



HAL
open science

MRAC-based dynamic consensus of linear systems with biased measurements, over directed networks

Maitreyee Dutta, Elena Panteley, Antonio Loria, Sukumar Srikant

► **To cite this version:**

Maitreyee Dutta, Elena Panteley, Antonio Loria, Sukumar Srikant. MRAC-based dynamic consensus of linear systems with biased measurements, over directed networks. 2022. hal-03869892

HAL Id: hal-03869892

<https://hal.science/hal-03869892>

Preprint submitted on 24 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MRAC-based dynamic consensus of linear systems with biased measurements, over directed networks

Maitreyee Dutta^{a,1} Elena Panteley,^{b,1,2} Antonio Loría,^{b,1,2} Srikant Sukumar^{a,1}

^a*Systems and Control Engineering Department, IIT Bombay, India*

^b*Laboratoire des signaux et systèmes, CNRS, 3, Rue Joliot Curie, 91192, Gif-sur-Yvette, France*

Abstract

Dynamic consensus pertains to the achievement of a common collective behavior among interconnected systems. In this paper we address the dynamic consensus control problem for generic linear systems (stable, unstable, marginally stable, . . .) interconnected over directed networks and under the influence of biased measurements. Essentially, the control problem consists in redesigning a standard distributed consensus controller which, for each system, relies on own state biased measurements and respective erroneous data received from a set of neighbors. The difficulty in such a scheme resides in the fact that the measurement bias is directly amplified by the control gain so it cannot be handled as an additive external disturbance. Instead, our control design relies, on one hand, on the solution of a Riccati equation and, on the other, on the design of an estimator reminiscent of a model-reference-adaptive control design. The estimator successfully computes a bias estimate and completely compensates for its effect if the bias is constant—indeed, in this case, we establish exponential stability of the consensus manifold and we show that the controller provides robustness with respect to time-varying biases.

Key words: Measurement bias, directed graphs, multi-agent systems, linear systems, Lyapunov stability

1 Introduction

It is not a platitude to say that *feedback* is the essence of automatic control, so using faulty sensor data can have major repercussions in the closed-loop stability of the physical systems and can lead to severe accidents—see [11]. Sensor faults may consist, *e.g.*, in bias, drift, scaling, noise, *etc.* [1]. Measurement bias, which is the one of interest here, is a persistent offset in the reading, that may affect all the samples gathered in a particular deployment.

In the context of networked multi-agent systems, while a bidirectional communication helps in decoupling the sensor biases from the state measurements, as seen in *e.g.*, [21], [22], the influence of one agent over its neighbor is not always reciprocal in nature, so decoupling of the bias from the measurements can turn out to be an intricate problem in directed-topology networks—see *e.g.*, [23], [14], [3], [25], [2], [27]. Different techniques are applied in the latter references to cope with constant [23] or time-varying bias [14]: adaptive control in [14], event-triggered control in [3], adaptive neural control in [25], and an LMI-based approach in [27]. In [2] a distributed piece-wise constant impulsive reference is used.

A general connected directed graph may consist in, or contain, a spanning tree, so leader-follower consensus is applicable if the communication topology is a spanning tree with the root node behaving as a leader. In [14], [3], and [25] the leader-follower consensus problem is addressed, under the assumption that the measurement of the follower state/output variables are unreliable but the measurement of the leader states is bias-free. However, such assumption appears conservative, specially in a leaderless consensus scenario.

A significant difficulty to deal with measurement bias is that, in general, it cannot be dealt with using *ad hoc* robust control techniques tailored to cope with additive disturbances and consisting, essentially, in increasing the gain—the bias, if anything, is amplified by the control gain. In [23] the leaderless consensus problem is tackled using measurements affected by constant bias; the synchronization error is shown to be ultimately ultimately bounded (UUB).

In general, bias cannot be dealt with as a common uncertainty in lumped parameters either. Yet, some works borrow inspiration from the classical model-reference adaptive control (MRAC) method—see *e.g.*, [6] to successfully design distributed model-reference adaptive controllers for cooperative tracking [18], [24], [10], as well as for cooperative regulation [13], and for different communication topologies—whether it is undirected [18], [24], [10] or directed [13].

The application of MRAC-inspired techniques for leader-follower consensus generally relies on assimilating the leader node as the singular reference model—*e.g.*, [18], [24], [10]. In [13], a framework for adaptive leaderless consensus of linear (homogeneous) agents

¹ This work was supported by the CEFIPRA under the grant number 6001-A

² E. Panteley and A. Loría's work is also supported by the ANR (project HANDY, contract number ANR-18-CE40-0010).

with uncertain dynamics is proposed. The method consists in designing a reference model for each system and having the latter track the output generated by the corresponding linear reference model with relative state measurement as input.

Now, in all of the above cited references, MRAC-based techniques are used to deal with uncertain systems, but none of them considers biased relative state measurements. In the continuous-time domain, the study of leader-less dynamical consensus among agents exchanging biased state measurements (even for root nodes) through unidirectional network is rather limited. In [17] an MRAC-based technique is proposed to accommodate constant sensor bias, but not in a multi-agent context.

Thus, the dynamic (leaderless) consensus problem under measurement bias for multi-agent systems interconnected over generic directed graphs is rather limited. In this paper, we address it for generic identical linear systems under the assumption that measurement bias is different and bounded for each agent. More precisely, our contributions are the following:

- (1) in contrast to [18], [24], [10], and [13] we provide an MRAC-based control law for leader-less consensus among agents, which can override the effect of biased measurements;
- (2) in contrast to [14], [3], and [25], we do not assume that the leader (root) nodes are equipped with bias-free sensors;
- (3) unlike [23], in which the synchronization error is only uniformly ultimately bounded in presence of constant biases, we establish sufficient conditions for exponential stability of the synchronization manifold in the presence of constant biases and UUB if the latter are time-varying.

The remainder of the paper is organized as follows. In the next section we provide a detailed problem formulation, in Section 3 we present our main results; in Section 4 we provide some simulation results to illustrate our theoretical findings and we conclude with some closing remarks in Section 5.

2 Problem Formulation: System Setup

We consider $N \in \mathbb{N}$ identical, linear, autonomous systems,

$$\dot{x}_i = Ax_i + Bu_i, \quad \forall i = \{1, 2, \dots, N\}, \quad (1)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^p$ is the control input and A, B are constant matrices of compatible dimensions satisfying the following:

- (A1) the pair (A,B) is stabilizable,
- (A2) the system matrix A is of full rank.

