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ASYMPTOTICALLY SELF-SIMILAR GLOBAL SOLUTIONS FOR HARDY-HÉNON PARABOLIC SYSTEMS

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Abstract. In this paper we study the nonlinear parabolic system
\[
\begin{align*}
\partial_t u &= \Delta u + a|x|^{-\gamma}|v|^{q-1}v, \\
\partial_t v &= \Delta v + b|x|^{-\gamma}|u|^{p-1}u,
\end{align*}
\]
with initial data
\[
\begin{align*}
u(0,x) &= \varphi_1(x), \\
v(0,x) &= \varphi_2(x),
\end{align*}
\]
where \(u, v \in \mathbb{R}^N \setminus \{0\}, N \geq 1, a, b \in \mathbb{R}, 0 \leq \gamma < \min(N, 2), 0 < p < \min(N, 2), p, q > 1\). Under conditions on the parameters \(p, q, \gamma\) and \(\rho\) we show the existence and uniqueness of global solutions for initial values small with respect to some norms. In particular, we show the existence of self-similar solutions with initial value \(\Phi = (\varphi_1, \varphi_2)\), where \(\varphi_1, \varphi_2\) are homogeneous initial data. We also prove that some global solutions are asymptotic for large time to self-similar solutions.

1. INTRODUCTION

In this paper we consider global in time solutions of the following nonlinear parabolic system
\[
(S) \begin{cases}
\partial_t u &= \Delta u + a|x|^{-\gamma}|v|^{q-1}v, \\
\partial_t v &= \Delta v + b|x|^{-\gamma}|u|^{p-1}u,
\end{cases}
\]
with initial data
\[
u(0,x) = \varphi_1(x), \\
v(0,x) = \varphi_2(x),
\]
where \( u = u(t, x) \in \mathbb{R} \), \( v = v(t, x) \in \mathbb{R} \), \( t > 0 \), \( x \in \mathbb{R}^N \), \( a, b \in \mathbb{R} \), \( 0 \leq \gamma < \min(N, 2) \), \( 0 < \rho < \min(N, 2) \), \( p, q > 1 \).

In what follows, we denote \( \| \cdot \|_{L^r(\mathbb{R}^N)} \) by \( \| \cdot \|_r \). For \( f, g : I \to \mathbb{R} \), we denote when there exists \( \sup_{t \in I} [f(t), g(t)] = \max [\sup_{t \in I} f(t), \sup_{t \in I} g(t)] \). For all \( t > 0 \), \( e^{t\Delta} \) denotes the heat semi-group, that is

\[
(e^{t\Delta} f)(x) = \int_{\mathbb{R}^N} G(t, x - y)f(y)dy,
\]

where

\[
G(t, x) = (4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^N,
\]

and \( f \in L^r(\mathbb{R}^N), \ r \in [1, \infty) \) or \( f \in C_0(\mathbb{R}^N) \). For \( f \in \mathcal{S}'(\mathbb{R}^N) \), \( e^{t\Delta} f \) is defined by duality.

A mild solution of the system \((S)-(1.1)\) is a solution of the integral system

\[
\begin{align*}
  u(t) &= e^{t\Delta} \varphi_1 + a \int_0^t e^{(t-\sigma)\Delta} \left( |\cdot|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma) \right) d\sigma, \\
  v(t) &= e^{t\Delta} \varphi_2 + b \int_0^t e^{(t-\sigma)\Delta} \left( |\cdot|^{-\rho}|u(\sigma)|^{q-1}u(\sigma) \right) d\sigma.
\end{align*}
\]

(1.2)

We investigate the existence of global solutions, including self-similar solutions for the semi-linear system (1.2). Moreover, we are concerned with estimating the decaying rate in time of some global solutions and their asymptotic behavior.

Using the key estimate established by Proposition 2.1 in [1] we can adapt the method in Fujita and Kato [9, 10] and recently used in [1, 3, 4, 5, 6, 7, 8, 17, 18, 19].

This method is based on a contraction mapping argument on the associated integral system (1.2). Precisely we transform the problem of existence and uniqueness of global solutions into a problem of a fixed point for a function defined in a suitable Banach space equipped with a norm chosen so that we obtain directly the global character of the solution.

In this paper we seek conditions for the following parameters \( p, q, \gamma \) and \( \rho \) such that we have the global existence of some class of solutions, including self-similar solutions and the nonlinear asymptotic self-similar behavior of these solutions. For this we define \( k, \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) by

\[
k = \frac{(2 - \gamma)q + (2 - \rho)}{(2 - \rho)p + (2 - \gamma)},
\]

(1.3)

\[
\alpha_1 = \frac{1}{2(pq - 1)} [(2 - \rho)p + (2 - \gamma)],
\]

(1.4)

\[
\alpha_2 = \frac{1}{2(pq - 1)} [(2 - \gamma)q + (2 - \rho)],
\]

(1.5)

\[
\beta_1 = \alpha_1 - N \frac{1}{2r_1} = \frac{1}{2(pq - 1)} [(2 - \rho)p + (2 - \gamma)] - \frac{N}{2r_1}, \quad r_1 > 1,
\]

(1.6)

\[
\beta_2 = \alpha_2 - N \frac{1}{2r_2} = \frac{1}{2(pq - 1)} [(2 - \gamma)q + (2 - \rho)] - \frac{N}{2r_2}, \quad r_2 > 1.
\]

(1.7)
Note that $\alpha_1$ and $\alpha_2$ verify the following system
\begin{align*}
2 - \gamma + 2\alpha_1 &= 2\alpha_2 p, \\
2 - \rho + 2\alpha_2 &= 2\alpha_1 q,
\end{align*}
and that
\[ kp > 1, \quad q > k \quad \text{and} \quad \frac{\alpha_2}{\alpha_1} = k. \]

Let us summarize the results of this paper. First of all if we suppose that the following conditions
\begin{align*}
2\alpha_1 < \min\left( N, \frac{p}{q} (N - \rho) \frac{(2 - \gamma)q + (2 - \rho)}{2 + (2 - \rho)p - \gamma pq} \right), \quad (1.9) \\
2\alpha_2 < \min\left( N, \frac{q}{p} (N - \gamma) \frac{(2 - \rho)p + (2 - \gamma)}{2 + (2 - \gamma)q - ppq} \right), \quad (1.10)
\end{align*}
are satisfied, then we prove the global existence of solutions for some initial data $\Phi = (\varphi_1, \varphi_2)$ small with respect to the norm $N$ defined by
\[ N(\Phi) := \sup_{t>0} \left[ e^{t\Delta} \varphi_1 \| e^{t\Delta} \varphi_1 \|_{r_1}, e^{t\Delta} \varphi_2 \| e^{t\Delta} \varphi_2 \|_{r_2} \right], \quad (1.11) \]
where $\beta_1$ and $\beta_2$ are given by (1.6) and (1.7), $r_1$ and $r_2$ are defined in Lemma 2.1 below. See Theorem 1 below. We also prove, for $\varphi_1$ homogeneous of degree $-2\alpha_1$ and $\varphi_2$ homogeneous of degree $-2\alpha_2$, where $\alpha_1$ and $\alpha_2$ are given by (1.4) and (1.5), that the initial data $\Phi = (\varphi_1, \varphi_2)$ gives rise to a global self-similar solution. See Theorem 2 below. Next we show as in [1], that solutions with initial data $\Psi$ which behaves asymptotically like $\Phi$ in some appropriate sense as $|x| \to \infty$, are asymptotically self-similar in the $L^\infty$-norm. See Theorem 3 below. The norm $N$ given in (1.11) is weak enough so that initial data $\Phi = (\varphi_1, \varphi_2)$ with homogeneous components have finite norm. We prove finally stronger uniqueness results in Lebesgue spaces for initial values small with respect of some norm. See Theorem 4 below.

