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MRAC-based dynamic consensus of linear systems with biased measurements over directed networks

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Abstract

In this paper we address the dynamic leaderless consensus control problem for generic linear systems (stable, unstable, marginally stable, . . .) interconnected over directed networks and under the influence of biased measurements. Essentially, the control problem consists in redesigning a standard distributed consensus controller which, for each system, relies on own state biased measurements and respective erroneous data received from a set of neighbors. The difficulty in such a scheme resides in the fact that the measurement bias is directly amplified by the control gain so it cannot be handled as an additive external disturbance. Our control design relies, on one hand, on the solution of a Riccati equation and, on the other, on the design of an estimator reminiscent of a model-reference-adaptive control design. The estimator successfully computes a bias estimate and completely compensates for its effect if the bias is constant—indeed, in this case, we establish exponential stability of the consensus manifold and we show that the controller provides robustness with respect to time-varying biases.

Key words: Measurement bias, directed graphs, multi-agent systems, linear systems, Lyapunov stability

1 Introduction

Dynamic consensus, as defined in [1], pertains to the state of synchronization of networked dynamical systems which, as a result of their interconnections, achieve asymptotically a common dynamic behavior. It generalizes the concept of consensus in which case all the systems reach a common stable *equilibrium* point³.

We address this problem under the assumption that measurements are faulty. Measurement bias, which is the one of interest here, is a persistent offset in the reading, that may affect all the samples gathered in a particular deployment. Constant, or very slowly-varying, bias appears in many applications, notably in aerospace [4] and in robotic systems—see, *e.g.*, the discussion in [5, Remark 3]. Time-varying bias may be related, to some extent, to false-data injection attacks, in which case the measurements are affected by an additive bounded offset that is deliberately injected into the network, by a hostile agent—see *e.g.*, [6], [7], [8].

Depending on the scenario, different techniques may apply to cope with constant [9] or time-varying bias [6]. For instance, adaptive control is used in [6,7], an LMI-based approach is proposed in [10], and in [11] a distributed piece-wise constant impulsive reference is used.

Articles that address consensus under false-data injection attacks include, *e.g.*, [6], [7], [8], [12]. In the first two the leader-follower consensus problem is addressed, but under the assumption that the data received from the leader is not under attack (unbiased). In [8] a leader-follower scenario is also studied, but in which multiple leaders are permitted and they may be compromised. In [12] more generic leaderless directed-graph networks are considered, but under the assumption that the biases are homogeneous and multiplicative rather than additive.

A significant difficulty to deal with additive measurement bias is that, in general, it cannot be dealt with using *ad hoc* robust control techniques tailored to cope with additive disturbances and consisting, essentially, in increasing the gain—the bias, if anything, is amplified by the control gain. In general, bias cannot be dealt with as a common uncertainty in lumped parameters either. Yet, some works successfully borrow inspiration from the classical model-reference adaptive control (MRAC) method—see *e.g.*, [13] to successfully design distributed model-reference adaptive controllers for cooperative tracking [14]–[16], as well as for cooperative regulation [17], and for different communication topologies—whether it is undirected [14]–[16] or directed [17]. In [14]–[16] the application of MRAC-inspired techniques for leader-follower consensus relies on assimilating the leader node as the singular reference model. In [17], a framework for adaptive leaderless consensus of linear (homogeneous) agents with uncertain dynamics is proposed. The method consists in designing a reference model for each system and having the latter track the output generated by the corresponding linear reference model with relative state measurement as input.

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³ The terminology “dynamic (average) consensus” is used, *e.g.*, in [2,3] to refer to a problem of consensus-tracking control, that is, in which the systems are required to follow a pre-defined reference.

In all of the references cited above MRAC-based techniques are used to deal with uncertain systems, but none of them considers biased relative state measurements. In [18] an MRAC-based technique is proposed to accommodate constant sensor bias, but not in a multi-agent context.

In this paper, which is the outgrowth of [19], we address the problem of dynamic consensus for linear systems over generic directed graphs and under the assumption that measurement bias is different and bounded for each agent. In [19], as in [9], only constant bias is considered; in the former only asymptotic convergence to the synchronization manifold is established and in the latter uniform ultimate boundedness. Relative to [6,7,8], as well as [14]–[17], we provide an MRAC-based control law for leader-less consensus among agents, which can override the effect of biased measurements. In comparison with [6] and [7], we assume that any agent’s measurement may be corrupted; relative to [7] and [8] we consider generic directed-graph networks, which include the leader-follower scenario. As in these references we ensure uniform ultimate boundedness under time-varying bias, but in contrast to them, as well as to [19] and [9], we establish global exponential stability of the synchronization manifold, under measurements corrupted by constant bias. This includes bias estimation.

The remainder of the paper is organized as follows. In the upcoming section we provide a detailed problem formulation, in Section 3 we present our main results, in Section 4 we provide some simulation results to illustrate our theoretical findings, and we conclude with some closing remarks in Section 5.

Notation. We use $\|\cdot\|$ to denote the Euclidean norm of vectors and the induced norm of matrices. For a square matrix Q we use σ_m^Q and σ_M^Q to denote, respectively, lower and upper bounds on $|Q|$.

2 Model and problem formulation

We consider $N \in \mathbb{N}$ identical, linear, autonomous systems modelled by the equation

$$\dot{x}_i = Ax_i + Bu_i, \quad \forall i = \{1, 2, \dots, N\}, \quad (1)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^p$ is the control input and A, B are constant matrices of compatible dimensions satisfying the following:

- (A1) the pair (A,B) is stabilizable,
- (A2) the system matrix A has full rank.

