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ON SOME ALGORITHMIC ASPECTS OF HYPERGRAPHIC MATROIDS

MOURAD BAÏOU AND FRANCISCO BARAHONA

ABSTRACT. Hypergraphic matroids were studied first by Lorea [18] and later by Frank et al [8]. They can be seen as generalizations of graphic matroids. Here we show that several algorithms developed for the graphic case can be extended to hypergraphic matroids. We treat the following: the separation problem for the associated polytope, testing independence, separation of partition inequalities, computing the rank of a set, computing the strength, computing the arboricity and network reinforcement.

1. INTRODUCTION

Hypergraphic matroids were introduced by Lorea [18] and later studied by Frank et al [8]. They showed that the notion of circuit-matroid of graphs can be generalized to hypergraphs. In [8] they generalized the notion of spanning trees to hypertrees and extended a theorem of Tutte [24] and Nash-Williams [20], to give the maximum number of disjoint hypertrees contained in a hypergraph.

1.1. Previous work and our contribution. For graphic matroids different algorithms have been developed based on the graph structure. Here we extend some of these algorithms to the hypergraphic case. More precisely, we give an algorithm for separating a vector from the associated matroid polytope. For the graphic case this had been treated in [22] and [21]. We also show that when applying the greedy algorithm to find a maximum weighted independent set [6], testing independence at each iteration can be done by finding a minimum cut in an associated graph. We also treat partition inequalities, this leads to an algorithm to compute the rank of a set. For the graphic case, partition inequalities were treated in [4] and [1]. The maximum number of disjoint spanning trees in a graph has been proposed in [14] as a measure of the *strength* of a network. An algorithm for computing the strength of a graph was given in [4], and later in [15, 3], here we give an algorithm to compute a similar measure in a hypergraph, namely the maximum number of disjoint hypertrees. The *arboricity* of a graph is the minimum number of edge-disjoint forests into which the edge-set can be decomposed. Algorithms to compute the arboricity were given in [22, 11, 10], here we give an algorithm to compute a similar measure in hypergraphs. Also the *reinforcement problem* was studied in [4], this consists of given an initial graph G , a target value k and a set of candidate edges, find a minimum cost set of candidate edges to be added to G so that the resulting graph has k disjoint spanning trees. Here we give an algorithm for a similar reinforcement problem in

a hypergraph. As pointed out in [8], some of these problems can be solved using general matroid algorithms, here we show that they can be reduced to a sequence of minimum cut problems.

1.2. Organization. This paper is organized as follows. In Section 2 we give some definitions and notation. In Section 3 we study the separation problem and the greedy algorithm. Section 4 is devoted to partition inequalities and to computing the rank of a set. The strength of a hypergraph is studied in Section 5. The arboricity is studied in Section 6. In Section 7 we study the reinforcement problem.

2. PRELIMINARIES

Let $H = (V, E)$ be hypergraph. For a non-empty set $X \subset V$ and $F \subseteq E$, $F[X]$ denotes the set of hyperedges in F contained in X . Let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a family of non-empty subsets of V with $V_i \cap V_j = \emptyset$ for $i \neq j$, we denote by $\delta_F(\mathcal{P})$ the set of hyperedges in F included in $\cup_i V_i$ and that intersect at least two sets in \mathcal{P} . For a hypergraph $H' = (V, E')$ sometimes we use $\delta_{H'}(\mathcal{P})$ instead of $\delta_{E'}(\mathcal{P})$. Also when there is no confusion we use $\delta(\mathcal{P})$ instead of $\delta_F(\mathcal{P})$.

A linear system $Ax \leq b$, where A and b are rational, is called *totally dual integral* (TDI) if for any $c \in \mathbb{Z}^n$ such that there is a feasible bounded solution to the linear program $\max\{c^T x \mid Ax \leq b\}$, there is an integer optimal dual solution. Edmonds and Giles [7] showed that if a polyhedron P is the solution set of a TDI system $Ax \leq b$, where b has all integer entries, then every vertex of P is integer-valued.

For a *multiset* $F \subseteq E$, its *incidence vector* $x^F \in \mathbb{R}^E$ is defined as follows, $x^F(e)$ is the *multiplicity* of e in F , if $e \in F$, and $x^F(e) = 0$ otherwise. For a vector $x \in \mathbb{R}^E$, and $S \subseteq E$ we use $x(S)$ to denote $\sum_{e \in S} x(e)$.

Let $D = (V, A)$ be a directed graph, for $S \subseteq V$, we denote by $\delta^+(S)$ the set $\delta^+(S) = \{(u, v) \in A \mid u \in S, v \notin S\}$. Given two distinguished vertices s and t , for a set $S \subset V$, with $s \in S$, $t \notin S$, the set of arcs $\delta^+(S)$ is called an *st-cut*. Given a capacity vector $c \in \mathbb{R}_+^A$, a *minimum st-cut* is an *st-cut* $\delta^+(S)$ such that $c(\delta^+(S))$ is minimum. A minimum *st-cut* can be found in $O(|V|^3)$ time, see [12].

A *hyperforest* in $H = (V, E)$ is a set $F \subseteq E$ such that $|F[X]| \leq |X| - 1$ for every non-empty $X \subseteq V$. A hyperforest F is called a *hypertree* of H if $|F| = |V| - 1$. If H is a graph, $F \subseteq E$ is a hypertree if and only if F is a spanning tree. It was proven by Lorea [18] that the hyperforests of a hypergraph form the family of independent sets of a matroid. These are called *hypergraphic matroids*. Frank et al. [8] further studied hypergraphic matroids. Hypertrees are bases of hypergraphic matroids.

3. BASIC ALGORITHMS

Here we study the separation problem for polytopes of hypergraphic matroids, and the application of the greedy algorithm to find a maximum weighted independent set.

