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# Fully automated physics-regularized model updating algorithm for vibration-based damage detection from sparse data

M. Diaz<sup>a,\*</sup>, P.-É. Charbonnel<sup>b</sup>, L. Chamoin<sup>a,c</sup>

<sup>a</sup>*Université Paris-Saclay, CentraleSupélec, ENS Paris-Saclay, LMPS - Laboratoire de Mécanique Paris-Saclay, 91190, Gif-sur-Yvette, France*

<sup>b</sup>*DES-Service d'Études Mécaniques et Thermiques (SEMT), CEA, Université Paris-Saclay, 91191 Gif-sur-Yvette, France*

<sup>c</sup>*IUF, Institut Universitaire de France*

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## Abstract

Digital twins efficiency lies in fast and representative solutions of inverse problems to accommodate models with physical observations. The quality of the solution of an inverse problem is conditioned by inherent features of the latter, in particular (i) the richness of available data, (ii) the *a priori* experimental and modeling knowledge that allows to regularize the ill-posedness nature of the problem, and (iii) the complexity of the space in which updated parameters are sought. We present in this contribution a fully automated robust model updating framework dedicated to the correction of finite element models from low-frequency dynamics measurements. The proposed methodology is based on the minimization of a modified Constitutive Relation Error (mCRE) functional, whose basic idea is to construct mechanical fields and identify material parameters that are a trade-off between all available information but without any further assumptions. The dependency into some expert-user's judgment is thus avoided. Dedicated rules are provided to automatically calibrate all mCRE internal tuning parameters as well as a strategy to optimize the space in which parameters are sought, leading to a fully autonomous algorithm. The performance and robustness of the proposed model updating methodology are illustrated using synthetic ground motion tests on a bending plate in which defects of various shapes are identified from noisy acceleration datasets, with inherent limitations due to sensors sparsity and defect observability.

*Keywords:* Modified Constitutive Relation Error, Automated Model Updating, Low-Frequency Dynamics, Vibration-based damage detection.

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\*Corresponding author

*Email address:* [matthieu.diaz@ens-paris-saclay.fr](mailto:matthieu.diaz@ens-paris-saclay.fr) (M. Diaz)

## 1. Introduction

The design, analysis and prediction of dynamical systems requires the construction of robust numerical models. These models can be directly built from measurements (*black-box modeling*) or derived after in-depth physical description of the involved phenomena (*white-box modeling*). In each case, as most of modern systems are now equipped with numerous sensors, those models are (at least) assessed by comparison with experimental data in order to define their degree of representativeness. Indeed, the validation, enrichment and exploitation of numerical models with experimental data are part of the *digital twin* paradigm, whose applications range from simple post-processing to state prediction, control of systems, or even decision-making process [1–3]. Earthquake engineering problems are no exception to the need of using experimental data to build, validate and operate robust numerical models, in particular when it comes to monitor and predict the structural state of civil engineering structures [4].

From the numerical viewpoint, the integration of a digital twin requires a complex numerical framework including a robust model updating algorithm able to operate in real-time. In this paper, we will address the possibilities to perform damage detection from sparse data collected in an (offline) low-frequency dynamics context, with an emphasis on the trade-off that must be found between:

- the richness of available measurements,
- the complexity of the parameter space,
- the inherent model updating limitations due to the inverse problem itself.

### 1.1. *Vibration-based damage detection techniques at a glance*

For structural health monitoring (SHM) applications, the calibration of stiffness parameters from experimental data allows to identify (*i.e.* locate and quantify) structural damage. Throughout the last decades, a wide panel of damage detection methods has been proposed [4–6]. All these techniques derive various model updating approaches and are often dedicated to the exploitation of kinematics measurements. For the sake of conciseness, we will exclusively focus on damage detection from sparse data (obtained from accelerometers, strain gauges, transducers, or Bragg optic fiber measurements for example) although dedicated approaches operating full-field measurements have also been developed, see [7–9] to cite a few. When dealing with spatially sparse datasets, the identification of damage in large structures is inherently difficult because of the relatively reduced amount of available measurements, leading to an ill-posed inverse problem [10].

Due to the fact that many SHM applications directly exploit modal data, a parametrized model is thus not mandatory as the changes in modal features such as eigenfrequencies, damping ratios or modeshapes can provide direct information about the damage state of a structure. Their evolution with time allows the user to track structural damage [11]. In particular, classical modal analysis techniques aim at identifying modal features through the identification of a state-space model from measurements. One of the fastest and most accurate methods is based on stochastic subspace identification [12, 13]. Since forced vibration testing of large structures in operational conditions is impractical, *Operational Modal Analysis* (OMA) techniques have been considerably developed to process output-only measurements [14–18] obtained from unknown excitations in an operational environment (*i.e.* wind or road traffic for bridges).

However, Finite Element (FE) models can still be updated according to experimental (possibly modal) data. In such a context, Bayesian approaches (whose extended comprehensive review is available in [5]) are classically distinguished from deterministic methods. Among the latter, the sensitivity method is one of the most popular techniques to solve inverse problems. Reviewed in [19], the sensitivity method is based upon linearization of the generally nonlinear relationship between measurable outputs and the parameters in need of correction. As most of deterministic inverse problems, regularization techniques are mandatory to prevent the ill-posedness of the problem [20]. Such techniques, although easy to implement, provide identification results that are strongly sensitive to the regularization term (whether it is a  $L_2$ -norm Tikhonov or a  $L_1$ -norm sparse regularization) whose definition totally depends on some user's *a priori* expertise; nevertheless, it is of crucial importance as it conditions the smoothness of the obtained solution [21–23]. Besides, such approaches may also be highly disturbed by measurement noise, with the possibility to obtain divergent or physical-meaningless results.

An alternative consists in using the concept of *modified Constitutive Relation Error* (mCRE) that exploits the reliability of information [24, 25]. This is the main driver behind its selection as a reference method for model updating in this paper. Initially proposed for model updating in dynamics by Ladevèze and co-workers [26, 27], the mCRE functional is defined as a quadratic model-to-measurements distance enriched with a term based on the concept of *Constitutive Relation Error* (CRE) [28], whose enriched physical meaning and strong mechanical content avoid the direct use of regularization terms based on *a priori* expert-user knowledge. This energy-based residual offers interesting advantages. First, local convexity properties are enhanced compared to classical deterministic functionals [29]. Moreover, the CRE part of the residual computed over the whole structure can be exploited as a spatial distribution of the modeling error: it allows for example to restrain the updating process to a few parameters [30], which can be computationally helpful and regularizing (in Tikhonov's sense) when the number of parameters to update becomes important. The relevance and robustness of the mCRE for model updating has been emphasized in many applications. Among other works, the mCRE was shown to be robust to highly noisy and corrupted measurements [31] and able to perform local defect detection [25, 32, 33], or full-field material identification from dense measurements [34, 35]. It has recently been successfully applied to Earthquake Engineering problems, with an implementation in an industrial FE software [36, 37]. As one can explicitly establish a link between mCRE with other deterministic and stochastic model updating functionals, it is also worth mentioning the comparative study between mCRE, Tikhonov-based and Bayesian damage detection using optical fiber strain measurements investigated in [38].

For the sake of completeness, let us finally mention damage detection techniques based on neural networks and machine learning, that are getting more and more popular due to their capability to process large amount of data [6, 39, 40], exploiting techniques such as statistical pattern recognition that are of growing interest in SHM [41]. However, they still lack of generality in the sense that the learning process conditions the performance of such methods.

## 1.2. Main contributions

A dedicated mCRE-based model updating framework has been recently developed by the authors in order to perform accurate identification of Finite Element (FE) models from sparse data collected in low-frequency dynamics [42]. Although the robustness to

measurement noise of the approach was proved by processing data from actual shaking-table experiments, the performance of the methodology remains limited by the complex setting of the mCRE framework. To the authors' best knowledge, the design of the parameter space and the tuning procedure that integrates measurements of varying SNR are still open questions in this framework, that have not been clearly addressed in the literature. The ambition of this work is to propose a mCRE-based fully automated model updating algorithm with the objective of identifying damaged areas optimally, *i.e.*, as accurately as possible at minimum computational cost. More precisely, in this contribution, an automated selection of the parameter space is proposed based on the modeling error discrepancies. A clustering strategy is carried out to identify in which areas model updating is of higher importance, generalizing the above-mentioned concept of localization of most erroneous areas. The call to the CRE as model updating indicator makes the tuning of mCRE internal parameters crucial to design an optimal parameter space for damage detection. Dedicated (automated) strategies are implemented and compared for optimal calibration of the confidence into measurements. Finally, physics-inherent model updating limitations are highlighted in this contribution, namely the impact of the sensor placement strategy (particularly when dealing with sparse data) and the observability of defects in problems where the strain energy distribution of the tested structure is strongly heterogeneous.

The remainder of the paper is structured as follows. Section 2 recalls basics on mCRE for FE stiffness model updating in low-frequency dynamics. Several strategies for the full-automation of the algorithm are discussed regarding the calibration of internal parameters in Section 3. The questions of the confidence into measurements and of the construction of an optimal parametrization using CRE-based clustering are particularly emphasized. A discussion tackling additional regularization approaches (whether in  $L_2$  or  $L_1$  metrics) is also included. In order to assess the relevance of the proposed approach, a numerical benchmark in which several defect shapes are identified is fully discussed in Section 4. Conclusions and prospects are finally drawn in Section 5, suggesting a future implementation of the automated model updating technique for experimental test campaigns and data assimilation.

## 2. The mCRE for damage detection in low-frequency dynamics

Updating a model is an optimization problem that consists in identifying a set of internal model parameters  $\theta \in \Theta$  that guarantee the relevance of model predictions  $\mathcal{M}(\theta)$  from collected measurements  $y$ . In a deterministic approach, in order to overcome the ill-posedness of the inverse problem, a regularization term  $\mathcal{G}$  must be often added. It guarantees uniqueness of the solution by enforcing local ellipticity properties of the functional to minimize  $\mathcal{J}$ . Without loss of generality, a deterministic model updating problem thus reads:

$$\hat{\theta} = \arg \min_{\theta \in \Theta} \left[ \mathcal{J}(\theta) \triangleq \alpha \mathcal{D}(\mathcal{M}(\theta), y) + \mathcal{G}(\theta, \theta^*) \right] \quad (1)$$

The cost-function  $\mathcal{J}$  is built as the weighted sum of a data-to-model distance term  $\mathcal{D}(\mathcal{M}(\theta), y)$  enriched with the so-called regularization term. The weighting parameter  $\alpha$  thus quantifies the reliability of the data-to-model distance  $\mathcal{D}(\mathcal{M}(\theta), y)$  with respect to the *a priori* knowledge  $\theta^*$  included in  $\mathcal{G}(\theta, \theta^*)$ . Its value may be tuned according to several techniques as the L-curve criterion [43, 44] or Morozov's discrepancy principle [45, 46].

In this section, we will show how the mCRE-based model updating problem is similar in its formulation to the general case (1) and we will briefly recall the fundamentals of this model updating framework for low-frequency dynamics, starting from a linear FE problem written in the frequency domain. Particular attention is paid to the construction of the *Constitutive Relation Error* (CRE) residual from the set of equations defining the reference mechanical problem. Then, details about the *modified Constitutive Relation Error* (mCRE) are explicitly given considering the correction of stiffness parameters. An algorithm allowing to conduct the solution of the associated inverse problem is also given. For additional details, the interested reader is referred to [42, 47] for extended overviews of the mCRE-based model updating framework in the low-frequency dynamics regime.

### 2.1. FE framework, measurements and stiffness parametrization

Let us consider the general case of an elastic structure  $\Omega$  spatially discretized in (non-overlapping) finite elements such that  $\Omega = \cup_{e=1}^E \Omega_e$ , and subjected to a given dynamical excitation  $F$ . We denote by  $K, D, M$  the stiffness, damping and mass FE matrices, respectively, while  $F_\omega$  and  $U_\omega$  are the frequency counterparts of nodal loading conditions and displacement field. With these notations, the dynamic equilibrium written in the frequency domain at a given angular frequency  $\omega$  reads:

$$[-\omega^2 M + i\omega D + K] U_\omega = F_\omega \quad (2)$$

In addition to the dynamic equilibrium, a set of sensors is used to measure the magnitude of some kinematic quantities (displacement, velocity and/or acceleration). In the frequency domain, assuming for the moment that measurements are perfect, such information can be written without loss of generality as:

$$\Pi U_\omega = Y_\omega \quad (3)$$

where  $Y_\omega$  refers to the frequency counterpart of measurements at angular frequency  $\omega$  and  $\Pi$  denotes a projection matrix that allows to extract the components of  $U_\omega$  that are measured. Its non-zero values are integer power of  $(i\omega)$ .