For these systems, we address the dynamic consensus control problem. This pertains to making all systems achieve a common dynamic behavior, hence, such that

$$\lim_{t \rightarrow \infty} |x_i(t) - x_j(t)| \rightarrow 0, \quad \forall i \neq j \in \{1, 2, \dots, N\}, \quad (2)$$

via a distributed consensus controller of the form

$$u_i = K \sum_{j \in \mathcal{N}_i} a_{ij} (\bar{x}_i - \bar{x}_j) + \nu_i. \quad (3)$$

In the expression above, the first term on the right-hand side corresponds to a common consensus control law, preceded by a suitable gain matrix $K \in \mathbb{R}^{p \times n}$ to be determined. It is assumed that the states x_i are not measurable, or, more precisely, that

- (A3) the i^{th} agent has access to its own state and that of its neighbors, but these measurements are biased by an unknown offset δ_i , that is, $\bar{x}_i := x_i + \delta_i$;
- (A4) the bias δ_i is either constant or time-varying with bounded derivative.

The term ν_i in (3) denotes an additional, redesign, control input to be determined, notably to compensate for the effect of the measurement bias.

Finally, the coefficients $a_{ij} \geq 0$ represent the existence (if $a_{ij} > 0$) or the absence (if $a_{ij} = 0$) of a one-way interaction from the j^{th} node to the i^{th} node. As it is customary, we consider that there are no self-loops, so $a_{ii} = 0$. More precisely, we assume that

- (A5) the network of interconnections form a directed connected graph with Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$, where

$$l_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}, \quad l_{ij} = -a_{ij}. \quad (4)$$

Using the interactions as per (4), the i^{th} node can also have access to the bias estimates $\hat{\delta}_j$ of the state measurement's unknown bias $\delta_j \forall j \in \mathcal{N}_i$. The Laplacian, which is positive semidefinite in the case that the graph is undirected and connected, may be used with relative easiness to construct Lyapunov functions that serve to analyze the stability of the consensus manifold $\{x_i = x_j\}$. For connected directed networks \mathcal{L} is non-symmetric and this hinders the construction of suitable Lyapunov function candidates. To that end, we recall the following useful Lemmata.

Lemma 1 [19,20] *If a directed network has a directed spanning tree, then the Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ has a singular zero eigenvalue and $N - 1$ eigenvalues have strictly positive real part. That is,*

$$\sigma_1(\mathcal{L}) = 0, \quad \Re\{\sigma_k(\mathcal{L})\} > 0, \quad k = \{2, \dots, N\}.$$

On the other hand, the right eigenvector of the zero eigenvalue is given by $\mathbf{1}_N = [1 \ 1 \ \dots \ 1]^T$ and the left eigenvector v_l satisfies $\sum_{k=1}^N v_{l_k} = 1$ and $v_l^T \mathcal{L} = 0_N^T$. \square

Lemma 2 [16] *Let us consider a directed graph \mathcal{G} of order N containing a spanning tree and its Laplacian matrix is $\mathcal{L} \in \mathbb{R}^{N \times N}$. Then, for any positive definite symmetric matrix $Q_{\mathcal{L}} \in \mathbb{R}^{N \times N}$ and $\alpha \in \mathbb{R}^+$, there exists another positive definite symmetric matrix $P_{\mathcal{L}} \in \mathbb{R}^{N \times N}$ such that*

$$P_{\mathcal{L}} \mathcal{L} + \mathcal{L}^T P_{\mathcal{L}} = Q_{\mathcal{L}} - \alpha [P_{\mathcal{L}} \mathbf{1}_N v_l^T + v_l \mathbf{1}_N^T P_{\mathcal{L}}]. \quad (5)$$

\square

3 Main results

3.1 Control design

The consensus controller proposed in this paper relies on a distributed estimator whose design, along similar lines as those in [18], [24], [10], and [13], borrows inspiration from model-reference adaptive control. The control strategy consists in coupling each agent's dynamics to that of an estimator of the form

$$\dot{\hat{x}}_i = A\hat{x}_i + \epsilon BF \sum_{j \in \mathcal{N}_i} a_{ij}(\bar{x}_i - \bar{x}_j) - \epsilon BF \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\delta}_i - \hat{\delta}_j) \quad (6)$$

where the coupling weight $\epsilon > 0$, the control gain $F \in \mathbb{R}^{p \times n}$, and the bias estimate $\hat{\delta}_i$ are yet to be defined. Then, to achieve the consensus control objective (2) the first control goal is set to make the estimators (6) achieve dynamic consensus, that is,

$$\lim_{t \rightarrow \infty} |\hat{x}_i(t) - \hat{x}_j(t)| \rightarrow 0, \quad \forall i \neq j \in \{1, 2, \dots, N\} \quad (7)$$

and, second, to steer each agent's state trajectories to its corresponding estimator's, that is,

$$\lim_{t \rightarrow \infty} |x_i(t) - \hat{x}_i(t)| \rightarrow 0 \quad \forall i \in \{1, 2, \dots, N\}. \quad (8)$$

Note that (7) and (8) imply (2) since

$$\{\hat{x}_i = \hat{x}_j\} \cap \{x_i = \hat{x}_i\} \quad \forall i, j \in \{1, 2, \dots, N\}$$

implies that $\{x_i = x_j\}$ for all $i \in \{1, 2, \dots, N\}$. However, since the states x_i are not measurable, we pose as objective to steer the measured state \bar{x}_i to the corresponding biased state of the reference model in (6), that is, to make $\tilde{e}_i \rightarrow 0$, where

$$\tilde{e}_i = \bar{x}_i - (\hat{x}_i + \hat{\delta}_i). \quad (9)$$

It is remarked that this task, on one hand, appears feasible since \tilde{e}_i is available and, on the other hand, is useful since

$$\tilde{e}_i = x_i - \hat{x}_i + \tilde{\delta}_i, \quad (10)$$

with $\tilde{\delta}_i := \delta_i - \hat{\delta}_i$, so (8) holds if $\tilde{e}_i \rightarrow 0$ and $\hat{\delta}_i$ is updated so as to have

$$\lim_{t \rightarrow \infty} |\hat{\delta}_i(t) - \delta_i| \rightarrow 0, \quad \forall i \in \{1, 2, \dots, N\}. \quad (11)$$

To steer $\tilde{e}_i \rightarrow 0$ the control law u_i in (12) is redesigned as follows. We start by differentiating on both sides of (10) to obtain

$$\begin{aligned} \dot{\tilde{e}}_i &= A\tilde{e}_i - A\tilde{\delta}_i + Bu_i + \dot{\tilde{\delta}}_i \\ &\quad - \epsilon BF \sum_{j \in \mathcal{N}_i} a_{ij}[(x_i - x_j) + (\tilde{\delta}_i - \tilde{\delta}_j)], \end{aligned} \quad (12)$$

for which we used (1) and

$$\dot{\hat{x}}_i = A\hat{x}_i + \epsilon BF \sum_{j \in \mathcal{N}_i} a_{ij}[(x_i - x_j) + (\tilde{\delta}_i - \tilde{\delta}_j)]. \quad (13)$$