3.1. Separating from the associated polytope. For a hypergraph $H = (V, E)$, let $P(H)$ be the convex hull of incidence vectors of independent sets of the associated matroid. Let $r(S)$ be the rank of S , for $S \subseteq E$. A fundamental theorem of Edmonds [6] gives an explicit description of $P(H)$ as below.

Theorem 1. $P(H) = \{x \in \mathbb{R}^E \mid x \geq 0, x(S) \leq r(S), \forall S \subseteq E\}$.

The *separation problem* for $P(H)$ consists of given a vector \bar{x} , find a hyperplane separating \bar{x} from $P(H)$, or decide that $\bar{x} \in P(H)$. For the graphic case this has been solved in [22] and [21]. A polynomial algorithm for the separation problem enables us to optimize a linear function over $P(H)$ in polynomial time, cf. [13]. Frank et al [8] proved the following.

Theorem 2. For $S \subseteq E$, the rank of S , $r(S)$ is given by the following formula:

$$r(S) = \min\{|V| - |\mathcal{P}| + |\delta_S(\mathcal{P})| : \mathcal{P} \text{ a partition of } V\}.$$

We assume that $0 \leq \bar{x}(e) \leq 1$ for all $e \in E$, otherwise we have a separating hyperplane immediately. Thus for $S \subseteq E$, let us suppose that there is a partition $\mathcal{P} = \{V_1, \dots, V_k\}$ of V such that

$$\bar{x}(S) > |V| - |\mathcal{P}| + |\delta_S(\mathcal{P})|.$$

Since $\bar{x}(e) \leq 1$, for all $e \in E$, we have $\bar{x}(\delta_S(\mathcal{P})) \leq |\delta_S(\mathcal{P})|$. And $\bar{x}(S) = \bar{x}(\delta_S(\mathcal{P})) + \sum_i \bar{x}(S[V_i])$, therefore $\sum_i \bar{x}(S[V_i]) > |V| - |\mathcal{P}| = \sum_{i=1}^k (|V_i| - 1)$. Thus $\bar{x}(S[V_r]) > |V_r| - 1$ for at least one index r . This suggest to look for a non-empty node-set $W \subset V$ that maximizes $\bar{x}(E[W]) - |W| + 1$. This can be reduced to a sequence of minimum cut problems as follows. This construction is inspired on a construction of Lawler [17].

We define a directed graph $D = (V', A)$, with $V' = \{s, t\} \cup V \cup \{e', e'' : e \in E\}$. The arc set is defined as follows:

- (a) For every node $v \in V$ we set an arc $(s, v) \in A$ with capacity equal to one.
- (b) For every hyperedge $e \in E$ we define an arc (e', e'') with capacity $\bar{x}(e)/2$, also for each node $u \in e$ we define the arcs (u, e') and (e'', u) with infinite capacity. We also define arcs (e', t) and (e'', t) with capacity $\bar{x}(e)/2$.
- (c) Finally we pick a node $\bar{v} \in V$ and add an arc (\bar{v}, t) with infinite capacity. This is to avoid having $\delta^+(V' \setminus \{t\})$ as a minimum st -cut.

We find a minimum st -cut, and we repeat for each $\bar{v} \in V$.

Assume that $\{s\} \cup T$ defines one of these minimum cuts. Let $T' = V \cap T$, $\bar{T}' = V \setminus T'$, $E_1 = \{e \mid e' \in T, e'' \notin T\}$, $E_2 = \{e \mid e', e'' \in T\}$, $E_3 = \{e \mid e', e'' \notin T\}$. The capacity of this cut is

$$|\bar{T}'| + \bar{x}(E_1) + \bar{x}(E_2) = |\bar{T}'| - \bar{x}(E_3) + \bar{x}(E).$$

Notice that E_3 is the set of hyperedges contained in \bar{T}' . Thus for each $\bar{v} \in V$ we have minimized $|W| - \bar{x}(E[W])$, with the constraint that $\bar{v} \in W$. If the value of the minimum is less than one we have the separating hyperplane $\bar{x}(E_3) > |T'| - 1$. Now we can state the following.

Theorem 3. For hypergraphic matroids the separation problem for the associated polytope reduces to $|V|$ minimum cut problems in a graph with $O(|V| + |E|)$ nodes.

3.2. Finding a maximum weighted hyperforest. For a weight vector $w \in \mathbb{R}_+^E$, we deal with the problem of finding an independent set I such that $w(I)$ is maximum. Edmonds [6] showed that this can be done with the greedy algorithm below.

Step 0. Order the elements so that $w(e_1) \geq w(e_2) \geq \dots w(e_m)$. Set $I = \emptyset$.

Step 1. For $i = 1, \dots, m$ do
 if $I \cup \{e_i\}$ is independent, set $I \leftarrow I \cup \{e_i\}$.

Here we need to show how to decide whether the set $I' = I \cup \{e_i\}$ in Step 1, remains independent. This is equivalent to look for the minimum of

$$|\cup \{e \mid e \in F\}| - |F|$$

for $F \subseteq I'$, with $e_i \in F$. Then I' is independent if and only if the minimum is at least 1.

Now we have to see that this reduces to a minimum cut problem. For that we build a directed graph with node-set $\{s, t\} \cup \{e \mid e \in I'\} \cup \{v \mid v \in V'\}$. Here $V' = \cup \{e \mid e \in I'\}$. The arc-set is as follows:

- There is an arc (s, e) for each $e \in I$, with capacity equal to one.
- There is an arc (v, t) for each $v \in V'$, with capacity equal to one.
- There is an arc (e, v) for each $e \in I'$, and for each $v \in e$, with infinite capacity.
- There is an arc (s, e_i) with infinite capacity.