As mentioned in the introduction, one can interpret damage as local stiffness loss. Therefore, a convenient manner to parametrize a linear FE problem for damage detection is to parametrize the FE stiffness matrix. Doing so, in this contribution, the  $n_\theta$  parameters to update  $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$  only affect the stiffness matrix  $K$ . More precisely, the FE stiffness matrix is decomposed into  $n_\theta$  non-overlapping subdomains and parametrized as follows:

$$K(\theta) = \sum_{i=1}^{n_\theta} \frac{\theta_i}{\theta_{0,i}} K_{0,i} \quad \text{with} \quad K(\theta_0) = \sum_{i=1}^{n_\theta} K_{0,i} \quad (4)$$

Note that the subdomains can perfectly match with finite elements or gather some of them to reduce the number  $n_\theta$  of parameters to identify.

### 2.2. The modified Constitutive Relation Error: a physics-based approach for inverse problems

Originally intended to perform model verification [28, 48], the key idea for the construction of the CRE residual lies into the distinction between reliable and unreliable information on the reference mechanical problem. The mCRE concept extends this distinction

to experimental data as well, allowing to build a functional able to handle measurement error and model uncertainty simultaneously. Indeed, the redundant information gathered in (2) and (3) means all equations cannot be exactly verified. Some of them must be considered less reliable, and thus relaxed as they will be only verified at best by minimizing the so-called modified CRE functional.

In other words, the fundamental idea of the mCRE concept is to identify mechanical fields and material parameters that are a trade-off between all available information without adding any other *a priori* assumption. Although this distinction between reliable and unreliable equations is non-unique and deeply relies on the case study and engineering expertise, it is also well-known that, in most applications, constitutive relations are subject to caution. The full separation of equations for the considered damage detection case is given in TAB. 1. Note that the damping modelling will not be subject to caution as damping variations are secondary for damage detection. This choice could be questioned if the stiffness properties were correctly updated. Once modeling equations have been labeled as reliable or unreliable, two admissibility spaces are implicitly defined:

- (i) a kinematic admissibility space  $\mathcal{U}_{ad}$  inside which any displacement field  $U$  verifies the reliable kinematic equations of the problem,
- (ii) an auxiliary dynamically admissibility space  $\mathcal{D}_{ad}$  defined as the set of displacement fields  $V$  derived from the stress field verifying the dynamic equilibrium.

The reciprocity gap between  $\mathcal{U}_{ad}$  and  $\mathcal{D}_{ad}$  can be measured using an energy norm - the CRE - that estimates the relevance of a solution couple  $s_\omega = (U_\omega, V_\omega) \in \mathcal{U}_{ad} \times \mathcal{D}_{ad}$  with respect to the mechanical problem. With the above notations, the CRE at a given angular frequency  $\omega$  reads:

$$\zeta_\omega^2(s_\omega, \theta) = \frac{1}{2}(U_\omega - V_\omega)^H K(\theta)(U_\omega - V_\omega) = \frac{1}{2}\|U_\omega - V_\omega\|_{K(\theta)}^2 \quad (5)$$

with  $\square^H$  referring to the Hermitian transpose.

	Reliable	Unreliable
Model	<ul style="list-style-type: none"> <li>• Geometry</li> <li>• Boundary conditions</li> <li>• Equilibrium equations</li> <li>• Dissipative constitutive relations</li> </ul>	<ul style="list-style-type: none"> <li>• Elastic constitutive relations</li> </ul>
Experiments	<ul style="list-style-type: none"> <li>• Loading frequencies <math>\omega/2\pi</math></li> <li>• Sensor locations</li> <li>• Measured inputs <math>F_\omega</math></li> </ul>	<ul style="list-style-type: none"> <li>• Measured outputs <math>Y_\omega</math></li> </ul>

TABLE 1: Distinction between reliable and unreliable information for damage detection from stiffness update in dynamics.

The extension of the CRE concept to unreliable experimental data (see TAB. 1) directly leads to the so-called *modified Constitutive Relation Error* (mCRE). The CRE is completed by a data-to-model distance between the predictions  $U_\omega$  and the frequency counterpart of measurements  $Y_\omega$  :

$$e_\omega^2(s_\omega, \theta, Y_\omega) \triangleq \zeta_\omega^2(s_\omega, \theta) + \frac{\alpha}{2}\|\Pi U_\omega - Y_\omega\|_G^2 \quad (6)$$

The choice of the symmetric positive-definite matrix  $G$  is not critical as it intends to ensure that  $\|\square\|_G$  is homogeneous to  $\zeta_\omega^2$  and equivalent in level. Herein,  $G$  is chosen proportional to the identity matrix and weighted by the strain energy stored in the first (involved) eigenmodes. The tuning factor  $\alpha \in \mathbb{R}^+$  enables one to give more or less confidence into the measurements; large values can be specified when measurements are considered reliable whereas close-to-zero values are better suited to corrupted or noisy recordings. The choice of  $\alpha$  is therefore crucial for providing relevant parameter estimates [49, 50]. Automated techniques for its optimal tuning will be discussed in the following.

It turns out that the analysis of a single angular frequency is too restrictive in low-frequency dynamics, particularly when several eigenmodes are simultaneously involved in the structure response. The model updating procedure must be conducted on a frequency bandwidth  $D_\omega$  which contains essential information about the response of the structure. The mCRE functional  $\mathcal{J}$  to be minimized is thus obtained by direct integration over  $D_\omega$ :

$$\hat{\theta} = \arg \min_{\theta \in \Theta} \left[ \mathcal{J}(\theta, Y) \triangleq \int_{D_\omega} z(\omega) e_\omega^2(\hat{s}(\theta, Y_\omega), \theta, Y_\omega) d\omega \right] \quad (7)$$

where  $z(\omega)$  is a frequency weighting normalized function such that  $\int_{D_\omega} z(\omega) d\omega = 1$  allowing to modulate the importance of specific frequencies of  $D_\omega$ . An appropriate manner to define  $z(\omega)$  in low-frequency dynamics is given in the following (see Section 3.1). We denote by  $\hat{s}(\theta, Y_\omega)$  the optimal solution in the mCRE sense for given parameters and measurements, which is defined as

$$\forall \omega \in D_\omega, \quad \hat{s}(\theta, Y_\omega) = \arg \min_{[-\omega^2 M + i\omega D]U_\omega + K(\theta)V_\omega = F_\omega} e_\omega^2(s, \theta, Y_\omega) \quad (8)$$

Introducing Lagrange multipliers  $\hat{\Lambda}_\omega$  and an augmented cost function, it is easy to show that this constrained minimization problem is equivalent to the solution of the following linear system:

$$A \begin{bmatrix} \hat{\Lambda}_\omega \\ \hat{U}_\omega \end{bmatrix} = b \text{ with } \begin{cases} \hat{\Lambda}_\omega &= \hat{U}_\omega - \hat{V}_\omega \\ A &= \begin{bmatrix} [K(\theta) + i\omega D - \omega^2 M]^H & \alpha \Pi^H G \Pi \\ & [K(\theta) + i\omega D - \omega^2 M] \end{bmatrix} \\ b &= \begin{bmatrix} \alpha \Pi^H G Y_\omega \\ F_\omega \end{bmatrix} \end{cases} \quad (9)$$

whose size can be drastically reduced using projection on reduced basis, *e.g.* a truncated modal basis enriched with Krylov vectors [42, 47].

### 2.3. Solution algorithm

The minimization of the mCRE is slightly more complex than standard deterministic cost-functions as it is formed of nested minimization problems addressed on mechanical fields (8) and parameters (7). Herein, the computational burden can be drastically reduced using (i) parallel computing on  $\omega \in D_\omega$  and projection on reduced basis when solving (9), (ii) an analytical expression of the mCRE gradient with respect to the parameters. With the above stiffness parametrization, the latter reads (for a given angular frequency  $\omega$ ):

$$\nabla_{\theta_j} e_\omega^2(s_\omega, \theta, Y_\omega) = \frac{1}{2} \|U_\omega - V_\omega\|_{K_{0,j}}^2 + \Re(U_\omega - V_\omega)^T K_{0,j} \Re(V_\omega) + \Im(U_\omega - V_\omega)^T K_{0,j} \Im(V_\omega) \quad (10)$$

where  $\Re(\square)$  and  $\Im(\square)$  respectively denote the extraction of real and imaginary components. This result directly derives from the adjoint-state problem written in the frequency domain. As shown in ALG. 1, typical stopping criteria of the model updating algorithm are based on the value of the mCRE functional and on the stationarity of the updated parameters. The associated thresholds  $(\epsilon_1, \epsilon_2)$  must be chosen consistently to guarantee relevant results and reasonable computational time. Normalizing the mCRE functional [42] allows to select  $(\epsilon_1, \epsilon_2)$  within  $[10^{-8}; 10^{-6}]$ .

The localization of most erroneous areas (in the sense of the modeling error) is the only regularizing process in Tikhonov's sense as the model updating is restrained to a limited number of parameters. This principle will be discussed in details regarding the full automation of the algorithm in the next section.

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**Algorithm 1:** Pseudo-code of the mCRE-based model updating strategy.

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**Data:**

- FE model including mesh and matrices  $K, D, M$
- Measured outputs  $y(t)$
- Subdomain decomposition and associated initial parameter guess  $\theta_0 \in \Theta$
- Frequency bandwidth  $D_\omega$
- Confidence into measurements scalar  $\alpha$
- Stopping criteria thresholds  $\epsilon_1, \epsilon_2$

**Result:** Updated set of parameters  $\hat{\theta}$ .

**Initialization:**

Data preprocessing in the frequency domain:  $Y_\omega, \forall \omega \in D_\omega$ ;

Computation of the frequency weighting function  $z(\omega)$ ;

Evaluation of the initial quality of the model:  $\mathcal{J}_0 = \mathcal{J}(\theta_0)$ ;

**Model updating algorithm:**

**while**  $\mathcal{J}(\theta_k) \leq \epsilon_1 \mathcal{J}_0$  and  $|\theta_k - \theta_{k-1}| \leq \epsilon_2 |\theta_{k-1}|$  **do**

**1. Mechanical fields and localization of erroneous areas**

Solution of (9)  $\forall \omega \in D_\omega$ ;

Identification of the most erroneous subdomains after computing

$\zeta_{\omega,e}^2 \forall (\omega, e) \in (D_\omega \times \llbracket 1; E \rrbracket)$ ;

**2. Correction of parameters**

Minimization of  $\mathcal{J}(\theta)$  with respect to the identified parameters:  $\theta_{k+1} \leftarrow \theta_k$

(BFGS method with supplied analytical gradient) ;

**3. Convergence tests**

Computation of criteria:  $\mathcal{J}(\theta_{k+1})$  and  $|\theta_{k+1} - \theta_k| / |\theta_k|$ ;

**end**

---

### 3. Full automation of the modified CRE for user-friendly model updating

In order to build a fully automated model updating strategy, all internal tuning parameters must be calibrated with rigorous and robust rules. One must then pay attention to the setting of:

- ▷ the frequency bandwidth  $D_\omega$ ,
- ▷ the frequency weighting function  $z(\omega)$ ,
- ▷ the confidence into measurements scaling coefficient  $\alpha$ ,
- ▷ the parameter space in which parameters are sought  $\Theta$ .

As all of these parameters have a significant impact on model updating results, a systematic tuning procedure would then be useful to handle potential non-convincing model updating results. If a large part of them has been already robustly handled in former studies, we will briefly recall how to automatically calibrate all these influent internal parameters so as to guarantee relevant model updating results without requiring any user's *a priori* knowledge or experience (once the CRE has been defined!). The selection of the parameter space and the tuning of  $\alpha$  are extensively discussed as no clear contribution has been found in that sense in the literature, out of empirical studies.

