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Correlation functions for open XXX spin 1/2 quantum chains with unparallel boundary magnetic fields

G. Niccoli¹

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Abstract

In this first paper, we start the analysis of correlation functions of quantum spin chains with general integrable boundary conditions. We initiate these computations for the open XXX spin 1/2 quantum chains with some unparallel magnetic fields allowing for a spectrum characterization in terms of homogeneous Baxter like TQ-equations, in the framework of the quantum separation of variables (SoV). Previous SoV analysis leads to the formula for the scalar products of the so-called separate states. Here, we solve the remaining fundamental steps allowing for the computation of correlation functions. In particular, we rederive the ground state density in the thermodynamic limit thanks to SoV approach, we compute the so-called boundary-bulk decomposition of boundary separate states and the action of local operators on these separate states in the case of unparallel boundary magnetic fields. These findings allow us to derive multiple integral formulae for these correlation functions similar to those previously known for the open XXX quantum spin chain with parallel magnetic fields.

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1 Introduction

The open integrable quantum spin chains with magnetic fields located at the boundaries [1] have attracted large scientific attention [1–29]. They have been used in connection to the studies of classical stochastic models, as asymmetric simple exclusion models [30], but also to modelling numerous applications in quantum condensed matter physics, as out-of-equilibrium and transport properties in the spin chains [31]. Sklyanin [2] has extended to them the quantum inverse scattering method (QISM) [32–41], by using the reflection equation introduced by Cherednik [42], in this way introducing the natural algebraic framework to handle these open spin chains. In this framework, the so-called boundary matrices, i.e. the scalar solutions of the reflection equation [3, 42, 43], allow to parametrize the magnetic fields at the boundaries of the quantum spin chain. In particular, parallel boundary magnetic fields along the z -direction correspond to diagonal boundary matrices while the unparallel cases correspond to non-simultaneously diagonalizable boundary matrices. The open spin chains associated to both diagonal boundary matrices have been first analyzed by means of coordinate Bethe ansatz [1] while Sklyanin has generalized the algebraic Bethe ansatz (ABA) approach to these boundary cases in its fundamental work [2].

The cases with non-diagonal boundary matrices have proven themselves to be more involved to analyze and, until recently, their spectrum has long remained a very challenging problem in quantum integrability. Let us recall that in [5] a first description of the spectrum of these open XXZ spin $1/2$ chains with non-diagonal boundary matrices has been obtained using the fusion procedure [44], under a special constrain relating the parameters of the two boundary matrices. There, the transfer matrix spectrum has been described in terms of polynomial solutions of ordinary TQ -equation of Baxter’s type [45], for the roots of unity points and later in [6] for general value of the inhomogeneity parameter. This constraint² is required also for generalized ABA-like constructions of eigenstates [4, 7, 8], which use Baxter’s gauge transformations [45, 46] to simplify the form of the boundary matrices.

Only more recently in [26], on the basis of analytic properties and functional relations satisfied by the transfer matrices a description of the spectrum of open chains for the unconstrained cases has been proposed in terms of polynomial solutions of inhomogeneous TQ -equations, i.e. admitting some extra term. These inhomogeneous TQ -equations have also emerged in the framework of the so-called modified algebraic Bethe ansatz [15–17, 19] to deal with unconstrained boundary conditions.

The quantum Separation of Variables (SoV), pioneered by Sklyanin [47–52] in the QISM framework, was introduced as an alternative to ABA approach for solving models in which a reference state cannot be identified, as the Toda model. Then, it has been shown to be applicable to a large class of models [23–25, 27–29, 53–78] and more recently reintroduced in [79] on the pure basis of the integrable structure of the models and widely extended even to higher rank cases in [79–86], see also [52, 55, 87, 88] for previous developments. The SoV approaches in their different presentations have the built-in advantage of the completeness of the characterization of eigenvalues and eigenstates of the models. This method has been used, in particular, for open spin chains with the most general unconstrained non-diagonal boundary matrices [23–25, 27, 28]. In [29], it has been proven that the complete SoV characterization of the spectrum of these open chains can be reformulated in terms of polynomial solutions to functional TQ -equations of Baxter type that have the aforementioned inhomogeneous extra term, for the most general unconstrained boundary matrices. Another advantage of the SoV approaches, of particular relevance for the correlation functions analysis, is the natural and universal emergence of determinant formulae for the scalar products of *separate*

²It has emerged also in several other approaches leading to description of the spectrum in terms of polynomial solutions of ordinary TQ equations, as by coordinate Bethe ansatz with elements of matrix product ansatz [11, 12], q -Onsager algebra [9, 10] etc.

states [25, 27, 28, 65, 67–70, 72, 74, 76–78]³. This is a class of states with factorized wave-functions in the SoV basis which includes the eigenstates of the transfer matrix.

The short recall here presented clarify the progresses achieved about the knowledge of the spectrum of these open spin 1/2 quantum chains for general integrable boundary conditions. One should, however, remark that the study of the ground state in the thermodynamic limit of these models presents still some open problems for non-diagonal boundary matrices and, in particular, in the unconstrained cases⁴.

Let us now focus our attention on the study of correlation functions which is the main subject of the current paper. In this area of research, one should stress that main results for interacting models are available for the XXX/XXZ spin 1/2 quantum chains or the quantum non-linear Schrödinger model with periodic boundary conditions. The current state of the art is quite unsatisfactory in connection to the integrable boundary conditions for which results are so far available.

Indeed, these correlation functions have been computed first for the case of closed chains with periodic boundary conditions⁵, for the zero-temperature cases [104–106] and non-zero global magnetic fields [107–113], and subsequently for the temperature cases [114–119]. Always in this periodic setting, further developments have led to the analytical study in the thermodynamic limit of long distances two-point and multi-point correlation functions [120–130] and the numerical study of the dynamical structure factors [131–133], accessible experimentally through neutron scattering [134].

So far, the main exception to periodic boundary conditions for a closed chain is our computation of correlation functions in the case of antiperiodic boundary conditions in the SoV framework [135]. While for XXX and XXZ quantum spin 1/2 chains with open boundaries the only available results on correlation functions are those derived for zero-temperature in the case of parallel magnetic fields on the boundary for the XXX chains and of parallel magnetic fields along the z-direction for the XXZ chains [13, 14, 21, 22].

Here, we start the analysis of correlation functions of quantum spin chains with general integrable boundary conditions beyond those so far analyzed. The aim of this paper is to derive correlation functions for the open XXX spin 1/2 quantum chains under unparallel boundary magnetic fields. This achievement represents on one side a first access to correlation functions for these more general boundary conditions and on the other hand is instrumental to introduce some technical ingenuity which will be then used also in more involved models like open XXZ/XYZ quantum spin 1/2 chains whose correlation functions will be derived in our forthcoming papers. We develop our analysis in the framework of the quantum separation of variables (SoV) considering a generic magnetic field in the first site 1 of the XXX spin 1/2 quantum chain. While, we adjust the magnetic field in the last site N of the chain such that it isn't parallel to the one in site 1 but it allows for an SoV complete description of the transfer matrix spectrum in terms of homogeneous Baxter like TQ -equations.

Once the transfer matrix spectral problem is characterized by SoV approach, we derive the following four main steps to compute correlation functions in the SoV framework: i. A decomposition formula, *boundary-bulk decomposition*, expressing the so-called boundary separate states (a class of states containing the transfer matrix eigenstates of the open chain) in term of analogous states generated by bulk operators, associated to the closed chain. ii. A decomposition formula over the boundary separate states for the action of local operators on a generic boundary separate state. iii. Simple determinant formulae defining the scalar products between left and right boundary separate states and their particularization when one of the states is a transfer matrix eigenstate [76]. iv. The density

³This is surely the case for the rank one models and in [89] we have proven it for the higher rank $gl(3)$ case under a special choice of the conserved charges generating the SoV bases. See also the interesting and recent papers [90,91] dealing with the computations of higher rank scalar products in a related SoV framework.

⁴See, however, [92] for a numerical analysis of the Bethe roots of these inhomogeneous equations.

⁵Interesting results on correlation functions related to the hidden Grassmann structure have been derived in [93–103].

distribution of the ground state Bethe's roots in the thermodynamic limit.

In our current analysis the built-in feature of completeness of the SoV method, both in its original Sklyanin's like generalization to open chain [25, 27, 28] as well as in our new SoV approach [79, 81], plays a fundamental role in the characterization of the ground state in the thermodynamic limit. Indeed, by using the complete SoV characterization of the transfer matrix spectrum, we can prove the isospectrality of the transfer matrices under consideration with specific ones with parallel boundary magnetic fields, in this way recovering the traditional thermodynamic results for the ground state. On the other hand, in the SoV framework, we have at our disposal the scalar products of the boundary separate states [74, 76, 78].

It is however worth mentioning that, the recent and interesting results on scalar products for open chains with general boundary conditions⁶ [136] put a basis for the computation of correlation functions in a generalized/modified Bethe Ansatz framework⁷.

The paper is arranged in the following sections. In Section 2, we make a brief introduction to the XXX quantum spin 1/2 chain and to the reflection algebra. In Section 3, we first recall and rework in the new SoV framework [79, 81] the characterization of the transfer matrix spectrum for general boundary conditions. Then, we setup the boundary conditions for the XXX model that we will use to determine correlation functions. More precisely, by the SoV approach, we show that we can leave the magnetic field on site 1 arbitrary while adjusting the one in site N such that they are kept unparallel, the transfer matrix is proven to be isospectral to one associated to parallel boundary magnetic field and their spectrum is completely characterized by a homogeneous Baxter's like TQ -equation. In Section 4, we derive the boundary-bulk decomposition of the boundary separate states and the action on them of local operators. In Section 5, we recall and rearrange the known results on the scalar products of separate states [76]. In Section 6, we compute the correlation functions in terms of multiple integral representations. The final section contain some conclusion and outlooks.

2 The open XXX quantum spin 1/2 chain

The Hamiltonian of the open spin-1/2 XXX quantum spin chain with the most general boundary magnetic fields reads:

$$H = \sum_{i=1}^{N-1} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right] + \frac{\eta}{\zeta_-} \left[\sigma_1^z + 2\kappa_- \left(e^{\tau_-} \sigma_1^+ + e^{-\tau_-} \sigma_1^- \right) \right] + \frac{\eta}{\zeta_+} \left[\sigma_N^z + 2\kappa_+ \left(e^{\tau_+} \sigma_N^+ + e^{-\tau_+} \sigma_N^- \right) \right]. \quad (2.1)$$

The local spin-1/2 operators (Pauli matrices) σ_i^α , for $\alpha = x, y, z$, acts on the local quantum space $\mathcal{H}_i \simeq \mathbb{C}^2$ at site i , η is a fixed arbitrary parameter, and the six complex boundary parameters ζ_\pm , κ_\pm and τ_\pm parametrize the coupling of the spin operators at site 1 and N with two arbitrary boundary magnetic fields.

Following the seminal Sklyanin's paper [2], this Hamiltonian can be obtained as the following

⁶See [137] and also [138] for the original idea developed first for periodic chains and see also [20] for a first conjecture on these determinant formulae and [139–141] for previous determinant representations under special boundary constraints.

⁷One should notice that a priori the scalar products analyzed in [136] are between left C -gauged Bethe like states and right B -gauged Bethe like states so a priori different w.r.t. to those of our papers [74, 76, 78], which are computed between left and right separate states. However, the completeness of the spectrum description by SoV approach [81] and the therein proven simplicity of the transfer matrix spectrum can be used to relate these scalar products once the left C -gauged Bethe like state is a transfer matrix eigenstate.

derivative:

$$H = \frac{2\eta^{1-2N}}{\text{tr}\{K_+(\eta/2)\} \text{tr}\{K_-(\eta/2)\}} \frac{d}{d\lambda} \mathcal{T}(\lambda) \Big|_{\lambda=\eta/2} + \text{constant}, \quad (2.2)$$

of the one-parameter family of commuting boundary transfer matrices

$$\begin{aligned} \mathcal{T}(\lambda) &= \text{tr}_0 \{ K_{0,+}(\lambda) M_0(\lambda) K_{0,-}(\lambda) \hat{M}_0(\lambda) \} \\ &= \text{tr}_0 \{ K_+(\lambda) \mathcal{U}_-(\lambda) \} = \text{tr}_0 \{ K_-(\lambda) \mathcal{U}_+(\lambda) \} \in \text{End}(\mathcal{H}), \end{aligned} \quad (2.3)$$

on the 2^N -dimensional linear space $\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n$, the physical space of states of the Hamiltonian (2.1), where we have defined the boundary monodromy matrices

$$\mathcal{U}_-(\lambda) = M_0(\lambda) K_-(\lambda) \hat{M}_0(\lambda) = \begin{pmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{pmatrix} \in \text{End}(\mathcal{H}_0 \otimes \mathcal{H}), \quad (2.4)$$

$$\mathcal{U}_+^{t_0}(\lambda) = M_0^{t_0}(\lambda) K_+^{t_0}(\lambda) \hat{M}_0^{t_0}(\lambda) = \begin{pmatrix} \mathcal{A}_+(\lambda) & \mathcal{C}_+(\lambda) \\ \mathcal{B}_+(\lambda) & \mathcal{D}_+(\lambda) \end{pmatrix} \in \text{End}(\mathcal{H}_0 \otimes \mathcal{H}). \quad (2.5)$$

and the bulk monodromy matrices $M_0(\lambda) \in \text{End}(\mathcal{H}_0 \otimes \mathcal{H})$ by

$$M_0(\lambda) = R_{0N}(\lambda - \xi_N - \eta/2) \dots R_{01}(\lambda - \xi_1 - \eta/2) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \quad (2.6)$$

$$\hat{M}_0(\lambda) = (-1)^N \sigma_0^y M_0^{t_0}(-\lambda) \sigma_0^y = (-1)^N \begin{pmatrix} D(-\lambda) & B(-\lambda) \\ C(-\lambda) & A(-\lambda) \end{pmatrix} \in \text{End}(\mathcal{H}_0 \otimes \mathcal{H}), \quad (2.7)$$

for arbitrary complex parameters ξ_n , $1 \leq n \leq N$, the so-called inhomogeneities. Moreover, the R -matrix of the model,

$$R(\lambda) = \begin{pmatrix} \lambda + \eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda + \eta \end{pmatrix} \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2), \quad (2.8)$$

is the 6-vertex polynomial solution of the Yang-Baxter equation and the two boundary K -matrices

$$K_-(\lambda) = K(\lambda - \eta/2; \zeta_-, \kappa_-, \tau_-), \quad K_+(\lambda) = K(\lambda + \eta/2; \zeta_+, \kappa_+, \tau_+), \quad (2.9)$$

with the boundary parameters appearing $\zeta_{\pm}, \kappa_{\pm}, \tau_{\pm}$, coinciding with those of (2.1), are defined in terms of [3, 43]

$$K(\lambda; \zeta, \kappa, \tau) = \frac{1}{\zeta} \begin{pmatrix} \zeta + \lambda & 2\kappa e^{\tau} \lambda \\ 2\kappa e^{-\tau} \lambda & \zeta - \lambda \end{pmatrix}, \quad (2.10)$$

the most general non-diagonal scalar solution $K(\lambda) \in \text{End}(\mathbb{C}^2)$ of the reflection equation [42]:

$$R_{ab}(\lambda - \mu) K_a(\lambda) R_{ab}(\lambda + \mu) K_b(\mu) = K_b(\mu) R_{ab}(\lambda + \mu) K_a(\lambda) R_{ab}(\lambda - \mu) \in \text{End}(\mathcal{H}_a \otimes \mathcal{H}_b). \quad (2.11)$$

Then, the bulk monodromy matrix is solution of the Yang-Baxter equation:

$$R_{ab}(\lambda - \mu) M_a(\lambda) M_b(\mu) = M_b(\mu) M_a(\lambda) R_{ab}(\lambda - \mu) \in \text{End}(\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}), \quad (2.12)$$

while the two boundary monodromy matrices $\mathcal{V}_-(\lambda) = \mathcal{U}_-(\lambda + \eta/2)$ and $\mathcal{V}_+(\lambda) = \mathcal{U}_+^{t_0}(-\lambda - \eta/2)$ are solutions of the reflection equation:

$$R_{ab}(\lambda - \mu) \mathcal{U}_a(\lambda) R_{ab}(\lambda + \mu) \mathcal{U}_b(\mu) = \mathcal{U}_b(\mu) R_{ab}(\lambda + \mu) \mathcal{U}_a(\lambda) R_{ab}(\lambda - \mu) \in \text{End}(\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}). \quad (2.13)$$

As shown in [2], the commutativity of the boundary transfer matrices is implied by these reflection equations as well as the following inversion relation for the boundary monodromy matrix $\mathcal{U}_\pm(\lambda)$,

$$\mathcal{U}_\pm(\lambda + \eta/2) \mathcal{U}_\pm(-\lambda + \eta/2) = \frac{\det_q \mathcal{U}_\pm(\lambda)}{2(\lambda \pm \eta)}, \quad (2.14)$$

where $\det_q \mathcal{U}_\pm(\lambda)$ are the quantum determinants, which are central elements of the corresponding boundary algebra:

$$[\det_q \mathcal{U}_\pm(\lambda), \mathcal{U}_\pm(\mu)] = 0. \quad (2.15)$$

They can be expressed as

$$\det_q \mathcal{U}_\pm(\lambda) = \det_q M(\lambda) \det_q M(-\lambda) \det_q K_\pm(\lambda), \quad (2.16)$$

where

$$\det_q M(\lambda) = a(\lambda + \eta/2) d(\lambda - \eta/2), \quad (2.17)$$

is the bulk quantum determinant, as well a central element, with

$$a(\lambda) \equiv \prod_{n=1}^N (\lambda - \xi_n + \eta/2), \quad d(\lambda) \equiv \prod_{n=1}^N (\lambda - \xi_n - \eta/2), \quad (2.18)$$

and

$$\det_q K_\pm(\lambda) = \pm 2(\lambda \pm \eta) \left(\frac{\lambda^2}{\bar{\zeta}_\pm^2} - 1 \right), \quad (2.19)$$

is the quantum determinant of the scalar boundary matrix $K_\pm(\lambda)$, where we have defined:

$$\bar{\zeta}_\pm = \epsilon_\pm \frac{\zeta_\pm}{\sqrt{1 + 4\kappa_\pm^2}}, \quad (2.20)$$

with ϵ_\pm plus or minus one at will. The following fundamental identities relate the transfer matrix at special values to the quantum determinant:

$$\mathcal{T}(\xi_a - \eta/2) \mathcal{T}(\xi_a + \eta/2) = \frac{\det_q K_\pm(\xi_a) \det_q \mathcal{U}_\mp(\xi_a)}{\eta^2 - 4\xi_a^2}, \quad (2.21)$$

here we also introduce the notation

$$k_n = (\xi_n + \eta)/(\xi_n - \eta) \quad (2.22)$$

and the function

$$\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\lambda) \equiv (-1)^N \frac{2\lambda + \eta}{2\lambda} \frac{(\lambda - \frac{\eta}{2} + \bar{\zeta}_+)(\lambda - \frac{\eta}{2} + \bar{\zeta}_-)}{\bar{\zeta}_+ \bar{\zeta}_-} a(\lambda) d(-\lambda), \quad (2.23)$$

which allows for the following explicit writing of the quantum determinant

$$\frac{\det_q K_{\pm}(\lambda) \det_q \mathcal{U}_{\mp}(\lambda)}{\eta^2 - 4\lambda^2} = A_{\bar{\zeta}_+, \bar{\zeta}_-}(\lambda + \eta/2) A_{\bar{\zeta}_+, \bar{\zeta}_-}(-\lambda + \eta/2), \quad (2.24)$$

that will be used in the following.

