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Distributed Bipartite Containment Tracking over Signed Networks with Multiple Leaders

Pelin Şekercioğlu Elena Panteley Ioannis Sarras Antonio Loría Julien Marzat

Abstract—We address the distributed bipartite containment tracking-control problem for second-order systems steered by multiple cooperative and antagonistic leaders. The influence of the former is represented via positive links, while negative interconnections are used for the latter, thereby generating a signed network. Because of the presence of diverse leaders, having positive or negative influence on the followers, complete consensus is impossible. Instead, the followers' respective states may only converge to a residual compact set, not predefined, but resulting from the agents' initial conditions and the network's topology. For this set, which is called bipartite containment set, we establish global exponential stability and we compute the specific equilibria to which all agents converge inside the containment set (this is called multiconsensus). In addition, we provide strict Lyapunov functions for signed-graph networks and establish robustness of the bipartite containment tracking control. Finally, we illustrate our theoretical findings via numerical simulations.

Index Terms—Multiconsensus, bipartite containment tracking, containment, autonomous vehicles, signed graphs

I. INTRODUCTION

In the study of coordination of multiagent networks, a large number of problems have been extensively addressed, such as consensus for first-order [1], second-order [2] and linear high-order dynamics [3]. In particular, when there is a leader in the network, all followers converge to the leader's state [4]. However, this so-called leader-follower consensus does not occur when the network contains more than one leader. For this case, in which multiple consensus equilibria may appear, it appears more appropriate to speak of *containment control* [5]. This problem consists in making all the followers' states converge to a convex hull that is determined by the leaders' initial conditions and is solvable if for each follower there exists at least one leader that has a directed path to that follower.

There are multiple studies on distributed containment control, e.g., for social networks [6] or for networks of single-integrators [7], [8], double-integrators [9], [10], and general linear autonomous systems [11]. However, most of the current research on the consensus or containment problems for multiagent systems is on cooperative networks, i.e., the coordination

of nodes is achieved only by cooperative interactions. Yet, there are many scenarii in which agents may be competitive or even disruptive. These may appear, e.g., in robotics applications, in the context of herding control [12], in aerospace applications involving control of multiple satellites that must avoid face debris represented as non-cooperative leaders, and in the context of social networks that include deceiving influencers injecting disinformation [13], to mention a few. In this case, the interconnections may be either positive or negative and at least two consensus equilibria appear—we speak of bipartite consensus [13], [14]. In the case that multiple leaders (cooperative and/or antagonistic) appear, moreover, the overall behavior is even more complex [15] and more than two consensus equilibria may appear [16]–[18]. Thus, signed interconnections and leader multiplicity are distinct aspects that may appear in a number of usual scenarios involving networked systems.

To analyze directed signed networks containing multiple leaders, in [19] the notion of containment control is extended to *bipartite containment tracking-control*. Succeeding [19], bipartite containment has been also studied in [20], [21] and [22]. In general, it consists in having all the followers' states converge to a convex hull determined by the network's topology and the initial conditions of the leaders and the mirror leaders. Moreover, in [19] and [22] limit points for the followers states are given explicitly. It is also important to mention that in [20]–[22] only structurally balanced networks are considered. Roughly speaking, these are networks whose representing graph contains positive-only cycles or none at all—see Section III and [23] for details. Such networks exclude important cases, such as certain agents capable of being friends with the enemies of their friends. Furthermore, in [20], only cooperative leaders are considered and in [22] interconnections between the followers are assumed to be only undirected.

As in [19], in this paper we study the bipartite containment tracking problem over structurally balanced and unbalanced signed networks with multiple cooperative or competitive leaders. Also, as in [19], we allow leaders to have neighbors. In contrast to the latter reference, however, our results apply to second-order systems, which better describe mechanical systems and a variety of (feedback linearizable) autonomous vehicles [24]. In contrast to [20]–[22] our main statements hold for the general case of structurally unbalanced networks and, beyond bipartite containment tracking, we provide explicit estimates of the limit points of the followers.

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Furthermore, we give sufficient conditions for exponential stability of the containment set and, in contrast to all references mentioned previously, our proofs are constructive; we provide a strict Lyapunov function regardless of whether the network is structurally balanced or unbalanced. Exponential stability is important because it covers the much weaker property of mere convergence to the limit points (or to the interior of a convex hull); providing strict Lyapunov functions is significant because they are a basis to establish, in addition, input-to-state stability. Thus, our main results guarantee robustness of the containment set with respect to additive perturbations.

From a technical viewpoint, our main results are based on the framework introduced in [25]. We recast the bipartite containment problem into one of stability of a set of the appropriately defined errors. Then, we generalize a statement in [26] on the Lyapunov characterization of exponential stability of sets, for linear systems with one pole at zero and others having negative real part. The technical result we provide here applies to systems with several poles at the origin. Then, in addition to exponential stability of the containment set, we provide the explicit limit values of the followers by a matrix determined by all eigenvectors associated to the zero eigenvalues. In particular, when the considered graph is a traditional cooperative network, the bipartite containment tracking results stay valid for containment tracking.

The remainder of the paper is organized as follows. In the next section, we describe the bipartite containment-tracking control problem and how it may be recast as one of stability analysis. In Section III we present the technical statements described above. In Section IV we present our main results. Some numerical examples are presented in Section V, and we wrap up the paper with some closing remarks in Section VI.

II. PROBLEM FORMULATION

Consider a group of n second-order dynamical systems modeled by

$$\dot{x}_{1_i} = x_{2_i}, \quad x_{1_i}, x_{2_i} \in \mathbb{R} \quad (1a)$$

$$\dot{x}_{2_i} = u_i, \quad u_i \in \mathbb{R}, i \in \mathcal{I}_{\mathcal{N}} \quad (1b)$$

where $\mathcal{I}_{\mathcal{N}} := \{1, 2, \dots, n\}$.

Remark 1: The system (1) is a basic representation of (feedback-linearizable) mechanical systems; it is for notational simplicity, and without loss of generality, that we assume that $x_{1_i} \in \mathbb{R}$, but the contents of this paper apply to systems of higher dimension—see Section V.

The consensus problem for (1), that is, to ensure that

$$\lim_{t \rightarrow \infty} [x_j(t) - x_i(t)] = 0, \quad \forall i, j \leq n, \quad (2)$$

where $x_i = [x_{1_i} \quad x_{2_i}]^\top$, via distributed control, is completely solved under various conditions on the interconnections and the resulting network's topology. For instance, for static directed networks, it is well-known that under the consensus-control law,

$$u_i = -k_1 \sum_{j=1}^n a_{ij}(x_{1_i} - x_{1_j}) - k_2 \sum_{j=1}^n a_{ij}(x_{2_i} - x_{2_j}), \quad (3)$$

where $a_{ij} \in \mathbb{R}$ is the adjacency weight between nodes i and j , with $k_1, k_2 > 0$, the expressions in (2) hold if and only if the underlying graph contains a directed spanning tree [27]. Moreover, the consensus equilibrium may be computed exactly. Indeed, if there exists a directed spanning tree, the resulting Laplacian matrix, $L := [\ell_{ij}] \in \mathbb{R}^{n \times n}$, where

$$\ell_{ij} = \begin{cases} \sum_{k \in \mathcal{I}_{\mathcal{N}}} a_{ik} & i = j \\ -a_{ij} & i \neq j, \end{cases} \quad (4)$$

has exactly one eigenvalue equal to zero. Therefore, the consensus equilibrium is uniquely determined by the left eigenvector v_l corresponding to that eigenvalue, $x_{1_m} := v_l^\top x_1(0)$ [27]. Furthermore, one can construct a strict Lyapunov function to establish exponential stability of the origin in the space of the synchronization errors $e := x_1 - v_r x_{1_m}$ [26], where v_r is the right eigenvector corresponding to zero eigenvalue.

Such is the case, at least, when all nodes in the network are *cooperative*, that is, if $a_{ij} \geq 0$ for all $i, j \leq n$. For networks in which some of the nodes are *competitive*, we have $a_{ij} < 0$ for *some* $i, j \leq n$. In such scenario, the distributed consensus control law (3) becomes

$$u_i = -k_1 \sum_{j=1}^n |a_{ij}|(x_{1_i} - \text{sgn}(a_{ij})x_{1_j}) - k_2 \sum_{j=1}^n |a_{ij}|(x_{2_i} - \text{sgn}(a_{ij})x_{2_j}) - k_3 x_{2_i}, \quad (5)$$

where $k_1, k_2 > 0$, and $k_3 \geq 0$. If $k_3 = 0$, bipartite containment is reached but the system's trajectories may grow unbounded. In the case that $k_3 > 0$, in addition to bipartite containment, the velocities tend to zero.