### 3.1. Frequency bandwidth - frequency weighting

The model updating procedure is conducted on a given frequency bandwidth  $D_\omega = [\omega_{min}; \omega_{max}]$  which contains the essential part of the mechanical energy of the system. For seismic applications, ground motions usually have a significant frequency content up to 50 Hz, which implies that  $D_\omega \subset [0 \text{ Hz}; 50 \text{ Hz}]$ . In practice, the integration over  $D_\omega$  also requires to introduce a frequency step  $\Delta f$ . The latter must be carefully chosen to correctly capture the frequency content associated to the sollicitated eigenmodes. A common engineering judgment one can recommend is to choose  $\Delta f$  such that the narrowest resonant peak is described by at least three points. Considering the 3 dB cut-off frequency, a simple rule of thumb for the choice of  $\Delta f$  is  $3\Delta f \approx \min_i(\xi_i f_i)$  where  $(\xi_i, f_i)$  refers to the damping ratio and natural frequency of mode  $i$ . In the upcoming earthquake engineering-inspired applications, with typical 5% damping ratios and first eigenfrequencies at around 2-5 Hz,  $\Delta f$  is thus chosen within  $[0.1 \text{ Hz}; 0.5 \text{ Hz}]$ .

The case of the frequency weighting  $z(\omega)$  has already been handled in [42] for the low-frequency dynamics case. Briefly, the frequency weighting function  $z(\omega)$  can be used as a modulation function to favor frequencies deemed to have the largest influence on the model updating procedure. In [42], an automated computation of  $z(\omega)$  from experimental datasets was proposed, based on the fact that the essential frequency content of the measurements is gathered around experimental natural frequencies. A normalized version of the *Complex Mode Indicator Function* (CMIF) [51] on transfer functions has been introduced to naturally emphasize eigenfrequencies. The transfer matrix  $H(\omega)$  must be first computed from the crossed input/output PSD matrices ( $\mathcal{S}_{OO}(\omega), \mathcal{S}_{OI}(\omega), \mathcal{S}_{IO}(\omega), \mathcal{S}_{II}(\omega)$ ):

$$H(\omega) = [\mathcal{S}_{OO}(\omega) \ \mathcal{S}_{OI}(\omega)] [\mathcal{S}_{IO}(\omega) \ \mathcal{S}_{II}(\omega)]^\dagger \quad (11)$$

where  $\bullet^\dagger$  refers to the Moore-Penrose pseudo-inverse. The dominant singular value of  $H(\omega)$  has the property of peaking in the vicinity of natural frequencies and can be advantageously used to define  $z(\omega)$ . This indicator based on the transfer matrix  $H$  is called *H-CMIF* in what follows (owing to its similarities with classical CMIF). Finally, one defines the frequency weighting function as

$$z(\omega) = \frac{H\text{-CMIF}(\omega)}{\int_{D_\omega} H\text{-CMIF}(\omega) d\omega} \quad (12)$$

### 3.2. Confidence into measurements

Historically, out of the specific continuation scheme proposed in [34], several approaches related to regularization techniques were employed to define an optimal value of  $\alpha$ . The influence of  $\alpha$  was particularly investigated in [24, 35, 52, 53], with empirical

conclusions that an equivalent confidence between the CRE and data-to-model distance terms provided relevant model updating results. In other words, one should intend to calibrate  $\alpha$  such that the two terms are in the same order of magnitude. This idea is closely related to the L-curve method [44].

Alternatively, another technique to choose  $\alpha$  lies in Morozov’s discrepancy principle [46, 54, 55]. Therefore, the latter could be used in order to integrate the *a priori* knowledge on measurement noise features appropriately, as it was done in [55, 56] to perform mCRE-based identification from imaging databases.

In this part, two tuning strategies based on the previously mentioned principles are discussed. Both require a preliminary parametric study on  $\alpha$  that can be done (at low-cost) in preamble of the model updating procedure (see ALG. 1) and intend to avoid a full parametric study on  $\alpha$  that would be prohibitive in terms of CPU time. Their performance will be assessed in Section 4.

### 3.2.1. *A priori balance between modeling and measurement errors*

As mentioned above, the (physics-based) CRE term allows to explicitly integrate modeling errors into the updating process. Its relative weight with respect to the modeling error term must be correctly set in order to guarantee relevant identification results. Without any additional *a priori* information, one can choose to calibrate  $\alpha$  to ensure a correct *a priori* balance between measurement error and modeling error. This approach, which is close to the L-curve principle in terms of formulation, has been proposed in recent works [35, 42] and provided relevant model updating results.

$$\hat{\alpha} = \arg \min_{\alpha \in \mathbb{R}^+} \left| \underbrace{\int_{D_\omega} z(\omega) \zeta_\omega^2(\hat{s}(\theta, Y_\omega), \theta) \, d\omega}_{\text{Modeling error}} - \underbrace{\int_{D_\omega} z(\omega) \frac{\alpha}{2} \|\Pi U_\omega - Y_\omega\|_G^2 \, d\omega}_{\text{Measurement error}} \right| \quad (13)$$

Note that this approach is non-trivial as the mechanical fields that needs to be computed ( $U_\omega, V_\omega$ ) are indirectly impacted by the value of  $\alpha$  following (9).

### 3.2.2. *Morozov’s discrepancy principle to integrate the knowledge of measurement noise*

In most cases, it seems relevant to take advantage of the knowledge of measurement noise features (if available) to enhance the model updating process. Morozov’s discrepancy principle permits to calibrate  $\alpha$  so that the measurement error term should not be lower than the noise level, denoted  $\delta_s$  in the following.

We owe to Bonnet and co-workers the only use of Morozov’s discrepancy principle from data obtained in the frequency domain within the mCRE framework [53, 56], although its implementation remains questionable. Indeed, as expressed in [56], the measurement noise is assumed to be proportional to the magnitude of measurements, meaning that during the same experiment, sensors measuring low- or high-amplitude data do not have the same noise level. Besides, the implementation of the criterion has only been proposed for harmonic analysis and from synthetic data with low noise levels (up to 5% only).

In this contribution, a Morozov’s discrepancy criterion for *a posteriori* optimal choice of  $\alpha$  dedicated to the considered mCRE-based model updating framework is proposed. If one (legitimately) assumes that the measurement noise level is additive and follows a zero-mean Gaussian probability density function, *i.e.*  $\eta(t) \sim \mathcal{N}(0, \delta_s^2) \forall t$ , and if  $G$  is

proportional to the identity matrix, *i.e.*  $G = G_0 I$ , then one can show that the mCRE measurement error term can be statistically approximated by:

$$\mathbf{E} \left( \int_{D_\omega} \frac{z(\omega)}{2} \|\Pi U_\omega - Y_\omega\|_G^2 d\omega \right) = \frac{1}{2} G_0 \delta_s^2 N_s \quad (14)$$

as the squared modulus of a zero-mean random process follows a non-centered  $\chi^2(2)$  probability distribution conditioned by its variance  $\delta_s^2$ , whatever the value of  $\omega$  (see Appendix A for the complete proof starting from Morozov's discrepancy main theorem). One can thus statistically bound the measurement error term, which directly corresponds to the adaptation of Morozov's discrepancy principle to the mCRE written in the frequency domain (when all sensors are supposed uncorrelated, which is legitimate in practice). The statistical upper-bound thus depends simultaneously into the number of sensors  $N_s$ , the measurement noise  $\delta_s$  and the scaling matrix  $G$ , which is intuitively consistent.

Morozov's discrepancy principle is originally intended to choose the optimal value  $\hat{\alpha}$  *a posteriori*. In other words, one should follow an iterative scheme, where the value of  $\alpha$  is tuned using a bisection method after assessing the quality of the solution once having fully minimized the mCRE functional  $\mathcal{J}(\theta)$ . Doing so, in a few iterations, one could hope to find a relevant value for  $\hat{\alpha}$ .

---

**Algorithm 2:** A posteriori calibration of the confidence into measurements coefficient according to Morozov's discrepancy principle.

---

**Data:**

- FE model including mesh and matrices  $K, D, M$
- Measured outputs  $y(t)$  and noise level of sensors  $\delta_s$
- Subdomain decomposition and associated initial parameter guess  $\theta_0 \in \Theta$
- Frequency bandwidth  $D_\omega$
- Initial value of confidence into data  $\alpha$  (recommended to be small)
- Threshold  $\epsilon$

**Result:** Optimal confidence into parameter coefficients  $\hat{\alpha}$ .

**while**  $\left| \int_{D_\omega} z(\omega) \|\Pi U_\omega(\alpha) - Y_\omega\|_G^2 d\omega - G_0 \delta_s^2 N_s \right| > \epsilon$  **do**

**1. Update the confidence into measurements  $\alpha$**

**if**  $\int_{D_\omega} z(\omega) \|\Pi U_\omega(\alpha) - Y_\omega\|_G^2 d\omega > G_0 \delta_s^2 N_s$  **then**

| Increase the value of  $\alpha$  ;

**else if**  $\int_{D_\omega} z(\omega) \|\Pi U_\omega(\alpha) - Y_\omega\|_G^2 d\omega < G_0 \delta_s^2 N_s$  **then**

| Decrease the value of  $\alpha$  ;

**2. Minimization of the mCRE functional**

Minimization of  $\mathcal{J}(\theta)$  with the updated value of  $\alpha$  starting from  $\theta_0$  using ALG. 1 ;

**3. Assessment**

Compute Morozov's discrepancy:  $\int_{D_\omega} z(\omega) \|\Pi U_\omega(\alpha) - Y_\omega\|_G^2 d\omega - G_0 \delta_s^2 N_s$  ;

**end**

---

However, one cannot afford to perform several minimization steps to update parameters correctly when performing model updating on-the-fly, as it is our intention in perspective of this work [57]. Similarly to the *a priori* balance criterion (13), one can also perform an *a priori* parametric study on  $\alpha$  that directly allows to propose a value for  $\hat{\alpha}$  that one should expect to be suboptimal but that should guarantee convergence of the forthcoming minimization:

$$\hat{\alpha} = \arg \min_{\alpha \in \mathbb{R}^+} \left| \int_{D_\omega} z(\omega) \|\Pi U_\omega(\alpha) - Y_\omega\|_G^2 d\omega - G_0 \delta_s^2 N_s \right| \quad (15)$$

Illustrations of the *a priori* choice of  $\alpha$  following (13) and (15) will be provided in the following applications, once combined to all the tools discussed in this section.

### 3.3. CRE-based stiffness parametrization for fully automated model updating

#### 3.3.1. CRE-based localization of most erroneous areas

The CRE provides a direct insight regarding the validity of the model itself, making it a relevant tool for identifying erroneous parts of the model as all finite element contributions to CRE can be computed independently. The CRE per subdomain  $S_i$  then reads:

$$\forall i \in \llbracket 1; n_\theta \rrbracket, \quad \zeta_{\omega,i}^2(s_\omega, \theta) = \sum_{e \subset S_i} \frac{1}{2} \|U_\omega - V_\omega\|_{K_e(\theta)}^2 \quad (16)$$

where  $K_e(\theta)$  is the stiffness matrix of element  $e$ . This asset can be seen as a Tikhonov regularization in the mCRE framework in the sense that a restrained number of parameters can be updated [30, 47]. Indeed, defining a threshold  $\beta \in [0; 1]$ , one can identify subdomains to update such that they satisfy the following inequality:

$$\forall i \in \llbracket 1; n_\theta \rrbracket, \quad \frac{1}{|S_i|} \int_{D_\omega} z(\omega) \zeta_{\omega,i}^2(s_\omega, \theta) \, d\omega \geq \beta \max_{i \in \llbracket 1; n_\theta \rrbracket} \left\{ \frac{1}{|S_i|} \int_{D_\omega} z(\omega) \zeta_{\omega,i}^2(s_\omega, \theta) \, d\omega \right\} \quad (17)$$

where the normalization by the size of subdomains  $|S_i|$  permits to consider subdomains of various sizes. However, if most references consider that  $\beta = 0.8$  is a convenient value, no clear parametric study has been performed in order to optimize the choice of  $\beta$  for optimal defect detection based on noisy datasets.

