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# Hybrid Observer-based Asymptotic Stabilization of Non-uniformly Observable Systems: a Case Study

M. Maghenem W. Pasillas-Lépine A. Loría M. Aguado-Rojas

**Abstract**—We solve an open problem of output-feedback stabilization for a nonlinear system that is not uniformly observable at the target equilibrium. The plant’s model is of interest not only because it is an instance of the well-studied class of bilinear non-uniformly observable systems, but also because it represents the dynamics of an antiblock-braking system (ABS). To stabilize the origin, where the system is unobservable, we use a switched observer previously reported and a novel hybrid controller. The former relies on making the system operate in a regime that induces enough excitation to ensure the state estimation, while the latter is based on tracking a vanishing piecewise-constant reference. Our main statement establishes asymptotic stability of the origin in a hybrid sense.

**Index Terms**—Non-uniformly observable systems, observers, hybrid control, ABS.

## I. INTRODUCTION

Observer-based output-feedback stabilization of systems that are non-uniformly observable, beyond its academic interest [1]–[4], is well-justified by concrete engineering applications in which the system is not observable at the target equilibrium. This is the case, for instance, in sensorless motor control [5], bioreactor systems [6], and automotive applications [7].

The problem remains largely open in full generality, even when restricted to the case of observer design only. A particularly well-studied class of systems affine in the unmeasurable state, *i.e.*, given by the equation  $\dot{z} = A(\cdot)z$ , where  $A$  is a nonlinear function of known variables and  $y = C(\cdot)z$  is a measured output. For such systems, under certain assumptions, one may successfully mimic a classical Luenberger observer and update the state’s estimate  $\hat{z}$  using  $\dot{\hat{z}} = A(\cdot)\hat{z} - L(\cdot)y$ , whether  $A$  is a function of time [8], of an input  $u$  [9], of an input and [10], [11] or of all of the above [12]. For instance, a strategy for observer design, recurrently used at least since [9], is to make the system operate in a way that the measured output inject persistency of excitation into the observer dynamics. Although efficient for state estimation, this method is in clear conflict with the control goal if the target equilibrium includes zeroing the output.

This is the case of the system of equations

$$\dot{z}_1 = -az_1z_2 + u \quad (1a)$$

$$\dot{z}_2 = (cz_2 + d)z_1, \quad z_1, z_2 \in \mathbb{R}, \quad (1b)$$

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where  $a, c$ , and  $d > 0$ ,  $u$  is the control input,  $z_2$  is the main state of interest to be controlled, which is unmeasurable, and  $y = Cz = [1 \ 0]z = z_1$  is the measured output.

The equations (1) model the dynamics of the so-called extended-braking stiffness [7], [13]. This variable, represented by  $z_2$  in (1b), is a state whose regulation translates into maximizing the braking force in the antiblock braking system (ABS) of a vehicle’s rolling tire [14]. Stabilizing the origin  $\{(z_1, z_2) = (0, 0)\}$  for (1), measuring only  $z_1$  (which tantamounts to the linear acceleration of the tire at the wheel-ground contact point, relative to the longitudinal acceleration of the vehicle) is the open problem that we solve here. In previous works [7], [13], [15] we presented a switched observer that successfully estimates the plant’s states. Essentially, the observer is of Luenberger type, with a gain designed to commute between values that are appropriate for either of two dynamics, depending on the sign of  $z_1$ . Then, persistency of excitation of the latter and a dwell condition are used.

A way to excite the system is via hybrid control, by constructing a switching reference for the plant’s state [13], [16], but this, of course, prevents stabilization. To overcome this conundrum, in this paper we propose a novel hybrid controller that relies on the exponentially convergent observer previously proposed in [7], [13] and on the construction of a piecewise-constant, decreasing reference (more precisely, a function taking values in a discrete set formed by the elements of a decreasing sequence). Thus, the controller is hybrid in nature and successfully stabilizes the origin semiglobally and asymptotically. This Technical Note builds upon a long-standing work by the second author, devoted entirely to the observer-design problem [7], [13], [15], [17]. Relative to these references, in addition to the hybrid controller, we provide explicit exponential bounds on the observer’s estimation errors. The reference [18] is a shorter version of this Note, containing numerical simulations in place of the proofs of our main results—see also the available preprint [19].

## II. SWITCHED-OBSERVER DESIGN

We revisit a switched observer for system (1) from [7], [17] and we provide an explicit estimation of the convergence rate. Solely for the purpose of observer design, we introduce a hypothesis on the measured output  $z_1$  that relaxes the hypotheses imposed in [17].

*Assumption 1:* There exist positive constants  $\tau_d, \tau_s, \underline{z}$ , and  $\bar{z}$ , and an infinite union of disjoint intervals, denoted  $I_d$ , such that:

- $|z_1(t)| \geq \underline{z}$  for all  $t \in I_d$ ,
- $|z_1(t)| \leq \bar{z}$  for all  $t \in \mathbb{R}_{\geq 0} \setminus I_d$ ,

- the length of each connected interval in  $I_d$  is no smaller than  $\tau_d$ , and
- the length of each connected interval in  $\mathbb{R}_{\geq 0} \setminus I_d$  is smaller than  $\tau_s$ .

Then, let  $\hat{z} := (\hat{z}_1, \hat{z}_2)$  denote the estimate of  $z := [z_1 \ z_2]^\top$  and consider the observer

$$\begin{aligned}\dot{\hat{z}}_1 &= -a z_1 \hat{z}_2 - u + k_1(z_1) z_1 (z_1 - \hat{z}_1) \\ \dot{\hat{z}}_2 &= c z_1 \hat{z}_2 + d z_1 + k_2(z_1) z_1 (z_1 - \hat{z}_1),\end{aligned}$$

where

$$k_i(z_1) := \begin{cases} k_i^+ & \text{if } z_1 > 0 \\ k_i^- & \text{if } z_1 < 0 \\ 0 & \text{if } z_1 = 0, \end{cases} \quad i \in \{1, 2\}.$$

and, following [13],  $k_1^+ > c$ ,  $k_2^+ < -\frac{c}{a} k_1^+$ ,  $k_2^- < -\frac{c}{a} k_1^-$ ,  $k_1^- = 2c - k_1^+$ ,  $k_1^+ + a k_2^+ = c k_1^- + a k_2^-$ . Then, the dynamics of the estimation error  $\tilde{z}_i := \hat{z}_i - z_i$  is given by the linear time-varying system

$$\begin{bmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{bmatrix} = z_1(t) \begin{bmatrix} -k_1(z_1(t)) & -a \\ -k_2(z_1(t)) & c \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}, \quad (3)$$

which depends on the measurable output trajectory  $t \mapsto z_1(t)$ .

Next, for the purpose of analysis, we introduce the new time scale—cf. [17],

$$\tau := \int_{t_o}^t |z_1(s)| ds =: f_{z_1}(t), \quad (4)$$

which, under Assumption 1, is well defined since there exist  $\mu$  and  $T > 0$  such that

$$\int_{t_o}^t |z_1(s)| ds \geq \mu(t - t_o) \quad \forall t \geq t_o + T, \quad \forall t_o \geq 0. \quad (5)$$

In the new time-scale, system (3) is equivalent to

$$\tilde{w}' := \frac{d\tilde{w}}{d\tau} = A(w_1(\tau)) \tilde{w} \quad \forall \tau \in \text{Im}(f_{z_1}), \quad (6)$$

where  $w_1 : \text{Im}(f_{z_1}) \rightarrow \mathbb{R}$  and  $\tilde{w} : \text{Im}(f_{z_1}) \rightarrow \mathbb{R}^2$  are given by

$$w_1(\tau) := \begin{cases} z_1(f_{z_1}^{-1}(\tau)) & \text{if } \text{card}(f_{z_1}^{-1}(\tau)) = 1 \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

$$\tilde{w}(\tau) := \begin{cases} \tilde{z}(f_{z_1}^{-1}(\tau)) & \text{if } \text{card}(f_{z_1}^{-1}(\tau)) = 1 \\ \tilde{z}(\min\{f_{z_1}^{-1}(\tau)\}) & \text{otherwise,} \end{cases} \quad (8)$$

and

$$A(w_1) := \begin{cases} A_1 & \text{if } w_1 > 0 \\ A_2 & \text{if } w_1 < 0 \\ 0 & \text{if } w_1 = 0, \end{cases}$$

where  $A_1$  and  $A_2$  correspond, respectively, to the matrix in (3) with  $k_i = k_i^+$  and  $k_i = k_i^-$ , and  $\text{card}(\cdot)$  denotes the cardinality of  $(\cdot)$ .

