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► **To cite this version:**

Toufik Mansour, Gökhan Yıldırım. Inversion sequences avoiding 021 and another pattern of length four. 2022. hal-03871544v1

HAL Id: hal-03871544

<https://hal.science/hal-03871544v1>

Preprint submitted on 25 Nov 2022 (v1), last revised 6 Nov 2023 (v3)

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INVERSION SEQUENCES AVOIDING 021 AND ANOTHER PATTERN OF LENGTH FOUR

TOUFIK MANSOUR AND GÖKHAN YILDIRIM

ABSTRACT. We study the enumeration of inversion sequences that avoid the pattern 021 and another pattern of length four. We determine the generating trees for all possible pattern pairs and compute the corresponding generating functions. We introduce the concept of d -regular generating trees and conjecture that for any 021-avoiding pattern τ , the generating tree $\mathcal{T}'(\{021, \tau\})$ is d -regular for some integer d .

1. INTRODUCTION

An integer sequence $e = e_0e_1 \cdots e_n$ of length n is called an *inversion sequence* if $0 \leq e_i \leq i$ for each $0 \leq i \leq n$. We use \mathbf{I}_n to denote the set of inversion sequences of length n . There is a bijection between \mathbf{I}_n and the set of permutations of length $n + 1$, denoted by S_{n+1} . Let τ be a word of length k over the alphabet $[k] := \{0, 1, \dots, k - 1\}$, we say that an inversion sequence $e \in \mathbf{I}_n$ contains the pattern τ if there is a subsequence of length k in e that has the same relative order with τ ; otherwise, we say that e avoids the pattern τ . For instance, $e = 010213 \in I_5$ avoids the pattern 201 because there is no subsequence $e_j e_k e_l$ of length three in e with $j < k < l$ and $e_k < e_l < e_j$. On the other hand, $e = 010213$ contains the patterns 010 and 0012. For a given pattern τ , we use $\mathbf{I}_n(\tau)$ to denote the set of all τ -avoiding inversion sequences of length n . Similarly, for a given set of patterns B , we set $\mathbf{I}_n(B) = \bigcap_{\tau \in B} \mathbf{I}_n(\tau)$. The first results on the pattern-avoiding inversion sequences were obtained by Mansour and Shattuck [20] and Corteel et.al [8] for the patterns of length three. Later, Martinez and Savage [24] generalized and extended the notion of pattern-avoidance for the inversion sequences to triples of binary relations that lead to new conjectures and open problems. Various other pattern-avoidance conditions such as vincular patterns, pairs of patterns, and longer patterns are also studied for inversion sequences, for relevant results see [1, 3, 4, 6, 7, 16, 17, 21, 27] and references therein. In the context of inversion sequences, two sets of patterns B_1 and B_2 are said to be Wilf equivalent if $|\mathbf{I}_n(B_1)| = |\mathbf{I}_n(B_2)|$ for all $n \geq 0$, that is, they have the same counting sequence. Note that there are basically 13 patterns of length three and 75 of length four up to order isomorphism; we use \mathcal{P}_k to denote the set of patterns of length k . Yan and Lin [27] completed the classification of the Wilf-equivalences for inversion sequences avoiding pairs of length-three patterns. Hong and Li [10] studied the Wilf classification for patterns of length four. In this paper, we study inversion sequences avoiding 021 and another pattern of length four. We will use the algorithmic approach based on generating trees developed by the authors in [23] and the kernel method. Our results mainly

2010 *Mathematics Subject Classification.* 05A05, 05A15, 05A16.

Key words and phrases. Pattern-avoiding inversion sequences, generating functions, generating trees, kernel method, Catalan numbers, Motzkin numbers.

G. Yildirim was partially supported by Tubitak-Ardeb-120F352.

follow the following five-step procedure: (i) an educated guess of the rules of the generating tree based on the algorithm output, (ii) verification of the previous step, (iii) translating the rules of the generating tree into a one-parameter infinite system of equations involving related generating functions, (iv) using bivariate generating functions and obtaining a finite system of equations, (v) using kernel method to obtain an expression for the generating function of the pattern class. In most cases, it is possible to obtain a nice closed formula for the enumerating sequence of the class. We will provide the details of this procedure in the following section.

The paper is organized as follows. In Section 2, we outline the details of our procedure that lead to the main result of this paper and present some examples. In Section 3, we study the generating functions for all possible pattern pairs of 021 and τ where τ is a pattern of length four that avoids 021. There are 33 patterns of length four that avoid 021. The pairs of patterns 021 and 0000 is one of the most intriguing cases with a long list of succession rules for the generating tree that requires some special techniques to obtain the corresponding generating function.

We use C_n and M_n to denote the Catalan and Motzkin numbers, respectively. Their generating functions are $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ and $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$. For a letter or number s , we use s^m to denote the string of m s , that is, $s^m = \underbrace{ss \cdots s}_{m \text{ times}}$.

2. AN ALGORITHMIC APPROACH BASED ON GENERATING TREES

For each class of inversion sequences avoiding a fixed set of patterns, there is a corresponding generating tree that encodes the elements of the class as on its vertices under some specific rules. For the details, we refer the reader to Section 2 of [23].

We use $\mathcal{I}_B = \cup_{n=0}^{\infty} \mathbf{I}_n(B)$ to denote the set of all inversion sequences avoiding the pattern set B . The corresponding generating tree $\mathcal{T}(B)$ is a rooted, labeled, plane tree whose vertices are the objects of \mathcal{I}_B with the following properties: (i) each element of \mathcal{I}_B appears exactly once in the tree; (ii) element of size n appears at level n in the tree (the root has level 0); (iii) there is a set of succession rules that determine the number of children and their labels for each vertex. The tree $\mathcal{T}(B)$ will be empty if there is no inversion sequence of arbitrary length avoiding the set B . Otherwise, the label of the root will be always 0, that is, $0 \in \mathcal{T}(B)$. From the root, whose level is zero, we construct the remainder of the tree $\mathcal{T}(B)$ in a recursive way where the n^{th} level of the tree consists exactly the elements of $\mathbf{I}_n(B)$ arranged in such a way that the parent of an inversion sequence $e_0e_1 \cdots e_n \in \mathbf{I}_n(B)$ is the unique inversion sequence $e_0e_1 \cdots e_{n-1} \in \mathbf{I}_{n-1}(B)$. We obtain the children of $e_0e_1 \cdots e_{n-1} \in \mathbf{I}_{n-1}(B)$ from the set $\{e_0e_1 \cdots e_{n-1}e_n \mid e_n = 0, 1, \dots, n\}$ by obeying the pattern-avoiding restrictions of the patterns in B . We arrange the nodes from the left to the right so that if $e = e_0e_1 \cdots e_{n-1}i$ and $e' = e_0e_1 \cdots e_{n-1}j$ are children of the same parent $e_1 \cdots e_{n-1}$, then e appears on the left of e' if $i < j$. Based on this initial tree $\mathcal{T}(B)$, we define an equivalence relation on the set of the nodes of this tree and obtain a second representation of the tree corresponding to the class \mathcal{I}_B which is more efficient for enumerated purposes. We relabel the vertices of the tree $\mathcal{T}(B)$ as follows. Define $\mathcal{T}(B; e)$ to be the subtree consisting of the inversion sequence e as the root and its descendants in $\mathcal{T}(B)$. We say that e is *equivalent* to e' , denoted by $e \sim e'$, if and only if $\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$ (in the sense of plan trees). Let $\mathcal{T}'(B)$ be the same tree $\mathcal{T}(B)$ where we replace each node $e \in \mathcal{T}(B)$ from top to bottom and from left to right in $\mathcal{T}(B)$ such that $\mathcal{T}(B; e) \cong \mathcal{T}(B; e')$. Clearly, the generating tree $\mathcal{T}'(B)$ has a root 0, for any B such that $0 \notin B$.

Let B be any set of patterns and let $\mathcal{T}'(B)$ be the generating tree for the class $\mathbf{I}_n(B)$. The *length* of a node $v \in \mathcal{T}'(B)$ is defined to be the number of letters in v and it is denoted by $len(v)$. For any $k \geq 1$, let $D_k(B)$ be the multiset of all nodes of length k at level $k - 1$ in $\mathcal{T}'(B)$. For each node $v \in D_k(B)$, we denote the multiset of all children of v at level k in $\mathcal{T}'(B)$ by $N_k(B; v)$. A generating tree $\mathcal{T}'(B)$ is said to be *d-regular* if there exists $k \geq 1$ such that

- the number different nodes in $D_r(B)$ equals d , for all $r > k$;
- for any $v \in D_r(B)$ and $w \in N_r(B; v)$, the number of occurrences of w in $N_r(B; v)$ does not depend on r , whenever $r > k$.

Clearly, $\mathcal{T}'(B)$ is 0-regular if and only if the generating tree $\mathcal{T}'(B)$ is finite. Now, we present an example of d -regular and non-regular trees to illustrate the definition.

Example 2.1. *If $B = \{021, 0123\}$, then the generating tree $\mathcal{T}'(B)$ has the following rules*

$$a_m \rightsquigarrow a_{m+1}b_m b_{m-1} \cdots b_1, \quad b_m \rightsquigarrow b_{m+1}^2 c^m, \quad c \rightsquigarrow c^2,$$

where $a_m = 0^m$, $b_m = 0^m 1$, and $c = 012$. Note that $D_1(B) = \{a_1\}$, $D_2(B) = \{a_2, b_1\}$, $D_3(B) = \{a_3, b_2, b_2, c\}$, and $D_k(B) = \{a_k, \underbrace{b_{k-1}, \dots, b_{k-1}}_{2^{k-1} - 1 \text{ times}}\}$ for all $k \geq 4$. Moreover, the

child a_{k+1} of $a_k \in D_k(B)$ occurs in $N_k(B; a_k)$ exactly once and the child b_k of $b_{k-1} \in D_k(B)$ occurs exactly twice. Hence, $\mathcal{T}'(B)$ is 2-regular.

Let $B = \{000, 0011\}$. The generating tree $\mathcal{T}'(B)$ has the following rules:

$$a_m \rightsquigarrow b_m^{m+1} a_{m+1}, \quad b_m \rightsquigarrow b_m^{m+2},$$

where $a_m = 012 \cdots m$ and $b_m = a_m 0$. Note that $D_k(B)$ contains a_{k-1} exactly once and b_{k-2} exactly $k - 1$ times, for all $k \geq 1$. Thus, $\mathcal{T}'(B)$ is not d -regular for any d .

Based on our results, see Table 1, we have the following conjecture.

Conjecture 2.2. *For any 021-avoiding pattern τ , the generating tree $\mathcal{T}'(\{021, \tau\})$ is d -regular for some d .*

We will use the following procedure to study the generating functions for the sequences $\{|I_n(\{021, \tau\})|\}_{n \geq 0}$ where τ is any pattern of length four that avoids 021.

The five-step Procedure: The main results of this paper are applications of the five-step procedure that we described briefly in the introduction. We will now outline the details of the procedure for the pattern set $B = \{021, 1002\}$. First, we obtain the rules of the generating tree $\mathcal{T}'(B)$ by using the algorithm developed in [23]. We then obtain an explicit formula for the generating function $F_B(x) = \sum_{n \geq 0} |I_n(B)|x^{n+1}$ from the rules of generating tree in a systematic way that can be programmed in software.

- **Step 1- An educated guess for the rules of $\mathcal{T}'(B)$:** Based on the output of the algorithm of [23], we guess that the rules of the generating tree $\mathcal{T}'(B)$ can be described as follows: the generating tree $\mathcal{T}'(B)$'s root is denoted by a_1 , and the succession rules

are:

$$\begin{aligned}
a_m &\rightsquigarrow a_{m+1}b_m b_{m-1} \cdots b_1, & b_m &\rightsquigarrow c_m b_{m+1} d_m d_{m-1} \cdots d_1, \\
c_m &\rightsquigarrow f c_{m+1} e_m e_{m-1} \cdots e_1 g, & d_m &\rightsquigarrow e_m d_{m+1} d_m \cdots d_1, \\
e_m &\rightsquigarrow h e_{m+1} e_m \cdots e_1 g, & f &\rightsquigarrow f^2, \\
g &\rightsquigarrow h e_1 g, & h &\rightsquigarrow h,
\end{aligned}$$

where $m \geq 1$, $a_m = 0^m$, $b_m = a_m 1$, $c_m = b_m 0$, $d_m = a_m 12$, $e_m = a_m 102$, $f = 0100$, $g = 0103$, and $h = 01020$. We use \mathcal{R} to denote the (proposed) set of rules of the tree.