Yamauchi in [24] studied the parabolic system (S). In [24, Theorem 2.1, p. 339] it is shown that for some nonnegative initial values under the conditions $\gamma < \min(N, 2), \rho < \min(N, 2), \quad \rho - 1 > 0$ and $\max(\alpha_1, \alpha_2) \geq \frac{N}{2}$, that no nonnegative nontrivial solutions exist.

The case $\gamma = \rho = 0$ has been already covered in [19]. In the case where $p = q$ and $\gamma = \rho > 0$, the parabolic system (S) behaves like a parabolic equation with singularity in the nonlinearity. For more reading about Hardy-Hénon equations see [1, 12, 14, 22].

The rest of the paper is organized as follows. In Section 2, we state the main results. In Section 3, we give the proofs of the main theorems. Finally, in Section 4, we give stronger uniqueness results. Throughout this paper $C$ will be a positive constant which may have different values at different places. We denote sometimes $u(t)$ by $u(t, \cdot)$. 
2. Main results

We now state the main results of the paper. Let \( e^{t\Delta} \) be the linear heat semi-group defined by
\[
(e^{t\Delta} \varphi)(x) = (G(t, .) * \varphi)(x),
\]
where \( G \) is the heat kernel
\[
G(t, x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^N.
\]
We recall the smoothing effect of the heat semi-group
\[
\|e^{t\Delta}f\|_{s_2} \leq (4\pi t)^{-\frac{N}{2}(\frac{1}{s_1} - \frac{1}{s_2})}\|f\|_{s_1}, \quad (2.1)
\]
for \( 1 \leq s_1 \leq s_2 \leq \infty, t > 0 \) and \( f \in L^{s_1}(\mathbb{R}^N) \). We recall also the following key estimate from [1]
\[
\|e^{t\Delta}(|.|^\gamma f)\|_{q_2} \leq C(N, \gamma, q_1, q_2)t^{-\frac{N}{2}(\frac{1}{q_1} - \frac{1}{q_2})}\|f\|_{q_1}, \quad (2.2)
\]
for \( 0 \leq \gamma < N, q_1 \) and \( q_2 \) such that \( 0 \leq \frac{1}{q_1} < \frac{\gamma}{N} + \frac{1}{q_2} < 1, t > 0 \) and \( f \in L^{q_1}(\mathbb{R}^N) \). We note that if \( q_2 = \infty \), then \( e^{t\Delta}(|.|^\gamma f) \in C_0(\mathbb{R}^N) \).

We begin with the following technical lemma.

**Lemma 2.1** (Technical lemma). Let \( N \) be a positive integer. Let \( p, q > 1 \). Let \( 0 \leq \gamma < \min(N, 2) \) and \( 0 < \rho < \min(N, 2) \). Let \( k \) be given by (1.3). Let \( \alpha_1, \alpha_2 \) defined by (1.4) and (1.5). Suppose that (1.9) and (1.10) are satisfied. Let \( \beta_1, \beta_2 \) be given by (1.6) and (1.7). Then there exist \( r_1 > 1 \) and \( r_2 > 1 \) satisfying
\[
r_1 = kr_2, \quad (2.3)
\]
such that
(i) \( \beta_1 > 0, \beta_2 > 0 \) and \( \beta_2 = k\beta_1 \),
(ii) \( \frac{1}{r_1} < \frac{\gamma}{N} + \frac{\rho}{r_2} < 1 \) and \( \frac{1}{r_1} < \frac{\rho}{N} + \frac{\rho}{r_2} < 1 \),
(iii) \( \beta_2 \rho < 1 \) and \( \beta_1 \gamma < 1 \),
(iv) \( \frac{N}{2r_1} \left(-1 + \frac{\rho}{r_2}\right) < \frac{2-\gamma}{2} \) and \( \frac{N}{2r_2} \left(-1 + \frac{\rho}{r_2}\right) < \frac{2-\gamma}{2} \),
(v) \( \frac{1}{r_1} < \frac{2\alpha_1}{N} < \frac{\gamma}{N} + \frac{\rho}{r_2} \) and \( \frac{1}{r_1} < \frac{2\alpha_2}{N} < \frac{\rho}{N} + \frac{\rho}{r_2} \),
(vi) \( -\frac{N}{2} \left(\frac{p}{r_1} - \frac{1}{r_2}\right) - \frac{p}{2} - \beta_2 \rho + 1 + \beta_1 = 0 \) and \( -\frac{N}{2} \left(\frac{q}{r_1} - \frac{1}{r_2}\right) - \frac{q}{2} - \beta_1 \gamma + 1 + \beta_2 = 0 \).

We prove this lemma in the appendix.

**Theorem 1** (Global existence and continuous dependence). Let \( N \) be a positive integer. Let \( p, q > 1 \). Let \( 0 \leq \gamma < \min(N, 2) \) and \( 0 < \rho < \min(N, 2) \). Let \( \alpha_1, \alpha_2 \) defined by (1.4)
and (1.5). Suppose that (1.9) and (1.10) are satisfied. Let \( \beta_1, \beta_2 \) be given by (1.6) and (1.7). Let \( r_1 \) and \( r_2 \) be as in Lemma 2.1. Let \( M > 0 \) be such that

\[
\nu = \max(M^{p-1} \nu_1, M^{q-1} \nu_2) < 1,
\]

(2.4)

where \( \nu_1 \) and \( \nu_2 \) are two positive constants given by (3.8) and (3.9) below. Choose \( R > 0 \) such that

\[
R + M \nu \leq M.
\]

(2.5)

Let \( \Phi = (\varphi_1, \varphi_2) \) be an element of \( S'(\mathbb{R}^N) \times S'(\mathbb{R}^N) \) such that

\[
\mathcal{N}(\Phi) := \sup_{t > 0} \left[ t^{\beta_1} \| e^{t \Delta} \varphi_1 \|_{r_1}, t^{\beta_2} \| e^{t \Delta} \varphi_2 \|_{r_2} \right] \leq R.
\]

(2.6)

Then there exists a unique global solution \( U = (u, v) \in C((0, \infty); L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N)) \) of the integral system (1.2) such that

\[
\sup_{t > 0} \left[ t^{\beta_1} \| u(t) \|_{r_1}, t^{\beta_2} \| v(t) \|_{r_2} \right] \leq M.
\]

(2.7)

Furthermore,

(a) \( \lim_{t \searrow 0} u(t) = \varphi_1 \) and \( \lim_{t \searrow 0} v(t) = \varphi_2 \) in the sense of tempered distributions,

(b) \( u(t) - e^{t \Delta} \varphi_1 \in C \left( [0, \infty), L^{r_1}(\mathbb{R}^N) \right) \) for \( r_1 \) satisfying \( \frac{2q_1}{N} < \frac{1}{r_1} < \frac{q}{2} + \frac{p}{r_2} \),

(c) \( v(t) - e^{t \Delta} \varphi_2 \in C \left( [0, \infty), L^{r_2}(\mathbb{R}^N) \right) \) for \( r_2 \) satisfying \( \frac{2q_2}{N} < \frac{1}{r_2} < \frac{q}{2} + \frac{p}{r_1} \),

(d) \( \sup_{t > 0} t^{q_1 - \frac{N}{r_1}} \| u(t) \|_r < \infty, \forall r \in [r_1, \infty], \) and \( u \in C \left( (0, \infty), L^r(\mathbb{R}^N) \cap C_0(\mathbb{R}^N) \right) \),

(e) \( \sup_{t > 0} t^{q_2 - \frac{N}{r_2}} \| v(t) \|_r < \infty, \forall r \in [r_2, \infty], \) and \( v \in C \left( (0, \infty), L^r(\mathbb{R}^N) \cap C_0(\mathbb{R}^N) \right) \).