Assume that $\{s\} \cup T$ defines a minimum st -cut. Then $F = T \cap I'$, and $\cup \{e \mid e \in F\} = T \cap V'$. If C is the capacity of this cut, then $C - |I'| = |\cup \{e \mid e \in F\}| - |F|$ is the minimum value needed. We can state the following.

Theorem 4. *Finding a maximum weighted hyperforest reduces to $|E|$ minimum cut problems in a graph with $O(|V| + |E|)$ nodes.*

4. PARTITION INEQUALITIES AND COMPUTING THE RANK

Here we study the separation problem for partition inequalities. This will be used to compute the rank of a set, and also it will be used in the next two sections. Frank et al [8] proved the following.

Theorem 5. *A hypergraph contains k disjoint hypertrees if and only if*

$$|\delta(\mathcal{P})| \geq k(|\mathcal{P}| - 1)$$

holds for every partition \mathcal{P} of V .

Thus for a hypergraph $H = (V, E)$, we can write the system of inequalities below.

- (1) $x(\delta(\mathcal{P})) \geq |\mathcal{P}| - 1$, for all partitions \mathcal{P} of V ,
- (2) $x \geq 0$.

Let P be the polyhedron defined by (1)-(2). It can have extreme points with components greater than one. For instance if $H = (V = \{v_1, v_2, v_3\}, E = \{e_1, e_2\})$ and $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_1, v_2, v_3\}$, then $P = \{(x_1, x_2) : x_1 + x_2 \geq 2, x \geq 0\}$. The extreme points are $(2, 0)$ and $(0, 2)$. If H is a graph, this is an unbounded polyhedron

whose extreme points are the incidence vectors of spanning trees. This follows from a theorem of Tutte [24] and Nash-Williams [20]. Theorem 5 extends this to hypergraphs.

Corollary 6. *The polyhedron P has integral extreme points.*

Proof. First notice that the incidence vector of any hypertree satisfies (1)-(2). Let \bar{x} be an extreme point of P . This is a rational vector, so there is a positive integer k such that $k\bar{x}$ is an integer vector. Consider the hypergraph $\bar{H} = (V, \bar{E})$, where for each $e \in E$ there are $k\bar{x}(e)$ copies of e in \bar{E} . Theorem 5 implies that \bar{H} contains k disjoint hypertrees. Thus there are k hypertrees T_1, \dots, T_k such that $k\bar{x} \geq (x^{T_1} + \dots + x^{T_k})$. Here each edge e is being used at most $k\bar{x}(e)$ times, and it could be used more than once in some hypertrees. Thus $\bar{x} \geq \frac{1}{k}(x^{T_1} + \dots + x^{T_k})$.

Since \bar{x} is an extreme point, we have $\bar{x} = \frac{1}{k}(x^{T_1} + \dots + x^{T_k})$, and $x^{T_1} = \dots = x^{T_k}$. \square

Now we discuss the separation problem for inequalities (1). We assume that $\bar{x} \in \mathbb{R}_+^E$ is an input vector. We are going to solve

$$(3) \quad \text{minimize} \quad \bar{x}(\delta(\mathcal{P})) - \beta(|\mathcal{P}| - 1).$$

Where the minimum is taken among all partitions \mathcal{P} of V . Since the partition $\mathcal{P} = \{V\}$ is also possible, the minimum is always less than or equal to zero. In this section we need $\beta = 1$, but in the following sections this number could be different.

We fix an arbitrary node $r \in V$, and for each non-empty $S \subseteq V$ we define

$$(4) \quad f(S) = \begin{cases} \beta + \bar{x}(E[S]), & \text{if } r \notin S, \\ \bar{x}(E[S]), & \text{if } r \in S. \end{cases}$$

The function f is intersecting supermodular, i.e., $f(S \cup T) + f(S \cap T) \geq f(S) + f(T)$ for $S, T \subseteq V$, with $S \cap T \neq \emptyset$. We associate a variable $y(v)$ to each node $v \in V$, and we propose to solve the linear program

$$(5) \quad \min y(V)$$

$$(6) \quad y(S) \geq f(S) \text{ for } S \subseteq V.$$

Edmonds [5] showed that this can be solved with the greedy algorithm below.

Step 0. Set $\bar{y}(v) = \beta + \bar{x}(E)$, for all $v \in V$; $\mathcal{F} = \emptyset$.

Step 1. Pick a node \bar{v} that does not belong to a set in \mathcal{F} . If such a node does not exist stop, otherwise let $\bar{S} = \operatorname{argmin}\{\bar{y}(S) - f(S) \mid \bar{v} \in S\}$, and let $\alpha = \bar{y}(\bar{S}) - f(\bar{S})$.

Set $\bar{y}(\bar{v}) \leftarrow \bar{y}(\bar{v}) - \alpha$. $\mathcal{F} \leftarrow \mathcal{F} \cup \{\bar{S}\}$.

Step 2. Uncross: while there are two sets S and T in \mathcal{F} with $S \cap T \neq \emptyset$, do $\mathcal{F} \leftarrow \mathcal{F} \setminus \{S, T\} \cup \{S \cup T\}$. Go to Step 1.

The vector \bar{y} is built so it satisfies (6). A set $S \subseteq V$ is called *tight* if $\bar{y}(S) = f(S)$. It follows from the supermodularity of f that if S and T are tight and $S \cap T \neq \emptyset$, then $S \cup T$ is also tight. This justifies the uncrossing operation in Step 2. At the end the family \mathcal{F} defines a partition of V so that $\bar{y}(S) = f(S)$ for every $S \in \mathcal{F}$. Now we discuss the two possible values of the solution.

