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Explicit construction of the minimum error variance estimator for stochastic LTI state-space systems.

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ABSTRACT

In this short article, we showcase the derivation of the optimal (minimum error variance) estimator, when one part of the stochastic LTI system output is not measured but is able to be predicted from the measured system outputs. Similar derivations have been done before but not using state-space representation.

1. Introduction

Realization theory of stochastic LTI state-space representations with exogenous inputs is a mature theory Lindquist and Picci (2015); Katayama (2005). In particular, there is a constructive theory for constructing a minimal stochastic LTI state-space representation of a process \mathbf{y} with exogenous input \mathbf{w} . The construction uses geometric ideas, and it is based on oblique projection of future outputs onto past inputs and outputs.

Note that in system identification it is often assumed that (\mathbf{y}, \mathbf{w}) has a realization by an autonomous stochastic LTI system driven by white noise. Indeed, if \mathbf{w} has a realization by a stochastic LTI state-space representation, and \mathbf{y} has a realization by a LTI state-space representation with exogenous input \mathbf{w} , then under some mild assumptions (absence of feedback from \mathbf{y} to \mathbf{w}) (\mathbf{y}, \mathbf{w}) will be the output of an autonomous stochastic LTI state-space representation. It is then natural to ask the question how to construct a minimal stochastic LTI realization of \mathbf{y} with input \mathbf{w} , from an LTI realization of the joint process (\mathbf{y}, \mathbf{w}) , instead of computing a realization of \mathbf{y} using oblique projections.

In this paper we present an explicit construction of a minimal stochastic LTI space- representation of \mathbf{y} with an exogenous input \mathbf{w} from an autonomous stochastic LTI state-representation of the joint process (\mathbf{y}, \mathbf{w}) . The basic idea is as follows: we will assume that there is no feedback from \mathbf{y} to \mathbf{w} and then use the result of Jozsa, Petreczky and Camlibel (2018) stating that there exists a minimal LTI realization of (\mathbf{y}, \mathbf{w}) matrices of which admit a

An explicit construction of a realization of \mathbf{y} with input \mathbf{w} could be useful for several reasons: it could be useful in proofs and it has the potential to provide an alternative to existing system identification algorithms. There are many subtleties in the consistency analysis of subspace identification algorithms with inputs Lindquist and Picci (2015); Katayama (2005); Chiuso and Picci (2004), so an alternative approach

involving the identification of an autonomous model of (\mathbf{y}, \mathbf{w}) could be advantageous in some cases.

Our motivation for developing an explicit construction of an LTI realization of \mathbf{y} with input \mathbf{w} from a LTI realization of (\mathbf{y}, \mathbf{w}) was that this construction turned out to be useful in providing non-asymptotic error bounds of PAC-Bayesian type Alquier, Ridgway and Chopin (2016) for LTI systems Eringis, Leth, Tan, Wisniewski, Esfahan and Petreczky (2021). The latter could be a first step towards extending the PAC-Bayesian framework for stochastic state-space representations, in particular, to recurrent neural networks.

More precisely, one of the byproducts of the construction of this paper is a one-to-one relationship between LTI systems which generate (\mathbf{y}, \mathbf{w}) and optimal linear predictors of future values of \mathbf{y} based on past values of \mathbf{w} . The latter predictor is just the deterministic part of a realization of \mathbf{y} with input \mathbf{w} in forward innovation form. That is, finding a realization of (\mathbf{y}, \mathbf{w}) boils down to finding the best predictor in a parametrized family of predictors. This is done by minimizing the empirical prediction error. In ? PAC-Bayesian error bounds are formulated relating the expected prediction error and the empirical prediction error for each predictor. In turn, the derivation of these error bounds require the use of LTI state-space representations of (\mathbf{y}, \mathbf{w}) and of minimal LTI realizations of \mathbf{y} with input \mathbf{w} in forward innovation form.

The contribution of the paper can also be viewed as follows. We wish to construct an estimator of $\mathbf{y}(t)$ given past ($s < t$) and present ($s = t$) measurements of $\mathbf{w}(s)$. We consider a specific class of relationships, specifically when the two processes are related by a common stochastic linear time invariant(LTI) state-space system, i.e. $[\mathbf{y}^T(t) \mathbf{w}^T(t)]^T$ is an output of an LTI system. The problem of find this estimator can also be thought of as trying to estimate non-measurable quantities of a system from measurable quantities.

Related work As it was pointed out above, stochastic realization theory with inputs is a mature topic with several publications, see the monographs Lindquist and Picci (2015); Katayama (2005); Caines (1988) and the references therein. However, we have not found in the literature an explicit procedure for constructing a stochastic LTI state-space realization in forward innovation form of \mathbf{y} with input

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\mathbf{w} from the joint stochastic LTI state-space realization of (\mathbf{y}, \mathbf{w}) . The current note is intended to fill this gap.