In addition, if \( \Phi = (\varphi_1, \varphi_2) \) and \( \Psi = (\psi_1, \psi_2) \) satisfy (2.6), and if \( U_1 = (u_1, v_1) \) and \( U_2 = (u_2, v_2) \) respectively are the solutions of the system (1.2) with initial values \( \Phi \) and \( \Psi \), then

\[
\sup_{t > 0} \left[ t^{\beta_1} \| u_1(t) - u_2(t) \|_{r_1}, t^{\beta_2} \| v_1(t) - v_2(t) \|_{r_2} \right] \leq (1 - \nu)^{-1} \mathcal{N}(\Phi - \Psi).
\]

(2.8)

Furthermore, if the initial data \( \Phi \) and \( \Psi \) are such that

\[
\mathcal{N}_\delta(\Phi - \Psi) = \sup_{t > 0} \left[ t^{\beta_1 + \delta} \| e^{t \Delta} (\varphi_1 - \psi_1) \|_{r_1}, t^{\beta_2 + \delta} \| e^{t \Delta} (\varphi_2 - \psi_2) \|_{r_2} \right] < \infty,
\]

(2.9)

for some \( 0 < \delta < \delta_0 \), where

\[
\delta_0 = \min \{ 1 - \beta_1 q, 1 - \beta_2 p \}.
\]

(2.10)

Then

\[
\sup_{t > 0} \left[ t^{\beta_1 + \delta} \| u_1(t) - u_2(t) \|_{r_1}, t^{\beta_2 + \delta} \| v_1(t) - v_2(t) \|_{r_2} \right] \leq (1 - \nu')^{-1} \mathcal{N}_\delta(\Phi - \Psi),
\]

(2.11)
where the positive constant $M$ is chosen small enough so that $0 < \nu' < 1$, where $\nu'$ is given by the relations (3.16)-(3.18) below.

Finally, if we suppose also that $\Phi = (\varphi_1, \varphi_2) \in L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ such that

$$N'(\Phi) := \max \left[ \| \varphi_1 \|_{L^\infty}, \| \varphi_2 \|_{L^\infty} \right] < R,$$

then the solution $U = (u, v)$ of the integral system (1.2) satisfies also $U \in C \left( [0, \infty), L^\infty(\mathbb{R}^N) \right) \times C \left( [0, \infty), L^\infty(\mathbb{R}^N) \right)$ and

$$\sup_{t \geq 0} \left[ \| u(t) \|_{L^\infty}, \| v(t) \|_{L^\infty} \right] \leq M.$$  \hspace{1cm} (2.13)

Where $M$ and $R$ are sufficiently small.

Now we give the following result which proves the existence of self-similar solutions.

**Theorem 2** (Self-similar solutions). Let $N$ be a positive integer. Let $p, q > 1$. Let $0 \leq \gamma < \min(N, 2)$ and $0 < \rho < \min(N, 2)$. Let $\alpha_1, \alpha_2$ defined by (1.4) and (1.5). Suppose that (1.9) and (1.10) are satisfied. Let $\varphi_1(x) = \omega_1(x)|x|^{-2\alpha_1}$, $\varphi_2(x) = \omega_2(x)|x|^{-2\alpha_2}$, where $\omega_1, \omega_2 \in L^\infty(\mathbb{R}^N)$ are homogeneous of degree 0 and $\| \omega_1 \|_\infty, \| \omega_2 \|_\infty$ are sufficiently small. Denote $\Phi = (\varphi_1, \varphi_2)$, then there exists a global self-similar solution $U_S = (u_S, v_S)$ of (1.2) with initial data $\Phi$. Moreover $U_S(t) \to \Phi$ in $S'(\mathbb{R}^N)$ as $t \to 0$.

We turn now to the asymptotic behavior.

**Theorem 3** (Asymptotic behavior). Let $N$ be a positive integer. Let $p, q > 1$. Let $0 \leq \gamma < \min(N, 2)$ and $0 < \rho < \min(N, 2)$. Let $\alpha_1, \alpha_2$ defined by (1.4) and (1.5). Suppose that (1.9) and (1.10) are satisfied. Let $\beta_1, \beta_2$ be given by (1.6) and (1.7). Let $r_1$ and $r_2$ be as in Lemma 2.1. Define $\beta_1(q)$ and $\beta_2(q)$ by

$$\beta_1(q) = \alpha_1 - \frac{N}{2q}, \quad \beta_2(q) = \alpha_2 - \frac{N}{2q}, \quad q > 1.$$ \hspace{1cm} (2.14)

Let $\Phi$ be given by

$$\Phi(x) = (\varphi_1(x), \varphi_2(x)) := (\omega_1(x)|x|^{-2\alpha_1}, \omega_2(x)|x|^{-2\alpha_2})$$

with $\omega_1, \omega_2$ homogeneous of degree 0, $\omega_1, \omega_2 \in L^\infty(\mathbb{R}^N)$ and $\| \omega_1 \|_\infty, \| \omega_1 \|_\infty$ are sufficiently small. Let

$$U_S(t, x) = \left( t^{-\alpha_1} u_S(1, \frac{x}{\sqrt{t}}), t^{-\alpha_2} v_S(1, \frac{x}{\sqrt{t}}) \right)$$

be the self-similar solution of (1.2) given by Theorem 2.
Let \( \Psi = (\psi_1, \psi_2) \in C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N) \) be such that
\[
|\psi_1(x)| \leq \frac{c}{(1 + |x|^2)^{\alpha_1}}, \quad \forall x \in \mathbb{R}^N, \quad \psi_1(x) = \omega_1(x)|x|^{-2\alpha_1}, \quad |x| \geq A,
\]
\[
|\psi_2(x)| \leq \frac{c}{(1 + |x|^2)^{\alpha_2}}, \quad \forall x \in \mathbb{R}^N, \quad \psi_2(x) = \omega_2(x)|x|^{-2\alpha_2}, \quad |x| \geq A,
\]
for some constant \( A > 0 \), where \( c \) is a small positive constant. (We take \( k_1, k_2 \) sufficiently small so that (2.6) is satisfied by \( \Phi \) and \( \Psi \)).

Let \( U = (u, v) \) be the global solution of (1.2) with initial data \( \Psi \) constructed by Theorem 1. Then there exists \( \eta > 0 \) sufficiently small such that
\[
\|u(t) - u_S(t)\|_{q_1} \leq C t^{-\beta_1(q_1) - \delta}, \quad \forall t > 0,
\]
\[
\|v(t) - v_S(t)\|_{q_2} \leq C t^{-\beta_2(q_2) - \delta}, \quad \forall t > 0,
\]
for all \( q_1 \in [r_1, \infty], \ q_2 \in [r_2, \infty] \). Also, we have
\[
\|t^{\alpha_1} u(t, \cdot, \sqrt{t}) - u_S(1, .)\|_{q_1} \leq C t^{-\delta}, \quad \forall t > 0,
\]
\[
\|t^{\alpha_2} v(t, \cdot, \sqrt{t}) - v_S(1, .)\|_{q_2} \leq C t^{-\delta}, \quad \forall t > 0,
\]
for all \( q_1 \in [r_1, \infty], \ q_2 \in [r_2, \infty] \).