Under the distributed control law (5), agents on a directed signed network containing a directed spanning tree or a leader achieve *bipartite consensus* if and only if the underlying graph is structurally balanced [13]. If the signed network is structurally balanced, the associated Laplacian matrix has a simple zero eigenvalue [13] and its associated right eigenvector has all entries equal to ± 1 , which results in agents converging to the same state in modulus but different in signs. In such networks, moreover, it appears natural to encounter several competitive leader nodes that inject disinformation into the network. In this particular case, bipartite consensus can no longer be achieved due to the existence of multiple leaders and antagonistic interactions in the network. Agents can then achieve *bipartite containment* [19], where followers converge to the convex hull spanned by all leaders' trajectories and their mirrored trajectories, in order to avoid competitive leaders in the system.

In this paper we analyze the behavior of the networked systems (1) in closed loop with the distributed control law (5) and under the assumption that several cooperative or competitive leaders interfere. Since consensus is unattainable, the control goal is set to achieving *bipartite containment tracking*, that is,

$$\lim_{t \rightarrow \infty} [|x_j(t)| - \max_{i \in \mathcal{L}} |x_i(t)|] \leq 0, \quad j \in \mathcal{F}, \quad (6)$$

where \mathcal{L} and \mathcal{F} are the sets of leaders and followers respectively—see the Standing Assumption below. This is the problem solved, e.g., in [19]–[22]. Beyond the inequality in (6), we give the explicit limit value of the followers’ states and establish exponential stability of the containment set. This is significant because it allows to give a tight bound depending only on the initial conditions of the leaders. This is done under the following

Standing Assumption:

- 1) The network contains m leaders, which can be organized into k groups of p_i leaders included in a strongly connected subgraph, where $1 \leq k \leq m \leq n$, $i \in \mathcal{I}_k$, and $\sum_{i=1}^k p_i = m$ —cf. [19, Definition 1].
- 2) Given each follower ν_j , i.e., with $j \in \mathcal{F}$ with $\mathcal{F} := \{m+1, m+2, \dots, n\}$, there exists at least one leader ν_i i.e., with $i \in \mathcal{L} := \{1, 2, \dots, m\}$, such that there exists at least one path from ν_i to ν_j —cf. [19, Condition 1].

The Standing Assumption is mild; in the case that there exists no more than one leader group, it boils down to the necessary condition for consensus that imposes the existence of a spanning tree. As the networks considered here contain, *a priori*, multiple leaders, the resulting Laplacian matrix has as many zero eigenvalues and associated eigenvectors as the number of groups of leaders [28]. This also results in multiple convergence points for the agents. Therefore, in contrast to the consensus equilibrium $x_{1,m} = v_l^\top x_1(0)$ for single-leader networks, the final states of the agents are determined by all eigenvectors associated to the zero eigenvalues. One of this paper’s contributions is to show that under the control law (5) and the Standing Assumption, the limit-values of the agents are given by

$$x_m := (\mathbb{V} \otimes I_2)x, \quad (7)$$

where \mathbb{V} is a matrix determined by all the eigenvectors associated to the k null eigenvalues of the Laplacian. More precisely, the matrix \mathbb{V} is given by,

$$\mathbb{V} := \sum_{i=1}^k \begin{bmatrix} v_{r_{i,1}} \\ v_{r_{i,2}} \\ \vdots \\ v_{r_{i,n}} \end{bmatrix} [v_{l_{i,1}} \quad v_{l_{i,2}} \quad \dots \quad v_{l_{i,n}}], \quad (8)$$

where v_{r_i} and v_{l_i} denote, respectively, the right and left eigenvectors of the matrix L corresponding to the i th 0 eigenvalue, and, for each $j \leq n$, $v_{r_{i,j}}$ and $v_{l_{i,j}}$ denote, respectively, the j th element of the i th right and left eigenvectors. In particular, as we demonstrate further below, a possible form of the right and left eigenvectors associated with the zero eigenvalues is

$$v_{r_1} = \begin{pmatrix} 1_{p_1} \\ 0_{p_2} \\ \vdots \\ 0_{p_k} \\ \xi_1 \end{pmatrix}, v_{r_2} = \begin{pmatrix} 0_{p_1} \\ 1_{p_2} \\ \vdots \\ 0_{p_k} \\ \xi_2 \end{pmatrix}, \dots, v_{r_k} = \begin{pmatrix} 0_{p_1} \\ 0_{p_2} \\ \vdots \\ 1_{p_k} \\ \xi_k \end{pmatrix}, \quad (9a)$$

$$v_{l_1} = \begin{pmatrix} \rho_{p_1} \\ 0_{p_2} \\ \vdots \\ 0_{p_k} \\ 0_{n-m} \end{pmatrix}, v_{l_2} = \begin{pmatrix} 0_{p_1} \\ \rho_{p_2} \\ \vdots \\ 0_{p_k} \\ 0_{n-m} \end{pmatrix}, \dots, v_{l_k} = \begin{pmatrix} 0_{p_1} \\ 0_{p_2} \\ \vdots \\ \rho_{p_k} \\ 0_{n-m} \end{pmatrix}, \quad (9b)$$

where $\xi_i \in \mathbb{R}^{n-m}$ is defined in Lemma 1 farther below, $1_{p_i} \in \mathbb{R}^{p_i}$ is a vector of ones, $0_{p_i} \in \mathbb{R}^{p_i}$ and $0_{n-m} \in \mathbb{R}^{n-m}$ are vectors of zeros, and $\rho_{p_i} \in \mathbb{R}^{p_i}$. If $p_i = 1$, where $i \leq k$, we have $\rho_{p_i} = 1$, so the corresponding left eigenvector v_{l_i} has a unique non-zero element equal to 1. If $p_i > 1$, v_{l_i} has p_i elements belonging to $(-1, 1)$, where $\sum_{l=1}^{p_i} |\rho_l| = 1$. The definition of the eigenvectors given in (9) covers that of [28, Corollary 4.2] and [16, Proposition 3], which are restricted to unsigned networks. With this under consideration, we establish bipartite containment of the system (1) and, more significantly, that $x \rightarrow x_m$ —as defined in (7)—exponentially.

III. ANALYSIS APPROACH

Our main results are based on original statements for networks with an associated Laplacian having multiple zero eigenvalues. First, following the framework laid in [25], we show how to construct the matrix \mathbb{V} in (7)–(8), to define the average states of the agents. Then, we extend the method of [26], to construct strict Lyapunov functions for linear systems with one zero eigenvalue, to the case of multiple null eigenvalues.

To that end, we first recall certain notations and definitions that will be used later. A signed graph is said to be *structurally balanced* if it may be split into two disjoint sets of vertices \mathcal{V}_1 and \mathcal{V}_2 , where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ such that for every $i, j \in \mathcal{V}_p$, $p \in [1, 2]$ if $[a_{ij}] \geq 0$ while for every $i \in \mathcal{V}_p$, $j \in \mathcal{V}_q$, $p, q \in [1, 2]$, $p \neq q$ if $[a_{ij}] \leq 0$. It is *structurally unbalanced*, otherwise. The following definition is also recalled from [28] to introduce some useful sets of vertices in a graph.

Definition 1: A set \mathcal{R} of vertices is called a *reach* if it is a maximal reachable set that consists of a leader group and its followers. For each reach \mathcal{R}_i of a graph, we define the *exclusive part* of \mathcal{R}_i as the set $\mathcal{H}_i = \mathcal{R}_i \setminus \bigcup_{j \neq i} \mathcal{R}_j$, i.e., the set of followers influenced only by the leader group i , and the *common part* of \mathcal{R}_i as the set $\mathcal{C}_i = \mathcal{R}_i \setminus \mathcal{H}_i$, i.e., the set of followers influenced by leaders other than the i th leader group.

The following statement, which leads to the construction of the matrix \mathbb{V} , is an original contribution of this paper and extends Corollary 4.2 of [28] to the case of signed networks.