- If the value of the optimum is $\bar{x}(E)$ we can take any partition $\{V_1, \dots, V_p\}$ of V , add the associated inequalities (6), and we have

$$\bar{x}(E) = \bar{y}(V) = \sum \bar{y}(V_i) \geq \sum f(V_i) = \sum_{i=1}^p \bar{x}(E(V_i)) + \beta(p-1).$$

Since $\bar{x}(E) = \sum \bar{x}(E(V_i)) + \bar{x}(\delta(V_1, \dots, V_p))$, we obtain

$$\bar{x}(\delta(V_1, \dots, V_p)) \geq \beta(p-1).$$

This shows that inequalities (1) are satisfied.

- Now assume that the value of the optimum is greater than $\bar{x}(E)$. We set $\bar{z}_S = 1$ if $S \in \mathcal{F}$ and $\bar{z}_S = 0$ otherwise, for each $S \subseteq V$. This vector is a feasible solution of the dual of (5)-(6). We have $\bar{y}(V) = \sum \{\bar{y}(S) \mid S \in \mathcal{F}\} = \sum \{f(S) \mid S \in \mathcal{F}\} = \sum \{f(S)\bar{z}_S \mid S \subseteq V\}$. This shows that \bar{z} is an optimal dual solution. Let $\mathcal{F} = \{W_1, \dots, W_p\}$. We have

$$\sum \{f(S) \mid S \in \mathcal{F}\} = \sum_i \bar{x}(E(W_i)) + \beta(p-1) > \bar{x}(E).$$

Then

$$0 > \bar{x}(E) - \sum_i \bar{x}(E(W_i)) - \beta(p-1) = \bar{x}(\delta(W_1, \dots, W_p)) - \beta(p-1).$$

Since \bar{z} is an optimal solution, we have a most violated partition inequality.

It remains to show how to find the set \bar{S} in Step 1. For that we define a directed graph $D = (V', A)$, with $V' = \{s, t\} \cup V \cup \{e', e'' : e \in E\}$. Let $\eta(v) = \bar{y}(v)$ for $v \neq r$, and $\eta(r) = \bar{y}(r) + \beta$. Let $\eta^+(v) = \max\{0, \eta(v)\}$, and $\eta^-(v) = \min\{0, \eta(v)\}$ for $v \in V$. The arc set is defined as follows.

- For every node $v \in V$ we set an arc $(s, v) \in A$ with capacity equal to $\eta^+(v)$, and we set an arc $(v, t) \in A$ with capacity equal to $-\eta^-(v)$.
- For every hyperedge $e \in E$ we define an arc (e', e'') with capacity $\bar{x}(e)/2$, also for each node $u \in e$ we define the arcs (u, e') and (e'', u) with infinite capacity. We also define arcs (e', t) and (e'', t) with capacity $\bar{x}(e)/2$.
- Finally we add an arc (\bar{v}, t) with infinite capacity.

Assume that $\{s\} \cup S$ defines a minimum st -cut. Let $S' = V \cap S$, $\bar{S}' = V \setminus S'$, $E_1 = \{e \mid e' \in S, e'' \notin S\}$, $E_2 = \{e \mid e', e'' \in S\}$, $E_3 = \{e \mid e', e'' \notin S\}$. The capacity of this cut is

$$\eta^+(\bar{S}') - \eta^-(S') + \bar{x}(E_1) + \bar{x}(E_2) = \eta(\bar{S}') - \bar{x}(E_3) + \bar{x}(E) - \eta^-(V).$$

Notice that E_3 is the set of hyperedges contained in \bar{S}' . Thus if γ is the capacity of the minimum cut obtained, then $\gamma - \beta - \bar{x}(E) + \eta^-(V)$ is the value α in Step 1. This minimizes $\bar{y}(W) - f(W)$, with the constraint that $\bar{v} \in W$. Now we can state the following.

Theorem 7. *The separation problem for inequalities (1) reduces to $|V|$ minimum cut problems in a graph with $O(|V| + |E|)$ nodes.*

Theorem 2 shows that given a set $F \subseteq E$ the rank of F is given by

$$\min\{|\delta_F(\mathcal{P})| - |\mathcal{P}| + |V| : \mathcal{P} \text{ a partition of } V\}.$$

Thus we just have to apply the separation algorithm above to the incidence vector of F . This leads to the following.

Corollary 8. *Computing the rank of a set $F \subseteq E$ reduces to $|V|$ minimum cut problems in a graph with $O(|V| + |E|)$ nodes.*

5. STRENGTH OF A NETWORK

For a network represented by a graph G , the maximum number of edge-disjoint spanning trees contained in G , has been proposed as a measure of the strength of the network, see [14]. Algorithms to compute the strength of a graph have been given in [4, 15, 3]. Here we give an algorithm to compute the maximum number of disjoint hypertrees in a hypergraph.

Based on Theorem 5, for a hypergraph $H = (V, E)$ and a capacity vector $c \in \mathbb{R}_+^E$ we give an algorithm to find the minimum of

$$(7) \quad \frac{c(\delta(\mathcal{P}))}{|\mathcal{P}| - 1},$$

among all partitions \mathcal{P} of V , with $|\mathcal{P}| \geq 2$. This gives the value of a maximum packing of hypertrees with capacities $c(e)$ for each hyperedge $e \in E$. We follow an approach similar to the one of [3] for the graphic case. We use Newton's method, cf. [23], as below.

Newton's method

Step 0. Pick any partition $\bar{\mathcal{P}}$ of V with $|\bar{\mathcal{P}}| \geq 2$. Set

$$\beta = \frac{c(\delta(\bar{\mathcal{P}}))}{|\bar{\mathcal{P}}| - 1}.$$

Step 1. Find $\hat{\mathcal{P}} = \operatorname{argmin} \left\{ c(\delta(\mathcal{P})) - \beta(|\mathcal{P}| - 1) \right\}$.