- **Step 2- Verifying the set of rules for $\mathcal{T}'(B)$:** We prove that the proposed set of rules \mathcal{R} hold for any level in the tree. Note that we have to show that when we add a letter $j = 0, 1, \dots, \text{len}(v)$ for the father v of a rule, we obtain a rule in \mathcal{R} . For instance, for the father of the rule $h \rightsquigarrow h$, we can add a letter $j = 0, 1, \dots, 5$:
 - since the inversion sequences avoid B , we see that $\mathcal{T}'(B; h0) \sim \mathcal{T}'(B; h)$.
 - $h1 = 010201$ contains 021 , so $h1$ is not a child of h ;
 - $hj = 01020j$, $j = 2, 3, 4, 5$, contains 1002 , so hj is not a child of h .
Thus the rule $h \rightsquigarrow h$ holds for $\mathcal{T}'(B)$. Similarly, we can show that the rules \mathcal{R} describe all the rules of $\mathcal{T}'(B)$.

- **Step 3- From the rules of $\mathcal{T}'(B)$ to a one-parameter infinite system of equations:** For each rule of the type $v \rightsquigarrow v^{(1)}v^{(2)} \cdots v^{(\ell)}$ in \mathcal{R} , we define the generating function $A_v(x)$ as the generating function for the number of nodes at level n in the subtree $\mathcal{T}'(B; v)$ of $\mathcal{T}'(B)$, where the root of this subtree is the vertex v that stays at level 0. Then each rule of the type $v \rightsquigarrow v^{(1)}v^{(2)} \cdots v^{(\ell)}$ can be translated into an equation for the generating functions as

$$A_v(x) = x + x \sum_{j=1}^{\ell} A_{v^{(j)}}(x).$$

Note that if we have a finite number of rules, then there will be a finite system of equations. For the above rules, we have

$$\begin{aligned}
A_m(x) &= x + xA_{m+1}(x) + xB_m(x) + \cdots + xB_1(x), \\
B_m(x) &= x + xC_m(x) + xB_{m+1}(x) + xD_m(x) + \cdots + xD_1(x), \\
C_m(x) &= x + xF(x) + xC_{m+1}(x) + xE_m(x) + \cdots + xE_1(x) + xG(x), \\
D_m(x) &= x + xE_m(x) + xD_{m+1}(x) + \cdots + xD_1(x), \\
E_m(x) &= x + xH(x) + xE_{m+1}(x) + \cdots + xE_1(x) + xG(x), \\
F(x) &= x + 2xF(x), \\
G(x) &= x + xH(x) + xE_1(x) + xG(x), \\
H(x) &= x + xH(x).
\end{aligned}$$

where $A_m(x) = A_{a_m}(x)$, $B_m(x) = A_{b_m}(x)$, $C_m(x) = A_{c_m}(x)$, $D_m(x) = A_{d_m}(x)$, $E_m(x) = A_{e_m}(x)$, $F(x) = A_f(x)$, $G(x) = A_g(x)$, and $H(x) = A_h(x)$.

- **Step 4- A finite system of equations for bivariate generating functions:** There are many different methods to solve a such system of recurrence relations.

In this paper, we observe that for each set of patterns $B = \{021, \tau\}$ with a 021-avoiding four-letter pattern τ , the system of recurrence relations is linear, and it is parametrized by one parameter m . We will use bivariate generating functions to solve the recurrence relations. Consider $A_m(x)$ parametrized by one parameter m , we define the bi-variate generating function as

$$A(x, v) = \sum_{m \geq s} A_m(x) v^{m-s},$$

where s indicates the minimal value of m such that the recurrence for the generating function $A_m(x)$ holds. From the above recurrence relations, we obtain

$$\begin{aligned} A(x, v) &= \frac{x}{1-v}(1 + B(x, v)) + \frac{x}{v}(A(x, v) - A(x, 0)), \\ B(x, v) &= \frac{x}{1-v}(1 + D(x, v)) + xC(x, v) + \frac{x}{v}(B(x, v) - B(x, 0)), \\ C(x, v) &= \frac{x}{1-v}(1 + F(x) + E(x, v) + G(x)) + \frac{x}{v}(C(x, v) - C(x, 0)), \\ D(x, v) &= \frac{x}{1-v}(1 + D(x, v)) + xE(x, v) + \frac{x}{v}(D(x, v) - D(x, 0)), \\ E(x, v) &= \frac{x}{1-v}(1 + H(x) + E(x, v) + G(x)) + \frac{x}{v}(E(x, v) - E(x, 0)), \\ F(x) &= \frac{x}{1-2x}, \\ G(x) &= \frac{x + x(1-x)E_1(x)}{(1-x)^2}, \\ H(x) &= \frac{x}{1-x}. \end{aligned}$$

- **Step 5- Obtaining the generating function $F_B(x)$ with the kernel method:** To solve a such system, we use the kernel method several times. We apply the algorithm in [11]. Let $v_0 = \frac{1-\sqrt{1-4x}}{2}$, we consider the equation of $E(x, v)$ with $v = v_0$, which leads to

$$E(x, 0) = \frac{1 - \sqrt{1 - 4x}}{(1-x)(1-2x + \sqrt{1-4x})}.$$

Then by setting this into the same equation of $E(x, v)$, we obtain

$$E(x, v) = \frac{x(1-2x - \sqrt{1-4x})}{(1-x)(1-2x + \sqrt{1-4x})(v^2 - v + x)}.$$

For the second step, we take the equation of $D(x, v)$ at limit $v = v_0$, which leads to

$$D(x, 0) = \frac{(1-2x)\sqrt{1-4x} - (1-x)(1-4x)}{(1-4x)(1-x)}.$$

Then by setting this into the same equation of $D(x, v)$, we obtain

$$\begin{aligned} D(x, v) &= \frac{(v-1)(4v^2x^2 - 6v^2x - 8vx^2 + 4x^3 + v^2 + 7vx - 2x^2 - v)}{2\sqrt{1-4x}(1-x)(v^2 - v + x)^2} \\ &+ \frac{(v-1+x)(2v^2x - v^2 - 4vx + 2x^2 + v)}{2(1-x)(v^2 - v + x)^2} \end{aligned}$$

Similarly, we solve other cases. Let $v_0 = x$, for $C(x, v)$ with $v = v_0$, we first obtain $C(x, 0)$ and then obtain an explicit formula for $C(x, v)$. The case of $B(x, v)$ and $A(x, v)$ follow with the same $v = v_0$. Hence we obtain

$$A(x, 0) = \frac{x}{1-x} + \frac{x}{1-x}B(x, x),$$

which leads to that the generating function $A(x, 0)$ is given by

$$\frac{(1-2x)(2x^2-4x+1)}{2x^2(1-x)\sqrt{1-4x}} - \frac{(2x^2-2x+1)(2x^6-12x^5+25x^4-34x^3+24x^2-8x+1)}{2x^2(1-x)^5(1-2x)}.$$

Clearly, $A(x, 0) = A_1(x)$ is the generating function for the number of inversion sequences in $I_n(B)$.

An explicit formula for the enumerating sequence $\{|\mathbf{I}_n(\{021, 1002\})|\}_{n \geq 1}$ follows from the generating function:

$$\frac{1}{2} \binom{2n+4}{n+2} - \frac{5}{2} \binom{2n+2}{n+1} + \frac{5}{2} \binom{2n}{n} + \frac{1}{2} \sum_{j=0}^{n-1} \binom{2j}{j} + 2^n - \frac{1}{24}(n^4 - 2n^3 + 11n^2 + 14n + 12).$$

All other cases are based on similar techniques as described above. For some $\{021, \tau\}$ -avoiding inversion sequence classes, we observe that finding the root of the kernel requires the solution of a third or fourth-order polynomial. We will provide the details of the proofs of such cases in the following sections.

3. FOUR LETTER PATTERNS

In this section, we consider the set of inversion sequences of length n that avoid both 021 and τ , where τ is any pattern of length four that avoids 021. We summarize our results in Table 1; the results follow from the procedure discussed in the previous section (for the Maple file which includes all the computations, we refer the reader to [19]). We obtain exact formulas for the generating functions

$$F_{\{021, \tau\}}(x) = \sum_{n \geq 0} |I_n(\{021, \tau\})| x^{n+1}.$$

Most of them are algebraic of degree at most 6 and only involve the square root of $1 - 4x$. We will give all the details of the above procedure for five cases to show how we find analytic expressions for the generating functions. In Corollary 3.9, we summarize the exact enumerating formulas for all pattern cases except the ones whose generating functions don't lead to a nice closed formula.

Table 1: Succession rules for the generating trees $\mathcal{T}'(B)$ and generating functions $F_B(x)$ for pattern set $B = \{021, \tau\}$, where τ is any pattern of length four that avoids 021.

Beginning of Table 1				
No.	τ	d	Rules of $\mathcal{T}'(\{021, \tau\})$	$F_{\{021, \tau\}}(x)$
1	0000	6	Theorem 3.10	Theorem 3.10