Thus, if the CRE can be a convenient local model indicator, the localization criterion (17) alone is limited to provide an automated parametrization (and associated subdomains). In the remainder of this subsection, we present the potentialities and inherent limitations of a CRE-based clustering technique for automated subdomain definition.

#### 3.3.2. CRE-based clustering of erroneous areas: key ideas

Clustering is one of the most widely used techniques for data analysis and classification [58]. The objective of cluster analysis is the classification of objects according to similarities among them, and therefore organizing datasets into groups (the so-called *clusters*). Without going into much details, clustering techniques allow to partition a space in the sense of a given metrics: within a cluster, the distance between objects remains small, whereas the distance between objects of different clusters is larger. There exists many clustering techniques: hard clustering, fuzzy clustering [59], spectral clustering [60] to only cite a few of them. Regarding civil engineering applications, clustering algorithms have been essentially applied for automated modal analysis purposes [13, 17, 61].

To recall algorithmic principles in a nutshell, in hard clustering techniques, the data to classify  $X \in \mathbb{R}^{d \times N}$  is divided into distinct clusters, where each data point can only belong to exactly one cluster. To do so, the cluster center locations  $\{C_j\}_{j=1}^k$  are optimized such that:

$$\{C\} = \arg \min_{C_j \in \mathbb{R}^d \forall j} \sum_{j=1}^k \sum_{i=1}^N \|X_i - C_j\|_2^2 \quad (18)$$

and the data samples are then directly classified into clusters according to their distance to centers. Although efficient, such algorithms tend to be less effective when dealing

with not-well separated, and non-spherical clusters. This motivated the development of fuzzy clustering (also called soft clustering) where membership functions are associated to each data point  $X_i$ , which somehow relaxes the belonging to a given cluster. The membership function of data point  $i$  to cluster  $j$  is denoted  $\mu_{ij}$  and takes values in  $[0; 1]$ . As a consequence, a data point  $X_i$  will be more likely to belong to a cluster  $j$  if  $\mu_{ij} \rightarrow 1$ . Using a non-necessary Euclidean norm  $\|\square\|_A$ , the centers and membership functions are sought according to the following constrained minimization problem:

$$\{C, \mu\} = \arg \min_{C, \mu} \underbrace{\sum_{j=1}^k \sum_{i=1}^N \mu_{ij}^m \|X_i - C_j\|_A^2}_{J(C, \mu; X)} \quad \text{with} \quad \sum_{j=1}^k \mu_{ij} = 1 \quad \forall i \in \llbracket 1; N \rrbracket \quad (19)$$

where the exponent  $m$  reflects the degree of fuzziness of the partition: if  $m = 1$  corresponds to hard clustering,  $m = 2$  is the classical (empirically) chosen value in many applications. The Hermitian norm  $\|\square\|_A$  can be defined using a positive definite matrix  $A$ , identically chosen for all clusters. Alternatively, the metric can be defined using a  $k$ -tuple  $\{A_j\}_{j=1}^k$ , with  $A_j$  a positive definite matrix. The objective function is then extended as  $J(C, \mu, A; X)$ . This choice allows better identification of elliptic and overlapping clusters. More details on that subtlety can be found in [13]. In any case, the constrained minimization of these methods is performed numerically using a fixed-point algorithm that iteratively optimizes all the arguments of the functional  $J$ . When non-convex clusters are expected to be identified, spectral clustering techniques often outperform the above-mentioned traditional approaches by performing clustering on a transformed dataset which hopefully provides better separated and more easily identifiable clusters. The interested reader can find a pedagogical and detailed tutorial in [60].

In this work, a fuzzy  $k$ -means clustering algorithm developed in [13] is reinvested. Originally intended to separate the physical modes from the spurious modes produced by a growing model-order identification algorithm in a modal analysis context, its robustness with respect to non-convex, poorly dissociated, and heterogeneous clusters makes it a relevant tool to define parameter distributions from the CRE map. Starting from the initial model, the key idea is to identify a cluster of most erroneous elements through the computation of the modeling error (CRE) map per element which naturally emphasizes the damaged area. In other words, we perform a two-cluster distinction from

$$X = \left\{ \int_{D_\omega} z(\omega) \zeta_{\omega,i}^2 d\omega \right\}_{i=1}^E \quad (20)$$

which allows to label each finite element as "healthy" or "erroneous". Note that this is not a computationally expensive procedure as it is a (vectorizable) post-processing operation once mechanical fields solution of (9) has been obtained. Finally, a new parameter space  $\hat{\Theta}$  is defined in which all erroneous elements are updated independently whereas the healthy elements remain unchanged. This can be interpreted as a generalization of the localization of most erroneous areas principle that was mentioned earlier, but in a more adaptive manner.

As a last remark, note that the absence of finite element barycentric coordinates in  $X$  allows to simultaneously identify damaged areas at different locations. It avoids the complex discussion about the robustness of clustering algorithms when the number of clusters to identify changes from an application to the other.

### 3.4. A sparse-regularized modified Constitutive Relation Error

Following the developments of [35], one could also imagine adding an explicit regularization term to the mCRE so as to favor some *a priori* knowledge on the parameter estimates that are sought. In particular, when it comes to detect localized damage, sparse regularization would allow to focus the model updating process to a reduced amount of parameters. The sparse-regularized mCRE  $\mathcal{J}_r$  reads as follows:

$$\mathcal{J}_r(\theta, Y, \theta^*, \gamma) = \int_{D_\omega} z(\omega) e_\omega^2(\widehat{s}(\theta, Y_\omega), \theta, Y_\omega) d\omega + \gamma \|\theta - \theta_0\|_1^2 \quad (21)$$

where  $\theta^*$  is the *a priori* information one has on the parameters (healthy area stiffness value for example),  $\gamma$  is the weighting parameter allowing to give more or less importance to the regularization, and  $\|\square\|_1$  refers to the  $L_1$ -norm. The minimization of this functional can also be performed using the same algorithmic structure and mathematical developments that have been given previously, except for the analytical gradient formulation that must be slightly modified to integrate the regularizing term into consideration.

However, the introduction of a new *a priori* information goes against the philosophy of the mCRE which intends to analyze the mechanical equations of the problem in order to exempt the inverse problem of any expert-user's judgment. Another disadvantage of this technique is the arbitrary nature of the choice of a regularizing parameter  $\gamma$  which impacts the smoothness of the solution [20]. Finally, one can notice that the purpose of the clustering step and the additive sparse regularization are similar. The use of the CRE to define an optimal parameter space seems a better integrated and more natural approach within the mCRE framework. If one performs model updating without clustering, the effect of the sparse regularization (once well calibrated) should be beneficial on the accuracy of the identified parameters, but it is likely that the clustering alone will be as efficient in terms of accuracy and more robust insofar as less calibration efforts are needed.

The impact of sparse regularization and a comparison with CRE-based clustering is proposed in the following application section.

## 4. Application to damage detection from sparse accelerometer datasets

### 4.1. Description of the problem and objectives of the study

In this section, numerical results are presented and discussed to assess the relevance of the automated mCRE-based model updating strategy for damage detection with typical earthquake engineering applications. To evaluate the performance of the methodology, academic examples are considered in which a simply supported plane rectangular floor is subjected to a vertical low-magnitude ground acceleration input. The reference meshes with the defects one hopes to identify are shown in FIG. 1. The optimality criterion for  $\alpha$  that have been previously discussed are compared and assessed, as well as the effect of an additional sparse regularization term.

A uniform sensor placement is considered: discrete accelerometers oriented in the outer plane direction are spread over the structure to collect data at the sampling frequency  $f_s = 1000$  Hz. The effect of the sensors density is illustrated in the following by comparing the sensor placement configurations shown in FIG. 2 and characterized by the distance between sensors  $d_s$ . In order to assess the robustness of the methodology with respect

to measurement noise, a white noise of known standard deviation is added to simulated data in order to process noisy synthetic measurements:

$$\ddot{y}_{noisy}(t) = \ddot{y}(t) + \delta_s \eta(t) \quad (22)$$

$\delta_s = \delta \cdot \text{std}(\ddot{u}_d(t))$  represents the noise level with  $\delta \in [0; 1]$  and  $\ddot{u}_d$  the input ground acceleration) while  $\eta(t)$  is a random vector normally distributed with zero mean and unit standard deviation. Therefore, the added measurement noise standard deviation is equal to  $\delta \cdot \text{std}(\ddot{u}_d)$ .

Two cases of damage defects are considered to assess the capability of the methodology: (i) a Y-shape defect (representative of a propagated crack with bifurcation) that is modeled with a local 50% Young modulus loss and (ii) a double circular inclusion problem, respectively modeled with local 50% and 70% Young's modulus losses (see FIG. 1). If the first case is much more representative of what could happen in actual experiments involving crack propagation, the double inclusion case will allow to assess the robustness of the clustering algorithm to identify several defects while the Y-shape case will permit to assess the possibility to perform relevant clustering of non-convex defects.

Meshes and FE matrices (reference and initial guess) are built using the CEA simulation software CAST3M<sup>©</sup> [62] before being uploaded in a MATLAB<sup>©</sup> environment [63]. In both cases, the initial model guess is made of regular quadrangular shell elements whose homogeneous Young's modulus is equal to the non-damaged reference, meaning one expects to recover stiffness parameter values between 1 (healthy element) or 0 (fully damaged element). Note that a modeling error bias is implicitly introduced as the mesh of the updated model (regular quadrangular elements) is not compatible with the defects to identify. The knowledge of the expected parameter estimate  $\theta^*$  enables to assess the model updating accuracy using the following criteria:

$$\eta(\theta, \theta^*) = \frac{\int_{\Omega} \|\theta - \theta^*\|^2 d\Omega}{\int_{\Omega} \|\theta^*\|^2 d\Omega} \quad ; \quad \eta_d(\theta, \theta^*) = \frac{\int_{\Omega_d} \|\theta - \theta^*\|^2 d\Omega}{\int_{\Omega_d} \|\theta^*\|^2 d\Omega} \quad (23)$$

which indicate the closeness of an estimate  $\theta$  to the expected parameter set  $\theta^*$ , respectively on the full structure  $\Omega$  or on the restriction to the damaged area  $\Omega_d$  (see FIG. 1).

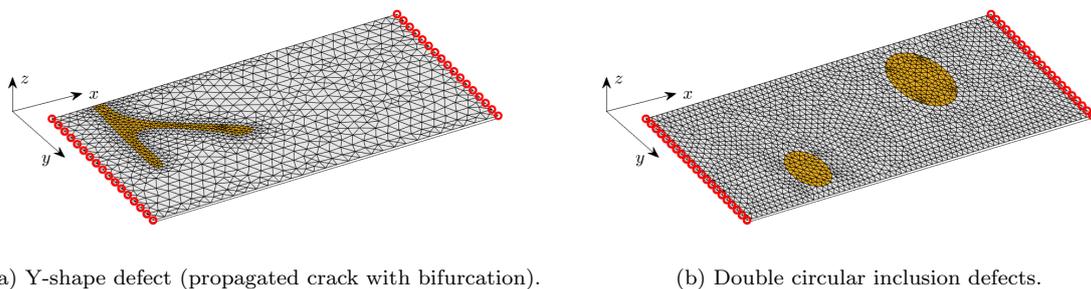


FIGURE 1: Reference FE meshes to simulate synthetic data with emphasis on the defect areas  $\Omega_d$  to identify (orange elements). Locked dofs are specified with red circles.