Step 2. If $c(\delta(\hat{\mathcal{P}})) - \beta(|\hat{\mathcal{P}}| - 1) < 0$, update β as

$$\beta = \frac{c(\delta(\hat{\mathcal{P}}))}{|\hat{\mathcal{P}}| - 1}.$$

and go to Step 1.

Otherwise $c(\delta(\hat{\mathcal{P}})) - \beta(|\hat{\mathcal{P}}| - 1) = 0$, and we stop.

The minimum in Step 1 is found among all partitions \mathcal{P} of V with $|\mathcal{P}| \geq 2$. Notice that because of the definition of β , in Step 2 we always have $c(\delta(\hat{\mathcal{P}})) - \beta(|\hat{\mathcal{P}}| - 1) \leq 0$. If $\mathcal{P}_1, \dots, \mathcal{P}_k$ is the sequence of partitions obtained we have $c(\delta(\mathcal{P}_i))/(|\hat{\mathcal{P}}_i| - 1) > c(\delta(\mathcal{P}_{i+1}))/(|\hat{\mathcal{P}}_{i+1}| - 1)$, for $i = 1, \dots, k-2$. It follows from results of [23] that $|\mathcal{P}_i| > |\mathcal{P}_{i+1}|$, for $i = 2, \dots, k-2$. Thus this algorithm converges in at most $|V|$ iterations.

Finding the minimum in Step 1 is similar to the separation problem of inequalities (1) in Section 4. The only difference is the number β that multiplies $|\mathcal{P}| - 1$. Notice that

after each iteration of Newton's method the number β is decreasing. In the algorithm of Section 4, for each node v through the iterations of Newton's method we have to solve a sequence of minimum st -cut problems where for each node w the capacities of the arcs (s, w) are not increasing, and the capacities of the arcs (w, t) are not decreasing, while the capacities of all the other arcs remain the same. This suggests the use of the parametric network flow algorithm of [11]. There they deal with the case where for each node w the capacities of the arcs (s, w) are not decreasing, and the capacities of the arcs (w, t) are not increasing. One can reverse the orientation of every arc and look for a minimum ts -cut in the new graph. Then each of these sequences can be solved with the same asymptotic complexity as one application of the push-preflow algorithm of [12]. This leads to the following.

Theorem 9. *The strength of a hypergraph can be computed with the same asymptotic complexity of $|V|$ applications of the push-preflow algorithm in a graph with $O(|V| + |E|)$ nodes.*

6. ARBORICITY

The *Arboricity* of a graph is the minimum number of edge-disjoint forests into which the edge-set can be decomposed. Nash-Williams [19] gave a characterization of this number, Frank et al. [8] extended this to hypergraphs as below.

Theorem 10. *A hypergraph $H = (V, E)$ can be partitioned into k disjoint hyperforests if and only if for every $X \subseteq V$*

$$|E[X]| \leq k(|X| - 1).$$

Algorithms for computing the arboricity of a graph have been given in [22, 10, 11]. Here we give an algorithm to compute the arboricity of a hypergraph. For that we need to compute

$$\max \frac{|E[X]|}{|X| - 1},$$

for $X \subseteq V$ and $|X| \geq 2$. As in the last section we use Newton's method as follows.

Newton's method

Step 0. Pick any $\bar{X} \subseteq V$ with $|\bar{X}| \geq 2$. Set

$$\beta = \frac{|E[\bar{X}]|}{|\bar{X}| - 1}.$$

Step 1. Find $\hat{X} = \operatorname{argmax} \left\{ |E[X]| - \beta(|X| - 1) \right\}$.

Step 2. If $|E[\hat{X}]| - \beta(|\hat{X}| - 1) > 0$, update β as

$$\beta = \frac{|E[\hat{X}]|}{|\hat{X}| - 1}.$$

and go to Step 1.

Otherwise $|E[\hat{X}]| - \beta(|\hat{X}| - 1) = 0$, and we stop.

The maximum in Step 1 is found among all $X \subseteq V$ with $|X| \geq 2$. Because of the definition of β , in Step 2 we always have $|E[\hat{X}]| - \beta(|\hat{X}| - 1) \geq 0$. If X_1, \dots, X_k is the sequence of node-sets obtained, we have $|E[X_i]|/(|X_i| - 1) > |E[X_{i+1}]|/(|X_{i+1}| - 1)$, for $i = 1, \dots, k-2$. It follows from the results of [23] that $|X_i| > |X_{i+1}|$, for $i = 2, \dots, k-2$. Thus this algorithm converges in at most $|V|$ iterations.

To find the maximum in Step 1, we proceed as in Subsection 3.1. We define a similar directed graph D . One difference is that in (a) the capacity of (s, v) is β , the other difference is that in (b) the capacity of (e', e'') , (e', t) and (e'', t) is $1/2$. We find a minimum st -cut, and we repeat for each $\bar{v} \in V$.

Assume that $\{s\} \cup T$ defines one of these minimum cuts. Let $T' = V \cap T$, $\bar{T}' = V \setminus T'$. As seen in Subsection 3.1, the set \bar{T}' minimizes $\beta|W| - |E[W]|$ with the constraint that $\bar{v} \in W$. At each iteration of Newton's method the value of β increases. Then for a particular node \bar{v} , we have to solve a sequence of minimum cut problems, where the capacities of the arcs (s, v) are increasing. Then the parametric network flow algorithm of [11] solves this sequence with the same asymptotic complexity as one application of the push-preflow algorithm of [12]. We summarize below.

Theorem 11. *The arboricity of a hypergraph can be computed with the same asymptotic complexity of $|V|$ applications of the push-preflow algorithm in a graph with $O(|V| + |E|)$ nodes.*

7. NETWORK REINFORCEMENT

The following problem was studied in [4]: Given a graph, a number k and a set of candidate edges, each of them with an associated cost, find a minimum cost set of candidate edges to be added to the network so it has strength equal to k . Algorithms for this have been given in [4, 9, 2]. Here we give an algorithm for a similar reinforcement problem in hypergraphs. The algorithm of Section 4 will be used as a subroutine, so we start with some properties of its solutions.