To close this section we give the conditions on \( p, q, \gamma, \rho \) which guarantee that the relations (1.9) and (1.10) are satisfied.

**Proposition 2.2.** Let \( N \) be a positive integer. Let the real numbers \( p, q > 1 \). Suppose that
\[
\max[p, q] + 1 < \frac{N}{2} (pq - 1).
\]
Then there exist \( \gamma_0, \rho_0 > 0 \) such that for all \( 0 \leq \gamma < \gamma_0, \ 0 < \rho < \rho_0, \) (1.9) and (1.10) are satisfied.

**Proposition 2.3.** Let \( N \) be a positive integer. Fix \( 0 < \gamma < \min(2, N) \) and \( 0 < \rho < \min(2, N) \). Let \( p, q > 1 \) such that
\[
p \geq \max \left( \frac{2 - \gamma}{N} + \frac{2 - \rho}{N} + 1, \frac{2 - \gamma}{\rho} + \frac{2}{\rho} \right),
\]
and
\[
q \geq \max \left( \frac{2 - \rho}{N} + \frac{2 - \gamma}{N} + 1, \frac{2 - \rho}{\gamma} + \frac{2}{\gamma} \right).
\]
Then (1.9) and (1.10) are satisfied.

The proof of those two propositions is given in the next section.
We look for global solutions of the system (1.2) via a fixed point argument. Let us denote $U = (u, v)$, $\Phi = (\varphi_1, \varphi_2)$ and
\[
F_\Phi(U) = (F_\Phi(U), G_\Phi(U)),
\]
where
\[
F_\Phi(U)(t) = e^{t\Delta} \varphi_1 + a \int_0^t e^{(t-s)\Delta} \left(|\cdot|^{-\gamma} |v(s)|^{p-1} v(s)\right) ds,
\]
\[
G_\Phi(U)(t) = e^{t\Delta} \varphi_2 + b \int_0^t e^{(t-s)\Delta} \left(|\cdot|^{-\rho} |u(s)|^{q-1} u(s)\right) ds,
\]
with $\varphi_1$ and $\varphi_2$ being two tempered distributions, $a, b \in \mathbb{R}$, $0 \leq \gamma < \min(N, 2)$, $0 < \rho < \min(N, 2)$, $p, q > 1$.

**Proof of Theorem 1.** Let $X$ be the set of continuous functions $U : (0, \infty) \rightarrow L^r(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, such that
\[
\|U\|_X := \sup_{t>0} \left[ t^{\beta_1} \|u(t)\|_{r_1}, t^{\beta_2} \|v(t)\|_{r_2} \right] < \infty,
\]
where $r_1, r_2$ are two positive real numbers satisfying conditions in Lemma 2.1 and $\beta_1, \beta_2$ are respectively given by (1.6) and (1.7).

Let $M > 0$ and define the closed ball in the Banach space $X$ by
\[
X_M = \{ U \in X, \|U\|_X \leq M \}.
\]
$X_M$, endowed with the metric $d(U_1, U_2) = \|U_1 - U_2\|_X$, is a complete metric space.

Consider the mapping $F_\Phi$ defined by (3.1), where $\Phi = (\varphi_1, \varphi_2) \in S'(\mathbb{R}^n) \times S'(\mathbb{R}^n)$ satisfies (2.6). We will show that $F_\Phi = (F_\Phi, G_\Phi)$ is a strict contraction mapping on $X_M$.

Let $\Phi = (\varphi_1, \varphi_2)$ and $\Psi = (\psi_1, \psi_2)$ belong to $S'(\mathbb{R}^n) \times S'(\mathbb{R}^n)$ satisfying (2.6). Let $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ be two elements of $X_M$. Then we have
\[
t^{\beta_1} \|F_\Phi(U_1)(t) - F_\Phi(U_2)(t)\|_{r_1} \leq t^{\beta_1} \|e^{t\Delta} (\varphi_1 - \psi_1)\|_{r_1} + |a| t^{\beta_1} \int_0^t \|e^{(t-s)\Delta} |\cdot|^{-\gamma} |v_1(s)|^{p-1} v_1(s) - |v_2(s)|^{p-1} v_2(s)\|_r ds.
\]
It follows, by the key estimate (2.2) with $(q_1, q_2) = (\frac{2N}{p}, r_1)$ that
\[
t^{\beta_1} \|F_\Phi(U_1)(t) - F_\Phi(U_2)(t)\|_{r_1} \leq t^{\beta_1} \|e^{t\Delta} (\varphi_1 - \psi_1)\|_{r_1} + |a| t^{\beta_1} \int_0^t C(t - \sigma)^{-\frac{N}{q_1}} \left(\frac{t}{r_1}\right)^{\frac{N}{q_1}} \|v_1(s)|^{p-1} v_1(s) - |v_2(s)|^{p-1} v_2(s)\|_{r_2} ds.
\]
Using the fact that, for $r > p > 1$,
\[
|||f||^p - |g||^p||| f - g||_{r/p} \leq p(||f||^p_{r} + ||g||^p_{r})||f - g||_r,
\]
we obtain by (3.4) and the fact that $U_1$ and $U_2$ belongs to $X_M$, that
\[
t^{\beta_1}||F_\phi(U_1)(t) - F_\phi(U_2)(t)||_{r_1} \leq t^{\beta_1}||e^{t\Delta}(\varphi_1 - \psi_1)||_{r_1} + 2|a|Ct^{\beta_1}M^{p-1}d\sigma \times \left[ \int_0^t (t - \sigma)^{-\frac{N}{\delta_1}(1+\frac{2\delta}{2})-\frac{2}{2} - \sigma - \beta_2 p M^{-1}d\sigma} \right] ||U_1 - U_2||_X.
\]

It follows that
\[
t^{\beta_1}||F_\phi(U_1)(t) - F_\phi(U_2)(t)||_{r_1} \leq t^{\beta_1}||e^{t\Delta}(\varphi_1 - \psi_1)||_{r_1} + 2|a|Ct^{\beta_1}M^{p-1}d\sigma \times \left[ \int_0^t (t - \sigma)^{-\frac{N}{\delta_1}(1+\frac{2\delta}{2})-\frac{2}{2} - \sigma - \beta_2 p M^{-1}d\sigma} \right] ||U_1 - U_2||_X. \tag{3.5}
\]

Similarly using estimate (2.2) with $(q_1, q_2) = (\frac{2}{q}, r)$, we obtain an analogous estimate of
\[
t^{\beta_2}||G_\phi(U_1)(t) - G_\phi(U_2)(t)||_{r_2} \leq t^{\beta_2}||e^{t\Delta}(\varphi_2 - \psi_2)||_{r_2} + 2|b|Ct^{\beta_2}M^{q-1}d\sigma \times \left[ \int_0^t (t - \sigma)^{-\frac{N}{\delta_2}(1+\frac{2\rho}{2})-\frac{2}{2} - \sigma - \beta_1 q M^{-1}d\sigma} \right] ||U_1 - U_2||_X. \tag{3.6}
\]

