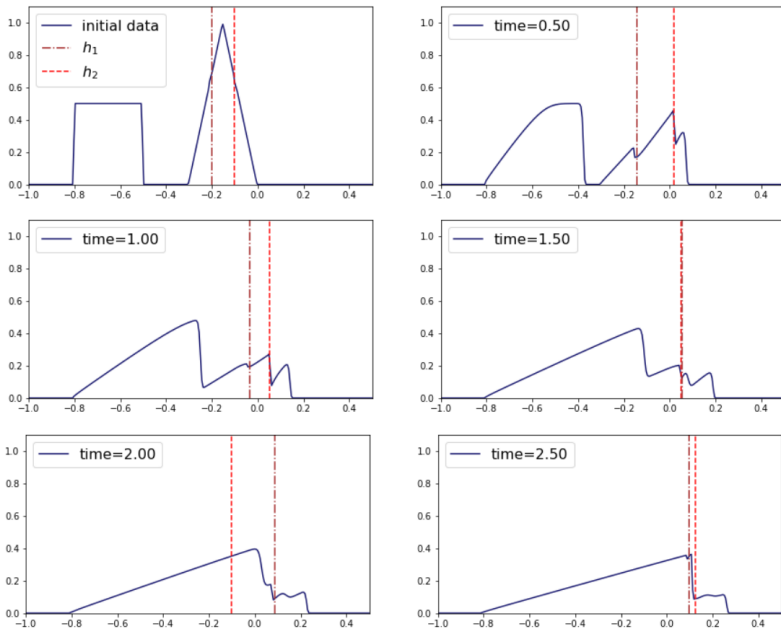
Note that, upon parametrization of  $\beta$  by  $p = k_- + k_+$ , the left-hand side of the above relation is a monotone increasing function of  $p$ . The model of [56] allows for the presence of multiple particles, with a “drafting-kissing-tumbling” phenomenon observed in simulations [57, 58].

We propose a numerical simulation whose purpose is merely to show that the scheme proposed is actually implementable and provides reasonable results. Consider two crossing trajectories  $h_1, h_2$  imitating the observed behavior, see Figure 3 below and also [58, Figure 4]. The idea is that the two particles are initially headed towards each other until colliding. In the full model the dynamics is then dictated by the fluid density and the ODEs. Here, we approximate

the density  $\rho$  in the model analogous to (31) with two particles located at  $x = h_1(t)$ ,  $x = h_2(t)$ , respectively. The evolution of the numerical simulation is presented in Figure 4.



**Fig. 3** Illustration of the local modifications of the mesh; when the curves are to close, they are replaced by their mean curve.



**Fig. 4** Evolution of the numerical solution of (31).

## Acknowledgment

This paper has been supported by the RUDN University Strategic Academic Leadership Program.

## References

- [1] Colombo, R.M., Perrollaz, V., Sylla, A.: Conservation Laws and Hamilton-Jacobi Equations with Space Inhomogeneity. Submitted (2022)
- [2] Kruzhkov, S.N.: First order quasilinear equations with several independent variables. *Math. USSR-Sbornik* **81**(123), 228–255 (1970)
- [3] Bressan, A., Guerra, G., Shen, W.: Vanishing viscosity solutions for conservation laws with regulated flux. *J. Differ. Equ.* **266**(1), 312–351 (2019)
- [4] Karlsen, K.H., Towers, J.D.: Convergence of the Lax-Friedrichs scheme and stability for conservation laws with a discontinuous space-time dependent flux. *Chinese Ann. Math. Ser. B* **25**(3), 287–318 (2004)
- [5] Panov, E.Y.: Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux. *Arch. Ration. Mech. Anal.* **195**(2), 643–673 (2010)
- [6] Gimse, T., Risebro, N.H.: Solution of the Cauchy problem for a conservation law with a discontinuous flux function. *SIAM J. Math. Anal.* **23**(3), 635–648 (1992)
- [7] Diehl, S.: On scalar conservation laws with point source and discontinuous flux function. *SIAM J. Math. Anal.* **26**(6), 1425–1451 (1995)
- [8] Diehl, S.: A uniqueness condition for nonlinear convection-diffusion equations with discontinuous coefficients. *J. Hyperbolic Differ. Equ.* **6**(1), 127–159 (2009)
- [9] Andreianov, B., Karlsen, K.H., Risebro, N.H.: A theory of  $L^1$ -dissipative solvers for scalar conservation laws with discontinuous flux. *Archive for Rational Mechanics and Analysis* **201**(1), 27–86 (2011)
- [10] Kaasschieter, E.F.: Solving the Buckley-Leverett equation with gravity in a heterogeneous porous medium. *Comput. Geosci.* **3**(1), 23–48 (1999)
- [11] Adimurthi, Mishra, S., Gowda, G.D.V.: Optimal entropy solutions for conservation laws with discontinuous flux-functions. *J. Hyperbolic Differ. Equ.* **2**(4), 783–837 (2005)
- [12] Cancès, C.: Asymptotic behavior of two-phase flows in heterogeneous porous media for capillarity depending only on space. I. Convergence to the optimal entropy solution. *SIAM J. Math. Anal.* **42**(2), 946–971 (2010)
- [13] Andreianov, B., Rosini, M.D.: Microscopic selection of solutions to scalar conservation laws with discontinuous flux in the context of vehicular traffic. In: *Semigroups of Operators – Theory and Applications.*, pp. 113–135. Cham: Springer, (2020)
- [14] Cancès, C.: On the effects of discontinuous capillarities for immiscible two-phase flows in porous media made of several rock-types. *Netw. Heterog. Media* **5**(3), 635–647 (2010)
- [15] Andreianov, B., Cancès, C.: Vanishing capillarity solutions of Buckley-Leverett equation with gravity in two-rocks’ medium. *Comput. Geosci.* **17**(3), 551–572 (2013)
- [16] Colombo, R.M., Goatin, P.: A well posed conservation law with a variable unilateral constraint. *J. Differ. Equ.* **234**(2), 654–675 (2007)