For these systems, we address the dynamic consensus control problem. This pertains to making all systems achieve a common dynamic behavior, hence, such that

$$\lim_{t \rightarrow \infty} |x_p(t) - x_q(t)| = 0, \quad \forall p \neq q \in \{1, 2, \dots, N\}, \quad (2)$$

for all initial conditions, via a distributed consensus controller of the form

$$u_i = K \sum_{j \in \mathcal{N}_i} a_{ij} (\bar{x}_i - \bar{x}_j) + \nu_i, \quad (3)$$

where $\bar{x}_i := x_i + \delta_i$ is the state measurement of the i^{th} agent, affected by a measurement bias δ_i . In that regard, it is assumed that

- (A3) the i^{th} agent has access to its own state and that of its neighbors, but these measurements are biased by an unknown offset δ_i , that is, $\bar{x}_i := x_i + \delta_i$;
- (A4) the bias δ_i is either constant or time-varying with bounded derivative.

Then, the term ν_i in (3) denotes an additional, re-design, control input to be determined, notably to compensate for the effect of the measurement bias.

The matrix K is a coupling gain to be defined and the coefficients $a_{ij} \geq 0$ represent the existence (if $a_{ij} > 0$) or the absence (if $a_{ij} = 0$) of a one-way interaction from the j^{th} node to the i^{th} node. As it is customary, we consider that there are no self-loops, so $a_{ii} = 0$. More precisely, we assume that

- (A5) the network’s topology is directed and contains a directed spanning tree. Hence, its associated Laplacian is defined by $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$, where

$$l_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}, \quad l_{ij} = -a_{ij}, \quad (4)$$

and, in general, $a_{ij} \neq a_{ji}$.

Under Assumption (A5), our results hold for varied network topologies, including those admitting a directed spanning-tree graph [6,7], connected undirected graphs [5,20], as well as strongly-connected directed graphs. In the latter two cases all the elements of v_l are strictly positive, so $x^\top [P\mathcal{L} + \mathcal{L}^\top P]x$, where $P := \text{diag}[v_{l_i}]$, qualifies as a Lyapunov function—cf. [21]. Topologies satisfying Assumption (A5) cover, but are not bound to, those of graphs for which all the elements of v_l are strictly positive. Therefore, none of the approaches cited above generally applies.

Furthermore, under Assumption (A5) the following two statements hold; they are instrumental to construct a Lyapunov function to analyze the stability of the consensus manifold $\{x_i = x_j\}$.

Lemma 1 [22,23] *If a directed network has a directed spanning tree, then the Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ has a unique zero eigenvalue σ_1 and $N - 1$ eigenvalues σ_k have strictly positive real part. That is,*

$$\sigma_1(\mathcal{L}) = 0, \quad \Re\{\sigma_k(\mathcal{L})\} > 0, \quad k = \{2, \dots, N\}.$$

On the other hand, the right eigenvector of the zero eigenvalue is given by $\mathbf{1}_N = [1 \ 1 \ \dots \ 1]^\top$ and the left eigenvector v_l satisfies $\sum_{k=1}^N v_{l_k} = 1$ and $v_l^\top \mathcal{L} = 0_N^\top$. \square

Lemma 2 [24] *Let us consider a directed graph \mathcal{G} of order N containing a spanning tree and its Laplacian matrix is $\mathcal{L} \in \mathbb{R}^{N \times N}$. Then, for any positive definite symmetric matrix $Q_{\mathcal{L}} \in \mathbb{R}^{N \times N}$ and $\alpha \in \mathbb{R}^+$, there exists another positive definite symmetric matrix $P_{\mathcal{L}} \in \mathbb{R}^{N \times N}$ such that*

$$P_{\mathcal{L}}\mathcal{L} + \mathcal{L}^\top P_{\mathcal{L}} = Q_{\mathcal{L}} - \alpha[P_{\mathcal{L}}\mathbf{1}_N v_l^\top + v_l \mathbf{1}_N^\top P_{\mathcal{L}}]. \quad (5)$$

\square

3 Main results

3.1 Control design

The consensus controller that we propose relies on a distributed estimator whose design, along similar lines

as those in [14]–[17], borrows inspiration from model-reference adaptive control. The control strategy consists in coupling each agent’s dynamics to that of an estimator, reminiscent of a reference model and defined as

$$\dot{\hat{x}}_i = A\hat{x}_i + \epsilon BF \sum_{j \in \mathcal{N}_i} a_{ij}(\bar{x}_i - \bar{x}_j) - \epsilon BF \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\delta}_i - \hat{\delta}_j), \quad (6)$$

where the coupling weight $\epsilon > 0$, the control gain $F \in \mathbb{R}^{p \times n}$, and the bias estimate of the i th agent, $\hat{\delta}_i$, is yet to be defined. Note that the estimator also depends on the bias estimates of its neighbours, $\hat{\delta}_j$.

Then, to achieve the consensus control objective (2) the first control goal is set to make the estimators (6) achieve dynamic consensus, that is,

$$\lim_{t \rightarrow \infty} |\hat{x}_i(t) - \hat{x}_j(t)| = 0, \quad \forall i \neq j \in \{1, 2, \dots, N\}. \quad (7)$$

The second objective is to steer each agent’s state trajectories to its corresponding estimator’s, that is,

$$\lim_{t \rightarrow \infty} |x_i(t) - \hat{x}_i(t)| = 0 \quad \forall i \in \{1, 2, \dots, N\}. \quad (8)$$

Note that (7) and (8) imply (2) since

$$\{\hat{x}_i = \hat{x}_j\} \cap \{x_i = \hat{x}_i\} \quad \forall i, j \in \{1, 2, \dots, N\}$$

implies that $x_i = x_j$ for all $i \in \{1, 2, \dots, N\}$. However, since the states x_i are not measurable, we pose as objective to steer the measured state \bar{x}_i to the corresponding biased state of the reference model in (6), that is, to make $\tilde{e}_i \rightarrow 0$, where

$$\tilde{e}_i = \bar{x}_i - (\hat{x}_i + \hat{\delta}_i). \quad (9)$$

It is remarked that this task, on one hand, appears feasible since \tilde{e}_i is available and, on the other hand, is useful since

$$\tilde{e}_i = x_i - \hat{x}_i + \tilde{\delta}_i, \quad (10)$$

with $\tilde{\delta}_i := \delta_i - \hat{\delta}_i$, so (8) holds if $\tilde{e}_i \rightarrow 0$ and $\hat{\delta}_i$ is updated so as to have

$$\lim_{t \rightarrow \infty} |\hat{\delta}_i(t) - \delta_i| = 0, \quad \forall i \in \{1, 2, \dots, N\}. \quad (11)$$