Note that, by definition,  $w_1(\tau) = 0$  only for  $\tau$  in a null-measure set. Hence,

$$A(w_1(\tau)) \in \{A_1, A_2\} \quad \text{for almost all } \tau \in \text{Im}(f_{z_1}).$$

Furthermore, under Assumption 1,  $\text{Im}(f_{z_1}) = \mathbb{R}_{\geq 0}$  and  $\tau \mapsto w_1(\tau)$  has the following property,

*Property 1:* There exist an infinite union of disjoint intervals, denoted  $\bar{I}_d$ , such that

- $|w_1(\tau)| > 0$  for all  $t \in \bar{I}_d$ ,
- the length of each connected interval in  $\bar{I}_d$  is no smaller than  $(\tau_d \bar{z})$ ,
- the length of each connected interval in  $\mathbb{R}_{\geq 0} \setminus \bar{I}_d$  is smaller than  $(\tau_s \bar{z})$ .

*Lemma 1:* If the non-autonomous system (6) possesses Property 1, the trajectories of (6) satisfy the uniform exponential stability bound

$$|\tilde{w}(\tau)| \leq \kappa_1 |\tilde{w}(0)| \exp^{-\kappa_2 \tau} \quad \forall \tau \geq 0,$$

where, given  $\lambda > 0$ ,  $\kappa_1$  and  $\kappa_2$  are defined as

$$\kappa_1 := \gamma \frac{\bar{k} + 2\bar{c}^2 \gamma}{\rho(1 + \bar{k})}, \quad \kappa_2 := -\frac{\ln(\rho)}{L}, \quad (9)$$

where

$$\gamma := \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}, \quad \bar{c} := \exp^{\max\{|A_1|, |A_2|\} \bar{z} \tau_s}, \quad (10a)$$

$$k := \max\{|K_1|, |K_2|\}, \quad \bar{k} := \frac{\lambda_{\max}(P) \bar{c}^2 k^2}{\lambda}, \quad (10b)$$

$L$  is such that

$$\rho := \frac{2\gamma \bar{c}^2 \exp^{-2\lambda L} + \bar{k}}{1 + \bar{k}} < 1, \quad (11)$$

and  $K_1, K_2 \in \mathbb{R}^2$  are such that, for each  $i \in \{1, 2\}$ , we have

$$|\exp^{(A_i + K_i C)\tau}| \leq \frac{1}{\bar{c}} \exp^{-2\lambda(\tau - \frac{\tau_d \bar{z}}{2})} \quad \forall \tau \geq \frac{\tau_d \bar{z}}{2}. \quad (12)$$

*Remark 1:* In view of Lemma 1, under Assumption 1—see (5), we have

$$\begin{aligned} |\tilde{z}(t)| &\leq \kappa_1 |\tilde{z}(t_o)| \exp^{-\kappa_2 \int_{t_o}^t |z_1(s)| ds} \\ &\leq \kappa_1 |\tilde{z}(t_o)| \exp^{-\kappa_2 \mu(t - t_o)} \quad \forall t \geq t_o + T, \quad t_o \geq 0. \end{aligned} \quad (13)$$

*Remark 2:* According to [20, Lemma 9] and when the pairs  $(A_1, C)$  and  $(A_2, C)$  are observable, we can always compute  $K_1 \in \mathbb{R}^2$  and  $K_2 \in \mathbb{R}^2$  such that, for each  $i \in \{1, 2\}$ , (12) holds.

*Proof of Lemma 1:* Let [17, Theorem 1] generate a positive definite matrix  $P \in \mathbb{R}^{2 \times 2}$  such that

$$A_i^\top P + P A_i = -C^\top C \quad \forall i \in \{1, 2\},$$

where  $C := [1 \ 0]$ . Then, the derivative of

$$V_{obs}(\tilde{w}) := \tilde{w}^\top P \tilde{w}$$

along the solutions to the switched linear system in (6) verifies

$$V'_{obs}(\tilde{w}) = -\tilde{w}^\top C^\top C \tilde{w},$$

which implies local stability of the origin for (6). As a result, system (6) is a linear system that switches between two modes defined by the matrices  $A_1$  and  $A_2$ , which are both Hurwitz, admit a common (non-strict) Lyapunov function  $V_{obs}$ , and the pairs  $(A_1, C)$  and  $(A_2, C)$  are observable.

After the proof of [20, Lemma 5], we conclude that there exists a map  $w_1 \mapsto K(w_1) \in \{K_1, K_2, 0\}$  such that, along each map  $\tau \mapsto w_1(\tau)$  enjoying Property 1, we have

$$|\Phi_{\bar{z}_1}(\tau, 0)| \leq \bar{c} \exp^{-\lambda \tau} \quad \forall \tau \geq 0,$$

where  $\Phi_{\bar{z}_1}$  is the transition matrix of the system

$$\tilde{w}' = [A(w_1(\tau)) + K(w_1(\tau))C] \tilde{w}.$$

Now, using the proof of [20, Theorem 4], we conclude that

$$|\tilde{w}(0)| \leq \kappa_1 \rho^{\frac{\tau}{L}} |\tilde{w}(0)| \quad \forall \tau \geq 0.$$

As a result, for  $\kappa_2 := -\frac{\ln(\rho)}{L}$ , we conclude that

$$|\tilde{w}(\tau)| \leq |\tilde{w}(0)| \kappa_1 \exp^{-\kappa_2 \tau} \quad \forall \tau \geq 0.$$

### III. OBSERVER-BASED HYBRID CONTROL ALGORITHM

Let  $z^*$  be a given, piecewise-constant and vanishing reference trajectory (to be defined) for  $z_1$  and consider the simple certainty-equivalence control law

$$u := az_1 \hat{z}_2 - kz_{1e}, \quad z_{1e} := z_1 - z^*. \quad (14)$$

Then, the tracking-error dynamics corresponds to

$$\dot{z}_{1e} = -(k + a\tilde{z}_2)z_{1e} + az^* \tilde{z}_2. \quad (15)$$

This system is input to state stable with respect to  $z^*$  uniformly in balls of initial conditions. To better see this, let  $R > 0$  be arbitrarily fixed. Then, after [17], one can compute a positive definite matrix  $P \in \mathbb{R}^{2 \times 2}$  such that the time derivative of

$$V_{obs}(\tilde{z}) := \tilde{z}^\top P \tilde{z}, \quad (16)$$

along the solutions to (3), verifies

$$\dot{V}_{obs}(\tilde{z}(t)) \leq 0 \quad \forall t \geq 0. \quad (17)$$

Therefore, for any  $R > 0$ , and all  $|z(0)| \leq R$ , we can find  $\tilde{R}$  known such that  $V_{obs}(\tilde{z}(0)) \leq \lambda_{\max}(P)\tilde{R}^2$ . In turn, we have

$$\tilde{z}(t)^2 \leq \gamma^2 \tilde{z}(0)^2 \leq \gamma^2 \tilde{R}^2 \quad \forall t \geq 0, \quad (18)$$

with  $\gamma := \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$ . Then, using the function  $V(z_{1e}) := (1/2)z_{1e}^2$  and setting