Performing an accurate model updating from sparse and noisy data in that case is a complex task, particularly for a non-convex defect to recover (such as the Y-shape defect). From these typical 2D earthquake engineering academical examples, the authors' aim is to emphasize the main underlying issues of model updating from discrete sensors and to validate the automated strategy presented previously. To do so, a complete numerical study is conducted with several objectives:

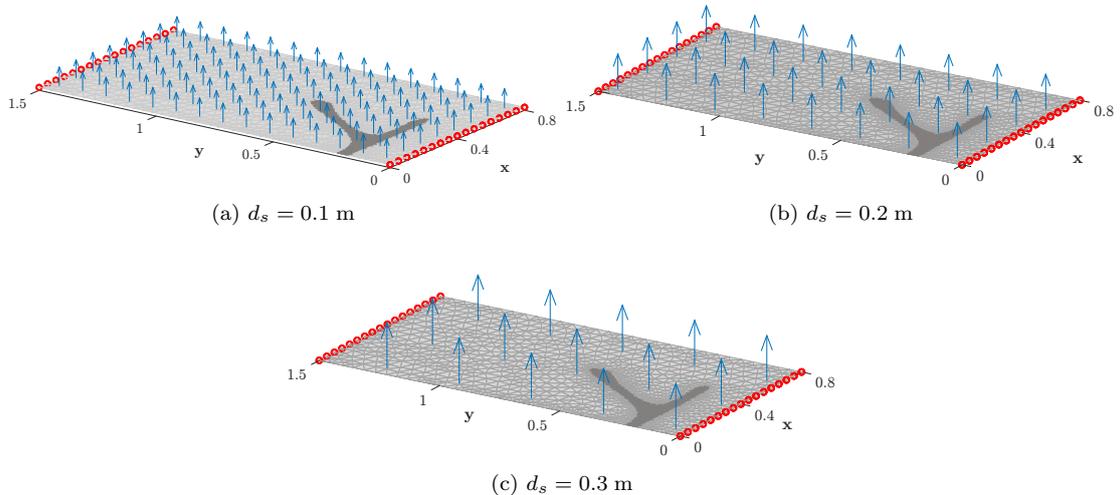


FIGURE 2: Three uniform sensor placement configurations with different densities (quantified by the distance between sensors  $d_s$ ). Blue arrows locate the discrete accelerometers and red circles emphasize on boundary conditions.

- ▷ illustrate the limits of the localization criteria for damage detection and emphasize the effectiveness of the CRE-based clustering step for automated parametrization;
- ▷ assess both effectiveness and soundness of the two criteria proposed in Section 3.2 for automated calibration of  $\alpha$  and evaluate the influence of the confidence into measurements coefficient  $\alpha$  on parameter estimates;
- ▷ evaluate the robustness of the automated mCRE-based model updating algorithm with respect to measurement noise;
- ▷ illustrate the damage detection expectations from limited sensor density;
- ▷ assess the benefits of the additional sparse regularization to the mCRE functional.

For the sake of conciseness, all similar numerical results will not be shown as most of them are redundant between the two configurations of FIG. 1 (in terms of results analysis). As the Y-shaped defect is the most challenging application, this report will mainly be focused on the latter, whereas the inclusion example will be used to drive complementary discussions.

In terms of mCRE calibration, as the first eigenmode (which is the most excited) is around 20 Hz, a frequency bandwidth  $D_\omega = [1 \text{ Hz}; 30 \text{ Hz}]$  with  $\Delta f = 0.1 \text{ Hz}$  has been chosen for the computation of all forthcoming results. The call to a reduced basis made of the first 20 eigenmodes of the structure allowed to achieve convergence in fast CPU times (between 0.5 and 2 minutes per mCRE-minimization with CRE-based clustering) on a personal laptop.

The remainder of this section is structured according to the (above-mentioned) major objectives of this numerical study in order to emphasize the potentialities and limitations of mCRE-based model updating algorithm from sparse noisy data.

#### 4.2. Limits of the localization criterion for optimal damage detection

First, a parametric study on the localization parameter  $\beta$  defined in (17) has been made in order to assess the capabilities of classical mCRE-based model updating to perform optimal damage detection, *i.e.*, accurate identification at low computational cost. It

considers the most convenient model updating setting: large amount of data (sensor placement (a) from FIG. 2) and perfect measurements ( $\delta = 0\%$ ) which yield a high confidence into measurements.

The criteria (23) are computed for each parameter estimate identified at convergence and stored in TAB. 2. Two typical model updating results are plotted in FIG. 3 and 4. One can observe that mCRE-based model updating with strong localization (FIG. 3) is accurate at the full structure scale as it does not correct many parameters (letting most of the undamaged area unchanged). However, the few parameters that are corrected do not permit to obtain an accurate defect shape. On the contrary, with a lower localization value (FIG. 4), the defect shape is accurately described, but unexpected corrections occur in the undamaged area, which leads to parameter estimates that are not optimal in the sense of  $\eta$ . Besides, note that the CPU time increases for low values of  $\beta$  as a (too) large amount of parameters is updated at once. Finally, one should notice that iterative model updating approaches with a high value of  $\beta$  do not improve identification results as they take more CPU time and often fall into equivalent solutions (at the specimen scale) that do not describe the defect accurately.

Therefore, the localization of most erroneous areas principle, which regularizes the model updating algorithm in the Tikhonov sense [30], is not sufficient to perform accurate damage detection from sparse data in reasonable computational times. Comparative results obtained with CRE-based clustering are given in FIG. 5 for the same model updating context. As observed, only a restrained group of finite elements is corrected (60 among 480). This group of elements correctly matches with the shape of the damaged area because of its CRE-based construction and permits to obtain an accurate estimate, as shown in TAB. 2. Besides, the comparison of modeling error (CRE) map before and after model updating confirms the efficiency of the proposed approach.

Localization parameter $\beta$	$\eta(\hat{\theta}, \theta^*)$	$\eta_d(\hat{\theta}, \theta^*)$	CPU time [s]
0.2	0.00859	1.897	> 5000
0.3	0.00880	1.829	1670
0.4	0.00932	1.794	526.2
0.5	0.01212	1.833	171.9
0.6	0.01041	1.966	192.5
0.7	0.01783	2.168	33.8
0.8	0.04982	2.431	39.6
with CRE-based clustering	0.00955	1.641	100.3

TABLE 2: Emphasis on the limitations of the localization of most erroneous areas for accurate and efficient damage detection in the most favorable damage detection case (non-noisy data - rich sensor placement).

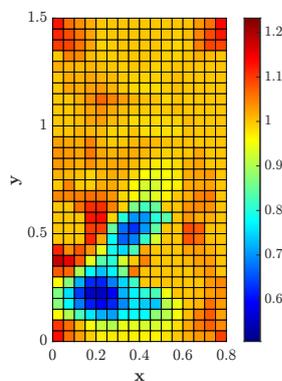


FIGURE 3:  $\hat{\theta}$  for  $\beta = 0.2$  - 92% of elements are individually updated.

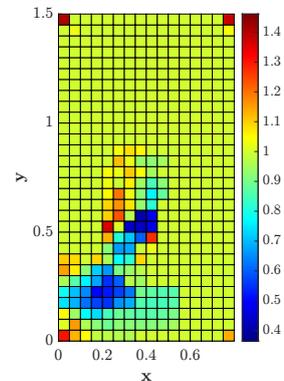


FIGURE 4:  $\hat{\theta}$  for  $\beta = 0.4$  - 25% of elements are individually updated.

### 4.3. Automated confidence into measurements

One of the objectives of this study is to check the validity of the criteria allowing to automatically define an optimal confidence into measurements parameter  $\hat{\alpha}$ . As a reminder, a first criterion defines  $\hat{\alpha}_1$  such that the *a priori* balance between model and measurement error is guaranteed (13). The second criterion lies on the *a priori* use of Morozov's discrepancy principle and chooses  $\hat{\alpha}_2$  in accordance with the noise level (15).

Using the most favorable (*i.e.*, the richest) sensor placement configuration and a given

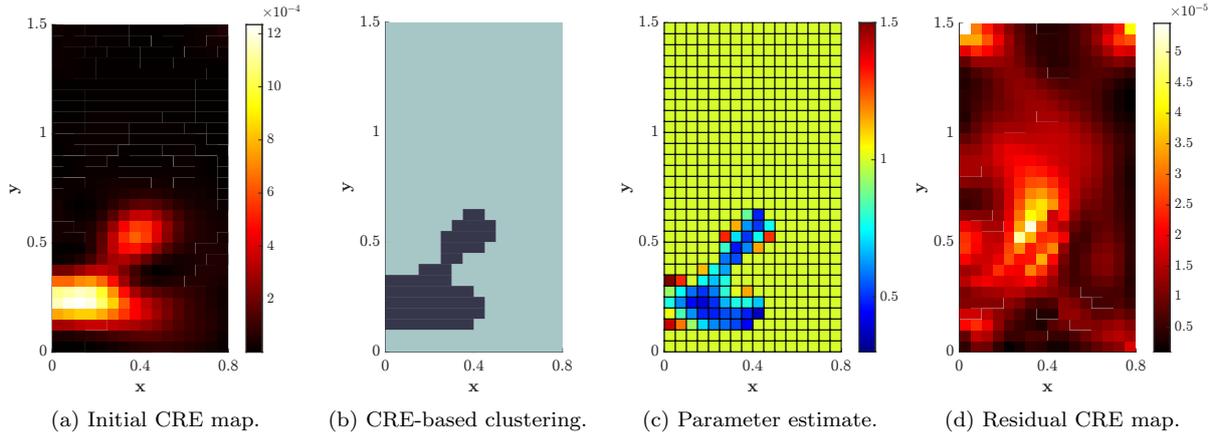


FIGURE 5: Fully automated mCRE-based model updating results obtained with sensor placement of FIG. 2a and non-noisy data.

random input acceleration signal of  $0.1g$  standard deviation, we show in FIG. 6 the natural decrease of confidence parameters ( $\hat{\alpha}_1, \hat{\alpha}_2$ ) with respect to the measurement noise level  $\delta$ . Both criteria thus react as expected to measurement noise: the more measurement noise, the less confidence into experimental data.

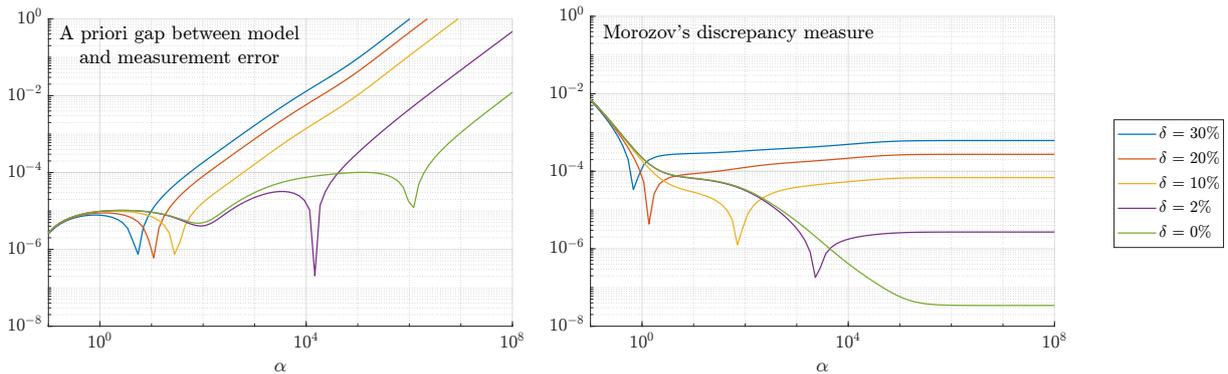


FIGURE 6: Automated choice of confidence into measurements coefficient: comparison of L-curve (*a priori* balance) and Morozov's discrepancy criteria for several noise levels  $\delta$ .

$\delta$ [%]	Sensor config. (a)		Sensor config. (b)		Sensor config. (c)	
	$d_s = 0.1$ m		$d_s = 0.2$ m		$d_s = 0.3$ m	
	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
0	962040	$10^8$	284	$10^8$	570	$10^8$
5	70.589	452.34	358	1444.32	908	2297.9
10	27.885	70.589	112	225.393	226	452.33
15	13.895	3.4497	70	11.0158	141	44.369
20	8.7386	1.3628	44	5.48903	70.58	13.895

TABLE 3: Values  $\hat{\alpha}_1$  (L-curve) and  $\hat{\alpha}_2$  (Morozov) of the confidence into measurements parameter with respect to noise level. The numerical values are implicitly conditioned by the choice for  $G$ .