Continuation of Table 1				
No.	τ	d	Rules of $\mathcal{T}'(\{021, \tau\})$	$F_{\{021, \tau\}}(x)$
2	0001	2	$a_0 \rightsquigarrow b_0 c_1, b_0 \rightsquigarrow e a_1 f, e \rightsquigarrow e, f \rightsquigarrow e g f,$ $g \rightsquigarrow e h a_1 f$ $h \rightsquigarrow e h, i \rightsquigarrow h i, a_m \rightsquigarrow e^3 b_m a_m a_{m-1} \cdots a_1 f$ $b_m \rightsquigarrow e^3 h a_{m+1} a_m \cdots a_1 f,$ $c_m \rightsquigarrow a_m d_m c_m c_{m-1} \cdots c_1,$ $d_m \rightsquigarrow i b_m c_{m+1} c_m \cdots c_1$ $a_m = 0^2 1^2 \cdots (m-1)^2 m, b_m = a_m m$ $c_m = 01^2 2^2 \cdots (m-1)^2 m, d_m = c_m m$ $e = 000, f = 002, g = 0022, h = 00111,$ $i = 0111$	$\frac{9x^5 - 6x^4 - x^3 + 5x^2 - 4x + 1}{2x^2(1+x)(1-x)^3 \sqrt{1-2x-3x^2}}$ $+ \frac{2x^6 - 6x^5 + x^4 + 3x^3 - 4x^2 + 3x - 1}{2x^2(1+x)(1-x)^3}$
3	0010	2	$a_1 \rightsquigarrow a_2 b_1, a_m \rightsquigarrow a_{m+1} a_m \cdots a_2 c$ $b_m \rightsquigarrow a_{m+2} b_{m+1} b_m \cdots b_1, c \rightsquigarrow a_2 c$ $a_m = 0^m, b_m = 01^m, c = 002$	$\frac{x}{\sqrt{1-4x}}$
4	0011	2	$a_1 \rightsquigarrow a_2 b_1, b_1 \rightsquigarrow a_3 b_2 b_1,$ $a_m \rightsquigarrow a_{m+1} a_m \cdots a_2 c$ $b_m \rightsquigarrow a_{m+2} b_{m+1} a_m a_{m-1} \cdots a_2 c, c \rightsquigarrow a_2 c$ $a_m = 0^m, b_m = 01^m, c = 002$	$\frac{2x^3 + 2x^2 - 4x + 1}{2x^2(1-x)}$ $- \frac{(1-2x)\sqrt{1-4x}}{2x^2(1-x)}$
5	0012	2	$a_1 \rightsquigarrow a_2 b_1, b_1 \rightsquigarrow a_3 b_2 b_1, a_m \rightsquigarrow a_{m+1} c^m$ $b_m \rightsquigarrow a_{m+2} b_{m+1} c^m, c \rightsquigarrow c^2$ $a_m = 0^m, b_m = 01^m, c = 001$	$\frac{x(2x^4 - 5x^3 + 8x^2 - 4x + 1)}{(1-x)^4(1-2x)}$
6	0100	3	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow c_m b_{m+1} b_m \cdots b_1$ $c_m \rightsquigarrow c_{m+1} c_m \cdots c_1 d, d \rightsquigarrow c_1 d$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 0, d = 0103$	$\frac{(1-3x)^2}{2x^2 \sqrt{1-4x}} - \frac{(1-3x)(1-x)}{2x^2}$
	0110	3	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow c_m b_{m+1} b_m \cdots b_1$ $c_m \rightsquigarrow c_{m+1} c_m \cdots c_1 d, d \rightsquigarrow c_1 d$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 1, d = 0113$	
7	0101	2	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow a_{m+1} b_{m+1} b_m \cdots b_1$ $a_m = 0^m, b_m = a_m 1$	Theorem 3.2
	0111	2		
8	0102	3	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow d b_{m+1} c_m c_{m-1} \cdots c_1$ $c_m \rightsquigarrow e c_{m+1} c_m \cdots c_1$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d = 010,$ $e = 0120$	$\frac{4x^6 - 12x^5 + 24x^4 - 27x^3 + 19x^2 - 7x + 1}{2x(1-2x)(1-x)^4}$ $- \frac{\sqrt{1-4x}}{2x(1-x)}$
9	0112	2	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow c b_{m+1} b_m \cdots b_1, c \rightsquigarrow c^2$ $a_m = 0^m, b_m = a_m 1, c = 011$	$\frac{1-4x+5x^2-4x^3-(1-x)^2\sqrt{1-4x}}{2x(1-2x)(1-x)}$
10	0120	3	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow b_{m+1}^2 c_1 \cdots c_m$ $c_m \rightsquigarrow c_1 \cdots c_{m+1}$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2$	$\frac{1-2x}{2(1-x)} \left(\frac{1}{\sqrt{1-4x}} - 1 \right)$
	0122	3		
11	0123	2	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1, b_m \rightsquigarrow b_{m+1}^2 c^m,$ $c \rightsquigarrow c^2$ $a_m = 0^m, b_m = a_m 1, c = 012$	$\frac{x(1-7x+21x^2-30x^3+22x^4-8x^5)}{(1-x)^3(1-2x)^3}$
12	1000	4	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow b_1 \cdots b_{m+1} c_m$ $c_m \rightsquigarrow c_1 \cdots c_{m+1} d_m e, d_m \rightsquigarrow d_1 \cdots d_{m+1} f g$ $e \rightsquigarrow e f c_1, f \rightsquigarrow d_1 f g, g \rightsquigarrow f g$	

Continuation of Table 1				
No.	τ	d	Rules of $\mathcal{T}'(\{021, \tau\})$	$F_{\{021, \tau\}}(x)$
			$a_m = 0^m, b_m = a_m 1, c_m = b_m 0, d_m = c_m 0$ $e = 0103, f = 01003, g = 01004$	
	1100	4	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow b_1 \cdots b_{m+1} c_m$ $c_m \rightsquigarrow c_1 \cdots c_{m+1} d_m e, d_m \rightsquigarrow d_1 \cdots d_{m+1} f g$ $e \rightsquigarrow e f c_1, f \rightsquigarrow d_1 f g, g \rightsquigarrow f g$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 1, d_m = c_m 0$ $e = 0113, f = 01103, g = 01104$	$\frac{(1-x)(1-2x)^2}{2x^3}$ $+ \frac{2x^5 - 56x^4 + 78x^3 - 44x^2 + 11x - 1}{2x^3 \sqrt{1-4x^3}}$
13	1001	3	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow c_m b_1 \cdots b_{m+1}$ $c_m \rightsquigarrow a_{m+2} b_1 \cdots b_{m+1} c_{m+1}$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 0$	
	1011	3	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow c_m b_1 \cdots b_{m+1}$ $c_m \rightsquigarrow a_{m+2} b_1 \cdots b_{m+1} c_{m+1}$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 1$	
	1101	3	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow c_m b_1 \cdots b_{m+1}$ $c_m \rightsquigarrow a_{m+2} b_1 \cdots b_{m+1} c_{m+1}$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 1$	Theorem 3.1
14	1002	5	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow c_m b_{m+1} d_m d_{m-1} \cdots d_1$ $c_m \rightsquigarrow f c_{m+1} e_m e_{m-1} \cdots e_1 g,$ $d_m \rightsquigarrow e_m d_{m+1} d_m \cdots d_1$ $e_m \rightsquigarrow h e_{m+1} e_m \cdots e_1 g, f \rightsquigarrow f^2$ $g \rightsquigarrow h e_1 g, h \rightsquigarrow h$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 0$ $d_m = a_m 12, e_m = a_m 102, f = 0100$ $g = 0103, h = 01020$	$\frac{(1-2x)(2x^2-4x+1)}{2x^2(1-x)\sqrt{1-4x}}$ $- \frac{x^2(2x^2-2x+1)(2x^2-12x+25)}{2(1-x)^5(1-2x)}$ $+ \frac{(2x^2-2x+1)(34x^3-24x^2+8x-1)}{2x^2(1-x)^5(1-2x)}$
15	1010	4	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow c_m b_1 \cdots b_{m+1}$ $c_m \rightsquigarrow c_{m+1} d_m b_1 \cdots b_{m+1},$ $d_m \rightsquigarrow d_1 \cdots d_{m+1} e f$ $e \rightsquigarrow d_1 e f, f \rightsquigarrow e f$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 0, d_m = c_m 1$ $e = 01013, f = 01014$	
	1110	4	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m, b_m \rightsquigarrow c_m b_1 \cdots b_{m+1}$ $c_m \rightsquigarrow c_{m+1} d_m b_1 \cdots b_{m+1},$ $d_m \rightsquigarrow d_1 \cdots d_{m+1} e f$ $e \rightsquigarrow d_1 e f, f \rightsquigarrow e f$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 1, d_m = c_m 1$ $e = 01113, f = 01114$	Theorem 3.3
20	1012	3	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow c_m b_{m+1} b_m \cdots b_1$ $c_m \rightsquigarrow d c_{m+1} b_{m+1} b_m \cdots b_1, d \rightsquigarrow d^2$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 0, d = 0101$	Theorem 3.5
21	1020	5	$a_m \rightsquigarrow a_{m+1} b_1 \cdots b_m,$ $b_m \rightsquigarrow c_m b_{m+1} d_1 \cdots d_m$ $c_m \rightsquigarrow c_{m+1}^2 e_1 \cdots e_{m-1} f g,$ $d_m \rightsquigarrow e_m d_1 \cdots d_{m+1}$ $e_m \rightsquigarrow e_1 \cdots e_{m+1} f g, f \rightsquigarrow f g, g \rightsquigarrow e_1 f g$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 0, d_m = b_m 2$ $e_m = d_m 0, f = 0103, g = 0102$	$\frac{2-15x+34x^2-22x^3}{2x^3(1-x)\sqrt{1-4x}}$ $+ \frac{2x^4+4x^3-16x^2+11x-2}{2x^3(1-x)}$
23	1023	5	$a_m \rightsquigarrow a_{m+1} b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow c_m b_{m+1} d_m d_{m-1} \cdots d_1$ $c_m \rightsquigarrow c_{m+1}^2 f^{m+1}, d_m \rightsquigarrow e_m d_{m+1} d_m \cdots d_1$ $e_m \rightsquigarrow e_{m+1} f^{m+2}, f \rightsquigarrow f^2$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 0, d_m = a_m 12$	$-\frac{(2x^2-2x+1)\sqrt{1-4x}}{2x(1-x)^2(1-2x)}$

Continuation of Table 1				
No.	τ	d	Rules of $\mathcal{T}'(\{021, \tau\})$	$F_{\{021, \tau\}}(x)$
			$e_m = d_m 0, f = 0102$ $e = 0110, f = 0113, g = 01120$	$-\frac{16x^9 - 76x^8 + 212x^7 - 342x^6}{2x(1-x)^5(1-2x)^3}$ $-\frac{374x^5 - 286x^4 + 151x^3 - 53x^2 + 11x - 1}{2x(1-x)^5(1-2x)^3}$
26	1102	4	$a_m \rightsquigarrow a_{m+1}b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow b_{m+1}b_m \cdots b_1 c_m$ $c_m \rightsquigarrow e c_{m+1} d_m d_{m-1} \cdots d_1 f,$ $d_m \rightsquigarrow g d_{m+1} d_m \cdots d_1 f$ $e \rightsquigarrow e^2, f \rightsquigarrow g d_1 f, g \rightsquigarrow g$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 1, d_m = c_m 2$ $e = 0110, f = 0113, g = 01120$	$-\frac{4x^5 + 31x^4 - 46x^3 + 30x^2 - 9x + 1}{2x^2(1-x)^2(1-2x)\sqrt{1-4x}}$ $-\frac{4x^6 - 15x^5 + 35x^4 - 40x^3 + 25x^2 - 8x + 1}{2x^2(1-x)^3(1-2x)}$
28	1120	4	$a_m \rightsquigarrow a_{m+1}b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow b_{m+1}b_m \cdots b_1 c_m$ $c_m \rightsquigarrow c_{m+1}^2 d_m d_{m-1} \cdots d_1,$ $d_m \rightsquigarrow d_{m+1} d_m \cdots d_1 e, e \rightsquigarrow d_1 e$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 1, d_m = c_m 2,$ $e = 0113$	$\frac{(3x-1)\sqrt{1-4x} + 5x^2 - 5x + 1}{x(1-4x)}$
29	1200	4	$a_m \rightsquigarrow a_{m+1}b_1 \cdots b_m, b_m \rightsquigarrow b_{m+1}^2 c_1 \cdots c_m$ $c_m \rightsquigarrow d_m c_1 \cdots c_{m+1}, d_m \rightsquigarrow d_1 \cdots d_{m+1} e,$ $e \rightsquigarrow d_1 e$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 0,$ $e = 01204$	$\frac{22x^3 - 29x^2 + 10x - 1}{2x(1-x)\sqrt{1-4x^3}} + \frac{(1-5x)(1-2x)}{2x(1-4x)}$
	1220	4	$a_m \rightsquigarrow a_{m+1}b_1 \cdots b_m, b_m \rightsquigarrow b_{m+1}^2 c_1 \cdots c_m$ $c_m \rightsquigarrow d_m c_1 \cdots c_{m+1}, d_m \rightsquigarrow d_1 \cdots d_{m+1} e,$ $e \rightsquigarrow d_1 e$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 2,$ $e = 01224$	
30	1202	4	$a_m \rightsquigarrow a_{m+1}b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow b_{m+1}^2 c_m c_{m-1} \cdots c_1$ $c_m \rightsquigarrow d_m c_{m+1} c_m \cdots c_1,$ $d_m \rightsquigarrow d_{m+1} d_m \cdots d_1$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 0$	Theorem 3.7
31	1203	4	$a_m \rightsquigarrow a_{m+1}b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow b_{m+1}^2 c_m c_{m-1} \cdots c_1$ $c_m \rightsquigarrow c_{m+1} d_m d_{m-1} \cdots d_1 e,$ $d_m \rightsquigarrow d_{m+1} d_m \cdots d_1 f$ $e \rightsquigarrow e^2, f \rightsquigarrow f$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 4$ $e = 0120, f = 01230$	$\frac{(1-2x)}{2(1-x)^2\sqrt{1-4x}}$ $-\frac{16x^8 - 76x^7 + 188x^6 - 270x^5}{2(1-x)^5(1-2x)^3}$ $-\frac{246x^4 - 145x^3 + 53x^2 - 11x + 1}{2(1-x)^5(1-2x)^3}$
32	1220	4	$a_m \rightsquigarrow a_{m+1}b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow b_{m+1}^2 c_m c_{m-1} \cdots c_1$ $c_m \rightsquigarrow d_m c_{m+1} c_m \cdots c_1,$ $d_m \rightsquigarrow d_{m+1} d_m \cdots d_1 e, e \rightsquigarrow d_1 e$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 2,$ $e = 01224$	$\frac{(1-2x)(1-5x)}{2x(1-4x)} + \frac{22x^3 - 29x^2 + 10x - 1}{2x(1-x)\sqrt{1-4x^3}}$
33	1230	4	$a_m \rightsquigarrow a_{m+1}b_m b_{m-1} \cdots b_1,$ $b_m \rightsquigarrow b_{m+1}^2 c_m c_{m-1} \cdots c_1$ $c_m \rightsquigarrow c_{m+1}^2 d_m d_{m-1} \cdots d_1,$ $d_m \rightsquigarrow d_{m+1} d_m \cdots d_1$ $a_m = 0^m, b_m = a_m 1, c_m = b_m 2, d_m = c_m 3$	$\frac{x(3-16x+26x^2-16x^3-(1-2x+2x^2)\sqrt{1-4x})}{2(1-x)^2(1-2x)(1-4x)}$
End of Table 1				