We will need to further analyse the relationship between \mathbf{y} and \mathbf{w} by feedback-free assumption. In Granger (1963), the author defines what it means for one process to cause another, a similar notion to feedback. In Caines (1976), the authors further extend the notion and define weak and strong feedback free processes. As strong feedback free condition implies weak feedback free, we consider the relaxed case of weak feedback free throughout the paper. Using causal real rational transfer function matrices to describe processes \mathbf{y} and \mathbf{w} , and analysing these processes with feedback free assumption, yields a straightforward construction of estimator of \mathbf{y} given \mathbf{w} in frequency domain, see Caines and Chan (1975) and Gevers and Anderson (1982). In this paper we study this problem in time domain, using LTI state-space representations.

Outline This paper is organised as follows. Below we start by defining the notation and terminology used in this paper, then in Section 2 we reformulate the state-space system driven by innovation of $[\mathbf{y}^T(t) \ \mathbf{w}^T(t)]^T$ into a state-space system, which yields a realisation of \mathbf{y} , driven by \mathbf{w} and the innovation of a purely non-deterministic part of \mathbf{y} . Afterwards in Section 2.1, given this new realisation we provide the optimal (in the sense of minimum error variance) predictor of \mathbf{y} . And finally we summarise by stating an algorithm to compute such predictors in Section 3

Notation and terminology Let \mathbf{F} denote a σ -algebra on the set Ω and \mathbf{P} be a probability measure on \mathbf{F} . Unless otherwise stated all probabilistic considerations will be with respect to the probability space $(\Omega, \mathbf{F}, \mathbf{P})$. In this paragraph let \mathbb{E} denote some euclidean space. We associate with \mathbb{E} the topology generated by the 2-norm $\|\cdot\|_2$, and the Borel σ -algebra generated by the open sets of \mathbb{E} . The closure of a set M is denoted clM . For $S \subseteq \mathbb{N}$ and stochastic variables $\mathbf{y}, \mathbf{z}_1, \mathbf{z}_2, \dots$ with values in \mathbb{R} we denote by $\mathbf{E}(\mathbf{y} \mid \{\mathbf{z}_i\}_{i \in S})$ the conditional expectation of \mathbf{y} with respect to the σ -algebra $\sigma(\{\mathbf{z}_i\})$ generated by the family $\{\mathbf{z}_i\}_{i \in S}$. Recall that $\mathbf{E}(\mathbf{z}\mathbf{x})$ define an inner product in $L^2(\Omega, \mathbf{F}, \mathbf{P})$ and that $\mathbf{E}(\mathbf{y} \mid \{\mathbf{z}_i\}_{i \in S})$ can be interpreted as the orthogonal projection onto the closed subspace $L^2(\Omega, \sigma(\{\mathbf{z}_i\}_{i \in S}), \mathbf{P})$ which also can be identified with the closure of the subspace generated by $\{\mathbf{z}_i\}_{i \in S}$. That is,

$$L^2(\Omega, \sigma(\{\mathbf{z}_i\}_{i \in S}), \mathbf{P}) = cl \left\{ \sum_{i \in S} \alpha_i \mathbf{z}_i \mid \alpha_i \in \mathbb{R} \right\} \quad (1)$$

with only a finite number of summands in (1) being nonzero when $S = \mathbb{N}$. Moreover, for a closed subspace H of $L^2(\Omega, \mathbf{F}, \mathbf{P})$ and a stochastic variable \mathbf{y} with values in \mathbb{E} and $\mathbf{E}(\|\mathbf{y}\|_2^2) < \infty$, we let $\mathbf{E}(\mathbf{y} \mid H)$ denote the $\dim(\mathbb{E})$ -dimensional vector with i th coordinate equal to $\mathbf{E}(y_i \mid H)$ with y_i denoting the i th coordinate of \mathbf{y} .

There are two closed subspaces of particular importance. Following Lindquist and Picci (2015), for a discrete time stochastic process $\mathbf{z}(t)$ with values in \mathbb{E} and $\mathbf{E}(\|\mathbf{z}(t)\|_2^2) < \infty$, we write $H_t^-(\mathbf{z})$ for the closure of the subspace in $L^2(\Omega, \mathbf{F}, \mathbf{P})$ generated by the coordinate functions $\mathbf{z}_i(s)$ of

$\mathbf{z}(s)$ for all $s < t$. That is,

$$H_t^-(\mathbf{z}) = cl \left\{ \sum_{i=-\infty}^{t-1} \alpha_i^T \mathbf{z}(i) \mid \alpha_i \in \mathbb{E} \right\} \quad (2)$$

with T indicating transpose and only a finite number of summands in (2) being nonzero. In a similar manner we define

$$H_t^+(\mathbf{z}) = cl \left\{ \sum_{i=t}^{\infty} \alpha_i^T \mathbf{z}(i) \mid \alpha_i \in \mathbb{E} \right\}, \quad (3)$$

$$H(\mathbf{z}) = cl \left\{ \sum_{i=-\infty}^{\infty} \alpha_i^T \mathbf{z}(i) \mid \alpha_i \in \mathbb{E} \right\}. \quad (4)$$