Although similar trends in the evolution of  $\hat{\alpha}$  with respect to measurement noise are observed, the values given by both criteria are quite different, and yet one cannot conclude about their validity so far. To assess which of the criteria is the most appropriate to use, a full parametric study on the joint influence of  $\alpha$  and  $\delta$  on the identified estimates has been conducted. Associated results are given in FIG. 7 where criteria  $\eta$  and  $\eta_d$  are plotted as

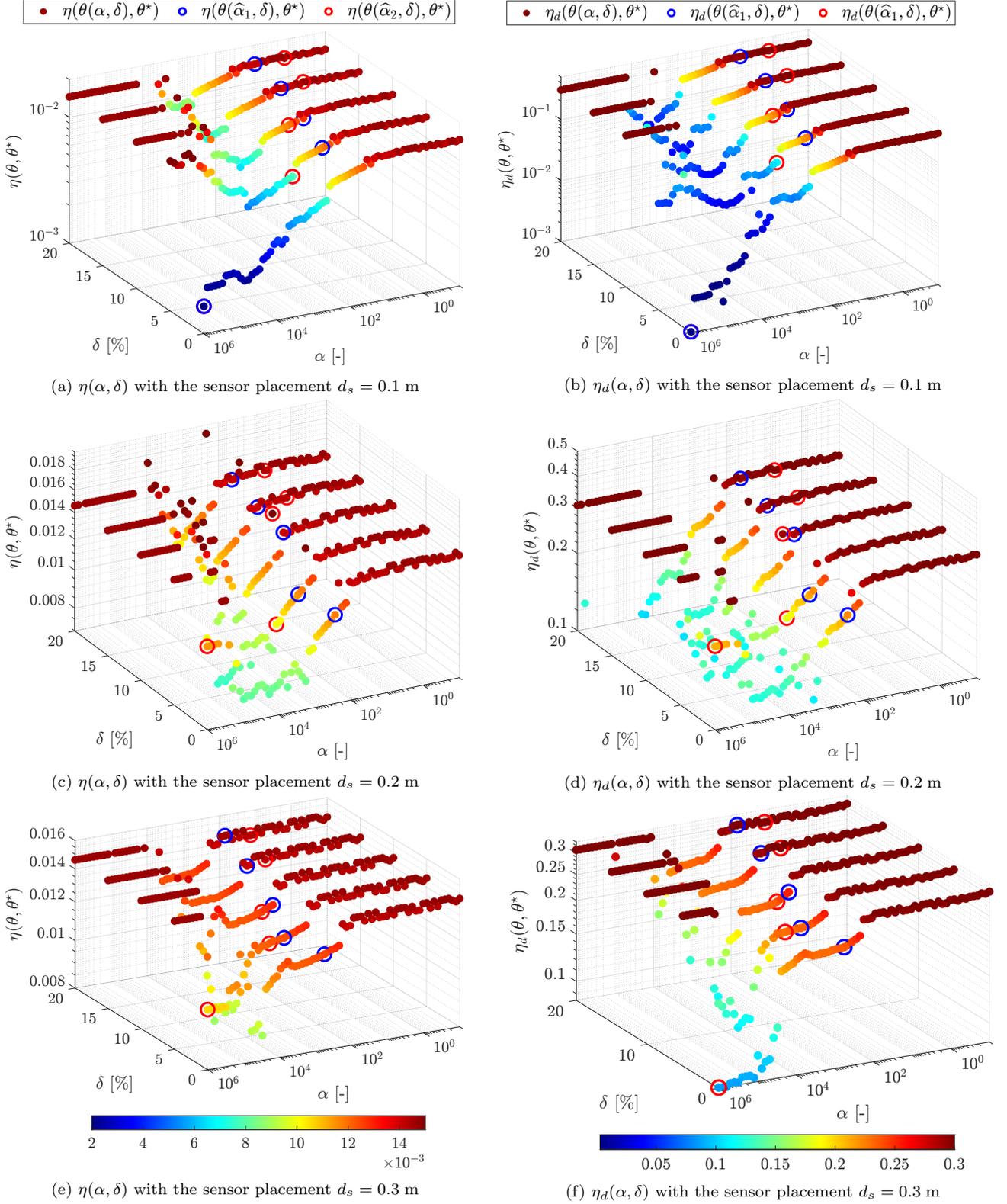


FIGURE 7: Automated mCRE-based model updating algorithm - Quality assessment of parameter estimates using  $\eta$  (left side) and  $\eta_d$  (right side) according to  $\alpha$ ,  $\delta$  and the sensor plans shown in FIG. 2. The positions of  $\hat{\alpha}_1$  (L-curve criterion - in blue) and  $\hat{\alpha}_2$  (Morozov's criterion - in red) are also given to evaluate the relevance of the automated methodology. Colormaps are identical in each column to compare the effect of sensors density more easily.

a function of  $\delta$  and  $\alpha$  for the three considered sensor placement configurations previously displayed in FIG. 2. The colormaps (that are identical for subfigures a-c-e and b-d-f) allow to appreciate more clearly the combined effects of sensor density, noise level and confidence into measurements on the values of  $\eta$  and  $\eta_d$ .

The location of  $\hat{\alpha}_1$  and  $\hat{\alpha}_2$  compared to the optimal  $\alpha$  values in the sense of  $\eta$  or  $\eta_d$  suggests that both criteria do not provide the optimal value of  $\alpha$  for damage detection in the considered case. They can thus be considered too "conservative" as they systematically underestimate the optimal confidence one should put into data. If one has to choose between them, the criterion based on Morozov's discrepancy principle should be preferred as it integrates additional information about the measurement noise.

In practice, the "conservative" choice of  $\alpha$  should also be seen as a way to guarantee robustness of the algorithm: excessive confidence into measurements could lead to diverging results (with non-physical negative values of stiffness parameters - see FIG. 8). The automated calibration of  $\alpha$  using L-curve-type or Morozov's discrepancy criteria allows to avoid such issues, at the cost of suboptimal identification results. Indeed, the "conservative" feature of  $\hat{\alpha}_1$  and  $\hat{\alpha}_2$  directly impacts the quality of identification: the sub-optimal choices in the sense of  $\eta$  and  $\eta_d$  lead to less accurate identification results, with a cluster of false-labeled elements larger than the defect to describe (see FIG. 9). However, the obtained results still remain consistent with the location of the defect whatever the sensor density.

As a last remark, please note that the values of  $\alpha$  are indirectly depending on the value given to  $G = G_0 I$ , showing once again how this scaling matrix is not that crucial if  $\alpha$  is afterwards well tuned.

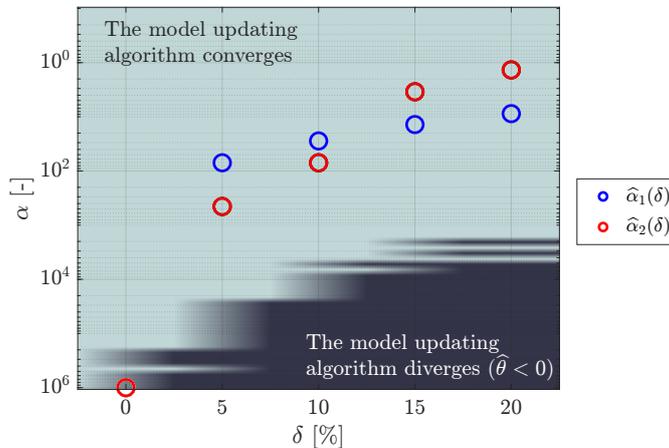


FIGURE 8: Convergence assessment of the automated mCRE-based model updating algorithm. If the algorithm converges most of the time (white area), overexcessive confidence into data can lead to non-physical results (dark area, where some estimated parameters have negative values). The automated calibrated values of  $\hat{\alpha}_1$  and  $\hat{\alpha}_2$  systematically lead to convergent model updating results.

#### 4.4. Impact of sensor placement on damage detection performance

As one could have expected, the more sensors are scattered, the more accurate identification results are. However, for real industrial applications, the amount of available sensors is constrained by instrumentation constraints and economic restrictions. What is thus interesting to observe from the identification results displayed in FIG. 10 is that one can coarsely identify damage locations from a restricted amount of sensors. It also

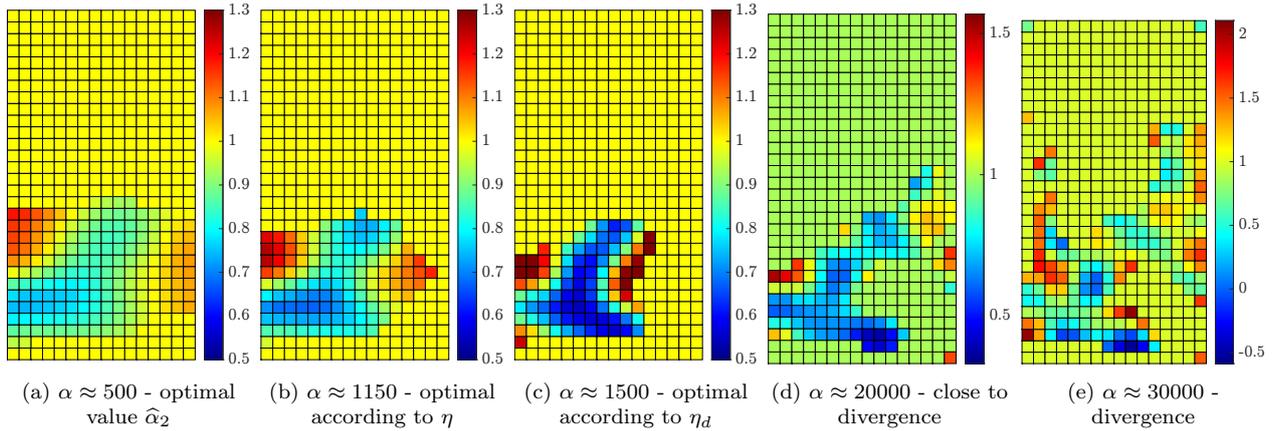


FIGURE 9: Automated mCRE-based model updating algorithm - Impact of  $\alpha$  on clustering and parameter estimates. Results obtained for sensor placement FIG. 2.a with  $\delta = 10\%$ .

emphasizes the fact that inverse problem parametrizations should be made in accordance with the associated sensor placement configuration. These results also highlight that optimal sensor placement strategies should be applied in such cases in order to get the most relevant information from a restricted amount of data.

As a remark, please note that very similar results have been obtained when identifying the two inclusions of FIG. 1, but as they remain convex-shaped defects, reducing the amount of sensors has less effects on the parameter estimate accuracy.

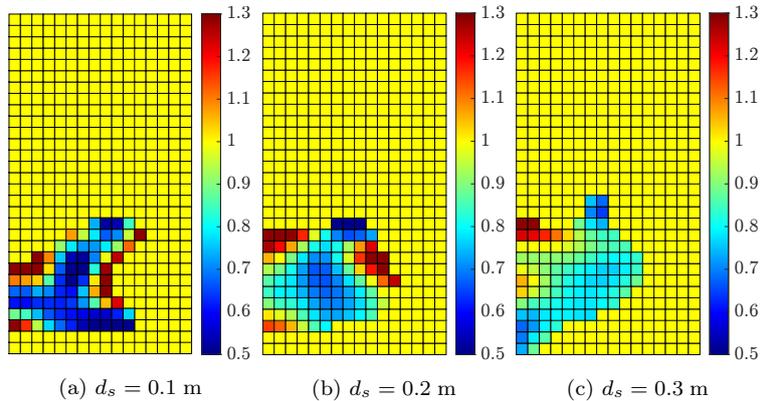


FIGURE 10: Automated mCRE-based model updating algorithm - Impact of the sensor placement configuration on parameter estimates. Results obtained for sensor placement FIG. 2 with  $\delta = 10\%$  with the optimal value of  $\alpha$  in the sense of  $\eta_d$ .