Lemma 1: Let \mathcal{G} denote a directed signed graph and let L denote the associated Laplacian matrix. Suppose \mathcal{G} has n vertices and k reaches. Then, the algebraic and geometric multiplicity of the eigenvalue 0 is equal to k .

Furthermore, if \mathcal{G} is structurally unbalanced, the eigenspace generated by the eigenvectors associated to the null eigenvalues, has a basis defined by the vectors $\gamma_i \in \mathbb{R}^n$, with $i \leq k$, whose elements satisfy the following:

- 1) $\gamma_{i,j} = 0$ for $j \notin \mathcal{R}_i$,
- 2) $|\gamma_{i,j}| = 1$ for $j \in \mathcal{H}_i$,
- 3) $|\gamma_{i,j}| \leq 1$ for $j \in \mathcal{C}_i$,
- 4) $\sum_j |\gamma_i| \leq \mathbf{1}_n$.

On the other hand, in the more restrictive case that \mathcal{G} is structurally balanced,

- 1') $\gamma_{i,j} = 0$ for $j \notin \mathcal{R}_i$
- 2') $\gamma_{i,j} = \begin{cases} 1, & \text{if } (\nu_j, \nu_i) \in \mathcal{V}_1 \\ -1, & \text{if } \nu_j \in \mathcal{V}_1, \nu_i \in \mathcal{V}_2 \end{cases}$ for $j \in \mathcal{H}_i$
- 3') $\gamma_{i,j} \in \begin{cases} (0, 1), & \text{if } (\nu_j, \nu_i) \in \mathcal{V}_1 \\ (-1, 0), & \text{if } \nu_j \in \mathcal{V}_1, \nu_i \in \mathcal{V}_2 \end{cases}$ for $j \in \mathcal{C}_i$
- 4') $\sum_j |\gamma_i| = \mathbf{1}_n$,

where \mathcal{V}_1 and \mathcal{V}_2 are the two disjoint sets of vertices, $i \in \mathcal{I}_K$, $j \in \mathcal{I}_N$, and $\gamma_{i,j}$ denotes the j th element of γ_i . \square

Proof: Let $L_+ := D - |A|$ denote the associated Laplacian matrix of \mathcal{G}_+ , where D is the in-degree matrix and A is the adjacency matrix of \mathcal{G} . \mathcal{G}_+ and \mathcal{G} have identical reaches. Then, the fact that the algebraic and geometric multiplicity of 0 equals k follows from [28, Theorem 3.2].

Now, for structurally-balanced graphs, Items 1) and 2), follow from Definition 1, under the Standing Assumption. Item 3) follows from [15], while Item 4) results from computing the null space of L , which is generated by $\gamma_i \in \mathbb{R}^n$ such that $L\gamma_i = 0$ for each $i \leq k$. This gives $L \sum_{i=1}^k \gamma_i = 0$. Then, under the Standing Assumption, the Laplacian matrix being equal to

$$L = \begin{bmatrix} 0 & 0 \\ -A_{lf} & L_f + \Delta_{|A_{lf}|} \end{bmatrix},$$

where $A_{lf} \in \mathbb{R}^{(n-m) \times m}$ and $L_f + \Delta_{|A_{lf}|} \in \mathbb{R}^{(n-m) \times (n-m)}$, we have $-A_{lf} \sum_{i=1}^k \gamma_{i,m} + (L_f + \Delta_{|A_{lf}|}) \sum_{i=1}^k \gamma_{i,n-m} = 0$. This gives us $(L_f + \Delta_{|A_{lf}|}) \sum_{i=1}^k \gamma_{i,n-m} = A_{lf} \sum_{i=1}^k \gamma_{i,m}$, where $\sum_{i=1}^k \gamma_{i,m} = \mathbf{1}_m$ from (9). The sum of the remaining rows of γ_i s gives the following, $\sum_{i=1}^k \gamma_{i,n-m} = (L_{nr} + \Delta_{|A_{rnr}|})^{-1} A_{rnr} \mathbf{1}_m$ and it follows that $\sum_j |\gamma_{i,n-m}| \leq \mathbf{1}_{n-m}$ from Lemma 6 of [19]. Thus, we conclude that $\gamma_{i,j} = 0$ for $j \notin \mathcal{R}_i$, $|\gamma_{i,j}| = 1$ for $j \in \mathcal{H}_i$, $|\gamma_{i,j}| \leq 1$ for $j \in \mathcal{C}_i$, and $\sum_j |\gamma_i| \leq \mathbf{1}_n$.

On the other hand, if the graph is structurally balanced, we may apply the *gauge transformation*, which consists in a change of coordinates performed by the matrix $D = \text{diag}(\sigma)$, where $\sigma = [\sigma_1, \dots, \sigma_n]$, $\sigma_j \in \{1, -1\}$, $j \in \mathcal{I}_N$ [13]. Let $L_D = DLD$ denote the unsigned Laplacian matrix of the transformed network. Then, we may express the Laplacian L in Jordan canonical form, as $L = DL_D D = DP_D \Lambda P_D^{-1} D$, where $P_D = [v_{D_1} \dots v_{D_k} P_{D_1}] \in \mathbb{C}^{n \times n}$, $P_D^{-1} = [w_{D_1}^\top \dots w_{D_k}^\top P_{D_1}^\dagger]^\top \in \mathbb{C}^{n \times n}$, and $\Lambda \in \mathbb{C}^{n \times n}$, with v_{D_i} and w_{D_i} , $i \leq m$ are the right and left eigenvectors associated to k zero eigenvalues of L_D . From the Jordan decomposition, we can see that the basis of the null space of L is given by $LD\gamma_{D_i} = LDv_{D_i} = 0$, $i \leq k$ and has a basis defined by the columns of $\gamma = D\gamma_D$, where $\gamma_D = [\gamma_{D_1} \dots \gamma_{D_k}]$ and the columns $\{\gamma_{D_i}\}$ constitute the basis of the associated eigenspace of L_D . Then, using

[28, Corollary 4.2], we obtain the following for a structurally balanced signed network: $\gamma_{i,j} = \sigma_j \gamma_{D_{i,j}} = 0$ for $j \notin \mathcal{R}_i$, $\gamma_{i,j} = \sigma_j \gamma_{D_{i,j}} = \sigma_j$ for $i \in \mathcal{H}_i$, $\gamma_{i,j} = \sigma_j \gamma_{D_{i,j}} \in \sigma_j(0, 1)$ for $j \in \mathcal{C}_i$ and $\sum_j \gamma_i = \sum_j D\gamma_{D_i} = D\mathbf{1}_n$, which gives $\sum_j |\gamma_i| = \mathbf{1}_n$. Items 1') to 4') follow. \blacksquare

From Lemma 1, setting $v_{r_{i,j}} = \gamma_{i,j}$, we obtain the form given in (9a) for the k right eigenvectors associated to the zero eigenvalues, such that for each $j \leq m$, $v_{r_{i,j}}$ is either equal to 1 or to 0, because the j th leader can only be in the exclusive part of its corresponding reach. The remaining rows $\xi_{i,j}$ of v_{r_i} belong, either to $\{-1, 1\}$ or to $(-1, 1)$, depending on network's topology and signs of the interconnections. Moreover, under the Standing Assumption and the given form in (9a) for right eigenvectors, as the Laplacian matrix has all entries equal to 0 for its first m rows, we obtain (9b) for k left eigenvectors associated to zero eigenvalues. Notice that, because of the form of the left eigenvectors, each column of the matrix \mathbb{V} has the same properties as the basis defined in Lemma 1. Therefore, we may split \mathbb{V} in four blocks, as follows:

$$\mathbb{V} = \sum_{i=1}^k v_{r_i} v_{l_i}^\top = \left[\begin{array}{c|c} V_l & 0_{m \times (n-m)} \\ \hline V_f & 0_{(n-m) \times (n-m)} \end{array} \right], \quad (10)$$

where $V_l \in \mathbb{R}^{m \times m}$ represents leaders' interactions and $V_f \in \mathbb{R}^{(n-m) \times m}$ represents leader-follower interactions.

It is important to remark that the elements of V_f have the properties stated in Lemma 1. This is significant because, owing to the fact that $x \rightarrow x_m$ where x_m is defined by (7), it is clear that V_f is the matrix that defines the limit point for the followers. Notice that even if during their trajectories, depending on network's topology, followers' states may be influenced by other followers' states, it follows from (10) that the final states of the followers are defined only by leaders' states.