Let A, B and C be closed subspaces of $L^2(\Omega, \mathbf{F}, \mathbf{P})$. We then define

$$A \vee B = cl\{a + b \mid a \in A, b \in B\} \quad (5)$$

and say that A and B are orthogonal given C , denoted $A \perp B \mid C$, if

$$\mathbf{E}\left((a - \mathbf{E}(a \mid C))(b - \mathbf{E}(b \mid C))\right) = 0 \quad (6)$$

for all $a \in A$ and $b \in B$.

We use the following notation, $\mathcal{Y} = \mathbb{R}^p$, $\mathcal{W} = \mathbb{R}^q$ and for the disjoint union $\mathcal{W}^* = \bigsqcup_{k=1}^{\infty} \mathcal{W}^k$ we write $w = (w_1, \dots, w_k)$ in place of the more correct $(w, k) = ((w_1, \dots, w_k), k)$ for an element in \mathcal{W}^* .

2. Formulating a Realisation

Suppose we want to construct an estimator of the output stochastic process $\mathbf{y}(t) : \Omega \rightarrow \mathcal{Y}$ given a sequence of measurements as inputs obtained from the stochastic process $\mathbf{w}(t) : \Omega \rightarrow \mathcal{W}$. In order to narrow down and formally describe the estimation problem, we assume that the processes $\mathbf{y}(t)$ and $\mathbf{w}(t)$ can be represented as outputs of an LTI system in forward innovation form:

Assumption 1. The processes $\mathbf{y}(t)$ and $\mathbf{w}(t)$ can be generated by a stochastic discrete-time minimal LTI system on the form

$$\mathbf{x}(t+1) = A_g \mathbf{x}(t) + K_g \mathbf{e}_g(t) \quad (7a)$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{w}(t) \end{bmatrix} = C_g \mathbf{x}(t) + \mathbf{e}_g(t) \quad (7b)$$

where $A_g \in \mathbb{R}^{n \times n}$, $K_g \in \mathbb{R}^{n \times m}$, $C_g \in \mathbb{R}^{(p+q) \times n}$ for $n \geq 0$, $m, p > 0$ and $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^p$, $\mathbf{w} \in \mathbb{R}^q$ and \mathbf{e}_g are stationary, square-integrable, zero-mean, and jointly Gaussian stochastic processes. The processes \mathbf{x} and \mathbf{e}_g are called state and noise process, respectively. Recall, that stationarity and square-integrability imply constant expectation and that the covariance matrix

$Cov(\mathbf{y}(t), \mathbf{y}(s)) = \mathbf{E}[(\mathbf{y}(t) - \mathbf{E}[\mathbf{y}(t)])(\mathbf{y}(s) - \mathbf{E}[\mathbf{y}(s)])^T]$ only depends on time lag $(t - s)$. Furthermore, we require that A_g is stable (all its eigenvalues are inside the open unit circle) and that for any $t, k \in \mathbb{Z}$, $k \geq 0$, $E[\mathbf{e}_g(t) \mathbf{e}_g^T(t-k-1)] = 0$, $E[\mathbf{e}_g(t) \mathbf{x}^T(t-k)] = 0$, i.e., the stationary Gaussian process $\mathbf{e}_g(t)$ is white noise and uncorrelated with $\mathbf{x}(t-k)$. We identify the system (7) with the tuple $(A_g, K_g, C_g, I, \mathbf{e}_g)$; note that the state process \mathbf{x} is uniquely defined by the infinite sum $\mathbf{x}(t) = \sum_{k=1}^{\infty} A_g^{k-1} K_g \mathbf{e}_g(t-k)$.

Before we can continue we have to consider the relationship between \mathbf{y} and \mathbf{w} . For technical reasons we can not have feedback from \mathbf{y} to \mathbf{w} , as \mathbf{w} would then be determined by a dynamical relation involving the past of the process \mathbf{y} . As such we have Assumption 2

Assumption 2. There is no feedback from \mathbf{y} to \mathbf{w} , following definition 17.1.1. from Lindquist and Picci (2015), i.e.,

$$H_t^-(\mathbf{y}) \perp H_t^+(\mathbf{w}) \mid H_t^-(\mathbf{w})$$

holds, i.e., the future of \mathbf{w} is conditionally uncorrelated with the past of \mathbf{y} , given the past of \mathbf{w} .