First, note that the terms on the second line of (12) can be canceled directly since they correspond exactly to the data available to the i^{th} node. Second, we may insert a control term of the form $F\tilde{e}_i$, such that $(A +$

$BF)$ is Hurwitz; this is possible since the pair (A, B) is stabilizable by assumption. More precisely, the latter implies the existence of a matrix $M = M^\top > 0$ that solves the algebraic Riccati equation

$$MA + A^\top M - MB B^\top M = -Q, \quad (14)$$

for any given $Q = Q^\top > 0$. Thus, for any such given Q that generates M as above, we define $F = -B^\top M$ —cf. [26, 28, 4, 8, 9] and we redefine u_i in (3) as

$$u_i = \epsilon F \sum_{j \in \mathcal{N}_i} a_{ij}(\bar{x}_i - \bar{x}_j) - \epsilon F \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{\delta}_i - \hat{\delta}_j) + F\tilde{e}_i. \quad (15)$$

Then, using (15) in (12), we obtain

$$\dot{\tilde{e}}_i = (A + BF)\tilde{e}_i - A\tilde{\delta}_i + \dot{\tilde{\delta}}_i. \quad (16)$$

Since $(A + BF)$ is Hurwitz by design, it is left to define the dynamics of the bias estimates $\hat{\delta}_i$ —hence that of $\tilde{\delta}_i$ —so as to ensure that $\tilde{\delta}_i$ and $\dot{\tilde{\delta}}_i$ vanish asymptotically. The design of the bias estimation law depends on whether δ_i is constant or time-varying, as per Assumption (A4), and is explained in the next sections.

3.2 Dynamic consensus under constant bias

In the case that δ_i is constant we define

$$\dot{\tilde{\delta}}_i = -M^{-1}A^\top M \tilde{e}_i, \quad (17)$$

where M solves (14) for any given positive definite Q . Such estimation law is distributed since it depends only on \tilde{e}_i , which is defined by variables that pertain to the i^{th} agent only—see (9). Then, we have the following.

Remark 3 (Notation) In the remainder of the paper, we use the notations σ_m^H and σ_M^H to represent the minimum and maximum eigenvalue of any square matrix H respectively. •

Proposition 4 (Dynamic consensus) Consider N identical linear systems as in (1) in closed loop with the distributed control input given by (15), with $\epsilon \geq \sigma_M^{P_{\mathcal{L}}}$ where $P_{\mathcal{L}}$ solves (5) for $Q_{\mathcal{L}} = I_N$, the feedback matrix $F = -B^\top M$ where M solves (14), the reference model (6), and the bias update law (17). Then, under Assumptions (A1)–(A5), (2) holds. □

The previous statement follows after a cascades argument and the rationale presented previously. The limit in (2) holds if so do those in (7) and (8). The latter is established in Proposition 5, which is presented next, and (7) is established in Proposition 7 farther below.

Proposition 5 (bias estimation) Consider N identical linear systems as in (1) in closed loop with (15), (6), and (17) under the conditions of Proposition 4. Then, there exist λ and $\kappa > 0$, such that

$$|\xi(t)| \leq \kappa |\xi(t_0)| e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0, \quad (18)$$

where $\xi := [\tilde{e}^\top \ \tilde{\delta}^\top]^\top$, $\tilde{e}^\top := [\tilde{e}_1^\top \ \dots \ \tilde{e}_N^\top]$, and $\tilde{\delta}^\top := [\tilde{\delta}_1^\top \ \dots \ \tilde{\delta}_N^\top]$. In turn, the limits in (8) and (11) hold. □

Proof. In compact form, the multi-agent closed-loop equations correspond to

$$\dot{\tilde{e}} = [I_N \otimes (A + BF)]\tilde{e} - [I_N \otimes A]\tilde{\delta},$$

$$+[I_N \otimes (M^{-1}A^\top M)]\tilde{e} \quad (19a)$$

$$\dot{\tilde{\delta}} = [I_N \otimes (M^{-1}A^\top M)]\tilde{e}. \quad (19b)$$

Then, consider the Lyapunov function candidate

$$V(\tilde{e}, \tilde{\delta}) = \tilde{e}^\top (I_N \otimes M)\tilde{e} + \tilde{\delta}^\top (I_N \otimes M)\tilde{\delta}, \quad (20)$$

which, since M is symmetric positive definite, is both positive definite and decrescent. Indeed, V satisfies

$$\sigma_m^M |\xi|^2 \leq V(\tilde{e}, \tilde{\delta}) \leq \sigma_M^M |\xi|^2. \quad (21)$$

Furthermore, in virtue of (14), the time-derivative of V along the trajectories of (19) is negative semidefinite. Indeed,

$$\begin{aligned} \dot{V}(\tilde{e}, \tilde{\delta}) &= \tilde{e}^\top [I_N \otimes (MA + A^\top M - 2MBB^\top M)]\tilde{e} \\ &\quad - 2\tilde{e}^\top [I_N \otimes (MA)]\tilde{\delta} + 2\tilde{e}^\top [I_N \otimes (A^\top M)]\tilde{e} \\ &\quad + 2\tilde{\delta}^\top [I_N \otimes (A^\top M)]\tilde{e} \\ &= 2\tilde{e}^\top [I_N \otimes (MA + A^\top M - MBB^\top M)]\tilde{e} \\ &= -2\tilde{e}^\top [I_N \otimes Q]\tilde{e} \leq 0. \end{aligned} \quad (22)$$

From (21)-(22), it follows that the origin, $\{(\tilde{e}, \tilde{\delta}) = (0, 0)\}$, for the system (19a)-(19b) is uniformly globally stable. In particular, it is uniformly stable and all solutions are uniformly globally bounded.

To establish uniform global exponential stability we use the following statement, which follows from [12, Lemma 3].

