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(School of Business and Governance)
& Department of Cybernetics (School of Science)
Tallinn University of Technology, Estonia

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Mathematical truth and mathematical intuition

Two theses have been widely held by philosophers in the second part of the twentieth century and the early part of the twenty-first; not by every philosopher by any means but nevertheless by a great number of them in the analytic tradition:

(T1) Language contacts reality through quantifiers, and so mathematical language likewise contacts mathematical reality through quantifiers.

(T2) The semantic interpretation or value of the sentences of a language is to be understood in terms of their truth conditions and so the semantic interpretation or value of the sentences of a mathematical language is also to be understood in terms of their truth conditions.

Some of these philosophers have then argued that they were facing a problem:

(PB) It is mysterious how we know anything abstract, in particular the abstracta that feature in the truth conditions of the sentences of mathematical languages. Or, under a different formulation involving propositional attitudes: it's hard to see how our belief in mathematical entities can so well reflect the facts about them, i.e., how these beliefs might be reliable.

The same philosophers have then believed they had to meet a challenge:

(C) Vindicate (T1)-(T2) and solve (PB).

Some have thought it preferable to tackle (PB) by starting with the semantic side of the issue on the assumption that the epistemological problem would be solved if one could countenance epistemologically secure items as values of the bound variables.

They have done so either by:

(1) Looking for abstractless truth conditions for whatever mathematical languages we might be concerned with so that no abstracta occur in the course of values of the bound variables,

or by :

(2) Looking for a substitute for the notion of truth, such as (i) provability, either *hic et nunc* or in principle, or (ii) conservativity, so that our use of mathematical existence assertions, e.g., our application of pure mathematical results, would give us no grounds for believing them to be either true without further epistemic qualification, independently of provability (case (i)), or to be true at all (case (ii)).

Why should anyone believe this and embark on such tasks? Although the agendas (1) and (2) are distinct, they share assumptions about mathematical knowledge, and these assumptions have determined a model of philosophical inquiry — “too obvious for words” to adapt Charles Taylor’s phrase —, a model of what must be done in the philosophy of mathematics; or at least they’ve yielded an organizing principle for these philosophers’ practice (see *Taylor 1984*: 20).

I want to consider a different way to look at (PB) that consists in giving an explanation of how we acquire mathematical knowledge and justify our mathematical beliefs that doesn’t rely on causal relations and disregards reliability constraints entirely, but still strives to explain how we acquire the knowledge and to account for the truth of the mathematical propositions we justifiably believe in.

The explanation appeals to intuition. I’ll be looking at Kant exclusively, and only in relation to Parsons’s interpretation of Kant’s proposal. (I’ll leave on the side Gödel’s claim that we have a *de re* mathematical intuition of the objects of transfinite set theory).

Kant’s dilemma. We only have intuitions of objects which are given to us, either through the perception of the senses (sight, typically), or in the imagination. But we also have pure or specifically mathematical knowledge (*Kant [1781, 1787] 1965*: B75). We cannot intuit *both* spontaneously [*ursprünglich*] and a priori because “an intuition is such a representation as would immediately depend on the presence [*Gegenwart*] of the object” (*Kant [1783] 1950*: §8).

Parsons's interpretation of the dilemma. Our best account of mathematical truth (Tarski's) involves postulating mathematical objects, while our best account of mathematical knowledge requires causal relations of the objects of this knowledge to us; but mathematical objects, provided there are any, are causally inert. This is Benacerraf's dilemma (*Benacerraf [1968] 2016, [1973] 1983*). Kant's dilemma is similar: mathematical truth requires applicability to the physical world, but our best account of mathematical knowledge makes it rest on intuition, which requires the prior presence of the object and this contradicts the a priori character of mathematics. In other words, no spontaneous intuition may yield a priori knowledge. "This is of interest because it is a form of the dilemma that does not require that the semantics of mathematics involve mathematical objects [...]. But of course it depends on other assumptions, in particular that mathematics is a priori" (*Parsons [1979-1980] 1996: 99, note 12*).

Ungenerous objection to Parsons. A dilemma which doesn't require that the semantics of mathematical languages involve quantification over abstract objects may not be a genuine variant of Benacerraf's original dilemma.

Rejoinder to the objection. The claim that mathematics taken at face value yield truths, or consists in a body of truths, is in conflict with an epistemological constraint that may be construed in two different ways: (i) as an empiricist one, either causalist or reliabilist in Benacerraf's case, or (ii) as one in which intuition is of one kind only, namely sensory, so that intuition only yields singular representations that relate to objects immediately in Kant's case. It might not be entirely preposterous, then, to consider the puzzles *conjointly*, at least from the epistemological point of view, and claim that, should we wish to preserve truth, we would end up either with abstracta we cannot access or with the presence of objects which can't be known a priori.

There is, in this particular sense, a genuine Benacerraf-Kant dilemma according to which a link must be provided between what it is for mathematical propositions to be true and what it is for us to recognize that they are true.

In order for the solution to the dilemma to be acceptable, the link between our cognitive faculties and the interpretation of the referential apparatus of mathematical theories

(obtained via Tarski), which is severed in Benacerraf's original dilemma, must be restored in the Kantian solution to the Kantian version of the puzzle suggested by Parsons.

Problem with Parsons's own suggestion.

