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# Some remarks on the stabilisation problem and the stabilisability radius for linear switched systems

P. Mason\*

**Abstract**—We discuss stabilisability issues for discrete-time linear switched systems, assuming that one can use the switching law as a control function either in feedback form or as an open loop control. Stabilisability properties may be characterised through suitable indices such as the joint spectral subradius and the stabilisability radius. The paper focuses on the numerical computation of these quantities, especially for what concerns lower bounds, and investigates conditions under which they are different. Our analysis shed light on some counterintuitive phenomena related to stabilisation of discrete-time linear switched systems. In particular, one can construct examples of systems which are (exponentially) stabilisable for every initial condition, yet no periodic stabilising switching law exists, except for a set of initial conditions of measure zero.

## I. INTRODUCTION

Joint spectral characteristics are numerical quantities that describe the asymptotic behaviour of matrix semigroups and linear discrete time switched systems. These quantities have attracted a lot of attention due to their numerous applications and in particular, in the control community, due to their links with stability issues.

Given a set of matrices  $\mathcal{M}$ , the first quantity, and perhaps the most well-known, represents the worst case rate of growth of the semigroup generated by  $\mathcal{M}$ :

$$\rho(\mathcal{M}) = \lim_{k \rightarrow \infty} \max_{A \in \mathcal{M}^k} \{|A|^{1/k}\}.$$

It is commonly referred to as the *joint spectral radius* of the set  $\mathcal{M}$ . It has been introduced by Rota and Strang [1] and it is nowadays very popular being strictly related with stability issues in the context of robust control. See [2] for a monograph on the topic. Since then, several other quantities were proposed, in order to describe other possible rates of growth of the system. Let us mention the *p-radius*, [3], [4] with motivations in mathematical analysis; the *Lyapunov exponent* (see [5], [6]) with motivations in randomly switching systems; or the *joint spectral subradius*, which represents the minimal rate of growth for the semigroup generated by  $\mathcal{M}$ .

In this paper, we are concerned with the analysis of two joint spectral characteristics in relation with the stabilisation problem, namely the joint spectral subradius  $\check{\rho}(\mathcal{M})$  mentioned above and the stabilisability radius  $\tilde{\rho}(\mathcal{M})$ .

While the joint spectral subradius may be interpreted as a measure of the stabilisability of a linear switched system by using a time dependent switching law common for all possible initial condition, in the definition of the stabilisability radius one is allowed to choose the switching law

depending on the initial condition. The stabilisability radius is thus smaller than the previously introduced subradius. It has only been introduced formally recently [7], where it was shown by an explicit example that the strict inequality  $\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M})$  may hold true, and further analysed in [8]. However, the reader can find earlier implicit studies of it in [9], [10], [11]. Concerning its computation, some techniques have been proposed in the control literature, which implicitly allow to derive an upper bound on the stabilisability radius, mainly based on semidefinite programming [12], [9], [13]. In [7], and especially in [8], some effort has been devoted in order to provide lower bounds for  $\tilde{\rho}(\mathcal{M})$ , although numerical methods capable of computing tight lower bounds are still missing. Similarly, numerical methods for the computation of tight upper bounds for  $\check{\rho}(\mathcal{M})$  can be easily derived from its definition, while it is not obvious how to devise algorithms providing tight estimates of  $\tilde{\rho}(\mathcal{M})$  from below.

The purpose of this paper is to pursue the analysis of  $\tilde{\rho}(\mathcal{M})$  initiated in [7], [8] and to explore more deeply its relationship with  $\check{\rho}(\mathcal{M})$ . First, we slightly refine the main result in [8] concerning the numerical computation of a lower bound for  $\tilde{\rho}(\mathcal{M})$ . We then provide a characterisation of  $\tilde{\rho}(\mathcal{M})$  in terms of stabilisation by means of periodic switching laws: the condition  $\tilde{\rho}(\mathcal{M}) < 1$  holds true if and only if there exists a periodic switching law globally stabilising the system, while, if  $\tilde{\rho}(\mathcal{M}) \geq 1$ , for almost every initial state  $x_0$  there is no periodic stabilising switching law (depending on  $x_0$ ). Finally, partially motivated by the previous results, we investigate sufficient conditions guaranteeing that  $\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M})$ . We will also quickly discuss the continuous-time case and the condition  $\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M})$  in case of discrete-time approximations of continuous-time linear switched systems.

*Notations:* We denote by  $\mathbb{N}_+$  the set of positive integers. We denote by  $M_n(\mathbb{R})$  the set of  $n \times n$  real matrices and by  $\text{Id}_n$  the identity matrix therein. The singular values of a matrix  $M \in M_n(\mathbb{R})$ , that is the square roots of the eigenvalues of the matrix  $M^T M$ , are denoted by  $s_1(M) \leq \dots \leq s_n(M)$ . The symbol  $e_i$  is used to denote a vector whose  $i$ -th component is equal to one, while all the other components are zero. We use the conventions  $\prod_{i=0}^k A_{\sigma(i)} = A_{\sigma(k)} \dots A_{\sigma(0)}$  and, given a set of matrices  $\mathcal{M}$ ,  $\mathcal{M}^k = \{\prod_{i=0}^{k-1} A_{\sigma(i)} \mid A_{\sigma(i)} \in \mathcal{M}, i = 0, \dots, k-1\}$ .