$$k := \gamma a \tilde{R} + k', \quad k' > 0, \quad (19)$$

we see that, in view of (18), the derivative of  $V$  along the trajectories of (15) satisfies

$$\dot{V}(z_{1e}) \leq -k' z_{1e}^2 + a|z^*|_\infty |\tilde{z}_2|_\infty |z_{1e}|, \quad (20)$$

where  $|\phi|_\infty := \text{ess sup}_{t \geq 0} |\phi(t)|$ . It follows from (20) that the tracking errors converge provided that so do  $|\tilde{z}_2|$  and  $|z^*|$ . On the other hand, under Assumption 1, for  $|\tilde{z}_2|$  to converge, it is required that  $|z^*(t)|$  dwells a certain amount of time separated from zero. To achieve these antagonistic objectives we design a succession of cycles indexed  $i \in \{1, 2, \dots\}$  during each of which  $z^*$  switches taking values in

$$S^* := \bigcup_{i=0}^{\infty} \left\{ -\frac{z_{in}^*}{2^i}, \frac{z_{in}^*}{2^i} \right\}, \quad (21)$$

where  $z_{in}^* > 0$  is fixed by design (see below). That is,  $z^*(t)$  undergoes a sequence of commutations between two constant values during each cycle (this guarantees the decrease of  $|\tilde{z}_2|$ ) and the said constants decrease as the index  $i$  increases.

**Initialization step:** Let  $z_{in}^*$  and  $R > 0$  be given. Then, initially, we set  $z^*(t) = z_{in}^*$  for all  $t \in [0, t_1]$ , where  $t_1$  is to be defined, and  $\hat{z}_2(0)$  is chosen such that (18) holds. Then,

$$|\tilde{z}_2(t)| \leq \gamma \tilde{R} \quad \forall t \geq 0.$$

In view of (18) and (20) there exists  $T > 0$  such that

$$|z_{1e}(t)| \leq \frac{2a\tilde{R}}{\sqrt{k'}} |z^*(t)| \quad \forall t \geq T,$$

so, by setting  $k' \geq 16a^2 \tilde{R}^2$ , it follows that

$$|z_{1e}(t)| \leq z_{in}^*/2 \quad \forall t \geq T,$$

$z_1(t) \in [\frac{z_{in}^*}{2}, z_{in}^*]$ , and, consequently, Assumption 1 holds. On the other hand, there exist  $\kappa_{1o}, \kappa_{2o} > 0$  such that—see (13),

$$|\tilde{z}(t)| \leq |\tilde{z}(0)| \kappa_{1o} \exp\left(-\frac{\kappa_{2o} z_{in}^* t}{2}\right) \quad \forall t \geq 0,$$

so, for any  $\varepsilon > 0$ , there exists  $T_o \geq T > 0$  such that

$$|\tilde{z}(t)| \leq g(0)(\varepsilon/\gamma) \quad \forall t \geq T_o, \quad g(0) := 1,$$

and, from the first inequality in (18), we have  $|\tilde{z}_2(t)| \leq \varepsilon$  for all  $t \geq T_o$ .

**First cycle:** From  $t_1 := T_o$ , we set  $z^*$  to satisfy  $|z^*| = \frac{z_{in}^*}{2}$ , moreover, the tracking error  $z_{1e}$  satisfies (20) with  $|\tilde{z}_2| \leq \varepsilon$ . Therefore, a limit cycle is generated by switching  $z^*$  between  $-z_{in}^*/2$  and  $z_{in}^*/2$  each time  $\hat{z}_2(t)$  reaches  $d/2c$  or  $-d/2c$ , as follows:

- 1) If  $\hat{z}_2(t_1) \leq 0$ ,  $z^*(t_1)$  is set to  $\frac{z_{in}^*}{2}$ . Then, at  $t'_1 \geq t_1$  such that  $\hat{z}_2(t'_1) = \frac{d}{2c}$ , which means that  $z_2(t'_1) \in [\frac{d}{2c} - \varepsilon, \frac{d}{2c} + \varepsilon]$ , the reference  $z^*$  is set to  $z^*(t'_1) = -\frac{z_{in}^*}{2}$ . Then, at  $t''_1 \geq t'_1$  such that  $\hat{z}_2(t''_1) = -\frac{d}{2c}$ , which means that  $z_2(t''_1) \in [-\varepsilon - \frac{d}{2c}, \varepsilon - \frac{d}{2c}]$ , the reference  $z^*$  is set back to  $\frac{z_{in}^*}{2}$ .
- 2) If  $\hat{z}_2(t_1) \geq 0$ , the reference is set to  $z^*(t_1) = -\frac{z_{in}^*}{2}$  and the same switching rules as above apply *mutatis mutandis*. Along the first cycle, Assumption 1 holds on  $[t_1, +\infty)$ ; thus, there exist positive constants  $(\kappa_{11}, \kappa_{21})$  such that

$$|\tilde{z}(t)| \leq \kappa_{11} |\tilde{z}(t_1)| \exp^{-\kappa_{21} \int_{t_1}^t |z_1(s)| ds} \quad \forall t \geq t_1.$$

The first cycle ends at  $t_2 := T_o + T_1 > 0$ , such that

$$|\hat{z}_2(t_2)| \leq d/2c \quad \text{and} \quad |\tilde{z}(t_2)| \leq (\varepsilon/\gamma)g(1), \quad g(1) \in (0, 1),$$

*idem* for each succeeding cycle indexed  $i \geq 2$ .

***i*th cycle:** From  $t_i = T_o + T_1 + \dots + T_{i-1}$ , the reference  $z^*$  is set to satisfy  $|z^*| = \frac{z_{in}^*}{2^i}$ , and  $z_{1e}$  satisfies (20) with  $|\tilde{z}_2| \leq \varepsilon g(i-1)$  for some  $g(i-1) \in (0, 1)$ . Hence, a limit cycle is generated by making  $z^*$  switch between  $-\frac{z_{in}^*}{2^i}$  and  $\frac{z_{in}^*}{2^i}$  each time  $\hat{z}_2(t) = d/2^i c$  or  $\hat{z}_2(t) = -d/2^i c$ , as follows:

- 1) If  $\hat{z}_2(t_i) \leq 0$ ,  $z^*(t_i)$  is set to  $\frac{z_{in}^*}{2^i}$ . Then, at  $t'_i \geq t_i$  such that  $\hat{z}_2(t'_i) = \frac{d}{2^i c}$ , which means that  $z_2(t'_i) \in [\frac{d}{2^i c} - \varepsilon, \frac{d}{2^i c} + \varepsilon]$ , the reference  $z^*$  is set to  $z^*(t'_i) = -\frac{z_{in}^*}{2^i}$ . Then, at  $t''_i \geq t'_i$  such that  $\hat{z}_2(t''_i) = -\frac{d}{2^i c}$ , which means that  $z_2(t''_i) \in [-\frac{d}{2^i c} - \varepsilon, -\frac{d}{2^i c} + \varepsilon]$ , the reference  $z^*$  is set to  $\frac{z_{in}^*}{2^i}$ .
- 2) If  $\hat{z}_2(t_i) \geq 0$ , the reference is set to  $z^*(t_i) = -\frac{z_{in}^*}{2^i}$ , etc.

During the *i*th cycle, Assumption 1 holds on  $[t_i, +\infty)$ , so there exist positive constants  $(\kappa_{1i}, \kappa_{2i})$  such that

$$|\tilde{z}(t)| \leq \kappa_{1i} |\tilde{z}(t_i)| \exp^{-\kappa_{2i} \int_{t_i}^t |z_1(s)| ds} \quad \forall t \geq t_i.$$

The cycle ends at  $t_{i+1} := T_o + T_1 + \dots + T_i > 0$ , such that

$$|\hat{z}_2(t_{i+1})| \leq d/2^i c \quad \text{and} \quad |\tilde{z}(t_{i+1})| \leq (\varepsilon/\gamma)g(i) \quad \forall t \geq t_{i+1},$$

with  $g(i) \in (0, 1)$ . A new cycle starts over and so on.