Lemma 6 *Consider the system $\dot{\xi} = f(t, \xi)$, where $f(\cdot, \xi)$ is locally integrable and $f(t, \cdot)$ is locally Lipschitz uniformly in t . Assume that there exist constants p , and $c > 0$ such that the solution $\xi(\cdot, t_o, \xi_o)$ of $\dot{\xi} = f(t, \xi)$ satisfies*

$$\max \left\{ \sup_{t \geq t_o} |\xi(t)|, \left[\int_{t_o}^{\infty} |\xi(\tau)|^p d\tau \right]^{1/p} \right\} \leq c|\xi_o|, \quad (23)$$

for all initial conditions³ $(t_o, x_o) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. Then, the origin $\{\xi = 0\}$ is uniformly globally exponentially stable and, moreover, (18) holds with $\kappa = ce^{1/p}$ and $\lambda = [pc^p]^{-1}$. \square

Thus, the rest of the proof consists in establishing the bounds in (23) for the closed-loop system (19). The first inequality in (23) follows by integrating along trajectories on both sides of the inequality $\dot{V} \leq 0$ in (22) and using (21) to obtain

$$V(\tilde{e}(t), \tilde{\delta}(t)) \leq V(\tilde{e}(t_o), \tilde{\delta}(t_o)) \quad (24)$$

$$\sigma_m^M |\xi(t)|^2 \leq V(\tilde{e}(t), \tilde{\delta}(t)) \leq \sigma_M^M |\xi(t_o)|^2, \quad \forall t \geq t_o \geq 0,$$

which, in turn, implies that

$$|\xi(t)| \leq \sqrt{\frac{\sigma_M^M}{\sigma_m^M}} |\xi(t_o)|, \quad \forall t \geq t_o, \quad (25)$$

so the first inequality in (23) holds with $p = 2$ and $c^2 := \sigma_M^M / \sigma_m^M$.

³ Considering $t_o \geq 0$ is a mere convention; the result certainly holds for all $t_o \in \mathbb{R}$.

On the other hand, integrating on both sides of $\dot{V} = -2\tilde{e}^\top (I_N \otimes Q)\tilde{e}$ in (22), along the trajectories, we obtain

$$V(\tilde{e}(t), \tilde{\delta}(t)) - V(\tilde{e}(t_o), \tilde{\delta}(t_o)) \leq -2\sigma_m^Q \int_{t_o}^t |\tilde{e}(\tau)|^2 d\tau.$$

Rearranging the letter and using (24) gives

$$2\sigma_m^Q \int_{t_o}^t |\tilde{e}(\tau)|^2 d\tau \leq 2V(\tilde{e}(t_o), \tilde{\delta}(t_o))$$

which, in view of (21), implies that

$$\lim_{t \rightarrow \infty} \int_{t_o}^t |\tilde{e}(\tau)|^2 d\tau \leq \frac{\sigma_M^M}{\sigma_m^Q} |\xi(t_o)|^2. \quad (26)$$

The latter establishes the second inequality in (23), for the \tilde{e} -part of ξ . To compute a similar inequality for $\tilde{\delta}$, consider the function $W : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, defined by

$$W(\xi) = \tilde{e}^\top [I_N \otimes A]\tilde{\delta} \quad (27)$$

and let $\sigma_M^A > 0$ be such that $|A| \leq \sigma_M^A$. Then, along the trajectories,

$$W(\xi(t)) \leq \frac{\sigma_M^A}{2} |\xi(t)|^2 \leq \frac{\sigma_M^A}{2} c^2 |\xi(t_o)|^2, \quad \forall t \geq t_o, \quad (28)$$

while the initial conditions satisfy

$$W(\xi(t_o)) \leq \frac{\sigma_M^A}{2} |\xi(t_o)|^2. \quad (29)$$

The directional derivative of W along the trajectories of (19) yields

$$\begin{aligned} \dot{W}(\xi(t)) &\leq \tilde{e}^\top [I_N \otimes AM^{-1}A^\top M]\tilde{e} \\ &\quad + \tilde{\delta}^\top [I_N \otimes (A^\top A + A^\top BF + A^\top M^{-1}A^\top M)]\tilde{e} \\ &\quad - \tilde{\delta}^\top [I_N \otimes A^\top A]\tilde{\delta}. \end{aligned} \quad (30)$$

Then, to compact the notation, we introduce

$$\mathcal{A} := A^\top A - A^\top BB^\top M + A^\top M^{-1}A^\top M$$

and, since A , B , and M are constant, we also introduce $\sigma_M^A > 0$ to denote an upper bound on the induced norm of \mathcal{A} , that is, $|\mathcal{A}| \leq \sigma_M^A$. Hence, the terms in the second line of (30) satisfy

$$\tilde{\delta}^\top [I_N \otimes \mathcal{A}]\tilde{e} \leq \frac{(\sigma_M^A)^2}{2\mu} |\tilde{e}|^2 + \frac{\mu}{2} |\tilde{\delta}|^2 \quad (31)$$

for any $\mu > 0$. For further development, we also introduce $\gamma' > 0$ such that $|AM^{-1}A^\top M| \leq \gamma'$ and $a'_m > 0$ such that $\sigma_m^{A^\top A} \geq a'_m$; the latter holds under Assumption (A2).

Next, replacing (31) and $|AM^{-1}A^\top M| \leq \gamma'$ in (30) yields, along trajectories,

$$\dot{W}(\xi(t)) \leq \left[\gamma' + \frac{(\sigma_M^A)^2}{2\mu} \right] |\tilde{e}(t)|^2 - \left[a'_m - \frac{\mu}{2} \right] |\tilde{\delta}(t)|^2.$$

Then, we set $\mu = a'_m$, we define $\gamma'' := \gamma' + \frac{(\sigma_M^A)^2}{2\mu}$, and we integrate on both sides of the latter inequality and rearrange terms to obtain

$$\frac{a'_m}{2} \int_{t_o}^t |\tilde{\delta}(\tau)|^2 d\tau \leq |W(\xi(t))| + |W(\xi(t_o))|$$

$$+ \gamma'' \int_{t_o}^t |\tilde{e}(\tau)|^2 d\tau. \quad (32)$$

Thus, using (28), (29), and (26), we obtain

$$\int_{t_o}^t |\tilde{\delta}(\tau)|^2 d\tau \leq \frac{2}{a'_m} \left[\gamma'' \frac{\sigma_M^M}{\sigma_m^Q} + \frac{\sigma_M^A}{2} \left[\frac{\sigma_M^M}{\sigma_m^M} + 1 \right] \right] |\xi(t_o)|^2, \quad \forall t \geq t_o. \quad (33)$$

The latter, together with (25) and (26), imply the condition (23) with $p = 2$ and

$$c^2 := \max \left\{ \frac{\sigma_M^M}{\sigma_m^M}, \frac{\sigma_M^M}{\sigma_m^Q} + \frac{2}{a'_m} \left[\gamma'' \frac{\sigma_M^M}{\sigma_m^Q} + \frac{\sigma_M^A}{2} \left[\frac{\sigma_M^M}{\sigma_m^M} + 1 \right] \right] \right\},$$

so the result follows with κ and λ as in Lemma 6. ■

Proposition 7 (Estimators' consensus) *Consider the estimators defined in (6), with $a_{ij} \geq 0$ such that the corresponding Laplacian matrix, whose elements are defined in (4), contains a directed spanning tree, and with $\epsilon \geq \sigma_M^{P_{\mathcal{L}}}$ where $P_{\mathcal{L}}$ solves (5) for $Q_{\mathcal{L}} = I_N$, the feedback matrix $F = -B^{\top}M$ where M solves (14). Then, on the manifold $\{i \leq N : (\tilde{e}_i, \tilde{\delta}_i) = (0, 0)\}$, the estimators (6) achieve dynamic consensus, that is, (7) holds. More precisely,*

$$\lim_{t \rightarrow \infty} |\hat{x}_i(t) - \hat{x}_m(t)| = 0, \quad \forall i \in \{1, 2, \dots, N\},$$

where \hat{x}_m corresponds to the solutions of $\dot{\hat{x}}_m = A\hat{x}_m$ with $\hat{x}_m(0) := (v_l^{\top} \otimes I_n)\hat{x}(0)$. □