- [17] Andreianov, B., Goatin, P., Seguin, N.: Finite volume schemes for locally constrained conservation laws. *Numer. Math.* **115**(4), 609–645 (2010)
- [18] Andreianov, B.: New approaches to describing admissibility of solutions of scalar conservation laws with discontinuous flux. *ESAIM: Proceedings and Surveys* **50**, 40–65 (2015)
- [19] Andreianov, B., Sbihi, K.: Well-posedness of general boundary-value problems for scalar conservation laws. *Trans. Amer. Math. Soc.* **367**(6), 3763–3806 (2015)
- [20] Crasta, G., De Cicco, V., De Philippis, G., Ghiraldin, F.: Structure of solutions of multidimensional conservation laws with discontinuous flux and applications to uniqueness. *Arch. Ration. Mech. Anal.* **221**(2), 961–985 (2016)
- [21] Andreianov, B., Mitrović, D.: Entropy conditions for scalar conservation laws with discontinuous flux revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32**(6), 1307–1335 (2015)
- [22] Audusse, E., Perthame, B.: Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies. *Proc. Roy. Soc. Edinburgh Sect. A* **135**(2), 253–265 (2005)
- [23] Panov, E.Y.: On existence and uniqueness of entropy solutions to the Cauchy problem for a conservation law with discontinuous flux. *J. Hyperbolic Differ. Equ.* **6**(3), 525–548 (2009)
- [24] Towers, J.D.: Convergence of a difference scheme for conservation laws with a discontinuous flux. *SIAM J. Numer. Anal.* **38**(2), 681–698 (2000)
- [25] Adimurthi, Gowda, G.D.V.: Conservation law with discontinuous flux. *J. Math. Kyoto Univ.* **43**(1), 27–70 (2003)
- [26] Bürger, R., Karlsen, K.H., Towers, J.D.: An Engquist-Osher-type scheme for conservation laws with discontinuous flux adapted to flux connections. *SIAM J. Numer. Anal.* **47**(3), 1684–1712 (2009)
- [27] Andreianov, B., Lagoutière, F., Seguin, N., Takahashi, T.: Well-posedness for a one-dimensional fluid-particle interaction model. *SIAM J. Math. Anal.* **46**(2), 1030–1052 (2014)
- [28] Karlsen, K.H., Risebro, N.H., Towers, J.D.:  $L^1$  stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients. *Skr. K. Nor. Vidensk. Selsk.* **1**(3), 1–49 (2003)
- [29] Coclite, G.M., Risebro, N.H.: Conservation laws with time dependent discontinuous coefficients. *SIAM J. Math. Anal.* **36**(4), 1293–1309 (2005)
- [30] Andreianov, B., Karlsen, K.H., Risebro, N.H.: On vanishing viscosity approximation of conservation laws with discontinuous flux. *Netw. Heterog. Media* **5**(3), 617–633 (2010)
- [31] Karlsen, K.H., Towers, J.D.: Convergence of a Godunov scheme for conservation laws with a discontinuous flux lacking the crossing condition. *J. Hyperbolic Differ. Equ.* **14**(4), 671–701 (2017)
- [32] Towers, J.D.: Convergence via OSLC of the godunov scheme for a scalar conservation law with time and space flux discontinuities. *Numerische Mathematik* **139**(4), 939–969 (2018)
- [33] Andreianov, B., Cancès, C.: On interface transmission conditions for conservation laws with discontinuous flux of general shape. *J. Hyperbolic Differ. Equ.* **12**(2), 343–384 (2015)
- [34] Bachmann, F., Vovelle, J.: Existence and uniqueness of entropy solution of scalar conservation laws with a flux function involving discontinuous coefficients. *Comm. Partial Differ. Equ.* **31**(1-3), 371–395 (2006)