#### IV. MAIN STATEMENT

The ABS system (1), under the previously described control algorithm may be implemented and formally analyzed as a hybrid system. To make a precise statement on asymptotic stability of the origin we rely on the framework of hybrid systems, as laid in [21]. Following the latter, we first show that the closed-loop composed of the plant (1), the controller (14), and the observer (2), may be expressed as a hybrid system that consists in the combination of a constrained differential and a constrained difference equations,

$$\mathcal{H} : \begin{cases} \dot{x} = F(x) & x \in C \\ x^+ = G(x) & x \in D, \end{cases} \quad (22)$$

where the state variable  $x \in \mathcal{X} \subset \mathbb{R}^n$  has a continuous evolution while in the flow set  $C \subset \mathcal{X}$  and it is allowed to jump if in the jump set  $D \subset \mathcal{X}$ . The continuous- and the discrete-time evolution of  $x$  are governed by the flow and the jump maps  $F : C \rightarrow \mathbb{R}_{\geq 0} \times \{0\} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \{0\}$  and  $G : D \rightarrow \mathcal{X}$ , respectively. Furthermore, the closed-loop state is defined as

$$\begin{aligned} x &:= (\tau, i, z, \tilde{z}, z^*) \in \mathcal{X}, \\ \mathcal{X} &:= \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R} \times \left(-\frac{d}{c}, +\infty\right) \times \mathbb{R}^2 \times S^*. \end{aligned}$$

Then, the jump and flow sets are defined as follows. The flow set  $C := \text{cl}(\mathcal{X} \setminus D)$ , where  $\text{cl}(\cdot)$  denotes *closure relative to*  $\mathcal{X}$  and the jump set  $D := D_c \cup D_{nc}$ . The set  $D_c$ , which determines the jump conditions *within* the  $i$ th cycle, is given by

$$D_c := \left\{ x \in \mathcal{X} : |\hat{z}_2| \geq \frac{d|z^*|}{cz_{in}^*}, \hat{z}_2 z^* \geq 0 \right\} \quad (23)$$

and the set  $D_{nc}$ , which determines the jump condition *from* the  $i$ th to the  $(i+1)$ th cycle, is given by

$$\begin{aligned} D_{nc} &:= \left\{ x \in \mathcal{X} : |\hat{z}_2| \leq \frac{d|z^*|}{cz_{in}^*}, \hat{z}_2 z^* \leq 0, \right. \\ &\quad \left. |\Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0)|^{\frac{1}{2}} \leq \lambda_{\min}(P)^{\frac{1}{2}} h(i) \right\}, \quad (24) \end{aligned}$$

where  $\Phi_i$  is the transition matrix corresponding to the system

$$\frac{d\tilde{w}}{d\tau} = A(w_1(\tau + \tau_i))\tilde{w} \quad \tau \geq 0, \quad (25)$$

and  $\tau_i := \int_0^{t_i} |z_1(s)| ds$ . Moreover  $h(0) := \varepsilon/(\gamma\tilde{R})$ , and

$$h(i) := \frac{g(i)}{g(i-1)} \in (0, 1) \quad \forall i \geq 1.$$

The definition of the jump sets  $D_c$  and  $D_{nc}$  follows the rationale developed in the previous section, but certain technical aspects are also considered in order to cast the analysis in the framework of [21]. The respective first inequalities in  $D_c$  and  $D_{nc}$  correspond to the switch conditions explained in Section III. The constraint  $\hat{z}_2 z^* \leq 0$ , which requires that the signs of  $\hat{z}_2$  and  $z^*$  be different, is imposed in the definition of  $D_{nc}$ , while the opposite is used to define  $D_c$ , to render the intersection of these sets empty (the apparent intersection  $\{\hat{z}_2 = z^* = 0\}$  is void since  $z^* \neq 0$  by design). A somewhat more natural manner to define the jump sets  $D_{nc}$  and  $D_c$ , which is in more strict concordance with the algorithm loosely

described in the previous section, would not include such condition and would simply impose a strict inequality in either set. However, such definition would lead to the hybrid system being not well-posed [21].

The third inequality in the definition of  $D_{nc}$ ,

$$|\Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0)| \leq \lambda_{\min}(P)h(i)^2, \quad (26)$$

is a conservative, yet verifiable, condition that essentially tests the size of the otherwise non-measurable estimation errors  $\tilde{z}(t) \equiv \tilde{w}(\tau)$ . To better see this, consider the function  $V_{obs}$  in (16). Its total derivative along the solutions to (6) satisfies  $\dot{V}_{obs}(\tilde{w}(\tau)) \leq 0$ , so  $V_{obs}(\tilde{w}(\tau)) \leq V_{obs}(\tilde{w}(0))$  for all  $\tau \geq 0$ . Hence, equivalently,

$$\tilde{w}(\tau)^\top P \tilde{w}(\tau) \leq \tilde{w}(0)^\top P \tilde{w}(0).$$

Therefore, using the fact that  $\tilde{w}(\tau) = \Phi_i(\tau, 0)\tilde{w}(0)$ , we see that (26) implies that, for any  $\tilde{w}(0) \in \mathbb{R}^2$ ,

$$\tilde{w}(\tau)^\top P \tilde{w}(\tau) \leq \lambda_{\min}(P)h(i)^2|\tilde{w}(0)|^2,$$

that is,  $|\tilde{w}(\tau)|^2 \leq h(i)^2|\tilde{w}(0)|^2$ .

Then, with these definitions of the flow and jump sets, we introduce the flow map

$$F(x) := \begin{bmatrix} |z_1| \\ 0 \\ \left[ \begin{array}{cc} -(k + a\tilde{z}_2)z_{1e} + az^*\tilde{z}_2 \\ (cz_2 + d)z_1 \end{array} \right] \\ z_1 \left[ \begin{array}{cc} -k_1(z_1) & -a \\ -k_2(z_1) & c \end{array} \right] \tilde{z} \\ 0 \end{bmatrix}. \quad (27)$$

Note that in the definition of  $F$ , the dynamics of the discrete variables  $(i, z^*)$  is null, the dynamics of  $\tau$  corresponds to (4). Finally, the dynamics of  $z$  and  $\tilde{z}$  are simply repeated from (15) and (3), respectively.

On the other hand, the jump map is given by

$$G(x) := \begin{bmatrix} \left\{ \begin{array}{ll} 0 & \text{if } x \in D_{nc} \\ \tau & \text{if } x \in D_c \\ i + 1 & \text{if } x \in D_{nc} \\ i & \text{if } x \in D_c \end{array} \right. \\ z \\ \tilde{z} \\ \left\{ \begin{array}{ll} z^*/2 & \text{if } x \in D_{nc} \\ -z^* & \text{if } x \in D_c \end{array} \right. \end{bmatrix}. \quad (28)$$

The map  $G$  is designed to reset the value of  $\tau$  to 0 each time a new cycle starts and updates the cycle index  $i$ . The variables  $z$  and  $\tilde{z}$  are continuous variables, so they do not change their values during jumps. According to the algorithm previously explained, the variable  $z^*$  halves its size in absolute value whenever a jump to a new cycle occurs, otherwise (while switching within a cycle),  $z^*$  only alternates sign. It is important to note that since  $D_c \cap D_{nc} = \emptyset$ , then the map  $G$  is continuous on  $D$ . This is important for the system to be well-posed [21].