It is import to remark that the transfer matrix $\mathcal{T}(\lambda)$ is a polynomial function of degree $N + 1$ in the variable λ^2 and its leading coefficient is given by,

$$t_{N+1} \lambda^{2(N+1)} \text{Id}, \quad \text{with} \quad t_{N+1} = \frac{2}{\zeta_+ \zeta_-} [1 + 4\kappa_+ \kappa_- \cosh(\tau_+ - \tau_-)], \quad (2.25)$$

and that its value in $\lambda = \pm\eta/2$ is central:

$$\mathcal{T}(\pm\eta/2) = 2(-1)^N \det_q M(0). \quad (2.26)$$

Then, we have the following interpolation formula

$$\mathcal{T}(\lambda) = t_{N+1} u_{\mathbf{h}}(\lambda) + \mathcal{T}(\eta/2) s_{\mathbf{h}}(\lambda) + \sum_{a=1}^N r_{a, \mathbf{h}}(\lambda) \mathcal{T}(\xi_a^{(h_a)}), \quad (2.27)$$

where

$$t_{N+1} = \frac{2}{\zeta_+ \zeta_-} [1 + 4\kappa_+ \kappa_- \cosh(\tau_+ - \tau_-)], \quad (2.28)$$

$$r_{a, \mathbf{h}}(\lambda) = \frac{\lambda^2 - (\eta/2)^2}{(\xi_a^{(h_a)})^2 - (\eta/2)^2} \prod_{b \neq a, b=1}^N \frac{\lambda^2 - (\xi_b^{(h_b)})^2}{(\xi_a^{(h_a)})^2 - (\xi_b^{(h_b)})^2}, \quad (2.29)$$

$$s_{\mathbf{h}}(\lambda) = \prod_{b=1}^N \frac{\lambda^2 - (\xi_b^{(h_b)})^2}{(\eta/2)^2 - (\xi_b^{(h_b)})^2}, \quad (2.30)$$

$$u_{\mathbf{h}}(\lambda) = (\lambda^2 - (\eta/2)^2) \prod_{b=1}^N (\lambda^2 - (\xi_b^{(h_b)})^2), \quad (2.31)$$

and

$$\xi_b^{(h_b)} = \xi_b^{(h_b)} + (1 - 2h_b)\eta/2. \quad (2.32)$$

3 Separation of variable spectrum characterization

In this section, we further develops known results about the transfer matrix spectrum characterization of the open XXX spin 1/2 quantum chains, with generic integrable boundaries, in the framework of the recent formulation of the quantum separation of variables (SoV) [79, 81]. This analysis allows us to give a uniform description of the transfer matrix spectrum independently from the boundary conditions. More in detail, we write explicitly the left and right SoV basis and the left and right transfer matrix eigenstates without the need to distinguish between the cases of parallel or unparallel magnetic fields. Distinction which is instead essential in Sklyanin's like SoV framework [47, 49, 50], holding only in the case of unparallel fields⁸. These SoV results allow us to prove that the transfer matrices associated to unparallel boundary magnetic fields satisfying one specific boundary condition are isospectral to those associated to the parallel case. This isospectrality holds up to the relative Hamiltonians and it is an important fact for the computation of the thermodynamic limit of the ground state.

⁸Observation which may lead to the false perception of a dichotomy unparallel case solvable by SoV method and parallel case by ordinary Algebraic Bethe Ansatz.

3.1 Discrete SoV spectrum characterization

3.1.1 Covector and vector SoV bases

Here, we complete the construction of the left and right SoV basis in our new SoV approach further deriving their left/right couplings, i.e. the SoV measure. As a corollary of the Theorem 2.1 of our paper [81] the following proposition holds:

Proposition 3.1. *Let us suppose that the inhomogeneity parameters are generic:*

$$\xi_j, \xi_j \pm \xi_k \notin \{0, -\eta, \eta\}, \quad \forall j, k \in \{1, \dots, N\}, \quad j \neq k, \quad (3.1)$$

and that the boundary matrices $K_-(\lambda)$ and $K_+(\lambda)$ are not both proportional to the identity, then

$$\langle \mathbf{h} | \equiv \langle S | \prod_{n=1}^N \left(\frac{\mathcal{T}(\xi_n - \eta/2)}{A_{\bar{\xi}_+, \bar{\xi}_-}(\eta/2 - \xi_n)} \right)^{1-h_n}, \quad \mathbf{h} \equiv (h_1, \dots, h_N) \in \{0, 1\}^N, \quad (3.2)$$

is a co-vector basis of \mathcal{H} for almost any choice of the co-vector $\langle S |$, of the inhomogeneity parameters satisfying (3.1) and of the boundary parameters. So, denoted with $|R\rangle$ the unique vector satisfying the following orthogonality conditions:

$$\langle h_1, \dots, h_N | R \rangle = \delta_{\mathbf{h}, \mathbf{0}} \frac{\widehat{V}(\xi_1, \dots, \xi_N)}{\widehat{V}(\xi_1^{(0)}, \dots, \xi_N^{(0)})}, \quad (3.3)$$

we have that the following set of vectors:

$$|\mathbf{h}\rangle \equiv \prod_{n=1}^N \left(\frac{\mathcal{T}(\xi_n + \eta/2)}{k_n A_{\bar{\xi}_+, \bar{\xi}_-}(\eta/2 - \xi_n)} \right)^{h_n} |R\rangle, \quad \mathbf{h} \in \{0, 1\}^N, \quad (3.4)$$

is a vector basis of \mathcal{H} and the two basis are orthogonal with respect to the canonical scalar product in the spin basis:

$$\langle \mathbf{h}' | \mathbf{h} \rangle = \delta_{\mathbf{h}, \mathbf{h}'} \frac{N_{\xi}}{\widehat{V}(\xi_1^{(h_1)}, \dots, \xi_N^{(h_N)})}, \quad (3.5)$$

with

$$N_{\xi} = \widehat{V}(\xi_1, \dots, \xi_N) \frac{\widehat{V}(\xi_1^{(0)}, \dots, \xi_N^{(0)})}{\widehat{V}(\xi_1^{(1)}, \dots, \xi_N^{(1)})}. \quad (3.6)$$

and

$$\widehat{V}(x_1, \dots, x_N) = \det_{1 \leq i, j \leq N} [x_i^{2(j-1)}] = \prod_{j < k} (x_k^2 - x_j^2), \quad (3.7)$$

for the Vandermonde determinant of a N -tuple of square variables (x_1^2, \dots, x_N^2) .

Proof. The proof that the set of covectors is a basis is detailed in Theorem 2.1 of [81]. We prove the rest of the proposition just proving the orthogonality conditions (3.5). These conditions are satisfied by definition by the vector $|R\rangle$ so we can prove them by induction. That is, we assume that they hold for a fixed \mathbf{h} such that

$$\sum_{n=1}^N h_n = m, \quad (3.8)$$

and then we prove them for the generic $\bar{\mathbf{h}}$ obtained changing one of the elements of \mathbf{h} from zero to 1, i.e. the $m + 1$ case. Then, there exists a permutation of $\{1, \dots, N\}$ such that:

$$h_{\pi(n)} = \{1 \text{ for } n \leq m \text{ and } 0 \text{ for } m < n\} \quad (3.9)$$

and we want to prove that for any $a \in \{\pi(m + 1), \dots, \pi(N)\}$ it holds:

$$\langle \mathbf{h}' | \bar{\mathbf{h}} \rangle = \delta_{\bar{\mathbf{h}}, \mathbf{h}'} \frac{N\xi}{\widehat{V}(\xi_1^{(\bar{h}_1)}, \dots, \xi_N^{(\bar{h}_N)})}, \quad (3.10)$$

where:

$$\bar{h}_b = h_b \forall b \neq a \text{ and } \bar{h}_a = 1. \quad (3.11)$$

Let us start proving the orthogonality condition for $h'_a = 0$, we have that it holds:

$$\begin{aligned} \langle \mathbf{h}' | \bar{\mathbf{h}} \rangle &= \langle h'_1, \dots, h''_a = 1, \dots, h'_N | \frac{\mathcal{T}(\xi_a - \eta/2)\mathcal{T}(\xi_a + \eta/2)}{k_a \mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 - \xi_a) \mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 - \xi_a)} | \mathbf{h} \rangle \\ &= \frac{\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 + \xi_a)}{k_a \mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 - \xi_a)} \langle h'_1, \dots, h''_a = 1, \dots, h'_N | \mathbf{h} \rangle = 0 \end{aligned} \quad (3.12)$$

where we have used the quantum determinant relation and we get zero by the induction being \mathbf{h} such that it holds $h_a = 0$. Let us now show the orthogonality condition for $h'_a = 1$ and $\mathbf{h}' \neq \bar{\mathbf{h}}$, this is the case if it exists a $b \neq a$ such that $h'_b = 1 - h_b$.

Then, to compute the action of $\mathcal{T}(\xi_a + \eta/2)$ on $\langle \mathbf{h}' |$, we use the following interpolation formula:

$$\mathcal{T}(\lambda) = t_{N+1} u_{\mathbf{h}'}(\lambda) + \mathcal{T}(\eta/2) s_{\mathbf{h}'}(\lambda) + \sum_{a=1}^N r_{a, \mathbf{h}'}(\lambda) \mathcal{T}(\xi_a^{(h'_a)}), \quad (3.13)$$

to get:

$$\begin{aligned} \langle \mathbf{h}' | \bar{\mathbf{h}} \rangle &= (t_{N+1} u_{\mathbf{h}'}(\xi_a + \eta/2) + \mathcal{T}(\eta/2) s_{\mathbf{h}'}(\xi_a + \eta/2)) \langle \mathbf{h}' | \mathbf{h} \rangle + \sum_{c=1}^N r_{c, \mathbf{h}'}(\xi_a + \eta/2) \\ &\quad \times \frac{\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 - \xi_c)}{k_a \mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 - \xi_a)} \langle h'_1, \dots, h''_c = 1 - h'_c, \dots, h'_N | \mathbf{h} \rangle, \end{aligned} \quad (3.14)$$

where we have used the simple identity:

$$\langle \mathbf{h}' | \frac{\mathcal{T}(\xi_c^{(h'_c)})}{k_a \mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 - \xi_a)} = \frac{\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 - \xi_c)}{k_a \mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 - \xi_a)} \langle h'_1, \dots, h''_c = h'_c - 1, \dots, h'_N | \mathbf{h} \rangle. \quad (3.15)$$

Then, by the induction we get:

$$\langle \mathbf{h}' | \bar{\mathbf{h}} \rangle = 0 \quad (3.16)$$

being

$$\langle \mathbf{h}' | \mathbf{h} \rangle = 0 \text{ and } \langle h'_1, \dots, h''_c = 1 - h'_c, \dots, h'_N | \mathbf{h} \rangle = 0 \quad \forall c \neq a \quad (3.17)$$

as by definition $h'_a = 1$ and $h_a = 0$ and

$$\langle h'_1, \dots, h''_a = 0, \dots, h'_N | \mathbf{h} \rangle = 0, \quad (3.18)$$

being by definition $h'_b = 1 - h_b$. Let us finally compute the last coupling:

$$\langle \bar{\mathbf{h}} | \bar{\mathbf{h}} \rangle = \langle \bar{\mathbf{h}} | \frac{\mathcal{T}(\xi_a + \eta/2)}{k_a \mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\eta/2 - \xi_a)} | \mathbf{h} \rangle \quad (3.19)$$

using once again the interpolation formula for $\mathcal{T}(\xi_a - \eta/2)$ for $\bar{\mathbf{h}}$ we get:

$$\langle \bar{\mathbf{h}} | \bar{\mathbf{h}} \rangle = \frac{r_{a, \bar{\mathbf{h}}}(\xi_a^{(0)})}{k_a} \langle \mathbf{h} | \mathbf{h} \rangle, \quad (3.20)$$

as all the others contributions are zero as one can prove following the same steps described above. From which the formula for the normalization follows. \square

3.1.2 Transfer matrix spectrum and their isospectrality

The previous proposition on the SoV bases directly implies the following complete characterization of the transfer matrix spectrum which represents a completion from the wave-functions to the eigenstates of the Theorem 2.2 of [81].

Theorem 3.1. *Let the inhomogeneity parameters be generic (3.1) and let the boundary matrices $K_-(\lambda)$ and $K_+(\lambda)$ not be both proportional to the identity, then, for almost any choice of the boundary parameters, the eigenvalue spectrum $\Sigma_{\mathcal{T}}$ of $\mathcal{T}(\lambda)$ is simple and it coincides with the set of polynomials*

$$\begin{aligned} t(\lambda) &= t_{N+1} \left(\lambda^2 - (\eta/2)^2 \right) \prod_{b=1}^N (\lambda^2 - \xi_b^2) + 2(-1)^N \det_q M(0) \prod_{b=1}^N \frac{\lambda^2 - \xi_b^2}{(\eta/2)^2 - \xi_b^2} \\ &\quad + \sum_{a=1}^N \frac{4\lambda^2 - \eta^2}{4\xi_a^2 - \eta^2} \prod_{\substack{b=1 \\ b \neq a}}^N \frac{\lambda^2 - \xi_b^2}{\xi_a^2 - \xi_b^2} t(\xi_a), \end{aligned} \quad (3.21)$$

satisfying the following discrete system of equations

$$\det \begin{pmatrix} t(\xi_n^{(0)}) & -\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\xi_n^{(0)}) \\ -\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(-\xi_n^{(1)}) & t(\xi_n^{(1)}) \end{pmatrix} = 0, \quad \forall n \in \{1, \dots, N\}. \quad (3.22)$$

The following vector and co-vectors

$$|\Psi_t\rangle = \sum_{\mathbf{h} \in \{0,1\}^N} \prod_{n=1}^N Q_t(\xi_n^{(h_n)}) \widehat{V}(\xi_1^{(h_1)}, \dots, \xi_N^{(h_N)}) | \mathbf{h} \rangle, \quad (3.23)$$

$$\langle \Psi_t | = \sum_{\mathbf{h} \in \{0,1\}^N} \prod_{n=1}^N \left[\left(\frac{\xi_n - \eta}{\xi_n + \eta} \frac{\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\xi_n^{(0)})}{\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(-\xi_n^{(1)})} \right)^{h_n} Q_t(\xi_n^{(h_n)}) \right] \widehat{V}(\xi_1^{(h_1)}, \dots, \xi_N^{(h_N)}) \langle \mathbf{h} |, \quad (3.24)$$

generate respectively the one-dimensional right and left $\mathcal{T}(\lambda)$ -eigenspaces associated with the eigenvalue $t(\lambda) \in \Sigma_{\mathcal{T}}$, where the Q_t is defined on the discrete set of values $\xi_n^{(h_n)}$, $n \in \{1, \dots, N\}$, $h_n \in \{0, 1\}$ by

$$\frac{Q_t(\xi_n^{(1)})}{Q_t(\xi_n^{(0)})} = \frac{t(\xi_n^{(0)})}{\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(\xi_n^{(0)})} = \frac{\mathbf{A}_{\bar{\zeta}_+, \bar{\zeta}_-}(-\xi_n^{(1)})}{t(\xi_n^{(1)})}, \quad n = 1, \dots, N. \quad (3.25)$$

Proof. This theorem follows from Theorem 2.2 of [81] and the decomposition of the identity induced from the previous proposition. \square

In the following we will use also the following notations:

$$g_n \equiv g_{\bar{\zeta}_+, \bar{\zeta}_-}(\xi_n) = \frac{(\xi_n + \bar{\zeta}_+)(\xi_n + \bar{\zeta}_-)}{(\xi_n - \bar{\zeta}_+)(\xi_n - \bar{\zeta}_-)}, \quad (3.26)$$

and

$$f_n \equiv f(\xi_n, \{\xi\}) = - \prod_{\substack{a=1 \\ a \neq n}}^N \frac{(\xi_n - \xi_a + \eta)(\xi_n + \xi_a + \eta)}{(\xi_n - \xi_a - \eta)(\xi_n + \xi_a - \eta)}, \quad (3.27)$$

from which

$$\frac{\xi_n - \eta}{\xi_n + \eta} \frac{A_{\bar{\zeta}_+, \bar{\zeta}_-}(\xi_n^{(0)})}{A_{\bar{\zeta}_+, \bar{\zeta}_-}(-\xi_n^{(1)})} = f_n g_n. \quad (3.28)$$

The previous theorem allows us to state the following corollary on the isospectrality of transfer matrices associated to different boundary conditions:

Corollary 3.1. *Let the inhomogeneity parameters be generic (3.1) let us consider two different sets of boundary parameters:*

$$(\zeta_{1,\pm}, \kappa_{1,\pm}, \tau_{1,\pm}) \neq (\zeta_{2,\pm}, \kappa_{2,\pm}, \tau_{2,\pm}) \quad (3.29)$$

with for both of them the associated boundary matrices are not both proportional to the identity. Then, if the following conditions holds:

$$\frac{1 + 4\kappa_{1,+}\kappa_{1,-} \cosh(\tau_{1,+} - \tau_{1,-})}{\zeta_{1,+}\zeta_{1,-}} = \frac{1 + 4\kappa_{2,+}\kappa_{2,-} \cosh(\tau_{2,+} - \tau_{2,-})}{\zeta_{2,+}\zeta_{2,-}}, \quad (3.30)$$

$$\bar{\zeta}_{1,\pm}^2 = \bar{\zeta}_{2,\pm\epsilon}^2 \text{ for a given } \epsilon = \{+, -\}, \quad (3.31)$$

the associated two transfer matrices are isospectral, i.e. there exists an invertible $\Gamma_{12} \in \text{End}\mathcal{H}$ such that:

$$\mathcal{T}(\lambda|\zeta_{1,\pm}, \kappa_{1,\pm}, \tau_{1,\pm}) = \Gamma_{12}^{-1} \mathcal{T}(\lambda|\zeta_{2,\pm}, \kappa_{2,\pm}, \tau_{2,\pm}) \Gamma_{12}. \quad (3.32)$$

Moreover, taken the set of boundary parameters $(\zeta_{\pm}, \kappa_{\pm}, \tau_{\pm})$ satisfying the condition:

$$\omega_{\epsilon}(\kappa_{\pm}, \tau_{\pm}) \equiv 1 + 4\kappa_{+}\kappa_{-} \cosh(\tau_{+} - \tau_{-}) - \epsilon \sqrt{(1 + 4\kappa_{+}^2)(1 + 4\kappa_{-}^2)} = 0, \quad (3.33)$$

for a given $\epsilon = \{+, -\}$, then the following isospectrality holds:

$$\mathcal{T}(\lambda|\zeta_{\pm}, \kappa_{\pm}, \tau_{\pm}) = \Gamma^{-1} \mathcal{T}(\lambda|\bar{\zeta}_{\pm}, 0, 0) \Gamma, \quad (3.34)$$

where $\mathcal{T}(\lambda|\bar{\zeta}_{\pm}, 0, 0)$ is the transfer matrix associated to diagonal boundary matrices with parameters $\bar{\zeta}_{\pm}$.