## II. JOINT SPECTRAL CHARACTERISTICS FOR THE STABILISABILITY PROBLEM

We consider the discrete-time switched system

$$x(k+1) = A_{\sigma(k)}x(k), \quad k \in \mathbb{N}, \quad (1)$$

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where  $x(\cdot)$  takes values in  $\mathbb{R}^n$ , the switching law  $\sigma(\cdot)$  takes values in a finite set of indices  $\{1, \dots, m\}$  and  $\mathcal{M} = \{A_1, \dots, A_m\} \subset M_n(\mathbb{R})$ .

The minimal rate of growth of the matrix products taking values in  $\mathcal{M}$  is described by the joint spectral subradius (see, e.g. [2]) defined as

$$\check{\rho}(\mathcal{M}) = \lim_{k \rightarrow \infty} \inf_{A \in \mathcal{M}^k} \|A\|^{1/k}.$$

Equivalently, we have the following characterisations of  $\check{\rho}(\mathcal{M})$

$$\check{\rho}(\mathcal{M}) = \inf_{A \in \cup_{k \geq 1} \mathcal{M}^k} \|A\|^{1/k} \quad (2)$$

$$= \lim_{k \rightarrow \infty} \inf_{A \in \mathcal{M}^k} \rho(A)^{1/k} \quad (3)$$

$$= \inf_{A \in \cup_{k \geq 1} \mathcal{M}^k} \rho(A)^{1/k}. \quad (4)$$

The joint spectral subradius can be thought of as a measure of the stabilisability of (1) by using an open loop control strategy. Equation (4) defines an algorithm to estimate from above the joint spectral subradius: an upper bound is computed by simply taking the minimum value  $\rho(A)^{1/k}$  among all matrix products  $A \in \mathcal{M}^k$  with  $k \leq \bar{k}$ , with  $\bar{k}$  as large as possible. However, there is no guaranteed rate of convergence to the actual value  $\check{\rho}(\mathcal{M})$  as  $\bar{k}$  goes to infinity. For this reason, algorithms providing tight estimates of  $\check{\rho}(\mathcal{M})$  from below may be very useful, but few results in this direction are known (see e.g. [2, Chapter 7]). A simple algebraic estimate from below is the following.

*Lemma 2.1:* A lower bound for the joint spectral subradius is given by

$$\check{\rho}(\mathcal{M}) \geq \min_{i=1, \dots, m} |\det(A_i)|^{1/n}.$$

*Proof:* For every matrix  $A \in M_n(\mathbb{R})$  one has  $\rho(A)^n \geq |\det(A)|$  and, by (3), for every  $\varepsilon > 0$  there exists  $k > 0$  and  $A \in \mathcal{M}^k$  such that

$$\begin{aligned} \check{\rho}(\mathcal{M}) + \varepsilon &\geq \rho(A)^{\frac{1}{k}} \\ &\geq |\det(A)|^{\frac{1}{nk}} \\ &\geq \min_{i=1, \dots, m} |\det(A_i)|^{\frac{1}{n}}. \end{aligned}$$

The lemma follows by letting  $\varepsilon$  tend to zero.  $\blacksquare$

A further measure of stabilisability for the switched system (1), introduced in [7], is the *stabilisability radius* which is defined as follows

$$\tilde{\rho}(\mathcal{M}) = \sup_{x \in \mathbb{R}^n} \tilde{\rho}_x(\mathcal{M}), \quad \text{where}$$

$$\tilde{\rho}_x(\mathcal{M}) = \inf \left\{ \lambda > 0 \mid \exists C > 0, \exists \sigma(\cdot) \text{ s.t.} \right. \\ \left. \left\| \prod_{i=0}^{k-1} A_{\sigma(i)} x \right\| \leq C \lambda^k \|x\| \right\}.$$

The stabilisability radius is related to the problem of stabilising (1) by means of a time dependent switching law possibly depending on the initial condition; roughly speaking,  $\tilde{\rho}_x(\mathcal{M})$

represents the best achievable exponential rate of stabilisation starting from  $x$ , while  $\tilde{\rho}(\mathcal{M})$  is a uniform estimate of such stabilisation rate. Equivalently, as shown in [7], the stabilisability radius can be used to characterise the switched systems stabilisable by a feedback law  $x \mapsto \sigma(x)$ . Similarly to the joint spectral subradius, algorithms estimating  $\tilde{\rho}(\mathcal{M})$  from above can be easily established from the definition, but there is no guaranteed rate of convergence to the actual value  $\tilde{\rho}(\mathcal{M})$ . Algorithms providing tight estimates of  $\tilde{\rho}(\mathcal{M})$  from below are currently unknown. Nevertheless, lower bounds for the stabilisability radius can be computed by using the following result.