In addition, the initial state  $x_o := (\tau_o, i_o, z_o, \tilde{z}_o, z_o^*) \in \mathcal{X}$  is defined as follows. By assumption, a number  $R$  is known such that  $|z_o| \leq R$ . Then, the estimates  $\tilde{z}_o$  are set so that  $|\tilde{z}_o| \leq \tilde{R}$  for some  $\tilde{R} > 0$  known. Hence, when a reliable



estimate of  $|z_o|$  is available, the Initialization step described on p. 3 may be skipped by defining the initial cycle index as  $i_o := \max\{0, \kappa_1(\tilde{R})\}$ , where

$$\kappa_1(\tilde{R}) := \max \left\{ i \in \mathbb{Z} : \tilde{R} \leq \frac{\varepsilon g(i-1)}{\gamma} \right\}.$$

Furthermore, according to (4),  $\tau_o = 0$ . Finally, the vanishing reference trajectory  $z^*$  is initialized to

$$z_o^* := \begin{cases} \begin{cases} \frac{z_{in}^*}{2^{i_o}} & \text{if } \hat{z}_{2o} < 0 \\ -\frac{z_{in}^*}{2^{i_o}} & \text{otherwise} \end{cases} & \text{if } i_o \geq 1 \\ z_{in}^* & \text{if } i_o = 0. \end{cases}$$

Our main statement establishes semi-global attractivity of the set  $\mathcal{A} := \{x \in \mathcal{X} : z = \tilde{z} = 0\}$  for the closed-loop system. That is, for any ball of initial conditions of radius  $R$ , there exists a control gain  $k(R)$ , as defined in (19), such that all trajectories converge to the set  $\mathcal{A}$ . In particular, the domain of attraction may be enlarged by increasing the control gain.

*Theorem 1:* Consider the closed-loop hybrid system  $\mathcal{H} = (C, F, D, G)$  defined by (22)–(24), (27), and (28). Let  $R, \tilde{R} > 0$  be such that  $|z_o| \leq R$  and  $|\tilde{z}_o| \leq \tilde{R}$ , and let  $(i_o, z_o^*, \tau_o)$  be defined as above. Then, for each  $R$  and  $\tilde{R}$ , we can find  $k > 0$  such that

- (i) each solution to<sup>1</sup>  $\mathcal{H}$  satisfies

$$\lim_{(t+j) \rightarrow +\infty} |x(t, j)|_{\mathcal{A}} = 0,$$

provided that  $\lim_{i \rightarrow \infty} g(i) = 0$ .

- (ii) the set  $\mathcal{A}$  is locally stable provided that  $\exists \delta > 0$ , sufficiently small, such that  $|\tilde{z}_o| \leq \delta$ ; in the sense that, there exists a class  $\mathcal{K}$  function  $\kappa^2$  such that  $|(z, \tilde{z})|_{\infty} \leq \kappa(|z_o| + \delta)$ .
- (iii) the system  $\mathcal{H}$  is well posed—see [21], and its solutions are uniformly non-Zeno, that is, there exist  $T > 0$  and  $J \in \mathbb{N}$  such that, on any time period of length  $T$ , at most  $J$  jumps can occur.

In a general setting, the assumption in item (ii) is restrictive. However, this is not the case for commercial ABS systems, for which the initial condition  $z_{1o}$  is usually known and there exists a small  $\delta$  such that  $z_{2o} \in [-\delta, \delta]$ . Note that, in this case, items (i) and (ii), together with item (iii) goes well beyond that of mere convergence of estimation errors  $\tilde{z}$  and the state variable  $z$  since it guarantees certain robustness—see the latter reference for details.

### Proof of Theorem 1

*Proof of item (i):* To guarantee asymptotic convergence of  $z$  to zero, we first show that Assumption 1 holds on the  $i$ th cycle, for all  $i \in \{1, 2, \dots\}$ . Let  $i \in \{1, 2, \dots\}$  be arbitrarily fixed and consider the behavior of the solutions to  $\mathcal{H}$  for all  $t \in \mathcal{I}_i$ , that is, during the duration of the  $i$ th cycle. Over the

<sup>1</sup>Note that  $(t, j) \mapsto x(t, j)$  are defined as absolutely continuous functions mapping their hybrid domain,  $\text{dom } x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ , into  $\mathbb{R}^2$ . See [21] for details.

<sup>2</sup>i.e.  $\kappa$  is continuous, strictly increasing, and  $\kappa(0) = 0$ .

interval  $\mathcal{I}_i$ , the solutions to  $\mathcal{H}$  coincide with those of the hybrid system  $\mathcal{H}_i := (F_i, G_i, C_i, D_i)$ , with state vector

$$x := (z, \tilde{z}, z^*) \in \mathcal{X}_i := \mathbb{R}^2 \times \mathbb{R}^2 \times \left\{ -\frac{z_{in}^*}{2^i}, \frac{z_{in}^*}{2^i} \right\},$$

with flow map

$$F_i(x) := \begin{bmatrix} \begin{bmatrix} -(k + a\tilde{z})z_{1e} + az^*\tilde{z}_2 \\ (cz_2 + d)z_1 \end{bmatrix} \\ z_1 \begin{bmatrix} -k_1(z_1) & -a \\ -k_2(z_1) & c \end{bmatrix} \tilde{z} \\ 0 \end{bmatrix},$$

with jump map  $G_i(x) := [z^\top \tilde{z}^\top - z^*]^\top$ , with jump set  $D_i := D_{i1} \cup D_{i2}$ , where

$$D_{i1} := \left\{ x \in \mathcal{X}_i : \hat{z}_2 \geq \frac{d/c}{2^i}, z^* = \frac{z_{in}^*}{2^i} \right\},$$

$$D_{i2} := \left\{ x \in \mathcal{X}_i : \hat{z}_2 \leq -\frac{d/c}{2^i}, z^* = -\frac{z_{in}^*}{2^i} \right\},$$

with  $C_i := \text{cl}(\mathcal{X}_i \setminus D_i)$ , and with initial conditions

$$x_o := (z_o, \tilde{z}_o, z_o^*),$$

such that

$$|\tilde{z}_o| \leq \frac{\varepsilon g(i-1)}{\gamma} \quad \text{and} \quad z_o^* = \begin{cases} \frac{z_{in}^*}{2^i} & \text{if } \hat{z}_{o2} < 0 \\ -\frac{z_{in}^*}{2^i} & \text{otherwise.} \end{cases} \quad (29)$$

The solutions to  $\mathcal{H}_i$  (and consequently to  $\mathcal{H}$  over  $\mathcal{I}_i$ ), within the  $i$ th cycle, jump according to the conditions defining  $D_{i1} \cup D_{i2}$  and satisfy the following.

*Lemma 2:* Consider the hybrid system  $\mathcal{H}_i$  such that (29) holds and let the parameter  $k$  satisfy (19) with  $2k' \geq a^2\varepsilon^2$ . Then,

- (i) the set

$$\mathcal{D}_i := \left\{ x \in \mathcal{X}_i : |z_1| \leq \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}}g(i-1) \right\} \quad (30)$$

is forward invariant and finite-time attractive;

- (ii) let  $x_o \in \mathcal{D}_i$  and let the parameter  $k$  satisfy (19) with  $2k' \geq a^2\varepsilon^2$ . Then, there exists  $T_{lmin} > 0$  (independent of  $i$ ) such that the time between each pair of consecutive jumps of the solution starting from  $x_o$  is larger than  $T_{lmin}$ .

Furthermore, after Lemma 2 the following also holds (see the appendix for the proofs).

*Lemma 3:* Consider the hybrid system  $\mathcal{H}_i(C_i, F_i, D_i, G_i)$  such that (29) holds and the parameter  $k$  satisfy (19). Then, for  $k'$  sufficiently large and independent of  $i$ , there exist positive constants  $(\tau_{di}, \tau_{si}, \tilde{z}_i, \underline{z}_i)$  so that Assumption 1 holds.