Theorem 3.1. *Let $B \in \{\{021, 1001\}, \{021, 1011\}, \{021, 1101\}\}$. Then, the generating function $F_B(x)$ satisfies $F_B(x) = x(1 + F_B(x))^2(1 + F_B^2(x))$. Moreover, for all $n \geq 1$,*

$$|\mathbf{I}_n(B)| = \frac{1}{n+1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{j} \binom{2n+2}{n-2j}.$$

Proof. By the generating tree succession rules for $\mathcal{T}'(\{021, 1001\})$, $\mathcal{T}'(\{021, 1011\})$, and $\mathcal{T}'(\{021, 1101\})$ in Table 1, it follows that for all x ,

$$F_{\{021,1001\}}(x) = F_{\{021,1011\}}(x) = F_{\{021,1101\}}(x).$$

Define $A_m(x)$, $B_m(x)$, and $C_m(x)$ to be the generating functions for the number of nodes at level n in $\mathcal{T}'(\{021, 1001\}; a_m)$, $\mathcal{T}'(\{021, 1001\}; b_m)$, and $\mathcal{T}'(\{021, 1001\}; c_m)$, respectively, where its root stays at level 0. Then

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + xB_1(x) + \cdots + xB_m(x), \\ B_m(x) &= x + xC_m(x) + xB_1(x) + \cdots + xB_{m+1}(x), \\ C_m(x) &= x + xA_{m+2}(x) + xB_1(x) + \cdots + xB_{m+1}(x) + xC_{m+1}(x). \end{aligned}$$

Define $G(x, v) = \sum_{n \geq 1} G_n(x)v^{n-1}$ for all $G \in \{A, B, C\}$. Then the above recurrences can be written as

$$\begin{aligned} A(x, v) &= \frac{x}{1-v} + \frac{x}{v}(A(x, v) - A_1(x)) + \frac{x}{1-v}B(x, v), \\ B(x, v) &= \frac{x}{1-v} + xC(x, v) + \frac{x}{1-v}B(x, v) + \frac{x}{v}(B(x, v) - B_1(x)), \\ C(x, v) &= \frac{x}{1-v} + \frac{x}{v^2}(A(x, v) - A_1(x) - vA_2(x)) + \frac{x}{1-v}B(x, v) \\ &\quad + \frac{x}{v}(B(x, v) + C(x, v) - B_1(x) - C_1(x)). \end{aligned}$$

By finding $A(x, v)$ from the first equation and $C(x, v)$ from the second equation, we obtain that the second equation can be written as

$$\begin{aligned} &\frac{v^4 - 2v^3x + 2v^2x^2 + (x-v)^3}{vx^2(1-v)(x-v)}B(x, v) \\ &= \frac{v}{x-v}A_1(x) + \frac{x-v-vx}{vx}B_1(x) - (A_2(x) + C_1(x)) + \frac{(vx+v-x)(v-x) + x^2}{x(1-v)(v-x)}. \end{aligned}$$

Let v_1, v_2, v_3, v_4 be the four roots of $K(v) = v^4 - 2v^3x + 2v^2x^2 + (x-v)^3 = 0$ as functions of x , where (here $i^2 = -1$)

$$\begin{aligned} v_1 &= x + x^{\frac{4}{3}} + \frac{2}{3}x^{\frac{5}{3}} + \frac{2}{3}x^2 + \frac{82}{81}x^{\frac{7}{3}} + \frac{349}{243}x^{\frac{8}{3}} + \cdots, \\ v_2 &= x - \frac{1 + \sqrt{3}i}{2}x^{\frac{4}{3}} - \frac{1 - \sqrt{3}i}{3}x^{\frac{5}{3}} + \frac{2}{3}x^2 - \frac{41(1 + \sqrt{3}i)}{81}x^{\frac{7}{3}} - \frac{349(1 - \sqrt{3}i)}{486}x^{\frac{8}{3}} + 2x^3 + \cdots, \\ v_3 &= x - \frac{1 - \sqrt{3}i}{2}x^{\frac{4}{3}} - \frac{1 + \sqrt{3}i}{3}x^{\frac{5}{3}} + \frac{2}{3}x^2 - \frac{41(1 - \sqrt{3}i)}{81}x^{\frac{7}{3}} - \frac{349(1 + \sqrt{3}i)}{486}x^{\frac{8}{3}} + 2x^3 + \cdots, \\ v_4 &= 1 - x - 2x^2 - 6x^3 - 21x^4 - 82x^5 - \cdots. \end{aligned}$$

By taking $v = v_j$, $j = 1, 2, 3$, we obtain

$$\frac{v_j}{x - v_j} A_1(x) + \frac{x - v_j - v_j x}{x v_j} B_1(x) - (A_2(x) + C_1(x)) + \frac{(v_j x + v_j - x)(v_j - x) + x^2}{x(1 - v_j)(v_j - x)} = 0$$

by solving this system for $A_1(x)$, $B_1(x)$, and $A_2(x) + C_1(x)$, we obtain

$$A_1 = \frac{v_1 v_2 v_3 (1 + x^2) + (v_1 + v_2 + v_3)(1 - x)x^2 + x^4 - (v_1 v_2 + v_1 v_3 + v_2 v_3)x}{x^2(1 - v_1)(1 - v_2)(1 - v_3)}.$$

Note that $v_1 + v_2 + v_3 + v_4 = 1 + 2x$, $v_1 v_2 v_3 v_4 = x^3$, and $v_4(v_1 + v_2 + v_3) + v_3(v_1 + v_2) + v_2 v_1 = x(3 + 2x)$. Thus, $v_3(v_1 + v_2) + v_2 v_1 = x(3 + 2x) - v_4(1 + 2x - v_4)$. Hence, the generating function A_1 can be written as

$$A_1 = \frac{x^2(x^2 + 1) - x(x^2 + x + 2)v_4 + (x^2 + x + 1)v_4^2 - v_4^3}{x(v_4^3 - 2xv_4^2 + (2x^2 + x)v_4 - x^3)}.$$

By using a computer programming, we can express v_4 in terms of A_1 , and by the fact that $K(v_4) = 0$, we see that the generating function A_1 satisfies $A_1 = x(1 + A_1)^2(1 + A_1^2)$. By Lagrange inversion formula, we complete the proof. \square

From Table 1, we see that the patterns 0101 and 0111 when paired with 021 have the same generating trees. Our next result enumerates these pattern pairs.

Theorem 3.2. *The generating functions $F_{\{021, 0101\}}(x)$ and $F_{\{021, 0111\}}(x)$ are given by $f(x)$, where $f(x) = \frac{x(1+f(x))}{1-x(1+f(x))^2}$. Moreover, for all $n \geq 1$,*

$$|\mathbf{I}_n(\{021, 0101\})| = |\mathbf{I}_n(\{021, 0111\})| = \sum_{i=1}^{n+1} \frac{1}{i} \binom{n}{i-1} \binom{2n+2-i}{i-1}.$$

Proof. By the generating tree succession rules for $\mathcal{T}'(\{021, 0101\})$ and $\mathcal{T}'(\{021, 0111\})$ in Table 1, we have

$$F_{\{021, 0101\}}(x) = F_{\{021, 0111\}}(x).$$

By translating the rules of the generating tree $\mathcal{T}'(\{021, 0101\})$ to generating function, we obtain

$$\begin{aligned} A(x, v) &= \frac{x}{1-v} + \frac{x}{v}(A(x, v) - A(x, 0)) + \frac{x}{1-v}B(x, v), \\ B(x, v) &= \frac{x}{1-v} + \frac{x}{v}(A(x, v) - A(x, 0)) + \frac{x}{v}(B(x, v) - B(x, 0)) + \frac{x}{1-v}B(x, v). \end{aligned}$$

Finding $A(x, v)$ from the first equation and substituting it into the second equation, we obtain

$$\begin{aligned} B(x, v) &= \frac{x}{1-v} + \frac{x}{v} \left(\frac{x(vA(x, 0) + vB(x, v) - A(x, 0) + v)}{(1-v)(v-x)} - A(x, 0) \right) \\ &\quad + \frac{x}{v}(B(x, v) - B(x, 0)) + \frac{x}{1-v}B(x, v). \end{aligned}$$

The kernel of this equation is given by $K(v) = \frac{v^3 - (1+x)v^2 + x(x+2)v - x^2}{(1-v)(v-x)v}$. The equation $K(v)$ has three roots, that is, $K(v_j) = 0$ for $j = 0, 1, 2$, where

$$v_j = \frac{2\sqrt{1-4x-2x^2}}{3} \cos \left(\frac{1}{3} \arccos \left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^3} \right) + \frac{(2j+1)\pi}{3} \right) + \frac{x+1}{3}.$$

By taking either $v = v_1$ or $v = v_2$ into the equation, we get a system of equations. By solving this system for $A(x, 0)$ and $B(x, 0)$, we obtain

$$A(x, 0) = \frac{(x+1)v_1v_2 - x(v_1+v_2)x}{x(1-v_1)(1-v_2)}.$$

Note that $v_1 + v_2 + v_3 = 1 + x$ and $v_1v_2v_3 = x^2$, we have

$$A(x, 0) = \frac{v_3^2 - (1+x)v_3 + x(1+x)}{v_3^2 - xv_3 + x^2}.$$

Since $K(v_3) = 0$, we have that $A(x, 0)$ satisfies the equation

$$A(x, 0) = \frac{x(1 + A(x, 0))}{1 - x(1 + A(x, 0))^2}.$$

Note that by Lagrange inversion formula, we obtain

$$A(x, 0) = \sum_{i \geq 1} \sum_{j \geq 0} \frac{1}{i} \binom{i-1+j}{i-1} \binom{i+2j}{i-1} x^{i+j},$$

which, by comparing coefficient of x^n , we complete the proof. \square

Theorem 3.3. *Let $B \in \{\{021, 1010\}, \{021, 1110\}\}$. Then,*

$$F_B(x) = \frac{x}{1-x} + \frac{x}{1-x}R(x),$$

with $R(x) = P(x)/Q(x)$ where

$$\begin{aligned} P(x) &= x(1-x)((x-1)\sqrt{1-4x}-3x+1) + ((1-x)^2\sqrt{1-4x}-x^2+4x-1)v_3 - x(\sqrt{1-4x}+1)v_3^2, \\ Q(x) &= x^2(x-1)(\sqrt{1-4x}-2x+1) - 2x((x-1)\sqrt{1-4x}-x^2+3x-1)v_3 - (1-4x+2x^2+(1-2x)\sqrt{1-4x})v_3^2 \text{ and } v_3 = \frac{2\sqrt{1-4x-2x^2}}{3} \cos\left(\frac{1}{3}\arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^3}\right) + \frac{\pi}{3}\right) + \frac{x+1}{3} = \\ &1-x-2x^2-\dots \text{ is one of the roots of the equation } v^3 - (vx+v-x)(v-x) = 0. \end{aligned}$$