Proof. Here, we have just to remark that the SoV characterization of the transfer matrix spectrum of the previous theorem implies that the spectrum depends by the boundary parameters only by $\bar{\zeta}_{\pm}^2$ and by t_{N+1} , so that the identities (3.30)-(3.31) assure that two different sets of boundary parameters $(\zeta_{1,\pm}, \kappa_{1,\pm}, \tau_{1,\pm})$ and $(\zeta_{2,\pm}, \kappa_{2,\pm}, \tau_{2,\pm})$ share the same values of $\bar{\zeta}_{\pm}^2$ and by t_{N+1} . The completeness of the transfer matrix spectrum description implies then the isospectrality. Finally, taking diagonal boundary matrices with parameters $\bar{\zeta}_{\pm}$, by definition it holds:

$$t_{N+1}(\bar{\zeta}_{\pm}, 0, 0) = \frac{2}{\bar{\zeta}_{+}\bar{\zeta}_{-}} = t_{N+1}(\zeta_{\pm}, \kappa_{\pm}, \tau_{\pm}), \quad (3.35)$$

under the condition (3.33) and the choice $\epsilon = \epsilon_{-}\epsilon_{+}$, where ϵ_{\pm} are the signs that we chose in (2.20), so that the isospectrality statement follows. \square

It is important to stress that this isospectrality goes far beyond that associated with the $GL(2)$ symmetry of the model, which just implies the isospectrality in the case one can go from the set of boundary matrices associated to $(\zeta_{1,\pm}, \kappa_{1,\pm}, \tau_{1,\pm})$ to the set associated to $(\zeta_{2,\pm}, \kappa_{2,\pm}, \tau_{2,\pm})$ by a similarity transformation. One simple example is the isospectrality of the transfer matrix associated to $(\zeta_{-}, \kappa_{-}, \tau_{-})$ in the site 1 and to $(\zeta_{+}, \kappa_{+}, \tau_{+})$ in the site N and the transfer matrix associated to $(\zeta_{+}, \kappa_{+}, \tau_{+})$ in the site 1 and to $(\zeta_{-}, \kappa_{-}, \tau_{-})$ in the site N . These two transfer matrices are isospectral but for general values of the boundary parameters one cannot pass from one set of boundary matrices to the other by a similarity transformation.

The main example of interest for us now of this beyond $GL(2)$ isospectrality is the case of unparallel boundary magnetic fields which satisfy the condition (3.33). The unparallel boundary magnetic fields case is equivalent to ask that the boundary matrices are non simultaneously diagonalizable, we will see that this condition is compatible with (3.33). So that we establish the isospectrality of these transfer matrices with those with parallel magnetic fields according to (3.34), which we will use in the computation of the correlation functions. Note that being the parallel boundary magnetic fields case equivalent to ask that the two boundary matrices are simultaneously diagonalizable, then our statement of beyond $GL(2)$ isospectrality follows as well as the fact that in general the similarity Γ in (3.34) is not easy to derive and it is not of tensor product type.

3.2 Functional TQ -equation spectrum characterization

The transfer matrix eigenvalues and eigenstates are characterized in the SoV framework in terms of the Q_t defined on the discrete set of shifted inhomogeneity parameters only. It is possible to show that this discrete characterization can be reformulated by a functional equation for a Q_t function defined on the whole complex plane \mathbb{C} .

In the case of the open XXZ spin-1/2 chain with generic integrable boundary conditions, this was first proven in [29] under the quite general boundary conditions allowing for the introduction of the Sklyanin's like SoV approach. Here, we generalized the results derived for the XXX case in [76] thanks to the wider SoV approach derived in [81] and further detailed in the previous section.

Theorem 3.2. *Let the inhomogeneity parameters be generic (3.1) and let the boundary matrices $K_{-}(\lambda)$ and $K_{+}(\lambda)$ not be both proportional to the identity, then, for almost any choice of the boundary parameters, defined*

$$F(\lambda) = \frac{2\omega_{\epsilon}(\kappa_{\pm}, \tau_{\pm})}{\bar{\zeta}_{-}\bar{\zeta}_{+}} \left(\lambda^2 - (\eta/2)^2 \right) \prod_{b=1}^N \prod_{h=0}^1 \left(\lambda^2 - (\xi_b^{(h)})^2 \right), \quad (3.36)$$

with $\epsilon = \epsilon_- \epsilon_+$ and ϵ_{\pm} the signs in (2.20), $t(\lambda) \in \Sigma_{\mathcal{T}}$ iff there exists a unique polynomial $Q_t(\lambda)$ of the form

$$Q_t(\lambda) = \prod_{b=1}^q (\lambda^2 - \lambda_b^2), \quad \lambda_1, \dots, \lambda_q \in \mathbb{C} \setminus \{ \pm \xi_1^{(0)}, \dots, \pm \xi_N^{(0)} \}, \quad (3.37)$$

satisfying

$$t(\lambda) Q_t(\lambda) = A_{\bar{\zeta}_+, \bar{\zeta}_-}(\lambda) Q_t(\lambda - \eta) + A_{\bar{\zeta}_+, \bar{\zeta}_-}(-\lambda) Q_t(\lambda + \eta) + F(\lambda), \quad (3.38)$$

or equivalently $t(\lambda) \in \Sigma_{\mathcal{T}}$ iff there exists a unique polynomial $P_t(\lambda)$ of the form

$$P_t(\lambda) = \prod_{b=1}^p (\lambda^2 - \mu_b^2), \quad \mu_1, \dots, \mu_p \in \mathbb{C} \setminus \{ \pm \xi_1^{(0)}, \dots, \pm \xi_N^{(0)} \}, \quad (3.39)$$

such that

$$t(\lambda) P_t(\lambda) = A_{-\bar{\zeta}_+, -\bar{\zeta}_-}(\lambda) P_t(\lambda - \eta) + A_{-\bar{\zeta}_+, -\bar{\zeta}_-}(-\lambda) P_t(\lambda + \eta) + F(\lambda), \quad (3.40)$$

where:

$$p = q = N \text{ if } \omega_{\epsilon}(\kappa_{\pm}, \tau_{\pm}) \neq 0, \quad p + q = N \text{ if } \omega_{\epsilon}(\kappa_{\pm}, \tau_{\pm}) = 0. \quad (3.41)$$

Proof. Starting from the SoV discrete characterization derived in the previous section, we can prove the current theorem just as done in [76], for the Theorem 3.2, 3.3 and Proposition 3.1. \square

It is important to remark that differently from the case of the Sklyanin's like SoV characterization⁹, here, on the one hand, we are doing a characterization holding in general both for parallel and unparallel boundary magnetic fields and, on the other hand, we do not need to implement any similarity transformation to bring the system in an appropriate form to use Sklyanin's like SoV characterization. These remarks are very important for the case of the XXZ and XYZ spin chains where these similarity transformations, required to make Sklyanin's like SoV applicable, are not of simple tensor product form but have a non-local form intrinsic of the Baxter's gauge transformations.

3.3 Bethe ansatz form of separate state and boundary-bulk decomposition

Here, we recall the rewriting in Bethe ansatz form of the SoV characterization of the transfer matrix eigenstates and separate states of the open XXX quantum chain following [76]. Then, we present the main result of the section, i.e. the boundary-bulk decomposition of these separate states.

3.3.1 Similarity transformation to triangular cases

We can define the following similarity transformed boundary monodromy matrices:

$$\bar{\mathcal{U}}_{\mp}(\lambda) = W_0 \Gamma_W \mathcal{U}_{\mp}(\lambda) \Gamma_W^{-1} W_0^{-1} = \begin{pmatrix} \bar{\mathcal{A}}_{\mp}(\lambda) & \bar{\mathcal{B}}_{\mp}(\lambda) \\ \bar{\mathcal{C}}_{\mp}(\lambda) & \bar{\mathcal{D}}_{\mp}(\lambda) \end{pmatrix}, \quad (3.42)$$

where $W_0 \in GL(2, \mathbb{C})$ acts on the auxiliary space, whereas $\Gamma_W \equiv \otimes_{n=1}^N W_n$ acts on the quantum space of states. By the $GL(2, \mathbb{C})$ invariance of the R-matrices:

$$R_{12}(\lambda) W_1 W_2 = W_2 W_1 R_{12}(\lambda), \quad (3.43)$$

⁹See for example Theorem 3.1 of our paper [76] for the open XXX spin chain.

we have that it holds:

$$\bar{U}_-(\lambda) = M(\lambda) \bar{K}_-(\lambda) \hat{M}(\lambda), \quad (3.44)$$

$$\bar{U}_+^{t_0}(\lambda) = M^{t_0}(\lambda) \bar{K}_+^{t_0}(\lambda) \hat{M}^{t_0}(\lambda), \quad (3.45)$$

where we have defined:

$$\bar{K}_\mp(\lambda) = W_0 K_\mp(\lambda) W_0^{-1} = \begin{pmatrix} \bar{a}_\mp(\lambda) & \bar{b}_\mp(\lambda) \\ \bar{c}_\mp(\lambda) & \bar{d}_\mp(\lambda) \end{pmatrix}. \quad (3.46)$$

The following similarity transformation holds:

$$\bar{\mathcal{T}}(\lambda) = \Gamma_W \mathcal{T}(\lambda) \Gamma_W^{-1} \quad (3.47)$$

where:

$$\bar{\mathcal{T}}(\lambda) = \text{tr}_0 \left\{ \bar{K}_+(\lambda) M(\lambda) \bar{K}_-(\lambda) \hat{M}(\lambda) \right\}. \quad (3.48)$$

Then, under the following choice:

$$W \equiv W_{\epsilon_+, \epsilon_-} = \begin{pmatrix} 1 & \frac{-1 + \epsilon_- \sqrt{1 + 4\kappa_-^2}}{2\kappa_- e^{-\tau_-}} \\ \frac{1 - \epsilon_+ \sqrt{1 + 4\kappa_+^2}}{2\kappa_+ e^{\tau_+}} & 1 \end{pmatrix}, \quad (3.49)$$

for a given $(\epsilon_+, \epsilon_-) \in \{-1, 1\}^2$, the boundary matrices take the triangular form

$$\bar{K}_-(\lambda) = \text{I} + \frac{\lambda + \eta/2}{\bar{\zeta}_-} (\sigma^z + \bar{c}_- \sigma^-), \quad \bar{K}_+(\lambda) = \text{I} + \frac{\lambda - \eta/2}{\bar{\zeta}_+} (\sigma^z + \bar{b}_+ \sigma^+), \quad (3.50)$$

with

$$\bar{c}_- = \frac{2\epsilon_- \kappa_- e^{-\tau_-}}{\sqrt{1 + 4\kappa_-^2}} \left[1 + \frac{(1 + \epsilon_- \sqrt{1 + 4\kappa_-^2})(1 - \epsilon_+ \sqrt{1 + 4\kappa_+^2})}{4\kappa_+ \kappa_- e^{\tau_+ - \tau_-}} \right], \quad (3.51)$$

$$\bar{b}_+ = \frac{2\epsilon_+ \kappa_+ e^{\tau_+}}{\sqrt{1 + 4\kappa_+^2}} \left[1 + \frac{(1 - \epsilon_- \sqrt{1 + 4\kappa_-^2})(1 + \epsilon_+ \sqrt{1 + 4\kappa_+^2})}{4\kappa_+ \kappa_- e^{\tau_+ - \tau_-}} \right]. \quad (3.52)$$

The interest in the above similarity transformation is that for $\bar{b}_+ \neq 0$ we can implement the Sklyanin's like SoV approach using as generator of the left and right SoV bases the monodromy matrix entry $\bar{B}_+(\lambda)$ (diagonalizable and with simple spectrum) to solve the spectral problem of the transfer matrix $\bar{\mathcal{T}}(\lambda)$. Then, the eigenstates $(|t\rangle, \langle t|)$ of the original transfer matrix $\mathcal{T}(\lambda)$ can be therefore expressed in terms of those of the new triangular one $(|\bar{t}\rangle, \langle \bar{t}|)$ using the tensor product similarity transformation Γ_W , i.e. it holds:

$$|t\rangle = \Gamma_W^{-1} |\bar{t}\rangle, \quad \langle t| = \langle \bar{t}| \Gamma_W. \quad (3.53)$$

Finally, let us remark that the Hamiltonian

$$\bar{H} = \Gamma_W H \Gamma_W^{-1}, \quad (3.54)$$

associated to the transfer matrix $\bar{\mathcal{T}}(\lambda)$ reads:

$$\bar{H} = \sum_{i=1}^{N-1} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right] + \frac{\eta}{\bar{\zeta}_-} \left[\sigma_1^z + \bar{c}_- \sigma_1^- \right] + \frac{\eta}{\bar{\zeta}_+} \left[\sigma_N^z + \bar{b}_+ \sigma_N^+ \right]. \quad (3.55)$$

3.3.2 Unparallel cases isospectral to the parallel ones

Here, we characterize the cases in which the original transfer matrix $\mathcal{T}(\lambda)$, associated to unparallel boundary magnetic fields, is isospectral to the transfer matrix $\hat{\mathcal{T}}(\lambda) \equiv \mathcal{T}(\lambda|\bar{\zeta}_{\pm}, 0, 0)$, associated to parallel along the z-direction boundary magnetic fields, i.e.

$$\hat{\mathcal{T}}(\lambda) = \text{tr}_0 \left\{ \hat{K}_+(\lambda) M(\lambda) \hat{K}_-(\lambda) \hat{M}(\lambda) \right\}, \quad (3.56)$$

with

$$\hat{K}_-(\lambda) = \text{I} + \frac{\lambda + \eta/2}{\bar{\zeta}_-} \sigma^z, \quad \hat{K}_+(\lambda) = \text{I} + \frac{\lambda - \eta/2}{\bar{\zeta}_+} \sigma^z. \quad (3.57)$$

Let us recall that by construction of the similarity transformation, $\mathcal{T}(\lambda)$ and $\bar{\mathcal{T}}(\lambda)$ are clearly isospectral. Then, let us now impose the following boundary conditions:

$$\bar{c}_- = 0, \quad \bar{b}_+ \neq 0, \quad (3.58)$$

which keep $\bar{\mathcal{T}}(\lambda)$ associated to a properly triangular boundary matrix in site N , i.e. $\mathcal{T}(\lambda)$ is properly associated to unparallel boundary magnetic fields. Then, the following identity:

$$\bar{c}_- \bar{b}_+ = \frac{\epsilon_+ \epsilon_- e^{\tau_- - \tau_+} \omega_{\epsilon_+ \epsilon_-}(\kappa_{\pm}, \tau_{\pm})}{4\kappa_+ \kappa_- \sqrt{(1 + 4\kappa_+^2)(1 + 4\kappa_-^2)}}, \quad (3.59)$$

implies that the condition (3.58) is equivalent to the following one

$$\omega_{\epsilon_+ \epsilon_-}(\kappa_{\pm}, \tau_{\pm}) = 0, \quad \bar{b}_+ \neq 0, \quad (3.60)$$

which by Corollary 3.1 implies the isospectrality of our original transfer matrix $\mathcal{T}(\lambda)$ (with unparallel boundary magnetic fields) to $\hat{\mathcal{T}}(\lambda)$ (with parallel boundary magnetic fields). We can state the following:

Lemma 3.1. *Let us fix a couple $(\epsilon_+, \epsilon_-) \in \{-1, 1\}^2$ and let us impose the following boundary conditions:*

$$e^{\tau_+} = e^{\tau_-} \frac{(\epsilon_- \sqrt{1 + 4\kappa_-^2} + 1)(\epsilon_+ \sqrt{1 + 4\kappa_+^2} - 1)}{4\kappa_+ \kappa_-}, \quad (3.61)$$

then for any choice of the boundary parameters such that:

$$\kappa_+ \neq \pm \kappa_-, \quad (3.62)$$

our original transfer matrix $\mathcal{T}(\lambda)$ has unparallel boundary magnetic fields and it is isospectral to $\hat{\mathcal{T}}(\lambda)$ with parallel z-oriented boundary magnetic fields. Moreover, taken $i\eta \in \mathbb{R}$, then for any choice of the boundary parameters $(\bar{\zeta}_{\pm}, \kappa_{\pm})$ such that:

$$i\bar{\zeta}_{\pm} \in \mathbb{R}, \quad (3.63)$$

the transfer matrix $\hat{\mathcal{T}}(\lambda)$ is self-adjoint. So, the ground state distribution of our original Hamiltonian H (associated to $\mathcal{T}(\lambda)$) coincides with that of the Hamiltonian

$$\hat{H} = \sum_{i=1}^{N-1} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right] + \frac{i}{\bar{\zeta}_-} \sigma_1^z + \frac{i}{\bar{\zeta}_+} \sigma_N^z, \quad (3.64)$$

(associated to $\hat{\mathcal{T}}(\lambda)$) and in the thermodynamic limit $N \rightarrow \infty$ it is known to be characterized by the following distribution on the positive real axis:

$$\rho(\lambda) = \frac{i}{\sinh \pi \lambda}, \quad (3.65)$$

once we fix $\eta = i$ and single roots are absent for $\bar{\zeta}_- \neq 0$.

3.3.3 The Bethe ansatz form of eigenstates and separate states

Let us here present the Bethe ansatz form of the transfer matrix eigenstates following from our SoV characterization of the spectrum

Proposition 3.2 ([76]). *Let the inhomogeneities ξ_1, \dots, ξ_N be generic (3.1) and under the condition $\bar{b}_+ \neq 0$, then, for any $t(\lambda) \in \Sigma_{\mathcal{T}}$, the corresponding (unique up to normalization) right and left $\mathcal{T}(\lambda)$ -eigenstates reads:*

$$\langle t| = \langle \underline{0}| \prod_{a=1}^p \bar{\mathcal{B}}_+(\mu_a) \Gamma_W, \quad |t\rangle = \Gamma_W^{-1} \prod_{a=1}^q \bar{\mathcal{B}}_+(\lambda_a) |0\rangle, \quad (3.66)$$

where $\lambda_1, \dots, \lambda_q$ are the roots of the polynomial $Q_t(\lambda)$ and μ_1, \dots, μ_p are those of the polynomial $P_t(\lambda)$, solutions of (3.38) and (3.38), respectively.

Proof. The Proposition 3.3 and 3.4 of [76] show that the SoV characterization of the eigenvectors is equivalent to the above Bethe ansatz like formulation. \square

Let us observe now that under the condition $\bar{b}_+ \neq 0$, the results of Section 2.3 of the paper [81], implies the identities:

$$\langle \mathbf{h}| = l \langle +, \mathbf{h}| \Gamma_W, \quad |\mathbf{h}\rangle = r \Gamma_W^{-1} |\mathbf{h}, +\rangle, \quad (3.67)$$

where l and r are some computable normalization constant and $\langle +, \mathbf{h}|$ and $|\mathbf{h}, +\rangle$ are the left and right eigenstates of $\bar{\mathcal{B}}_+(\lambda)$. So that we can equivalently use one or the other SoV basis leading to the same above results.

In separation of variable the so-called *separate states* read:

$$|\gamma\rangle = \sum_{\mathbf{h} \in \{0,1\}^N} \prod_{n=1}^N \gamma_n^{(h_n)} \widehat{V}(\xi_1^{(h_1)}, \dots, \xi_N^{(h_N)}) |\mathbf{h}\rangle, \quad (3.68)$$

for some coefficients $\gamma_n^{(h_n)}$, $n \in \{1, \dots, N\}$, $h_n \in \{0, 1\}$. They are a set of states containing as special elements the transfer matrix eigenvectors and playing a fundamental role in the computation of correlation functions. Indeed, scalar products of the separate states universally admit [65, 67] determinat representations, shown in [76] for the open XXX case to be equivalent to determinants generalizing the Slavnov's determinants [146], results previously known only in the case of parallel boundary magnetic fields [13, 147]. Moreover, as we will show in the next section, the action of local operators on separate states can be efficiently rewritten in terms of linear combinations of separate states in this way allowing to implement the calculation of correlation functions.