Now, due to Part (vi) of Lemma 2.1, inequalities (3.5) and (3.6) we obtain
\[
||F_\phi(U_1) - F_\phi(U_2)||_X \leq N(\Phi - \Psi) + \nu ||U_1 - U_2||_X, \tag{3.7}
\]
where
\[
\nu = \max(M^{p-1}\nu_1, M^{q-1}\nu_2),
\]
with
\[
\nu_1 = 2|a|Ct \int_0^1 (1 - \sigma)^{-\frac{N}{\delta_1}(1+\frac{2\delta}{2})-\frac{2}{2} - \sigma - \beta_2 p d\sigma}, \tag{3.8}
\]
\[
\nu_2 = 2|b|Ct \int_0^1 (1 - \sigma)^{-\frac{N}{\delta_2}(1+\frac{2\rho}{2})-\frac{2}{2} - \sigma - \beta_1 q d\sigma}. \tag{3.9}
\]
Finally, from Parts (iii)-(iv) of Lemma 2.1, we see that both quantities \( \nu_1 \) and \( \nu_2 \) are finite. Setting \( \Psi = 0 \) and \( U_2 = 0 \), the inequality (3.7) becomes

\[
\| \mathcal{F}_\Phi(U_1) \|_X \leq N(\Phi) + \nu \| U_1 \|_X. \tag{3.10}
\]

If we choose \( M \) and \( R \) such that (2.5) and (2.6) are satisfied then by (3.10), \( \mathcal{F}_\Phi \) maps \( X_M \) into itself. Letting \( \Phi = \Psi \), we observe that (3.7) becomes

\[
\| \mathcal{F}_\Phi(U_1) - \mathcal{F}_\Phi(U_2) \|_X \leq \nu \| U_1 - U_2 \|_X.
\]

Hence inequality (2.4) gives that \( \mathcal{F}_\Phi \) is a strict contraction mapping from \( X_M \) into itself. So \( \mathcal{F}_\Phi \) has a unique fixed point \( U = (u, v) \) in \( X_M \) which is solution of (1.2). This achieves the proof of the existence of a unique global solution of (1.2) in \( X_M \).

We now prove the statements (a)-(c). Let \( \tau_1 \) be a positive real number satisfying

\[
\frac{2\alpha_1}{N} < \frac{1}{\tau_1} < \frac{\gamma}{N} + \frac{p}{r_2}, \tag{3.11}
\]

then by (2.2) with \((q_1, q_2) = (\frac{p}{r}, \tau_1)\), we have

\[
\| u(t) - e^{t \Delta} \varphi_1 \|_{\tau_1} \leq |a| \int_0^t \| e^{(t-s)\Delta} \left( |.|^{-\gamma} |v(\sigma)|^{p-1} v(\sigma) \right) \|_{\tau_1} d\sigma
\leq |a| \int_0^t C(t - \sigma)^{-\frac{N}{2} \left( \frac{p}{r} - \frac{1}{r_1} \right) - \frac{N}{2}} \| v(\sigma) \|_{\tau_1} d\sigma
\leq |a| C M P \int_0^t (1 - \sigma)^{-\frac{N}{2} \left( \frac{p}{r} - \frac{1}{r_1} \right) - \frac{N}{2}} d\sigma.
\]

Therefore

\[
\| u(t) - e^{t \Delta} \varphi_1 \|_{\tau_1} \leq C_1 \left( -\frac{N}{2} \left( \frac{p}{r} - \frac{1}{r_1} \right) - \frac{N}{2} \right)^{-1} \sigma^{-\frac{N}{2} \left( \frac{p}{r} - \frac{1}{r_1} \right) - \frac{N}{2}} d\sigma,
\]

where

\[
C_1 = |a| C M P \int_0^1 (1 - \sigma)^{-\frac{N}{2} \left( \frac{p}{r} - \frac{1}{r_1} \right) - \frac{N}{2}} d\sigma,
\]

is a positive constant. Owing to (3.11) and Part (iii) of Lemma 2.1, the constant \( C_1 \) is finite. Similarly using (2.2) with \((q_1, q_2) = (\frac{q}{r}, \tau_2)\), we obtain for \( \tau_2 \) satisfying

\[
\frac{2\alpha_2}{N} < \frac{1}{\tau_2} < \frac{p}{N} + \frac{q}{r_1}, \tag{3.13}
\]

the following inequality

\[
\| v(t) - e^{t \Delta} \varphi_2 \|_{\tau_2} \leq C_2 \left( -\frac{N}{2} \left( \frac{q}{r} - \frac{1}{r_2} \right) - \frac{N}{2} \right)^{-1} \sigma^{-\frac{N}{2} \left( \frac{q}{r} - \frac{1}{r_2} \right) - \frac{N}{2}} d\sigma,
\]

where \( C_2 \) is a positive constant given by

\[
C_2 = |b| C M P \int_0^1 (1 - \sigma)^{-\frac{N}{2} \left( \frac{q}{r} - \frac{1}{r_2} \right) - \frac{N}{2}} d\sigma,
\]
which is finite by (3.13) and Part (iii) of Lemma 2.1.
Owing to the conditions (3.11) and (3.13), the right hand sides of (3.12) and (3.14) converges to zero as \( t \to 0 \). This proves statements (a)-(c) of Theorem 1.

Finally, the continuous dependence relation (2.8) of Theorem 1 follows by considering \( F(U_1) = U_1 \) and \( F(U_2) = U_2 \) in the inequality (3.7).

Now, if in addition \( \Phi \) and \( \Psi \) satisfy (2.9), then following the same steps as above but with the norm

\[
\|U\| = \|u, v\|_{X, \delta} = \sup_{t \geq 0} \left[ t^{\beta_1 + \delta} \|u(t)\|_{r_1}, t^{\beta_2 + \delta} \|v(t)\|_{r_2} \right],
\]

we obtain by the key estimate (2.2) with \((q_1, q_2) = \left( \frac{p}{2}, r_1 \right)\), the fact that \( U_1 \) and \( U_2 \) belongs to \( X_M \) and the estimate \( \|v_1(\sigma) - v_2(\sigma)\|_{r_2} \leq \sigma^{-\beta_2 - \delta} \|U_1 - U_2\|_{X, \delta} \)

\[
t^{\beta_1 + \delta} \|F(\Phi(U_1))(t) - F(\Phi(U_2))(t)\|_{r_1} \leq t^{\beta_1 + \delta} \|e^{t\Delta} (\phi_1 - \psi_1)\|_{r_1} + |a| t^{\beta_1 + \delta}
\]

\[
\times \int_0^t \left( C(t - \sigma) - \frac{N}{T_2} \frac{\frac{d}{d\sigma} - \frac{\delta}{2}}{\frac{d}{d\sigma} - \frac{\delta}{2} - \frac{\beta_1}{2} - \frac{\beta_2}{2}} \right) \left( v_1(\sigma) - v_2(\sigma) \right) \left( \sigma^{-\beta_2 - \delta} \right) d\sigma
\]

\[
\leq t^{\beta_1 + \delta} \|e^{t\Delta} (\phi_1 - \psi_1)\|_{r_1} + 2|a|Ct^{p-1}t^{\beta_1 + \delta}
\]

\[
\times \left[ \int_0^t (t - \sigma)^{-\frac{N}{T_2} \left( -1 + \frac{\delta}{2} \right)} \sigma^{-\beta_2 - \delta} d\sigma \right] \|U_1 - U_2\|_{X, \delta}
\]

\[
\leq t^{\beta_1 + \delta} \|e^{t\Delta} (\phi_1 - \psi_1)\|_{r_1}
\]

\[
+ 2|a|Ct^{p-1}t^{\beta_1 + \delta}
\]

\[
\times \left[ \int_0^t (1 - \sigma)^{-\frac{N}{T_2} \left( -1 + \frac{\delta}{2} \right)} \sigma^{-\beta_2 - \delta} d\sigma \right] \|U_1 - U_2\|_{X, \delta}.
\]