*Theorem 2.2 ([7], [8]):* Consider System (1) and assume that  $\mathcal{M}$  only contains nonsingular matrices. Then, the stabilisability radius satisfies

- $\tilde{\rho}(\mathcal{M}) \geq \tilde{\rho}_1(\mathcal{M}) \triangleq \min_{k=1, \dots, m} s_1(A_k)$ ,
- $\tilde{\rho}(\mathcal{M}) \geq \tilde{\rho}_2(\mathcal{M}) \triangleq \left( \sum_{k=1}^m |\det A_k|^{-1} \right)^{-1/n}$ .

A refinement of Theorem 2.2 is given as follows.

*Theorem 2.3 ([8]):* Consider System (1) and assume that  $\mathcal{M}$  only contains nonsingular matrices. Let us denote, for simplicity,  $\delta_i = s_1(A_i)$  and  $\Delta_i = |\det A_i|$ . Consider the simplex  $\Sigma_m$  defined as

$$\Sigma_m = \{ \nu \in [0, 1]^m \mid \sum_{k=1}^m \nu_k = 1 \},$$

the map

$$\Psi : \Sigma_m \rightarrow \mathbb{R}, \quad \Psi(\nu) = \sum_{k=1}^m \nu_k \log \left( \frac{\nu_k \Delta_k}{\delta_k^n} \right)$$

and the element  $\bar{\nu} \in \Sigma_m$  whose components are defined by  $\bar{\nu}_k = \frac{\Delta_k^{-1}}{\sum_{h=1}^m \Delta_h^{-1}}$ . Then we have the following alternative:

- If  $\Psi(\bar{\nu}) \geq 0$ , then  $\tilde{\rho}(\mathcal{M}) \geq \tilde{\rho}_2(\mathcal{M}) \geq \tilde{\rho}_1(\mathcal{M})$ .
- If  $\Psi(\bar{\nu}) < 0$ , then the set  $\mathcal{Z} = \{ \nu \in \Sigma_m \mid \Psi(\nu) = 0 \}$  is nonempty and, setting

$$\tilde{\rho}_3(\mathcal{M}) \triangleq \min_{\nu \in \mathcal{Z}} \prod_{k=1}^m \delta_k^{\nu_k}, \quad (5)$$

we have  $\tilde{\rho}_3(\mathcal{M}) \geq \tilde{\rho}_1(\mathcal{M})$  and

$$\tilde{\rho}(\mathcal{M}) \geq \tilde{\rho}_3(\mathcal{M}) > \tilde{\rho}_2(\mathcal{M}). \quad (6)$$

### III. ON THE COMPUTATION OF $\tilde{\rho}_3(\mathcal{M})$

The following result concerns the numerical computation of the lower bound  $\tilde{\rho}_3(\mathcal{M})$  of  $\tilde{\rho}(\mathcal{M})$  defined by (5). We follow the notations of Theorem 2.3, and set

$$S \triangleq \{ i \in \{1, \dots, m\} \mid \delta_i = \tilde{\rho}_1(\mathcal{M}) \}.$$

*Theorem 3.1:* Assume that we are in the case (b) of Theorem 2.3. If  $\delta_1 = \dots = \delta_m = \tilde{\rho}_1(\mathcal{M})$  or if  $\sum_{i \in S} \Delta_i^{-1} \geq \tilde{\rho}_1(\mathcal{M})^{-n}$  then  $\tilde{\rho}_3(\mathcal{M}) = \tilde{\rho}_1(\mathcal{M})$ . Otherwise, the argument of the minimum in (5) belongs to the interior of  $\Sigma_m$  and takes the form

$$\hat{\nu}_k(\beta) = \frac{\delta_k^\beta \Delta_k^{-1}}{\sum_{h=1}^m \delta_h^\beta \Delta_h^{-1}} \quad (7)$$

for some real value  $\beta$ . In this case  $\tilde{\rho}_3(\mathcal{M})$  is strictly larger than  $\tilde{\rho}_1(\mathcal{M})$  and it may be calculated numerically by solving the scalar equations  $\Psi(\hat{\nu}_1(\beta), \dots, \hat{\nu}_m(\beta)) = 0$  and substituting the corresponding solution(s) in (5).

*Proof:* The result is obvious in the case  $\delta_1 = \dots = \delta_m = \tilde{\rho}_1(\mathcal{M})$ . Hence we assume in the following that the  $\delta_i$ 's are not all equal. Suppose that the minimum in (5) is attained at the boundary of  $\Sigma_m$ , at  $\hat{\nu}$ . If  $\hat{\nu} = e_i$  for some  $i$  then  $\delta_i = \tilde{\rho}_1(\mathcal{M})$ . Indeed, if  $\Psi(e_i) = 0$ ,  $\delta_i > \tilde{\rho}_1(\mathcal{M})$  and  $\delta_k = \tilde{\rho}_1(\mathcal{M})$  for some  $k$  then it is easy to see that there exists  $\lambda \in [0, 1]$  such that  $\Psi(\lambda e_i + (1 - \lambda)e_k) = 0$ , so that  $\tilde{\rho}_3(\mathcal{M}) \leq \tilde{\rho}_1(\mathcal{M})^{1-\lambda} \delta_i^\lambda < \delta_i$  contradicting the fact that the minimum in (5) is attained at  $\hat{\nu} = e_i$ . Furthermore,  $\Psi(e_i) = 0$  is equivalent to the condition  $\Delta_i = \delta_i^n = \tilde{\rho}_1(\mathcal{M})^n$ , so that the assumption  $\sum_{i \in S} \Delta_i^{-1} \geq \tilde{\rho}_1(\mathcal{M})^{-n}$  is satisfied. The theorem is thus proven in the case  $\hat{\nu} = e_i$ . We assume in the following that  $\hat{\nu} \neq e_i$  for every  $i$ . Note that  $\hat{\nu}$  minimises the linear function