Proof. In compact multi-variable form, the equations (6) are written as

$$\dot{\hat{x}} = [I_N \otimes A]\hat{x} - \epsilon[\mathcal{L} \otimes BB^{\top}M]\hat{x} - \epsilon[\mathcal{L} \otimes BB^{\top}M]\tilde{\delta}, \quad (34)$$

and, on the manifold $\{(\tilde{e}, \tilde{\delta}) = (0, 0)\}$, in view of (10), we have

$$\dot{\hat{x}} = [I_N \otimes A]\hat{x} - \epsilon[\mathcal{L} \otimes BB^{\top}M]\hat{x} \quad (35)$$

Now, following [15], to assess that the estimators (35) reach dynamic consensus, we verify that \hat{x}_i , for each $i \leq N$, tends asymptotically to the trajectories of a weighted-average system with state $\hat{x}_m := (v_l^{\top} \otimes I_n)\hat{x}$. To that end, we also define the synchronization errors $s_{\hat{x}_i} := \hat{x}_i - \hat{x}_m$. In multi-variable form, we have

$$\begin{bmatrix} \hat{x}_m \\ s_{\hat{x}} \end{bmatrix} = \begin{bmatrix} (v_l^{\top} \otimes I_n)\hat{x} \\ [(I_N - 1_N v_l^{\top}) \otimes I_n]\hat{x} \end{bmatrix}, \quad (36)$$

where $s_{\hat{x}} = [s_{\hat{x}_1}^{\top} \dots s_{\hat{x}_N}^{\top}]^{\top}$. Then, differentiating on both sides of (36) and using (35), we obtain the system

$$\begin{bmatrix} \dot{\hat{x}}_m \\ \dot{s}_{\hat{x}} \end{bmatrix} = \begin{bmatrix} A\hat{x}_m \\ [(I_N \otimes A) - \epsilon(\mathcal{L} \otimes BB^{\top}M)]s_{\hat{x}} \end{bmatrix}. \quad (37)$$

The latter is a dichotomous representation of the multi-agent estimators (35). On one hand, one has the decoupled average dynamics $\dot{\hat{x}}_m = A\hat{x}_m$ and, on the other, the synchronization error dynamics. Therefore, establishing that $s_{\hat{x}} \rightarrow 0$ implies that all estimators behave as the averaged model $\dot{\hat{x}}_m = A\hat{x}_m$.

Remark 8 In an ordinary consensus problem in which all systems attain a common *equilibrium* point, x_m , such that $\dot{x}_m \equiv 0$. On the other hand, replacement of suitable control input can ensure that the averaged model of the multi-agent system is $\dot{x}_m = Ax_m$, whence the terminology *dynamic consensus*. •

For assessing convergence of individual reference systems to the concerned averaged model, we use procedure outlined in [5]. To that end, we consider the Lyapunov function candidate

$$V(s_{\hat{x}}) = s_{\hat{x}}^{\top} (P_{\mathcal{L}} \otimes M) s_{\hat{x}}, \quad (38)$$

where $P_{\mathcal{L}}$ is generated by Lemma 2 with $Q_{\mathcal{L}} = I_N$. This function is positive definite under Assumption (A5) and Lemma 2.

The time derivative of $V(s_{\hat{x}})$, along the trajectories of (37) yields

$$\begin{aligned} \dot{V}(s_{\hat{x}}) &= s_{\hat{x}}^{\top} \left[P_{\mathcal{L}} \otimes (MA + A^{\top}M) \right. \\ &\quad \left. - \epsilon(P_{\mathcal{L}}\mathcal{L} + \mathcal{L}^{\top}P_{\mathcal{L}}) \otimes (M B B^{\top}M) \right] s_{\hat{x}}. \end{aligned} \quad (39)$$

Then, we use (5), with $Q_{\mathcal{L}} = I_N$, in (39), we apply the property

$$\alpha[P_{\mathcal{L}}1_N v_l^{\top} \otimes I_n]s_{\hat{x}} = \alpha[P_{\mathcal{L}}1_N v_l^{\top} \otimes I_n] \times [(I_N - 1_N v_l^{\top}) \otimes I_n]\hat{x} = 0_{Nn},$$

which holds under Assumption (A4)—see Lemma 1, and the orthogonal decomposition of $P_{\mathcal{L}} = P_{\mathcal{L}}^{\top} > 0$, $P_{\mathcal{L}} = T\Lambda T^{\top}$ where $T \in \mathbb{R}^{N \times N}$ is an orthogonal matrix. Thus, (39) becomes

$$\begin{aligned} \dot{V}(s_{\hat{x}}) &= \sigma_M^{P_{\mathcal{L}}} s_{\hat{x}}^{\top} \left[T T^{\top} \otimes [MA + A^{\top}M] \right. \\ &\quad \left. - \frac{\epsilon}{\sigma_M^{P_{\mathcal{L}}}} M B B^{\top}M \right] s_{\hat{x}}, \end{aligned}$$

so for any $\epsilon \geq \sigma_M^{P_{\mathcal{L}}}$, as posed in the Proposition 4, we have

$$\dot{V}(s_{\hat{x}}) \leq \sigma_M^{P_{\mathcal{L}}} s_{\hat{x}}^{\top} [I_N \otimes (MA + A^{\top}M - M B B^{\top}M)] s_{\hat{x}}$$

and, after the algebraic Riccati equation (14), we get

$$\dot{V}(s_{\hat{x}}) \leq -\sigma_M^{P_{\mathcal{L}}} s_{\hat{x}}^{\top} (I_N \otimes Q) s_{\hat{x}},$$

so global exponential stability of $\{s_{\hat{x}} = 0\}$ follows invoking standard Lyapunov theory. ■

3.3 Practical dynamic consensus under time-varying measurement bias

To take into account the time-varying unknown bounded biases $\delta_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n, i = \{1, \dots, N\}$, the corresponding dynamics of the bias estimates is now modified as

$$\dot{\delta}_i = -(M^{-1}A^{\top}M)\tilde{e}_i - \beta M^{-1}\hat{\delta}_i, \quad (40)$$

where $\beta \in \mathbb{R}^+$. Then, we have the following.