Here, we recall that along the same lines of the previous proposition in [76] we have shown that also the separate states naturally admit Bethe ansatz representations. In the following we will use the following representation of separate states:

$$|\beta\rangle = \Gamma_W^{-1} \prod_{a=1}^{n_\beta} \bar{\mathcal{B}}_+(b_a) |0\rangle, \quad (3.69)$$

where

$$\beta(\lambda) = \prod_{m=1}^{n_\beta} (\lambda^2 - b_m^2), \quad (3.70)$$

which coincides with the separate state $|\gamma\rangle$ under the identification

$$\gamma_n^{(h_n)} = (-1)^N \beta(\xi_n^{(h_n)}) \bar{b}_+(\xi_n^{(h_n)}). \quad (3.71)$$

3.3.4 Boundary-bulk decomposition of separate states

Here, we compute the boundary-bulk decomposition for the separate states. We observe that it holds:

$$\bar{\mathcal{B}}_+(\lambda) = \hat{\mathcal{B}}_+(\lambda) + D(\lambda) D(-\lambda) \bar{b}_+(\lambda) \quad (3.72)$$

where $\hat{\mathcal{B}}_+(\lambda)$ is the operator associated to the diagonal part of $\bar{K}_+(\lambda)$:

$$\bar{\mathcal{B}}_+(\lambda) = B(\lambda) D(-\lambda) \bar{a}_+(\lambda) - D(\lambda) B(-\lambda) \bar{d}_+(\lambda), \quad (3.73)$$

then the next proposition follows:

Proposition 3.3. *The following boundary-bulk decomposition of separate states holds:*

$$\begin{aligned} \prod_{j=1}^M \mathcal{B}_+(\lambda_j) |0\rangle &= \sum_{a=0}^M \sum_{\substack{\mathcal{X} \cup \mathcal{Y} = I_M \\ \mathcal{X} \cap \mathcal{Y} = \emptyset, |\mathcal{Y}|=a}} \frac{\bar{b}_+(\lambda_{\mathcal{X}}) d(\lambda_{\mathcal{X}}) d(-\lambda_{\mathcal{X}})}{\lambda_{\mathcal{X}}^2 - \lambda_{\mathcal{Y}}^2} \\ &\times \sum_{\sigma_{\mathcal{Y}} = \pm} [\lambda_{\mathcal{X}}^2 - (\lambda_{\mathcal{Y}}^{(\sigma)} - \eta)^2] H_{(\sigma_{\mathcal{Y}})}^{\mathcal{B}_+}(\lambda_{\mathcal{Y}}) B(\lambda_{\mathcal{Y}}^{(\sigma)}) |0\rangle, \end{aligned} \quad (3.74)$$

where we have used the short notations $I_M = \{1, \dots, M\}$ and

$$B(\lambda_{\mathcal{Y}}^{(\sigma)}) = \prod_{j \in \mathcal{Y}} B(\sigma_j \lambda_j), \quad \lambda_{\mathcal{X}}^2 - \lambda_{\mathcal{Y}}^2 = \prod_{i \in \mathcal{X}} \prod_{j \in \mathcal{Y}} (\lambda_i^2 - \lambda_j^2), \quad \text{etc} \quad (3.75)$$

and

$$\begin{aligned} H_{(\sigma_1, \dots, \sigma_R)}^{\mathcal{B}_+}(\lambda_1, \dots, \lambda_R) &= \prod_{j=1}^R d(-\sigma_j \lambda_j) \frac{\sinh(2\lambda_j + \eta)}{\sinh(2\lambda_j)} \sinh(\lambda_j + \sigma_j(\bar{\zeta}_+ - \eta/2)) \\ &\times \prod_{1 \leq r < s \leq R} \frac{\sinh(\sigma_s \lambda_s + \sigma_r \lambda_r - \eta)}{\sinh(\sigma_s \lambda_s + \sigma_r \lambda_r)}. \end{aligned} \quad (3.76)$$

Proof. In the following, we use the commutativity of the following three sets of operator families, $\bar{\mathcal{B}}_+(\lambda)$, $\hat{\mathcal{B}}_+(\lambda)$ and $D(\lambda)$. We clearly have the following identity:

$$\prod_{j=1}^M \bar{\mathcal{B}}_+(\lambda_j) |0\rangle = \sum_{a=0}^M \sum_{\substack{\mathcal{X} \cup \mathcal{Y} = I_M \\ \mathcal{X} \cap \mathcal{Y} = \emptyset, |\mathcal{Y}|=a}} \sum_{\sigma_{\mathcal{Y}} = \pm} C_{\mathcal{X}, \mathcal{Y}}^{(\sigma)}(\lambda) B(\lambda_{\mathcal{Y}}^{(\sigma)}) |0\rangle, \quad (3.77)$$

as a consequence of the boundary-bulk decomposition of the $\bar{\mathcal{B}}_+$ -operator family, the Yang-Baxter commutation relations between $D(\lambda)$ and $B(\lambda)$ and the following identity:

$$D(\lambda) |0\rangle = |0\rangle d(\lambda). \quad (3.78)$$

So we are left with the proof that the coefficients take the above defined form. Let us fix a couple of sets $\mathcal{X} \cup \mathcal{Y} = I_M$ then by using the commutativity we can take the following rewriting:

$$\prod_{j=1}^M \bar{\mathcal{B}}_+(\lambda_j) |0\rangle = \bar{\mathcal{B}}_+(\lambda_{\mathcal{X}}) \bar{\mathcal{B}}_+(\lambda_{\mathcal{Y}}) |0\rangle, \quad (3.79)$$

then it is easy to understand that the terms

$$\sum_{\sigma_{\mathcal{Y}} = \pm} C_{\mathcal{X}, \mathcal{Y}}^{(\sigma)}(\lambda) B(\lambda_{\mathcal{Y}}^{(\sigma)}) |0\rangle \quad (3.80)$$

in the state (3.77) can be generated only by the following term

$$\bar{b}_+(\lambda_{\mathcal{X}}) D(\lambda_{\mathcal{X}}) D(-\lambda_{\mathcal{X}}) \hat{\mathcal{B}}_+(\lambda_{\mathcal{Y}}) |0\rangle \quad (3.81)$$

and only by taking the direct action of the operators $D(\pm\lambda_{\mathcal{X}})$ on the state:

$$\hat{\mathcal{B}}_+(\lambda_{\mathcal{Y}}) |0\rangle = \sum_{\sigma_{\mathcal{Y}}=\pm} H_{(\sigma_{\mathcal{Y}})}^{\mathcal{B}_+}(\lambda_{\mathcal{Y}}) B(\lambda_{\mathcal{Y}}^{(\sigma)}) |0\rangle, \quad (3.82)$$

where the above boundary-bulk decomposition has been derived in Proposition 3.4 of our paper [13] and it holds being $\hat{\mathcal{B}}_+(\lambda)$ associated to the diagonal part of $\bar{K}_+(\lambda)$. So all we have to do to compute the coefficients $C_{\mathcal{X},\mathcal{Y}}^{(\sigma)}(\lambda)$ is to compute this direct action on the above state, which, by the Yang-Baxter commutation relations and the property (3.78), clearly give:

$$\begin{aligned} \bar{b}_+(\lambda_{\mathcal{X}}) [D(\lambda_{\mathcal{X}}) D(-\lambda_{\mathcal{X}})]_{\text{Direct action}} \hat{\mathcal{B}}_+(\lambda_{\mathcal{Y}}) |0\rangle &= \bar{b}_+(\lambda_{\mathcal{X}}) d(\lambda_{\mathcal{X}}) d(-\lambda_{\mathcal{X}}) \\ &\times \sum_{\sigma_{\mathcal{Y}}=\pm} \frac{\lambda_{\mathcal{X}}^2 - (\lambda_{\mathcal{Y}}^{(\sigma)} - \eta)^2}{\lambda_{\mathcal{X}}^2 - \lambda_{\mathcal{Y}}^2} H_{(\sigma_{\mathcal{Y}})}^{\mathcal{B}_+}(\lambda_{\mathcal{Y}}) B(\lambda_{\mathcal{Y}}^{(\sigma)}) |0\rangle, \end{aligned} \quad (3.83)$$

in this way completing our proof. \square

4 Action of local operators on boundary separate states

In order to compute correlation functions we have to be able to compute the action of local operators on transfer matrix eigenstates and so on separate states. The first fundamental step in these computations is the reconstruction of local operators [107] in terms of the bulk generators of the Yang-Baxter algebra and, in particular, their simplified version derived in [13]. Indeed, thanks to these reconstructions and the previously derived boundary-bulk decomposition of separate states, these actions can be computed by using the known Yang-Baxter algebra or SoV representations.

Here, we compute the action of the local and quasi-local operators on separate states associated to the transfer matrix $\bar{\mathcal{T}}(\lambda)$, i.e. the one with diagonal $\bar{K}_-(\lambda)$ and properly triangular ($b_+ \neq 0$) $\bar{K}_+(\lambda)$ boundary matrices. These results then translate directly in action of local and quasi-local operators on separate states associated to the original transfer matrix $\mathcal{T}(\lambda)$. Indeed, the similarity transformation relating the two transfer matrices is of pure tensor product type and so it transforms local operators in local operators.

4.1 Action of the quasi-local operator $Q_m(\kappa)$

Let us recall that the first nontrivial correlation functions for the open quantum spin chains are the one point functions, i.e. the ground state average of a local spin operators, which measure the correlation with the boundaries. Following, the presentation of [14], we first compute the action of the quasi-local operator:

$$Q_m(\kappa) = \prod_{a=1}^m (E_a^{11} + \kappa E_a^{22}) = \prod_{a=1}^m (A(\xi_a^{(0)}) + \kappa D(\xi_a^{(0)})) \prod_{a=1}^m (A(\xi_a^{(1)}) + D(\xi_a^{(1)})), \quad (4.1)$$

which gives access to the one-point function of the local operator:

$$\sigma_m^z = 1 + 2 [(\partial_{\kappa} Q_m(\kappa))_{\kappa=1} - (\partial_{\kappa} Q_{m+1}(\kappa))_{\kappa=1}]. \quad (4.2)$$

Here, we use the implicit multiplication notation introduced in (3.75) and the following further notations:

$$R_n^\kappa(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}|\mu_{\alpha_-}) = R(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}|\mu_{\alpha_-})\bar{S}_n^\kappa(\xi_{\gamma_+}|\mu_{\alpha_+}), \quad (4.3)$$

where:

$$R(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}|\mu_{\alpha_-}) = \frac{\tau(\mu_{\alpha_+}|\mu_{\alpha_-})\tau(\xi_{\gamma_-}^{(0)}|\xi_{\gamma_+}^{(0)})}{\tau(\xi_{\gamma_+ \cup \gamma_-}^{(0)}|\mu_{\alpha_+})f(\mu_{\alpha_-}|\xi_{\gamma_+}^{(0)})}, \quad (4.4)$$

$$\tau(x|y) = a(x)f(y|x), \quad f(x|y) = \frac{x-y+\eta}{x-y}, \quad (4.5)$$

and $\bar{S}_n^\kappa(\xi_{\gamma_+}|\mu_{\alpha_+})$ is defined by

$$\bar{S}_n^\kappa(\xi_{\gamma_+}|\mu_{\alpha_+}) = \frac{\xi_{\gamma_+}^{(0)} - \mu_{\alpha_+} + \eta}{\prod_{\substack{a>b \\ a,b \in \gamma_+}} \xi_{ab} \prod_{\substack{a>b \\ a,b \in \alpha_+}} \mu_{ba}} \det_{k \in \gamma_+, j \in \alpha_+} \bar{M}_{jk}^\kappa, \quad (4.6)$$

with the $n \times n$ matrix M^κ defined by:

$$\bar{M}_{jk}^\kappa = r(\xi_k^{(0)}|\mu_j) - \kappa r(\mu_j|\xi_k^{(0)}) \frac{f(\mu_{\alpha_+ - \{j\}}|\mu_j) f(\mu_j|\xi_{\gamma_+}^{(0)})}{f(\mu_j|\mu_{\alpha_+ - \{j\}}) f(\xi_{\gamma_+}^{(0)}|\mu_j)}, \quad (4.7)$$

and

$$r(x|y) = \frac{\eta}{(x-y)(x-y+\eta)}. \quad (4.8)$$

Then, the following proposition holds:

Proposition 4.1. *The action of the following local operators on boundary separate states read:*

$$Q_m^\kappa \bar{\mathcal{B}}_+(\mu_{I_M})|0\rangle = \sum_{n=0}^{l(m,M)} \sum_{\substack{I_M = \alpha_+ \cup \alpha_-, \alpha_+ \cap \alpha_- = \emptyset \\ I_m = \gamma_+ \cup \gamma_-, \gamma_+ \cap \gamma_- = \emptyset \\ |\gamma_+| = |\alpha_+| = n}} \mathcal{R}_n^\kappa(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}|\mu_{\alpha_-}) \bar{\mathcal{B}}_+(\mu_{\alpha_-} \cup \xi_{\gamma_+}^{(0)})|0\rangle, \quad (4.9)$$

where we have defined $l(m, R) = \min(m, R)$,

$$\mathcal{R}_n^\kappa(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}|\mu_{\alpha_-}) = \sum_{\sigma_{\alpha_+} = \pm} \frac{H_{\sigma_{\alpha_+}}^{\mathcal{B}_+}(\mu_{\alpha_+})}{H^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)})} R_n^\kappa(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}^\sigma|\pm\mu_{\alpha_-}), \quad (4.10)$$

and $H^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)})$ stays for the coefficient¹⁰ (3.76) with $\sigma_{\gamma_+} = (1, \dots, 1)$ while we have defined $\mu_{\alpha_+}^\sigma \equiv \{\sigma_i \mu_i \forall i \in \alpha_+\}$ and $\pm\mu_{\alpha_-} \equiv \mu_{\alpha_-} \cup (-\mu_{\alpha_-})$.

Proof. We use first the boundary-bulk decomposition of the separate states given in the previous section, so that it holds:

$$Q_m^\kappa \bar{\mathcal{B}}_+(\mu_{I_M})|0\rangle = \sum_{a=0}^M \sum_{\substack{\mathcal{X} \cup \mathcal{Y} = I_M \\ \mathcal{X} \cap \mathcal{Y} = \emptyset, |\mathcal{Y}| = a}} \frac{\bar{b}_+(\mu_{\mathcal{X}}) d(\mu_{\mathcal{X}}) d(-\mu_{\mathcal{X}})}{\mu_{\mathcal{X}}^2 - \mu_{\mathcal{Y}}^2} \sum_{\sigma_{\mathcal{Y}} = \pm} [\mu_{\mathcal{X}}^2 - (\mu_{\mathcal{Y}}^{(\sigma)} - \eta)^2] \\ \times H_{(\sigma_{\mathcal{Y}})}^{\mathcal{B}_+}(\mu_{\mathcal{Y}}) \left(Q_m^\kappa \mathcal{B}(\mu_{\mathcal{Y}}^{(\sigma)})|0\rangle \right), \quad (4.11)$$

¹⁰Note that these are the only nonzero coefficients in such arguments.

Now, we use the known action of the operator Q_m^κ on a generic bulk state, as derived in Proposition 4.1 of [14], to write:

$$Q_m^\kappa B(\mu_{\mathcal{Y}}^{(\sigma)})|0\rangle = \sum_{n=0}^{l(m,a)} \sum_{\substack{\mathcal{Y}=\mathcal{Y}_+\cup\mathcal{Y}_- \\ \xi_{I_m}=\xi_{\gamma_+}\cup\xi_{\gamma_-} \\ |\gamma_+|=|\mathcal{Y}_+|=n}} R_n^\kappa(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\mathcal{Y}_+}^{(\sigma)}|\mu_{\mathcal{Y}_-}^{(\sigma)})B(\mu_{\mathcal{Y}_-}^{(\sigma)}\cup\xi_{\gamma_+})|0\rangle, \quad (4.12)$$

then by expanding the coefficients, we get:

$$[\mu_{\mathcal{X}}^2 - (\mu_{\mathcal{Y}}^{(\sigma)} - \eta)^2]H_{(\sigma_{\mathcal{Y}})}^{\mathcal{B}_+}(\mu_{\mathcal{Y}})R_n^\kappa(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\mathcal{Y}_+}^{(\sigma)}|\mu_{\mathcal{Y}_-}^{(\sigma)}) \times \left(\frac{\xi_{\gamma_+}^{(0)} + \mu_{\mathcal{Y}_-}^{(\sigma)} - \eta}{\xi_{\gamma_+}^{(0)} + \mu_{\mathcal{Y}_-}^{(\sigma)}} \right)^{\pm 1} \quad (4.13)$$

$$\begin{aligned} &= \bar{S}_n^\kappa(\xi_{\gamma_+}|\mu_{\mathcal{Y}_+}^{(\sigma)}) \frac{a(\mu_{\mathcal{Y}_+}^{(\sigma)})}{a(\xi_{\gamma_+})} \frac{\mu_{\mathcal{Y}_-}^2 - (\mu_{\mathcal{Y}_+}^{(\sigma)} - \eta)^2}{\mu_{\mathcal{Y}_-}^2 - \mu_{\mathcal{Y}_+}^2} \frac{\mu_{\mathcal{Y}_-}^2 - \xi_{\gamma_+}^{(0)2}}{\mu_{\mathcal{Y}_-}^2 - \xi_{\gamma_+}^{(1)2}} \frac{\xi_{\gamma_+} - \xi_{\gamma_-} + \eta}{\xi_{\gamma_+} - \xi_{\gamma_-}} \frac{H_{(\sigma_{\mathcal{Y}_+})}^{\mathcal{B}_+}(\mu_{\mathcal{Y}_+})}{H^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)})} \\ &\times \frac{\mu_{\mathcal{Y}_+}^{(\sigma)} - \xi_{\gamma_+}^{(0)}}{\mu_{\mathcal{Y}_+}^{(\sigma)} - \xi_{\gamma_+}^{(1)}} [\mu_{\mathcal{X}}^2 - (\mu_{\mathcal{Y}_+}^{(\sigma)} - \eta)^2] [\mu_{\mathcal{X}}^2 - (\mu_{\mathcal{Y}_-}^{(\sigma)} - \eta)^2] H_{(1_{\gamma_+}\cup\sigma_{\mathcal{Y}_-})}^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)}\cup\mu_{\mathcal{Y}_-}). \end{aligned} \quad (4.14)$$

