



**HAL**  
open science

# A novel formulation of the unintegrated gluon distribution for DIS

Renaud Boussarie, Yacine Mehtar-Tani

► **To cite this version:**

Renaud Boussarie, Yacine Mehtar-Tani. A novel formulation of the unintegrated gluon distribution for DIS. Physics Letters B, 2022, 831, pp.137125. 10.1016/j.physletb.2022.137125 . hal-03872064

**HAL Id: hal-03872064**

**<https://hal.science/hal-03872064>**

Submitted on 22 Jul 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

# A novel formulation of the unintegrated gluon distribution for DIS

Renaud Boussarie<sup>1,\*</sup> and Yacine Mehtar-Tani<sup>2,3,†</sup>

<sup>1</sup>CPHT, CNRS, Ecole polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau, France

<sup>2</sup>Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA.

<sup>3</sup>RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA

We provide a semi-classical description of the inclusive gluon induced Deep Inelastic Scattering cross section in a way that accounts for the leading powers in both the Regge and Bjorken limits. Our approach thus allows a systematic matching of small and moderate  $x_{\text{Bj}}$  regimes of gluon proton structure functions. We find a new unintegrated gluon distribution with an explicit dependence on the longitudinal momentum fraction  $x$  which entirely spans both the dipole operator and the gluonic Parton Distribution Function. Computing this gauge invariant gluon operator on the lattice could allow to probe the energy dependence of the saturation scale from first principles.

## I. INTRODUCTION

At asymptotically short distances the proton behaves as a collection of free quarks and gluons (partons). This regime of QCD is probed for example in Deep Inelastic Scattering (DIS) experiments such as electron-proton collisions, where a highly virtual photon of momentum  $q$  and virtuality  $Q^2 = -q^2 \gg \Lambda_{\text{QCD}}^2$  is exchanged between the electron and the hadronic target. At small  $x_{\text{Bj}} \equiv Q^2/2(q \cdot P)$ , where  $P$  is the 4-momentum of the proton, the number of gluons probed in the proton rises rapidly and it is expected to reach saturation at very high energies due to gluon recombination effects. This takes place at the saturation scale  $Q_s$  [1, 2], which increases with decreasing  $x_{\text{Bj}}$ . This remarkable yet elusive emergent phenomenon is the subject of active experimental research.

The probability for a parton to carry a fraction  $x$  of the proton momentum, known as the parton distribution function (PDF), is encompassed by the structure functions probed in DIS. PDF's obey renormalization group equations, the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [3], which appear in the Bjorken limit,  $Q \rightarrow \infty$  at fixed  $x_{\text{Bj}}$ . In that limit, an expansion in powers of  $1/Q$  separates short-distance physics, i.e. the hard subprocess, from the long-distance physics encoding the non-perturbative dynamics of confinement in the proton via the PDF. To leading logarithmic accuracy the dominant contribution arises from diagrams that connect the target to the photon with a strong ordering from small to large transverse momenta  $\mu \ll k_{\perp,1} \ll k_{\perp,2} \ll \dots \ll Q$ , while the longitudinal components along the dominant light cone direction of the proton are of similar magnitude:  $x_{\text{Bj}}P^- \sim k_1^- \sim k_2^- \sim \dots \sim P^-$ , where  $P^-$  is the large momentum component of the proton. We are using light cone variables  $k^+ = (k_0 + k_3)/\sqrt{2}$  and  $k^- = (k_0 - k_3)/\sqrt{2}$  in addition to the transverse components  $\mathbf{k} \equiv (k^1, k^2)$ , in the frame

where the proton is moving in the  $-z$  direction its dominant momentum component is  $P^-$ .

In addition to the Bjorken limit, attention has been given to the Regge limit,  $x_{\text{Bj}} \rightarrow 0$  at fixed  $Q$ . Here, an expansion in powers of  $x_{\text{Bj}}$  is to be performed and gluon saturation is expected to emerge for very small values of this variable.

In DIS, the dominant process at small  $x_{\text{Bj}}$  is that of the virtual photon splitting into a quark-antiquark dipole which subsequently interacts with the target by the  $t$ -channel exchange of gluons with small longitudinal momenta in the light cone direction of the photon. In this case, the quantum mechanical time associated with the dipole formation,  $(x_{\text{Bj}}P^-)^{-1}$  is much larger than the target longitudinal size of order  $1/P^-$ . This opens up a large phase space for long lived quantum fluctuations that are enhanced by potentially large logarithms of the form  $\alpha_s \log 1/x_{\text{Bj}} \sim 1$ . The latter are generated by strong ordering in the exchanged longitudinal gluon momenta  $k_i^+$ . In contrast to DGLAP evolution, the transverse components are assumed to be of the same order, i.e.,  $k_{\perp,1} \sim k_{\perp,2} \sim \dots \sim k_{\perp,n}$  along the cascade. This implies a strong inverse ordering the  $k_i^- \sim k_{i\perp}^2/k_i^+$  components.

However, it was observed in Ref. [4] that the Next-to-Leading Order (NLO) corrections (see [5–7]) to the Balitsky-Fadin-Kuraev-Lipatov (BFKL) [8], Balitsky-Kovchegov (BK) [9, 10] and Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK) [11] equations that govern small- $x_{\text{Bj}}$  physics, generate large collinear logarithms that have to be resummed to ensure numerical stability of the equations [12, 13].