We obtain also

\[
t^{\beta_2 + \delta} \|G(\Phi(U_1))(t) - G(\Phi(U_2))(t)\|_{r_2} \leq t^{\beta_2 + \delta} \|e^{t\Delta} (\phi_2 - \psi_2)\|_{r_2}
\]

\[
+ 2|b|Cq t^{q-1}t^{\beta_2 + \delta}
\]

\[
\times \left[ \int_0^t (1 - \sigma)^{-\frac{N}{T_2} \left( -1 + \frac{\delta}{2} \right)} \sigma^{-\beta_2 - \delta} d\sigma \right] \|U_1 - U_2\|_{X, \delta}.
\]

Then

\[
\|F(\Phi(U_1)) - F(\Phi(U_2))\|_{X, \delta} \leq N_\delta (\Phi - \Psi) + \nu' \|U_1 - U_2\|_{X, \delta},
\]  

(3.15)

where

\[
\nu' = \max (M^{p-1} \nu_1', M^{q-1} \nu_2'),
\]

(3.16)

with

\[
\nu_1' = 2|a|Cp \int_0^t (1 - \sigma)^{-\frac{N}{T_2} \left( -1 + \frac{\delta}{2} \right)} \sigma^{-\beta_2 - \delta} d\sigma,
\]

(3.17)

\[
\nu_2' = 2|b|Cq \int_0^t (1 - \sigma)^{-\frac{N}{T_2} \left( -1 + \frac{\delta}{2} \right)} \sigma^{-\beta_2 - \delta} d\sigma.
\]

(3.18)
Since \( \mathcal{F}_r(U_1) = U_1 \) and \( \mathcal{F}_r(U_2) = U_2 \), then (3.15) becomes
\[
\sup_{t > 0} \left[ t^{\beta_1 + \delta} \| u_1(t) - u_2(t) \|_{r_1}, t^{\beta_2 + \delta} \| v_1(t) - v_2(t) \|_{r_2} \right] \leq (1 - \nu')^{-1} \mathcal{N}_\delta(\Phi - \Psi).
\]
Now, since \( 0 < \delta < \delta_0 \) with \( \delta_0 \) given by (2.10), \( \nu'_1 \) and \( \nu'_2 \) are finite. Thus, (2.11) holds by choosing \( \nu' < 1 \) (this choice is possible for \( M \) small enough), where \( \nu' \) is given by (3.16)-(3.18).

We now prove statements (d)-(e) of Theorem 1 for \( r = \infty \), we use some arguments of [18]. Let us consider two real numbers \( r \) and \( r' \) such that \( r = kr' \) and
\[
1 < r_1 < r \leq \infty, \quad 1 < r_2 < r' \leq \infty, \quad 0 < \frac{N}{2} \left( \frac{p}{r_2} - 1 \right) < \frac{2}{2^p}, \quad 0 < \frac{N}{2} \left( \frac{p}{r_1} - 1 \right) < \frac{2}{2^p}.
\]
(3.19)
Remark that a such choice is possible owing to Lemma A.1. Write now,
\[
\eta(t) = e^{\frac{1}{2} \Delta} u(t/2) + a \int_{\frac{1}{2}}^{t} e^{(t-s)\Delta} (|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)) \, d\sigma.
\]
Then by using the smoothing properties of the heat semigroup (2.1), the estimate (2.2) with \( (q_1, q_2) = (\frac{2}{p}, r) \), (3.19) and the estimate (2.7), we obtain
\[
\begin{align*}
\| u(t) \|_r &\leq C \sup_{t > 0} \left[ t^{\alpha_1 - \frac{N}{2}} \| u(t) \|_{r_1} \right] \\
&\quad + \left| a \right| t^{\alpha_1 - \frac{N}{2}} \int_{\frac{1}{2}}^{t} \| e^{(t-s)\Delta} (|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)) \|_r \, d\sigma \\
&\leq CM + C t^{\alpha_1 - \frac{N}{2}} \int_{\frac{1}{2}}^{t} (t-s)^{-\frac{N}{2} \left( \frac{p}{r_2} - 1 \right) - \frac{\gamma}{2}} \| v(\sigma) \|_{r_2} \, d\sigma \\
&\leq CM + C M \mu t^{\alpha_1 - \frac{N}{2}} \int_{\frac{1}{2}}^{t} (t-s)^{-\frac{N}{2} \left( \frac{p}{r_2} - 1 \right) - \frac{\gamma}{2}} \sigma^{-\beta_2 \mu} \, d\sigma \\
&\leq CM + C M \mu \int_{\frac{1}{2}}^{1} (1-s)^{-\frac{N}{2} \left( \frac{p}{r_2} - 1 \right) - \frac{\gamma}{2}} \sigma^{-\beta_2 \mu} \, d\sigma,
\end{align*}
\]
which leads to
\[
\sup_{t > 0} \left[ t^{\alpha_1 - \frac{N}{2}} \| u(t) \|_r \right] \leq C(M) < \infty.
\]
Analogously, we obtain the following estimate on the second component \( v \):
\[
\sup_{t > 0} \left[ t^{\alpha_2 - \frac{N}{2}} \| v(t) \|_{r'} \right] \leq C(M) < \infty.
\]
We iterate this procedure, for the next step we replace in (3.19) $r_1$ by $r$, $r_2$ by $r'$ and we consider two real numbers $s_2$ and $s_2'$ such that $s_2 = ks'_2$ and

$$1 < r < s_2 \leq \infty, \quad 0 < \frac{N}{2} \left(\frac{p}{r} - \frac{1}{s_2} \right) < \frac{2 - \gamma}{2}, \quad 0 < \frac{N}{2} \left(\frac{p}{r} - \frac{1}{s_2'} \right) < \frac{2 - \gamma}{2}.$$  

We obtain

$$\sup_{t > 0} \left[ t^{\alpha_1 - \frac{N}{2s_2} \| u(t) \|_{s_2}}, t^{\alpha_2 - \frac{N}{2s_2'} \| v(t) \|_{s_2'}} \right] \leq C(M) < \infty.$$  

We therefore construct two sequences $(s_i)_i$ and $(s'_i)_i$ with $s_0 = r_1, s'_0 = r_2, s_1 = r, s'_1 = r'$ and such that $s_i = ks'_i, \forall i = 0, 1, 2, ...$ and

$$1 < s_i < s_{i+1} \leq \infty, \quad 1 < s'_i < s'_{i+1} \leq \infty, \quad 0 < \frac{N}{2} \left(\frac{p}{r} - \frac{1}{s_{i+1}} \right) < \frac{2 - \gamma}{2}, \quad 0 < \frac{N}{2} \left(\frac{p}{r} - \frac{1}{s'_{i+1}} \right) < \frac{2 - \gamma}{2}.$$  

We prove that

$$\sup_{t > 0} \left[ t^{\alpha_1 - \frac{N}{2s_i} \| u(t) \|_{s_i}}, t^{\alpha_2 - \frac{N}{2s'_i} \| v(t) \|_{s'_i}} \right] \leq C(M) < \infty, \quad \forall i = 0, 1, 2, ...$$

Now by Lemma A.1, one can choose the sequences $(s_i)_i$ and $(s'_i)_i$ such that they reach $\infty$ for some finite $i$. We finally obtain

$$\sup_{t > 0} \left[ t^{\alpha_1 \| u(t) \|_\infty}, t^{\alpha_2 \| v(t) \|_\infty} \right] \leq C(M) < \infty,$$

with $C(M) \searrow 0$ as $M \searrow 0$.