By Lemma 3, Assumption 1 and, consequently, Property 1, hold. Therefore, from Lemma 1 it follows that there exist positive constants  $\kappa_{1i}, \kappa_{2i}, T_i$ , and  $\mu_i$  such that

$$|\tilde{z}(t)| \leq \kappa_{1i} |\tilde{z}(t_i)| \exp^{-\kappa_{2i}\mu_i(t-t_o)} \quad \forall t \geq t_i + T_i,$$

where  $t_i$  is the beginning the interval  $\mathcal{I}_i$ . Hence, in view of the second condition in (24), the interval of duration of the  $i$ th cycle,  $\mathcal{I}_i$ , is finite. Now, we use Lemmata 2 and 3 to complete

the proof of Item (i) of the theorem. We show that, for each  $i_o \in \{1, 2, \dots\}$ , there exists  $i^* \geq i_o$  and  $t_{i^*} \in \mathcal{I}_{i^*}$ , i.e., during the Cycle  $i^*$ , such that  $x(t_{i^*}) \in \mathcal{D}_{i^*}$ . By the definition of  $\mathcal{D}_i$ , the convergence of  $z_1(t)$  follows.

Let  $i_o \in \{1, 2, \dots\}$  and  $t_{i_o} \geq 0$  be the time at which Cycle  $i_o$  starts. Assume, without loss of generality, that  $z_1(t_{i_o}) > 0$ , but  $x(t) \notin \mathcal{D}_{i_o}$  for all  $t \in \mathcal{I}_{i_o}$ , that  $z_2(t_{i_o}) \leq 0$ , and that  $z^*(t_{i_o}) > 0$ —the same reasoning that will follow applies to any other choice of initial conditions. For the considered choice of initial conditions,  $\hat{z}_2(t)$  increases until one of the following two scenarios occurs:

- 1) There exist a time instant when  $\hat{z}_2(t) = \frac{d/c}{2^{i_o}}$ , in which case,  $\text{sign}(z^*(t))$  becomes negative, so the jump to Cycle  $i_o + 1$  does not occur before  $\hat{z}_2(t)$  becomes, again, smaller or equal than  $\frac{d/c}{2^{i_o}}$ . For this to happen,  $z_2(t)$  must decrease, that is,  $z_1(t)$  must become negative—see (1b)—and, consequently,  $x(t)$  must enter the set  $\mathcal{D}_{i_o}$ .
- 2) A jump to Cycle  $i_o + 1$  occurs before  $\hat{z}_2$  passes  $\frac{d/c}{2^{i_o}}$ . In this case, either the previous scenario occurs with  $i_o$  replaced by  $i_o + 1$  and  $x(t)$  enters  $\mathcal{D}_{i_o+1}$  within Cycle  $i_o + 1$ , or a jump to Cycle  $i_o + 2$  occurs before  $\hat{z}_2$  passes  $\frac{d/c}{2^{i_o+1}}$ . However, at some point, there must exist  $i^* \geq i_o$  such that  $x(t)$  enters  $\mathcal{D}_{i^*}$  within Cycle  $i^*$ .

Next, we show that  $z_2(t)$  also converges, by establishing an upperbound in the latter for all  $t \in \mathcal{I}_{i^*}$  such that  $z_1(t) \in \mathcal{D}_{i^*}$  and when  $\hat{z}_2(t_{i^*}) = \frac{d/c}{2^{i^*-1}}$  and  $z^*(t_{i^*}) = -\frac{z_{in}^*}{2^{i^*}}$ . The latter must happen at some point while in Cycle  $i^*$ . Note that, using the proof of Lemma 3, the overshoot of  $z_2(t)$  occurs during the interval  $[0, T_{lmin}]$ , where  $T_{lmin}$  corresponds to the time it takes  $z_1(t)$  to acquire the same sign as  $z^*(t)$ —in this case, to become negative. By virtue of the comparison lemma, it is enough to construct a bound on the solution of

$$\dot{\hat{z}}_2 = \left[ \max_{z_1 \in \mathcal{D}_{i^*}} |z_1| \right] [cz_2 + d], \quad z_2(0) = \frac{d/c + \varepsilon}{2^{i^*-1}},$$

over the interval  $[0, T_{lmin}]$ . Clearly, we deduce an upperbound on  $z_2$  that converges to zero as  $i^*$  goes to infinity.

*Proof of item (ii):* By definition, the control algorithm is initiated at Cycle  $i_o$  with  $i_o := \max\{0, \kappa_1(\delta)\}$ . Furthermore, when  $\delta$  is sufficiently small, we conclude that  $i_o := \kappa_1(\delta)$ . Therefore, by definition of  $\kappa_1$  and (18), we conclude that

$$|\tilde{z}(t, j)| \leq \min\{\varepsilon g(i_o - 1), \gamma \delta\} \quad \forall (t, j) \in \text{dom } \tilde{z}.$$

Next, to find an upper bound for  $z_1$ , we distinguish between two cases:

- If  $x_o \in \mathcal{D}_{i_o}$ , where  $\mathcal{D}_{i_o}$  is defined in (30) and is forward invariant. Then, we know that, there exists a class  $\mathcal{K}$  function  $\kappa_2$  such that

$$\mathcal{D}_{i_o} \subset \kappa_2(|\delta|)[-1, 1].$$

Indeed, it is easy to see that when  $\delta$  goes to zero,  $i_o$  goes to infinity, and thus  $\mathcal{D}_{i_o}$  goes  $\{0\}$ .

- If  $\mathcal{D}_{i_o} \subset \{x \in \mathcal{X} : z_1 \in [-|z_{1o}|, |z_{1o}|]\}$ , we use the fact that  $\mathcal{D}_{i_o}$  is finite-time attractive — see Item (i) in Lemma 2. Furthermore, since the flows are unique and  $z_1$  is a continuous variable, we conclude that  $[-|z_{1o}|, |z_{1o}|]$  must be forward invariant. Hence, we obtain that

$$|z_1(t, j)| \leq |z_{1o}| \quad \forall (t, j) \in \text{dom } z_1. \quad (31)$$

Finally, to complete the proof, we establish an upper bound on  $z_2$ . Assume, without loss of generality, that  $z_{2o} > 0$  and

$$z_o^* = -2^{-i_o} z_{in}^* = -2^{-\kappa_1(\delta)} z_{in}^* =: \kappa_3(\delta).$$

Then, consider the following two cases:

- If  $z_{1o} \in \mathcal{D}_{i_o}$ , we conclude that the overshoot of  $|z_2|$  occurs only on the interval  $[0, T_{1*}]$ , on which  $|z_1| \leq \kappa_2(\delta)$ , and before  $z_1$  becomes negative.
- When  $\mathcal{D}_{i_o} \subset \{x \in \mathcal{X} : z_1 \in [-|z_{1o}|, |z_{1o}|]\}$ , we conclude that (31) holds. Hence, the overshoot of  $z_2$  occurs only on the interval  $[0, T_{2*}]$ , on which  $|z_1| \leq |z_{1o}|$ , and before  $z_1$  becomes negative.

Hence, after the comparison Lemma, it suffices to assess the behavior of the solutions of

$$\dot{z}_2 = \max\{|z_{1o}|, \kappa_2(|\delta|)\} (cz_2 + d) \quad \text{with } z_2(0) = |z_{2o}|,$$

over the interval  $[0, T_*]$ , where  $T_* := \max\{T_{1*}, T_{2*}\}$  is an upper bound on the time that  $|z_1|$  takes to flow from  $\max\{|z_{1o}|, \kappa_2(|\delta|)\}$  to zero.