Proposition 9 (robust dynamic consensus) *Consider N linear systems defined by (1) under the conditions laid in Proposition 4 and the bias update law (40) in place of (17). Then, the tracking and estimation errors $(\tilde{e}, \tilde{\delta})$ are uniformly ultimately bounded. Consequently, so*

are the tracking errors $x_e = x_i - \hat{x}_i$, as well as the consensus errors for the reference models, $\hat{x}_i - \hat{x}_j$, and for the original systems (1), $x_i - x_j$, for all $i \neq j$.

Proof. The closed-loop equations, now using the update law (40), correspond to

$$\begin{aligned} \dot{\tilde{e}} &= [I_N \otimes (A + BF)]\tilde{e} - [I_N \otimes A]\tilde{\delta} + \dot{\tilde{\delta}} \\ &\quad + \beta(I_N \otimes M^{-1})\tilde{\delta} + [I_N \otimes (M^{-1}A^\top M)]\tilde{e} \end{aligned} \quad (41a)$$

$$\dot{\tilde{\delta}} = [I_N \otimes (M^{-1}A^\top M)]\tilde{e} + \dot{\tilde{\delta}} + \beta(I_N \otimes M^{-1})\tilde{\delta}. \quad (41b)$$

Reconsider the Lyapunov function V in (20). After (22), its total derivative along the solutions of (41a)-(41b) yields

$$\begin{aligned} \dot{V} &= -2\tilde{e}^\top [I_N \otimes Q]\tilde{e} + 2(\tilde{\delta} + \tilde{e})^\top (I_N \otimes M)\dot{\tilde{\delta}} \\ &\quad + 2\beta(\tilde{\delta} + \tilde{e})^\top \tilde{\delta} - 2\beta\tilde{e}^\top \tilde{\delta} - 2\beta|\tilde{\delta}|^2. \end{aligned} \quad (42)$$

Therefore,

$$\dot{V} \leq -[2\sigma_m^Q - \beta]|\tilde{e}|^2 - \beta|\tilde{\delta}|^2 + k[|\tilde{e}| + |\tilde{\delta}|] \quad (43)$$

where $k := 2(\sigma_M^M + \beta)\delta_M$, with

$$\delta_M := \max \left\{ \operatorname{ess\,sup}_{t \geq 0} |\delta(t)|, \operatorname{ess\,sup}_{t \geq 0} |\dot{\delta}(t)| \right\}.$$

Global ultimate boundedness of \tilde{e} and $\tilde{\delta}$ follow from standard Lyapunov theory [7, p. 169]. To assess ultimate boundedness of the tracking errors, $x_e = x - \hat{x}$, we analyze the corresponding dynamics equation

$$\dot{x}_e = [I_N \otimes (A + BF)]x_e + (I_N \otimes BF)\tilde{\delta}. \quad (44)$$

To that end, we evaluate the total derivative of the Lyapunov function $V(x_e) = x_e^\top (I_N \otimes M)x_e$ along the trajectories of (44), to obtain

$$\begin{aligned} \dot{V}(x_e) &= x_e^\top [I_N \otimes (MA + A^\top M - 2MBB^\top M)]x_e \\ &\quad - 2x_e^\top (I_N \otimes (MBB^\top M))\tilde{\delta} \\ &\leq x_e^\top [I_N \otimes (MA + A^\top M - 2MBB^\top M)]x_e \\ &\quad + x_e^\top (I_N \otimes (MA + A^\top M))x_e + \tilde{\delta}^\top (I_N \otimes \overline{M})\tilde{\delta} \\ &\leq -2x_e^\top (I_N \otimes Q)x_e + |\overline{M}||\tilde{\delta}|^2, \end{aligned} \quad (45)$$

where we introduced

$$\overline{M} := MBB^\top M(MA + A^\top M)^{-1}MBB^\top M.$$

Note that $(MA + A^\top M)$ is non-singular under Assumption (A2). Global ultimate boundedness of x_e follows from the ultimate boundedness of $\tilde{\delta}$, after integrating on both sides of (45).

Finally, to verify that the consensus error amongst the reference systems is also UUB, we reconsider the synchronization error $s_{\hat{x}}$ and the mean field value \hat{x}_m of the reference systems, as defined in (36). The corresponding dynamics, in this case, is

$$\dot{\hat{x}}_m = A\hat{x}_m \quad (46a)$$

$$\begin{aligned} \dot{s}_{\hat{x}} &= [(I_N \otimes A) - \epsilon(\mathcal{L} \otimes BB^\top M)]s_{\hat{x}} \\ &\quad - \epsilon[\mathcal{L} \otimes BB^\top M][(I_N - 1_N v_l^\top) \otimes I_n]\tilde{\delta}. \end{aligned} \quad (46b)$$

Global uniform ultimate boundedness follows from the fact that the origin for (46b) with $\tilde{\delta} = 0$ is globally exponentially stable and $\tilde{\delta}(t)$ is globally uniformly ultimately bounded. ■

4 Simulation Results

We provide some numerical simulation results, generated using Matlab R2021a. As case-study, we consider the dynamic consensus control of five harmonic oscillators, that is, linear systems modeled as in (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (47)$$

so Assumptions (A1) and (A2) hold. The oscillators are taken to form the graph showed in Figure 1, so Assumption (A5) also holds.

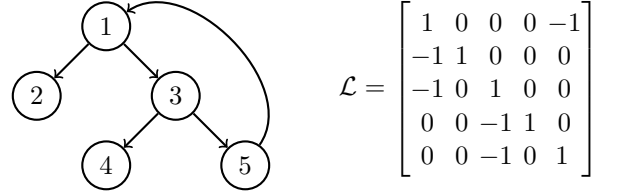


Fig. 1. Directed connected graph and corresponding Laplacian.

We present the results for two tests, one in which the biases are piecewise-constant and one in which they are periodic functions. Irrespective of the nature of the measurement biases, constant or otherwise, the initial conditions assigned for the five agents are set to

$$\begin{aligned} x_1(0) &= [3 \ -1 \ -2 \ 1.5 \ 2]^\top, \\ \hat{x}_1(0) &= [-1.5 \ 1 \ -1.75 \ -0.5 \ 2.75]^\top, \\ x_2(0) &= [2 \ -2 \ -3 \ 3 \ 2.5]^\top, \\ \hat{x}_2(0) &= [0.5 \ -2 \ 1 \ 0.75 \ 1.5]^\top, \end{aligned}$$

and the initial conditions for the bias estimates are set to

$$\begin{aligned} \hat{\delta}_1(0) &= [1.5 \ -1.5 \ 2 \ -2.5 \ 0.5]^\top, \\ \hat{\delta}_2(0) &= [0.5 \ -0.5 \ -2 \ 2 \ 3]^\top. \end{aligned}$$

In the first numerical test, the biases are modeled as piecewise-constant functions taking random values within $[-1, 1]$ and changing at random instants separated by no less than 10s; see Figure 2 below.