The origin of this problem can be traced back to the so-called shock wave approximation which assumes that the target longitudinal extent, as perceived by the photon, is equal to zero. For instance two successive gluons with  $k_1^+ \gg k_2^+$  should also obey the following ordering  $k_1^- \ll k_2^-$ . However, the transverse integrations are not explicitly constrained through the evolution, resulting in the extension of the  $k^-$  phase-space in the non-physical region  $k_1^- > k_2^-$ . More importantly, when  $k_1^- \sim k_2^-$ , with  $k_1^+ \gg k_2^+$  we have  $k_{\perp,1} \gg k_{\perp,2}$ , which corresponds to the DGLAP region.

In light of these recent developments, one can reason-

\* rboussarie@polytechnique.edu

† mehtartani@bnl.gov

ably hope that a reformulation of small  $x$  physics may cure this problem without resorting to an order by order resummation of secular terms. This is the goal of the present letter.

We will use a semi-classical approach [14] similar to the shock wave approach [9] but we will refrain from making assumptions about the extent of the target. The leading powers in both the Bjorken and the Regge limits are instead obtained by performing a gradient expansion around the transverse position of quantum fluctuations. In is approach, light cone time ordering between long-lived quantum fluctuations such as the photon splitting into a  $q\bar{q}$  dipole and the interaction of the dipole with the target field, is always satisfied implying an ordering in the  $k^+$  variable in addition to the  $k^-$  ordering. This feature will obviously hold at any order in perturbation theory.

This Letter is organized as follows. We first present

$$xG^{ij}(x, \mathbf{k}) = \int \frac{d\xi^+ d^2\mathbf{r}}{(2\pi)^3 P^-} e^{ixP^- \xi^+ - i\mathbf{k}\cdot\mathbf{r}} \int_0^1 ds \int_0^1 ds' \langle P | \text{Tr} \mathcal{U}_0(s\mathbf{r}, s'\mathbf{r}) F^{i-}(\xi^+, s'\mathbf{r}) \mathcal{U}_r(s'\mathbf{r}, s\mathbf{r}) F^{j-}(0, s\mathbf{r}) | P \rangle, \quad (1)$$

where  $\mathbf{k}$  is a transverse momentum and  $i, j = 1, 2$  label two orthogonal transverse directions. Also,  $F^{i-} = \partial^i A^- - \partial^- A^i - ig[A^i, A^-]$  is the field strength tensor and

$$\begin{aligned} \mathcal{U}_0(s\mathbf{r}, s'\mathbf{r}) &= [s\mathbf{r}, \mathbf{0}]_{0+} [0, \xi^+]_0 [\mathbf{0}, s'\mathbf{r}]_{\xi^+}, \\ \mathcal{U}_r(s'\mathbf{r}, s\mathbf{r}) &= [s'\mathbf{r}, \mathbf{r}]_{\xi^+} [\xi^+, 0]_{\mathbf{r}} [\mathbf{r}, s\mathbf{r}]_{0+}, \end{aligned} \quad (2)$$

are two finite length staple-shaped gauge links that connect  $F^{i-}(\xi^+, s'\mathbf{r}) \equiv F^{i-}(\xi^+, 0^-, s'\mathbf{r})$  to  $F^{i-}(0^+, s\mathbf{r}) \equiv F^{i-}(0^+, 0^-, s\mathbf{r})$  as depicted in Fig. 2, and where

$$[\xi^+, 0^+]_{\mathbf{r}} \equiv \mathcal{P} \exp \left[ ig \int_{0^+}^{\xi^+} dx^+ A^-(x^+, \mathbf{r}) \right] \quad (3)$$

and

$$[\mathbf{x}, \mathbf{y}]_{\xi^+} \equiv \mathcal{P} \exp \left[ -ig \int_{\mathbf{y}}^{\mathbf{x}} dz \cdot \mathbf{A}(\xi^+, z) \right] \quad (4)$$

are path ordered Wilson lines in the  $-$  and  $\perp$  directions, respectively, with  $\mathbf{z} \equiv \mathbf{z}(s) = s\mathbf{x} + (1-s)\mathbf{y}$ .

One can readily verify that the distribution (1) encompasses both the gluon PDF at large  $x$  and the dipole unintegrated distribution at small  $x$ . Integrating over  $\mathbf{k}$  yields a  $\delta(\mathbf{r})$  and one recovers the gluon PDF

$$\begin{aligned} \int d^2\mathbf{k} xG^{ii}(x, \mathbf{k}) &= xg(x) \equiv \int \frac{d\xi^+}{(2\pi)P^-} e^{ixP^- \xi^+} \\ &\times \langle P | \text{Tr} [0, \xi^+] F^{i-}(\xi^+) [\xi^+, 0] F^{i-}(0) | P \rangle, \end{aligned} \quad (5)$$

where the gluonic operator is implicitly evaluated at  $\mathbf{r} = 0$ . The small  $x_{\text{Bj}}$  limit is slightly more cumbersome,

the operator definition for a novel unintegrated gluon distribution that appears in inclusive DIS within our approach, and which interpolates between the Bjorken and Regge limits. Then we explicitly define our scheme and derive the general DIS cross section. We finally perform the classical expansion to recover the interpolating expression for both kinematic limits and show how the aforementioned gluon distribution appears in inclusive DIS.