Finally, if in addition $\Phi$ satisfies (2.12), the fact that the solution $U = (u, v)$ of the integral system (1.2) with initial value $\Phi$ belongs to $C \left( [0, \infty), L^\frac{N}{2\alpha_1} (\mathbb{R}^N) \right) \times C \left( [0, \infty), L^\frac{N}{2\alpha_2} (\mathbb{R}^N) \right)$ and the proof of the affirmation (2.13) are based on a contraction mapping argument in the set

$$Y_M = \left\{ U = (u, v) \in C \left( [0, \infty), L^\frac{N}{2\alpha_1} (\mathbb{R}^N) \right) \times C \left( [0, \infty), L^\frac{N}{2\alpha_2} (\mathbb{R}^N) \right) \cap C \left( (0, \infty), L^{\alpha_1} (\mathbb{R}^N) \right) \times C \left( (0, \infty), L^{\alpha_2} (\mathbb{R}^N) \right) ; \max \left[ \sup_{t \geq 0} \| u(t) \|_{\frac{N}{2\alpha_1}}, \| v(t) \|_{\frac{N}{2\alpha_2}}, \sup_{t \geq 0} t^{\beta_1} \| u(t) \|_{\alpha_1}, \| v(t) \|_{\alpha_2} \right] \leq M \right\}.$$  

Endowed with the metric

$$d(U_1, U_2) := d \left( (u_1, v_1), (u_2, v_2) \right) = \max \left[ \sup_{t \geq 0} \| u_1(t) - u_2(t) \|_{\frac{N}{2\alpha_1}}, \| v_1(t) - v_2(t) \|_{\frac{N}{2\alpha_2}}, \sup_{t \geq 0} t^{\beta_1} \| u_1(t) - u_2(t) \|_{\alpha_1}, \| v_1(t) - v_2(t) \|_{\alpha_2} \right],$$

$Y_M$ is a nonempty complete metric space.
Consider the mapping $F_\Phi$ defined by (3.2)-(3.3), where $\Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{2r_2}}(\mathbb{R}^N) \times L^{\frac{N}{2r_2}}(\mathbb{R}^N)$ satisfies (2.12). We will show that $F_\Phi = (F_\Phi, G_\Phi)$ is a strict contraction mapping on $Y_M$.

Let $\Phi = (\varphi_1, \varphi_2)$ and $\Psi = (\psi_1, \psi_2)$ belong to $L^{\frac{N}{2r_2}}(\mathbb{R}^N) \times L^{\frac{N}{2r_2}}(\mathbb{R}^N)$ satisfying (2.12). Let $U_1 = (u_1, v_1)$ and $U_2 = (u_2, v_2)$ be two elements of $Y_M$. Then we have

$$
\|F_\Phi(U_1)(t) - F_\Phi(U_2)(t)\| \leq \left|a\right| \int_0^t \|e^{(t-\sigma)\Delta} - (1 - \sigma)\|_{\frac{N}{2r_2}} \|v_1(\sigma)\|_{\frac{N}{2r_2}} - \|v_2(\sigma)\|_{\frac{N}{2r_2}} d\sigma.
$$

It follows, by the key estimate (2.2) with $(q_1, q_2) = \left(\frac{p}{r_2}, \frac{N}{2r_1}\right)$ that

$$
\|F_\Phi(U_1)(t) - F_\Phi(U_2)(t)\| \leq \left|a\right| \int_0^t C(t - \sigma)^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2q_2}{q_1}) - \frac{2}{q_2}} \|v_1(\sigma)\|_{\frac{N}{2r_2}} - \|v_2(\sigma)\|_{\frac{N}{2r_2}} d\sigma,
$$

we obtain by (3.20) and the fact that $U_1$ and $U_2$ belongs to $Y_M$, that

$$
\|F_\Phi(U_1)(t) - F_\Phi(U_2)(t)\| \leq \left|a\right| \int_0^t \left[\int_0^1 (1 - \sigma)^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2q_2}{q_1}) - \frac{2}{q_2}} \|v_1(\sigma)\|_{\frac{N}{2r_2}} - \|v_2(\sigma)\|_{\frac{N}{2r_2}} d\sigma\right] d(U_1, U_2),
$$

it follows that

$$
\|F_\Phi(U_1)(t) - F_\Phi(U_2)(t)\| \leq \left|a\right| \int_0^t \left[\int_0^1 (1 - \sigma)^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2q_2}{q_1}) - \frac{2}{q_2}} \|v_1(\sigma)\|_{\frac{N}{2r_2}} - \|v_2(\sigma)\|_{\frac{N}{2r_2}} d\sigma\right] d(U_1, U_2).
$$

Owing to (1.7), we get

$$
\|F_\Phi(U_1)(t) - F_\Phi(U_2)(t)\| \leq \left|a\right| \int_0^t \left[\int_0^1 (1 - \sigma)^{-\frac{N}{2}(\frac{p}{r_2} - \frac{2q_2}{q_1}) - \frac{2}{q_2}} \|v_1(\sigma)\|_{\frac{N}{2r_2}} - \|v_2(\sigma)\|_{\frac{N}{2r_2}} d\sigma\right] d(U_1, U_2).
$$

Since $\alpha_1$, $\alpha_2$ satisfy (1.4) and (1.5), using the fact that $r_1 > \frac{N}{2r_2}$ and due to Part (iv) of Lemma 2.1, it follows that

$$
\alpha_1 - p\alpha_2 + \frac{2 - \gamma}{2} = 0, \quad \frac{N}{2} \left(\frac{p}{r_2} - \frac{2\alpha_1}{N}\right) + \frac{\gamma}{2} < \frac{N}{2} \left(\frac{p}{r_2} - \frac{1}{r_1}\right) + \frac{\gamma}{2} < 1.
$$
Using also the fact that $\beta_2 p < 1$, we get
\[
\|F_\Phi(U_1)(t) - F_\Phi(U_2)(t)\|_{\frac{N}{p+1}} \leq \mathcal{N}'(\Phi - \Psi) + M^{p-1}\nu_1'' d(U_1, U_2),
\]
with $\nu_1''$ is a finite positive constant defined by
\[
\nu_1'' = 2|a|C_p \times \left[ \int_0^1 (1 - \sigma)^{-\frac{N}{2}} (N - 2p) - \frac{\sigma}{2} \sigma^{-\beta_2 p} d\sigma \right].
\]
Similarly, we get
\[
\|G_\Phi(U_1)(t) - G_\Phi(U_2)(t)\|_{\frac{N}{p+1}} \leq \mathcal{N}'(\Phi - \Psi) + M^{q-1}\nu_2'' d(U_1, U_2),
\]
with $\nu_2''$ is a finite positive constant defined by
\[
\nu_2'' = 2|b|C_q \times \left[ \int_0^1 (1 - \sigma)^{-\frac{N}{2}} (N - 2q) - \frac{\sigma}{2} \sigma^{-\beta_1 q} d\sigma \right].
\]
Owing to (3.21) and (3.22) we get
\[
\sup_{t \geq 0} \left[ \|F_\Phi(U_1)(t) - F_\Phi(U_2)(t)\|_{\frac{N}{p+1}}, \|G_\Phi(U_1)(t) - G_\Phi(U_2)(t)\|_{\frac{N}{p+1}} \right] \leq \mathcal{N}'(\Phi - \Psi) + \nu'' d(U_1, U_2),
\]
where
\[
\nu'' = \max(M^{p-1}\nu_1'', M^{q-1}\nu_2'').
\]
We can conclude now from (3.7) and (3.23) and from the estimate $\mathcal{N}'(\Phi - \Psi) \leq \mathcal{N}''(\Phi - \Psi)$, that
\[
d(\mathcal{F}_\Phi(U_1), \mathcal{F}_\Phi(U_2)) \leq \mathcal{N}'(\Phi - \Psi) + \max(\nu, \nu'')d(U_1, U_2).
\]
It is clear that if $U \in Y_M$, then $\mathcal{F}_\Phi(U) \in C \left( (0, \infty), L^{\frac{N}{2+2}}(\mathbb{R}^N) \right) \times C \left( (0, \infty), L^{\frac{N}{2+2}}(\mathbb{R}^N) \right)$, $C \left( (0, \infty), L^{1}(\mathbb{R}^N) \right) \times C \left( (0, \infty), L^{2}(\mathbb{R}^N) \right)$. Hence, by choosing $M$ and $R$ such that
\[
R + M \max(\nu, \nu'') \leq M,
\]
it follows that $\mathcal{F}_\Phi$ is a strict contraction from $Y_M$ into itself. So $\mathcal{F}_\phi$ has a unique fixed point in $Y_M$ which is solution of (1.2).