Now, akin to the case of networks containing one leader, in which case the error is defined as $e = x_1 - v_r x_{1,m}$, with $x_{1,m} := v_l^\top x_1$, for multi-leader networks we define the consensus errors as

$$e := ([I - \mathbb{V}] \otimes I_2)x, \quad (11)$$

where $e = [e_1 \ e_2]^\top$. Then, to establish beyond the convergence statements that $x_1 \rightarrow x_{1,m}$ and $x_2 \rightarrow x_{2,m}$ and, consequently, the bipartite containment objectives defined by (6), we will prove the stronger property of global exponential stability of the set $\{(e_1, e_2) = (0, 0)\}$. For that, we shall show how to construct strict—in the space of (e_1, e_2) —Lyapunov functions, based on the following proposition, which is another original contribution of this paper and extends Proposition 1 of [26] to the case of signed networks with multiple leaders.

Proposition 1: Let \mathcal{G} be a directed signed network with multiple leaders. Then the following are equivalent:

- (i) the graph has k groups of leaders and m leaders, and given each follower $\nu_j, \forall j \in \mathcal{F}$, there exists at least one leader $\nu_i, \forall i \in \mathcal{L}$, such that there exists at least one path from ν_i to ν_j ,

(ii) for any $Q \in \mathbb{R}^{n \times n}$, $Q = Q^\top > 0$ and for any $\alpha_i > 0$, there exists a matrix $P(\alpha_i) \in \mathbb{R}^{n \times n}$, $P = P^\top > 0$ such that

$$PL + L^\top P = Q - \sum_{i=1}^k \alpha_i (Pv_{ri}v_{li}^\top + v_{li}v_{ri}^\top P), \quad (12)$$

where $v_{ri}, v_{li} \in \mathbb{R}$ are the right and left eigenvectors of L associated with the i th 0 eigenvalue. \square

Remark 2: Note that the Proposition provides a Lyapunov characterization of the second part of the Standing Assumption. \bullet

Proof: (i) \Rightarrow (ii): By assumption, the graph \mathcal{G} has k leader groups and is connected. Then, from Lemma 1, it follows that L has k zero eigenvalues, and the rest of its eigenvalues have positive real parts: $0 = \lambda_1 = \dots = \lambda_k < \Re e(\lambda_{k+1}) \leq \dots \leq \Re e(\lambda_n)$. Following the lines of proof as for Lemma 2 in [26], we can write the Jordan decomposition of L as $L = U\Lambda U^{-1} = \sum_{i=1}^k \lambda_i(L)v_{ri}v_{li}^\top + U_1\Lambda_1U_1^\top$ with $\Lambda_1 \in \mathbb{C}^{n-k \times n-k}$, $U = \begin{bmatrix} v_{r_1} & \dots & v_{r_k} & U_1 \end{bmatrix} \in \mathbb{C}^{n \times n}$ and $U^{-1} = \begin{bmatrix} v_{l_1}^\top & \dots & v_{l_k}^\top & U_1^\top \end{bmatrix}^\top \in \mathbb{C}^{n \times n}$. For any $\alpha_i > 0$ define $R(\alpha_i) = L + \sum_{i=1}^k \alpha_i v_{ri}v_{li}^\top$. From this decomposition and the properties of Λ_1 , $\Re e\{\lambda_j(R)\} > 0$ for all $j \leq n$. $-R(\alpha_i)$ is Hurwitz, therefore for any $Q = Q^\top > 0$ and $\alpha_i > 0$, $i \leq m$, there exists $P = P^\top > 0$ such that

$$\begin{aligned} -PR(\alpha_i) - R(\alpha_i)^\top P &= -Q, \\ -P(L + \sum_{i=1}^k \alpha_i v_{ri}v_{li}^\top) - (L + \sum_{i=1}^k \alpha_i v_{ri}v_{li}^\top)^\top P &= -Q, \\ PL + L^\top P &= Q - \sum_{i=1}^k \alpha_i (Pv_{ri}v_{li}^\top + v_{li}v_{ri}^\top P). \end{aligned}$$

(ii) \Rightarrow (i): Let statement (ii) hold and assume that the Laplacian matrix has $k + 1$ zero eigenvalues and the rest of its eigenvalues have positive real parts. In view of Lemma 1, the assumption that the system has k groups of leaders does not hold. Now, the Jordan decomposition of L has the form $L = U\Lambda U^{-1} = \sum_{i=1}^{k+1} \lambda_i(L)v_{ri}v_{li}^\top + U_1\Lambda_1U_1^\top$ with $U = \begin{bmatrix} v_{r_1} & \dots & v_{r_{k+1}} & U_1 \end{bmatrix}$ and $U^{-1} = \begin{bmatrix} v_{l_1}^\top & \dots & v_{l_{k+1}}^\top & U_1^\top \end{bmatrix}^\top$. Next let us consider $R(\alpha_i) = L + \sum_{i=1}^k \alpha_i v_{ri}v_{li}^\top$ which admits the Jordan decomposition $R := U\Lambda_R U^{-1}$, where

$$\Lambda_R := \begin{bmatrix} \alpha_1 & & & & \\ & \ddots & & & \\ & & \alpha_k & & \\ & & & 0 & \\ & & & & \Lambda_1 \end{bmatrix}.$$

Clearly, R is not positive definite because one of its eigenvalues is equal to zero. Then, there exists a matrix $Q = Q^\top$ for which there does not exist a matrix $P = P^\top$ such that $-PR - R^\top P = -Q$, which contradicts statement (ii). \blacksquare

IV. BIPARTITE CONTAINMENT OF SECOND-ORDER SYSTEMS

In this section, we present our main results. We will consider bipartite containment protocols for second-order systems, with and without absolute velocity damping. Furthermore, we establish robustness of the bipartite containment tracking in the sense of input-to-state stability with respect to external bounded perturbations.

A. Exponential stability

Consider the system (1), interconnected with the bipartite containment control law (5). We analyze the dynamics of the errors in (11). Differentiating the latter on both sides, to obtain

$$\dot{e} = ([I - \mathbb{V}] \otimes I_2)\dot{x}$$

and using (1) and (5), we obtain the closed-loop dynamical equations

$$\dot{e}_1 = e_2 \quad (13a)$$

$$\dot{e}_2 = -k_1 L e_1 - k_2 L e_2 - k_3 e_2. \quad (13b)$$

Guaranteeing the bipartite containment problem is now recast as a problem of stability analysis of the dynamical system (13). Thus, relying on Proposition 1, our next statement provides sufficient conditions on the controller gains to achieve global exponential stability of the set $\{(e_1, e_2) = (0, 0)\}$, which covers the bipartite containment tracking objective (6).

Proposition 2: Consider the system (13) and let P be generated by (12) with $Q = I_n$. Then, under the Standing Assumption, the set $\{(e_1, e_2) = (0, 0)\}$ is exponentially stable if

$$k_1 > 0, \quad (14a)$$

$$k_2 > 2\sqrt{k_1 \bar{\lambda}_P}, \quad (14b)$$

$$\frac{k_2}{k_3} \geq \bar{\lambda}_P, \quad k_3 \geq 0, \quad (14c)$$

where $\bar{\lambda}_P \geq |P|$ is the largest eigenvalue of P . \square

Remark 3: For conciseness, we consider in one statement two interesting and distinct cases, in which $k_3 > 0$ and in which $k_3 = 0$. Even though the value of $k_3 \geq 0$ is inconsequential for exponential stability statement to hold, it plays a role on performance and the solutions' behavior. Indeed, $k_3 > 0$ implies the injection of velocity damping, hence, it guarantees that the agents states x_{1_i} converge to constant values, while $x_{2_i} \rightarrow 0$. In the case that $k_3 = 0$, containment is achieved in both variables x_{1_i} and x_{2_i} but the solutions grow unbounded. In the particular case of vehicles moving on a plane, these cases are tantamount to the vehicles converging to a rendezvous zone or permanently moving in formation and in containment, steered by the leaders. See Proposition 3 further below. \bullet