Now, to steer $\tilde{e}_i \rightarrow 0$ the control law u_i in (3) is redesigned as follows. First, we rewrite (6) as

$$\dot{\hat{x}}_i = A\hat{x}_i + \epsilon BF \sum_{j \in \mathcal{N}_i} a_{ij}[(x_i - x_j) + (\tilde{\delta}_i - \tilde{\delta}_j)]. \quad (12)$$

Then, we differentiate on both sides of (10) and we use (12) and (1) to obtain

$$\begin{aligned} \dot{\tilde{e}}_i &= A\tilde{e}_i - A\tilde{\delta}_i + Bu_i + \dot{\tilde{\delta}}_i \\ &\quad - \epsilon BF \sum_{j \in \mathcal{N}_i} a_{ij}[(x_i - x_j) + (\tilde{\delta}_i - \tilde{\delta}_j)], \end{aligned} \quad (13)$$

Now, note that the terms on the second line of (13) can be canceled directly since they correspond exactly to the data available to the i th node. Second, we may insert a control term of the form $F\tilde{e}_i$, such that $(A + BF)$ is Hurwitz; this is possible since the pair (A, B) is stabilizable by assumption. More precisely, the latter

implies the existence of a matrix $M = M^\top > 0$ that solves the algebraic Riccati equation

$$MA + A^\top M - MB B^\top M = -Q, \quad (14)$$

for any given $Q = Q^\top > 0$. Thus, for any such given Q that generates M as above, we define $F = -B^\top M$ —cf. [25,26] and we redefine u_i in (3) as

$$u_i = \epsilon F \sum_{j \in \mathcal{N}_i} a_{ij}(\bar{x}_i - \bar{x}_j) - \epsilon F \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\delta}_i - \hat{\delta}_j) + F\tilde{e}_i. \quad (15)$$

Then, using (15) in (13), we obtain

$$\dot{\tilde{e}}_i = (A + BF)\tilde{e}_i - A\tilde{\delta}_i + \dot{\tilde{\delta}}_i. \quad (16)$$

Since $(A + BF)$ is Hurwitz by design, it is left to define the dynamics of the bias estimates $\hat{\delta}_i$ —hence that of $\tilde{\delta}_i$ —so as to ensure that $\tilde{\delta}_i$ and $\dot{\tilde{\delta}}_i$ vanish asymptotically. The design of the bias estimation law depends on whether δ_i is constant or time-varying, as per Assumption (A4), and is presented in the next sections.

3.2 Dynamic consensus under constant bias

In the case that δ_i is constant we define

$$\dot{\hat{\delta}}_i = -M^{-1}A^\top M \tilde{e}_i, \quad (17)$$

where M solves (14) for any given positive definite Q . Such estimation law is distributed since it depends only on \tilde{e}_i , which is defined by variables that pertain to the i th agent only—see (9). Then, we have the following.

Proposition 3 (Dynamic consensus) *Consider N identical linear systems as in (1) in closed loop with the distributed control input given by (15), with $\epsilon \geq \sigma_M^{P_{\mathcal{L}}}$ and $P_{\mathcal{L}}$ solving (5) for $Q_{\mathcal{L}} = I_N$; the feedback matrix $F = -B^\top M$, where M solves (14); the reference model (6), and the bias update law (17). Then, under Assumptions (A1)–(A5), (2) holds uniformly and exponentially. \square*

Proof. The statement follows after a cascades argument and the rationale presented previously. More precisely, the limit in (2) holds if so do those in (7) and (8). Indeed, $\{\hat{x}_i = \hat{x}_j\}$ and $\{x_i = \hat{x}_i\}$ for all i and j imply that $\{x_i = x_j\}$ for all i, j . The limit in (8) is established in Proposition 4, which is presented next, and (7) is established in Proposition 6 farther below. \blacksquare

Proposition 4 (bias estimation) *Consider N identical linear systems as in (1) in closed loop with (15), (6), and (17) under the conditions of Proposition 3. Then, there exist λ and $\kappa > 0$, such that*

$$|\xi(t)| \leq \kappa |\xi(t_0)| e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0, \quad (18)$$

where $\xi := [\tilde{e}^\top \ \tilde{\delta}^\top]^\top$, $\tilde{e}^\top := [\tilde{e}_1^\top \ \dots \ \tilde{e}_N^\top]$, and $\tilde{\delta}^\top := [\tilde{\delta}_1^\top \ \dots \ \tilde{\delta}_N^\top]$. In turn, the limits in (8) and (11) hold. \square

Proof. In compact form, the multi-agent closed-loop equations correspond to

$$\begin{aligned} \dot{\tilde{e}} &= [I_N \otimes (A + BF)]\tilde{e} - [I_N \otimes A]\tilde{\delta} \\ &\quad + [I_N \otimes (M^{-1}A^\top M)]\tilde{e} \end{aligned} \quad (19a)$$

$$\dot{\tilde{\delta}} = [I_N \otimes (M^{-1}A^\top M)]\tilde{\delta}. \quad (19b)$$

Then, consider the Lyapunov function candidate

$$V(\tilde{e}, \tilde{\delta}) = \tilde{e}^\top (I_N \otimes M)\tilde{e} + \tilde{\delta}^\top (I_N \otimes M)\tilde{\delta}, \quad (20)$$

which, since M is symmetric positive definite, is both positive definite and decrescent. Indeed, V satisfies

$$\sigma_m^M |\xi|^2 \leq V(\tilde{e}, \tilde{\delta}) \leq \sigma_M^M |\xi|^2. \quad (21)$$