- [35] Shen, W.: On the uniqueness of vanishing viscosity solutions for Riemann problems for polymer flooding. *NoDEA Nonlinear Differential Equations Appl.* **24**(4), 37–25 (2017)
- [36] Bürger, R., Karlsen, K.H., Klingenberg, C., Risebro, N.H.: A front tracking approach to a model of continuous sedimentation in ideal clarifier-thickener units. *Nonlin. Anal. Real World Appl.* **4**(3), 457–481 (2003)
- [37] Garavello, M., Natalini, R., Piccoli, B., Terracina, A.: Conservation laws with discontinuous flux. *Netw. Heterog. Media* **2**(1), 159–179 (2007)
- [38] Sylla, A.: A LWR model with constraints at moving interfaces. *ESAIM: Math. Model. Numer. Anal.* **56**(3), 1081–1114 (2022)
- [39] Sylla, A.: Influence of a slow moving vehicle on traffic: Well-posedness and approximation for a mildly nonlocal model. *Networks & Heterogeneous Media* **16**(2), 221–256 (2021)
- [40] Andreianov, B., Girard, T.: Existence of solutions to a class of one-dimensional models for pedestrian evacuation. Preprint hal-03937464 (2023)
- [41] Vasseur, A.: Strong traces for solutions of multidimensional scalar conservation laws. *Arch. Ration. Mech. Anal.* **160**(3), 181–193 (2001)
- [42] Panov, E.Y.: Existence of strong traces for quasi-solutions of multidimensional conservation laws. *J. Hyperbolic Differ. Equ.* **4**(4), 729–770 (2007)
- [43] Cancès, C., Gallouët, T.: On the time continuity of entropy solutions. *J. Evol. Equ.* **11**(1), 43–55 (2011)
- [44] Colombo, R.M., Mercier, M., Rosini, M.D.: Stability and total variation estimates on general scalar balance laws. *Comm. Math. Sci.* **7**(1), 37–65 (2009)
- [45] Aleksić, J., Mitrović, D.: Strong traces for averaged solutions of heterogeneous ultra-parabolic transport equations. *J. Hyperbolic Differ. Equ.* **10**(4), 659–676 (2013)
- [46] Neves, W., Panov, E.Y., Silva, J.: Strong traces for conservation laws with general nonautonomous flux. *SIAM J. Math. Anal.* **50**(6), 6049–6081 (2018)
- [47] Crasta, G., De Cicco, V., De Philippis, G.: Kinetic formulation and uniqueness for scalar conservation laws with discontinuous flux. *Comm. Partial Differential Equations* **40**(4), 694–726 (2015)
- [48] Amadori, D., Goatin, P., Rosini, M.D.: Existence results for Hughes’ model for pedestrian flows. *J. Math. Anal. Appl.* **420**(1), 387–406 (2014)
- [49] Cancès, C., Seguin, N.: Error estimate for Godunov approximation of locally constrained conservation laws. *SIAM J. Numer. Anal.* **50**(6), 3036–3060 (2012)
- [50] Panov, E.Y.: On strong precompactness of bounded sets of measure-valued solutions of a first order quasilinear equation. *Sbornik: Mathematics* **186**(5), 729 (1995)
- [51] Eymard, R., Gallouët, T., Herbin, R.: *Finite Volume Methods. Handbook of Numerical Analysis*, vol. VII, pp. 713–1020. North-Holland, Amsterdam, ??? (2000)
- [52] Andreianov, B., Coclite, G.M., Donadello, C.: Well-posedness for vanishing viscosity solutions of scalar conservation laws on a network. *Discrete & Continuous Dynamical Systems* **37**(11), 5913–5942 (2017)
- [53] Hughes, R.L.: A continuum theory for the flow of pedestrians. *Transportation Research Part B: Methodological* **36**(6), 507–535 (2002)
- [54] El-Khatib, N., Goatin, P., Rosini, M.D.: On entropy weak solutions of Hughes’ model for pedestrian motion. *Z. Angew. Math. Phys.* **64**(2), 223–251 (2013)



- [55] Andreianov, B., Rosini, M.D., Stivaletta, G.: On existence, stability and many-particle approximation of solutions of 1D Hughes model with linear costs. submitted (2021)
- [56] Lagoutière, F., Seguin, N., Takahashi, T.: A simple 1D model of inviscid fluid-solid interaction. *J. Differential Equations* **245**(11), 3503–3544 (2008)
- [57] Andreianov, B., Lagoutière, F., Seguin, N., Takahashi, T.: Small solids in an inviscid fluid. *Networks & Heterogeneous Media* **5**(3), 385 (2010)
- [58] Towers, J.D.: The Lax-Friedrichs scheme for interaction between the inviscid Burgers equation and multiple particles. *Netw. Heterog. Media* **15**(1), 143–169 (2020)
- [59] Gyamfi, K.A.: Analysis of entropy solutions to conservation laws with discontinuous flux in space and time. PhD thesis, Univ. Degli Studi dell’Aquila (2021)
- [60] Aguillon, N., Lagoutière, F., Seguin, N.: Convergence of finite volume schemes for the coupling between the inviscid Burgers equation and a particle. *Math. Comp.* **86**(303), 157–196 (2017)
- [61] Towers, J.D.: A fixed grid, shifted stencil scheme for inviscid fluid-particle interaction. *Appl. Numer. Math.* **110**, 26–40 (2016)
- [62] Delle Monache, M.L., Goatin, P.: Scalar conservation laws with moving constraints arising in traffic flow modeling: an existence result. *J. Differential Equations* **257**(11), 4015–4029 (2014)