Proof of Proposition 2: Let $Q = Q^\top > 0$ and $\alpha > 0$ be arbitrarily fixed. Since by the Standing Assumption and Proposition 1, $\exists P = P^\top > 0$ such that (12) holds. Then, consider the following Lyapunov function candidate

$$V(e) = \frac{1}{2}|e_1|^2 + \epsilon e_1^\top P e_2 + \mu e_2^\top P e_2, \quad (15)$$

which is positive definite (for all e as in (11)), under the condition $\epsilon \leq \sqrt{\frac{2\mu}{|P|}}$, $\mu > 0$. The total time derivative of V along the trajectories yields

$$\begin{aligned}\dot{V}(e) &= -k_1\epsilon e_1^\top P L e_1 \\ &\quad - e_2^\top [k_2\mu(PL + L^\top P) + 2k_3\mu P - \epsilon P] e_2 \\ &\quad + e_1^\top [I - k_2\epsilon PL - 2k_1\mu L^\top P - k_3\epsilon P] e_2.\end{aligned}$$

Let $\mu > 0$ be such that $2k_1\mu = k_2\epsilon = 1$. Using (12) with $Q = I_n$ and the identity $P \sum_{i=1}^k v_{r_i} v_{l_i}^\top e_1 = P \sum_{i=1}^k v_{r_i} v_{l_i}^\top e_2 = 0$ we obtain the following:

$$\begin{aligned}-k_1\epsilon e_1^\top P L e_1 &= -k_1\epsilon e_1^\top (PL + L^\top P) e_1 = -\frac{1}{2}k_1\epsilon |e_1|^2, \\ -e_2^\top \mu k_2 (PL + L^\top P) e_2 &= -\mu k_2 e_2^\top e_2, \\ e_1^\top [I - (L^\top P + PL)] e_2 &= -e_1^\top [I - I] e_2 = 0.\end{aligned}$$

Hence, in compact form, we have

$$\begin{aligned}\dot{V}(e) &= -\frac{1}{2}e^\top \begin{bmatrix} \frac{1}{2}k_1\epsilon I & k_3\epsilon P \\ k_3\epsilon P & 4k_3\mu P \end{bmatrix} e \\ &\quad - \frac{1}{4}k_1\epsilon |e_1|^2 - e_2^\top [k_2\mu I - \epsilon P] e_2,\end{aligned}\quad (16)$$

so \dot{V} is negative definite if the matrix in (16) is positive semi-definite and $[k_2\mu I - \epsilon P]$ is positive definite, which, using $2k_1\mu = k_2\epsilon = 1$ we see that it is so if and only if

$$k_2 I - 2\frac{k_1}{k_2}\bar{\lambda}_P I > 0,$$

which, in turn, holds if $k_2 > \sqrt{2k_1\bar{\lambda}_P}$, that is, under condition (14b). Then, computing the Schur complement, the condition for the matrix in (16) to be positive semi-definite, is $4k_3\mu P - (k_3\epsilon P)^\top \frac{2}{k_1\epsilon} k_3\epsilon P \geq 0$. If $k_3 = 0$ the latter holds trivially. Otherwise, considering $k_3 > 0$, and using $2\frac{k_1}{k_2} = \frac{\epsilon}{\mu}$, we see that the latter inequality is equivalent to

$$2\frac{\mu}{\mu} I - \frac{k_3}{k_1} \frac{\epsilon}{\mu} P = 2\frac{\mu}{\mu} I - \frac{k_3}{k_1} \frac{2k_1}{k_2} P = 1 - \frac{k_3}{k_2} \bar{\lambda}_P \geq 0,$$

which is satisfied under (14c). We conclude that

$$\dot{V}(e) \leq -\frac{1}{4} \left[\epsilon k_1 |e_1|^2 + [k_2\mu - \epsilon \bar{\lambda}_P] |e_2|^2 \right], \quad (17)$$

so the statement of the proposition follows. \blacksquare

The following statement provides explicit expressions for the limit values of the followers' states and emphasizes the role of the gain k_3 .

Proposition 3: Consider the system (1) and the bipartite containment control law (5). Under the Standing Assumptions, the bipartite containment objective is achieved, that is, the inequalities (6) hold. Furthermore, if $k_3 > 0$, the final states of the followers satisfy

$$\lim_{t \rightarrow \infty} x_{1_f}(t) = V_f x_{1_l}(0) + \frac{1}{k_3} V_f x_{2_l}(0), \quad (18a)$$

$$\lim_{t \rightarrow \infty} x_{2_f}(t) = 0. \quad (18b)$$

On the other hand, if $k_3 = 0$,

$$\lim_{t \rightarrow \infty} x_{1_f}(t) = V_f x_{1_l}(0) + t V_f x_{2_l}, \quad (19a)$$

$$\lim_{t \rightarrow \infty} x_{2_f}(t) = V_f x_{2_l}, \quad (19b)$$

where x_l and x_f are leaders' and followers' states respectively and $V_f \in \mathbb{R}^{(n-m) \times m}$ is given in (10). \square

Proof: Differentiating the weighted average of the system (7), we obtain the dynamical equations below

$$\dot{x}_{1_m} = \mathbb{V} \dot{x}_1 = \mathbb{V} x_2 = x_{2_m}, \quad (20a)$$

$$\begin{aligned}\dot{x}_{2_m} &= \mathbb{V} \dot{x}_2 \\ &= \mathbb{V}(-k_1 L x_1 - k_2 L x_2 - k_3 x_2) = -k_3 x_{2_m},\end{aligned}\quad (20b)$$

with $v_{l_i}^\top L = 0$ for $i \leq k$. Their solutions give the following

$$x_{2_m}(t) = x_{2_m}(0) e^{-k_3 t}, \quad (21a)$$

$$x_{1_m}(t) = x_{1_m}(0) + x_{2_m}(0) \int_0^t e^{-k_3 s} ds. \quad (21b)$$

From Proposition 2, we have $\lim_{t \rightarrow \infty} e(t) = 0$, which results in $\lim_{t \rightarrow \infty} x(t) = x_m(t)$. From (21), we obtain $\lim_{t \rightarrow \infty} x_2(t) = 0$ and $\lim_{t \rightarrow \infty} x_1(t) = x_{1_m}(0) + \frac{1}{k_3} x_{2_m}(0)$. Then, using (10), we obtain the relations in (18). Under the Standing Assumption and from items 4) and 4') of Lemma 1, we can write

$$\begin{aligned}\lim_{t \rightarrow \infty} |x_{1_{fj}}(t)| &= \sum_{i=1}^m |v_{i,m+j} x_{1_{li}}(0) + \frac{1}{k_3} v_{i,m+j} x_{2_{li}}(0)| \\ &\leq \sum_{i=1}^m |v_{i,m+j}| |x_{1_{li}}(0) + \frac{1}{k_3} x_{2_{li}}(0)| \\ &\leq \sum_{i=1}^m |v_{i,m+j}| |x_{1_{li}}(t)| \leq \max_{1 \leq i \leq m} |x_{1_{li}}(t)|,\end{aligned}$$

where $v_{i,j}$ are the elements of the matrix \mathbb{V} in (10). In the case when $k_3 = 0$, (20) becomes

$$\dot{x}_{1_m} = x_{2_m}, \quad \dot{x}_{2_m} = 0,$$

so

$$x_{2_m}(t) = x_{2_m}(0), \quad (22a)$$

$$x_{1_m}(t) = x_{1_m}(0) + t x_{2_m}(0). \quad (22b)$$

From (22), we obtain $\lim_{t \rightarrow \infty} x_2(t) = x_{2_m}(0)$ and $\lim_{t \rightarrow \infty} x_1(t) = x_{1_m}(0) + x_{2_m}(0)t$. Then using (10), we obtain the relations in (19). Under the Standing Assumption and from items 4) and 4') of Lemma 1, we have

$$\begin{aligned}\lim_{t \rightarrow \infty} |x_{2_{fj}}(t)| &= \sum_{i=1}^m |v_{i,m+j} x_{2_{li}}| \leq \sum_{i=1}^m |v_{i,m+j}| \max_{1 \leq i \leq m} |x_{2_{li}}| \\ &\leq \max_{1 \leq i \leq m} |x_{2_{li}}|, \\ \lim_{t \rightarrow \infty} |x_{1_{fj}}(t)| &= \sum_{i=1}^m |v_{i,m+j} x_{1_{li}}(0) + t v_{i,m+j} x_{2_{li}}| \\ &\leq \max_{1 \leq i \leq m} |x_{1_{li}}(0)| + t \max_{1 \leq i \leq m} |x_{2_{li}}| \\ &\leq \max_{1 \leq i \leq m} |x_{1_{li}}(t)|,\end{aligned}$$

so (6) follows. \blacksquare

It is worth noting that for unsigned networks, the achievable objective is *containment* [5], that is,

$$\lim_{t \rightarrow \infty} [x_j(t) - \max_{i \in \mathcal{L}} x_i(t)] [x_j(t) - \min_{i \in \mathcal{L}} x_i(t)] \leq 0. \quad (23)$$

Then, for such networks, we recover the following statement from Proposition 3.