7.1. Properties of optimal partitions.

Lemma 12. *Let $\Phi = \{S_1, \dots, S_p\}$ be a solution of (3), and let $\{T_1, \dots, T_q\}$ be a partition of S_i , for some i , $1 \leq i \leq p$. Then*

$$\bar{x}(\delta(T_1, \dots, T_q)) - \beta(q - 1) \geq 0.$$

Proof. If $\bar{x}(\delta(T_1, \dots, T_q)) - \beta(q - 1) < 0$ one could improve the solution of (3) by removing S_i from Φ and adding $\{T_1, \dots, T_q\}$. \square

Lemma 13. *Let $\Phi = \{S_1, \dots, S_p\}$ be a solution of (3), and let $\{S_{i_1}, \dots, S_{i_l}\}$ be a subfamily of Φ . Then*

$$\bar{x}(\delta(S_{i_1}, \dots, S_{i_l})) - \beta(l - 1) \leq 0.$$

Proof. If $\bar{x}(\delta(S_{i_1}, \dots, S_{i_l})) - \beta(l - 1) > 0$, one could improve the solution of (3) by removing $\{S_{i_1}, \dots, S_{i_l}\}$ from Φ and adding their union. \square

Lemma 14. *Let $\Phi = \{S_1, \dots, S_p\}$ be a solution of (3) in H . Let H' be the hypergraph obtained by adding one new hyperedge e to H . The following are possible.*

- (i) *If there is an index i such that $e \subseteq S_i$ then Φ is a solution of (3) in H' ,*
- (ii) *otherwise there exists a solution Φ' of (3) in H' that is either of the form*
 - a) $\Phi' = (\Phi \setminus \{S_i : i \in I\}) \cup \{U = \cup_{i \in I} S_i\}$, *for some index set $I \subseteq \{1, \dots, p\}$, and $e \in \delta(S_{i_1}, S_{i_2}, \dots, S_{i_r})$, with $\{i_1, i_2, \dots, i_r\} \subseteq I$, see Figure 1, or*
 - b) $\Phi' = \Phi$.

Proof. (i) Suppose that $e \subseteq S_i$ for some index i . If Φ is not a solution of (3) in H' , there is a partition Φ' such that

$$\bar{x}(\delta_{H'}(\Phi')) - \beta(|\Phi'| - 1) < \bar{x}(\delta_{H'}(\Phi)) - \beta(|\Phi| - 1) = \bar{x}(\delta_H(\Phi)) - \beta(|\Phi| - 1).$$

If $e \notin \delta_{H'}(\Phi)$ we obtain $\bar{x}(\delta_H(\Phi')) - \beta(|\Phi'| - 1) < \bar{x}(\delta_H(\Phi)) - \beta(|\Phi| - 1)$, which is impossible. If $e \in \delta_{H'}(\Phi)$, we have $\bar{x}(\delta_H(\Phi')) - \beta(|\Phi'| - 1) \leq \bar{x}(\delta_{H'}(\Phi')) - \beta(|\Phi'| - 1) < \bar{x}(\delta_H(\Phi)) - \beta(|\Phi| - 1)$, a contradiction.

To prove (ii). Let $\{T_1, \dots, T_q\}$ be a solution of (3) in H' . Assume that there is a set S_i such that $S_i \subseteq \cup_{l=1}^k T_{j_l}$, $k \geq 2$, and $S_i \cap T_{j_l} \neq \emptyset$ for $1 \leq l \leq k$. Lemma 12 implies that

$$\bar{x}(\delta_H(T_{j_1} \cap S_i, \dots, T_{j_k} \cap S_i)) - \beta(k - 1) \geq 0,$$

and $\bar{x}(\delta_{H'}(T_{j_1}, \dots, T_{j_k})) - \beta(k - 1) \geq 0$. Therefore $\{T_{j_1}, \dots, T_{j_k}\}$ can be replaced by their union. So we can assume that for all i there is an index $j(i)$ such that $S_i \subseteq T_{j(i)}$.

Now suppose that for some index j , $T_j = \cup_{r=1}^l S_{i_r}$, $l > 1$. If $e \notin \delta_{H'}(S_{i_1}, \dots, S_{i_l})$, from Lemma 13 we have that

$$\bar{x}(\delta_{H'}(S_{i_1}, \dots, S_{i_l})) - \beta(l - 1) = \bar{x}(\delta_H(S_{i_1}, \dots, S_{i_l})) - \beta(l - 1) \leq 0,$$

and we could replace T_j by $\{S_{i_1}, \dots, S_{i_l}\}$.

If $e \in \delta_{H'}(S_{i_1}, \dots, S_{i_l})$ and

$$\bar{x}(\delta_{H'}(S_{i_1}, \dots, S_{i_l})) - \beta(l - 1) > 0,$$

we should keep $T_j \in \Phi'$, otherwise we can replace T_j by $\{S_{i_1}, \dots, S_{i_l}\}$. □

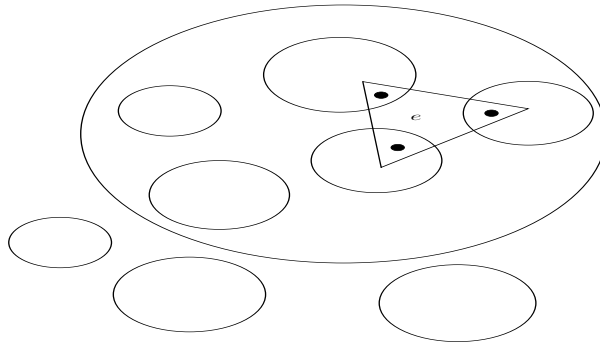


FIGURE 1. The family Φ' is obtained by combining some sets in Φ .