## II. UNINTEGRATED GLUON DISTRIBUTION FUNCTION AT SMALL $x$ AND BEYOND: OPERATOR DEFINITIONS

Correcting for the  $x$  dependence of the dipole scattering amplitude, which naturally arises in the small- $x$  limit, yields a novel unintegrated gluon distribution which shall be derived in the context of DIS in the next section. It reads

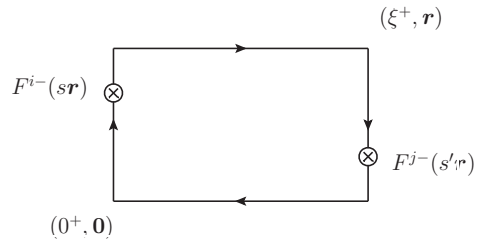


Figure 1. Diagrammatic depiction of the nonlocal operator that defines the unintegrated gluon distribution in Eq. (1). The horizontal and vertical lines represent path ordered Wilson lines along the  $+$  and transverse directions, respectively.

and its derivation is written explicitly in an accompanying longer article [30]. It is obtained from Eq. (1) by first setting  $x = 0$ , and neglecting the transverse gauge links that can be gauged away along with all  $A^i$  fields. One can get a more symmetric form for the operator by using translational invariance and the fact that for protons normalized as  $\langle P' | P \rangle = 2P^- (2\pi)^3 \delta(P'^- - P^-) \delta^2(\mathbf{P}' - \mathbf{P})$  one has  $\int d\mathbf{b} d\xi^+ = (2P^-)^{-1} \langle P' | P \rangle$  to write  $0 \rightarrow \xi^+$  and  $\mathbf{0} \rightarrow \mathbf{b}$  with the price of the proton normalization in the denominator. To make contact with the dipole amplitude in the small  $x_{\text{Bj}}$  limit one has to contract our distribution with  $\mathbf{r}^i \mathbf{r}^j$ . Heuristically, this projection is reminiscent of the standard dipole cross-section formula  $\sigma(|\mathbf{r}|) \propto r^2 xg(x, 1/|\mathbf{r}|)$  [29]. In the DIS cross sec-

tion, the projector  $r^i r^j$  appears naturally, see the derivatives acting on wave functions in momentum space, which would become dipole sizes in coordinate space in Eq. (17). The dipole scattering amplitude is recovered by noticing that  $-r^i \int_0^1 ds F^{i-}(s\mathbf{r}) = A^-(\mathbf{r}) - A^-(\mathbf{0})$ . These differences of  $A^-$  terms result from taking the derivative of the operator  $\text{Tr}[\xi^+, \zeta^+]_r[\zeta^+, \xi^+]_0$  w.r.t.  $\xi^+$  and  $\zeta^+$ , respectively, which allows to rewrite the whole operator as a total differential. Upon integration over  $\xi^+$  and  $\zeta^+$ , one finally obtains

$$\int d^2\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} [r^i r^j x G^{ij}(x, \mathbf{k})]_{x=0} \quad (6)$$

$$= \frac{2}{\alpha_s} \int \frac{d^2\mathbf{b}}{(2\pi)^2} \text{Re}\langle N_c - \text{tr}(U_{\mathbf{b}+\mathbf{r}} U_{\mathbf{r}}^\dagger) \rangle$$

where  $U_{\mathbf{r}} = [+∞, -∞]_{\mathbf{r}}$  and  $\langle \rangle \equiv \langle P|P \rangle / \langle P|P \rangle$ . Note that, in the absence of the phase factor  $e^{ixP^-\xi^+}$ , the interactions with the target encoded in the Wilson line  $U_{\mathbf{r}}$  can take place at any light cone time and are thus not restricted by the kinematics. This is the origin of the violation of the  $k^+$  ordering in the shock wave approximation that we alluded to in the introduction. The distribution from Eq. (1) thus spans both the gluon PDF in the Bjorken limit and the dipole distribution in the Regge limit. It appears in Deep Inelastic Scattering when one interpolates between both limits, as we will prove in the next Section. Whether it appears in other observables is left for future studies.

### III. DIS BEYOND THE SHOCK WAVE APPROXIMATION

In this section we will demonstrate how the unintegrated gluon distribution introduced in the previous section emerges in a physical observable, namely, inclusive DIS, but first let us summarize the three main approximations that will be made in the following derivation. First of all, we will only focus on gluon contributions to the cross section, since we want to improve a small- $x$  inspired scheme where gluons dominate. Secondly, we adopt  $k^+$  as a factorization variable between the target fields and the quantum fluctuations which allows to deal with powers of  $s \rightarrow \infty$  without any further specification on  $Q^2$ . It allows to resum both collinear and rapidity logarithms when  $Q^2 \sim s$  and  $Q^2 \ll s$ , respectively [18]. We will initially restrict ourselves to these two assumptions, but eventually a classical expansion in powers of an intrinsic transverse momentum in the proton,  $k_\perp/\sqrt{s}$ , will be performed.