Proof. By the generating tree succession rules for $\mathcal{T}'(\{021, 1010\})$ and $\mathcal{T}'(\{021, 1110\})$ in Table 1, we have

$$F_{\{021, 1010\}}(x) = F_{\{021, 1110\}}(x).$$

Define $A_m(x)$, $B_m(x)$, $C_m(x)$, $E(x)$, and $F(x)$ to be the generating functions for the number of nodes at level $n \geq 0$ in $\mathcal{T}'(\{021, 1010\}; a_m)$, $\mathcal{T}'(\{021, 1010\}; b_m)$, $\mathcal{T}'(\{021, 1010\}; c_m)$, $\mathcal{T}'(\{021, 1010\}; d_m)$, $\mathcal{T}'(\{021, 1010\}; e)$, and $\mathcal{T}'(\{021, 1010\}; f)$, respectively, where its root stays at level 0. Then

$$\begin{aligned} A_m(x) &= x + xA_{m+1}(x) + xB_1(x) + \dots + xB_m(x), \\ B_m(x) &= x + xC_m(x) + xB_1(x) + \dots + xB_{m+1}(x), \\ C_m(x) &= x + xC_{m+1}(x) + xD_m(x) + xB_1(x) + \dots + xB_{m+1}(x), \\ D_m(x) &= x + xD_1(x) + \dots + xD_{m+1}(x) + xE(x) + xF(x), \\ E(x) &= x + xD_1(x) + xE(x) + xF(x), \\ F(x) &= x + xE(x) + xF(x). \end{aligned}$$

Define $G(x, v) = \sum_{n \geq 1} G_n(x) v^{n-1}$ for all $G \in \{A, B, C, D\}$. Then the above recurrence can be written as

$$\begin{aligned} A(x, v) &= \frac{x}{1-v} + \frac{x}{v}(A(x, v) - A_1(x)) + \frac{x}{1-v}B(x, v), \\ B(x, v) &= \frac{x}{1-v} + xC(x, v) + \frac{x}{1-v}B(x, v) + \frac{x}{v}(B(x, v) - B_1(x)), \\ C(x, v) &= \frac{x}{1-v} + \frac{x}{v}(C(x, v) - C_1(x)) + xD(x, v) + \frac{x}{1-v}B(x, v) + \frac{x}{v}(B(x, v) - B_1(x)), \\ D(x, v) &= \frac{x}{1-v} + \frac{x}{v}(D(x, v) - D_1(x)) + \frac{x}{1-v}D(x, v) + \frac{x}{1-v}(E(x) + F(x)), \\ E(x) &= x + xD_1(x) + xE(x) + xF(x), \\ F(x) &= x + xE(x) + xF(x). \end{aligned}$$

By taking $v = \frac{1-\sqrt{1-4x}}{2}$, the last three equations lead to

$$\begin{aligned} D_1(x) &= \frac{(2x-1)\sqrt{1-4x} + 2x^2 - 4x + 1}{2x^3}, \\ E(x) &= \frac{(x-1)\sqrt{1-4x} - 3x + 1}{2x^2}, \\ F(x) &= \frac{1-2x-\sqrt{1-4x}}{2x}. \end{aligned}$$

By finding $C(x, v)$ from the third equation, then substituting it into the second equation, we obtain

$$\begin{aligned} &\frac{v^3 - (vx + v - x)(v - x)}{v(1-v)(x-v)}B(x, v) \\ &= \frac{x(vx + v - x)}{v(v-x)}B_1(x) + \frac{x^2}{x-v}C_1(x) - \frac{(vx - v - 2x + 1)v\sqrt{1-4x}}{2(v-x)(v^2 - v + x)} \\ &+ \frac{(2v^3x^2 - v^3x - 6v^2x^2 + 2vx^3 + v^3 + 5v^2x + 6vx^2 - 2x^3 - 2v^2 - 4vx + v)}{2(v^2 - v + x)(1-v)(v-x)}. \end{aligned}$$

Let v_1, v_2, v_3 be the four roots of $v^3 - (vx + v - x)(v - x) = 0$ as functions of x ,

$$v_j = \frac{2\sqrt{1-4x-2x^2}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^3}\right) + \frac{(2j+1)\pi}{3}\right) + \frac{x+1}{3}.$$

By taking $v = v_j$, $j = 1, 2$, we obtain

$$\begin{aligned} &\frac{x(v_jx + v_j - x)}{v_j(v_j - x)}B_1(x) + \frac{x^2}{x-v_j}C_1(x) - \frac{(v_jx - v_j - 2x + 1)v_j\sqrt{1-4x}}{2(v_j-x)(v_j^2 - v_j + x)} \\ &+ \frac{(2v_j^3x^2 - v_j^3x - 6v_j^2x^2 + 2v_jx^3 + v_j^3 + 5v_j^2x + 6v_jx^2 - 2x^3 - 2v_j^2 - 4v_jx + v_j)}{2(v_j^2 - v_j + x)(1-v_j)(v_j - x)} = 0. \end{aligned}$$

by solving this system for $B_1(x)$ and $C_1(x)$, we have explicit formulas for $B_1(x)$ and $C_1(x)$:

$$B_1(x) = \frac{x(2x^2(x^2 - 2x + 2) + x(x - 2)(\sqrt{1 - 4x} - 2x + 3)v_3 - (x - 1)(\sqrt{1 - 4x} - 2x + 3)v_3^2)(2x - 3 + \sqrt{1 - 4x})}{2(x^2 - 2x + 2)(x^2(x - 1)(\sqrt{1 - 4x} - 2x + 1) - 2x((x - 1)\sqrt{1 - 4x} - x^2 + 3x - 1)v_3 + (4x - 2x^2 - 1 + (2x - 1)\sqrt{1 - 4x})v_3^2)},$$

$$C_1(x) = \frac{x((1 + x)\sqrt{1 - 4x} + x - 1 - x(\sqrt{1 - 4x} + 1)v_3 + (\sqrt{1 - 4x} + 1)v_3^2)}{x^2(1 - x)(\sqrt{1 - 4x} - 2x + 1) + 2x((x - 1)\sqrt{1 - 4x} - x^2 + 3x - 1)v_3 + (1 - 4x + 2x^2 + (1 - 2x)\sqrt{1 - 4x})v_3^2}.$$

Hence, by the equation of $B(x, v)$, we obtain explicit formula for $B(x, v)$. In particular, we have $B(x, x) = R(x)$. Therefore, by equation of $A(x, v)$ with $v = x$, we have that $A_1(x) = \frac{x}{1-x} + \frac{x}{1-x}B(x, x)$, which completes the proof. \square

Remark 3.4. *By the proof of Theorem 3.3 and using that $v = v_3$ is a root of $v^3 - (vx + v - x)(v - x) = 0$, we get that $f = F_{\{021, 1010\}}(x)$ satisfies*

$$\begin{aligned} & x(x^6 - 6x^5 + 19x^4 - 32x^3 + 27x^2 - 9x + 1) \\ & + (x - 1)(6x^6 - 28x^5 + 58x^4 - 57x^3 + 35x^2 - 10x + 1)f \\ & + x(15x^6 - 76x^5 + 159x^4 - 170x^3 + 94x^2 - 23x + 2)f^2 \\ & + x(20x^6 - 84x^5 + 139x^4 - 117x^3 + 50x^2 - 11x + 1)f^3 + x^3(15x^4 - 46x^3 + 50x^2 - 21x + 3)f^4 \\ & + x^5(6x^2 - 10x + 3)f^5 + x^7f^6 = 0. \end{aligned}$$

Theorem 3.5. *The generating function $F_{\{021, 1012\}}(x)$ is given by*

$$\frac{((3x^2 - 3x + 1)v_3^2 + (-2x^3 + 2x - 1)v_3 + 2x^4 - 2x^2 + x)}{(1 - x)(1 - 2x)(v_3^2 - v_3x + x^2)},$$

where $v_3 = \frac{2\sqrt{1-4x-2x^2}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^3}\right) + \frac{\pi}{3}\right) + \frac{x+1}{3} = 1 - x - 2x^2 - \dots$ is a root of $v^3 - (1 + x)v^2 + x(2 + x)v - x^2 = 0$.

Proof. By translating the rules of the generating tree $\mathcal{T}'(\{021, 1012\})$ in Table 1 to generating functions, we have

$$\begin{aligned} A(x, v) &= \frac{x}{1-v} + \frac{x}{v}(A(x, v) - A(x, 0)) + \frac{x}{1-v}B(x, v), \\ B(x, v) &= \frac{x}{1-v} + xC(x, v) + \frac{x}{v}(B(x, v) - B(x, 0)) + \frac{x}{1-v}B(x, v), \\ C(x, v) &= \frac{x}{1-v} + \frac{x}{v}(C(x, v) - C(x, 0)) + \frac{x}{1-v}D(x) + \frac{x}{v}(B(x, v) - B(x, 0)) + \frac{x}{1-v}B(x, v), \\ D(x) &= \frac{x}{1-2x}. \end{aligned}$$

Finding $C(x, v)$ from the third equation and substituting its expression into the second equation, we obtain

$$\begin{aligned} B(x, v) &= \frac{x}{1-v} + \frac{x^2}{(1-\frac{x}{v})(1-v)} \left(1 + \frac{x}{1-2x} + \frac{1-v}{v}(B(x, v) - B(x, 0) - C(x, 0)) + B(x, v)\right) \\ &+ \frac{x}{v}(B(x, v) - B(x, 0)) + \frac{x}{1-v}B(x, v), \end{aligned}$$

The kernel of this equation is given by $K(v) = \frac{v^3 - (1+x)v^2 + x(2+x)v - x^2}{(1-v)(v-x)v}$. The equation $K(v) = 0$ has three roots, that is, $K(v_j) = 0$ for $j = 1, 2, 3$, where

$$v_j = \frac{2\sqrt{1-4x-2x^2}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^3}\right) + \frac{(2j+1)\pi}{3}\right) + \frac{x+1}{3}.$$

By taking either $v = v_1$ or $v = v_2$ into the equation, we get a system of equations. By solving this system for $B(x, 0)$ and $C(x, 0)$, we obtain

$$B(x, 0) = \frac{(x-1)^2 v_1 v_2}{x(1-2x)(1-v_1)(1-v_2)},$$

$$C(x, 0) = \frac{(2x^2-1)v_1 v_2 + x(1-x-x^2)(v_1+v_2) + 2x^3 - x^2}{x^2(1-2x)(1-v_1)(1-v_2)}.$$

Note that $v_1 + v_2 + v_3 = 1 + x$ and $v_1 v_2 v_3 = x^2$, we have

$$A(x, 0) = -\frac{((3x^2-3x+1)v_3^2 + (-2x^3+2x-1)v_3 + 2x^4-2x^2+x)}{(1-x)(1-2x)(v_3^2 - v_3x + x^2)},$$

as claimed. \square

Remark 3.6. *By the proof of Theorem 3.5 and using that $v = v_3$ is a root of $v^3 - v^2x + vx^2 - v^2 + 2vx - x^2 = 0$, we get that $f = F_{\{021, 1012\}}(x)$ satisfies*

$$\begin{aligned} & x(2x^2 - 2x + 1)(x^6 - 2x^5 + 13x^4 - 24x^3 + 19x^2 - 7x + 1) \\ & + (1-2x)(1-x)(x^7 - x^6 + 24x^5 - 51x^4 + 50x^3 - 27x^2 + 8x - 1)f \\ & + x(x^3 + 6x^2 - 6x + 2)(1-2x)^2(1-x)^2f^2 + x(1-2x)^3(1-x)^3f^3 = 0. \end{aligned}$$

Theorem 3.7. *The generating function $F_{\{021, 1202\}}(x)$ is given by*

$$\frac{x(2(1-3x)v_3^2 - (1-6x^2)v_3 + x(1+x)(1-3x))}{(1-x)(1-6x)(v_3^2 - xv_3 + x^2)},$$

where $v_3 = \frac{2\sqrt{1-4x-2x^2}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^3}\right) + \frac{\pi}{3}\right) + \frac{x+1}{3} = 1 - x - 2x^2 - \dots$ is a root of $v^3 - (1+x)v^2 + x(x+2)v - x^2 = 0$.