We perceive by ordinary means strings of stroke-tokens: |, ||, ||| and so on, which are isomorphic to the natural numbers. At the next level up, we have singular propositions such as “|| is the successor of |.” Such singular propositions are about types. Parsons construes the propositional knowledge *that* || is the successor of | as being justified by a single unique intuition (*Parsons op. cit.*: 105). It is also a general proposition, but only insofar as it has implications for *any* token. So we go from intuitions *of* to intuitions *that* because we take any instance of both the kind of situation and of the kind of assertion that correspond to it as being paradigmatic. It is unclear in Parsons's account how we may justify this particular move (i.e., the move relying on the recognition that the singular propositions have implications, whatever these might turn out to be).

We also have general propositions about types, such as “Each string of strokes can be extended by one more” and such general propositions “have in their scope indefinitely many *different* types” (*Parsons loc. cit.*). No actual perception or sensory input is available here, which would act as a warrant for the proposition. As Parsons notes, the idea that we have an intuition of types “faces serious objections because of the timelessness, acausality or incompleteness of types as abstract entities” (*Parsons op. cit.*: 109). What we have to do in this case is to imagine an arbitrary string of strokes either as a vague object, or in such a way that its internal structure is entirely irrelevant to our new concern about types. Parsons remarks that such imaginings or *Gedankenexperimente* count as warrants (“verification” is the word he uses in that respect) of the general statement about types. Obviously they do if and only if certain conditions as to *how* an arbitrary string of strokes *must* be imagined are met, namely in this case, either vaguely or in such a way that the internal structure is “seen” or “understood” or construed in some way as irrelevant. According to this picture, then, there is a link between the way in which we justify our claims about tokens by means of ordinary sense perception and the *Gedankenexperimente* we are legitimately appealing to when justifying claims about timeless types of such tokens. What one then needs, then, is an explanation of how such

means of justification are related.

Parsons's proposal is of course quite different from, say, Maddy's. (Maddy argues that we can acquire perceptual beliefs about sets of physical objects by construing the belief that, say, there are three physical objects at a given location (three eggs in a box) as a belief about a *set* of physical things and not about a *physical aggregate* (Maddy [1980] 1996: 126-131).) We do not have such direct intuition of abstracta (e.g., sets) in Parsons's analysis. What we have with Parsons is what he calls a "moderate position" to the effect that "intuition gives objects which form a model of arithmetic," this model being "as good as any, both for the foundations of arithmetic and for applications" (Parsons *op. cit.*: 111).

It is clear, on the Kantian side, that the limits of what we are able to establish as true in mathematics is determined by subjective conditions which are proper to us as human beings. We are limited to that which can be represented a priori in intuition, i.e. space and time and change in time. We may then ask the following question: What would determine such limits according to theories which hold that we perceive mathematical objects directly, or *de re*, so that the perception contains something contentual, utterly different from the *form* of sensibility? Such limits must also be linked to our particular cognitive constitution. But they must be so in a way that is radically different from the one envisaged by any transcendental philosophy.

Consider again the abstract object stroke-string-type. What we have here as warrants for the general propositions about types are properties of the abstract object. The object is abstract because, although it might be instantiated, it cannot be located anywhere. It possesses properties such as vagueness or lack of internal structure insofar as it is an object of our *intuition* (through the imagination). One might perhaps say that it *necessarily* possesses them *as intuitions*, in the sense that we may not intuit the object otherwise. In other words, the stroke-string type is arbitrary or vague or without structure insofar as it is intuited in this way *by us*. It isn't intrinsically so.

Why Kant's solution is preferable to Parsons's.

It is at once more general and unified, if only because the distinction between the intuition *of* and the intuition *that* plays no role in the Kantian explanation.

If it explains anything, it explains why our knowledge of mathematical truths *must* be a

priori, so that the a prioricity isn't as it were a matter of factual contingency, a feature that our mathematical knowledge could very well *not* have.

Consider the case of arithmetic. Under the Kantian assumption that knowledge of arithmetic is a priori, the causal dependence of our *de re* intuition of the objects, required under the causal efficacy version of the semantic horn of Benacerraf's dilemma, must of course be renounced. The requirement that an explanation of how it is that our mathematical beliefs can so well reflect the mathematical facts, which is essential under the reliability constraint version of the same horn, may in a sense be fulfilled provided the so-called "facts" are construed non-platonistically.

The Kantian rejoinder does indeed attribute to arithmetical propositions conditions of which we can clearly know that they obtain whenever they do while connecting them with an analysis of the conditions that are indeed conditions for their a priori *truth*, and not conditions of their warranted assertability or provability (either effective or in principle), thus addressing the key point of both versions of Benacerraf's dilemma against so-called "combinatorial" or purely proof-theoretic accounts of mathematical knowledge. In other words, on the view that there are indeed mathematical truths, the Kantian rejoinder to Benacerraf's incompatibility claim is that the attribution to mathematical propositions of conditions we can know to obtain when they do *succeeds* in connecting these conditions with an analysis of the propositions which shows how the assigned conditions are conditions that we may only know independently of experience.

The ontological price to pay is a denial that mathematical objects are either *sui generis* or abstract (in the sense of not being connected to any relevant feature of our cognitive makeup). The epistemological benefit is that we do not have to postulate either *de re* intuition or intellectual non-sensory intuition, i.e., any special kind of intuition that might be specially fitted, as it were miraculously, to the special mathematical case.

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