Corollary 1: Consider the system (1) with the containment control law (5) and let the associated digraph be unsigned. Under the Standing Assumption, the containment objective is achieved, that is the inequality (23) holds. \square

For structurally balanced signed networks, the bipartite containment set is refined so that the achievable objective is

$$\lim_{t \rightarrow \infty} [x_j(t) - \max_{i \in \mathcal{L}}(\sigma_i \sigma_j x_i(t))] [x_j(t) - \min_{i \in \mathcal{L}}(\sigma_i \sigma_j x_i(t))] \leq 0, \quad (24)$$

where $j \in \mathcal{F}$ and $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_n]$ with $\sigma_i = 1$ if $(i, j) \in \mathcal{V}_1$ or $\sigma_i = -1$ if $i \in \mathcal{V}_1, j \in \mathcal{V}_2$. See Section V-A for an example.

In the next section, we use the strict Lyapunov functions provided in this section in order to conduct a robustness analysis of the control law (5), in the sense of input-to-state stability of the bipartite containment tracking.

B. Robustness Analysis

Consider the perturbed second-order systems

$$\dot{x}_{1_i} = x_{2_i} \quad (25a)$$

$$\dot{x}_{2_i} = u_i + d_i(t), \quad (25b)$$

where the disturbances $d_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ are assumed to be essentially bounded locally integrable functions. Under the action of the control law (5), the system (25) becomes

$$\dot{x}_1 = x_2 \quad (26a)$$

$$\dot{x}_2 = -k_1 L x_1 - k_2 L x_2 - k_3 x_2 + d(t), \quad (26b)$$

where $d := [d_i] \in \mathbb{R}^n$. Differentiating the errors in (11) on both sides and using (26) we obtain

$$\dot{e}_1 = e_2 \quad (27a)$$

$$\dot{e}_2 = -k_1 L e_1 - k_2 L e_2 - k_3 e_2 + [I - \mathbb{V}]d(t). \quad (27b)$$

Then, we have the following.

Proposition 4: The closed-loop system (27), under the Standing Assumption, is input-to-state stable with respect to essentially bounded, locally integrable external disturbances if the conditions in (14) hold. \square

Proof: Consider the Lyapunov function (15) which is positive definite, under the condition $\epsilon \leq \sqrt{\frac{2\mu}{|P|}}$, $\mu > 0$. The total time derivative of V along the trajectories of (27) yields

$$\begin{aligned} \dot{V}(e) &= \frac{\partial V}{\partial e_1} e_2 + \frac{\partial V}{\partial e_2} (-k_1 L e_1 - k_2 L e_2 - k_3 e_2) \\ &\quad + \frac{\partial V}{\partial e_2} [I - \mathbb{V}]d. \end{aligned} \quad (28)$$

Then, from (17), we obtain

$$\begin{aligned} \dot{V}(e) &\leq -\frac{1}{4}[\epsilon k_1 |e_1|^2 + (k_2 \mu - \epsilon \bar{\lambda}_P) |e_2|^2] + \frac{\partial V}{\partial e_2} |[I - \mathbb{V}]| |d| \\ &\leq -\frac{1}{4} \epsilon k_1 |e_1|^2 - \frac{1}{4} (k_2 \mu - \epsilon \bar{\lambda}_P) |e_2|^2 \\ &\quad + \epsilon \bar{\lambda}_P e_1^\top |[I - \mathbb{V}]| |d| + 2\mu \bar{\lambda}_P e_2^\top |[I - \mathbb{V}]| |d|. \end{aligned}$$

We know that $0 \leq |[I - \mathbb{V}]| \leq |I| + |\mathbb{V}| \leq 2$, because all eigenvalues of I are equal to 1 and all eigenvalues of $|\mathbb{V}|$ are either 1 or 0. Let $\delta > 0$ such that $c_1 := \frac{1}{4}(\epsilon k_1 - \frac{4}{\delta} \epsilon \bar{\lambda}_P) > 0$ and $c_2 := \frac{1}{4}(k_2 \mu - \epsilon \bar{\lambda}_P - \frac{\delta}{8} \mu \bar{\lambda}_P) > 0$. Then,

$$\dot{V}(e) \leq -c_1 |e_1|^2 - c_2 |e_2|^2 + c_3 |d|^2,$$

with $c_3 = \bar{\lambda}_P \delta (\epsilon + 2\mu) > 0$. Thus, the system (27) is input-to-state stable with respect to the bounded external input. \blacksquare

Corollary 2: The system (27), under the Standing Assumption and over an unsigned network, is input-to-state stable with respect to an essentially bounded, locally integrable external disturbance if the conditions in (14) hold. \square

V. SIMULATION RESULTS

We provide some numerical examples on a system of nonholonomic unicycle mobile robots over both structurally balanced and unbalanced networks. Let $r_i = [r_{x_i} \ r_{y_i}]^\top$ be the inertial position, θ_i the orientation, s_i the linear speed, ω_i the angular speed, m_i the mass, J_i the moment of inertia, F_i the applied force and τ_i the applied torque. Each robot has the dynamical equation

$$\begin{pmatrix} \dot{r}_{x_i} \\ \dot{r}_{y_i} \\ \dot{\theta} \\ \dot{s}_i \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} s_i \cos \theta_i \\ s_i \sin \theta_i \\ \omega_i \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{m_i} & 0 \\ 0 & \frac{1}{J_i} \end{pmatrix} \eta_i, \quad (29)$$

where $\eta_i := [F_i \ \tau_i]^\top$. We choose a reference point $p_i = r_i + \delta_i \begin{pmatrix} \cos \theta_i \\ \sin \theta_i \end{pmatrix}$ located at a distance $\delta_i = 0.1\text{m}$ along the line that is perpendicular to the wheels' axis and we define

$$\zeta := \begin{bmatrix} r_{x_i} + \delta_i \cos \theta_i \\ r_{y_i} + \delta_i \sin \theta_i \\ s_i \cos \theta_i - \delta_i \omega_i \sin \theta_i \\ s_i \sin \theta_i + \delta_i \omega_i \cos \theta_i \\ \theta_i \end{bmatrix}. \quad (30)$$

In transformed coordinates, with $p_i = [\zeta_{1_i}^\top \ \zeta_{2_i}^\top]^\top$, we have

$$\begin{bmatrix} \dot{\zeta}_{1_i} \\ \dot{\zeta}_{2_i} \end{bmatrix} = \begin{bmatrix} \zeta_{3_i} \\ \zeta_{4_i} \end{bmatrix}, \quad (31a)$$

$$\begin{bmatrix} \dot{\zeta}_{3_i} \\ \dot{\zeta}_{4_i} \end{bmatrix} = \begin{bmatrix} -s_i \omega_i \sin \theta_i - L_i \omega_i^2 \cos \theta_i \\ s_i \omega_i \cos \theta_i - L_i \omega_i^2 \sin \theta_i \end{bmatrix} + \begin{bmatrix} \frac{1}{m_i} \cos \theta_i & -\frac{L_i}{J_i} \sin \theta_i \\ \frac{1}{m_i} \sin \theta_i & \frac{L_i}{J_i} \cos(\theta_i) \end{bmatrix} \eta_i, \quad (31b)$$

$$\dot{\zeta}_{5_i} = -\frac{1}{2\delta_i} \zeta_{3_i} \sin \zeta_{5_i} + \frac{1}{2\delta_i} \zeta_{4_i} \cos \zeta_{5_i}. \quad (31c)$$

The feedback linearizing control η_i is given by

$$\eta_i = \begin{bmatrix} \frac{1}{m_i} \cos \theta_i & -\frac{L_i}{J_i} \sin \theta_i \\ \frac{1}{m_i} \sin \theta_i & \frac{L_i}{J_i} \cos(\theta_i) \end{bmatrix}^{-1} \times \begin{bmatrix} u_i - \begin{bmatrix} -s_i \omega_i \sin \theta_i - L_i \omega_i^2 \cos \theta_i \\ s_i \omega_i \cos \theta_i - L_i \omega_i^2 \sin \theta_i \end{bmatrix} \end{bmatrix}, \quad (32)$$

which gives $[\dot{\zeta}_{1_i} \ \dot{\zeta}_{2_i}]^\top = [\zeta_{3_i} \ \zeta_{4_i}]^\top$ and $[\dot{\zeta}_{3_i} \ \dot{\zeta}_{4_i}]^\top = u_i$. Thus, we implemented (32) with u_i as in (5), $x_{1_i} = [\zeta_{1_i} \ \zeta_{2_i}]^\top$, $x_{2_i} = [\zeta_{3_i} \ \zeta_{4_i}]^\top$, $m_i = 8\text{kg}$ and $J_i = 0.12\text{kg/m}^2$ for each $i \in \mathcal{I}_{\mathcal{N}}$.