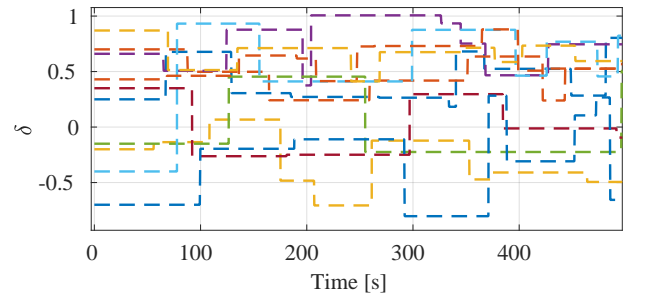


Fig. 2. Plot of piecewise-constant biases, with random changes of random amplitude.

The control gain $F = -B^\top M$ is computed by solving the algebraic Riccati equation (14) for M , with $Q = I_2$. This yields

$$M = \begin{bmatrix} 1.9123 & 0.4142 \\ 0.4142 & 1.3522 \end{bmatrix}. \quad (48)$$

The systems' trajectories are shown in Figure 3—it is noted that they all achieve dynamic consensus, as they enter in synchrony with the average oscillator. The controlled system is robust to the sudden changes in the biases.

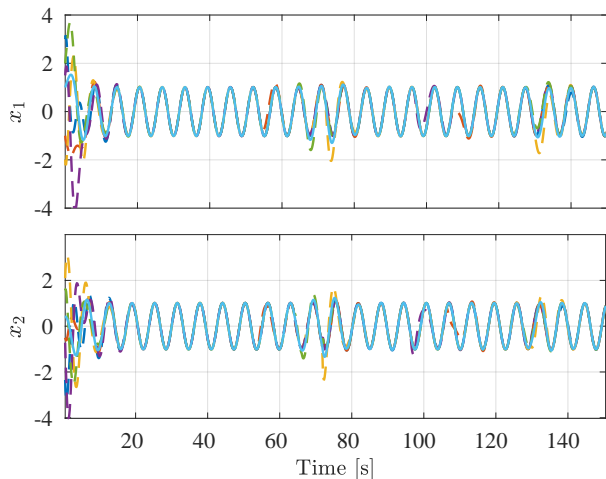


Fig. 3. Oscillators' trajectories reaching synchronization. The solid curve in cyan represents the average oscillator.

For completeness, we show the bias estimation errors in Figure 4 below.

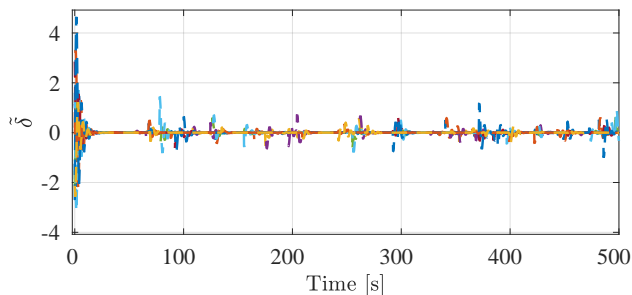


Fig. 4. Estimation errors for the piecewise constant biases for all agents.

With a second numerical test, we illustrate the robustness of our consensus controller *vis a vis* of time-varying biases. These are of the form $\delta_i(t) := c_i + \delta'_i(t)$, where $c_i \in \mathbb{R}$ and $\delta'_i(t)$ is a sinusoidal or cosinusoidal signal whose amplitude and frequency is varied over time using the $\text{rand}(\cdot)$ function of Matlab, so Assumption (A4) holds. See Figure 5. We set $Q = 3I_2$ and $\beta = 1.7$ so the factor of \tilde{e}_i in (43) be positive and we solve (14) for M with this new value. This yields

$$\begin{bmatrix} 4.4721 & 1 \\ 1 & 2.2361 \end{bmatrix}.$$

As per Proposition 9, the bias estimation errors are uniformly ultimately bounded—see Figure 6. Consequently, the multi-agent system achieves practical dynamic consensus—see Figure 7.

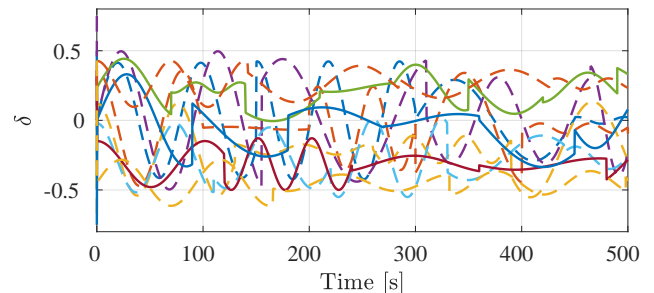


Fig. 5. Time-varying biases present in the state measurements of the system.

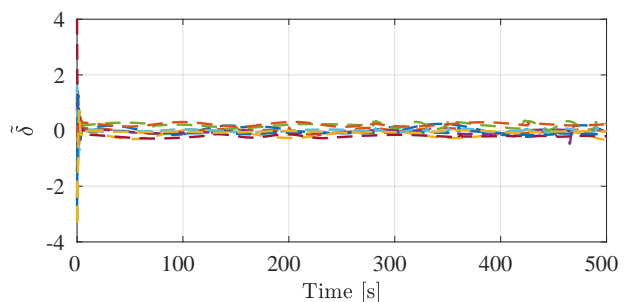


Fig. 6. Error in estimation of the time-varying biases as in Figure 5.

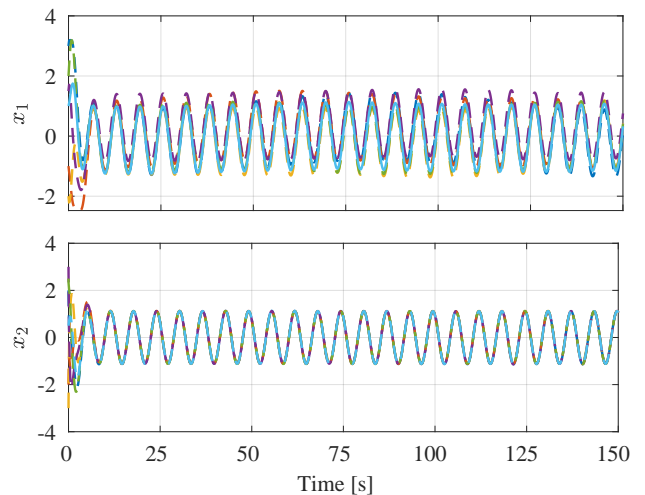


Fig. 7. Trajectories of all oscillators exhibiting practical dynamic consensus. The solid curve in cyan represents the average oscillator.