As a passing remark, the no feedback assumption is equivalent to weak feedback free assumption Caines (1976) or Granger non-causality Granger (1963). Thus the no feedback assumption can be stated as \mathbf{y} does not Granger cause \mathbf{w} . Several results can now be deduced from Assumption 2. First, by (Lindquist and Picci, 2015, Proposition 2.4.2), we obtain the following relation between projections

$$E[\mathbf{y}(t) \mid H(\mathbf{w})] = E[\mathbf{y}(t) \mid H_{t+1}^-(\mathbf{w})], \quad (8)$$

$$E[\mathbf{w}(t) \mid H_t^-(\mathbf{w}) \vee H_t^-(\mathbf{y})] = E[\mathbf{w}(t) \mid H_t^-(\mathbf{w})]. \quad (9)$$

Secondly, from (Lindquist and Picci, 2015, Ch. 17) it follows that the process \mathbf{y} can then be decomposed into a deterministic part \mathbf{y}_d and a stochastic part \mathbf{y}_s , as follows

$$\mathbf{y}(t) = \mathbf{y}_d(t) + \mathbf{y}_s(t), \quad (10)$$

$$\mathbf{y}_d(t) = E[\mathbf{y}(t) \mid H(\mathbf{w})] = E[\mathbf{y}(t) \mid H_{t+1}^-(\mathbf{w})], \quad (11)$$

$$\mathbf{y}_s(t) = \mathbf{y}(t) - \mathbf{y}_d(t). \quad (12)$$

Note that, as a consequence of (11) and (12)

$$E[\mathbf{y}_d(t) \mathbf{y}_s^T(\tau)] = 0 \quad \forall t, \tau,$$

i.e., \mathbf{y}_d and \mathbf{y}_s are uncorrelated. Moreover, the process \mathbf{y}_s can be realised by a state-space system in forward innovation form

$$\mathbf{x}_s(t+1) = A_s \mathbf{x}_s(t) + K_s \mathbf{e}_s(t), \quad (13a)$$

$$\mathbf{y}_s(t) = C_s \mathbf{x}_s(t) + \mathbf{e}_s(t), \quad (13b)$$

$$\mathbf{e}_s = \mathbf{y}_s(t) - E[\mathbf{y}_s(t) \mid H_t^-(\mathbf{y}_s)]. \quad (13c)$$

Finally, from (Lindquist and Picci, 2015, Proposition 17.1.3.) we get

$$H_t^-(\mathbf{y}) \vee H_{t+1}^-(\mathbf{w}) = H_t^-(\mathbf{y}_s) \oplus H_{t+1}^-(\mathbf{w}), \quad (14)$$

where \oplus denotes orthogonal sum, and

$$\mathbf{e}_s(t) = \mathbf{y}(t) - E[\mathbf{y}(t) \mid H_t^-(\mathbf{y}) \vee H_{t+1}^-(\mathbf{w})]. \quad (15)$$

Now consider a similarity transformation T of (7) such that $\bar{A}_g = T A_g T^{-1}$, $\bar{K}_g = T K_g$ and $\bar{C}_g = C_g T^{-1}$ are upper block triangular (see also Section 3)

$$\begin{bmatrix} \bar{x}_{g,1}(t+1) \\ \bar{x}_{g,2}(t+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}}_{\bar{A}_g} \begin{bmatrix} \bar{x}_{g,1}(t) \\ \bar{x}_{g,2}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} K_{1,1} & K_{1,2} \\ 0 & K_{2,2} \end{bmatrix}}_{\bar{K}_g} \begin{bmatrix} \mathbf{e}_1(t) \\ \mathbf{e}_2(t) \end{bmatrix} \quad (16a)$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{w}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{2,2} \end{bmatrix}}_{\bar{C}_g} \begin{bmatrix} \bar{x}_{g,1}(t) \\ \bar{x}_{g,2}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{e}_1(t) \\ \mathbf{e}_2(t) \end{bmatrix} \quad (16b)$$

where $[\mathbf{e}_1^T(t) \ \mathbf{e}_2^T(t)]^T = \mathbf{e}_g(t)$, and such that $(A_3, C_{2,2})$ is observable. Moreover, $A_{i,j} \in \mathbb{R}^{p_i \times p_j}$, $K_{i,j} \in \mathbb{R}^{p_i \times r_j}$, $C_{i,j} \in \mathbb{R}^{r_i \times p_j}$, with $r_1 = p$ and $r_2 = q$. From Jozsa et al. (2018) it then follows that $(A_{2,2}, K_{2,2}, C_{2,2}, \mathbf{e}_2)$ is a minimal Kalman representation of \mathbf{w} , and hence $\mathbf{e}_2(t)$ is the innovation process of \mathbf{w} i.e.,

$$\begin{aligned} \mathbf{e}_2(t) &= \mathbf{w}(t) - E[\mathbf{w}(t) \mid H_t^-(\mathbf{w})] \\ &= \mathbf{w}(t) - E[\mathbf{w}(t) \mid H_t^-(\mathbf{y}) \vee H_t^-(\mathbf{w})]. \end{aligned} \quad (17)$$