Consider the DIS subprocess  $\gamma^*(q) + \text{proton}(P) \rightarrow X$ . Owing to the optical theorem, the total cross-section is related to the forward scattering amplitude  $\gamma^*(q) + P \rightarrow \gamma^*(q) + P$ . We shall use light cone variables  $(k^+, k^-, \mathbf{k})$ , defined by the 4-vector decomposition  $k^\mu = k^+ n^\mu + k^- \bar{n}^\mu + k_\perp^\mu$ , where the light cone vectors  $n$  and  $\bar{n}$  satisfy  $n^2 = \bar{n}^2 = 0$  and  $n \cdot \bar{n} = 1$ . We choose the

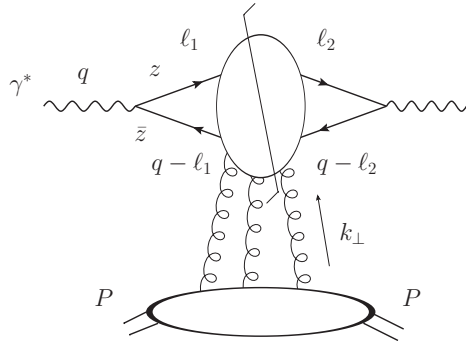


Figure 2. Diagrammatic representation of the process  $\gamma(q) + A \rightarrow q\bar{q}$ .

frame in which the photon and proton momenta are aligned with the  $z$  axis. The leading contributions in both Regge and Bjorken limits stems from gluons with a negligible  $+$  component of their momentum. This reflects itself in coordinate space as an expansion on the null plane  $x^- = 0$ , which implies for the target gluon field  $A^\mu \simeq A^\mu(x^+, 0^-, \mathbf{x})$  only depends on  $x^+$  and the transverse coordinate  $\mathbf{x}$ . This leading field is generated by a color charge current whose only non-vanishing component is  $J^-(x^+, \mathbf{x})$ . It is straightforward to see that the Yang-Mills equations in covariant gauge admit the solution  $A^-(x^+, \mathbf{x})$  and  $A^+ = A^i = 0$  where  $-\partial^2 A^-(x^+, \mathbf{x}) = J^-(x^+, \mathbf{x})$  [19]. This solution is also common to the light cone gauge  $A^+ = 0$ . A more general solution can be obtained by an arbitrary gauge rotation  $\Omega(x^+, \mathbf{x})$  which generates a transverse pure gauge field  $igA^i = -\Omega\partial^i\Omega^{-1}$ . In the shock wave approximation, one would assume that the current is very peaked around  $x^+ = 0$  and build effective Feynman rules by expanding around this point [20, 21]. Here, we will refrain from using this assumption from the get-go, and we will instead perform a gradient expansion in transverse position space in the final expressions for the cross-section.

The transverse and longitudinal cross-sections are related to the hadronic tensor as follows  $\sigma_T(x, Q^2) = \frac{2\pi m^2}{s-m^2} e^2 \varepsilon_\lambda^{*\mu} W_{\mu\nu} \varepsilon_\lambda^\nu$ , with  $\lambda = \pm 1$  and  $\sigma_L(x, Q^2) = \frac{2\pi m^2}{s-m^2} e^2 \varepsilon_L^{*\mu} W_{\mu\nu} \varepsilon_L^\nu$ . The longitudinal polarization vector may be chosen to be  $\varepsilon_L^\mu = \frac{1}{Q} \left( q^+ n + \frac{Q^2}{2q^+} \bar{n} \right)$ , while the transverse polarizations satisfy  $\varepsilon_{+1} \cdot \varepsilon_{-1}^* = 0$  and  $\sum_{\lambda=\pm 1} \varepsilon_\lambda^\mu \cdot \varepsilon_\lambda^{*\nu} = g_\perp^{\mu\nu}$ .

The first and second working assumptions, namely considering a gluonic target boosted on the light cone, allow us to write the hadronic tensor in the form  $\text{Tr}[\gamma^\mu D_F(\ell_2, \ell_1) \gamma^\nu D_F(q - \ell_1, q - \ell_2)]$ , where  $D_F$  is the Dirac propagator in the target background field  $A^-$ . See Fig. 2.

Because the background (target) field does not depend on  $x^-$ , its Fourier transform is proportional to  $\delta(k^+)$ . As a result, the  $+$  components of the quark and antiquark momenta are conserved. This implies the fol-

lowing Dirac decomposition for the quark propagator in momentum space:

$$D_F(\ell_2, \ell_1) = \frac{i\gamma^+}{2\ell_1^+} (2\pi)^4 \delta^4(\ell_2 - \ell_1) + \frac{\ell_2 \gamma^+ \ell_1}{2\ell_1^+} G_{\text{scal}}(\ell_2, \ell_1), \quad (7)$$

where the scalar propagator obeys the Klein-Gordon equation  $(-\square_x + 2igA^-(x)\partial_x^+) G_{\text{scal}}(x, x_0) = \delta(x - x_0)$ . The instantaneous term of the propagator, i.e. the first term in the r.h.s. of Eq. (7), does not contribute to the DIS cross section. We are therefore left with the scalar propagator term.

Making use of its independence on  $x^-$ , the scalar propagator can be expressed as follows in Schwinger notations:

$$G_{\text{scal}}(\ell_2, \ell_1) = \frac{2\pi}{2i\ell_1^+} \delta(\ell_2^+ - \ell_1^+) \int d\xi_1^+ \int d\xi_2^+ \times (\ell_2 | \mathcal{G}_{\ell_1^+}(\xi_2^+, \xi_1^+) | \ell_1) e^{i\ell_2^- \xi_2^+ - i\ell_1^- \xi_1^+}. \quad (8)$$

$\mathcal{G}_{\ell_1^+}(\xi_2^+, \xi_1^+)$  is nothing but the propagator of a non-relativistic particle in 2+1 dimension. It satisfies the Schrödinger equation for the final time

$$\left[ i \frac{\partial}{\partial \xi_2^+} - \frac{\hat{\mathbf{P}}^2}{2\ell^+} + gA(\xi_2^+, \hat{\mathbf{x}}) \right] \mathcal{G}_{\ell^+}(\xi_2^+, \xi_1^+) = i\delta(\xi_2^+ - \xi_1^+), \quad (9)$$

and a similar equation for the initial time. Here,  $\hat{\mathbf{P}} = i\partial$  is the momentum operator. The free propagator is recovered when setting  $A^- = 0$  in these equations, and

reads  $\mathcal{G}_{\ell^+}^{(0)}(\xi_2^+, \xi_1^+) = e^{-i\hat{\mathbf{P}}^2/2\ell^+(\xi_2 - \xi_1)^+}$ . For more insight about the  $\mathcal{G}$  operator, the reader is referred to [22].