Proof. By translating the rules of the generating tree $\mathcal{T}'(\{021, 1202\})$ in Table 1 to generating functions, we have

$$\begin{aligned} A(x, v) &= \frac{x}{1-v} + \frac{x}{v}(A(x, v) - A(x, 0)) + \frac{x}{1-v}B(x, v), \\ B(x, v) &= \frac{x}{1-v} + \frac{x}{1-v}C(x, v) + \frac{2x}{v}(B(x, v) - B(x, 0)), \\ C(x, v) &= \frac{x}{1-v} + \frac{x}{v}(C(x, v) - C(x, 0)) + \frac{x}{1-v}C(x, v) + xD(x, v), \\ D(x, v) &= \frac{x}{1-v} + \frac{x}{1-v}C(x, v) + \frac{x}{v}(C(x, v) + D(x, v) - C(x, 0) - D(x, 0)). \end{aligned}$$

Finding $D(x, v)$ from the fourth equation and substituting its expression into the third equation, we obtain

$$C(x, v) = \frac{x}{1-v}(1 + C(x, v)) + \frac{x^2}{(1-v)(v-x)}((v-1)(C(x, 0) + D(x, 0)) + C(x, v) + v) + \frac{x}{v}(C(x, v) - C(x, 0)).$$

The kernel of this equation is given by $K(v) = \frac{v^3 - (1+x)v^2x + x(2+x)v - x^2}{(1-v)(v-x)v}$. The equation $K(v) = 0$ has three roots, namely, $K(v_j) = 0$ for $j = 1, 2, 3$, where

$$v_j = \frac{2\sqrt{1-4x-2x^2}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{2-12x+6x^2-7x^3}{2\sqrt{1-4x-2x^2}^3}\right) + \frac{(2j+1)\pi}{3}\right) + \frac{x+1}{3}.$$

By taking either $v = v_1$ or $v = v_2$ into the equation, we get a system of equations. By solving this system for $B(x, 0)$ and $C(x, 0)$, we obtain

$$C(x, 0) = \frac{v_1v_2}{x(1-v_1)(1-v_2)},$$

$$D(x, 0) = -\frac{((1+x)v_1-x)((1+x)v_2-x)}{x^2(1-v_1)(1-v_2)}.$$

Thus, by expressions of $C(x, 0)$ and $D(x, 0)$, we have explicit formulas for $C(x, v)$ and $D(x, v)$.

Hence, by considering the equation of $B(x, v)$ with $v = 2x$, we obtain

$$B(x, 0) = \frac{(1+3x)v_1v_2 - 3x(v_1+v_2) + x}{(1-6x)(1-v_1)(1-v_2)},$$

which, by using this expression and $C(x, v)$, we obtain an explicit formula for $B(x, v)$.

By considering the equation of $A(x, v)$ with $v = x$, we obtain

$$A(x, 0) = \frac{x}{1-x} + \frac{x}{1-x}B(x, x)$$

$$= \frac{(1-2x-6x^2)v_1v_2 - 2x(1-3x)(v_1+v_2) + x(1-4x)}{(1-x)(1-6x)(1-v_1)(1-v_2)}.$$

Note that $v_1 + v_2 + v_3 = 1 + x$ and $v_1v_2v_3 = x^2$, we have

$$A(x, 0) = \frac{x(2(1-3x)v_3^2 - (1-6x^2)v_3 - x(1+x)(1-3x))}{(1-x)(1-6x)(v_3^2 - v_3x + x^2)},$$

as claimed. \square

Remark 3.8. By the proof of Theorem 3.7 and using that v_3 is a root of $v^3 - (vx + v - x)(v - x) = 0$, we get that $f = F_{\{021, 1202\}}(x)$ satisfies

$$x^2(2x^2 - 2x + 1) - x(1-x)(8x^2 - 3x + 1)f + x(12x - 1)(1-x)^2f^2 + (1-6x)(1-x)^3f^3 = 0.$$

Before we study the pair of 021 and 0000 in Section 3.1, we present the following corollary that states explicit formulas for $|I_n(\{021, \tau\})|$ where τ is any 021-avoiding four-letter pattern except three cases, 1110, 1012 and 1202. For these three patterns, we could not succeed to find a nice explicit formulas. The formulas follow from determining the coefficient of the term x^n in the generating functions $F_{\{021, \tau\}}(x)$, where we omit the details.

Corollary 3.9. *For all $n \geq 0$,*

$$\begin{aligned}
|\mathbf{I}_n(\{021, 0001\})| &= \frac{(4n-25)(-1)^n}{32} - \frac{n(n+1)-1}{4} + \frac{1}{32}3^{n+4} \\
&\quad + \sum_{j=0}^{n+1} \left(\frac{(4j-39)(-1)^j}{32} + \frac{1}{4}j^2 - j + \frac{1}{2} - \frac{1}{32}3^{j+2} \right) M_{n+1-j}, \\
|\mathbf{I}_n(\{021, 0010\})| &= \binom{2n}{n}, \\
|\mathbf{I}_n(\{021, 0011\})| &= C_{n+2} + 1 - \sum_{j=0}^{n+1} C_j, \\
|\mathbf{I}_n(\{021, 0012\})| &= 2^{n+3} - \frac{(n+1)(2n^2+7n+24)}{6} - 3, \\
|\mathbf{I}_n(\{021, 0100\})| &= |\mathbf{I}_n(\{021, 0110\})| = \frac{n^2+n+6}{8(2n+3)(2n+5)} \binom{2n+6}{n+3}, \\
|\mathbf{I}_n(\{021, 0101\})| &= |\mathbf{I}_n(\{021, 0111\})| = \sum_{i=1}^{n+1} \frac{1}{i} \binom{n}{i-1} \binom{2n+2-i}{i-1}, \\
|\mathbf{I}_n(\{021, 0102\})| &= 2^{n+1} - \frac{(n+1)(n^2+2n+12)}{6} - 1 + \sum_{j=0}^{n+1} C_j, \\
|\mathbf{I}_n(\{021, 0112\})| &= C_{n+1} - 2^{n+1} + 1 + \sum_{j=0}^n 2^{n-j} C_j, \\
|\mathbf{I}_n(\{021, 0120\})| &= |\mathbf{I}_n(\{021, 0122\})| = \frac{1}{2} \binom{2n+2}{n+1} - \frac{1}{2} \sum_{j=1}^n \binom{2j}{j}, \\
|\mathbf{I}_n(\{021, 0123\})| &= 2^{n-1}(n^2-3n+4) + \frac{n(n+1)}{2} - 1, \\
|\mathbf{I}_n(\{021, 1000\})| &= |\mathbf{I}_n(\{021, 1100\})| \\
&= \frac{n^5+2n^4+23n^3+46n^2+120n+48}{2(n+1)(n+2)(n+3)(n+4)} \binom{2n}{n}, \\
|\mathbf{I}_n(\{021, 1001\})| &= |\mathbf{I}_n(\{021, 1011\})| = |\mathbf{I}_n(\{021, 1101\})| \\
&= \frac{1}{n+1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{j} \binom{2n+2}{n-2j}, \\
|\mathbf{I}_n(\{021, 1002\})| &= \frac{1}{2} \binom{2n+6}{n+3} - \frac{5}{2} \binom{2n+4}{n+2} + \frac{5}{2} \binom{2n+2}{n+1} + \frac{1}{2} \sum_{j=0}^n \binom{2j}{j} \\
&\quad + 2^{n+1} - \frac{1}{24}(n^4+2n^3+11n^2+34n+36), \\
|\mathbf{I}_n(\{021, 1020\})| &= |\mathbf{I}_n(\{021, 1022\})|
\end{aligned}$$

$$\begin{aligned}
&= \binom{2n+8}{n+4} - \frac{13}{2} \binom{2n+6}{n+3} + \frac{21}{2} \binom{2n+3}{n+2} - \frac{1}{2} \sum_{j=0}^{n+1} \binom{2j}{j} - \frac{1}{2}, \\
|\mathbf{I}_n(\{021, 1023\})| &= \sum_{j=0}^{n+1} (2^{j+1} - j - 1) C_{n+1-j} \\
&\quad + \frac{n(3n^3 + 22n^2 + 129n + 398)}{24} + 2^{n-1}(n^2 - 3n - 52) + 24, \\
|\mathbf{I}_n(\{021, 1102\})| &= \frac{1}{2} \binom{2n+6}{n+3} - \frac{21}{4} \binom{2n+4}{n+2} + \binom{2n+2}{n+1} + \frac{(n+1)^2}{2} - 2^n + 1 \\
&\quad + \frac{1}{2} \sum_{j=1}^{n+3} (2^{j-2} - 3j + 8) \binom{2n+6-2j}{n+3-j} \\
|\mathbf{I}_n(\{021, 1120\})| &= 4^n - \frac{n}{2(2n+3)} \binom{2n+4}{n+2}, \\
|\mathbf{I}_n(\{021, 1200\})| &= |\mathbf{I}_n(\{021, 1220\})| \\
&= \frac{n+4}{2(n+2)} \binom{2n+2}{n+1} + \sum_{j=0}^{n-1} (2j+1) \binom{2j}{j} - 4^n, \\
|\mathbf{I}_n(\{021, 1203\})| &= \frac{n+1}{24} (n^3 + n^2 - 2 * n - 108) + 2^{n-1}(n^2 - 11n + 28) - \frac{19}{2} \\
&\quad + \frac{1}{2} \binom{2n+2}{n+1} - \frac{1}{2} \sum_{j=2}^{n+1} (j-1) \binom{2n+2-2j}{n+1-j}, \\
|\mathbf{I}_n(\{021, 1220\})| &= \frac{n+4}{n+2} \binom{2n+1}{n} - 4^n + \sum_{j=0}^n (2j+1) \binom{2j}{j}, \\
|\mathbf{I}_n(\{021, 1230\})| &= \frac{1}{3} (2 \cdot 4^n + 1) + \sum_{j=1}^n \left(\frac{j}{2} - 2^{j-1} \right) \binom{2n+2-2j}{n+1-j}.
\end{aligned}$$