7.2. Reinforcement. We study a slightly different formulation. For a hypergraph $H = (V, E)$, we assume that each hyperedge e has a non-negative per-unit cost $d(e)$ and a non-negative integer capacity $u(e)$, that gives the maximum number of copies allowed of e . For a non-negative integer k , we solve the linear program

$$\begin{aligned} (8) \quad & \min dx \\ (9) \quad & x(\delta(\mathcal{P})) \geq k(|\mathcal{P}| - 1), \text{ for all partitions } \mathcal{P} \text{ of } V, \\ (10) \quad & 0 \leq x(e) \leq u(e). \end{aligned}$$

Its dual is

$$\begin{aligned} (11) \quad & \max \sum_{\mathcal{P}} \gamma_{\mathcal{P}} k(|\mathcal{P}| - 1) - \sum u(e) \beta(e) \\ (12) \quad & \sum_{\mathcal{P}: e \in \delta(\mathcal{P})} \gamma_{\mathcal{P}} \leq d(e) + \beta(e), \quad \text{for all } e, \\ (13) \quad & \gamma \geq 0, \quad \beta \geq 0. \end{aligned}$$

Here for each partition \mathcal{P} we have a dual variable $\gamma_{\mathcal{P}}$. Also for each hyperedge e there is a dual variable $\beta(e)$. We use a dual algorithm, i.e., constraints (12)-(13) will always be satisfied, and we are going to maximize (11). For the primal problem, constraints (10) will always be satisfied and (9) will be satisfied at the end. Complementary slackness will be satisfied at the end. We start with an informal description of the algorithm.

At the beginning we set to zero all dual variables. At each iteration we choose a partition \mathcal{P} and increase the value of $\gamma_{\mathcal{P}}$ by ϵ . We have to make sure that constraints (12) are not violated for hyperedges in $\delta(\mathcal{P})$. We say that a hyperedge e is *tight* if its constraint (12) is satisfied as equation. For a tight hyperedge $e \in \delta(\mathcal{P})$ we have to increase the value of $\beta(e)$ by ϵ . Let H' be the hypergraph defined by the tight hyperedges, then the dual objective increases by

$$\epsilon(k(|\mathcal{P}| - 1) - u(\delta_{H'}(\mathcal{P}))).$$

Thus we need to find a partition \mathcal{P} of V so that $k(|\mathcal{P}| - 1) - u(\delta_{H'}(\mathcal{P}))$ is positive. Thus we solve

$$(14) \quad \min u(\delta_{H'}(\mathcal{P})) - k(|\mathcal{P}| - 1),$$

among all partitions of V , as in Section 4. Let $\bar{\mathcal{P}} = \{S_1, \dots, S_p\}$ be the partition obtained. Let $(\bar{\gamma}, \bar{\beta})$ be the current dual solution. If the minimum in (14) is negative we look for the largest value of ϵ so that a new hyperedge becomes tight, this is

$$(15) \quad \bar{\epsilon} = \min \left\{ d(e) - \sum_{\mathcal{P}: e \in \delta(\mathcal{P})} \bar{\gamma}_{\mathcal{P}} \mid e \in \delta_H(\bar{\mathcal{P}}) \setminus \delta_{H'}(\bar{\mathcal{P}}) \right\}.$$

If this minimum is taken over the empty set we say that $\bar{\epsilon} = \infty$. In this case the dual problem is unbounded and the primal problem is infeasible.

Let \bar{e} be an hyperedge giving the minimum in (15). If there is more than one hyperedge giving this minimum we pick arbitrarily one to be added to H' . Then in the next iteration H' will contain a subset of the tight hyperedges, and the next value for $\bar{\epsilon}$ will be zero.

Let \mathcal{P}' be the solution of (14) after adding \bar{e} to H' . If $\mathcal{P}' = \bar{\mathcal{P}}$ then $\beta(\bar{e})$ could increase and $x(\bar{e})$ takes the value $u(\bar{e})$ to satisfy complementary slackness. We call this *Case 1*. If $\mathcal{P}' \neq \bar{\mathcal{P}}$ then $\beta(\bar{e})$ remains equal to zero and $x(\bar{e})$ can take a value less than $u(\bar{e})$. This is called *Case 2*. The algorithm stops when the minimum in (14) is zero.

Initially we set $\bar{x} = 0$. We have to discuss how to update \bar{x} in Cases 1 and 2 above.

In Case 1, we set $\bar{x}(\bar{e}) = u(\bar{e})$.

In Case 2, from Lemma 14 we have that

$$\mathcal{P}' = (\bar{\mathcal{P}} \setminus \{S_i : i \in I\}) \cup \{U = \cup_{i \in I} S_i\},$$

for some index set $I \subseteq \{1, \dots, p\}$, and $\bar{e} \in \delta(S_{i_1}, \dots, S_{i_r})$, with $\{i_1, \dots, i_r\} \subseteq I$. Let $\mathcal{P}_I = \{S_i \mid i \in I\}$, Lemma 12 implies

$$(16) \quad u(\delta_{H'}(\mathcal{P}_I)) - k(|I| - 1) \geq 0.$$

At this point we have $\bar{x}(e) = u(e)$ for $e \in \delta_{H'}(\mathcal{P}_I) \setminus \{\bar{e}\}$, and $\bar{x}(e) = 0$ for all other hyperedges $e \in \delta_H(\mathcal{P}_I)$. Let

$$\lambda = k(|I| - 1) - \bar{x}(\delta_H(\mathcal{P}_I) \setminus \{\bar{e}\}),$$

Inequality (16) implies $\lambda \leq u(\bar{e})$. Then we set $\bar{x}(e) = \lambda$, and we have $\bar{x}(\delta_H(\mathcal{P}_I)) = k(|I| - 1)$.