We are then free to split the sum over $\sigma_{\mathcal{Y}} = \pm$ in sum over $\sigma_{\mathcal{Y}_+} = \pm$ and $\sigma_{\mathcal{Y}_-} = \pm$ and reverse the order of the sum in the following way:

$$\begin{aligned} Q_m^\kappa \bar{B}_+(\mu_{I_M})|0\rangle &= \sum_{a=0}^M \sum_{\substack{\mathcal{X}\cup\mathcal{Y}=I_M \\ \mathcal{X}\cap\mathcal{Y}=\emptyset, |\mathcal{Y}|=a}} \frac{\bar{b}_+(\mu_{\mathcal{X}})d(\mu_{\mathcal{X}})d(-\mu_{\mathcal{X}})}{\mu_{\mathcal{X}}^2 - \mu_{\mathcal{Y}}^2} \sum_{n=0}^{l(m,a)} \sum_{\substack{\mathcal{Y}=\mathcal{Y}_+\cup\mathcal{Y}_- \\ \xi_{I_m}=\xi_{\gamma_+}\cup\xi_{\gamma_-} \\ |\gamma_+|=|\mathcal{Y}_+|=n}} \\ &\left\{ \sum_{\sigma_{\mathcal{Y}_+}=\pm} [\mu_{\mathcal{X}}^2 - (\mu_{\mathcal{Y}_+}^{(\sigma)} - \eta)^2] \bar{S}_n^\kappa(\xi_{\gamma_+}|\mu_{\mathcal{Y}_+}^{(\sigma)}) \frac{a(\mu_{\mathcal{Y}_+}^{(\sigma)})}{a(\xi_{\gamma_+})} \frac{H_{(\sigma_{\mathcal{Y}_+})}^{\mathcal{B}_+}(\mu_{\mathcal{Y}_+})}{H^{\mathcal{B}_+}(\xi_{\gamma_+})} \right. \\ &\times \left. \frac{\mu_{\mathcal{Y}_-}^2 - (\mu_{\mathcal{Y}_+}^{(\sigma)} - \eta)^2}{\mu_{\mathcal{Y}_-}^2 - \mu_{\mathcal{Y}_+}^2} \frac{\xi_{\gamma_+} - \xi_{\gamma_-} + \eta}{\xi_{\gamma_+} - \xi_{\gamma_-}} \frac{\mu_{\mathcal{Y}_+}^{(\sigma)} - \xi_{\gamma_+}^{(0)}}{\mu_{\mathcal{Y}_+}^{(\sigma)} - \xi_{\gamma_+}^{(1)}} \right\} \frac{\mu_{\mathcal{Y}_-}^2 - \xi_{\gamma_+}^{(0)2}}{\mu_{\mathcal{Y}_-}^2 - \xi_{\gamma_+}^{(1)2}} \\ &\times \left\{ \sum_{\sigma_{\mathcal{Y}_-}=\pm} [\mu_{\mathcal{X}}^2 - (\mu_{\mathcal{Y}_-}^{(\sigma)} - \eta)^2] H_{(1_{\gamma_+}\cup\sigma_{\mathcal{Y}_-})}^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)}\cup\mu_{\mathcal{Y}_-}) B(\xi_{\gamma_+}^{(0)}\cup\mu_{\mathcal{Y}_-}^{(\sigma)})|0\rangle \right\}, \end{aligned} \quad (4.15)$$

and we can, moreover, multiply each term for:

$$1 = \frac{\mu_{\mathcal{X}}^2 - \xi_{\gamma_+}^{(0)2}}{\mu_{\mathcal{X}}^2 - \xi_{\gamma_+}^{(1)2}} \frac{\mu_{\mathcal{X}}^2 - \xi_{\gamma_+}^{(1)2}}{\mu_{\mathcal{X}}^2 - \xi_{\gamma_+}^{(0)2}}, \quad (4.16)$$

so that the previous expansion take the following form:

$$\begin{aligned}
Q_m^\kappa \bar{\mathcal{B}}_+(\mu_{I_M})|0\rangle &= \sum_{a=0}^M \sum_{\substack{\mathcal{X} \cup \mathcal{Y} = I_M \\ \mathcal{X} \cap \mathcal{Y} = \emptyset, |\mathcal{Y}| = a}} \frac{\bar{b}_+(\mu_{\mathcal{X}}) d(\mu_{\mathcal{X}}) d(-\mu_{\mathcal{X}})}{\mu_{\mathcal{X}}^2 - (\xi_{\gamma_+}^{(0)} \cup \mu_{\mathcal{Y}_-})^2} \sum_{n=0}^{l(m,a)} \sum_{\substack{\mathcal{Y} = \mathcal{Y}_+ \cup \mathcal{Y}_- \\ \xi_{I_m} = \xi_{\gamma_+} \cup \xi_{\gamma_-} \\ |\gamma_+| = |\mathcal{Y}_+| = n}} \\
&\left\{ \sum_{\sigma_{\mathcal{Y}_-} = \pm} [\mu_{\mathcal{X}}^2 - (\xi_{\gamma_+}^{(1)} \cup (\mu_{\mathcal{Y}_-}^{(\sigma)} - \eta))^2] H_{(1_{\gamma_+} \cup \sigma_{\mathcal{Y}_-})}^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)} \cup \mu_{\mathcal{Y}_-}) B(\xi_{\gamma_+}^{(0)} \cup \mu_{\mathcal{Y}_-}^{(\sigma)})|0\rangle \right\} \\
&\left\{ \sum_{\sigma_{\mathcal{Y}_+} = \pm} \frac{(\mu_{\mathcal{X}} \cup \mu_{\mathcal{Y}_-})^2 - (\mu_{\mathcal{Y}_+}^{(\sigma)} - \eta)^2}{(\mu_{\mathcal{X}} \cup \mu_{\mathcal{Y}_-})^2 - \mu_{\mathcal{Y}_+}^2} \bar{S}_n^\kappa(\xi_{\gamma_+} | \mu_{\mathcal{Y}_+}^{(\sigma)}) \frac{a(\mu_{\mathcal{Y}_+}^{(\sigma)})}{a(\xi_{\gamma_+}^{(0)})} \frac{H_{(\sigma_{\mathcal{Y}_+})}^{\mathcal{B}_+}(\mu_{\mathcal{Y}_+})}{H^{\mathcal{B}_+}(\xi_{\gamma_+})} \right. \\
&\times \left. \frac{\xi_{\gamma_+} - \xi_{\gamma_-} + \eta \mu_{\mathcal{Y}_+}^{(\sigma)} - \xi_{\gamma_+}^{(0)}}{\xi_{\gamma_+} - \xi_{\gamma_-}} \frac{\mu_{\mathcal{Y}_+}^{(\sigma)} - \xi_{\gamma_+}^{(0)}}{\mu_{\mathcal{Y}_+}^{(\sigma)} - \xi_{\gamma_+}^{(1)}} \frac{(\mu_{\mathcal{X}} \cup \mu_{\mathcal{Y}_-})^2 - \xi_{\gamma_+}^{(0)2}}{(\mu_{\mathcal{X}} \cup \mu_{\mathcal{Y}_-})^2 - \xi_{\gamma_+}^{(1)2}} \right\}. \tag{4.17}
\end{aligned}$$

Let us now remark that the factor associated to the sum over $\sigma_{\mathcal{Y}_+}$ depends only by the variable on the sets $\mathcal{Y}_+, \bar{\mathcal{Y}}_- = \mathcal{Y}_- \cup \mathcal{X}, \gamma_-$ and γ_+ , i.e. it does not distinguish between $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}_-}$ so that we are free to rewrite our result as it follows:

$$\begin{aligned}
Q_m^\kappa \bar{\mathcal{B}}_+(\mu_{I_M})|0\rangle &= \sum_{n=0}^{l(m,M)} \sum_{\substack{\mathcal{Y}_+ \cup \bar{\mathcal{Y}}_- = I_M \\ \xi_{I_m} = \xi_{\gamma_+} \cup \xi_{\gamma_-} \\ |\gamma_+| = |\mathcal{Y}_+| = n}} \left\{ \sum_{\sigma_{\mathcal{Y}_+} = \pm} \frac{\mu_{\bar{\mathcal{Y}}_-}^2 - (\mu_{\mathcal{Y}_+}^{(\sigma)} - \eta)^2}{\mu_{\bar{\mathcal{Y}}_-}^2 - \mu_{\mathcal{Y}_+}^2} \bar{S}_n^\kappa(\xi_{\gamma_+} | \mu_{\mathcal{Y}_+}^{(\sigma)}) \frac{a(\mu_{\mathcal{Y}_+}^{(\sigma)})}{a(\xi_{\gamma_+}^{(0)})} \right. \\
&\times \left. \frac{H_{(\sigma_{\mathcal{Y}_+})}^{\mathcal{B}_+}(\mu_{\mathcal{Y}_+})}{H^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)})} \frac{\xi_{\gamma_+} - \xi_{\gamma_-} + \eta \mu_{\mathcal{Y}_+}^{(\sigma)} - \xi_{\gamma_+}^{(0)}}{\xi_{\gamma_+} - \xi_{\gamma_-}} \frac{\mu_{\bar{\mathcal{Y}}_-}^2 - \xi_{\gamma_+}^{(0)2}}{\mu_{\bar{\mathcal{Y}}_-}^2 - \xi_{\gamma_+}^{(1)2}} \right\} \left\{ \sum_{a=0}^{M-n} \sum_{\substack{\mathcal{X} \cup \mathcal{Y}_- = \bar{\mathcal{Y}}_- \\ \mathcal{X} \cap \mathcal{Y}_- = \emptyset, |\mathcal{Y}_-| = a}} \right. \\
&\times \left. \frac{\bar{b}_+(\mu_{\mathcal{X}}) d(\mu_{\mathcal{X}}) d(-\mu_{\mathcal{X}})}{\mu_{\mathcal{X}}^2 - (\xi_{\gamma_+}^{(0)} \cup \mu_{\mathcal{Y}_-})^2} \sum_{\sigma_{\mathcal{Y}_-} = \pm} [\mu_{\mathcal{X}}^2 - (\xi_{\gamma_+}^{(1)} \cup (\mu_{\mathcal{Y}_-}^{(\sigma)} - \eta))^2] H_{(1_{\gamma_+} \cup \sigma_{\mathcal{Y}_-})}^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)} \cup \mu_{\mathcal{Y}_-}) \right. \\
&\left. B(\xi_{\gamma_+}^{(0)} \cup \mu_{\mathcal{Y}_-}^{(\sigma)})|0\rangle \right\}. \tag{4.18}
\end{aligned}$$

From which we get our result just remarking the identities:

$$\begin{aligned}
\mathcal{R}_n^\kappa(\xi_{\gamma_+} | \xi_{\gamma_-} | \mu_{\mathcal{Y}_+} | \mu_{\bar{\mathcal{Y}}_-}) &= \sum_{\sigma_{\mathcal{Y}_+} = \pm} \frac{\mu_{\bar{\mathcal{Y}}_-}^2 - (\mu_{\mathcal{Y}_+}^{(\sigma)} - \eta)^2}{\mu_{\bar{\mathcal{Y}}_-}^2 - \mu_{\mathcal{Y}_+}^2} \bar{S}_n^\kappa(\xi_{\gamma_+} | \mu_{\mathcal{Y}_+}^{(\sigma)}) \frac{a(\mu_{\mathcal{Y}_+}^{(\sigma)})}{a(\xi_{\gamma_+}^{(0)})} \frac{H_{(\sigma_{\mathcal{Y}_+})}^{\mathcal{B}_+}(\mu_{\mathcal{Y}_+})}{H^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)})} \\
&\times \frac{\xi_{\gamma_+} - \xi_{\gamma_-} + \eta \mu_{\mathcal{Y}_+}^{(\sigma)} - \xi_{\gamma_+}^{(0)}}{\xi_{\gamma_+} - \xi_{\gamma_-}} \frac{\mu_{\bar{\mathcal{Y}}_-}^2 - \xi_{\gamma_+}^{(0)2}}{\mu_{\bar{\mathcal{Y}}_-}^2 - \xi_{\gamma_+}^{(1)2}} \tag{4.19}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathcal{B}}_+(\xi_{\gamma_+} \cup \mu_{\bar{\mathcal{Y}}_-})|0\rangle &= \sum_{a=0}^{M-n} \sum_{\substack{\mathcal{X} \cup \mathcal{Y}_- = \bar{\mathcal{Y}}_- \\ \mathcal{X} \cap \mathcal{Y}_- = \emptyset, |\mathcal{Y}_-| = a}} \frac{\bar{b}_+(\mu_{\mathcal{X}}) d(\mu_{\mathcal{X}}) d(-\mu_{\mathcal{X}})}{\mu_{\mathcal{X}}^2 - (\xi_{\gamma_+}^{(0)} \cup \mu_{\mathcal{Y}_-})^2} \\
&\times \sum_{\sigma_{\mathcal{Y}_-} = \pm} [\mu_{\mathcal{X}}^2 - (\xi_{\gamma_+}^{(1)} \cup (\mu_{\mathcal{Y}_-}^{(\sigma)} - \eta))^2] H_{(1_{\gamma_+} \cup \sigma_{\mathcal{Y}_-})}^{\mathcal{B}_+}(\xi_{\gamma_+}^{(0)} \cup \mu_{\mathcal{Y}_-}) B(\xi_{\gamma_+}^{(0)} \cup \mu_{\mathcal{Y}_-}^{(\sigma)})|0\rangle, \tag{4.20}
\end{aligned}$$

as all the terms which correspond to a choice of $\bar{\mu}_\mathcal{X} \subset \{\pm \xi_{\gamma_+}^{(0)} \cup \pm \mu_{\bar{y}_-}\}$ with $\bar{\mu}_\mathcal{X} \cap \xi_{\gamma_+}^{(0)} \neq \emptyset$ are zero being $d(\bar{\mu}_\mathcal{X})d(-\bar{\mu}_\mathcal{X}) = 0$. \square

4.2 Action of local operators

Here, we present the action of a basis of local operators at the generic site m of the chain on the boundary separate states associated to the transfer matrix $\bar{T}(\lambda)$. In order, to do so we introduce some further notations:

$$\begin{aligned} \mathcal{R}_n^{2,2}(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}|\mu_a|\mu_{\alpha_-}) &= \sum_{\sigma_{\alpha_+}, \sigma_a = \pm} \frac{\eta \tau(\mu_a^\sigma | \pm \mu_{I_N - \{a\}} \cup \xi_{I_m}^{(1)})}{(\mu_a^\sigma - \xi_{m+1}^{(1)}) \tau(\xi_{m+1}^{(0)} | \pm \mu_{I_N - \{a\}} \cup \xi_{I_m}^{(1)})} \\ &\times \frac{H_{\sigma_{\alpha_+}}^{\mathcal{B}^+}(\mu_{\alpha_+}) H_{\sigma_a}^{\mathcal{B}^+}(\mu_a)}{H^{\mathcal{B}^+}(\xi_{\gamma_+}^{(0)}) H^{\mathcal{B}^+}(\xi_{m+1}^{(0)})} R(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}^\sigma | \pm \mu_{\alpha_-}) S_n^{2,2}(\xi_{\gamma_+}|\mu_{\alpha_+}^\sigma | \mu_a^\sigma), \end{aligned} \quad (4.21)$$

$$\mathcal{R}_n^-(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}|\mu_{\alpha_-}) = \frac{\sum_{\sigma_{\alpha_+} = \pm} H_{\sigma_{\alpha_+}}^{\mathcal{B}^+}(\mu_{\alpha_+}) R(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}^\sigma | \pm \mu_{\alpha_-}) S_n^-(\xi_{\gamma_+}|\mu_{\alpha_+}^\sigma)}{H^{\mathcal{B}^+}(\xi_{\gamma_+}^{(0)}) H^{\mathcal{B}^+}(\xi_{m+1}^{(0)}) \tau(\xi_{m+1}^{(0)} | \pm \mu_{I_N} \cup \xi_{I_m}^{(1)})}, \quad (4.22)$$

and

$$\begin{aligned} \mathcal{R}_n^+(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}|\mu_a|\mu_p|\mu_{\alpha_-}) &= \sum_{\sigma_{\alpha_+}, \sigma_a, \sigma_p = \pm} R(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}^\sigma | \pm \mu_{\alpha_-}) S_n^+(\xi_{\gamma_+}|\mu_{\alpha_+}^\sigma | \mu_a^\sigma | \mu_p^\sigma) \\ &\times \frac{H_{\sigma_{\alpha_+}}^{\mathcal{B}^+}(\mu_{\alpha_+}) H_{\sigma_a}^{\mathcal{B}^+}(\mu_a) H_{\sigma_p}^{\mathcal{B}^+}(\mu_p)}{H^{\mathcal{B}^+}(\xi_{\gamma_+}^{(0)}) H^{\mathcal{B}^+}(\xi_{m+1}^{(0)})} \frac{\eta^2 \tau(\mu_a^\sigma | \pm \mu_{I_N - \{a\}} \cup \xi_{I_m}^{(1)}) \tau(\mu_p^\sigma | \pm \mu_{I_{N+1} - \{a,p\}} \cup \xi_{I_m}^{(1)})}{(\mu_a^\sigma - \xi_{m+1}^{(1)}) (\xi_{m+1}^{(0)} - \mu_p^\sigma + \eta) \tau(\xi_{m+1}^{(0)} | \pm \mu_{I_N - \{a\}} \cup \xi_{I_m}^{(1)})}, \end{aligned} \quad (4.23)$$

where

$$S_n^x = \frac{\xi_{\gamma_+}^{(0)} - \mu_{\alpha_+} + \eta}{\prod_{\substack{a>b \\ a,b \in \gamma_+}} \xi_{ab} \prod_{\substack{a>b \\ a,b \in \alpha_+}} \mu_{ba}} \det_{k \in \gamma_+, j \in \alpha_+} M_{jk}^x, \quad (4.24)$$

with the $n \times n$ matrix M^x defined by:

$$M_{jk}^{2,2} = r(\xi_k^{(0)}|\mu_j) - (1 - \delta_{j,N+1}) r(\mu_j|\xi_k^{(0)}) \frac{f(\mu_{\alpha_+ - \{j\}} \cup \mu_a | \mu_j) f(\mu_j | \xi_{\gamma_+}^{(0)} \cup \xi_{m+1}^{(0)})}{f(\mu_j | \mu_{\alpha_+ - \{j\}} \cup \mu_a) f(\xi_{\gamma_+}^{(0)} \cup \xi_{m+1}^{(0)} | \mu_j)}, \quad (4.25)$$

$$M_{jk}^- = r(\xi_k^{(0)}|\mu_j) - (1 - \delta_{j,N+1}) r(\mu_j|\xi_k^{(0)}) \frac{f(\mu_{\alpha_+ - \{j\}} | \mu_j) f(\mu_j | \xi_{\gamma_+}^{(0)} \cup \xi_{m+1}^{(0)})}{f(\mu_j | \mu_{\alpha_+ - \{j\}}) f(\xi_{\gamma_+}^{(0)} \cup \xi_{m+1}^{(0)} | \mu_j)}, \quad (4.26)$$

$$M_{jk}^+ = r(\xi_k^{(0)}|\mu_j) - (1 - \delta_{j,N+1}) r(\mu_j|\xi_k^{(0)}) \frac{f(\mu_{\alpha_+ - \{j\}} \cup \mu_a \cup \mu_p | \mu_j) f(\mu_j | \xi_{\gamma_+}^{(0)} \cup \xi_{m+1}^{(0)})}{f(\mu_j | \mu_{\alpha_+ - \{j\}} \cup \mu_a \cup \mu_p) f(\xi_{\gamma_+}^{(0)} \cup \xi_{m+1}^{(0)} | \mu_j)}. \quad (4.27)$$

Then, the following proposition holds:

Proposition 4.2. *The action of the following local operators on boundary separate states read:*

$$E_{m+1}^{2,2} \bar{\mathcal{B}}_+(\mu_{I_M}) |0\rangle = \sum_{a=1}^M \sum_{n=0}^{l(m,M)} \sum_{\substack{I_{M+1}=\alpha_+ \cup \alpha_- \cup \{a\}, \\ \{a\} \cap (\alpha_+ \cup \alpha_-) = \emptyset, \alpha_+ \cap \alpha_- = \emptyset \\ I_m = \gamma_+ \cup \gamma_-, \gamma_+ \cap \gamma_- = \emptyset \\ |\gamma_+| = |\alpha_+| = n}} \mathcal{R}_n^{2,2}(\xi_{\gamma_+} | \xi_{\gamma_-} | \mu_{\alpha_+} | \mu_a | \mu_{\alpha_-}) \\ \times \bar{\mathcal{B}}_+(\mu_{\alpha_-} \cup \xi_{\gamma_+}^{(0)}) |0\rangle, \quad (4.28)$$

$$\sigma_{m+1}^- \bar{\mathcal{B}}_+(\mu_{I_M}) |0\rangle = \sum_{n=0}^{l(m,M)} \sum_{\substack{I_{M+1}=\alpha_+ \cup \alpha_-, \alpha_+ \cap \alpha_- = \emptyset \\ I_m = \gamma_+ \cup \gamma_-, \gamma_+ \cap \gamma_- = \emptyset \\ |\gamma_+| = |\alpha_+| = n}} \mathcal{R}_n^-(\xi_{\gamma_+} | \xi_{\gamma_-} | \mu_{\alpha_+} | \mu_{\alpha_-}) \\ \times \bar{\mathcal{B}}_+(\mu_{\alpha_-} \cup \xi_{\gamma_+}^{(0)}) |0\rangle, \quad (4.29)$$

and

$$\sigma_{m+1}^+ \bar{\mathcal{B}}_+(\mu_{I_M}) |0\rangle = \sum_{a=1}^M \sum_{\substack{p=1 \\ p \neq a}}^{M+1} \sum_{n=0}^{l(m,M)} \sum_{\substack{I_{M+1}=\alpha_+ \cup \alpha_- \cup \{a\} \cup \{p\}, \\ \{a,p\} \cap (\alpha_+ \cup \alpha_-) = \emptyset, \alpha_+ \cap \alpha_- = \emptyset \\ I_m = \gamma_+ \cup \gamma_-, \gamma_+ \cap \gamma_- = \emptyset \\ |\gamma_+| = |\alpha_+| = n}} \mathcal{R}_n^+(\xi_{\gamma_+} | \xi_{\gamma_-} | \mu_{\alpha_+} | \mu_a | \mu_p | \mu_{\alpha_-}) \\ \times \bar{\mathcal{B}}_+(\mu_{\alpha_-} \cup \xi_{\gamma_+}^{(0)}) |0\rangle, \quad (4.30)$$

where we have defined $\mu_{M+1} = \xi_{m+1}$.