$$\Lambda(\nu) \triangleq \log\left(\prod_{i=1}^m \delta_i^{\nu_i}\right) = \sum_{i=1}^m \nu_i \log \delta_i$$

on the set  $\mathcal{Z} = \{\nu \in \Sigma_m \mid \Psi(\nu) = 0\}$ . We first claim that  $\Lambda(\hat{\nu}) \geq \Lambda(e_i) = \log \delta_i$  for each  $i$  such that  $\hat{\nu}_i \neq 0$ . To prove the claim, we define  $\xi^i(\lambda) = (1 - \lambda)\hat{\nu} + \lambda e_i$ . Since  $\Psi(e_i) \geq 0$ ,  $\Psi(\hat{\nu}) = 0$  and due to the convexity of  $\Psi$  there are two alternatives:

- (i) either  $\Psi(\xi^i(\lambda)) < 0$  for  $\lambda \in (0, \bar{\lambda})$ ,  $\Psi(\xi^i(\bar{\lambda})) = 0$  and  $\Psi(\xi^i(\lambda)) > 0$  for  $\lambda \in (\bar{\lambda}, 1]$ , for some  $\bar{\lambda} \in (0, 1]$ .
- (ii) or  $\Psi \geq 0$  along  $\xi^i$  for all  $\lambda \in [0, 1]$ ,

Consider first the case (i), and therefore suppose  $\Psi(\xi^i(\bar{\lambda})) = 0$ . By assumption,  $\Lambda(\xi^i(\bar{\lambda})) \geq \Lambda(\hat{\nu})$ , which by linearity gives  $\Lambda(e_i) \geq \Lambda(\hat{\nu})$ . Assume now that we are in the second case, and take  $k$  such that  $\hat{\nu}_k = 0$  (the existence of such a  $k$  is guaranteed by the fact that  $\hat{\nu}$  is at the boundary of  $\Sigma_m$ ). The derivative of  $\lambda \mapsto \Psi(\xi^i(\lambda))$  is bounded around  $\lambda = 0$ , while for any  $K > 0$  the directional derivative of  $\Psi$  along the direction  $e_k - \hat{\nu}$  is smaller than  $-K$  for a small enough neighborhood  $U_K$  of  $\hat{\nu}$  in  $\Sigma_m$ . We deduce that for every small  $\lambda > 0$  there exists  $\lambda' = \lambda'(\lambda) \in [0, 1]$  such that

$$\Psi((1 - \lambda')\xi^i(\lambda) + \lambda' e_k) = 0$$

and  $\lambda'/\lambda$  goes to 0 as  $\lambda$  goes to 0. Using the fact that  $\hat{\nu}$  minimises  $\Lambda$  over  $\mathcal{Z}$  we get

$$\begin{aligned} 0 &\leq -\Lambda(\hat{\nu}) + \Lambda((1 - \lambda')\xi^i(\lambda) + \lambda' e_k) \\ &= -\Lambda(\hat{\nu}) + \Lambda(\xi^i(\lambda)) + \lambda'(-\Lambda(\xi^i(\lambda)) + \Lambda(e_k)) \\ &= -\lambda(\Lambda(\hat{\nu}) - \Lambda(e_i)) + \lambda'(-\Lambda(\xi^i(\lambda)) + \Lambda(e_k)). \end{aligned} \quad (8)$$

Dividing by  $\lambda$  and letting  $\lambda$  tend to 0 we get again  $\Lambda(e_i) \geq \Lambda(\hat{\nu})$ . We have proved the claim:  $\Lambda(e_i) \geq \Lambda(\hat{\nu})$  for every  $i$  such that  $\hat{\nu}_i \neq 0$ . But then

$$\Lambda(\hat{\nu}) = \sum_{i=1}^m \hat{\nu}_i \Lambda(e_i) \geq \sum_{i=1}^m \hat{\nu}_i \Lambda(\hat{\nu}) = \Lambda(\hat{\nu})$$

implies  $\Lambda(e_i) = \Lambda(\hat{\nu})$  for every  $i$  such that  $\hat{\nu}_i \neq 0$ .