Furthermore, in virtue of (14), the time-derivative of V along the trajectories of (19) is negative semidefinite. Indeed,

$$\begin{aligned} \dot{V}(\tilde{e}, \tilde{\delta}) &= \tilde{e}^\top [I_N \otimes (MA + A^\top M - 2MBB^\top M)]\tilde{e} \\ &\quad - 2\tilde{e}^\top [I_N \otimes (MA)]\tilde{\delta} + 2\tilde{e}^\top [I_N \otimes (A^\top M)]\tilde{\delta} \\ &\quad + 2\tilde{\delta}^\top [I_N \otimes (A^\top M)]\tilde{e} \\ &= 2\tilde{e}^\top [I_N \otimes (MA + A^\top M - MBB^\top M)]\tilde{e} \\ &= -2\tilde{e}^\top [I_N \otimes Q]\tilde{e} \leq 0. \end{aligned} \quad (22)$$

From (21)-(22), it follows that the origin, $\{(\tilde{e}, \tilde{\delta}) = (0, 0)\}$, for the system (19a)-(19b) is uniformly globally stable. That is, the origin is uniformly stable and all solutions are uniformly globally bounded.

To establish uniform global exponential stability we use the following statement, which follows from [27, Lemma 3].

Lemma 5 *Consider the system $\dot{\xi} = f(t, \xi)$, where $f(\cdot, \xi)$ is locally integrable and $f(t, \cdot)$ is locally Lipschitz uniformly in t . Assume that there exist constants p , and $c > 0$ such that the solution $\xi(\cdot, t_o, \xi_o)$ of $\dot{\xi} = f(t, \xi)$ satisfies*

$$\max \left\{ \sup_{t \geq t_o} |\xi(t)|, \left[\int_{t_o}^{\infty} |\xi(\tau)|^p d\tau \right]^{1/p} \right\} \leq c|\xi_o|, \quad (23)$$

for all initial conditions⁴ $(t_o, x_o) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. Then, the origin $\{\xi = 0\}$ is uniformly globally exponentially stable and, moreover, (18) holds with $\kappa = ce^{1/p}$ and $\lambda = [pc^p]^{-1}$. \square

Thus, the rest of the proof consists in establishing the bounds in (23) for the closed-loop system (19). The first inequality in (23) follows by integrating along trajectories on both sides of the inequality $\dot{V} \leq 0$ in (22) and using (21) to obtain

$$V(\tilde{e}(t), \tilde{\delta}(t)) \leq V(\tilde{e}(t_o), \tilde{\delta}(t_o)) \quad (24)$$

$$\sigma_m^M |\xi(t)|^2 \leq V(\tilde{e}(t), \tilde{\delta}(t)) \leq \sigma_M^M |\xi(t_o)|^2, \quad \forall t \geq t_o \geq 0,$$

which, in turn, implies that

$$|\xi(t)| \leq \sqrt{\frac{\sigma_M^M}{\sigma_m^M}} |\xi(t_o)|, \quad \forall t \geq t_o, \quad (25)$$

so the first inequality in (23) holds with $c^2 := \sigma_M^M / \sigma_m^M$.

⁴ Considering $t_o \geq 0$ is a mere convention; the result certainly holds for all $t_o \in \mathbb{R}$.

On the other hand, integrating on both sides of $\dot{V} = -2\tilde{e}^\top (I_N \otimes Q)\tilde{e}$ in (22), along the trajectories, we obtain

$$V(\tilde{e}(t), \tilde{\delta}(t)) - V(\tilde{e}(t_o), \tilde{\delta}(t_o)) \leq -2\sigma_m^Q \int_{t_o}^t |\tilde{e}(\tau)|^2 d\tau.$$

Rearranging the terms on both sides of the latter and using $V(\tilde{e}(t), \tilde{\delta}(t)) \geq 0$ we obtain

$$2\sigma_m^Q \int_{t_o}^t |\tilde{e}(\tau)|^2 d\tau \leq 2V(\tilde{e}(t_o), \tilde{\delta}(t_o))$$

which, in view of (21), implies that

$$\lim_{t \rightarrow \infty} \int_{t_o}^t |\tilde{e}(\tau)|^2 d\tau \leq \frac{\sigma_M^M}{\sigma_m^Q} |\xi(t_o)|^2. \quad (26)$$

The latter establishes the second inequality in (23), for the \tilde{e} -part of ξ . To compute a similar inequality for $\tilde{\delta}$, consider the function $W : \mathbb{R}^{2nN} \rightarrow \mathbb{R}$, defined by

$$W(\xi) = \tilde{e}^\top [I_N \otimes A]\tilde{\delta} \quad (27)$$

and let $\sigma_M^A > 0$ be such that $|A| \leq \sigma_M^A$. Then, $W(\xi) \leq (1/2)[\sigma_M^A |\tilde{e}|^2 + \sigma_M^A |\tilde{\delta}|^2]$ and, after (25) we see that, along the trajectories,

$$W(\xi(t)) \leq \frac{\sigma_M^A}{2} |\xi(t)|^2 \leq \frac{\sigma_M^A}{2} c^2 |\xi(t_o)|^2, \quad \forall t \geq t_o, \quad (28)$$

where $c^2 := \sigma_M^M / \sigma_m^M$. On the other hand,

$$W(\xi(t_o)) \leq \frac{\sigma_M^A}{2} |\xi(t_o)|^2. \quad (29)$$

The total derivative of W along the trajectories of (19) yields

$$\begin{aligned} \dot{W}(\xi(t)) &\leq \tilde{\delta}^\top [I_N \otimes (A^\top A + A^\top BF + A^\top M^{-1}A^\top M)]\tilde{e} \\ &\quad + \tilde{e}^\top [I_N \otimes AM^{-1}A^\top M]\tilde{\delta} \\ &\quad - \tilde{\delta}^\top [I_N \otimes A^\top A]\tilde{\delta}. \end{aligned} \quad (30)$$

Then, to compact the notation, we introduce

$$\mathcal{A} := A^\top A - A^\top BB^\top M + A^\top M^{-1}A^\top M$$

and, since A , B , and M are constant matrices, we also introduce $\sigma_M^A > 0$ to denote an upper bound on the induced norm of \mathcal{A} , that is, $|\mathcal{A}| \leq \sigma_M^A$. Hence, the terms in the first line of (30) satisfy

$$\tilde{\delta}^\top [I_N \otimes \mathcal{A}]\tilde{e} \leq \frac{(\sigma_M^A)^2}{2\mu} |\tilde{e}|^2 + \frac{\mu}{2} |\tilde{\delta}|^2 \quad (31)$$

for any $\mu > 0$. For further development, we also introduce $\gamma' > 0$ such that $|AM^{-1}A^\top M| \leq \gamma'$ and $a'_m > 0$ such that $\sigma_m^{A^\top A} \geq a'_m$; the latter holds under Assumption (A2).