Proof. The proof of this proposition is done exactly along the same lines of the previous one, we have to use the boundary-bulk decomposition of the boundary separate states and then using the bulk formulae of Proposition 4.2 of our paper [14] for the action of these local operators on bulk states is computed. \square

These formulae allows the computation of all one-point functions and we can use them to derive the action of a monomial of two local operators, for example one at the site 1 and one at the generic site m of the chain, here we present one instance of this:

Corollary 4.1. *The previous proposition implies the following form of the right action of $\sigma_{1+m}^+ \sigma_1^-$ on an arbitrary separate state:*

$$\sigma_{1+m}^+ \sigma_1^- \bar{\mathcal{B}}_+(\mu_{I_M}) |0\rangle = \sum_{a=1}^M \sum_{\substack{p=1 \\ p \neq a}}^{M+1} \sum_{n=0}^{m-1} \sum_{\substack{I_{M+1}=\alpha_+ \cup \alpha_- \cup \{a\} \cup \{p\}, \\ \{\alpha_+ \cup \alpha_-\} \cap \{a,p\} = \emptyset, \alpha_+ \cap \alpha_- = \emptyset \\ \{2, \dots, m\} = \gamma_+ \cup \gamma_-, \gamma_+ \cap \gamma_- = \emptyset \\ |\gamma_+| = |\alpha_+| = n}} \bar{\mathcal{B}}_+(\mu_{\alpha_-} \cup \xi_1^{(0)} \cup \xi_{\gamma_+}^{(0)}) |0\rangle \\ \times \mathcal{R}_n^{+-}(\xi_1 | \xi_{\gamma_+} | \xi_{\gamma_-} | \mu_{\alpha_+} | \mu_a | \mu_p | \mu_{\alpha_-}), \quad (4.31)$$

where:

$$\begin{aligned}
\mathcal{R}_n^{+-}(\xi_1|\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}|\mu_a|\mu_p|\mu_{\alpha_-}) &= \sum_{\sigma_{\alpha_+}, \sigma_a, \sigma_p = \pm} \frac{H_{\sigma_{\alpha_+}}^{\mathcal{B}_+}(\mu_{\alpha_+})H_{\sigma_a}^{\mathcal{B}_+}(\mu_a)H_{\sigma_p}^{\mathcal{B}_+}(\mu_p)}{H^{\mathcal{B}_+}(\xi_1^{(0)} \cup \xi_{\gamma_+}^{(0)})H^{\mathcal{B}_+}(\xi_{m+1}^{(0)})} \\
&\times \frac{R(\xi_{\gamma_+}|\xi_{\gamma_-}|\mu_{\alpha_+}^\sigma|\pm\mu_{\alpha_-})S_n^+(\xi_{\gamma_+}|\mu_{\alpha_+}^\sigma|\mu_a^\sigma|\mu_p^\sigma)\eta^2}{\tau(\xi_1^{(0)}|\pm\mu_{\alpha_-} \cup \mu_{\alpha_+}^\sigma)(\mu_a^\sigma - \xi_{m+1}^{(1)})(\xi_{m+1}^{(0)} - \mu_p^\sigma + \eta)} \\
&\times \frac{\tau(\mu_a^\sigma|\pm\mu_{I_M-\{a\}} \cup \xi_{I_m}^{(1)})\tau(\mu_p^\sigma|\pm\mu_{I_{M+1}-\{a,p\}} \cup \xi_{I_m}^{(1)})}{\tau(\xi_{m+1}^{(0)}|\pm\mu_{I_M-\{a\}} \cup \xi_{I_m}^{(1)})}, \tag{4.32}
\end{aligned}$$

and $\mu_{M+1} = \xi_{m+1}$.

4.3 Action of the quasi-local generators of elementary blocks

Let us present now the action of the basis $\prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j}$ of quasi-local operators from site one to the generic site m of the chain on the boundary separate states associated to the the transfer matrix $\bar{\mathcal{T}}(\lambda)$. Following the exposition of our previous paper [13], for any element $\prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j}$ of this basis, we can introduce the following sets of integers:

$$\{i_p\}_{p \in \{1, \dots, s\}}, \text{ with } i_k < i_h \text{ for } 0 < k < h \leq s, \tag{4.33}$$

$$\{i_p\}_{p \in \{s+1, \dots, s+s'\}}, \text{ with } i_k > i_h \text{ for } s < k < h \leq s+s', \tag{4.34}$$

defined by the conditions $\varepsilon_j = 1$ iff $j \in \{i_p\}_{p \in \{s+1, \dots, s+s'\}}$ and $\varepsilon'_j = 2$ iff $j \in \{i_p\}_{p \in \{1, \dots, s\}}$. Then, we can formulate the following proposition:

Proposition 4.3. *Taken the generic element of the quasi-local basis $\prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j}$, then its action on a boundary separate state reads:*

$$\prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \bar{\mathcal{B}}_+(\mu_{I_M})|0\rangle = \sum_{\beta_{s+s'}} \mathcal{F}_{\beta_{s+s'}}^+(\mu_{I_{M+m}}) \bar{\mathcal{B}}_+(\mu_{I_{M+m} \setminus \beta_{s+s'}})|0\rangle, \tag{4.35}$$

here, we have defined $\mu_{M+j} := \xi_{m+1-j}^{(0)}$ for $j \in \{1, \dots, m\}$, the sum run over all the possible sets of integers $\beta_{s+s'} = \{b_1, \dots, b_{s+s'}\}$ whose elements satisfy the conditions

$$\begin{cases} b_p \in \{1, \dots, M\} \setminus \{b_1, \dots, b_{p-1}\} & \text{for } 0 < p \leq s, \\ b_p \in \{1, \dots, M+m+1-i_p\} \setminus \{b_1, \dots, b_{p-1}\} & \text{for } s < p \leq s+s', \end{cases} \tag{4.36}$$

and the coefficient reads:

$$\begin{aligned}
\mathcal{F}_{\beta_{s+s'}}^+(\mu_{I_{M+m}}) &= \sum_{\sigma_{\alpha_+} = \pm} \frac{a(\mu_{\beta_{s+s'}}^\sigma)}{a(\xi_{I_M}^{(0)})} \frac{H_{\sigma_{\alpha_+}}^{\mathcal{B}^+}(\mu_{\alpha_+})}{H^{\mathcal{B}^+}(\xi_{\gamma_+}^{(0)})} \frac{(\xi_{\gamma_+}^{(0)} - \mu_{\alpha_+}^\sigma)}{(\xi_{\gamma_+}^{(1)} - \mu_{\alpha_+}^\sigma)} \prod_{1 \leq i < j \leq s+s'} \frac{\mu_{b_i b_j}^\sigma}{\mu_{b_i b_j}^\sigma + \eta} \\
&\times \prod_{\epsilon = \pm} \left(\frac{(\mu_{\alpha_+}^\sigma + \epsilon \mu_{\alpha_-} - \eta)(\xi_{\gamma_+}^{(0)} + \epsilon \mu_{\alpha_-})}{(\mu_{\alpha_+}^\sigma + \epsilon \mu_{\alpha_-})(\xi_{\gamma_+}^{(1)} + \epsilon \mu_{\alpha_-})} \right) \prod_{i \in \alpha_+} \frac{\prod_{j \in \alpha_+} (\mu_{j_i}^\sigma + \eta)}{\prod_{j \in \alpha_+ - \{i\}} (\mu_{j_i}^\sigma)} \\
&\times \prod_{p=1}^s \frac{\prod_{k=i_p+1}^m (\mu_{b_p}^\sigma - \xi_k^{(1)})}{\prod_{k=i_p}^m (\mu_{b_p}^\sigma - \xi_k^{(0)})} \prod_{p=s+1}^{s+s'} \frac{\prod_{k=i_p+1}^m (\xi_k^{(0)} - \mu_{b_p}^\sigma + \eta)}{\prod_{\substack{k=i_p \\ k \neq N+m+1-b_p}}^m (\xi_k^{(0)} - \mu_{b_p}^\sigma)}. \tag{4.37}
\end{aligned}$$

The sum is over all $\sigma_h \in \{+, -\}$ for $h \in \alpha_+$ and we have defined $\mu_j^\sigma := \sigma_j \mu_j$ for $j \in \beta_{s+s'}$, with $\sigma_j = 1$ if $j > M$, and

$$\gamma_- = \{M + m + 1 - j\}_{j \in \beta_{s+s'} \cap \{M+1, \dots, M+m\}}, \quad \gamma_+ = \{1, \dots, m\} \setminus \gamma_-, \tag{4.38}$$

$$\alpha_+ = \beta_{s+s'} \cap \{1, \dots, M\}, \alpha_- = \{1, \dots, M\} \setminus \alpha_+. \tag{4.39}$$

Proof. Here, we have to use the boundary-bulk decomposition of the boundary separate states and then we have to use the following formula:

$$\prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} B(\mu_{I_M}) |0\rangle = \sum_{\beta_{s+s'}} \mathcal{F}_{\beta_{s+s'}}(\mu_{I_{M+m}}) B(\mu_{I_{M+m} \setminus \beta_{s+s'}}) |0\rangle, \tag{4.40}$$

for these actions on bulk states, where the coefficient \mathcal{F}_{β_m} reads:

$$\begin{aligned}
\mathcal{F}_{\beta_{s+s'}}(\mu_{I_{M+m}}) &= \frac{a(\mu_{\beta_{s+s'}}^\sigma)}{a(\xi_{I_M}^{(0)})} \prod_{j=1}^{s+s'} \frac{\prod_{k=1}^M (\mu_k b_j + \eta)}{\prod_{\substack{k=1 \\ k \neq b_j}}^M \mu_k b_j} \frac{\mu_{I_M} - \xi_{I_m}^{(0)}}{\mu_{I_M} - \xi_{I_m}^{(1)}} \prod_{1 \leq i < j \leq m} \frac{\mu_{b_i b_j}}{\mu_{b_i b_j} + \eta} \\
&\times \prod_{p=1}^s \frac{\prod_{k=i_p+1}^m (\mu_{b_p} - \xi_k + \eta)}{\prod_{k=i_p}^m (\mu_{b_p} - \xi_k)} \prod_{p=s+1}^{s+s'} \frac{\prod_{k=i_p+1}^m (\xi_k - \mu_{b_p} + \eta)}{\prod_{\substack{k=i_p \\ k \neq N+m+1-b_p}}^m (\xi_k - \mu_{b_p})}, \tag{4.41}
\end{aligned}$$

as proven in Proposition 5.1 of our paper [13]. The proof of this proposition is done exactly along the same lines of the Proposition 4.1, \square

5 Scalar products of separate states

Here, we present the scalar products of boundary separate states with the transfer matrix eigenstates directly in the framework for which we want to compute correlation functions, i.e. for the case

$$\bar{c}_- = 0, \quad \bar{b}_+ \neq 0, \tag{5.1}$$

where our original transfer matrix $\mathcal{T}(\lambda)$ is associated to unparallel boundary magnetic fields and it is isospectral to the transfer matrix $\hat{\mathcal{T}}(\lambda)$ associated to parallel boundary magnetic fields. Here, we first rewrite the known scalar products results, derived in our previous paper [76] on the XXX antiperiodic chain, and then we analyze them in the thermodynamic limit for the case of the ground state thanks to the knowledge of its density root distribution achieved thanks to the derived isospectrality.

5.1 The scalar product of separate states with transfer matrix eigenstates

In order to compute correlation functions, we have to be able to compute the following type of ratios of scalar products of separate states with the transfer matrix eigenstates:

$$\frac{\langle \bar{t} | \bar{\mathcal{B}}_+(v_\gamma \cup \mu_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\mu_{I_q}) | 0 \rangle}, \text{ with } \alpha \subset I_q, \gamma \subset I_m, \quad (5.2)$$

where $\langle \bar{t} | (= \langle t | \Gamma_W^{-1})$ is the unique (up normalization) eigenvector of $\bar{\mathcal{T}}(\lambda)$ associated to the eigenvalue $t(\lambda)$ solving with $Q_t(\lambda)$, λ^2 -polynomial of degree q and roots $\lambda_1^2, \dots, \lambda_q^2$, the homogeneous version

$$t(\lambda) Q_t(\lambda) = A_{\bar{\zeta}_+, \bar{\zeta}_-}(\lambda) Q_t(\lambda - \eta) + A_{\bar{\zeta}_+, \bar{\zeta}_-}(-\lambda) Q_t(\lambda + \eta) \quad (5.3)$$

of the Baxter TQ -equation (3.38). Denoted $\bar{\mu}_{I_{p+n}} = v_\gamma \cup \mu_\alpha$ with

$$\bar{\mu}_a = \mu_{\alpha_a} \text{ for } a \in \{1, \dots, p\}, \quad \bar{\mu}_{p+a} = \mu_{\gamma_a} \text{ for } a \in \{p+1, \dots, p+n\}, \quad (5.4)$$

$\gamma = \{\gamma_1, \dots, \gamma_n\}$ and $\alpha = \{\alpha_1, \dots, \alpha_p\}$, then the following proposition holds:

Proposition 5.1. *Let the inhomogeneity parameters ξ_1, \dots, ξ_N be generic (3.1),*

$$\bar{c}_- = 0, \quad \bar{b}_+ \neq 0, \quad (5.5)$$

and let $t(\lambda)$ be a generic $\bar{\mathcal{T}}$ -eigenvalue and $\langle \bar{t} |$ the associated unique (up normalization) $\bar{\mathcal{T}}$ -eigenvector, then, the following representations hold:

$$\begin{aligned} \frac{\langle \bar{t} | \bar{\mathcal{B}}_+(v_\gamma \cup \mu_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\mu_{I_q}) | 0 \rangle} &= 0, \text{ if } p+n < q, \quad (5.6) \\ \frac{\langle \bar{t} | \bar{\mathcal{B}}_+(v_\gamma \cup \mu_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\mu_{I_q}) | 0 \rangle} &= \frac{\Gamma((\bar{\zeta}_+ + \bar{\zeta}_-)/\eta + N - (q+p+n)) Q_t(v_\gamma) (4\mu_\beta^2 - \eta^2)}{\Gamma((\bar{\zeta}_+ + \bar{\zeta}_-)/\eta + N - 2q) (4v_\gamma^2 - \eta^2) Q_t(\mu_\beta)} \\ &\times \frac{\widehat{V}(\mu_\beta) \mu_\beta^2 - \mu_\alpha^2 \det_{p+n} \mathcal{S}_t(\lambda_{I_q} | \bar{\mu}_{I_{p+n}})}{\widehat{V}(v_\gamma) v_\gamma^2 - \mu_\alpha^2 \det_q \mathcal{S}_t(\lambda_{I_q} | \mu_{I_q})}, \text{ if } p+n \geq q, \quad (5.7) \end{aligned}$$

where we have defined the set $\beta = I_q \setminus \alpha$ and, if $p+n \geq q$, the $(p+n) \times (p+n)$ square matrix $\mathcal{S}_t(\lambda_{I_q} | \omega_{I_{p+n}})$ is defined by:

$$\mathcal{S}_t(\lambda_{I_q} | \omega_{I_{p+n}})_{i,k} = \begin{cases} \partial t(\omega_i) / \partial \lambda_k & \text{if } k \leq q, \\ \sum_{\epsilon \in \{+, -\}} \epsilon A_{\bar{\zeta}_+, \bar{\zeta}_-}(-\epsilon \omega_i) \frac{Q_t(\omega_i + \epsilon \eta)}{Q_t(\omega_i)} (\omega_i + \epsilon \frac{\eta}{2})^{2(k-q)-1} & \text{if } k > q. \end{cases} \quad (5.8)$$

Then, the following identity holds

$$\frac{\langle \bar{t} | \bar{\mathcal{B}}_+(v_\gamma \cup \mu_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\mu_{I_q}) | 0 \rangle} = \frac{\langle \hat{t} | \hat{\mathcal{B}}_+(v_\gamma \cup \mu_\alpha) | 0 \rangle}{\langle \hat{t} | \hat{\mathcal{B}}_+(\mu_{I_q}) | 0 \rangle}, \text{ if } p+n \leq q, \quad (5.9)$$

where $\langle \hat{t} |$ is the unique (up normalization) eigenvector associated to the eigenvalue $t(\lambda)$ of the transfer matrix $\hat{\mathcal{T}}(\lambda)$ defined in (3.56).

Proof. This proposition is a direct corollary of Theorem 4.2 of our previous paper [76], we have just to use the formula (4.59) there to compute this ratio. Note that we do not need to specify the normalization of the eigencovector $\langle \bar{t} |$ as it appear to numerator and denominator simultaneously. The second part of the proposition is then a corollary on the Slavnov's type formulae [146] for the scalar products of Bethe's like states [13, 147]. In particular, we can use the formula (4.9) of Theorem 4.1 of our paper [13] to compute the following ratio and derive the identity:

$$\frac{\langle \hat{t} | \hat{\mathcal{B}}_+(v_\gamma \cup \mu_\alpha) | 0 \rangle}{\langle \hat{t} | \hat{\mathcal{B}}_+(\mu_{I_q}) | 0 \rangle} = \delta_{q, M+n} \frac{\langle \bar{t} | \bar{\mathcal{B}}_+(v_\gamma \cup \mu_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\mu_{I_q}) | 0 \rangle}. \quad (5.10)$$

□

The above proposition also point out that the scalar products for the model associated to the diagonal boundary matrices $\hat{K}_-(\lambda)$ and $\hat{K}_+(\lambda)$ and those associated to the diagonal $\bar{K}_-(\lambda) = \hat{K}_-(\lambda)$ and triangular $\bar{K}_+(\lambda) \neq \hat{K}_+(\lambda)$ boundary matrices do not coincide for $p+n > q$ being in general:

$$\frac{\langle \bar{t} | \bar{\mathcal{B}}_+(v_\gamma \cup \mu_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\mu_{I_q}) | 0 \rangle} \neq 0 \text{ for } p+n > q. \quad (5.11)$$

Here, our main results will be to prove that if $t(\lambda)$ is the ground state for the Hamiltonian associated to $\bar{\mathcal{T}}(\lambda)$ these scalar products for $\lambda_{I_q} = \mu_{I_q}$ go to zero quickly enough to make their contribution to correlation functions zero in the thermodynamic and homogeneous limit. More precisely, the following result holds:

Proposition 5.2. *Let the inhomogeneity parameters ξ_1, \dots, ξ_N be generic (3.1),*

$$\bar{c}_- = 0, \quad \bar{b}_+ \neq 0, \quad (5.12)$$

and let us fix $t(\lambda)$ ($\bar{\mathcal{T}}$ -eigenvalue) and $Q_t(\lambda)$ (associated solution of the homogeneous Baxter's equation (5.3)) such that the Q_t -roots $\{\lambda_{I_q}\}$ are distributed on the positive real axis according to the ground state density:

$$\rho(\lambda) = \frac{i}{\sinh \pi \lambda}, \quad (5.13)$$

in the thermodynamic limits, then in this limit it holds:

$$\frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\xi_\gamma^{(0)} \cup \lambda_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = 0, \text{ if } p+n < q, \quad (5.14)$$

$$\frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\xi_\gamma^{(0)} \cup \lambda_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = o(1/N^{(q-p)}), \text{ if } p+n > q, \quad (5.15)$$

if $q-p$ finite in the thermodynamic limit, and finally if $p+n = q$:

$$\begin{aligned} \frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\xi_\gamma^{(0)} \cup \lambda_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} &= \frac{\lambda_\beta \lambda_\beta^{(1)} (\xi_\gamma + \eta) y(\xi_\gamma^{(0)}; \{\lambda_{I_q}\}; \bar{\zeta}_\pm)}{\xi_\gamma \xi_\gamma^{(0)} \lambda_\beta^{(0)} y(\lambda_\beta; \{\lambda_{I_q}\}; \bar{\zeta}_\pm)} \frac{\widehat{V}(\lambda_\beta)}{\widehat{V}(\xi_\gamma^{(0)})} \frac{\lambda_\beta^2 - \lambda_\alpha^2}{\xi_\gamma^{(0)2} - \lambda_\alpha^2} \\ &\times \det_n \left[\frac{\rho(\lambda_{\beta_l} - \xi_{\gamma_k}^{(1)}) - \rho(\lambda_{\beta_l} + \xi_{\gamma_k}^{(0)})}{2N\rho(\lambda_{\beta_l})} \right] + o(1/N^n), \end{aligned} \quad (5.16)$$

where, we have defined

$$y(\mu; \{\lambda\}; \bar{\zeta}_\pm) = a(\mu) d(-\mu) (\mu + \bar{\zeta}_+ - \eta/2) (\mu + \bar{\zeta}_- - \eta/2) Q_t(\mu - \eta) \quad (5.17)$$

and $\beta = \{\beta_1, \dots, \beta_n\} = I_q \setminus \alpha$.