A. Structurally Balanced Networks

Consider a network with five leaders $\nu_i, i \leq 5$, organized in three leader groups $\{\nu_1, \nu_2, \nu_3\}, \{\nu_4\}, \{\nu_5\}$, and four followers $\nu_j, 6 \leq j \leq 9$, communicating over a directed graph as the one depicted in Figure 1, below.

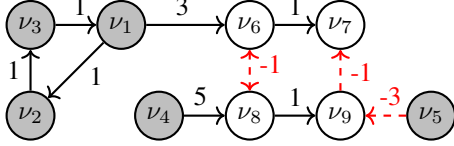


Fig. 1. Network 1: A network of nine mobile robots

The Laplacian matrix corresponding to the graph is

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & -5 & 0 & 1 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & -1 & 0 & 4 & 0 \end{bmatrix}$$

and its eigenvalues are $\lambda_L = 0, 0, 0, 1.5 \pm 0.86i, 2, 3.59, 4, 6.41$. The network may be bipartitioned into two subgroups, $\mathcal{V}_1 = \{\nu_1, \nu_2, \nu_3, \nu_5, \nu_6, \nu_7\}$, $\mathcal{V}_2 = \{\nu_4, \nu_8, \nu_9\}$, so is structurally balanced, and

$$V_l = \begin{bmatrix} 0.33 & 0.33 & 0.33 & 0 & 0 \\ 0.33 & 0.33 & 0.33 & 0 & 0 \\ 0.33 & 0.33 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (33)$$

$$V_f = \begin{bmatrix} 0.26 & 0.26 & 0.26 & -0.22 & 0 \\ 0.14 & 0.14 & 0.14 & -0.22 & 0.37 \\ -0.04 & -0.04 & -0.04 & 0.87 & 0 \\ -0.01 & -0.01 & -0.01 & 0.22 & -0.75 \end{bmatrix}. \quad (34)$$

We notice that V_l and V_f have the properties stated on Items 1') – 4') of Lemma 1, as the network is structurally balanced. For V_l , we have two elements equal to one, corresponding to isolated leaders ν_4 and ν_5 , as they are in the exclusive part of the reaches. Leaders ν_1, ν_2 and ν_3 are in the exclusive part of a reach so the corresponding elements of the right eigenvector are equal to one. Thus, as they are interconnected within a strongly-connected graph, the absolute value of the elements of the left eigenvector are less than one. V_f does not have an element equal to ± 1 since all the followers are influenced by more than one leader. Followers ν_6 and ν_8 are not influenced by the leader ν_5 , so the corresponding elements on the fifth column are equal to zero. We also remark that the sum of the absolute value of the terms on each row is equal to 1.

Now, let P be generated by (12) with $Q = I_n$ and $\alpha = 20$, then we obtain $\bar{\lambda}_P = 0.5193$. Consider the system (29) and the bipartite containment law (32) with $k_1 = 0.8, k_2 = 1.5$, and $k_3 = 1.2$, which satisfy the conditions in (14). The respective agents' initial states are $r_x(0) = [3.5 \ 1 \ -0.5 \ -1.6 \ 1 \ -6.5 \ 5.5 \ -3.5 \ 6]^\top$, $r_y(0) = [3 \ 2.5 \ -1.5 \ 1.3 \ -3 \ -1 \ -3 \ -3 \ -2.5]^\top$, $\theta(0) = [0.25 \ 0.61 \ -1.27 \ 0.32 \ 1.42 \ 0.46 \ 0.98 \ 1.19 \ 1.03]^\top$ and $s(0) = [1.2 \ 1.2 \ -1.4 \ -1.6 \ 1.3 \ -2.2 \ 0.4 \ -1.1 \ 1.2]^\top$. The simulation results are showed in Figures 2 and 3. The followers converge to the bipartite containment set spanned by cooperative leaders' final states and competitive leaders' mirrored final states and all agents' velocities converge to zero. Using the relations in (18) and the coordinate transformation, we obtain the following limit values for followers' states, $\lim_{t \rightarrow \infty} r_x(t) = [1.99 \ 1.89 \ -2.68 \ -1.46]^\top$, $\lim_{t \rightarrow \infty} r_y(t) = [1.31 \ -0.15 \ 0.44 \ 1.44]^\top$, and $\lim_{t \rightarrow \infty} s(t) = [0 \ 0 \ 0 \ 0]^\top$.

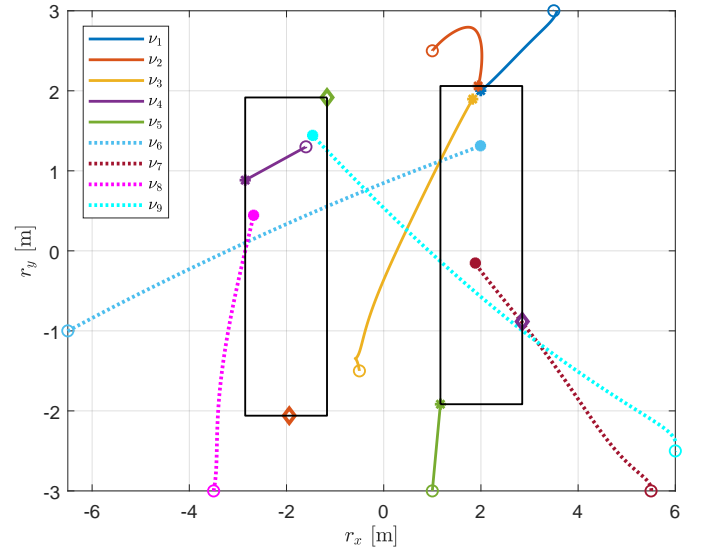


Fig. 2. Bipartite containment tracking of (29) with the control (5) on the plane. The filled dots are the final states of the agents. The diamonds represent the mirrored final states of the leaders.

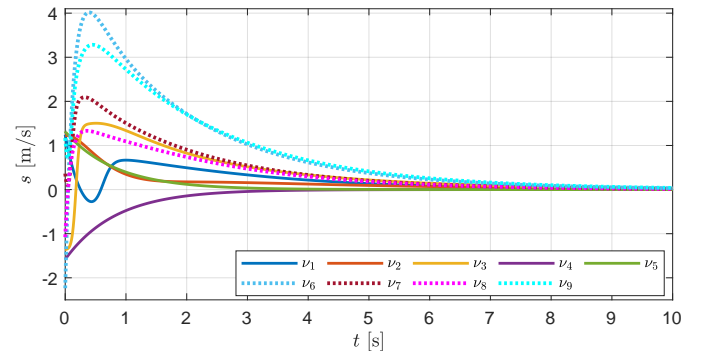


Fig. 3. Bipartite containment tracking of (29) with (5) on velocity.

In a second run of simulations, we tested the bipartite-containment control law (32) on the system (29), using the same initial conditions and controller gains as above. Therefore, in the simulations we consider that the system is subject

to a disturbance $d_i = \sigma_i(t) [1 \ 1]^\top$, where $\sigma_i(t)$ is given by

$$\sigma_i(t) = \begin{cases} \tanh(t-10) - 1 + \frac{1}{(t+10)} & i \in \{7, 9\} \\ -\tanh(t-10) + 1 - \frac{1}{(t+10)} & i = 6 \\ 0 & i \in \{1, 2, 3, 4, 5, 8\}. \end{cases}$$

The simulation results can be appreciated in Figures 4 and 5. During the first 10s, the perturbation $d(t)$ prevents the achievement of bipartite containment tracking but, after the perturbation vanishes, the trajectories of the followers towards the bipartite containment set spanned by cooperative leaders final states and competitive leaders' mirrored final states and all agents' velocities converge to zero. We obtain the same limit values as before for the followers.