Lemma 15. *For the set U defined in Case 2, we have*

- (a) $\bar{x}(E[U]) = k(|U| - 1)$, and
- (b) $\bar{x}(\delta_H(T_1, \dots, T_q)) \geq k(q - 1)$ for each partition $\{T_1, \dots, T_q\}$ of U .

Proof. We use induction, so we assume that the lemma holds for each S_i , $i \in I$.

Consider (a). We have $\bar{x}(E[S_i]) = k(|S_i| - 1)$ for $i \in I$, by the induction hypothesis. And since $\bar{x}(\delta_H(\mathcal{P}_I)) = k(|I| - 1)$, we have $\bar{x}(E[U]) = k(|U| - 1)$.

To prove (b) assume that $\{U_1, \dots, U_r\}$ is a solution of

$$(17) \quad \min \bar{x}(\delta_H(\mathcal{Q})) - k(|\mathcal{Q}| - 1)$$

over all partitions \mathcal{Q} of U . The induction hypothesis implies that for each $i \in I$, $S_i \subseteq U_{j(i)}$ for some index $j(i)$. If $\bar{x}(\delta_H(U_1, \dots, U_r)) - k(r - 1) < 0$, and since $\bar{x}(\delta_H(\mathcal{P}_I)) - k(|I| - 1) = 0$, there is a family $\{S_{i_1}, \dots, S_{i_m}\} \subset \mathcal{P}_I$, such that $\cup_{j=1}^{j=m} S_{i_j} = U_l$ for some index l , and

$$\bar{x}(\delta_H(S_{i_1}, \dots, S_{i_m})) > k(m - 1).$$

This contradicts Lemma 13 and the definition of \mathcal{P}' . □

The first partition of V consists of all singletons. At each iteration a new hyperedge is added to H' . In some cases some sets in the family $\bar{\mathcal{P}}$ are combined into one. Each time that several sets of a partition are combined into one set U , the update of \bar{x} implies $\bar{x}(E[U]) = k(|U| - 1)$, as shown in Lemma 15. When the minimum in (14) is zero, then $\mathcal{P} = \{V\}$ is a solution. Thus $U = V$, and we have $\bar{x}(E) = k(|V| - 1)$. Then Lemma 15 shows that we have a primal feasible solution.

At the end, consider a partition $\mathcal{P} = \{S_1, \dots, S_p\}$ with $\bar{\gamma}_{\mathcal{P}} > 0$. We have $\sum_i \bar{x}(E[S_i]) + \bar{x}(\delta_H(\mathcal{P})) = \bar{x}(E) = k(|V| - 1)$. Lemma 15 shows that $\bar{x}(E[S_i]) = k(|S_i| - 1)$, for all i . Therefore $\bar{x}(\delta_H(\mathcal{P})) = k(|V| - 1) - \sum_i k(|S_i| - 1) = k(|\mathcal{P}| - 1)$. Thus at the end the vectors \bar{x} and $(\bar{\gamma}, \bar{\beta})$ satisfy the complementary slackness conditions. Now we give a formal description of the algorithm.

Reinforcement

- **Step 0.** Start with $\bar{\gamma} = 0$, $\bar{\beta} = 0$, $\bar{x} = 0$, $\bar{d}(e) = d(e)$ for all $e \in E$. $\bar{\mathcal{P}}$ consisting of all singletons, and $H' = (V, \emptyset)$.
- **Step 1.** Compute

$$(18) \quad \bar{\epsilon} = \min\{\bar{d}(e) \mid e \in \delta_H(\bar{\mathcal{P}}) \setminus \delta_{H'}(\bar{\mathcal{P}})\}.$$

If $\bar{\epsilon} = \infty$ stop, the problem is infeasible.

Otherwise update $\bar{\beta}(e) \leftarrow \bar{\beta}(e) + \bar{\epsilon}$ for all $e \in \delta_{H'}(\bar{\mathcal{P}})$,

$\bar{\gamma}_{\bar{\mathcal{P}}} \leftarrow \bar{\gamma}_{\bar{\mathcal{P}}} + \bar{\epsilon}$,

$\bar{d}(e) \leftarrow \bar{d}(e) - \bar{\epsilon}$ for all $e \in \delta_H(\bar{\mathcal{P}}) \setminus \delta_{H'}(\bar{\mathcal{P}})$.

- **Step 2.** Let \bar{e} be a hyperedge giving the minimum in (18), add \bar{e} to H' . Solve problem (14) in H' to obtain a partition \mathcal{P}' .
- **Step 3.** If $\bar{\mathcal{P}} = \mathcal{P}'$ update \bar{x} as in Case 1. Otherwise update as in Case 2. If the minimum in (14) is zero stop, otherwise set $\bar{\mathcal{P}} \leftarrow \mathcal{P}'$ and go to Step 1.

This algorithm takes at most $|E|$ iterations, where each of them requires to solve (14). For the case when H is a graph, $k = 1$, and $u(e) = \infty$ for every edge e , this algorithm is similar to Kruskal's algorithm for minimum spanning trees [16]. Now we summarize the results of this section.

Theorem 16. *The reinforcement problem reduces to $|E||V|$ minimum cut problems in a graph with $O(|V| + |E|)$ nodes.*

Theorem 17. *The vector \bar{x} is an optimal solution of (8)-(10). If k is a nonnegative integer and the capacities u are integer, then \bar{x} is integer valued. Also if d is integer valued then there is an optimal solution of (11)-(13) that is also integer valued. Thus the system (9)-(10) is totally dual integral.*

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