Now, let us observe that, for any given  $\lambda \in [0, 1]$ , we can write  $\hat{\nu}$  as the convex combination  $\hat{\nu} = \sum_{i=1}^m \hat{\nu}_i \xi^i(\lambda)$ , so that

$$0 = \Psi(\hat{\nu}) \leq \sum_{i=1}^m \hat{\nu}_i \Psi(\xi^i(\lambda)). \quad (9)$$

By taking  $\lambda$  small enough, we have that  $\Psi(\xi^i(\lambda)) < 0$  whenever  $\hat{\nu}_i \neq 0$  and (i) holds true. Hence (9) implies the existence of an index  $\iota$  such that  $\hat{\nu}_\iota \neq 0$  and (ii) holds true. Considering again (8) with  $i = \iota$  and any  $k$  such that  $\hat{\nu}_k = 0$ , using the fact that  $\Lambda(e_\iota) = \Lambda(\hat{\nu})$ , we get  $\Lambda(\hat{\nu}) \leq \Lambda(e_k) = \log \delta_k$ . We deduce that  $\hat{\nu} = \sum_{i \in S} \hat{\nu}_i e_i$  (i.e.,  $\hat{\nu}$  has nonzero component along  $e_i$  only if  $i \in S$ ) and  $\Lambda(\hat{\nu}) = \log(\tilde{\rho}_1(\mathcal{M}))$ . Note that

$$\Psi(\nu) = \sum_{i \in S} \nu_i \log(\nu_i \Delta_i) - n \log(\tilde{\rho}_1(\mathcal{M}))$$

on the simplex  $\Sigma' = \{\nu \in \Sigma_m \mid \nu_i = 0 \text{ if } i \notin S\}$  and the minimum of  $\Psi$  on  $\Sigma'$  may be easily computed by means of Lagrange multipliers: it corresponds to the values  $\nu_i = \frac{\Delta_i^{-1}}{\sum_{k \in S} \Delta_k^{-1}}$  if  $i \in S$  and  $\nu_i = 0$  if  $i \notin S$ , and it is equal to  $-\log(\sum_{i \in S} \Delta_i^{-1}) - n \log(\tilde{\rho}_1(\mathcal{M}))$ . Summing up, if the minimum of  $\prod_{i=1}^m \delta_i^{\nu_i}$  constrained to  $\mathcal{Z} = \{\nu \in \Sigma_m \mid \Psi(\nu) = 0\}$  occurs at a point  $\hat{\nu}$  at the boundary of  $\Sigma_m$  then it is necessarily equal to  $\tilde{\rho}_1(\mathcal{M})$  and  $\hat{\nu} \in \Sigma'$ . Since  $\Psi(\hat{\nu}) = 0$  this implies that  $-\log(\sum_{i \in S} \Delta_i^{-1}) - n \log(\tilde{\rho}_1(\mathcal{M})) \leq 0$ , i.e.  $\sum_{i \in S} \Delta_i^{-1} \geq \tilde{\rho}_1(\mathcal{M})^{-n}$ . Conversely, if such an inequality holds true then  $\Psi$  takes both non-negative and negative values when evaluated on  $\Sigma'$ . As  $\Sigma'$  is connected, it is not contained in the union of the disjoint, nonempty and (relatively) open sets  $\{\nu \in \Sigma_m \mid \Psi(\nu) > 0\}$  and  $\{\nu \in \Sigma_m \mid \Psi(\nu) < 0\}$ , that is, there exists  $\hat{\nu} \in \Sigma'$  such that  $\Psi(\hat{\nu}) = 0$ , and obviously  $\prod_{i=1}^m \delta_i^{\hat{\nu}_i} = \prod_{i \in S} \delta_i^{\hat{\nu}_i} = \tilde{\rho}_1(\mathcal{M})$ .

It remains to consider the case in which the minimum of  $\Lambda$  restricted to  $\mathcal{Z}$  is attained in the interior of  $\Sigma_m$ . Again, this minimum can be computed by using Lagrange multipliers. In particular, using the fact that the values  $\delta_i$  are not all equal, one finds that

$$\nu_k = \alpha \delta_k^\beta \Delta_k^{-1},$$

where  $\alpha, \beta$  depend on the parameters  $\delta_i, \Delta_i, i = 1, \dots, m$ . Since  $\nu \in \Sigma_m$ , we get  $\alpha = \alpha(\beta) = (\sum_k \delta_k^\beta \Delta_k^{-1})^{-1}$ , so that, setting  $\hat{\nu}_k(\beta) = \alpha(\beta) \delta_k^\beta \Delta_k^{-1}$  the value  $\beta$  may be found numerically by solving the equation

$$\Psi(\hat{\nu}(\beta)) = 0.$$

This concludes the proof of the theorem.  $\blacksquare$

#### IV. PERIODIC STABILISATION AND THE GAP BETWEEN $\tilde{\rho}(\mathcal{M})$ AND $\check{\rho}(\mathcal{M})$

The following result describes how the joint spectral subradius relates to the stabilisability of (1) by means of periodic switching laws possibly depending on the initial condition.

*Theorem 4.1:* Let  $\mathcal{M} = \{A_1, \dots, A_m\} \subset M_n(\mathbb{R})$ . The following conditions are equivalent.

- 1) The joint spectral subradius  $\check{\rho}(\mathcal{M})$  satisfies  $\check{\rho}(\mathcal{M}) < 1$ ;

- 2) there exists  $k \in \mathbb{N}$  and a matrix  $A \in \mathcal{M}^k$  such that  $\|A\| < 1$ ;
- 3) there exists  $k \in \mathbb{N}$  and a matrix  $A \in \mathcal{M}^k$  such that  $\rho(A) < 1$ ;
- 4) there exists a periodic switching law  $\sigma : \mathbb{N} \rightarrow \{1, \dots, m\}$  such that  $\lim_{k \rightarrow \infty} \prod_{i=0}^k A_{\sigma(i)} = 0$ .