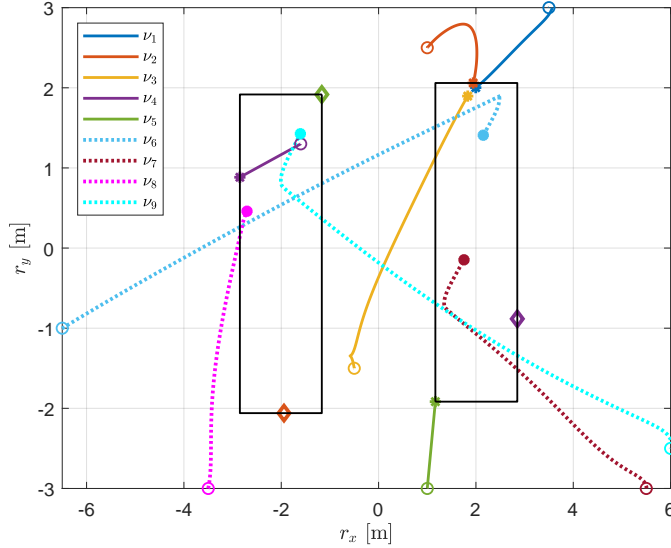


Fig. 4. Bipartite containment tracking of (25) on the plane.

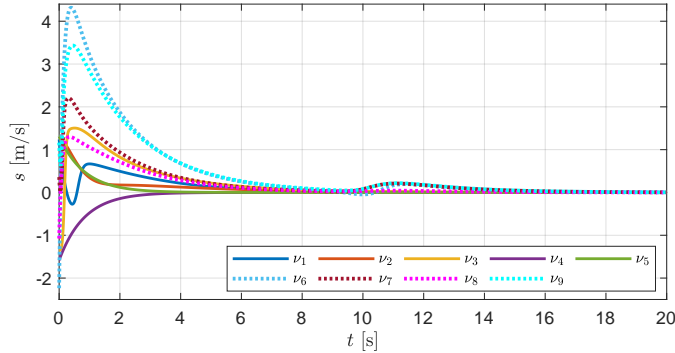


Fig. 5. Bipartite containment tracking of (25) on velocity.

B. Structurally Unbalanced Networks

Consider a network containing five leaders $\nu_i, i \leq 5$, three leader groups $\{\nu_1, \nu_2, \nu_3\}, \{\nu_4\}, \{\nu_5\}$, and four followers $\nu_j, 6 \leq j \leq 9$, provided in Figure 6.

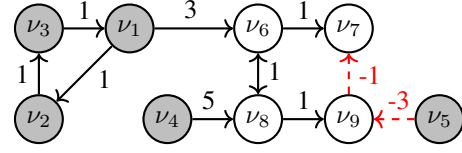


Fig. 6. Network 2: A network of nine mobile robots

The Laplacian matrix corresponding to the graph is

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & -5 & 0 & -1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & -1 & 0 & 4 \end{bmatrix}$$

and its eigenvalues are $\lambda_L = 0, 0, 0, 1.5 \pm 0.86i, 2, 3.59, 4, 6.41$. The network may not be bipartitioned into two subgroups, so is structurally unbalanced. The matrix V_l is calculated as in (33) and V_f is calculated as

$$V_f = \begin{bmatrix} 0.26 & 0.26 & 0.26 & 0.22 & 0 \\ 0.125 & 0.125 & 0.125 & 0 & 0.375 \\ 0.04 & 0.04 & 0.04 & 0.87 & 0 \\ 0.01 & 0.01 & 0.01 & 0.22 & -0.75 \end{bmatrix}.$$

We notice that V_f has the properties stated on Items 1)–4) of Lemma 1, since the network is structurally unbalanced. Since all followers are influenced by more than one leader, there are no agents corresponding to the exclusive part of a reach and as a result V_f does not have an element equal to ± 1 but the absolute value of each element is less than one. We also remark that the sum of the absolute value of the terms on each row is less than 1.

Now, let P be generated by (12) with $Q = I_n$ and $\alpha = 20$, then we obtain $\bar{\lambda}_P = 0.4839$. Consider the system (29) and the bipartite containment law (32) with the same controller gains as before. The respective agents' inertial positions and linear speeds are the same as before, while the orientations are $\theta(0) = [0.25 \ 0.61 \ -1.27 \ 0.32 \ 1.42 \ 0.46 \ 0.98 \ 1.19 \ 1.03]^\top$. The simulation results are depicted in Figures 7 and 8.

Note that all followers converge to the convex hull spanned by leaders' final states and mirrored final states and all agents' velocities converge to zero. Using relations in (18) and the coordinate transformation, we obtain the following limit values for followers' states, $\lim_{t \rightarrow \infty} r_x(t) = [1.5 \ 1.1 \ 0.01 \ -0.94]^\top$, $\lim_{t \rightarrow \infty} r_y(t) = [1.79 \ -0.01 \ 0.89 \ 1.54]^\top$, $\lim_{t \rightarrow \infty} s(t) = [0 \ 0 \ 0 \ 0]^\top$.

We remark that for structurally balanced signed networks, the followers' final states converge to the bipartite containment set spanned by cooperative leaders' final states and antagonistic leaders' mirrored states. However, in the case of a structurally unbalanced network, we cannot observe this as we cannot bipartition the agents into two disjoint subsets.

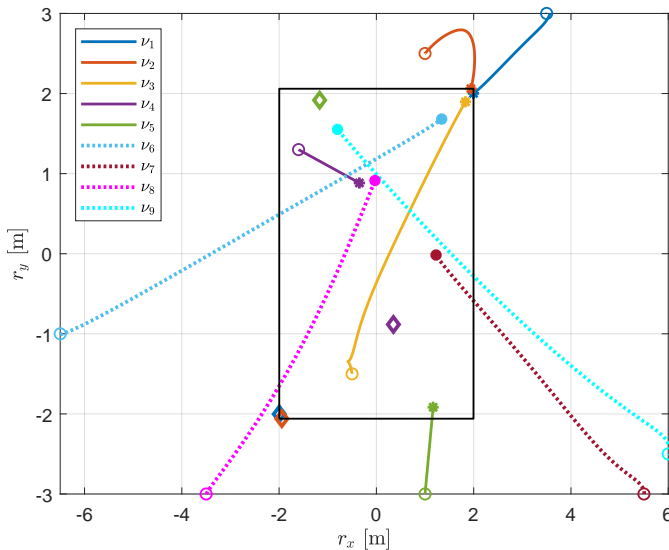


Fig. 7. Bipartite containment tracking of (1) on position. The filled dots are the final states of the agents. The diamonds represent the mirrored final states of the leaders.

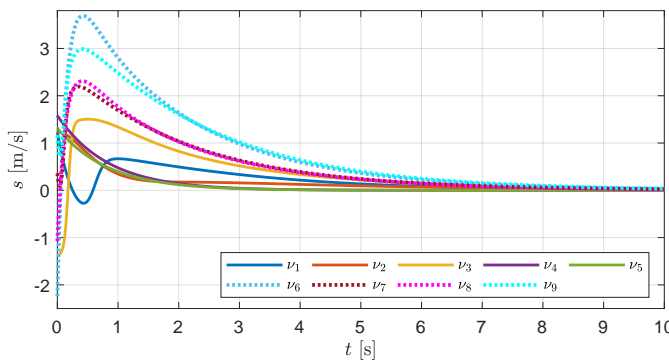


Fig. 8. Bipartite containment tracking of (1) on velocity.

VI. CONCLUSIONS

We presented a Lyapunov approach to analyze the exponential stability of the bipartite containment tracking problem of double-integrators over multiple-leader signed networks. Via a change of coordinates, we have shown the existence and characterized a bound for the convergence of the followers. Moreover, we have generalized the Lyapunov equation characterization of the Hurwitz property of a matrix to matrices having more than one zero eigenvalue, which allowed us to construct strict Lyapunov functions. Disposing of strict Lyapunov functions allowed us to establish the robustness of the system with a bounded disturbance. Even though, in this paper, we only address bipartite containment tracking of second-order systems, the stability and robustness properties may serve for more complex systems.

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