Proof. Thanks to the previous proposition we already know the validity of the statement for $p+n < q$ while for $p+n = q$ this can be proven exactly along the same lines with which we derived it in the Bethe ansatz framework, see Section 4.4 of our previous paper [13]. So we are left with the proof in the case $n+p > q$.

Let us observe that we can write the result of the previous proposition for the case $n+p > q$ as it follows:

$$\begin{aligned} \frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\xi_\gamma^{(0)} \cup \lambda_\alpha) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} &= \frac{\lambda_\beta \lambda_\beta^{(1)} (\xi_\gamma + \eta) y(\xi_\gamma^{(0)}; \{\lambda_{I_q}\}; \bar{\zeta}_\pm) \widehat{V}(\lambda_\beta) \lambda_\beta^2 - \lambda_\alpha^2}{\xi_\gamma \xi_\gamma^{(0)} \lambda_\beta^{(0)} y(\lambda_\beta; \{\lambda_{I_q}\}; \bar{\zeta}_\pm) \widehat{V}(\xi_\gamma^{(0)}) \xi_\gamma^{(0)2} - \lambda_\alpha^2} \\ &\times \frac{\Gamma((\bar{\zeta}_+ + \bar{\zeta}_-)/\eta + N - (q+p+n)) \det_{n+p} \mathcal{M}_t(\lambda_{I_q} | \xi_\gamma^{(0)} \cup \lambda_\alpha)}{\Gamma((\bar{\zeta}_+ + \bar{\zeta}_-)/\eta + N - 2q) \det_q \mathcal{N}_t(\lambda_{I_q})}, \end{aligned} \quad (5.18)$$

where \mathcal{N}_t is the matrix related to the Gaudin norm¹¹ and the $(n+p) \times (n+p)$ matrix $\mathcal{M}_t(\lambda_{I_q} | \xi_\gamma^{(0)} \cup \lambda_\alpha)$ has the following representation

$$\mathcal{M}_t(\lambda_{I_q} | \xi_\gamma^{(0)} \cup \lambda_\alpha) = \begin{pmatrix} \mathcal{M}_{q \times p}^{(1,1)} & \mathcal{M}_{q \times n}^{(1,2)} \\ \mathcal{M}_{(n+p-q) \times p}^{(2,1)} & \mathcal{M}_{(n+p-q) \times n}^{(2,2)} \end{pmatrix} \quad (5.19)$$

where

$$\begin{aligned} \mathcal{M}_{i,k}^{(1,1)} &= \mathcal{N}_{i,k}, & i \leq q, k \leq p, \\ \mathcal{M}_{i,k}^{(1,2)} &= i[r(\lambda_{\alpha_i}, \xi_{\gamma_k}^{(0)}) - r(\lambda_{\alpha_i}, -\xi_{\gamma_k}^{(1)})], & i \leq q, k \leq n, \\ \mathcal{M}_{i,k}^{(2,1)} &= \left(\lambda_{\alpha_k}^{(1)}\right)^{2i-1} + \left(\lambda_{\alpha_k}^{(0)}\right)^{2i-1}, & i \leq n+p-q, k \leq p, \\ \mathcal{M}_{i,k}^{(2,2)} &= \xi_{\gamma_k}^{2j-1}, & i \leq n+p-q, k \leq n, \end{aligned} \quad (5.20)$$

with $r(\lambda, \xi)$ defined in (4.8). Let us now rewrite this ratio of determinants as a single determinant:

$$\frac{\det_{n+p} \mathcal{M}_t(\lambda_{I_q} | \xi_\gamma^{(0)} \cup \lambda_\alpha)}{\det_q \mathcal{N}_t(\lambda_{I_q})} = \det_{M+n} \mathcal{W}_t(\lambda_{I_q} | \xi_\gamma^{(0)} \cup \lambda_\alpha), \quad (5.21)$$

where

$$\mathcal{W}_t(\lambda_{I_q} | \xi_\gamma^{(0)} \cup \lambda_\alpha) = \begin{pmatrix} I_{p \times p} & \mathcal{W}_{p \times n}^{(1,2)} \\ \mathcal{W}_{n \times p}^{(2,1)} & \mathcal{W}_{n \times n}^{(2,2)} \end{pmatrix} \quad (5.22)$$

where $I_{p \times p}$ is the $p \times p$ identity matrix and $\mathcal{W}_{x \times y}^{(a,b)}$ are matrices of size $x \times y$, defined by:

$$\begin{aligned} \mathcal{W}_{i,k}^{(1,2)} &= \left[\mathcal{N}^{-1} \mathcal{M}^{(1,2)} \right]_{i,k}, & i \leq p, k \leq n, \\ \mathcal{W}_{i,k}^{(2,1)} &= 0, & i \leq q-p, k \leq p, \\ \mathcal{W}_{i,k}^{(2,1)} &= \mathcal{M}_{i-(q-p),k}^{(2,1)}, & q-p+1 \leq i \leq n, k \leq p, \\ \mathcal{W}_{i,k}^{(2,2)} &= \left[\mathcal{N}^{-1} \mathcal{M}^{(1,2)} \right]_{p+i,k}, & i \leq q-p, k \leq n, \end{aligned} \quad (5.23)$$

$$\mathcal{M}_{i,k}^{(2,2)} = \xi_{\gamma_k}^{2i-1}, \quad i \leq n+p-q, k \leq n. \quad (5.24)$$

¹¹See formula (4.29) of [13] for its explicit expression.

Let us recall now that from the analysis of the ground state and the discussion in Section 4.4 of our previous paper [13], we have that it holds:

$$\left[\mathcal{N}^{-1} \mathcal{M}^{(1,2)} \right]_{i,k} = \frac{\rho(\lambda_{\alpha_i} - \xi_{\gamma_k}^{(1)}) - \rho(\lambda_{\alpha_i} + \xi_{\gamma_k}^{(0)})}{2N \rho(\lambda_{\alpha_i})} + o\left(\frac{1}{N}\right) \quad \text{if } i \leq q-p, k \leq n \quad (5.25)$$

so that we can write:

$$\det_{M+n} \mathcal{W}_t(\lambda_{I_q} | \xi_{\gamma}^{(0)} \cup \lambda_{\alpha}) = \det_n \mathcal{S}'_t(\xi_{\gamma}^{(0)}, \lambda_{\alpha}), \quad (5.26)$$

with $\mathcal{S}'_t = \mathcal{W}^{(2,2)} - \mathcal{W}^{(2,1)} \mathcal{W}^{(1,2)}$ the $n \times n$ matrices of elements defined by the following formulae up $o(1/N)$ terms:

$$\mathcal{S}'_{i,k} = \frac{\rho(\lambda_{\beta_i} - \xi_{\gamma_k}^{(1)}) - \rho(\lambda_{\beta_i} + \xi_{\gamma_k}^{(0)})}{2N \rho(\lambda_{\beta_i})} \quad \text{if } i \leq q-p, k \leq n,$$

$$\mathcal{S}'_{q-p+i,k} = \xi_{\gamma_k}^{2i-1} - \sum_{l=1}^p \mathcal{W}_{i,l}^{(2,1)} \left[\mathcal{N}^{-1} \mathcal{M}^{(1,2)} \right]_{l,k} \quad \text{if } 1 \leq i \leq n+p-q, k \leq p, \quad (5.27)$$

$$= \xi_{\gamma_k}^{2i-1} - \sum_{a=1}^p \left((\lambda_{\alpha_a}^{(1)})^{2i-1} + (\lambda_{\alpha_a}^{(0)})^{2i-1} \right) \frac{\rho(\lambda_{\alpha_i} - \xi_{\gamma_k}^{(1)}) - \rho(\lambda_{\alpha_i} + \xi_{\gamma_k}^{(0)})}{2N \rho(\lambda_{\alpha_i})}. \quad (5.28)$$

Now, let us observe that $p = |\alpha|$ is of the same order of q by assumption, i.e. it goes to infinity for N going to infinity and we can write:

$$\begin{aligned} & \sum_{a=1}^p \left((\lambda_{\alpha_a}^{(1)})^{2i-1} + (\lambda_{\alpha_a}^{(0)})^{2i-1} \right) \frac{\rho(\lambda_{\alpha_i} - \xi_{\gamma_k}^{(1)}) - \rho(\lambda_{\alpha_i} + \xi_{\gamma_k}^{(0)})}{2N \rho(\lambda_{\alpha_i})} \\ &= \sum_{l=1}^q \left((\lambda_l^{(1)})^{2j-1} + (\lambda_l^{(0)})^{2j-1} \right) \frac{\rho(\lambda_l - \xi_{\gamma_k}^{(1)}) - \rho(\lambda_l + \xi_{\gamma_k}^{(0)})}{2N \rho(\lambda_l)} + o(1/N). \end{aligned} \quad (5.29)$$

and these sums are finite for any $j \geq 1$. Here, we just compute the one associated to $j = 1$. Let us first notice that it holds:

$$\begin{aligned} & (\lambda^{(1)} + \lambda^{(0)}) \frac{\rho(\lambda - \xi^{(1)}) - \rho(\lambda + \xi^{(0)})}{\rho(\lambda)} = 2\lambda \frac{\rho(\lambda + i/2 - \xi) - \rho(\lambda + i/2 + \xi)}{\rho(\lambda)} \\ &= (\lambda + i/2) \frac{\rho(\lambda + i/2 - \xi)}{\rho(\lambda)} - (\lambda - i/2) \frac{\rho(\lambda + i/2 + \xi)}{\rho(\lambda)} + (\lambda - i/2) \frac{\rho(\lambda + i/2 - \xi)}{\rho(\lambda)} \\ & - (\lambda + i/2) \frac{\rho(\lambda + i/2 + \xi)}{\rho(\lambda)} \end{aligned} \quad (5.30)$$

$$= \sum_{\sigma=\pm} (\sigma\lambda + i/2) \frac{\rho(\sigma\lambda + i/2 - \xi_{\pi})}{\rho(\sigma\lambda)} \sigma - \sum_{\sigma=\pm} (\sigma\lambda - i/2) \frac{\rho(\sigma\lambda - i/2 - \xi)}{\rho(\sigma\lambda)} \sigma \quad (5.31)$$

where we have used the parity and periodicity proprieties of $\rho(\lambda)$, i.e.

$$\rho(-\lambda) = -\rho(\lambda), \quad \rho(\lambda \pm i) = -\rho(\lambda), \quad (5.32)$$

to get the identities:

$$-(\lambda - i/2) \frac{\rho(\lambda + i/2 + \xi)}{\rho(\lambda)} = -(-\lambda + i/2) \frac{\rho(-\lambda + i/2 - \xi)}{\rho(-\lambda)} \quad (5.33)$$

$$(\lambda - i/2) \frac{\rho(\lambda + i/2 - \xi)}{\rho(\lambda)} = -(\lambda - i/2) \frac{\rho(\lambda - i/2 - \xi)}{\rho(\lambda)} \quad (5.34)$$

$$-(\lambda + i/2) \frac{\rho(\lambda + i/2 + \xi)}{\rho(\lambda)} = (-\lambda - i/2) \frac{\rho(-\lambda - i/2 - \xi)}{\rho(-\lambda)}. \quad (5.35)$$

Now, we can use the following identity holding in the thermodynamic limit:

$$\sum_{\sigma=\pm} \left(\frac{1}{N} \sum_{j=1}^q \sigma f(\sigma \lambda_j) \right) \xrightarrow{N \rightarrow \infty} \sum_{\sigma=\pm} \int_0^{\infty} \sigma f(\sigma \lambda) \rho(\lambda) d\lambda = \int_{-\infty}^{\infty} f(\lambda) \rho(\lambda) d\lambda, \quad (5.36)$$

to get in this limit

$$\frac{1}{N} \sum_{l=1}^q \lambda_l \frac{\rho(\lambda_l + i/2 - \xi_{\gamma_k}) - \rho(\lambda_l + i/2 + \xi_{\gamma_k})}{\rho(\lambda_l)} \xrightarrow{N \rightarrow \infty} \sum_{\sigma=\pm} \frac{\sigma}{2} \int_{-\infty + \sigma i/2}^{\infty + \sigma i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda, \quad (5.37)$$

which can be computed by the Residue Theorem. Indeed, we can write:

$$\int_{-\infty + i/2}^{\infty + i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda - \int_{-\infty - i/2}^{\infty - i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda = \lim_{R \rightarrow +\infty} \left[\int_{-R + i/2}^{R + i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda \right] \quad (5.38)$$

$$- \int_{-R - i/2}^{R - i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda + \int_{-R - i/2}^{-R + i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda - \int_{R - i/2}^{R + i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda \quad (5.39)$$

$$= -2i\pi \operatorname{Res} \lambda \rho(\lambda, \xi_{\gamma_k}) \Big|_{\lambda=\xi} = 2\xi_{\gamma_k}, \quad (5.40)$$

for $|\operatorname{Im}(\xi_{\pi_k})| \leq 1/2$. Indeed, observing that it holds

$$|\sinh \pi(\lambda - \xi_{\gamma_k})|^2 = \frac{\cosh 2\pi R - \cos \pi(x + i\xi_{\gamma_k})}{2} \geq \frac{(\cosh 2\pi R) - 1}{2}, \quad \forall x \in [-1/2, 1/2] \quad (5.41)$$

for $\lambda = \pm R + ix \in [\pm R - i/2, \pm R + i/2]$, we get the estimates:

$$\left| \int_{\pm R - i/2}^{\pm R + i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda \right| \leq \sup_{\lambda \in [\pm R - i/2, \pm R + i/2]} |\lambda \rho(\lambda, \xi_{\gamma_k})| \leq \left(\frac{4R^2 + 1}{2[(\cosh 2\pi R) - 1]} \right)^{1/2}, \quad (5.42)$$

which imply

$$\lim_{R \rightarrow +\infty} \left[\int_{-R - i/2}^{-R + i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda - \int_{R - i/2}^{R + i/2} \lambda \rho(\lambda, \xi_{\gamma_k}) d\lambda \right] = 0. \quad (5.43)$$

So that we get our result:

$$\frac{1}{N} \sum_{l=1}^{M_0} \lambda_l \left[\frac{\rho(\lambda_l + i/2 - \xi_{\gamma_k}) - \rho(\lambda_l + i/2 + \xi_{\gamma_k})}{\rho(\lambda_l)} \right] \xrightarrow{N \rightarrow \infty} \xi_{\gamma_k}, \quad (5.44)$$

which proves that the line $S'_{q-p+i,k}$ goes to zero for $i = 1$ in the thermodynamic limit and so the proposition is proven. \square

6 Correlation functions

In the following we first develop the analysis of the correlation functions in the case of diagonal $\bar{K}_-(\lambda)$ and upper triangular $\bar{K}_+(\lambda)$, by imposing

$$\bar{c}_- = 0, \quad \bar{b}_+ \neq 0, \quad (6.1)$$

then hereon when we refer to the transfer matrix $\bar{\mathcal{T}}(\lambda)$, it is associated to these boundary conditions, as well as the associated Hamiltonian reads:

$$\bar{H} = \sum_{i=1}^{N-1} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right] + \frac{\eta}{\bar{\zeta}_-} \sigma_1^z + \frac{\eta}{\bar{\zeta}_+} \left[\sigma_N^z + \bar{b}_+ \sigma_N^+ \right]. \quad (6.2)$$

We have proven the isospectrality of the transfer matrix $\bar{\mathcal{T}}(\lambda)$ and Hamiltonian \bar{H} , associated to unparallel boundary magnetic fields, with the transfer matrix $\hat{\mathcal{T}}(\lambda)$ and Hamiltonian

$$\hat{H} = \sum_{i=1}^{N-1} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right] + \frac{\eta}{\bar{\zeta}_-} \sigma_1^z + \frac{\eta}{\bar{\zeta}_+} \sigma_N^z, \quad (6.3)$$

associated to parallel boundary magnetic fields along the z-direction. Then, in the next section, we show that in the thermodynamic limit they share the same zero-temperature correlation functions. While in a subsequent section, these results are used to state the results for correlation functions of the original open chain with the following transfer matrix and Hamiltonian:

$$\mathcal{T}(\lambda) = \Gamma_W^{-1} \bar{\mathcal{T}}(\lambda) \Gamma_W, \quad \bar{H} = \Gamma_W^{-1} \bar{H} \Gamma_W, \quad (6.4)$$

with Γ_W defined in Section 3.3.1.

6.1 The case of diagonal and triangular boundary matrices

The following theorem holds:

Theorem 6.1. *Let us assume that the following boundary conditions are satisfied:*

$$\bar{c}_- = 0, \quad \bar{b}_+ \neq 0, \quad (6.5)$$

and let us take the following reality conditions:

$$\eta = i, \quad i\bar{\zeta}_\pm \in \mathbb{R}$$

then, in the thermodynamic limit, all the zero-temperature correlation functions relative to the Hamiltonian (6.2) with unparallel magnetic fields coincide with the correlation functions (associated to the same quasi-local operator) relative to the Hamiltonian (6.3) with parallel magnetic fields along the z-direction.