With the help of the effective propagators, non-trivial algebra (detailed in [30]) and multiple uses of the integral form of the Schrödinger equation, the cross section can be cast into:

$$\begin{aligned} \sigma &= 8\alpha_s \alpha_{\text{em}} \sum_f q_f^2 \text{Re} \int_0^1 \frac{dz}{2\pi} \int dx_2^+ dx_1^+ d^2\mathbf{r} d^2\mathbf{r}' \\ &\times [\varphi_L(\mathbf{r}) \varphi_L^*(\mathbf{r}') + \frac{1}{2} \sum_{\lambda, \lambda'} \varphi_T^{\lambda, h}(\mathbf{r}) \varphi_T^{\lambda', h*}(\mathbf{r}')] \quad (10) \\ &\times e^{iq^-(x_2 - x_1)^+} \int d^2\mathbf{x}_2 d^2\mathbf{x}_1 \frac{\langle P | \mathcal{O}(x_2^+, x_1^+; \mathbf{x}_2, \mathbf{x}_1, \mathbf{r}, \mathbf{r}') | P \rangle}{\langle P | P \rangle}. \end{aligned}$$

This expression involves the longitudinal and transverse photon wave functions, respectively (see e.g. [23]):

$$\varphi_L(\mathbf{r}) = 2z\bar{z}QK_0(\sqrt{z\bar{z}Q^2\mathbf{r}^2}), \quad (11)$$

and

$$\varphi_T^{\lambda, h}(\mathbf{r}) = iQ\sqrt{z\bar{z}}(z - \bar{z} + 2\lambda h) \frac{(\epsilon_T^\lambda \cdot \mathbf{r})}{|\mathbf{r}|} K_1(\sqrt{z\bar{z}Q^2\mathbf{r}^2}), \quad (12)$$

where  $z = \ell_1^+/q^+$  is the loop quark's longitudinal momentum fraction,  $\bar{z} = 1 - z$ , and  $h = \pm 1/2$  is the quark helicity. In the present scheme, the complicated dependence of the  $t$ -channel operator has the consequence that the wave functions are evaluated a different dipole sizes, contrary to what the dipole or shock wave framework would have led to. The operator in Eq. (10) reads:

$$\begin{aligned} \mathcal{O}(x_2^+, x_1^+; \mathbf{x}_2, \mathbf{x}_1, \mathbf{r}, \mathbf{r}') &= \text{tr} \{ (\mathbf{x}_2 | \mathcal{G}_{zq^+}(x_2^+, x_1^+) | \mathbf{x}_1) [A^-(x_1^+, \mathbf{x}_1 + \mathbf{r}) - A^-(x_1^+, \mathbf{x}_1)] \\ &\times (\mathbf{x}_1 + \mathbf{r} | \mathcal{G}_{-\bar{z}q^+}(x_1^+, x_2^+) | \mathbf{x}_2 + \mathbf{r}') [A^-(x_2^+, \mathbf{x}_2 + \mathbf{r}') - A^-(x_2^+, \mathbf{x}_2)] \}. \quad (13) \end{aligned}$$

It describes the Brownian motion in the external field between the first and last interactions with the target. Each of these interactions can occur on the quark or on the antiquark, hence the four terms. One contribution is depicted in Fig. 3. Using

$$A^-(\mathbf{x} + \mathbf{r}) - A^-(\mathbf{x}) = -\mathbf{r}^i \int_0^1 ds F^{i-}(\mathbf{x} + s\mathbf{r}), \quad (14)$$

allows to combine and simplify all contributions. It is worth noting that the wave functions are unmodified by the inclusion of presumably  $1/\sqrt{s}$ -suppressed terms into the shock wave picture. The notion of the perceived extent of the target is very natural here: it can be understood as the time difference between the first and the last interaction. In the shock wave approximation, this difference would be assumed to be close to 0 and an expansion around that point would be performed [20, 21].

Let us turn now to our third approximation, the modified shock wave. When considering quantum diffusion in the external field, the propagator between interactions at points  $x_i, x_j$  is a Gaussian with the exponent  $-i\ell_1^+ \mathbf{x}_{ij}^2 / (2x_{ij}^+)$ , where double subscripts denote differences, e.g.  $\mathbf{x}_{ij} \equiv \mathbf{x}_i - \mathbf{x}_j$ . Parametrically,  $x_{ij}^+ \sim 1/P^-$  and  $\ell^+ \sim q^+$ , which means  $\mathbf{x}_{ij}^2 \sim x_{Bj}/Q^2$ . In other words, the main contribution for the Brownian motion through the external field conserves the transverse position, up to corrections which are  $x_{Bj}$ -suppressed in the Regge limit and  $1/Q^2$ -suppressed in the Bjorken limit. We can thus perform part of the classical expansion from [20] in a way which is consistent for both small and large  $x_{Bj}$ .