To complete the proof, we show that  $T_*$  can be chosen as a class  $\mathcal{K}$  function of  $|(z_o, \tilde{z}_o)|$ . To do so, we use the Lyapunov function  $v(z_{1e}) := z_{1e}^2$ , whose time derivative along the solutions to  $\mathcal{H}_i$  satisfies

$$\begin{aligned} \dot{v} &= -2k'v + 2az^* \tilde{z}_2 z_{1e} \\ &\leq -2k'v + 2a\varepsilon \frac{z_{in}^*}{2^{i_o}} g(i_o - 1) |z_{1e}|. \end{aligned}$$

By assuming, for example that  $k' \geq 1$ , we conclude that

$$\begin{aligned} \dot{v} &\leq -k'v + z^{*2} \frac{a^2 \varepsilon^2}{4^{i_o}} g(i_o + 1)^2 \\ &\leq -k'v + z^{*2} \frac{a^2 \varepsilon^2}{4^{\kappa_1(\delta)}} \\ &\leq -k'v + a^2 \kappa_3(\delta)^4. \end{aligned}$$

Hence, to obtain  $T_*$ , we can solve the system  $\dot{v} = -k'v + a^2 \kappa_3(\delta)^4$ . From  $v(0) := [\max\{|z_{1o}|, \kappa_2(\delta)\} + z^*]^2$  until  $v(t) \leq z^{*2}$ . Hence, we can see that  $T_*$  is upper bounded by a class  $\mathcal{K}$  function of  $(|z_o| + \delta)$ .

*Proof of item (iii):* After [21], system  $\mathcal{H}$  is well-posed if the sets  $C$  and  $D$  are closed relative to  $\mathcal{X}$  and  $F$  and  $G$  are continuous on  $C$  and  $D$ , respectively. It is easy to conclude that our closed-loop hybrid system  $\mathcal{H}$  satisfies the hybrid basic conditions which require the sets  $C$  and  $D$  to be closed and the maps  $F$  and  $G$  to be continuous. Note that both  $C$  and  $D$  are closed subsets relative to  $\mathcal{X}$ ,  $F$  is smooth and  $G$  is continuous on  $D = D_c \cup D_{nc}$  since both  $D_c$  and  $D_{nc}$  are closed relative to  $\mathcal{X}$  and their intersection is empty.

Next, we show that the closed-loop solutions are uniformly non-Zeno. To do so, we note that within a same Cycle  $i$ , and between each two consecutive jumps,  $\hat{z}_2(t)$  flows from  $-\frac{d}{2^{i-1}c}$  to  $\frac{d}{2^{i-1}c}$  back and forth. The latter flow phase takes a time we denote by  $T_{li}$ . After Item (ii) in Lemma 2, there exists a uniform lower bound  $T_{lmin} > 0$  such that  $T_{li} \geq T_{lmin}$  for all  $i \in \{1, 2, \dots\}$ , provided that  $z_{1o} \in \mathcal{D}_i$ . In general, after Item (i) of Lemma 2,  $z_1$  must reach  $\mathcal{D}_i$  in finite time while in Cycle  $i$ ; otherwise, only one jump occurs within Cycle  $i$ .

On the other hand, for a jump from Cycle  $i$  to Cycle  $i+1$  to occur, the variable  $\tau$  must flow so that  $|\Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0)|^{\frac{1}{2}}$  decreases from  $|P|^{\frac{1}{2}}$  to  $\frac{\lambda_{\min}(P)^{\frac{1}{2}}}{2}$ , where  $\Phi_i$  is the transition matrix associated with system (25). We show the existence of a strictly positive lower bound on the time the latter decrease process takes. To that end, we use  $V = |\tilde{w}|^2$  and the fact that

$$\tilde{w}^\top [A(w_1(\tau)) + A(w_1(\tau))^\top] \tilde{w} \geq -\eta |\tilde{w}|^2 \quad \forall \tau \geq 0, \tilde{w} \in \mathbb{R}^2,$$

where  $\eta := \max_{i \in \{1,2\}} \{|A_i + A_i^\top|\}$ . Then,  $V'(\tau) \geq -aV(\tau)$  for all  $\tau \geq 0$ , which, by virtue of the comparison Lemma, implies that  $V(\tau) \geq \exp^{-a\tau} V(0)$  and, in turn, for each  $\tilde{w}(0) \in \mathbb{R}^2$ ,

$$\tilde{w}(0)^\top \Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0) \tilde{w}(0) \geq \exp^{-a\tau} |\tilde{w}(0)|^2,$$

hence, since for our case  $|P| > \lambda_{\min}(P)$  there exists  $\tau^* > 0$  such that, for each  $i \in \{1, 2, \dots\}$ , and for all  $\tau \in [0, \tau^*]$ ,

$$|\Phi_i(\tau, 0)^\top P \Phi_i(\tau, 0)| \geq \lambda_{\min}(P) \geq \lambda_{\min}(P)h(i).$$

## V. CONCLUSION AND FUTURE WORK

Simultaneous estimation and stabilization at an equilibrium where observability is lost is challenging, but not impossible. Our main results set a basis broaden the applicability of our switching-observer-based hybrid controller to other bilinear and, more generally, non-uniformly observable systems. Another interesting open problem is to design a *smooth* output-feedback controller—note that in [19] we establish the *existence* of such smooth controller. Beyond these remaining open theoretical questions, a deeper study regarding control implementation is required to determine different cycle-jump conditions that deliver good performance while satisfying the technical conditions imposed by the analysis.

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## APPENDIX

### A. Proof of Lemma 2

1) *Proof of Item (i)*: We first use the fact that  $|\tilde{z}_o| \leq \frac{\varepsilon g(i-1)}{\gamma}$ , together with (18), to conclude that

$$|\tilde{z}(t, j)| \leq \varepsilon g(i-1) \quad \forall (t, j) \geq \text{dom } \tilde{z}.$$

Next, we use the Lyapunov function  $v(z_1) := z_1^2$ , whose time derivative along the solutions to  $\mathcal{H}_i$  satisfies

$$\begin{aligned} \dot{v} &= -2k' z_1^2 + 2k' z_1 z^* + 2a z_1 z^* \tilde{z}_2 \\ &\leq -k' v + 2k' z^{*2} + 2a^2 z^{*2} \tilde{z}_2^2 \\ &\leq -k' v + \frac{2k' z_{in}^{*2}}{4^i} + \frac{2a^2 \varepsilon^2 z_{in}^{*2}}{4^i} g(i-1)^2. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} v(t, j) &\leq v(0, 0) \exp^{-k't} + \frac{2z_{in}^{*2}}{2^{2i}} \left[ 1 - \exp^{-k't} \right] \\ &\quad + \frac{a^2 \varepsilon^2 z_{in}^{*2}}{(2k')4^{(i-1)}} g(i-1)^2 \left[ 1 - \exp^{-k't} \right], \end{aligned}$$

so, by choosing  $k'$  such that  $k' \geq \frac{a^2 \varepsilon^2}{2}$ , we get

$$\begin{aligned} v(t, j) &\leq v(0, 0) \exp^{-k't} \\ &\quad + \left[ \frac{z_{in}^{*2}}{2^{2(i-1)}} + \frac{z_{in}^{*2}}{4^{(i-1)}} g(i-1)^2 \right] \left[ 1 - \exp^{-k't} \right] \\ &\leq \max \left\{ v(0, 0), \left[ \frac{z_{in}^{*2}}{2^{2(i-1)}} + \frac{z_{in}^{*2}}{2^{2(i-1)}} g(i-1)^2 \right] \right\}. \end{aligned}$$

Hence, the set  $\mathcal{D}_i$  is finite-time attractive and forward invariant.



2) *Proof of Item (ii)*: If  $x_{oi} \in \mathcal{D}_i$ , using the comparison Lemma, a lower bound on the time between each two consecutive jumps of  $\mathcal{H}_i$  can be obtained by computing the time the solution to

$$\dot{z}_2 = - \left( \frac{2z_{in}^*}{2^{i-1}} \right) (cz_2 + d), \quad (32)$$

with initial conditions  $z_{o2} = \frac{d/c-\varepsilon}{2^{i-1}}$ , takes to reach  $\frac{-d/c+\varepsilon}{2^{i-1}}$ .