Moreover, the transformed system (16) induce a relation between the output \mathbf{y} and input \mathbf{w} . In detail, from (16b) we also have

$$\mathbf{e}_2(t) = \mathbf{w}(t) - C_{2,2} \bar{x}_{g,2}(t). \quad (18)$$

Hence, substituting (18) in (16) yields the following realisation of \mathbf{y}

$$\begin{bmatrix} \bar{x}_{g,1}(t+1) \\ \bar{x}_{g,2}(t+1) \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} - K_{1,2} C_{2,2} \\ 0 & A_{2,2} - K_{2,2} C_{2,2} \end{bmatrix} \begin{bmatrix} \bar{x}_{g,1}(t) \\ \bar{x}_{g,2}(t) \end{bmatrix} \quad (19a)$$

$$+ \begin{bmatrix} K_{1,2} \\ K_{2,2} \end{bmatrix} \mathbf{w}(t) + \begin{bmatrix} K_{1,1} \\ 0 \end{bmatrix} \mathbf{e}_1(t) \quad (19b)$$

$$\mathbf{y}(t) = \begin{bmatrix} C_{1,1} & C_{1,2} \end{bmatrix} \begin{bmatrix} \bar{x}_{g,1}(t) \\ \bar{x}_{g,2}(t) \end{bmatrix} + \mathbf{e}_1(t) \quad (19c)$$

Note that $\mathbf{e}_1(t)$ is the innovation process of \mathbf{y} (with respect to \mathbf{w}), i.e.,

$$\mathbf{e}_1(t) = \mathbf{y}(t) - E[\mathbf{y}(t) \mid H_t^-(\mathbf{y}) \vee H_t^-(\mathbf{w})]. \quad (20)$$

2.1. Optimal predictor

The goal in this section is to derive an optimal predictor (in the sense of minimum variance). Firstly, we claim that

$$\mathbf{e}_s(t) = \mathbf{e}_1(t) - E[\mathbf{y}(t) \mid \mathbf{e}_2(t)] = \mathbf{e}_1(t) - D_0 \mathbf{e}_2(t) \quad (21)$$

where $D_0 = (E[\mathbf{y}(t) \mathbf{e}_2^T(t)])^T (E[\mathbf{e}_2(t) \mathbf{e}_2^T(t)])^{-1}$ is the minimum variance linear estimator of $\mathbf{y}(t)$ given $\mathbf{e}_2(t)$, see (Lindquist and Picci, 2015, Proposition 2.2.3.). In order to show (21), we first show that

$$H_t^-(\mathbf{y}) \vee H_{t+1}^-(\mathbf{w}) = (H_t^-(\mathbf{y}) \vee H_t^-(\mathbf{w})) \oplus H(\mathbf{e}_2(t)), \quad (22)$$

where $H(\mathbf{e}_2(t)) = \{\alpha^T \mathbf{e}_2(t) \mid \alpha \in \mathbb{R}^q\}$, is the space spanned by innovation process $\mathbf{e}_2(t)$, considered only at the time t . By definition we have

$$\begin{aligned} &(H_t^-(\mathbf{y}) \vee H_t^-(\mathbf{w})) \vee H(\mathbf{e}_2(t)) = \\ &cl \left\{ \sum_{i=-\infty}^{t-1} \gamma_i^T \mathbf{y}(i) + \sum_{i=-\infty}^{t-1} \eta_i^T \mathbf{w}(i) \right. \\ &\quad \left. + \lambda_t^T \mathbf{e}_2(t) \mid \gamma_i \in \mathbb{R}^p, \eta_i \in \mathbb{R}^q, \lambda_t \in \mathbb{R}^q \right\} \end{aligned} \quad (23)$$

However, using definition of $\mathbf{e}_2(t)$ from (17) we have

$$(H_t^-(\mathbf{y}) \vee H_t^-(\mathbf{w})) \vee H(\mathbf{e}_2(t)) = cI \left\{ \sum_{i=-\infty}^{t-1} \gamma_i^T \mathbf{y}(i) + \sum_{i=-\infty}^{t-1} \eta_i^T \mathbf{w}(i) + \lambda_t^T \mathbf{w}(t) \mid \gamma_i \in \mathbb{R}^p, \eta_i \in \mathbb{R}^q, \lambda_t \in \mathbb{R}^q \right\}, \quad (24)$$

which equals $H_t^-(\mathbf{y}) \vee H_{t+1}^-(\mathbf{w})$ and therefore

$$(H_t^-(\mathbf{y}) \vee H_t^-(\mathbf{w})) \vee H(\mathbf{e}_2(t)) = H_t^-(\mathbf{y}) \vee H_{t+1}^-(\mathbf{w}) \quad (25)$$

Again from (17) it follows that $\mathbf{e}_2(t) \perp H_t^-(\mathbf{w}) \vee H_t^-(\mathbf{y})$, thus (22) holds.