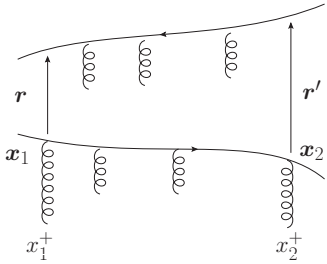


Figure 3. One contribution to the operator from Eq. (14). The quark and the antiquark undergo the propagation in the external field between their first and final interaction, respectively at times  $x_1^-$  and  $x_2^-$ , here both on the quark at respective transverse positions  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .

We readily find

$$(\mathbf{x}_2 | \mathcal{G}_{zq^+}(x_2^+, x_1^+) | \mathbf{x}_1) = \quad (15)$$

$$\mathcal{G}_{zq^+}^{(0)}(x_{21}^+, x_{21}) [x_2^+, x_1^+]_{\mathbf{b}} + \mathcal{O}(|x_{21}|/|\mathbf{b}|)$$

$$(\mathbf{x}_1 + \mathbf{r} | \mathcal{G}_{-\bar{z}q^+}(x_1^+, x_2^+) | \mathbf{x}_2 + \mathbf{r}') = \quad (16)$$

$$\mathcal{G}_{-\bar{z}q^+}^{(0)}(x_{12}^+, \mathbf{v}) [x_1^+, x_2^+]_{\mathbf{b}+\mathbf{R}} + \mathcal{O}(|\mathbf{v}|/|\mathbf{b}+\mathbf{R}|).$$

Here,  $\mathbf{b} = (\mathbf{x}_1 + \mathbf{x}_2)/2$ ,  $\mathbf{v} = \mathbf{x}_{12} + \mathbf{r} - \mathbf{r}'$  and  $\mathbf{R} = (\mathbf{r} + \mathbf{r}')/2$ . For the same parametric reasons, we can expand the arguments of the gluon fields around  $\mathbf{x}_1 \simeq \mathbf{x}_2 \simeq \mathbf{b}$ ,  $\mathbf{r} \simeq \mathbf{r}' \simeq \mathbf{R}$ . Going to momentum space and integrating out  $\mathbf{b}$  and  $\mathbf{R}$  yields our final result for the cross section, for photon helicity  $\lambda = L, +1, -1$ :

$$\sigma_\lambda(x_{Bj}, Q^2) = 4\alpha_{em}\alpha_s \sum_f q_f^2 \int_0^1 \frac{dx}{2\pi} \int_0^1 \frac{dz}{2\pi} \int d^2\mathbf{k} d^2\ell \partial^i \phi_\lambda \left( \ell - \frac{\mathbf{k}}{2} \right) \partial^j \phi_\lambda^* \left( \ell + \frac{\mathbf{k}}{2} \right) \delta \left[ x - x_{Bj} \left( 1 + \frac{\ell^2}{Q^2} \right) \right] x G^{ij}(x, \mathbf{k}), \quad (17)$$

where the unintegrated gluon distribution is defined in Eq. (1) and the wave functions  $\phi_\lambda$  are the Fourier transforms of the  $\varphi_\lambda$  functions from Eq. (11), Eq. (12). Note the derivatives acting of the wave functions, which would yield powers of the dipole size  $r^i r^j$  in the small  $x_{Bj}$  limit, see Eq. (6). These powers of the dipole sizes stem from the integral in Eq. (14). In terms of the original variables,  $\mathbf{k} = \ell_2 - \ell_1$  and  $\ell = (\ell_2 + \ell_1)/2$ . This equation is exact up to corrections of relative order  $p_\perp/\sqrt{s}$ , where  $p_\perp$  is an intrinsic transverse momentum in the proton. Such corrections are suppressed in the Bjorken regime as well as in the Regge limit. The explicit  $x$  dependence in Eqs. (17) and (10) results in a non-locality in transverse dipole sizes which is not compatible with the dipole model as previously noted in [24]. Only when  $x$  is neglected do we get  $\delta(\mathbf{r} - \mathbf{r}')$ . Eq. (17) together with Eq. (1) is our main result.

Finally, it is worth noting that when  $|\mathbf{k}| \ll |\ell|$ , Eq. (17) reproduces the DGLAP logarithm associated with the creation of the quark (resp. antiquark) from the splitting of a collinear gluon in the target. This limit is achieved for  $z \simeq 0$  (resp.  $z \simeq 1$ ). Integrating over  $z = y/(1-y)\ell^2/Q^2$  using the delta function in Eq. (17) with  $y = x_{Bj}/x$  and neglecting  $\mathbf{k}$  in the wave functions one recovers the gluon PDF upon integration over  $\mathbf{k}$ , multiplied by the Altarelli-Parisi splitting function  $P_{qg}(y) \sim y^2 + (1-y)^2$  and  $\int_{\mu^2}^{Q^2} d\ell^2/\ell^2 = \log Q^2/\mu^2$ . See [30] for the explicit proof.

#### IV. SUMMARY AND OUTLOOK

By investigating the small- $x$ -inspired semi-classical description of an observable beyond the “naive” high

energy limit, aka the shock wave approximation, we found an unintegrated gluon distribution with explicit dependence on the longitudinal fraction which spans both the collinear and the small  $x$  limits. This distribution, remarkably, does not involve infinite-length Wilson lines, hence its evaluation on the Euclidean Lattice [25] may be simplified when compared to the usual small- $x$  distributions. Such a lattice evaluation would directly confirm or infirm the existence and energy dependence [16, 26] of the saturation scale from first principles. Until we overcome the great numerical challenge of calculating parton distributions at small  $x$ , it will be interesting to investigate the transverse momentum dependence of this gluon distribution at currently accessible  $x$  values.