#### 4.5. Multiple defect detection

Although the convex shape of the inclusions makes their identification easier compared to the Y-shape defect, the fact that there are two defects to identify simultaneously is an interesting challenge to face with CRE-based clustering to perform optimal parametrization. We present in FIG. 11 and FIG. 12 the initial CRE maps and identification results obtained from both unnoisy and noisy measurements using the sensor placement of FIG. 2.b. The expected parameter value of the small bottom left inclusion and of the larger top right inclusion have been respectively chosen at 0.5 and 0.7. These results thus illustrate the robustness of the methodology to identify accurately multiple defects at once with an optimal parametrization, as attested by the emphasis of the CRE map on the defects, even

with a significant level of measurement noise and model bias. As mentioned previously, performing clustering without considering the barycentric coordinates of finite elements allows to avoid the (iterative) process on the optimal number of clusters to distinguish. Note that the value of  $\alpha$  had to be divided by 100 between non-noisy and 10% noisy data to obtain these results, highlighting once again the crucial role handled by the coefficient of confidence into measurements to get relevant identification results.

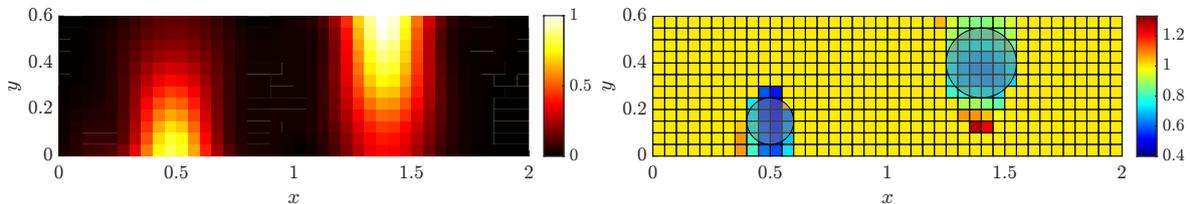


FIGURE 11: Identification of the two circular inclusions from non-noisy sparse accelerometer data. The initial CRE map used for clustering is plotted on the left. The parameter estimate is plotted on the right with the exact defects shapes on it to qualitatively assess its accuracy.

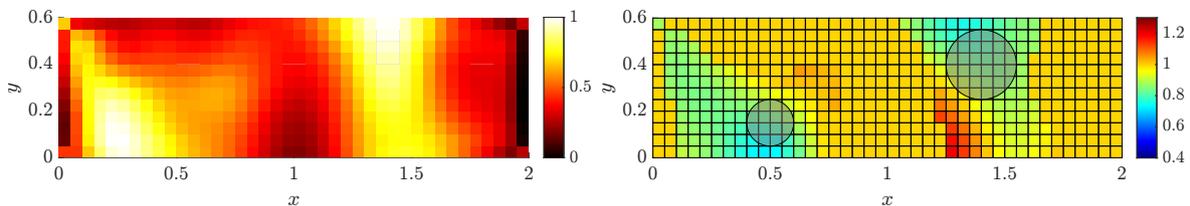


FIGURE 12: Identification of the two circular inclusions from 10% noisy sparse accelerometer data. The initial CRE map used for clustering is plotted on the left. The parameter estimate is plotted on the right with the exact defects shapes on it to qualitatively assess its accuracy.

#### 4.6. Effects of additional sparse regularisation

As written above, the addition of a sparse regularization term to the mCRE should be seen as a backup to the method for identifying localized defects more accurately. Identification results using sparse regularization with variable weighing  $\gamma$  and fixed *a priori* knowledge  $\theta_0 = 1$  are shown in FIG. 13. Several parameter estimates obtained from the same 10%-noisy measurements are plotted in the latter.

One can observe that the sparse regularization term permits to provide more accurate results only if the weighting is correctly calibrated. In this example, the weighting  $\gamma = 10^{-4}$  has been identified after several tests as a close-to-optimal value. The comparison of parameter estimates obtained with  $\gamma = 10^{-4}$  and  $\gamma = 10^{-3}$  illustrates how the calibration of the sparse regularization term is sensitive when it is employed. Besides, the reader should keep in mind that, in this academic example, the initial guess perfectly matches with the expected parameter value on non-damaged areas, thus making the chosen *a priori* value  $\theta_0 = 1$  highly relevant. This may not be the case for other applications, in particular considering industrial applications where the stiffness distribution might not be homogeneous.

As one can remark from FIG. 13 as well, the use of the automated CRE-based clustering step makes the effect of the additional sparse regularization less relevant as corrections are already focused on parameters in need of correcting actions.

In spite of these limitations, the additional sparse regularization term, if well calibrated, can be beneficial for getting more accurate parameter estimates. However, due to

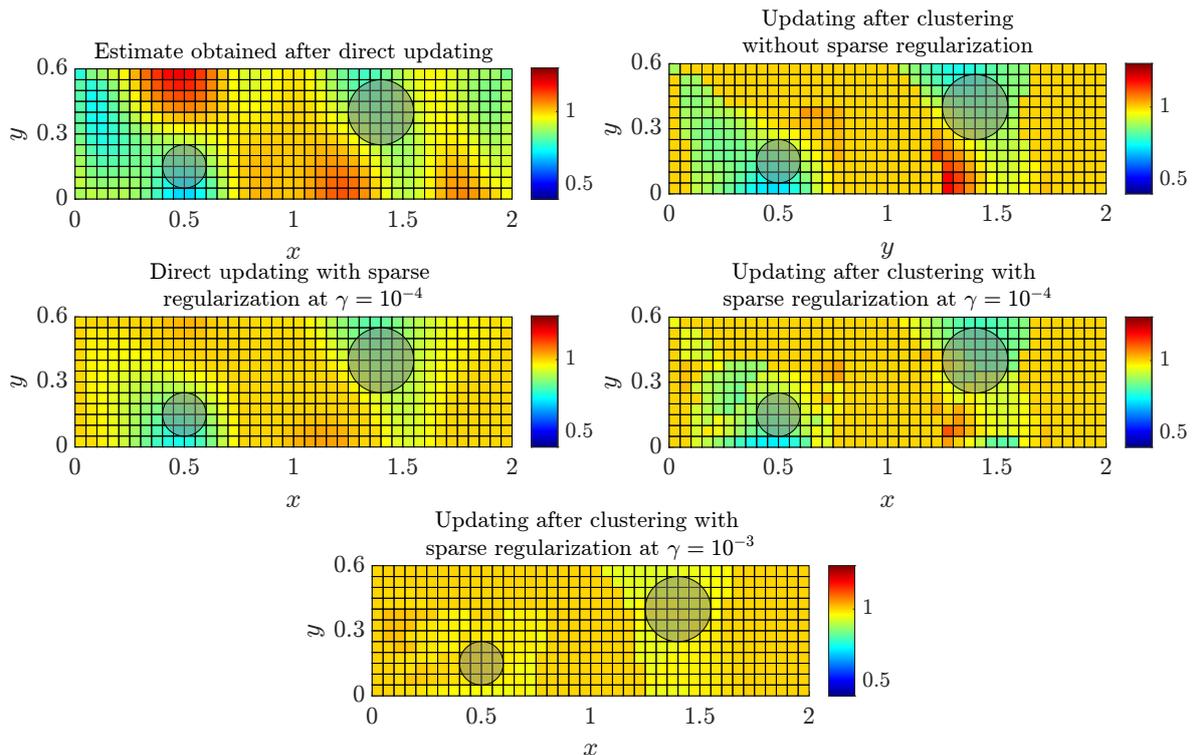


FIGURE 13: Effect of the sparse regularization effect over the identification of two inclusions from sparse 10% noisy acceleration datasets, with comparison to CRE-based clustering and direct mCRE-based model updating.

the fact it necessitates some *a priori* knowledge, this regularization should be seen as an option that can be considered and calibrated in a case-by-case approach.

#### 4.7. Remark on the unobservability issue

If the model updating algorithm has been able to identify defects so far, it should be noted that it is because the sensor placement and the input solicitation allowed their observability. If these conditions are not met, then one should not hope to identify defects with high accuracy. Actually, this remark has to be made for all inverse problem strategies.

As the mCRE is strongly related to strain energy, one could not expect to identify accurately field parameters in areas where little strain energy is present. To illustrate the observability concept on an academical example, let us consider the case of a cantilevered wall containing two rectangular defects submitted to a plane ground motion (see FIG. 14) on which accelerometers oriented in the outer plane direction are scattered. This structure has a highly heterogeneous strain energy distribution. Indeed, most of it is stored at the bottom of the structure, making the identification of the top defect impossible. In practice, the CRE map would be totally insensitive to the top defect, even if sensors are present. To confirm this claim, we plotted in FIG. 15 the mCRE functional surface assuming that both damaged areas have been correctly identified. The identification of the top area is not feasible, as the mCRE totally lacks of sensitivity into the associated parameters.

The authors thus recommend future mCRE users to plot relative confidence intervals at convergence [30]. This low-cost operation will allow to assess (at least) the possible difficulties in identifying some parameters.

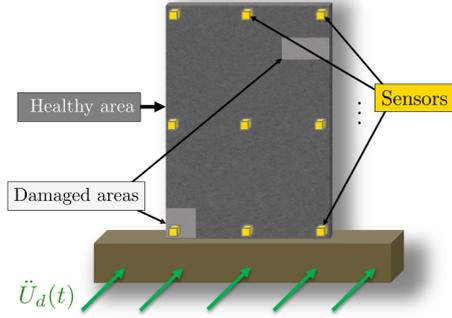


FIGURE 14: Cantilevered wall subjected to random ground motion with two damage defects to recover.

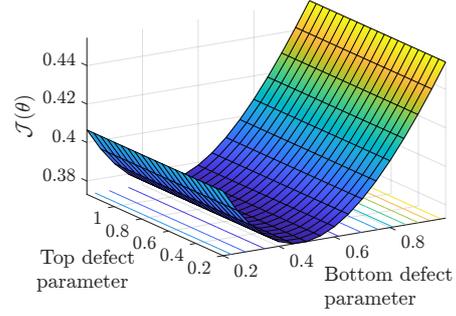


FIGURE 15: Evidence of the unobservability issue: areas storing few strain energy are less prone to relevant mCRE-based accurate identification.

#### 4.8. Conclusions on the fully automated mCRE-based model updating algorithm

From the previously obtained results, we would like to sum up the main advantages and limitations of the methodology presented in this paper:

- ✓ CRE-based clustering generalizes and extends the concept of localization of most erroneous areas. It facilitates damage detection as it strongly restrains the number of parameters to update from the same number of sparse measurements. This is particularly well suited for damage detection as such defects are supposed to appear locally (at the structural scale).
- ✓ The proposed model updating strategy is fully autonomous with all the implemented procedures for its automation. As the selection of  $\alpha$ , the computation of the CRE map and the clustering steps are fast operations, significant CPU time savings are made without losing accuracy. The robustness with respect to measurement noise has also been illustrated, particularly due to the selection of  $\hat{\alpha}$ .
- ✓ Sparse regularization may be helpful for getting for more accurate results in cases defects are strongly localized, but requires some *a priori* knowledge and dedicated calibration. Its effect is less significant if combined to CRE-based clustering.
- ✗ The performance of the clustering step and of the model updating process is conditioned by the quantity and quality of available sensors. This limitation has to be kept in mind when trying to identify damage from sparse datasets.
- ✗ Due to the energy-based definition of the mCRE, the observability of defects will be difficult in non-sensitive or heterogeneous areas (in the sense of the strain energy distribution). This problem-dependent issue should not be decisive for the case of damage detection from ground motion testings, but it could be the case when trying to identify inherent defects, *e.g.* due to material uncertainty.

If the core of the model updating algorithm remains the same as the one presented in [42], the automated setting of all the internal tuning parameters without any additional *a priori* engineering judgment and the low computational resources required by the algorithm make it an interesting tool for robust model updating problems in dynamics.

However, the ingredients for the full automation of the methodology do not lead to optimal results, regarding in particular the selection of  $\hat{\alpha}_1$  and  $\hat{\alpha}_2$  in FIG. 7. We thus

provide several recommendations on the application of the model updating algorithm according to the context of use:

- ▷ If real-time constraints exist or if the model updating is exploited for data assimilation, a first precalibration test (with a typical low-magnitude random input) should be made to calibrate all mCRE internal parameters at best. Besides, the initial model guess would be already well calibrated regarding the initial damage state of the specimen under study. This way, the MDKF developed in [57] could integrate all the proposed improvements. The obtained results will not be optimal (as evidenced before), but they should be relevant enough to provide accurate results in real-time.
- ▷ If there is no particular time constraint, *i.e.* if the algorithm is only exploited as a post-processing tool, then one can afford running several mCRE minimizations around the suboptimal values of  $\hat{\alpha}$  to find optimal results. Indeed, all the minimizations performed in this contribution lasted less than 2 minutes on a personal laptop.