Next, replacing (31) and $|AM^{-1}A^\top M| \leq \gamma'$ in (30) yields, along trajectories,

$$\dot{W}(\xi(t)) \leq \left[\gamma' + \frac{(\sigma_M^A)^2}{2\mu} \right] |\tilde{e}(t)|^2 - \left[a'_m - \frac{\mu}{2} \right] |\tilde{\delta}(t)|^2.$$

Then, we set $\mu = a'_m$, we define $\gamma'' := \gamma' + \frac{(\sigma_M^A)^2}{2\mu}$, and we integrate on both sides of the latter inequality and rearrange terms to obtain

$$\frac{a'_m}{2} \int_{t_o}^t |\tilde{\delta}(\tau)|^2 d\tau \leq |W(\xi(t))| + |W(\xi(t_o))| + \gamma'' \int_{t_o}^t |\tilde{e}(\tau)|^2 d\tau.$$

Thus, using (28), (29), and (26), we obtain

$$\int_{t_o}^t |\tilde{\delta}(\tau)|^2 d\tau \leq \frac{2}{a'_m} \left[\gamma'' \frac{\sigma_M^M}{\sigma_M^Q} + \frac{\sigma_M^A}{2} \left[\frac{\sigma_M^M}{\sigma_M^M} + 1 \right] \right] |\xi(t_o)|^2,$$

for all $t \geq t_o$. The latter, together with (25) and (26), imply the condition (23) with $p = 2$ and

$$c^2 := \max \left\{ \frac{\sigma_M^M}{\sigma_M^M}, \frac{\sigma_M^M}{\sigma_M^Q} + \frac{2}{a'_m} \left[\gamma'' \frac{\sigma_M^M}{\sigma_M^Q} + \frac{\sigma_M^A}{2} \left[\frac{\sigma_M^M}{\sigma_M^M} + 1 \right] \right] \right\},$$

so the result follows with κ and λ as in Lemma 5. \blacksquare

Proposition 6 (Estimators' consensus) *Consider the estimators defined in (6), with $a_{ij} \geq 0$ such that the corresponding Laplacian matrix, whose elements are defined in (4), contains a directed spanning tree, and with $\epsilon \geq \sigma_M^{P_{\mathcal{L}}}$ where $P_{\mathcal{L}}$ solves (5) for $Q_{\mathcal{L}} = I_N$, the feedback matrix $F = -B^{\top}M$ where M solves (14). Then, on the manifold $\{i \leq N : (\tilde{e}_i, \tilde{\delta}_i) = (0, 0)\}$, the estimators (6) achieve dynamic consensus, that is, (7) holds. More precisely, in the space of the synchronization errors $s_{\hat{x}_i} := \hat{x}_i - \hat{x}_m$, where \hat{x}_m corresponds to the solutions of $\dot{\hat{x}}_m = A\hat{x}_m$ with $\hat{x}_m(0) := (v_l^{\top} \otimes I_n)\hat{x}(0)$, the origin $\{s_{\hat{x}_i} = 0\}$ is uniformly exponentially stable. \square*

Proof. In compact multi-variable form, the equations (6) are written as

$$\dot{\hat{x}} = [I_N \otimes A]\hat{x} - \epsilon[\mathcal{L} \otimes BB^{\top}M]\hat{x} - \epsilon[\mathcal{L} \otimes BB^{\top}M]\tilde{\delta},$$

and, on the manifold $\{(\tilde{e}, \tilde{\delta}) = (0, 0)\}$, in view of (10), we have

$$\dot{\hat{x}} = [I_N \otimes A]\hat{x} - \epsilon[\mathcal{L} \otimes BB^{\top}M]\hat{x}. \quad (32)$$

Now, following [1], to assess that the estimators (32) reach dynamic consensus, we verify that \hat{x}_i , for each $i \leq N$, tends asymptotically to the trajectories of a weighted-average system with state $\hat{x}_m := (v_l^{\top} \otimes I_n)\hat{x}$. In multi-variable form, we have

$$\begin{bmatrix} \hat{x}_m \\ s_{\hat{x}} \end{bmatrix} = \begin{bmatrix} (v_l^{\top} \otimes I_n)\hat{x} \\ [(I_N - 1_N v_l^{\top}) \otimes I_n]\hat{x} \end{bmatrix}, \quad (33)$$

where $s_{\hat{x}} = [s_{\hat{x}_1}^{\top} \dots s_{\hat{x}_N}^{\top}]^{\top}$. Then, differentiating on both sides of (33) and using (32), we obtain the system

$$\begin{bmatrix} \dot{\hat{x}}_m \\ \dot{s}_{\hat{x}} \end{bmatrix} = \begin{bmatrix} A\hat{x}_m \\ [(I_N \otimes A) - \epsilon(\mathcal{L} \otimes BB^{\top}M)]s_{\hat{x}} \end{bmatrix}. \quad (34)$$

The latter is a dichotomous representation of the multi-agent estimators (32). On one hand, one has the decoupled average dynamics $\dot{\hat{x}}_m = A\hat{x}_m$ and, on the other, the synchronization error dynamics. Therefore,

establishing that $s_{\hat{x}} \rightarrow 0$ implies that all estimators behave as the averaged model $\dot{\hat{x}}_m = A\hat{x}_m$.