Phenomenology for semi-classical small- $x$  physics at NLL accuracy has revealed a consistency issue with the standard scheme [4]. Colossal efforts were made in order to address it, mostly by modifying the evolution equation without changing the evolved quantity [12, 13]. The origin of this inconsistency is easy to identify in Eq. (17). Indeed, it is widely believed that the smallness of  $x_{Bj}$  leads to the smallness of the longitudinal fraction  $x$ . This fraction can however be enhanced by loop integrals even at small values of  $x_{Bj}$ : the  $z \rightarrow 0, 1$  limits in Eq. (17) yield the full DGLAP logarithms when  $\ell \gg \mathbf{k}$ . By implicitly requiring  $x \simeq x_{Bj} \simeq 0$ , the shock wave approximation leads to an inconsistent treatment of such collinear logarithms. This is entirely due to the absence of light cone time ordering in the strict  $x = 0$  limit, while this ordering and thus an ordering in longitudinal momenta are restored in our classical picture. Studying the quantum evolution of the new distribution derived in this article should provide an alternative to the BK equation and

solve this conundrum in a natural way. In a similar fashion, the limit  $x \simeq 0$  tends to suppress target spin effects [27, 28]. We can actually conclude from the present analysis that spin effects can occur even in the small  $x_{Bj}$  limit because of the collinear corner of the phase space where  $x \gg x_{Bj}$ .

## ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, un-

der contract No. DE-SC0012704, and by Laboratory Directed Research and Development (LDRD) funds from Brookhaven Science Associates. Y. M.-T. acknowledges support from the RHIC Physics Fellow Program of the RIKEN BNL Research Center. Figures were drawn using Jaxodraw.

- 
- [1] L. Gribov, E. Levin and M. Ryskin, *Phys. Rept.* **100**, 1-150 (1983) doi:10.1016/0370-1573(83)90022-4.
- [2] A. H. Mueller and J. w. Qiu, *Nucl. Phys. B* **268**, 427-452 (1986) doi:10.1016/0550-3213(86)90164-1.
- [3] V. Gribov and L. Lipatov, *Sov. J. Nucl. Phys.* **15**, 438-450 (1972) IPTI-381-71 ; Y. L. Dokshitzer, *Sov. Phys. JETP* **46**, 641-653 (1977) ; G. Altarelli and G. Parisi, *Nucl. Phys. B* **126**, 298-318 (1977) doi:10.1016/0550-3213(77)90384-4.
- [4] T. Lappi and H. Mäntysaari, *Phys. Rev. D* **91**, no.7, 074016 (2015) doi:10.1103/PhysRevD.91.074016 [arXiv:1502.02400 [hep-ph]] ; E. Avsar, A. Stasto, D. Triantafyllopoulos and D. Zaslavsky, *JHEP* **10**, 138 (2011) doi:10.1007/JHEP10(2011)138 [arXiv:1107.1252 [hep-ph]].
- [5] V. S. Fadin and L. Lipatov, *Phys. Lett. B* **429**, 127-134 (1998) doi:10.1016/S0370-2693(98)00473-0 [arXiv:hep-ph/9802290 [hep-ph]] ; M. Ciafaloni and G. Camici, *Phys. Lett. B* **430**, 349-354 (1998) doi:10.1016/S0370-2693(98)00551-6 [arXiv:hep-ph/9803389 [hep-ph]].
- [6] I. Balitsky and G. A. Chirilli, *Phys. Rev. D* **77**, 014019 (2008) doi:10.1103/PhysRevD.77.014019 [arXiv:0710.4330 [hep-ph]].
- [7] A. Kovner, M. Lublinsky and Y. Mulian, *Phys. Rev. D* **89**, no.6, 061704 (2014) doi:10.1103/PhysRevD.89.061704 [arXiv:1310.0378 [hep-ph]] ; A. Kovner, M. Lublinsky and Y. Mulian, *JHEP* **08**, 114 (2014) doi:10.1007/JHEP08(2014)114 [arXiv:1405.0418 [hep-ph]].
- [8] I. Balitsky and L. Lipatov, "The Pomernanchuk Singularity in Quantum Chromodynamics," *Sov.J.Nucl.Phys.* **28** (1978) 822-829 ; E. A. Kuraev, L. N. Lipatov and V. S. Fadin, *Sov. Phys. JETP* **44**, 443-450 (1976) ; E. Kuraev, L. Lipatov, and V. S. Fadin, "The Pomernanchuk Singularity in Nonabelian Gauge Theories," *Sov.Phys.JETP* **45** (1977) 199-204.
- [9] I. Balitsky, *Nucl. Phys. B* **463** (1996), 99-160 doi:10.1016/0550-3213(95)00638-9 [arXiv:hep-ph/9509348 [hep-ph]].
- [10] Y. V. Kovchegov, *Phys. Rev. D* **60** (1999), 034008 doi:10.1103/PhysRevD.60.034008 [arXiv:hep-ph/9901281 [hep-ph]].
- [11] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, *Phys. Rev. D* **59** (1998), 014014 doi:10.1103/PhysRevD.59.014014 [arXiv:hep-ph/9706377 [hep-ph]] ; E. Iancu, A. Leonidov and L. D. McLerran, *Nucl. Phys. A* **692** (2001), 583-645 doi:10.1016/S0375-9474(01)00642-X [arXiv:hep-ph/0011241 [hep-ph]].
- [12] G. Beuf, *Phys. Rev. D* **89**, no.7, 074039 (2014) doi:10.1103/PhysRevD.89.074039 [arXiv:1401.0313 [hep-ph]].
- [13] E. Iancu, J. Madrigal, A. Mueller, G. Soyez and D. Triantafyllopoulos, *Phys. Lett. B* **744**, 293-302 (2015) doi:10.1016/j.physletb.2015.03.068 [arXiv:1502.05642 [hep-ph]] ; E. Iancu, J. Madrigal, A. Mueller, G. Soyez and D. Triantafyllopoulos, *Phys. Lett. B* **750**, 643-652 (2015) doi:10.1016/j.physletb.2015.09.071 [arXiv:1507.03651 [hep-ph]] ; B. Ducloué, E. Iancu, A. Mueller, G. Soyez and D. Triantafyllopoulos, *JHEP* **04**, 081 (2019) doi:10.1007/JHEP04(2019)081 [arXiv:1902.06637 [hep-ph]] ; B. Ducloué, E. Iancu, G. Soyez and D. Triantafyllopoulos, *Phys. Lett. B* **803**, 135305 (2020) doi:10.1016/j.physletb.2020.135305 [arXiv:1912.09196 [hep-ph]].
- [14] L. D. McLerran and R. Venugopalan, *Phys. Rev. D* **49**, 2233-2241 (1994) doi:10.1103/PhysRevD.49.2233 [arXiv:hep-ph/9309289 [hep-ph]] ; L. D. McLerran and R. Venugopalan, *Phys. Rev. D* **49**, 3352-3355 (1994) doi:10.1103/PhysRevD.49.3352 [arXiv:hep-ph/9311205 [hep-ph]] ; L. D. McLerran and R. Venugopalan, *Phys. Rev. D* **50**, 2225-2233 (1994) doi:10.1103/PhysRevD.50.2225 [arXiv:hep-ph/9402335 [hep-ph]].
- [15] K. Kutak and A. Stasto, *Eur. Phys. J. C* **41**, 343-351 (2005) doi:10.1140/epjc/s2005-02223-0 [arXiv:hep-ph/0408117 [hep-ph]].
- [16] S. Munier and R. B. Peschanski, *Phys. Rev. Lett.* **91**, 232001 (2003) doi:10.1103/PhysRevLett.91.232001 [arXiv:hep-ph/0309177 [hep-ph]].
- [17] Y. V. Kovchegov, *Phys. Rev. D* **61** (2000), 074018 doi:10.1103/PhysRevD.61.074018 [arXiv:hep-ph/9905214 [hep-ph]].
- [18] I. Balitsky and A. Tarasov, *JHEP* **10**, 017 (2015) doi:10.1007/JHEP10(2015)017 [arXiv:1505.02151 [hep-ph]].
- [19] F. Gelis and Y. Mehtar-Tani, *Phys. Rev. D* **73** (2006), 034019 doi:10.1103/PhysRevD.73.034019 [arXiv:hep-ph/0512079 [hep-ph]].
- [20] T. Altinoluk, N. Armesto, G. Beuf, M. Martínez and C. A. Salgado, *JHEP* **07**, 068 (2014) doi:10.1007/JHEP07(2014)068 [arXiv:1404.2219 [hep-ph]] ; T. Altinoluk, N. Armesto, G. Beuf and A. Moscoso,