The relation (21) now follows since

$$\begin{aligned} E[\mathbf{y}(t) \mid H_t^-(\mathbf{y}) \vee H_{t+1}^-(\mathbf{w})] \\ = E[\mathbf{y}(t) \mid H_t^-(\mathbf{y}) \vee H_t^-(\mathbf{w})] + E[\mathbf{y}(t) \mid \mathbf{e}_2(t)] \\ = E[\mathbf{y}(t) \mid H_t^-(\mathbf{y}) \vee H_t^-(\mathbf{w})] + D_0 \mathbf{e}_2(t), \end{aligned} \quad (26)$$

and therefore

$$\mathbf{e}_s(t) = \mathbf{y}(t) - E[\mathbf{y}(t) \mid H_t^-(\mathbf{y}) \vee H_t^-(\mathbf{w})] - D_0 \mathbf{e}_2(t) \quad (27)$$

$$= \mathbf{e}_1(t) - D_0 \mathbf{e}_2(t) \quad (28)$$

by using (20).

Now from (21) and (18) we get

$$\mathbf{e}_1(t) = \mathbf{e}_s(t) + D_0 \mathbf{w}(t) - D_0 C_{2,2} \bar{x}_{g,2}(t), \quad (29)$$

which can be applied to (19) to obtain the following realization of \mathbf{y}

$$\begin{aligned} \begin{bmatrix} \bar{x}_{g,1}(t+1) \\ \bar{x}_{g,2}(t+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} A_{1,1} & A_{1,2} - K_{1,2} C_{2,2} - K_{1,1} D_0 C_{2,2} \\ 0 & A_{2,2} - K_{2,2} C_{2,2} \end{bmatrix}}_A \begin{bmatrix} \bar{x}_{g,1}(t) \\ \bar{x}_{g,2}(t) \end{bmatrix} \\ &+ \underbrace{\begin{bmatrix} K_{1,2} + K_{1,1} D_0 \\ K_{2,2} \end{bmatrix}}_{\tilde{K}} \mathbf{w}(t) + \begin{bmatrix} K_{1,1} \\ 0 \end{bmatrix} \mathbf{e}_s(t) \quad (30a) \\ \mathbf{y}(t) &= \underbrace{\begin{bmatrix} C_{1,1} & C_{1,2} - D_0 C_{2,2} \end{bmatrix}}_C \begin{bmatrix} \bar{x}_{g,1}(t) \\ \bar{x}_{g,2}(t) \end{bmatrix} + D_0 \mathbf{w}(t) + \mathbf{e}_s(t) \quad (30b) \end{aligned}$$

Finally we are in a position to derive a formula for the minimum variance predictor $E[\mathbf{y}(t) \mid H_{t+1}^-(\mathbf{w})]$. That is, a formula for the orthogonal projection of $\mathbf{y}(t)$ given past and present values of \mathbf{w} . First define $\hat{x}_g(t) = E[\bar{x}_g(t) \mid H_{t+1}^-(\mathbf{w})]$, then from (30b) we get

$$E[\mathbf{y}(t) \mid H_{t+1}^-(\mathbf{w})] = E[\tilde{C} \bar{x}_g(t) + D_0 \mathbf{w}(t) + \mathbf{e}_s(t) \mid H_{t+1}^-(\mathbf{w})] \quad (31)$$

$$= \tilde{C} \hat{x}_g(t) + D_0 \mathbf{w}(t) + E[\mathbf{e}_s(t) \mid H_{t+1}^-(\mathbf{w})] \quad (32)$$

$$= \tilde{C} \hat{x}_g(t) + D_0 \mathbf{w}(t) \quad (33)$$

where (33) follows from (15). Now (30a) can be used to derive a dynamical expression for \hat{x}_g as follows

$$\begin{aligned} E[\bar{x}_g(t+1) \mid H_{t+2}^-(\mathbf{w})] \\ = E \left[\tilde{A} \bar{x}_g(t) + \tilde{K} \mathbf{w}(t) + \begin{bmatrix} K_{1,1} \\ 0 \end{bmatrix} \mathbf{e}_s(t) \mid H_{t+2}^-(\mathbf{w}) \right] \quad (34) \\ = \tilde{A} E[\bar{x}_g(t) \mid H_{t+2}^-(\mathbf{w})] + \tilde{K} E[\mathbf{w}(t) \mid H_{t+2}^-(\mathbf{w})] \\ + \begin{bmatrix} K_{1,1} \\ 0 \end{bmatrix} E[\mathbf{e}_s(t) \mid H_{t+2}^-(\mathbf{w})] \quad (35) \end{aligned}$$