To compute the latter time, we introduce the new time scale  $\tau := \left( \frac{2z_{in}^*}{2^{i-1}} \right) t$ , to obtain in the new time scale  $z'_2 = -cz_2 - d$ . By solving the latter equation, we obtain

$$z_2(\tau) = \sigma_i \exp^{-c\tau} - \frac{d}{c} [1 - \exp^{-c\tau}], \quad \sigma_i := \left[ \frac{d/c - \varepsilon}{2^{i-1}} \right]$$

and we use the latter to solve  $z_2(\tau) = \sigma_i$  for  $\tau$ . Reordering terms, we obtain  $\exp^{-c\tau} [d/c + \sigma_i] = d/c - \sigma_i$ . Hence,

$$\tau = \frac{1}{c} \ln \left[ \frac{d/c + \sigma_i}{d/c - \sigma_i} \right] = \frac{1}{c} \ln \left[ 1 + 2 \frac{\sigma_i}{d/c - \sigma_i} \right].$$

This implies that, in the original time scale, the length of the interval  $[t_i, t_{i+1}]$  between two jumps of the solution to (32), denoted  $T_{li}$ , satisfies

$$T_{li} \geq \frac{2^{i-1}}{2cz_{in}^*} \ln \left[ 1 + 2 \frac{\sigma_i}{d/c - \sigma_i} \right],$$

which is separated from zero, *i.e.*,

$$\lim_{i \rightarrow \infty} T_{li} \geq \frac{1}{dz_{in}^*} \left[ \frac{d}{c} - \varepsilon \right] > 0.$$

### B. Proof of Lemma 3

Let  $x_o \in \mathcal{D}_i$  and let  $\hat{z}_{o2} = \frac{d/c}{2^{i-1}}$ . There is no loss of generality since if  $x_o \notin \mathcal{D}_i$  Assumption 1 trivially holds over the  $i$ th cycle and if  $\hat{z}_{o2} \neq \frac{d/c}{2^{i-1}}$ ,  $\hat{z}_2(t, j) = \frac{d/c}{2^{i-1}}$  for some  $t + j < \infty$ . Moreover, the following reasoning applies *mutatis mutandis* if  $\hat{z}_{o2} = -\frac{d/c}{2^{i-1}}$ . We also use the fact that  $|\tilde{z}_o| \leq g(i-1) \frac{\varepsilon}{\gamma}$  together with (18) to conclude that

$$|\tilde{z}(t, j)| \leq \varepsilon g(i-1) \quad \forall (t, j) \in \text{dom } \tilde{z}. \quad (33)$$

- 1) At this point, we estimate a lower bound on the flow time that  $\hat{z}_2$  takes to flow from  $\frac{d/c}{2^{i-1}}$  to  $\frac{-d/c}{2^{i-1}}$ . Using (33), we conclude that such a time is lower bounded by the time  $z_2$  takes to flow from  $\frac{d/c}{2^{i-1}} - \frac{\varepsilon}{2^{i-1}}$  to  $\frac{-d/c}{2^{i-1}} + \frac{\varepsilon}{2^{i-1}}$  when

$$z_1 = - \left( \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1) \right).$$

Let us denote such time by  $T_{li}$ , which can be easily obtained by solving the ordinary differential equation

$$\dot{z}_2 = - \left( \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1) \right) (cz_2 + d),$$

with  $z_{2o} = \frac{d/c-\varepsilon}{2^{i-1}}$ . After Item (ii) of Lemma 2 there exists  $T_{lmin} > 0$  such that  $T_{li} \geq T_{lmin}$  for all  $i \in \{1, 2, \dots\}$ .

- 2) During the phase when  $\hat{z}_2$  flows from  $\frac{d/c}{2^{i-1}}$  to  $\frac{-d/c}{2^{i-1}}$ ,  $z^* = -\frac{z_{in}^*}{2^i}$ . Next, we show how to choose  $k' > 0$  to conclude that  $z_{1e}$  must take at most  $\frac{T_{li}}{2}$  units of time to enter the

ball of radius  $\frac{|z^*|}{2}$ . To that end, we use the Lyapunov function

$$v(z_{1e}) := z_{1e}^2,$$

whose time derivative along the solutions to  $\mathcal{H}_i$  satisfies

$$\dot{v} \leq -2k'v + \frac{2az_{in}^*}{2^i} \varepsilon g(i-1) |z_{1e}|.$$

By assuming, for example that  $k' \geq 1$ , we conclude that

$$\dot{v} \leq -k'v + \frac{a^2 \varepsilon^2 z_{in}^{*2}}{2^{2i}} g(i-1)^2,$$

Then, for each  $t \geq 0$  such that  $(t, 0) \in \text{dom } x$ ,

$$v(t, 0) \leq v(0, 0) \exp^{-k't} + \left[ \frac{a \varepsilon z_{in}^*}{2^i \sqrt{k'}} g(i-1) \right]^2 [1 - \exp^{-k't}]$$

so, choosing  $k'$  large such that  $k' \geq 4a^2 \varepsilon^2$ , we obtain

$$\begin{aligned} v(t, 0) &\leq v(0, 0) \exp^{-k't} + \left[ \frac{z_{in}^*}{2^{i+1}} g(i-1) \right]^2 [1 - \exp^{-k't}] \\ &\leq \left[ \frac{2z_{in}^*}{2^i} + \frac{2z_{in}^*}{2^i} g(i-1) \right]^2 \exp^{-k't} \\ &\quad + \left[ \frac{z_{in}^*}{2^{i+1}} g(i-1) \right]^2 [1 - \exp^{-k't}] \\ &\leq 9z^{*2} \exp^{-k't} + \frac{z^{*2}}{4} [1 - \exp^{-k't}]. \end{aligned}$$

Next, by taking  $k'$  such that

$$k' \geq -\frac{2 \ln(2^{-5})}{T_{lmin}},$$

we conclude that

$$z_{1e}(t, 0)^2 \leq \left( \frac{z^*}{2} \right)^2 \quad \forall t \in \left[ \frac{T_{li}}{2}, T_{li} \right]. \quad (34)$$

Hence, during the interval where  $\hat{z}_2$  flows from  $\frac{d/c}{2^{i-1}}$  to  $-\frac{d/c}{2^{i-1}}$ , we have  $z_1 \in [z^*, z^*/2]$  for all  $t$  belonging to a sub interval of length larger than  $T_{li}/2$ .

- 3) Next, we estimate an upper bound on the time that  $\hat{z}_2$  takes to flow from  $\frac{d/c}{2^{i-1}}$  to  $-\frac{d/c}{2^{i-1}}$ . Using (33) and (34), we conclude that such a time is upper bounded by the time that  $z_2$  takes to flow from  $\frac{d/c}{2^{i-1}} + \frac{\varepsilon}{2^{i-1}}$  to  $-\frac{d/c}{2^{i-1}} - \frac{\varepsilon}{2^{i-1}}$  when

$$z_1(t) = \begin{cases} \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1) & \forall t \in [0, \frac{T_{li}}{2}] \\ -\frac{|z^*|}{2} & \forall t \geq \frac{T_{li}}{2}. \end{cases}$$

Let us denote such a time by  $T_{ui}$ . Note that  $T_{ui}$  can be easily obtained by solving the linear switched dynamics:

$$\dot{z}_2 = \begin{cases} \left[ \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1) \right] (cz_2 + d) & \forall t \in [0, \frac{T_{li}}{2}] \\ -\frac{|z^*|}{2} (cz_2 + d) & \forall t \geq \frac{T_{li}}{2}, \end{cases}$$

from the initial condition  $z_{2o} = \frac{d/c+\varepsilon}{2^{i-1}}$ .

Finally, we conclude that Assumption 1 holds on Cycle  $i$  with

$$\begin{aligned} \tau_{di} &:= \frac{T_{li}}{2}, & \tau_{si} &:= T_{ui} - \frac{T_{li}}{2}, \\ \bar{z}_i &:= \frac{z_{in}^*}{2^{i-1}} + \frac{z_{in}^*}{2^{i-1}} g(i-1), & \underline{z}_i &:= \frac{|z^*|}{2}, \end{aligned}$$

which completes the proof.  $\blacksquare$