- JHEP **01**, 114 (2016) doi:10.1007/JHEP01(2016)114 [arXiv:1505.01400 [hep-ph]].
- [21] G. A. Chirilli, JHEP **01**, 118 (2019) doi:10.1007/JHEP01(2019)118 [arXiv:1807.11435 [hep-ph]].
- [22] J. P. Blaizot and Y. Mehtar-Tani, Int. J. Mod. Phys. E **24** (2015) no.11, 1530012 doi:10.1142/S021830131530012X [arXiv:1503.05958 [hep-ph]].
- [23] G. Beuf, Phys. Rev. D **85**, 034039 (2012) doi:10.1103/PhysRevD.85.034039 [arXiv:1112.4501 [hep-ph]].
- [24] A. Bialas, H. Navelet and R. B. Peschanski, Nucl. Phys. B **593** (2001), 438-450 doi:10.1016/S0550-3213(00)00640-4 [arXiv:hep-ph/0009248 [hep-ph]].
- [25] X. Ji, Phys. Rev. Lett. **110** (2013), 262002 doi:10.1103/PhysRevLett.110.262002 [arXiv:1305.1539 [hep-ph]].
- [26] A. Mueller and D. Triantafyllopoulos, Nucl. Phys. B **640**, 331-350 (2002) doi:10.1016/S0550-3213(02)00581-3 [arXiv:hep-ph/0205167 [hep-ph]].
- [27] Y. Hatta, Y. Nakagawa, F. Yuan, Y. Zhao and B. Xiao, Phys. Rev. D **95**, no.11, 114032 (2017) doi:10.1103/PhysRevD.95.114032 [arXiv:1612.02445 [hep-ph]].
- [28] Y. V. Kovchegov, D. Pitonyak and M. D. Sievert, JHEP **10**, 198 (2017) doi:10.1007/JHEP10(2017)198 [arXiv:1706.04236 [nucl-th]].
- [29] Y. V. Kovchegov and E. Levin, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **33**, 1-350 (2012) doi:10.1017/CBO9781139022187
- [30] R. Boussarie and Y. Mehtar-Tani, [arXiv:2112.01412 [hep-ph]].