Remark 7 In an ordinary consensus problem, *i.e.*, in which all systems attain a common *equilibrium* point x_m , we have $\dot{x}_m \equiv 0$. In the present setting the systems do not stabilize at an equilibrium, but adopt the behavior of a non-prespecified dynamic average model; whence the terminology *dynamic consensus*. \bullet

For assessing convergence of individual reference systems to the concerned averaged model, we use procedure outlined in [28]. To that end, we consider the Lyapunov function candidate

$$V(s_{\hat{x}}) = s_{\hat{x}}^{\top} (P_{\mathcal{L}} \otimes M) s_{\hat{x}}, \quad (35)$$

where $P_{\mathcal{L}}$ is generated by Lemma 2 with $Q_{\mathcal{L}} = I_N$. This function is positive definite under Assumption (A5) and Lemma 2.

The time derivative of $V(s_{\hat{x}})$, along the trajectories of (34) yields

$$\begin{aligned} \dot{V}(s_{\hat{x}}) &= s_{\hat{x}}^{\top} \left[P_{\mathcal{L}} \otimes (MA + A^{\top}M) \right. \\ &\quad \left. - \epsilon(P_{\mathcal{L}}\mathcal{L} + \mathcal{L}^{\top}P_{\mathcal{L}}) \otimes (MBB^{\top}M) \right] s_{\hat{x}}. \end{aligned} \quad (36)$$

Then, we set $Q_{\mathcal{L}} = I_N$ and use (5) to rewrite the second term on the right-hand side of (36). Also, we use

$$\alpha[P_{\mathcal{L}}1_N v_l^{\top} \otimes I_n]s_{\hat{x}} = \alpha[P_{\mathcal{L}}1_N v_l^{\top} \otimes I_n] \times [(I_N - 1_N v_l^{\top}) \otimes I_n]\hat{x} = 0_{Nn},$$

which holds under Assumption (A5)—see Lemma 1, and the orthogonal decomposition of $P_{\mathcal{L}} = P_{\mathcal{L}}^{\top} > 0$, $P_{\mathcal{L}} = T\Lambda T^{\top}$ where $T \in \mathbb{R}^{N \times N}$ is an orthogonal matrix. After all the latter, (36) becomes

$$\begin{aligned} \dot{V}(s_{\hat{x}}) &= \sigma_M^{P_{\mathcal{L}}} s_{\hat{x}}^{\top} \left[\frac{T\Lambda T^{\top}}{\sigma_M^{P_{\mathcal{L}}}} \otimes [MA + A^{\top}M \right. \\ &\quad \left. - \frac{\epsilon}{\sigma_M^{P_{\mathcal{L}}}} MBB^{\top}M] \right] s_{\hat{x}}, \end{aligned}$$

so for any $\epsilon \geq \sigma_M^{P_{\mathcal{L}}}$, as posed in Proposition 3, we have

$$\dot{V}(s_{\hat{x}}) \leq \sigma_M^{P_{\mathcal{L}}} s_{\hat{x}}^{\top} [I_N \otimes (MA + A^{\top}M - MBB^{\top}M)] s_{\hat{x}}$$

and, after the algebraic Riccati equation (14), we get

$$\dot{V}(s_{\hat{x}}) \leq -\sigma_M^{P_{\mathcal{L}}} s_{\hat{x}}^{\top} (I_N \otimes Q) s_{\hat{x}},$$

so global exponential stability of $\{s_{\hat{x}} = 0\}$ follows invoking standard Lyapunov theory. \blacksquare

3.3 Practical dynamic consensus under time-varying measurement bias

To take into account the time-varying unknown bounded biases $\delta_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n, i = \{1, \dots, N\}$, the corresponding dynamics of the bias estimates is now modified as

$$\dot{\hat{\delta}}_i = -(M^{-1}A^{\top}M)\tilde{e}_i - \beta M^{-1}\hat{\delta}_i, \quad (37)$$

where $\beta \in \mathbb{R}^+$. Then, we have the following.

Proposition 8 (robust dynamic consensus) *Consider N linear systems defined by (1) under the conditions laid in Proposition 3 and the bias update law (37) in*

place of (17). Then, the tracking and estimation errors $(\tilde{e}, \tilde{\delta})$ are uniformly ultimately bounded. Consequently, so are the tracking errors $x_e = x_i - \hat{x}_i$, as well as the consensus errors for the reference models, $\hat{x}_i - \hat{x}_j$, and for the original systems (1), $x_i - x_j$, for all $i \neq j$.

Proof. The closed-loop equations, now using the update law (37), correspond to

$$\begin{aligned} \dot{\tilde{e}} &= [I_N \otimes (A + BF)]\tilde{e} - [I_N \otimes A]\tilde{\delta} + \dot{\delta} \\ &\quad + \beta(I_N \otimes M^{-1})\tilde{\delta} + [I_N \otimes (M^{-1}A^T M)]\tilde{e} \end{aligned} \quad (38a)$$

$$\dot{\tilde{\delta}} = [I_N \otimes (M^{-1}A^T M)]\tilde{e} + \dot{\delta} + \beta(I_N \otimes M^{-1})\tilde{\delta}. \quad (38b)$$

Reconsider the Lyapunov function V in (20). After (22), its total derivative along the solutions of (38a)-(38b) yields

$$\begin{aligned} \dot{V} &= -2\tilde{e}^T [I_N \otimes Q]\tilde{e} + 2(\tilde{\delta} + \tilde{e})^T (I_N \otimes M)\dot{\delta} \\ &\quad + 2\beta(\tilde{\delta} + \tilde{e})^T \delta - 2\beta\tilde{e}^T \tilde{\delta} - 2\beta|\tilde{\delta}|^2. \end{aligned} \quad (39)$$

Therefore,

$$\dot{V} \leq -[2\sigma_m^Q - \beta]|\tilde{e}|^2 - \beta|\tilde{\delta}|^2 + k[|\tilde{e}| + |\tilde{\delta}|] \quad (40)$$

where $k := 2(\sigma_M^M + \beta)\delta_M$, with

$$\delta_M := \max \left\{ \operatorname{ess\,sup}_{t \geq 0} |\delta(t)|, \operatorname{ess\,sup}_{t \geq 0} |\dot{\delta}(t)| \right\}.$$

Global ultimate boundedness of \tilde{e} and $\tilde{\delta}$ follow from standard Lyapunov theory.