## 5. Conclusions and prospects

In this paper, a fully automated physics-based model updating algorithm has been proposed and assessed for vibration-based damage detection. It is based on the modified Constitutive Relation Error functional, whose potentialities for model updating have been shown in the last 30 years. We addressed in this work a complex inverse problem framework, where field parameters were to be identified at best from sparse noisy data. The main contributions of this study rely in (i) the full-analysis of the calibration of the confidence into measurements coefficient, that is crucial for guaranteeing relevant results; (ii) the original automated manner to exploit at best the modeling error map to design the parameter space before identification.

Although optimal results for damage detection have not been systematically obtained, the methodology has proven robustness through the automated choice of tuning parameters and numerical efficiency as one completed model updating (including data preprocessing) in a couple of minutes on a personal laptop. Those two aspects are essential in the perspective of proposing an efficient model updating toolbox for digital twin applications. Finally, clear inherent limitations of mCRE-based model updating performance have been highlighted due to the problem setting itself.

The optimal definition of the parameter space could be easily integrated in an iterative process where the growth of damage could be observed at minimal computational effort. We refer here to a new data assimilation algorithm called Modified Dual Kalman Filter (MDKF) [57] that incorporates mCRE into a Dual Kalman Filter. The latter could benefit from an optimal evolutive parametrization that would adapt to damage growth.

Besides, the impact of sensors placement has been evidenced as critical in the quality of the obtained identification results. Current investigations are conducted to perform optimal sensor placement dedicated to mCRE-based model updating. The integration of the mCRE within the information entropy concept seems a natural approach to do so as one can recover the mCRE in the Bayesian inference framework assuming the decoupling of modeling and measurement errors.

Finally, the authors intend to perform mCRE-based model updating using full-field (digital image correlation) or locally rich (optic fibers) measurements. In such a context,

not all the available data can be processed at once due to the computational limitations. The updated model could then be progressively enriched to integrate data and refine the parameter space only where needed, whether using model selection or mesh adaptation techniques. Coupling the automated model updating algorithm with mesh adaptation and model selection techniques is a promising idea, that will require dedicated efforts in a long-term perspective of this work.

## Appendix A. Morozov’s discrepancy principle for the mCRE framework - mathematical developments

*In order to go through technical details without carrying a burden due to notation complexity, please note that the notations used in the following appendices are not necessarily consistent with the ones of the contribution.*

### A.1 Theorem: Morozov’s discrepancy principle [54]

Let us consider an inverse problem  $F(x) = y$  with its associated regularized cost-function  $\mathcal{J} = \alpha\|F(x) - y^\delta\|^2 + R(x)$ ,  $R$  being the regularization function (in Tikhonov’s sense). Assuming that a scalar  $\delta$  quantifies measurement noise such that noisy data  $y$  satisfies

$$\|F(x) - y\|^2 \leq \delta^2 \leq \|y\|^2 \quad (\text{A.1})$$

and that  $R$  is strictly convex, non-negative, weakly coercive and weakly-lower semi-continuous, then there exists an optimal weighting  $\hat{\alpha}(\delta)$  associated to an inverse problem solution  $x_{\hat{\alpha}}^\delta$  such that

$$\|F(x_{\hat{\alpha}}^\delta) - y\|^2 = \delta^2 \quad (\text{A.2})$$

In other words, it is possible to calibrate the weighting coefficient  $\alpha$  according to the noise level quantifier  $\delta$ .

### A.2 Adaptation to the mCRE-based model updating framework

As all the conditions on  $R$  are met by the modeling error term of the mCRE functional, some attention must be paid to the definition of the upper bound of the data-to-model distance. Let us start from the fact that time-histories can be legitimately modeled such that:

$$y(t) = y_{ex}(t) + \eta(t) \quad (\text{A.3})$$

where  $y_{ex}(t)$  is the exact measurement time-series that would have been obtained without any measurement noise and  $\eta(t)$  is a white-noise signal, whose value at each acquisition time step follows a standard Gaussian random variable:  $\eta(t) \sim \mathcal{N}(0, \delta_s^2) \forall t$  with  $\delta_s$  the noise amplitude (standard deviation of measurement noise). Then, the frequency domain pre-processing step requires to take the (discrete) Fourier transform of measurements. Focusing on the frequency range  $D_\omega$ , and using the linearity property of the Fourier transform, one has:

$$Y_\omega = Y_{ex,\omega} + H_\omega, \quad \forall \omega \in D_\omega \quad (\text{A.4})$$

where  $H_\omega$  is the Fourier transform of the random process  $\eta(t)$ . From here, one can properly introduce the mCRE measurement error by replacing  $Y_{ex,\omega}$  with  $\Pi U_\omega$  :

$$\|\Pi U_\omega - Y_\omega\|_G^2 = \|H_\omega\|_G^2, \quad \forall \omega \in D_\omega \quad (\text{A.5})$$

$$\Rightarrow \int_{D_\omega} \frac{z(\omega)}{2} \|\Pi U_\omega - Y_\omega\|_G^2 d\omega = \int_{D_\omega} \frac{z(\omega)}{2} \|H_\omega\|_G^2 d\omega \quad (\text{A.6})$$

If one knows the statistics of the  $\|H_\omega\|^2$  square modulus of the discrete Fourier transform of a random process, then the  $\delta$  scalar of the Morozov discrepancy theorem would be explicitly made available in a formulation dedicated to the mCRE.

### A.2.1 Probability distribution function (pdf) of a random process

$H_\omega$  is the amplitude of the discrete Fourier transform of  $\{\eta(t_n) = \eta_n\}_{n=0}^{N-1}$  at angular frequency  $\omega$ . Mathematically, it means that:

$$H_\omega \triangleq \mathcal{F}(\{\eta_n\}_{n=0}^{N-1}, \omega) = \sum_{n=0}^{N-1} \eta_n e^{-2i\pi\omega n/N} \quad (\text{A.7})$$

Focusing on the real part of  $H_\omega$  denoted  $\Re(H_\omega)$ , one has:

$$\Re(H_\omega) = \sum_{n=0}^{N-1} \eta_n \cos(2\pi\omega n/N) \quad (\text{A.8})$$

Denoting  $y_n = \eta_n \cos(2\pi\omega n/N)$ , it implies that the pdf of  $y_n$  verifies:

$$\pi_{y_n}(y) = \frac{1}{\cos(2\pi\omega n/N)} \pi_{\eta_n} \left( \frac{y}{\cos(2\pi\omega n/N)} \right) \quad (\text{A.9})$$

In other words,  $y_n \sim \mathcal{N}(0, \delta_s^2 \cos^2(2\pi\omega n/N)^2)$ . For the sake of conciseness, we will abbreviate  $\cos(2\pi\omega n/N) \equiv c_{\omega,n}$ . With such a notation, one has  $y_n \sim \mathcal{N}(0, \delta_s^2 c_{\omega,n}^2) \forall n \in \llbracket 0; N-1 \rrbracket$ .

The pdf of  $\Re(H_\omega)$  is the convolution of all the random variables  $y_n$  (that have to be summed in the discrete Fourier transform).

$$\pi_{\Re(H_\omega)}(x) = \bigotimes_{n=0}^{N-1} \pi_{y_n}(x) \quad (\text{A.10})$$

Taking the Fourier transform of the last equation allows to simplify the convolutional product:

$$\mathcal{F}(\pi_{\Re(H_\omega)}, k) = \prod_{n=0}^{N-1} \mathcal{F}(\pi_{y_n}, k) \quad (\text{A.11})$$

Knowing that the Fourier transform of a Gaussian function of standard deviation  $\sigma$  is also a Gaussian of standard deviation  $1/\sigma$ , one can write

$$\mathcal{F}(\pi_{\Re(H_\omega)}, k) \propto \exp \left[ -\frac{k^2 \delta_s^2}{2} \sum_{n=0}^{N-1} \cos^2(2\pi\omega n/N) \right] = \exp \left[ -\frac{k^2 \delta_s^2 N}{4} \right] \quad (\text{A.12})$$

Therefore, one finally gets that the Fourier transform of  $\Re(H_\omega)$  is a random variable following a centered Gaussian pdf of standard deviation  $\sqrt{2}/(\delta_s \sqrt{N})$ . One can conclude that  $\Re(H_\omega) \sim \mathcal{N}(0, \delta_s^2 N/2)$ . The same conclusion holds for the imaginary part of  $H_\omega$  denoted  $\Im(H_\omega)$ .

We have thus proved that the discrete Fourier transform of a Gaussian random process is also a random variate whose pdf is a multivariate Gaussian law, both its real and imaginary parts taken alone being zero-mean Gaussian random variables of standard deviation  $\sigma \sqrt{N/2}$ . From this result, it is worth mentioning that:

- the measurement noise level directly impacts the value of  $H_\omega$ ;
- there is no dependency on the considered angular frequency  $\omega$ , which is consistent with the well-known constant Power Spectral Density (PSD) of a white-noise signal;
- the dependency on the number of sampling points in the time domain  $N$  seems intuitively logical as the more data points, the more accurate a Fourier transform of a white noise. This dependency can be cancelled by normalizing the discrete Fourier transform by  $1/\sqrt{N}$ , which is done numerically in the MATLAB<sup>®</sup> `fft` function.

### A.2.2 Probability distribution function of the squared modulus of a random process

Now that the distribution of  $H_\omega$  is known, to identify which law follows  $\|H_\omega\|^2$ , we can exploit the following theorem:

Be  $[X_1; \dots; X_d]$  a vector of  $d$  random variables such that  $X_i \sim \mathcal{N}(0, 1) \forall i \in \llbracket 1; d \rrbracket$  and all couples of random variables  $(X_i, X_j)$ ,  $i \neq j$  are independent. Then the squared norm of vector  $[X_1; \dots; X_d]$ , denoted  $Y$ , follows a chi-squared distribution of degree  $d$ . In other words:

$$Y = \sum_{i=0}^d X_i^2 \sim \chi^2(d) \text{ with } \pi_Y(y) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

where  $\Gamma : d \mapsto \int_{\mathbb{R}^+} t^{d-1} e^{-t} dt$  is the Gamma function.

The real and imaginary components of a Fourier transform indeed correspond to the individual Fourier transforms even and odd components of the time domain function. As all functions can be decomposed as a sum of an even and odd function and since the Fourier transform is a one-to-one mapping between the time and frequency domains, the lack of correlation between even and odd parts in the time domain would imply a lack of correlation in the frequency domain too. As we deal with a zero-mean white-noise time series, the real and imaginary parts of its Fourier transform are thus uncorrelated.

As  $\Re(H_\omega)$  and  $\Im(H_\omega)$  both are centered Gaussian random variables of standard deviation  $\delta_s \sqrt{N/2}$ , one can deduce that the pdf of  $2N/\sigma^2 \|H_\omega\|^2$  is a noncentered chi-squared distribution of degree 2 with  $\lambda = 0$ . After variable change, one finally obtains an analytical formulation for the pdf of  $\|H_\omega\|^2$ :

$$\pi_{\|H_\omega\|^2}(x) = \frac{e^{-x/(\delta_s^2/N)}}{\delta_s^2/N} \quad (\text{A.13})$$

As mentioned in the remarks of the previous paragraph, a  $1/\sqrt{N}$  normalization of the Fourier transforms allows to take off the dependency into the number of time points  $N$ .

Therefore, the squared modulus of a zero-mean random process  $\|H_\omega\|^2$  follows a non-centered  $\chi^2(2)$  probability distribution conditioned by the measurement noise variance  $\delta_s^2$ , whatever the value of  $\omega$ . In particular, one can note that

$$\mathbf{E}(\|H_\omega\|^2) = \delta_s^2 \quad (\text{A.14})$$

where  $\mathbf{E}(\square)$  is the mathematical expectation operator. Assuming  $G$  is proportional to the identity matrix, *i.e.*  $G = G_0 I$ , then one recovers (14), which is the adaptation of Morozov's discrepancy principle to the mCRE framework (in the frequency domain).

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