Clearly $E[\mathbf{w}(t) \mid H_{t+2}^-(\mathbf{w})] = \mathbf{w}(t)$. For the projection in (35) we have

$$E[\bar{x}_g(t) \mid H_{t+2}^-(\mathbf{w})] = E[\bar{x}_g(t) \mid H_{t+1}^-(\mathbf{w})]$$

since the state vector $\bar{x}_g(t)$ can be expressed as an infinite sum using (30a), where (15) is used for $\mathbf{e}_s(t)$

$$\begin{aligned} \bar{x}_g(t) &= \sum_{i=1}^{\infty} M_i \mathbf{w}(t-i) + \sum_{i=1}^{\infty} N_i \mathbf{y}(t-i) \\ &+ \sum_{i=1}^{\infty} N_i E[\mathbf{y}(t-i) \mid H_{t-i}^-(\mathbf{y}) \vee H_{t-i+1}^-(\mathbf{w})] \quad (36) \end{aligned}$$

Finally, from (13c) we observe that

$$\begin{aligned} E[\mathbf{e}_s(t) \mid H_{t+2}^-(\mathbf{w})] \\ = E[\mathbf{y}_s(t) \mid H_{t+2}^-(\mathbf{w})] - E[E[\mathbf{y}_s(t) \mid H_t^-(\mathbf{y}_s)] \mid H_{t+2}^-(\mathbf{w})] \\ = 0 - 0 \quad (37) \end{aligned}$$

since $H(\mathbf{y}_s) \perp H(\mathbf{w})$ by (12). In summary we obtain the following formula for the minimum variance predictor

$$\hat{x}_g(t+1) = \tilde{A} \hat{x}_g(t) + \tilde{K} \mathbf{w}(t) \quad (38a)$$

$$E[\mathbf{y}(t) \mid H_{t+1}^-(\mathbf{w})] = \tilde{C} \hat{x}_g(t) + D_0 \mathbf{w}(t) \quad (38b)$$

In order to predict $\mathbf{y}(t)$ based on $H_{t+1}^-(\mathbf{w})$, the system (38) would yield the minimum prediction error variance

3. Summary

Given (A, B, C, D) , which describe the system

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + Bv(t), \quad (39a)$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{w}(t) \end{bmatrix} = C\mathbf{x}(t) + Dv(t), \quad (39b)$$

$$v(t) \sim \mathcal{N}(0, I), \quad (39c)$$

and assume that there is no feedback from \mathbf{y} to \mathbf{w} .

Step 1 Compute forward innovation representation (Lindquist and Picci, 2015, Section 6.9), by first solving the discrete Lyapunov equation

$$P = APA^T + BB' \quad (40)$$

then compute

$$\bar{C} = CPA^T + DB^T \quad (41)$$

$$\Lambda_{0y} = CPC^T + DD^T \quad (42)$$

Then solve a discrete algebraic Riccati equation

$$\Pi = A\Pi A^T + (\bar{C}^T - A\Pi C^T)(\Lambda_{0y} - C\Pi C^T)^{-1}(\bar{C}^T - A\Pi C^T)^T \quad (43)$$

Finally compute the covariance of innovation process $\mathbf{e}_g(t)$ and the Kalman gain

$$Q_e = \Lambda_{0y} - C\Pi C^T \quad (44)$$

$$K = (\bar{C}^T - A\Pi C^T)Q_e^{-1} \quad (45)$$

Then we obtain a realisation in forward innovation form

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + K\mathbf{e}_g(t) \quad (46a)$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{w}(t) \end{bmatrix} = C\mathbf{x}(t) + \mathbf{e}_g(t) \quad (46b)$$

Step 2 Find a similarity transformation to obtain the system in upper block triangular form (16).

First compute the svd of the observability matrix related to \mathbf{w} .

$$C = \begin{bmatrix} C_y \\ C_w \end{bmatrix} \quad (47)$$

$$USV^T = \begin{bmatrix} C_w \\ C_w A \\ \vdots \\ C_w A^n \end{bmatrix} \quad (48)$$

where S is diagonal matrix with entries $[\sigma_1, \sigma_2, \dots, \sigma_n]$ and $\sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_n$. Note that the order of the singular values has been reversed compared to standard practice. The non-singular matrix T , which performs the desired similarity transformation is given by $T = V^T$. Moreover, p_1 (from (16)) is given by the number of singular values which are zero, i.e. the nullity of the observability matrix in (48), and $p_2 = n - p_1$. Now compute the upper block triangular representation by

$$T^{-1}AT = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0_{p_2 \times p_1} & A_{2,2} \end{bmatrix}, \quad TK = \begin{bmatrix} K_{1,1} & K_{1,2} \\ 0_{p_2 \times p_1} & K_{2,2} \end{bmatrix} \quad (49)$$

$$CT^{-1} = \begin{bmatrix} C_{1,1} & C_{1,2} \\ 0_{q \times p_1} & C_{1,2} \end{bmatrix} \quad (50)$$

and partition

$$Q_e = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \quad (51)$$

with $Q_2 \in \mathbb{R}^{p \times q}$, $Q_3 \in \mathbb{R}^{q \times q}$.