Proof. In order to prove the theorem it is enough to prove it for the correlation functions of a basis of quasi-local operators, so that to prove it we can use the so-called elementary blocks, i.e. the ground state average of the basis of monomial of elementary matrices:

$$\frac{\langle \bar{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}. \quad (6.6)$$

Let us observe that the following commutation relations hold:

$$\left[\prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j}, S_z \right] = \theta_{\varepsilon, \varepsilon'} \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j}, \quad \text{for } S_z = \sum_{j=1}^N \sigma_j^z, \quad (6.7)$$

where the parity $\theta_{\varepsilon, \varepsilon'}$ is associated to the integer $s + s'$ defined in Section 4.3 by:

$$\theta_{\varepsilon, \varepsilon'} = m - (s + s'), \quad (6.8)$$

and by the Proposition 4.3, we have that on the finite lattice it holds:

$$\frac{\langle \bar{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = \sum_{b_1=1}^q \dots \sum_{b_s=1}^q \sum_{b_{s+1}=1}^{q+m} \dots \sum_{b_{s+s'}=1}^{q+m} \mathcal{F}_{\beta_{s+s'}}^+(\lambda_{I_{q+m}}) \frac{\langle t | \bar{\mathcal{B}}_+(\lambda_{\alpha_- \cup \xi_{\gamma_+}^{(0)}}) | 0 \rangle}{\langle t | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}, \quad (6.9)$$

where $\beta_{s+s'} = \{b_1, \dots, b_{s+s'}\}$, $\lambda_{q+j} := \xi_{m+1-j}^{(0)}$ for $j \in \{1, \dots, m\}$,

$$\alpha_- = I_q \setminus \alpha_+, \quad \alpha_+ = \beta_{s+s'} \cap I_q, \quad \gamma_+ = \{1, \dots, m\} \setminus \gamma_-, \quad (6.10)$$

$$\gamma_- = \{N + m + 1 - j\}_{j \in \beta_{s+s'} \cap \{N+1, \dots, N+m\}}, \quad (6.11)$$

and the coefficients $\mathcal{F}_{\beta_{s+s'}}^+(\lambda_{I_{q+m}})$ are defined as in Proposition 4.3. We can use now the results on the scalar products of Proposition 5.2 to state that:

$$\frac{\langle \bar{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = 0 \quad \text{for } \theta_{\varepsilon, \varepsilon'} < 0, \quad (6.12)$$

already on the finite chains, being

$$\frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{\alpha_- \cup \xi_{\gamma_+}^{(0)}}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = 0 \quad \text{for } \theta_{\varepsilon, \varepsilon'} < 0, \quad (6.13)$$

as $|\alpha_-| + |\gamma_+| < q$, for any compatible choice of α_- and γ_+ with $\theta_{\varepsilon, \varepsilon'} < 0$.

In the case $\theta_{\varepsilon, \varepsilon'} = 0$, here we have just to point out that from Proposition 4.3, it follows that the action of an elementary monomial $\prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j}$ on the eigenstate $\bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle$ of transfer matrix $\bar{\mathcal{T}}(\lambda)$ has identical form of the action of the same monomial on the eigenstate $\hat{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle$ of transfer matrix $\hat{\mathcal{T}}(\lambda)$. This observation implies that for $\theta_{\varepsilon, \varepsilon'} = 0$ the elementary blocks associated to the Hamiltonian

\bar{H} and \hat{H} coincidence already for the finite chains, due to the following chain of identity:

$$\frac{\langle \bar{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = \sum_{b_1=1}^q \cdots \sum_{b_s=1}^q \sum_{b_{s+1}=1}^{q+m} \cdots \sum_{b_{s+s'}=1}^{q+m} \mathcal{F}_{\beta_{s+s'}}^+(\lambda_{I_{q+m}}) \times \frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{\alpha_- \cup \xi_{\gamma_+}^{(0)}}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} \quad (6.14)$$

$$= \sum_{b_1=1}^q \cdots \sum_{b_s=1}^q \sum_{b_{s+1}=1}^{q+m} \cdots \sum_{b_{s+s'}=1}^{q+m} \mathcal{F}_{\beta_{s+s'}}^+(\lambda_{I_{q+m}}) \times \frac{\langle \hat{t} | \hat{\mathcal{B}}_+(\lambda_{\alpha_- \cup \xi_{\gamma_+}^{(0)}}) | 0 \rangle}{\langle \hat{t} | \hat{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} \quad (6.15)$$

$$= \frac{\langle \hat{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \hat{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}{\langle \hat{t} | \hat{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}, \quad (6.16)$$

where we have used that by the Proposition 5.2 the following scalar products coincide:

$$\frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{\alpha_- \cup \xi_{\gamma_+}^{(0)}}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = \frac{\langle \hat{t} | \hat{\mathcal{B}}_+(\lambda_{\alpha_- \cup \xi_{\gamma_+}^{(0)}}) | 0 \rangle}{\langle \hat{t} | \hat{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} \text{ for } \theta_{\varepsilon, \varepsilon'} = 0. \quad (6.17)$$

Let us stress that up to here we have shown that the elementary blocks associated to the Hamiltonian \bar{H} and \hat{H} coincidence already for the finite chains both for $\theta_{\varepsilon, \varepsilon'} < 0$ and for $\theta_{\varepsilon, \varepsilon'} = 0$.

Instead, in the remaining case $\theta_{\varepsilon, \varepsilon'} > 0$, we will show that these elementary blocks coincide in the thermodynamic limit as for the finite chains it holds:

$$\frac{\langle \hat{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \hat{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}{\langle \hat{t} | \hat{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = 0, \text{ being } \frac{\langle \hat{t} | \hat{\mathcal{B}}_+(\lambda_{\alpha_- \cup \xi_{\gamma_+}^{(0)}}) | 0 \rangle}{\langle \hat{t} | \hat{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = 0 \text{ } \theta_{\varepsilon, \varepsilon'} > 0, \quad (6.18)$$

as $|\alpha_-| + |\gamma_+| > q$, for any compatible choice of α_- and γ_+ with $\theta_{\varepsilon, \varepsilon'} > 0$, while it may hold

$$\frac{\langle \bar{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} \neq 0, \text{ being } \frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{\alpha_- \cup \xi_{\gamma_+}^{(0)}}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} \neq 0. \quad (6.19)$$

Then, in the current case we have to show the coincidence just in the thermodynamic limit, i.e.

$$\lim_{N \rightarrow \infty} \frac{\langle \bar{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = 0 \text{ for } \theta_{\varepsilon, \varepsilon'} > 0. \quad (6.20)$$

In order to prove this we reorder the sums in these elementary block as it follows:

$$\frac{\langle \bar{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = \sum_{g=s}^{s+s'} \sum_{b_1=1}^q \cdots \sum_{b_g=1}^q \sum_{b_{g+1}=q+1}^{q+m} \cdots \sum_{b_{s+s'}=q+1}^{q+m} \mathcal{F}_{\beta_{s+s'}}^+(\lambda_{I_{q+m}}) \times \frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{\alpha_- \cup \xi_{\gamma_+}^{(0)}}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle}, \quad (6.21)$$

now in the thermodynamic limit the q diverges as N , so that for any fixed $s \leq g \leq s + s'$ the sums

$$\sum_{b_1=1}^q \dots \sum_{b_g=1}^q \quad (6.22)$$

leads to a finite g multiple integrals in the thermodynamic limit provided the integrand is of order $O(1/N^g)$, while the other sums contribute to order 1 to the thermodynamic limit. It is now enough to recall that in Proposition 5.2 we have shown

$$\frac{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{\alpha_-} \cup \xi_{\gamma_+}^{(0)}) | 0 \rangle}{\langle \bar{t} | \bar{\mathcal{B}}_+(\lambda_{I_q}) | 0 \rangle} = o(1/N^{(g-q-|\alpha_-|)}), \quad \text{if } |\alpha_-| + |\gamma_+| > q,$$

to conclude that these sums goes to zero and to prove our statement. \square

In the thermodynamic and homogeneous limit, then, these elementary blocks can be computed as done in our previous paper [13], leading to the following multiple integral representations:

$$\begin{aligned} \frac{\langle \bar{t} | \prod_{j=1}^m E_j^{\varepsilon_j, \varepsilon'_j} | \bar{t} \rangle}{\langle \bar{t} | \bar{t} \rangle} &= \delta_{\theta_{\varepsilon, \varepsilon'}, 0} (-1)^{m-s+\frac{m(m-1)}{2}} \bar{\zeta}_-^m \pi^{m(m+1)} \int_{-\infty}^{\infty} \prod_{j=1}^s \frac{d\lambda_j}{2} \int_{-\infty-i}^{\infty-i} \prod_{j=s+1}^m \frac{d\lambda_j}{2} \\ &\times \prod_{p=1}^s \frac{(\lambda_p + i/2)^{m+i_p-1} (\lambda_p - i/2)^{m-i_p}}{\cosh^{2m}(\pi \lambda_p)} \prod_{p=s+1}^m \frac{(\lambda_p + i/2)^{m+i_p-1} (\lambda_p + 3i/2)^{m-i_p}}{\cosh^{2m}(\pi \lambda_p)} \\ &\times \prod_{k < l} \frac{\sinh(\pi \lambda_{kl}) \sinh(\pi \bar{\lambda}_{kl})}{(\lambda_{kl} - i)(\bar{\lambda}_{kl} + i)} \prod_{k=1}^m \frac{\sinh(\pi \lambda_k)}{\lambda_k + i/2 + \bar{\zeta}_-}, \end{aligned} \quad (6.23)$$

where the $\{i_p\}$ have been defined in (4.33)-(4.34).

6.2 Non-diagonal case isospectral to the diagonal and triangular one

Let us consider here our original open XXX spin 1/2 quantum chain with non-diagonal and non-commutative boundary matrices $K_{\pm}(\lambda)$, satisfying the following boundary condition:

$$e^{\tau_+} = e^{\tau_-} \frac{(\varepsilon_- \sqrt{1+4\kappa_-^2} + 1)(\varepsilon_+ \sqrt{1+4\kappa_+^2} - 1)}{4\kappa_+ \kappa_-}, \quad (6.24)$$

fix a couple $(\varepsilon_+, \varepsilon_-) \in \{-1, 1\}^2$, with otherwise general boundary parameters

$$\kappa_+ \neq \pm \kappa_-, \quad (6.25)$$

satisfying the following reality condition:

$$\eta = i, \quad i\bar{\zeta}_{\pm}, \quad i\xi_a \in \mathbb{R}, \quad (6.26)$$

then by the Lemma 3.1 its transfer matrix $\mathcal{T}(\lambda)$ and Hamiltonian H , defined in (2.3) and (2.1), are isospectral to the transfer matrix $\hat{\mathcal{T}}(\lambda)$ and Hamiltonian \hat{H} , both self-adjoint, respectively. While, by tensor product similarity transformation Γ_W , $\mathcal{T}(\lambda)$ and H reduce to the transfer matrix $\bar{\mathcal{T}}(\lambda)$ and Hamiltonian \bar{H} .

Then these similarity transformations and the previous Theorem 6.1 allow us to compute the correlation functions/elementary blocks in the original model associated to the non-diagonal and non-commuting $K_{\pm}(\lambda)$ boundary matrices as simple linear combinations of those of the model associated to the $\tilde{K}_{\pm}(\lambda)$ ones.

More in detail, the gauge transformation can be explicitly written, only in terms of the $K_{-}(\lambda)$ boundary parameters, as it follows:

$$W = \begin{pmatrix} 1 & \frac{-1 + \sqrt{1 + 4\kappa_-^2}}{2\kappa_- e^{-\tau_-}} \\ \frac{1 - \sqrt{1 + 4\kappa_-^2}}{2\kappa_- e^{-\tau_-}} & 1 \end{pmatrix}, \quad (6.27)$$

and so defined

$$\check{E}_{1,m}^{\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}} \equiv \Gamma_W E_{1,m}^{\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}} \Gamma_W^{-1} = \prod_{a=1}^m \check{E}_a^{\{\varepsilon_a, \varepsilon'_a\}}, \quad (6.28)$$

with:

$$\check{E}_a^{\{\varepsilon_a, \varepsilon'_a\}} = W_a E_a^{\{\varepsilon_a, \varepsilon'_a\}} W_a^{-1}, \quad (6.29)$$

the generic m -site elementary block in the original model

$$\langle E_{1,m}^{\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}} \rangle_{ND} \equiv \frac{\langle t | \prod_{j=1}^m E_j^{\{\varepsilon_j, \varepsilon'_j\}} | t \rangle}{\langle t | t \rangle}, \quad (6.30)$$

coincides with

$$\begin{aligned} \frac{\langle \bar{t} | \Gamma_W E_{1,m}^{\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}} \Gamma_W^{-1} | \bar{t} \rangle}{\langle \bar{t} | \bar{t} \rangle} &= \frac{\langle \hat{t} | \Gamma_W E_{1,m}^{\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}} \Gamma_W^{-1} | \hat{t} \rangle}{\langle \hat{t} | \hat{t} \rangle} \\ &\equiv \langle \check{E}_{1,m}^{\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}} \rangle_D, \end{aligned} \quad (6.31)$$

i.e. a sum of up to 2^{2m} elementary blocks of the open XXX spin 1/2 chain with $\hat{K}_{\pm}(\lambda)$ diagonal boundary matrices. Let us write explicitly these decompositions of the one-point and two-point correlation functions of the original model in terms of those of the diagonal one:

Corollary 6.1. *The following one-point functions of the original model:*

$$\langle E_m^{(1,1)} \rangle_{ND} = \frac{2\kappa_-^2 \langle E_m^{(1,1)} \rangle_D}{1 + 4\kappa_-^2 - \sqrt{1 + 4\kappa_-^2}} + \frac{\sqrt{1 + 4\kappa_-^2} - 1}{2\sqrt{1 + 4\kappa_-^2}} \langle E_m^{(2,2)} \rangle_D, \quad (6.32)$$

$$\langle E_m^{(2,1)} \rangle_{ND} = \frac{e^{\tau_-} \kappa_- \langle \sigma_m^z \rangle_D}{\sqrt{1 + 4\kappa_-^2}}, \quad \langle E_m^{(1,2)} \rangle_{ND} = \frac{\kappa_- \langle \sigma_m^z \rangle_D}{e^{\tau_-} \sqrt{1 + 4\kappa_-^2}}, \quad (6.33)$$

$$\langle E_m^{(2,2)} \rangle_{ND} = \frac{2\kappa_-^2 \langle E_m^{(2,2)} \rangle_D}{1 + 4\kappa_-^2 - \sqrt{1 + 4\kappa_-^2}} + \frac{\sqrt{1 + 4\kappa_-^2} - 1}{2\sqrt{1 + 4\kappa_-^2}} \langle E_m^{(1,1)} \rangle_D, \quad (6.34)$$

admit the above representations in terms of the nonzero one-point functions of the diagonal model, so that the magnetization profile reads:

$$\langle \sigma_m^z \rangle_{ND} = \frac{\langle \sigma_m^z \rangle_D}{\sqrt{1 + 4\kappa_-^2}}. \quad (6.35)$$

Moreover, for the following two-point functions we have¹²:

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{ND} = \frac{\langle \sigma_1^z \sigma_{m+1}^z \rangle_D}{1 + 4\kappa_-^2} + \frac{4\kappa_-^2 \left(\langle \sigma_1^+ \sigma_{m+1}^- \rangle_D + \langle \sigma_1^- \sigma_{m+1}^+ \rangle_D \right)}{1 + 4\kappa_-^2}, \quad (6.36)$$

$$\langle \sigma_1^- \sigma_{m+1}^+ \rangle_{ND} = \frac{\kappa_-^2 \langle \sigma_1^z \sigma_{m+1}^z \rangle_D}{1 + 4\kappa_-^2} + \frac{2\kappa_-^4 \langle \sigma_1^+ \sigma_{m+1}^- \rangle_D}{1 + 4\kappa_-^2 - \sqrt{1 + 4\kappa_-^2}} + \frac{\sqrt{1 + 4\kappa_-^2} - 1}{\sqrt{1 + 4\kappa_-^2}} \langle \sigma_1^- \sigma_{m+1}^+ \rangle_D, \quad (6.37)$$

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle_{ND} = \frac{\kappa_-^2 \langle \sigma_1^z \sigma_{m+1}^z \rangle_D}{1 + 4\kappa_-^2} + \frac{2\kappa_-^4 \langle \sigma_1^- \sigma_{m+1}^+ \rangle_D}{1 + 4\kappa_-^2 - \sqrt{1 + 4\kappa_-^2}} + \frac{\sqrt{1 + 4\kappa_-^2} - 1}{\sqrt{1 + 4\kappa_-^2}} \langle \sigma_1^+ \sigma_{m+1}^- \rangle_D. \quad (6.38)$$

It is worth commenting that the elementary blocks of correlation functions can be used as a basis to decompose any correlation function in terms of them. From this point of view the natural elementary block basis to be used in our original open XXX spin 1/2 quantum chain with non-diagonal and non-commuting boundary matrices $K_{\pm}(\lambda)$ satisfying (6.24) and the reality conditions (6.26) is defined in the following:

Corollary 6.2. *Let us define the following quasi-local operators:*

$$\bar{E}_{1,m}(\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}) = \Gamma_W^{-1} E_{1,m}(\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}) \Gamma_W = \prod_{a=1}^m \bar{E}_a(\varepsilon_a, \varepsilon'_a), \quad (6.39)$$

where:

$$\bar{E}_a(\varepsilon_a, \varepsilon'_a) = W_a^{-1} E_a(\varepsilon_a, \varepsilon'_a) W_a. \quad (6.40)$$

Then the following identities holds:

$$\langle \bar{E}_{1,m}(\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}) \rangle_{ND} = \langle E_{1,m}(\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\}) \rangle_D, \quad (6.41)$$

which, in particular means, that the non-zero elementary blocks are only those associated to the $\bar{E}_{1,m}(\{(\varepsilon_1, \varepsilon'_1), \dots, (\varepsilon_m, \varepsilon'_m)\})$ commuting with:

$$\bar{E}_z = \sum_{a=1}^N \bar{\sigma}_a^z, \quad (6.42)$$

with

$$\bar{\sigma}_a^z = W_a^{-1} E_a(\varepsilon_a, \varepsilon'_a) W_a = \frac{\sigma_a^z + 2\kappa_- (e^{\tau_-} \sigma_a^+ + e^{-\tau_-} \sigma_a^-)}{\sqrt{1 + 4\kappa_-^2}}. \quad (6.43)$$

¹²Here, we have written only the standard three two point functions but the other possible five are also easily written as linear combinations.

7 Conclusion

We have shown that the correlation functions for the open XXX spin 1/2 chain with unparallel boundary magnetic fields are written as linear combination of those of the open XXX spin 1/2 chain with parallel boundary magnetic fields, whose multiple integral formulae were derived in [13, 14].

The main technical novelties here developed are the computations of the boundary-bulk decomposition of the boundary separate states together with the computation of the action of local operators on separate states for these open XXX quantum chains with unparallel magnetic fields. Let us comment that the boundary-bulk decomposition of boundary separate states is required as we have at our disposal only a reconstruction of local operators in terms of generators of the Yang-Baxter algebra and not in terms of the boundary generators of the reflection algebra [13, 14]. Then, the boundary-bulk decomposition allows us to compute the action of local operators over boundary separate states by acting on analogue bulk states. Here, the main difficulty is to rearrange this action as a linear combination of purely boundary separate states. Note that while the solution of these fundamental steps to correlation functions are here solved for the XXX open spin 1/2 quantum chains, they provide some exemplified versions of what will be required in the analysis of correlation functions of the XXZ/XYZ quantum spin 1/2 chains, where we will derive them by using the gauged generators of the Yang-Baxter algebra and reflection algebra.

Let us comment that the results of Section 6.1 provide a proof of the physical expectation that correlation functions of a quasi-local operator, on finite number of sites, should coincide in the thermodynamic limit for two open chains that share the same boundary magnetic fields on the site 1. In fact, we have proven it here in the special constrained case, for which the two open chains share the same boundary matrix in site 1 but one diagonal and the other triangular in the site N . It is worth commenting that the main technical reason for our choice to compute correlation functions under this constrain (allowing for an homogeneous TQ -equation formulation of the transfer matrix spectrum) is due to the reduced knowledge of the thermodynamic limit of the ground state for the most general unconstrained non-diagonal boundary matrices. To achieve the full control of this ground state distribution can then make possible the computation of correlation functions in this most general unconstrained case and will be of central interest for future research in the open chain framework.

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