5 Conclusion

In a network containing a directed spanning tree, agents with linear homogeneous stabilizable dynamics can reach dynamic consensus, relying on a distributed-estimation algorithm that, on one hand, consists in coupling the oscillators to the estimators and, on the other,

aims at making the estimators achieve dynamic consensus themselves. The approach relies on the ability of estimating the measurement bias exactly. The latter, in turn, is guaranteed only in the case that the said bias is constant. If time-varying, ultimate boundedness is still guaranteed. That is, the original controller renders the system robust.

Current research is aimed at extending the framework to the case of heterogeneous systems; to investigate the possibility of compensating for the effects of heterogeneity via adaptive control and estimation laws. Ensuring asymptotic dynamic consensus in the presence of time-varying bias requires further study, notably, regarding a functional realistic model of the bias dynamics.

References

- [1] E. Balaban, A. Saxena, P. Bansal, K. F. Goebel, and S. Curran. Modeling, detection, and disambiguation of sensor faults for aerospace applications. *IEEE Sensors Journal*, 9(12):1907–1917, 2009.
- [2] Tommaso Borzone, I-C Morărcscu, Marc Jungers, Michael Boc, and Christophe Janneteau. Hybrid formalism for consensus of a general class of multi-agent systems with biased measurements. In *2019 18th European Control Conference (ECC)*, pages 3016–3021. IEEE, 2019.
- [3] L. Cao, H. Li, G. G. Dong, and R. Lu. Event-triggered control for multiagent systems with sensor faults and input saturation. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 51(6):3855–3866, 2019.
- [4] C. Chen, F. L. Lewis, and X. Li. Event-triggered coordination of multi-agent systems via a lyapunov-based approach for leaderless consensus. *Automatica*, page 109936, 2021.
- [5] M. Dutta, E. Panteley, A. Loría, and S. Sukumar. Strict Lyapunov functions for dynamic consensus in linear systems interconnected over directed graphs. *IEEE Control Systems Letters*, 6(1):2323–2328, 2022.
- [6] P. A. Ioannou and J. Sun. *Robust adaptive control*. PTR Prentice-Hall, Upper Saddle River, NJ, 1996.
- [7] H. Khalil. *Nonlinear systems*. Prentice Hall, 3rd ed., New York, 2002.
- [8] Z. Li, W. Ren, X. Liu, and M. Fu. Consensus of multi-agent systems with general linear and lipschitz nonlinear dynamics using distributed adaptive protocols. *IEEE transactions on automatic control*, 58(7):1786–1791, 2012.
- [9] Z. Li, G. Wen, Z. Duan, and W. Ren. Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs. *IEEE transactions on Automatic control*, 60(4):1152–1157, 2014.
- [10] Y. Liu and Y. Jia. Adaptive leader-following consensus control of multi-agent systems using model reference adaptive control approach. *IET Control Theory & Applications*, 6(13):2002–2008, 2012.
- [11] T. Lombaerts, P. Chu, J. A. B. Mulder, and O. Stroosma. Fault tolerant flight control, a physical model approach. *Advances in Flight Control Systems*, 2011.
- [12] A. Loría and E. Panteley. Uniform exponential stability of linear time-varying systems: revisited. *Systems & Control Letters*, 47(1):13–24, 2002.
- [13] J. Mei, W. Ren, and Y. Song. A unified framework for adaptive leaderless consensus of uncertain multiagent systems under directed graphs. *IEEE Transactions on Automatic Control*, 66(12):6179–6186, 2021.
- [14] M. Meng, G. Xiao, and B. Li. Adaptive consensus for heterogeneous multi-agent systems under sensor and actuator attacks. *Automatica*, 122:109242, 2020.
- [15] E. Panteley and A. Loría. Synchronization and dynamic consensus of heterogeneous networked systems. *IEEE Trans. on Automatic Control*, 62(8):3758–3773, 2017.
- [16] E. Panteley, A. Loría, and S. Sukumar. Strict lyapunov functions for consensus under directed connected graphs. In *Proc. European Control Conference (ECC)*, pages 935–940, St. Petersburg, Russia, 2020.
- [17] P. Patre and S. M. Joshi. Accommodating sensor bias in MRAC for state tracking. In *AIAA Guidance, Navigation, and Control Conference*, page 6605, 2011.
- [18] Z. Peng, D. Wang, H. Zhang, G. Sun, and H. Wang. Distributed model reference adaptive control for cooperative tracking of uncertain dynamical multi-agent systems. *IET Control Theory and Applications*, 7(8):1079–1087, 2012.
- [19] W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on automatic control*, 50(5):655–661, 2005.
- [20] W. Ren and R. W. Beard. *Distributed consensus in multivehicle cooperative control*, volume 27 of 2. Springer verlag, London: Springer London, 2008.
- [21] M. Shi, C. De Persis, P. Tesi, and N. Monshizadeh. Bias estimation in sensor networks. *IEEE Transactions on Control of Network Systems*, 7(3):1534–1546, 2020.
- [22] Pallavi Sinha, Srikant Sukumar, and Himani Sinhmar. Consensus of networked double integrator systems under sensor bias. *arXiv preprint arXiv:2204.08666*, 2022.
- [23] S. Sukumar, E. Panteley, A. Loría, and W. Pasillas-Lépine. On consensus of double integrators over directed graphs and with relative measurement bias. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 4147–4152, Miami Beach, FL, USA, Dec 2018.
- [24] Kouichi Sumizaki, Lu Liu, and Shinji Hara. Adaptive consensus on a class of nonlinear multi-agent dynamical systems. In *Proceedings of SICE Annual Conference 2010*, pages 1141–1145. IEEE, 2010.
- [25] M. Tan, Z. Liu, C. P. Chen, and Y. Zhang. Neuroadaptive asymptotic consensus tracking control for a class of uncertain nonlinear multiagent systems with sensor faults. *Information Sciences*, 584:685–700, 2022.
- [26] F. Xiao and T. Chen. Adaptive consensus in leader-following networks of heterogeneous linear systems. *IEEE Transactions on Control of Network Systems*, 5(3):1169–1176, 2017.
- [27] B. Yan, C. Wu, and P. Shi. Formation consensus for discrete-time heterogeneous multi-agent systems with link failures and actuator/sensor faults. *Journal of the Franklin Institute*, 356(12):6547–6570, 2019.
- [28] D. Yang, W. Ren, X. Liu, and W. Chen. Decentralized event-triggered consensus for linear multi-agent systems under general directed graphs. *Automatica*, 69:242–249, 2016.