Step 3 Compute the minimum variance predictor (38).

$$\hat{\mathbf{x}}(t+1) = \begin{bmatrix} A_{1,1} & A_{1,2} - K_{1,2}C_{2,2} - K_{1,1}D_0C_{2,2} \\ 0 & A_{2,2} - K_{2,2}C_{2,2} \end{bmatrix} \hat{\mathbf{x}}(t) + \begin{bmatrix} K_{1,2} + K_{1,1}D_0 \\ K_{2,2} \end{bmatrix} \mathbf{w}(t) \quad (52a)$$

$$\hat{\mathbf{y}}(t) = [C_{1,1} \quad C_{1,2} - D_0C_{2,2}] \hat{\mathbf{x}}(t) + Q_2Q_3^{-1}\mathbf{w}(t) \quad (52b)$$

4. Illustrative Example

To illustrate the findings consider the system

$$\mathbf{x}(t+1) = \begin{bmatrix} 0.85 & 1 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v(t), \quad (53a)$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{w}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v(t), \quad (53b)$$

$$v(t) \sim \mathcal{N}(0, I), \quad (53c)$$

From system (53) we can see that \mathbf{w} is feedback free from \mathbf{y} . To make the example interesting we will not work with the system (53) but instead apply a similarity transformation to the system such that it is not in a nice and clean form. We use

$$T = \begin{bmatrix} 0.5 & 0.9 \\ 0.5 & 0.1 \end{bmatrix} \quad (54)$$

and obtain

$$\mathbf{x}(t+1) = \begin{bmatrix} 1.08 & -0.23 \\ 0.58 & 0.26 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.5 & 1.4 \\ 0.5 & 0.6 \end{bmatrix} v(t), \quad (55a)$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{w}(t) \end{bmatrix} = \begin{bmatrix} -0.25 & 2.25 \\ 1.25 & -1.255 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v(t) \quad (55b)$$

Note that if we had started with system (55), it would not be clear if \mathbf{w} was feedback free of \mathbf{y} . In this example we know that \mathbf{w} is feedback free, therefore we proceed by computing the forward innovation form as in **Step 1** we get

$$\mathbf{x}(t+1) = \begin{bmatrix} 1.08 & -0.23 \\ 0.58 & 0.26 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.5 & 0.9 \\ 0.5 & 0.1 \end{bmatrix} e(t), \quad (56a)$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{w}(t) \end{bmatrix} = \begin{bmatrix} -0.25 & 2.25 \\ 1.25 & -1.25 \end{bmatrix} \mathbf{x}(t) + e(t) \quad (56b)$$

$$Qe = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad (56c)$$

The triangular form is obtained by applying SVD to the observability matrix

$$U \begin{bmatrix} 0 & 0 \\ 0 & 1.9764 \end{bmatrix} \begin{bmatrix} -0.7 & -0.707 \\ -0.7 & 0.7 \end{bmatrix} = \begin{bmatrix} 1.25 & -1.25 \\ 0.62 & -0.62 \end{bmatrix} \quad (57)$$

Note that there is one singular value equal to zero, therefore there is one unobservable state and as such $p_1 = 1$. Taking $T = V^T$ we obtain a system in triangular form

$$\mathbf{x}(t+1) = \begin{bmatrix} 0.85 & 0.81 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -0.7 & -0.7 \\ 0 & -0.56 \end{bmatrix} e(t), \quad (58a)$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \mathbf{w}(t) \end{bmatrix} = \begin{bmatrix} -1.41 & 1.76 \\ 0 & -1.76 \end{bmatrix} \mathbf{x}(t) + e(t) \quad (58b)$$

and finally we can compute the estimator

$$\hat{\mathbf{x}}(t+1) = \begin{bmatrix} 0.85 & -1.6875 \\ 0 & -0.5 \end{bmatrix} \hat{\mathbf{x}}(t) + \begin{bmatrix} -1.41 \\ -0.56 \end{bmatrix} \mathbf{w}(t) \quad (59a)$$

$$\hat{\mathbf{y}}(t) = [-1.41 \quad 3.53] \hat{\mathbf{x}}(t) + 1\mathbf{w}(t) \quad (59b)$$

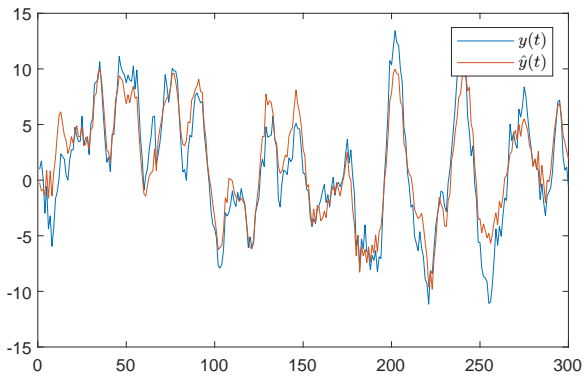


Figure 1: process y and prediction \hat{y}

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