3.1. **Case $B = \{021, 0000\}$.** In this section, we study the pair of patterns 021 and 0000. The generating tree $\mathcal{T}'(\{021, 0000\})$ has a long list of succession rules and the solution to the equations for the generating functions requires some technical steps. By applying our algorithm, we found that the generating tree $\mathcal{T}'(\{021, 0000\})$ has the following set of rules. The root has the label 0 and the succession rules are the following:

$$\begin{array}{ll}
0 \rightsquigarrow 0^2, 01, & 0^2 \rightsquigarrow 0^3, 0^21, 0^22, \\
01 \rightsquigarrow 0^21, 01^2, 01, & 0^3 \rightsquigarrow 0^31, 0^32, 0^33, \\
0^21 \rightsquigarrow 0^31, 0^21^2, 0^21, 0^22, & 0^22 \rightsquigarrow 0^32, 0^22^2, 0^22, \\
01^2 \rightsquigarrow 0^21^2, 01^3, a_{121}(2), 01, & 0^31 \rightsquigarrow 0^31^2, 0^31, 0^32, 0^33, \\
0^32 \rightsquigarrow 0^32^2, 0^32, 0^33, & 0^33 \rightsquigarrow 0^33^2, 0^33, \\
0^21^2 \rightsquigarrow 0^31^2, 0^21^3, a_{221}(2), 0^21, 0^22, & 0^22^2 \rightsquigarrow 0^32^2, 0^22^3, 0^21, 0^22, \\
01^3 \rightsquigarrow 0^21^3, a_{131}(2), a_{121}(2), 01, & 0^31^2 \rightsquigarrow 0^31^3, a_{321}(2), 0^31, 0^32, 0^33,
\end{array}$$

$$\begin{aligned}
 0^3 2^2 &\rightsquigarrow 0^3 2^3, 0^3 1, 0^3 2, 0^3 3, & 0^3 3^2 &\rightsquigarrow 0^3, 0^3 2, 0^3 3, \\
 0^2 1^3 &\rightsquigarrow 0^3 1^3, a_{231}(2), a_{221}(2), 0^2 1, 0^2 2, & 0^2 2^3 &\rightsquigarrow 0^3 2^3, a_{221}(2), 0^2 1, 0^2 2, \\
 0^3 1^3 &\rightsquigarrow a_{331}(2), a_{321}(2), 0^3 1, 0^3 2, 0^3 3, & 0^3 2^3 &\rightsquigarrow a_{321}(2), 0^3 1, 0^3 2, 0^3 3
 \end{aligned}$$

such that for all $m \geq 2$,

$$\begin{aligned}
 a_{121}(m) &\rightsquigarrow a_{221}(m), a_{122}(m), a_{121}(m), \dots, a_{121}(2), a_{131}(m-1), \dots, a_{131}(2), 01, \\
 a_{122}(m) &\rightsquigarrow a_{222}(m), a_{123}(m), a_{131}(m), \dots, a_{131}(2), a_{121}(m), \dots, a_{121}(2), 01, \\
 a_{123}(m) &\rightsquigarrow a_{223}(m), a_{121}(m+1), \dots, a_{121}(2), a_{131}(m), \dots, a_{131}(2), 01, \\
 a_{131}(m) &\rightsquigarrow a_{231}(m), a_{132}(m), a_{121}(m), \dots, a_{121}(2), a_{131}(m), \dots, a_{131}(2), 01, \\
 a_{132}(m) &\rightsquigarrow a_{232}(m), a_{133}(m), a_{121}(m+1), \dots, a_{121}(2), a_{131}(m), \dots, a_{131}(2), 01, \\
 a_{133}(m) &\rightsquigarrow a_{233}(m), a_{121}(m+1), \dots, a_{121}(2), a_{131}(m+1), \dots, a_{131}(2), 01, \\
 a_{221}(m) &\rightsquigarrow a_{321}(m), a_{222}(m), a_{221}(m), \dots, a_{221}(2), a_{231}(m-1), \dots, a_{231}(2), 0^2 1, 0^2 2, \\
 a_{222}(m) &\rightsquigarrow a_{322}(m), a_{223}(m), a_{221}(m), \dots, a_{221}(2), a_{231}(m), \dots, a_{231}(2), 0^2 1, 0^2 2, \\
 a_{223}(m) &\rightsquigarrow a_{323}(m), a_{221}(m+1), \dots, a_{221}(2), a_{231}(m), \dots, a_{231}(2), 0^2 1, 0^2 2, \\
 a_{231}(m) &\rightsquigarrow a_{331}(m), a_{232}(m), a_{221}(m), \dots, a_{221}(2), a_{231}(m), \dots, a_{231}(2), 0^2 1, 0^2 2, \\
 a_{232}(m) &\rightsquigarrow a_{332}(m), a_{233}(m), a_{221}(m+1), \dots, a_{221}(2), a_{231}(m), \dots, a_{231}(2), 0^2 1, 0^2 2, \\
 a_{233}(m) &\rightsquigarrow a_{333}(m), a_{221}(m+1), \dots, a_{221}(2), a_{231}(m+1), \dots, a_{231}(2), 0^2 1, 0^2 2, \\
 a_{321}(m) &\rightsquigarrow a_{322}(m), a_{321}(m), \dots, a_{321}(2), a_{331}(m-1), \dots, a_{331}(2), 0^3 1, 0^3 2, 0^3 3, \\
 a_{322}(m) &\rightsquigarrow a_{323}(m), a_{321}(m), \dots, a_{321}(2), a_{331}(m), \dots, a_{331}(2), 0^3 1, 0^3 2, 0^3 3, \\
 a_{323}(m) &\rightsquigarrow a_{321}(m+1), \dots, a_{321}(2), a_{331}(m), \dots, a_{331}(2), 0^3 1, 0^3 2, 0^3 3, \\
 a_{331}(m) &\rightsquigarrow a_{332}(m), a_{321}(m), \dots, a_{321}(2), a_{331}(m), \dots, a_{331}(2), 0^3 1, 0^3 2, 0^3 3, \\
 a_{332}(m) &\rightsquigarrow a_{333}(m), a_{321}(m+1), \dots, a_{321}(2), a_{331}(m), \dots, a_{331}(2), 0^3 1, 0^3 2, 0^3 3, \\
 a_{333}(m) &\rightsquigarrow a_{331}(m+1), a_{321}(m+1), \dots, a_{321}(2), a_{331}(m), \dots, a_{331}(2), 0^3 1, 0^3 2, 0^3 3.
 \end{aligned}$$

where $a_{ijk}(m) = 0^i 1^3 \dots (m-2)^3 (m-1)^j m^k$ for all $m \geq 2$, $1 \leq i, k \leq 3$ and $j = 2, 3$.

Define

$$\begin{aligned}
 A_r &= A_r(x) = \sum_{n \geq 0} (\text{number of nodes at level } n \text{ in } \mathcal{T}'(B; r)) x^{n+1}, \\
 A_{ijk}(v) &= A_{ijk}(x; v) = \sum_{n \geq 0} (\text{number of nodes at level } n \text{ in } \mathcal{T}'(B; a_{ijk}(m))) x^{n+1} v^{m-2}.
 \end{aligned}$$

Then, by translating each rule of the generating tree $T'(B)$, we obtain the following sets of equations (we group them into three sets of equations):

System S1:

$$\begin{aligned}
 A_0 &= x + xA_{00} + xA_{01}, & A_{00} &= x + xA_{000} + xA_{001} + xA_{002}, \\
 A_{01} &= x + xA_{001} + xA_{011} + xA_{01}, & A_{000} &= x + xA_{0001} + xA_{0002} + xA_{0003}, \\
 A_{001} &= x + xA_{0001} + xA_{0011} + xA_{001} + xA_{002}, & A_{002} &= x + xA_{0002} + xA_{0022} + xA_{002}, \\
 A_{011} &= x + xA_{0011} + xA_{0111} + xA_{121}(0) + xA_{01}, & A_{0001} &= x + xA_{00011} + xA_{0001} + xA_{0002} + xA_{0003}, \\
 A_{0002} &= x + xA_{00022} + xA_{0002} + xA_{0003}, & A_{0003} &= x + xA_{00033} + xA_{0003}, \\
 A_{0011} &= x + xA_{00011} + xA_{00111} + xA_{221}(0) & A_{0022} &= x + xA_{00022} + xA_{00222} + xA_{001} + xA_{002},
 \end{aligned}$$

$$\begin{aligned}
& + xA_{001} + xA_{002}, \\
A_{00111} &= x + xA_{000111} + x(A_{131}(0) + A_{121}(0)) + xA_{01}, & A_{00011} &= x + xA_{000111} + xA_{321}(0) + xA_{0001} \\
& & & + xA_{0002} + xA_{0003}, \\
A_{00022} &= x + xA_{000222} + xA_{0001} + xA_{0002} + xA_{0003}, & A_{00033} &= x + xA_{000} + xA_{0002} + xA_{0003}, \\
A_{000111} &= x + xA_{000111} + x(A_{231}(0) + A_{221}(0)) & A_{000222} &= x + xA_{000222} + xA_{221}(0) + xA_{001} \\
& + xA_{001} + xA_{002}, & & + xA_{002}, \\
A_{000111} &= x + x(A_{331}(0) + A_{321}(0)) + xA_{0001} & A_{000222} &= x + xA_{321}(0) + xA_{0001} + xA_{0002} \\
& + xA_{0002} + xA_{0003}, & & + xA_{0003},
\end{aligned}$$

System S2:

$$\begin{aligned}
A_{121}(v) &= \frac{x}{1-v} + xA_{221}(v) + xA_{122}(v) + \frac{x}{1-v}(A_{121}(v) + vA_{131}(v) + A_{01}), \\
A_{122}(v) &= \frac{x}{1-v} + xA_{222}(v) + xA_{123}(v) + \frac{x}{1-v}(A_{131}(v) + A_{121}(v) + A_{01}), \\
A_{131}(v) &= \frac{x}{1-v} + xA_{231}(v) + xA_{132}(v) + \frac{x}{1-v}(A_{131}(v) + A_{121}(v) + A_{01}), \\
A_{133}(v) &= \frac{x}{1-v} + xA_{233}(v) + \frac{x}{v}(A_{131}(v) + A_{121}(v) - A_{131}(0) - A_{121}(0)) \\
& + \frac{x}{1-v}(A_{131}(v) + A_{121}(v) + A_{01}), \\
A_{221}(v) &= \frac{x}{1-v} + xA_{321}(v) + xA_{222}(v) + \frac{x}{1-v}(A_{221}(v) + vA_{231}(v) + A_{001} + A_{002}), \\
A_{222}(v) &= \frac{x}{1-v} + xA_{322}(v) + xA_{223}(v) + \frac{x}{1-v}(A_{231}(v) + A_{221}(v) + A_{001} + A_{002}), \\
A_{231}(v) &= \frac{x}{1-v} + xA_{331}(v) + xA_{232}(v) + \frac{x}{1-v}(A_{231}(v) + A_{221}(v) + A_{001} + A_{002}), \\
A_{233}(v) &= \frac{x}{1-v} + xA_{333}(v) + \frac{x}{v}(A_{231}(v) + A_{221}(v) - A_{231}(0) - A_{221}(0)) \\
& + \frac{x}{1-v}(A_{231}(v) + A_{221}(v) + A_{001} + A_{002}), \\
A_{321}(v) &= \frac{x}{1-v} + xA_{322}(v) + \frac{x}{1-v}(A_{321}(v) + vA_{331}(v) + A_{0001} + A_{0002} + A_{0003}), \\
A_{322}(v) &= \frac{x}{1-v} + xA_{323}(v) + \frac{x}{1-v}(A_{331}(v) + A_{321}(v) + A_{0001} + A_{0002} + A_{0003}), \\
A_{331}(v) &= \frac{x}{1-v} + xA_{332}(v) + \frac{x}{1-v}(A_{331}(v) + A_{321}(v) + A_{0001} + A_{0002} + A_{0003}), \\
A_{333}(v) &= \frac{x}{1-v} + \frac{x}{v}(A_{331}(v) + A_{321}(v) - A_{331}(0) - A_{321}(0)) \\
& + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0001} + A_{0002} + A_{0003}),
\end{aligned}$$

and

System S3:

$$(3.1) \quad A_{123}(v) = \frac{x}{1-v} + xA_{223}(v) + \frac{x}{v}(A_{121}(v) - A_{121}(0)) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{01}),$$

$$A_{132}(v) = \frac{x}{1-v} + xA_{232}(v) + xA_{133}(v) + \frac{x}{1-v}A_{121}(v) + \frac{x}{v}(A_{121}(v) - A_{121}(0))$$

$$(3.2) \quad + \frac{x}{1-v}(A_{131}(v) + A_{01}),$$

$$(3.3) \quad A_{223}(v) = \frac{x}{1-v} + xA_{323}(v) + \frac{x}{v}(A_{221}(v) - A_{221}(0)) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{001} + A_{002}),$$

$$A_{232}(v) = \frac{x}{1-v} + xA_{332}(v) + xA_{233}(v) + \frac{x}{v}(A_{221}(v) - A_{221}(0))$$

$$(3.4) \quad + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{001} + A_{002}),$$

$$(3.5) \quad A_{323}(v) = \frac{x}{1-v} + \frac{x}{v}(A_{321}(v) - A_{321}(0)) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0001} + A_{0002} + A_{0003}),$$

$$(3.6) \quad \begin{aligned} A_{332}(v) &= \frac{x}{1-v} + xA_{333}(v) + \frac{x}{v}(A_{321}(v) - A_{321}(0)) \\ &+ \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0001} + A_{0002} + A_{0003}). \end{aligned}$$

In order to find the generating function $A_0 = F_{\{021,0000\}}(x)$, we have to solve a system that is obtained from S1-S2-S3. Since the expressions are too long to present here, we only describe the algorithm that leads to an explicit formula for A_0 . For a Maple worksheet file for finding A_0 , we refer the reader to [18].