On the other hand, if  $\check{\rho}(\mathcal{M}) \geq 1$  then, for almost every  $x \in \mathbb{R}^n$ , there exists no periodic switching law  $\sigma_x : \mathbb{N} \rightarrow \{1, \dots, m\}$  such that  $\lim_{k \rightarrow \infty} \prod_{i=0}^k A_{\sigma_x(i)} x = 0$ , i.e. the system is not stabilisable from  $x$  with a periodic switching law.

*Proof:* Equations (2)–(4) imply the equivalence between 1), 2) and 3). Also, it is clear that 4) implies 2).

We are left to show that 2) implies 4). If 2) holds true and  $\|A\| < 1$  with  $A = \prod_{i=0}^{\bar{k}-1} A_{\sigma(i)}$  for some positive integer  $\bar{k}$  and  $\sigma(\cdot)$  then we can extend  $\sigma(\cdot)$  to a periodic switching law by setting  $\sigma(h\bar{k} + j) = \sigma(j)$  for  $h \in \mathbb{N}$  and  $j = 0, \dots, \bar{k} - 1$ . Then

$$\begin{aligned} \left\| \prod_{i=0}^{h\bar{k}+j} A_{\sigma(i)} \right\| &= \left\| \left( \prod_{i=0}^j A_{\sigma(i)} \right) A^h \right\| \\ &\leq \left( \max_{M \in \mathcal{M}} \|M\| \right)^{\bar{k}-1} \|A\|^h, \end{aligned}$$

which implies 4).

Let us prove the second part. Consider all the finite products of matrices, i.e. the countable set  $\mathcal{M}_* = \cup_{k>0} \mathcal{M}^k$ . By assumption and the equivalence between 1) and 3) any matrix  $A \in \mathcal{M}_*$  is such that  $\rho(A) \geq 1$ , that is, it possesses at least one eigenvalue with absolute value larger or equal than one and  $\lim_{h \rightarrow \infty} A^h x = 0$  may only hold on a proper subset  $V_A$  of  $\mathbb{R}^n$ . It is clear that  $V_A$  is a proper (possibly trivial) linear subspace of  $\mathbb{R}^n$  and it coincides with the set of points from which the switching law periodically repeating the sequence of matrices of  $\mathcal{M}$  appearing in  $A$  stabilises the system. We deduce that the set of points from which it is possible to stabilise the system via a periodic switching law coincides with the countable union  $\cup_{A \in \mathcal{M}_*} V_A$ . Since every  $V_A$  has measure zero, we deduce that  $\cup_{A \in \mathcal{M}_*} V_A$  has measure zero as well, concluding the proof. ■

Following [7], [8] we consider now the switched system corresponding to the set of matrices  $\mathcal{M} = \{A_1, A_2\}$  with

$$A_1 = \begin{pmatrix} \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ -\sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

By Lemma 2.1 and the fact that  $\check{\rho}(\mathcal{M}) \leq \rho(A_1) = 1$  it follows that  $\check{\rho}(\mathcal{M}) = 1$ . Furthermore, in [7] (see also [11]) it has been shown that  $\check{\rho}(\mathcal{M}) < 0.975$ . Hence the system is stabilisable by using a switching law  $k \mapsto \sigma_{x_0}(k)$  depending on the initial condition  $x_0$ , however, as a consequence of Theorem 4.1, for a generic initial condition  $x_0$  no such stabilising switching law exists with the additional property of being periodic. Note that this is not a “pathological phenomenon” since the same property holds true if we replace  $\mathcal{M}$  with  $\{A_1/\alpha, A_2/\beta\}$  for every  $\alpha, \beta \in [0.975, 1]$ . More precisely, this phenomenon happens whenever  $\check{\rho}(\mathcal{M}) < 1 \leq \check{\rho}(\mathcal{M})$ .

Related to the phenomenon described above, but also interesting on its own, is the following question: under which conditions the strict inequality  $\check{\rho}(\mathcal{M}) < \bar{\rho}(\mathcal{M})$  holds true? Since there is no closed form for  $\check{\rho}(\mathcal{M})$  and  $\bar{\rho}(\mathcal{M})$  and both quantities are in general hard to compute, a global answer to this question does not seem to be obvious. The intuition is that, whenever the system is “controllable enough”, a switching law depending on the initial condition may be chosen so that the state remains as close as possible to the most contractive directions and the norm may be reduced as much as possible in most of the iterations of the dynamics. If this is the case, then it is natural to expect that  $\check{\rho}(\mathcal{M}) < \bar{\rho}(\mathcal{M})$ . Unfortunately, formalising this idea seems to be a nontrivial problem in general. In the particular case of two-dimensional systems, some simple sufficient conditions guaranteeing that  $\check{\rho}(\mathcal{M}) < \bar{\rho}(\mathcal{M})$  are given as follows.