Remark 9 An ultimate bound may be computed as follows. Let Q be such that $\sigma_m^Q := \beta$. Then, for sufficiently large t ,

$$|[\tilde{e}(t)^T \quad \tilde{\delta}(t)^T]^T| \leq \sqrt{2k/\beta} \quad (41)$$

Note that both the numerator and denominator in (41) are of order $\mathcal{O}(\beta)$ for any fixed M and Q . Therefore, as in other works (e.g., [7]), the ultimate bound cannot be diminished at will. This is reasonable because the uncertainty that one wants to compensate for appears in the measurements used in the control loop. •

To assess ultimate boundedness of the tracking errors, $x_e = x - \hat{x}$, we analyze the corresponding dynamics equation

$$\dot{x}_e = [I_N \otimes (A + BF)]x_e + (I_N \otimes BF)\tilde{\delta}. \quad (42)$$

To that end, we evaluate the total derivative of the Lyapunov function $V(x_e) = x_e^T (I_N \otimes M)x_e$ along the trajectories of (42), to obtain

$$\begin{aligned} \dot{V}(x_e) &= x_e^T [I_N \otimes (MA + A^T M - 2MBB^T M)]x_e \\ &\quad - 2x_e^T (I_N \otimes (MBB^T M))\tilde{\delta} \\ &\leq x_e^T [I_N \otimes (MA + A^T M - 2MBB^T M)]x_e \\ &\quad + x_e^T (I_N \otimes (MA + A^T M))x_e + \tilde{\delta}^T (I_N \otimes \overline{M})\tilde{\delta} \\ &\leq -2x_e^T (I_N \otimes Q)x_e + |\overline{M}|\tilde{\delta}|^2, \end{aligned} \quad (43)$$

where we introduced

$$\overline{M} := MBB^T M(MA + A^T M)^{-1}MBB^T M.$$

Note that $(MA + A^T M)$ is non-singular under Assumption (A2). Global ultimate boundedness of x_e follows from the ultimate boundedness of $\tilde{\delta}$ —see Ineq. (41).

Finally, to verify that the consensus error amongst the reference systems is also UUB, we reconsider the synchronization error $s_{\hat{x}}$ and the mean field value \hat{x}_m of the reference systems, as defined in (33). The corresponding dynamics, in this case, is

$$\dot{\hat{x}}_m = A\hat{x}_m \quad (44a)$$

$$\begin{aligned} \dot{s}_{\hat{x}} &= [(I_N \otimes A) - \epsilon(\mathcal{L} \otimes BB^T M)]s_{\hat{x}} \\ &\quad - \epsilon[\mathcal{L} \otimes BB^T M][(I_N - 1_N v_l^T) \otimes I_n]\tilde{\delta}. \end{aligned} \quad (44b)$$

Global uniform ultimate boundedness follows from the fact that the origin for (44b) with $\tilde{\delta} = 0$ is globally exponentially stable and $\tilde{\delta}(t)$ is globally uniformly ultimately bounded. ■

4 Simulation Results

We provide some numerical simulation results, generated using Matlab R2021a. As case-study, we consider the dynamic consensus control of five harmonic oscillators, that is, linear systems modeled as in (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (45)$$

so Assumptions (A1) and (A2) hold. The oscillators are taken to form the graph showed in Figure 1, so Assumption (A5) also holds.

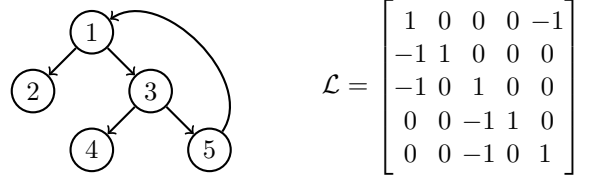


Fig. 1. Directed connected graph and corresponding Laplacian.

We present the results for two tests, one in which the biases are piecewise-constant and one in which they are periodic functions. Irrespective of the nature of the measurement biases, constant or otherwise, the initial conditions assigned for the five agents are set to

$$\begin{aligned} x_1(0) &= [3 \ -1 \ -2 \ 1.5 \ 2]^T, \\ \hat{x}_1(0) &= [-1.5 \ 1 \ -1.75 \ -0.5 \ 2.75]^T, \\ x_2(0) &= [2 \ -2 \ -3 \ 3 \ 2.5]^T, \\ \hat{x}_2(0) &= [0.5 \ -2 \ 1 \ 0.75 \ 1.5]^T, \end{aligned}$$

and the initial conditions for the bias estimates are set to

$$\begin{aligned} \hat{\delta}_1(0) &= [1.5 \ -1.5 \ 2 \ -2.5 \ 0.5]^T, \\ \hat{\delta}_2(0) &= [0.5 \ -0.5 \ -2 \ 2 \ 3]^T. \end{aligned}$$

In the first numerical test, the biases are modeled as piecewise-constant functions taking random values within $[-1, 1]$ and changing at random instants separated by no less than 10s; see Figure 2 below.

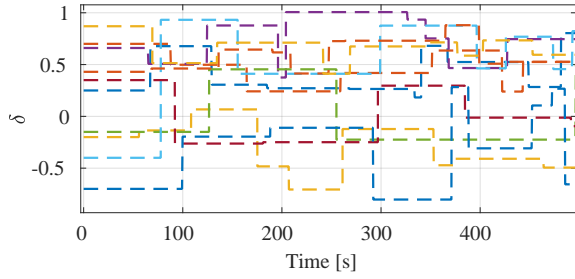


Fig. 2. Plot of piecewise-constant biases, with random changes of random amplitude.