Step 1: We use System **S2** to find formulas for $A_{121}(v)$, $A_{122}(v)$, $A_{131}(v)$, $A_{133}(v)$, $A_{221}(v)$, $A_{222}(v)$, $A_{231}(v)$, $A_{233}(v)$, $A_{321}(v)$, $A_{322}(v)$, $A_{331}(v)$, and $A_{333}(v)$ in terms of $A_{123}(v)$, $A_{123}(0)$, $A_{132}(v)$, $A_{132}(0)$, $A_{223}(v)$, $A_{223}(0)$, $A_{232}(v)$, $A_{232}(0)$, $A_{323}(v)$, $A_{323}(0)$, $A_{332}(v)$, and $A_{332}(0)$. We denote this set of expressions by **E1**.

Step 2: By substituting expressions of **E1** into all equations of System **S3**, and then using Gauss elimination method with respect to the six variables $A_{123}(v)$, $A_{132}(v)$, $A_{223}(v)$, $A_{232}(v)$, $A_{323}(v)$, and $A_{332}(v)$, we obtain System **NS3**.

Step 3: Define

$$K(v) = x^6 + x^3(x-2)v + (1-2x-x^2)v^2 - v^3.$$

The equation $A_{332}(v)$ in System **NS3** is given by

$$(3.7) \quad \begin{aligned} K(v)A_{332}(v) &= \frac{(v+2x-1)vx^3}{1-2x}A_{323}(0) + \frac{(-2x^4+x^3+v^2+2vx-v)x^3}{1-2x}A_{332}(0) \\ &+ \frac{xv^2}{1-2x}(1+A_{0001}+A_{0002}+A_{0003}). \end{aligned}$$

We use the kernel method to solve this equation. Note that the roots of $K(v) = 0$ are given by

$$v_j = \frac{2\sqrt{4x^4-2x^3+2x^2-4x+1}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{16x^6-12x^5+27x^4-10x^3+18x^2-12x+2}{2\sqrt{4x^4-2x^3+2x^2-4x+1}^3}\right) + \frac{2\pi j}{3}\right) + \frac{1-2x+x^2}{3},$$

where $j = 0, 1, 2$. Note that the first terms in the power series (in the variable \sqrt{x}) of v_j are given by

$$\begin{aligned} v_0 &= 1 - 2x - x^2 - 2x^3 - 3x^4 - 8x^5 - 22x^6 - 62x^7 - 182x^8 - 548x^9 + \dots, \\ v_1 &= x^3 - x^{\frac{7}{2}} + \frac{3}{2}x^4 - \frac{21}{8}x^{\frac{9}{2}} + 4x^5 - \frac{839}{128}x^{\frac{11}{2}} + 11x^6 - \frac{18733}{1024}x^{\frac{13}{2}} + 31x^7 - \dots, \\ v_2 &= x^3 + x^{\frac{7}{2}} + \frac{3}{2}x^4 + \frac{21}{8}x^{\frac{9}{2}} + 4x^5 + \frac{839}{128}x^{\frac{11}{2}} + 11x^6 + \frac{18733}{1024}x^{\frac{13}{2}} + 31x^7 + \dots. \end{aligned}$$

Note that there are two relations for these three roots

$$(3.8) \quad v_0 + v_1 + v_2 = 1 - 2x - x^2 \text{ and } v_0v_1v_2 = x^6.$$

By taking either $v = v_1$ or $v = v_2$ into (3.7), we obtain a system of two equations with variables $A_{323}(0)$ and $A_{332}(0)$. By solving the system and using (3.8), we obtain

$$A_{323}(0) = \frac{(1+A_{0001}+A_{0002}+A_{0003})(v_0(1-2x-x^2)-v_0^2-x^3)}{x^2v_0(x^2+v_0)}$$

and

$$A_{332}(0) = \frac{x(1+A_{0001}+A_{0002}+A_{0003})}{x^2v_0(x^2+v_0)}.$$

By substituting expressions of $A_{323}(0)$ and $A_{332}(0)$ into equations of $A_{323}(v)$ and $A_{332}(v)$ of **NS3** and solving for $A_{323}(v)$ and $A_{332}(v)$, we obtain

$$\begin{aligned}
A_{323}(v) &= \frac{(x^6 - 2x^5 - x^3 - (2x^3 - 2x^2 + 3x - 1)v - (x^2 - x + 1)v^2)(1 + A_{0001} + A_{0002} + A_{0003})}{xK(v)} \\
&\quad - \frac{(2x^6 - 2x^5 + 3x^4 - x^3 + (x^4 + 6x^2 - 5x + 1)v - (x^2 - 3x + 1)v^2)(1 + A_{0001} + A_{0002} + A_{0003})v_0}{x^4K(v)} \\
&\quad - \frac{(x^5 - x^4 + x^3 - (x^2 - 3x + 1)v + (1 - x)v^2)(1 + A_{0001} + A_{0002} + A_{0003})v_0^2}{x^4K(v)} \\
A_{332}(v) &= \frac{(x^6 - 2x^4 + 2x^3 - (2x^3 - 2x^2 - 2x + 1)v + (1 - x^2)v^2)(1 + A_{0001} + A_{0002} + A_{0003})}{xK(v)} \\
&\quad - \frac{(2x^5 - 2x^4 - 2x^3 + x^2 - (4x^2 - 4x + 1)v + (1 - 2x)v^2)(1 + A_{0001} + A_{0002} + A_{0003})v_0}{x^3K(v)} \\
&\quad - \frac{(x^4 - x^2 + (1 - 2x)v - v^2)(1 + A_{0001} + A_{0002} + A_{0003})v_0^2}{x^3K(v)}
\end{aligned}$$

Step 4: We will not the presentation exact expressions because most of them are quite long. For the next step, we consider the equation of $A_{232}(v)$ in System **NS3**. After using the expressions of $A_{323}(v)$, $A_{323}(0)$, $A_{332}(v)$, and $A_{332}(0)$, we obtain an equation of the form

$$K(v)^2 A_{232}(v) = \alpha(v) + \beta(v)A_{232}(0) + \gamma(v)A_{223}(0),$$

which leads to

$$2K(v)K'(v)A_{232}(v) + K^2(v)A'_{232}(v) = \alpha'(v) + \beta'(v)A_{232}(0) + \gamma'(v)A_{223}(0),$$

where $f'(v)$ denotes the derivative of f with respect to v . By substituting either $v = u_1$ or $v = u_2$, we obtain a system of two equations with the variables $A_{232}(0)$ and $A_{223}(0)$. By solving this system and using (3.8), we obtain explicit formulas for $A_{232}(0)$ and $A_{223}(0)$ in terms of the root v_0 and x . Then, by substituting expressions of $A_{323}(0)$, $A_{332}(0)$, $A_{223}(0)$, $A_{232}(0)$, $A_{323}(v)$, and $A_{332}(v)$ into the equations of $A_{223}(v)$ and $A_{232}(v)$ of **NS3** and solving for $A_{223}(v)$ and $A_{232}(v)$, we derive explicit formulas for $A_{223}(v)$ and $A_{232}(v)$ in terms of the root v_0 and v, x .

Step 5: Now, we consider the equation of $A_{132}(v)$ in System **NS3**. After using the expressions of $A_{323}(v)$, $A_{323}(0)$, $A_{332}(v)$, $A_{332}(0)$, $A_{223}(v)$, $A_{223}(0)$, $A_{232}(v)$, and $A_{232}(0)$, we obtain an equation of the form

$$K(v)^3 A_{132}(v) = \alpha(v) + \beta(v)A_{132}(0) + \gamma(v)A_{123}(0),$$

which leads to

$$(K^3(v)A_{132}(v))'' = \alpha''(v) + \beta''(v)A_{132}(0) + \gamma''(v)A_{123}(0).$$

By substituting either $v = u_1$ or $v = u_2$, we obtain a system of two equations with the variables $A_{132}(0)$ and $A_{123}(0)$. By solving this system and using (3.8), we obtain explicit formulas for $A_{132}(0)$ and $A_{123}(0)$ in terms of the root v_0 and x . Then, by substituting expressions of $A_{323}(0)$, $A_{332}(0)$, $A_{223}(0)$, $A_{232}(0)$, $A_{132}(0)$, $A_{123}(0)$, $A_{323}(v)$, $A_{332}(v)$, $A_{223}(v)$, and $A_{232}(v)$ into the equations of $A_{123}(v)$ and $A_{132}(v)$ of **NS3** and solving for $A_{123}(v)$ and $A_{132}(v)$, we derive explicit formulas for $A_{123}(v)$ and $A_{132}(v)$ in terms of the root v_0 and v, x .

All the explicit expressions of these generating functions can be found in the following Maple worksheet [18].

Step 6: Up to now, we have explicit formulas for the generating functions $A_r(v)$ (also for $A_r(0)$), for all $r \in \{123, 132, 223, 232, 323, 332\}$. By the expressions of **E1**, we get explicit formulas for all $A_r(v)$ which leads to explicit expressions for $A_r(0)$, where $r = ijk$ and $1 \leq i, k \leq 3$ and $j = 1, 2$.

Step 7: At the last step, we solve System **S1** by using expressions $A_r(0)$ from the previous step. In particular, we obtain an explicit formula for A_0 .

Theorem 3.10. *The generating function $F_{\{021,0000\}}(x)$ is given by*

$$\begin{aligned} & - \frac{256x^9 + 560x^7 + 132x^6 + 792x^5 + 168x^4 + 235x^3 - 115x^2 - 31x + 10}{x^2(16x^3 + 8x^2 + 11x - 4)^2} \\ & + \frac{(256x^9 - 576x^8 - 440x^7 - 1306x^6 - 104x^5 - 85x^4 + 335x^3 + 18x^2 - 62x + 10)v_0}{x^5(16x^3 + 8x^2 + 11x - 4)^2} \\ & + \frac{(256x^8 + 112x^6 - 452x^5 - 198x^4 - 191x^3 + 56x^2 + 42x - 10)v_0^2}{x^5(16x^3 + 8x^2 + 11x - 4)^2} \\ & = x + 2x^2 + 6x^3 + 21x^4 + 78x^5 + 296x^6 + 1126x^7 + 4285x^8 + 16281x^9 + 61690x^{10} \\ & + 233078x^{11} + 878164x^{12} + 3299936x^{13} + 12370320x^{14} + \dots \end{aligned}$$

The computations used in Steps 1-7 are programmed in [18].

4. CONCLUDING REMARKS

Table 1 presents the generating trees for all the cases $\mathbf{I}_n(\{021, \tau\})$ whenever τ is a four-letter pattern that avoids 021. Moreover, the table includes the explicit formulas for the corresponding generating functions $F_{\{021, \tau\}}(x)$. We see that Conjecture 2.2 holds for length-four patterns.

There are 106 patterns of length five that avoid 021. We applied our algorithm to each class and found that the Conjecture 2.2 holds for 021-avoiding five-letter patterns. More precisely, we determined the generating trees and obtained explicit formulas for the generating functions $F_{\{021, \tau\}}(x)$ whenever τ is a five-letter pattern that avoids 021. Since the computations are too long, especially for the following four cases $\tau = 00000, 00001, 00011, 00012$, we decided not to present them here. But the explicit formulas of the generating functions for five-letter pattern cases are available in [22].

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