*Theorem 4.2:* Assume that  $n = 2$  and that there exists a coordinate transformation  $T$  such that

$$\mathcal{M}' = T\mathcal{M}T^{-1} = \{TMT^{-1} \mid M \in \mathcal{M}\}$$

satisfies

- (a) there exists a matrix  $A_1 \in \mathcal{M}'$  with  $A_1 = \alpha R_\varphi$ , where  $\alpha \neq 0$  and  $R_\varphi$  is a rotation of an angle  $\varphi$  irrational with  $\pi$  (in other words, there exists a matrix in  $\mathcal{M}$  with a complex eigenvalue whose argument is irrational with  $\pi$ );
- (b)  $|\det(A_1)| = \alpha^2 = \min_{A \in \mathcal{M}'} |\det(A)|$ ;
- (c) there exists a matrix  $A_2 \in \mathcal{M}'$  with real eigenvalues such that  $s_1(A_2) < s_1(A_1) = |\alpha|$ .

Then  $\check{\rho}(\mathcal{M}) < \bar{\rho}(\mathcal{M}) = |\alpha|$ .

*Proof:* Note that  $\rho(A_1) = |\alpha|$ . Then, from (a)–(b) and by applying (4) and Lemma 2.1, it follows that  $\check{\rho}(\mathcal{M}) = |\alpha|$ . We will next show that  $\bar{\rho}(\mathcal{M}) < |\alpha|$ . Since  $s_1(A_2) < s_1(A_1)$ , for every  $\lambda \in (s_1(A_2), s_1(A_1))$  there exists a nonempty open set  $\mathcal{C}$  of the circle  $S^1$  such that  $\|A_2 x\| < \lambda \|x\|$  for every  $x \in \mathcal{C}$ . In particular, identifying a point  $(\cos \theta, \sin \theta)$  of  $S^1$  with the corresponding angle  $\theta$  (modulo  $2\pi$ ) there exists an angle  $\bar{\theta}$  and  $\delta > 0$  such that  $[\bar{\theta}, \bar{\theta} + \delta] \subset \mathcal{C}$ . We claim that there exists a positive integer  $N_{\varphi, \delta}$  such that, for every  $\varphi_0 \in S^1$  there exists  $k \in \{0, \dots, N_{\varphi, \delta}\}$  such that  $\varphi_0 + k\varphi \in [\bar{\theta}, \bar{\theta} + \delta]$ . Indeed, since  $\varphi$  is irrational with  $\pi$ , the sequence  $\{\varphi_k\}_{k \in \mathbb{N}}$  where  $\varphi_k = \varphi_0 + k\varphi$  constitutes a dense set in  $S^1$ , so that, for a large enough  $N_{\varphi, \delta}$ , the union  $\cup_{k=0}^{N_{\varphi, \delta}} (\varphi_k - \frac{\delta}{2}, \varphi_k + \frac{\delta}{2})$  covers  $S^1$ . (Note that the value  $N_{\varphi, \delta}$  defined in this way is independent of the “shift”  $\varphi_0$ .) Hence  $\bar{\theta} + \frac{\delta}{2} \in (\varphi_k - \frac{\delta}{2}, \varphi_k + \frac{\delta}{2})$  for some  $k \leq N_{\varphi, \delta}$ , that is  $\varphi_k \in (\bar{\theta}, \bar{\theta} + \delta) \subset \mathcal{C}$ . The claim is proved and, for every  $x \neq 0$ , there exists  $k \leq N_{\varphi, \delta}$  such that

$$\|A_2 A_1^k x\| \leq \lambda |\alpha|^k \|x\| \leq \left( \frac{\lambda}{|\alpha|} \right)^{k+1} |\alpha|^{k+1} \|x\|$$

which easily implies  $\bar{\rho}(\mathcal{M}) \leq \left( \frac{\lambda}{|\alpha|} \right)^{\frac{1}{N_{\varphi, \delta} + 1}} |\alpha| < |\alpha| = \check{\rho}(\mathcal{M})$ . ■

For continuous-time switched systems of the form

$$\dot{x}(t) = B_{\sigma(t)} x(t), \quad B_\sigma \in \{B_1, \dots, B_m\} \subset M_n(\mathbb{R}) \quad (10)$$

the fact that the dependence of the switching law on the initial condition  $x_0$  may enhance the contraction rate of the system is easier to assess. For instance, under some controllability conditions of the system, it is possible to reach in finite time, starting from any initial condition, the subspace corresponding to the eigenvalues with the lowest real part among all the matrices  $A_i$ . This provides a contraction rate independent of  $x_0$  which, in some cases, may not be achievable by a switching law independent of  $x_0$ . Furthermore, one can show that this property persists after a time discretisation of the continuous-time system. In particular, we have the following result.