The control gain $F = -B^\top M$ is computed by solving the algebraic Riccati equation (14) for M , with $Q = I_2$. This yields

$$M = \begin{bmatrix} 1.9123 & 0.4142 \\ 0.4142 & 1.3522 \end{bmatrix}. \quad (46)$$

The systems' trajectories are showed in Figure 3—they all achieve dynamic consensus, as they enter in synchrony with the average oscillator. The controlled system is robust to the sudden changes in the biases.

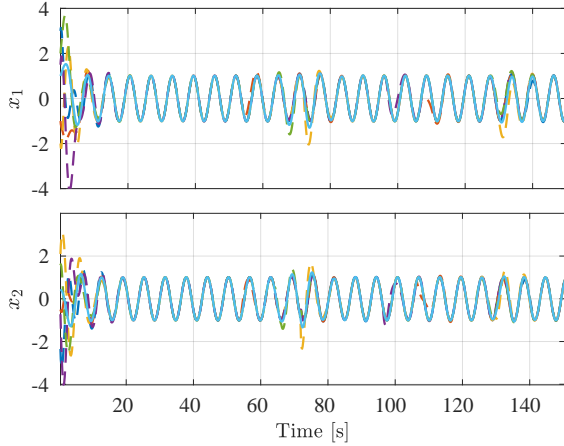


Fig. 3. Oscillators' trajectories reaching synchronization. The solid curve in cyan represents the average oscillator.

For completeness, we show the bias estimation errors in Figure 4 below.

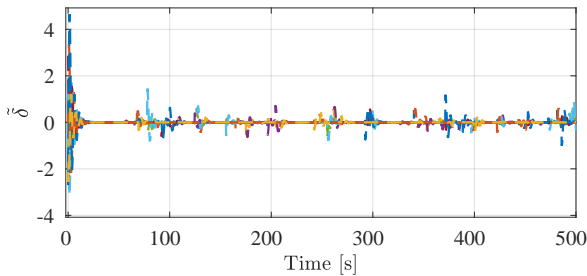


Fig. 4. Estimation errors for the piecewise constant biases for all agents.

With a second numerical test, we illustrate the robustness of our consensus controller *vis a vis* of time-varying biases. These are of the form $\delta_i(t) := c_i + \delta'_i(t)$,

where $c_i \in \mathbb{R}$ and $\delta'_i(t)$ is a sinusoidal or cosinusoidal signal whose amplitude and frequency is varied over time using the $\text{rand}(\cdot)$ function of Matlab, so Assumption (A4) holds—see Figure 5 below.

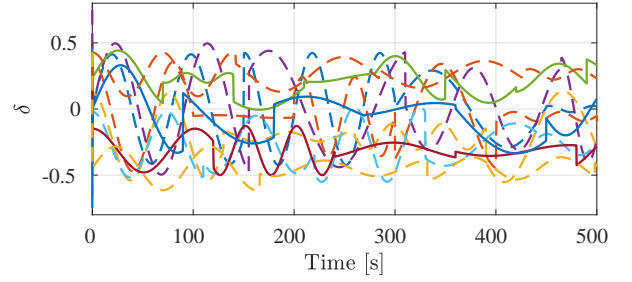


Fig. 5. Time-varying biases present in the state measurements of the system.

We set $Q = 3I_2$ and $\beta = 1.7$ so the factor of \tilde{e}_i in (40) be positive and we solve (14) for M with this new value. This yields

$$\begin{bmatrix} 4.4721 & 1 \\ 1 & 2.2361 \end{bmatrix}.$$

As per Proposition 8, the bias estimation errors are uniformly ultimately bounded—see Figure 6—and the multi-agent system achieves practical dynamic consensus—see Figure 7.

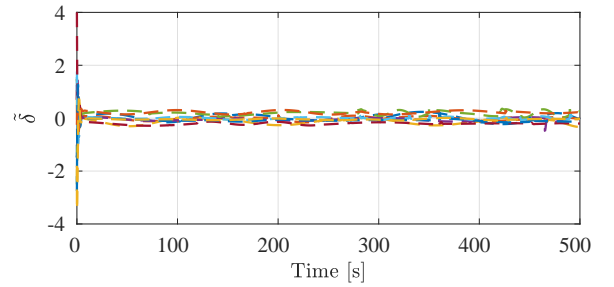


Fig. 6. Estimation errors corresponding to the time-varying biases in Figure 5.

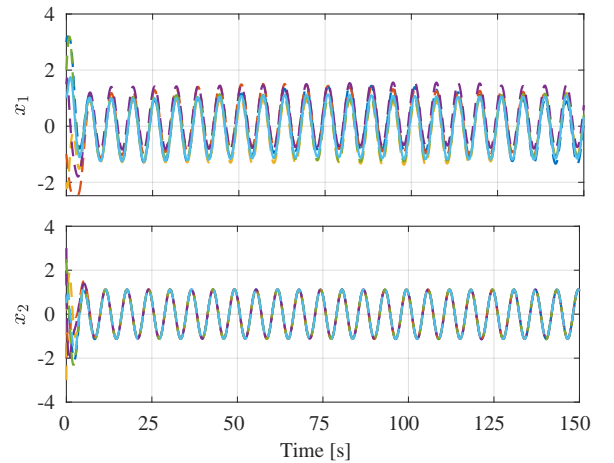


Fig. 7. Oscillators trajectories in practical dynamic consensus; the solid curve in cyan represents the average oscillator.

5 Conclusion

In a network containing a directed spanning tree, agents under measurement bias can reach dynamic leaderless consensus using a distributed-estimation-based controller reminiscent of model-reference adaptive controllers. However, consensus may be achieved only if the measurement bias is exactly estimated, which is possible only for piecewise-constant bias. If the bias is time-varying, as *e.g.*, due to a false-data-injection attack, only ultimate boundedness is guaranteed.

Current research is aimed at extending the framework to the case of heterogeneous systems; to investigate the possibility of compensating for the effects of heterogeneity via adaptive control and estimation laws. Ensuring asymptotic dynamic consensus in the presence of time-varying bias requires further study, notably, regarding a functional realistic model of the bias dynamics.

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