*Theorem 4.3:* Assume that the continuous-time switched system (10) is controllable in the projective space in finite time, in the sense that there exists  $T_0 > 0$  such that for every initial condition  $x_0 \in \mathbb{R}^n \setminus \{0\}$  and target state  $x_1 \in \mathbb{R}^n \setminus \{0\}$  there exists a trajectory  $x(\cdot)$  of (10) satisfying  $x(0) = x_0$  and  $x(T) = \alpha x_1$  for some  $T \in [0, T_0]$  and  $\alpha \in \mathbb{R}$ . Assume moreover that one of the matrices  $B_1, \dots, B_m$  possesses an eigenvalue of real part  $r$  satisfying

$$r < \frac{1}{n} \min_{i=1, \dots, m} \text{trace}(B_i).$$

Then for any  $\delta > 0$  small enough, setting  $\mathcal{M} = \{e^{\delta B_1}, \dots, e^{\delta B_m}\}$  we have  $\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M})$ .

*Proof:* In the following we present a sketch of the proof of the theorem. By Lemma 2.1 we have that

$$\check{\rho}(\mathcal{M}) \geq \min_{i=1, \dots, m} e^{\frac{\delta}{n} \text{trace}(B_i)}.$$

On the other hand it is easy to see from the assumptions of the theorem that, given  $\lambda \in (r, \min_{i=1, \dots, m} \frac{1}{n} \text{trace}(B_i))$ , for every  $x_0 \in \mathbb{R}^n$  there exists a trajectory  $x(\cdot)$  of (10) satisfying

$$\begin{aligned} \|x(T_1)\| &< e^{\lambda T_1} \|x_0\|, \\ \|x(t)\| &< C e^{\lambda t} \|x_0\|, \quad \forall t \in [0, T_1] \end{aligned} \quad (11)$$

for some  $C \geq 1$  and  $T_1 > 0$  independent of  $x_0$ . Furthermore, by a standard compactness argument, we can select a finite number of switching laws  $\sigma_1(\cdot), \dots, \sigma_N(\cdot)$  such that (11) is satisfied for a certain  $\sigma_i(\cdot)$ . By classical approximation results, the fundamental matrix at time  $t \leq T_1$  for (10) associated with  $\sigma_i(\cdot)$  can be approximated with arbitrary precision by the fundamental matrix at time  $[\frac{t}{\delta}]\delta$  associated with a switching law whose switching times are multiple of  $\delta$ , for  $\delta > 0$  small enough. Hence, from (11) and the definitions of  $\mathcal{M}$  and of  $\tilde{\rho}(\mathcal{M})$ , it is easy to see that

$$\tilde{\rho}(\mathcal{M}) \leq e^{\lambda \delta} < \min_{i=1, \dots, m} e^{\frac{\delta}{n} \text{trace}(B_i)}.$$

This concludes the proof of the theorem.  $\blacksquare$

## V. NUMERICAL EXAMPLE

In this section we show an example of application for Theorem 4.2. Let  $\mathcal{M} = \{M_1, M_2, M_3\}$ , where

$$M_1 = \begin{pmatrix} -2 & 3 \\ -6 & 4 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 4 & -1 \\ -4 & -3 \end{pmatrix}.$$

The matrix  $M_1$  has complex nonreal eigenvalues while  $M_2, M_3$  have real eigenvalues. We apply the transformation  $T = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$  and set  $A_i = T^{-1}M_iT$  for  $i = 1, 2, 3$  and

$$\mathcal{M}' = \{A_1, A_2, A_3\} = \left\{ \begin{pmatrix} -1 & 3 \\ -3 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 8 & -4 \end{pmatrix}, \begin{pmatrix} -5 & 2 \\ -7 & 6 \end{pmatrix} \right\}.$$

The matrix  $A_1$  satisfies Condition (a) of Theorem 4.2 with  $\alpha = \sqrt{10}$ . Indeed, if this condition is not verified then there exists a positive integer  $k$  such that  $A_1^k = \alpha^k \text{Id}_2$ . In this case all the eigenvalues of  $A_1$  are zeroes of the polynomial  $p_k(x) = x^k - \alpha^k$  and, as a consequence, of the polynomial  $q_k(x) = x^{2k} - 10^k$ . But it is easy to check that there exists no positive integer  $k$  such that the characteristic polynomial  $p(x) = x^2 - 2x + 10$  of  $A_1$  divides  $q_k(x)$ , leading to a contradiction. Also, Conditions (b)-(c) of Theorem 4.2 can be readily checked. We thus get  $\tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M}) = \sqrt{10} \approx 3.1623$ . Furthermore, one easily checks that, in the notations of Theorem 2.3,  $\delta_1 = \alpha = \sqrt{10}$ ,  $\delta_2 \approx 1.657$ ,  $\delta_3 \approx 1.514$ ,  $\Delta_1 = 10$ ,  $\Delta_2 = 16$ ,  $\Delta_3 = 16$  and  $\bar{\nu} = (\frac{4}{9}, \frac{5}{18}, \frac{5}{18})$ . Hence, Condition (b) of Theorem 2.3 holds true, i.e.  $\Psi(\bar{\nu}) < 0$ , and by using Theorem 3.1 one obtains  $\tilde{\rho}_3(\mathcal{M}) \approx 2.11$ . Summing up, we get

$$2.11 \approx \tilde{\rho}_3(\mathcal{M}) \leq \tilde{\rho}(\mathcal{M}) < \check{\rho}(\mathcal{M}) = \sqrt{10} \approx 3.1623.$$

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