Remark finally when the initial data $\Phi$ belongs to $L^{\frac{N}{2+1}}(\mathbb{R}^N) \times L^{\frac{N}{2+2}}(\mathbb{R}^N)$ with respect to the norm $\mathcal{N}'$, that the condition (2.6) is satisfied, since $\mathcal{N}'(\Phi) \leq \mathcal{N}''(\Phi)$. We note also that by the previous calculations, precisely (3.24) we have the following continuous dependence property: Let $\Phi = (\varphi_1, \varphi_2)$, $\Psi = (\psi_1, \psi_2) \in L^{\frac{N}{2+1}}(\mathbb{R}^N) \times L^{\frac{N}{2+2}}(\mathbb{R}^N)$ and let $U_\Phi = (u_\Phi, v_\Phi)$
and \( U_\Psi = (u_\Psi, v_\Psi) \) be the solutions of (1.2) with initial values \( \Phi \) and respectively \( \Psi \), with 
\[
\sup_{t \geq 0} \left[ \| u_\Phi(t) \|_{N^{r_1}}, \| v_\Phi(t) \|_{N^{r_2}} \right] \leq M \quad \text{and} \quad \sup_{t \geq 0} \left[ \| u_\Psi(t) \|_{N^{r_1}}, \| v_\Psi(t) \|_{N^{r_2}} \right] \leq M.
\]
Then 
\[
\sup_{t \geq 0} \left[ \| u_\Phi(t) - u_\Psi(t) \|_{N^{r_1}}, \| v_\Phi(t) - v_\Psi(t) \|_{N^{r_2}} \right] \leq (1 - K)^{-1} \times \max \left[ \| \varphi_1 - \psi_1 \|_{N^{r_1}}, \| \varphi_2 - \psi_2 \|_{N^{r_2}} \right],
\]
for some positive constant \( K = \max(\nu, \nu') \). This finishes the proof of Theorem 1.

Let us define the scaling operator \( d_\lambda \) by 
\[
[d_\lambda \varphi](x) = \varphi(\lambda x).
\]
It follows that 
\[
e^{t \Delta} d_\lambda = d_\lambda e^{t \lambda^2 \Delta}, \forall \lambda > 0.
\]

**Proof of Theorem 2.** We now construct self-similar solution with initial data \( \Phi \). We adapt the method used in [1]. Let us define \( \Phi_\lambda \), for \( \lambda > 0 \), by 
\[
\Phi_\lambda(x) := (\lambda^{2r_1} \varphi_1(\lambda x), \lambda^{2r_2} \varphi_2(\lambda x)).
\]
It is clear that \( \Phi_\lambda \) satisfies 
\[
\Phi_\lambda(x) = \Phi(x), \forall \lambda > 0.
\]

Let \( U \) be the solution of the integral system (1.2) with initial data \( \Phi \) constructed by Theorem 1 ( remark that \( N(\Phi) < \infty \), since \( r_1 \) satisfies Parts (i)-(ii) of Lemma A.1 below and by homogeneity, also \( N(\Phi) \) is sufficiently small since \( \| \varpi_1 \|_\infty \) and \( \| \varpi_2 \|_\infty \) are sufficiently small). That is \( U \) belong to \( X_M \). We want to prove that \( U_\lambda = U, \forall \lambda > 0 \), where \( U_\lambda(t, x) := (u_\lambda(t, x), v_\lambda(t, x)) \), \( \forall \lambda > 0 \), with 
\[
u_\lambda(t, x) = \lambda^{2r_2} v(\lambda^2 t, \lambda x),
\]
and 
\[
u_\lambda(t, x) = \lambda^{2r_1} u(\lambda^2 t, \lambda x).
\]
To do this it suffice to prove that \( U_\lambda \) is also a solution of (1.2) with the same initial data \( \Phi_\lambda = \Phi \) and that \( U_\lambda \) belong to \( X_M \). On one hand due the homogeneity properties of the
system (1.2), if $U = (u, v)$ solves this system, then the scaled function solve it also. In fact
\[
\begin{align*}
d_{l}u(\lambda^2 t) &= d_{l}e^{\lambda^2 t}\varphi_1 + a \int_{0}^{\lambda^2 t} d_{l}e^{(\lambda^2 t - \sigma)\Delta} (|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma)) \, d\sigma \\
&= e^{\lambda^2 t}d_{l}\varphi_1 + a \int_{0}^{\lambda^2 t} e^{(t - \frac{\sigma}{\lambda^2})\Delta} (d_{l}(|.|^{-\gamma}|v(\sigma)|^{p-1}v(\sigma))) \, d\sigma \\
&= e^{\lambda^2 t}d_{l}\varphi_1 + a \int_{0}^{\lambda^2 t} \lambda^{-\gamma}e^{(t - \frac{\sigma}{\lambda^2})\Delta} (|.|^{-\gamma}|d_{l}v(\sigma)|^{p-1}d_{l}v(\sigma)) \, d\sigma \\
&= e^{\lambda^2 t}d_{l}\varphi_1 + a \int_{0}^{t} \lambda^{2-\gamma}e^{(t - \sigma)\Delta} (|.|^{-\gamma}|d_{l}v(\lambda^2 \sigma)|^{p-1}d_{l}v(\lambda^2 \sigma)) \, d\sigma.
\end{align*}
\]
Hence by (1.8), we get
\[
\begin{align*}
\lambda^{2a_1}d_{l}u(\lambda^2 t) &= e^{\lambda^2 t}d_{l}(\lambda^{2a_1} \varphi_1) + a \int_{0}^{t} e^{(t - \sigma)\Delta} (|.|^{-\gamma}\lambda^{2-\gamma+2a_1}|d_{l}v(\lambda^2 \sigma)|^{p-1}d_{l}v(\lambda^2 \sigma)) \, d\sigma \\
&= e^{\lambda^2 t}d_{l}(\lambda^{a_1} \varphi_1) + a \int_{0}^{t} e^{(t - \sigma)\Delta} (|.|^{-\gamma}\lambda^{2a_2}d_{l}v(\lambda^2 \sigma)|^{p-1}\lambda^{2a_2}d_{l}v(\lambda^2 \sigma)) \, d\sigma.
\end{align*}
\]
we conclude finally that
\[
u_{l}(t) = e^{\lambda^2 t} \varphi_2 + a \int_{0}^{t} e^{(t - \sigma)\Delta} (|.|^{-\gamma}|v_{l}(\sigma)|^{p-1}v_{l}(\sigma)) \, d\sigma. \tag{3.27}
\]
Similarly we obtain
\[
u_{l}(t) = e^{\lambda^2 t} \varphi_2 + b \int_{0}^{t} e^{(t - \sigma)\Delta} (|.|^{-\gamma}|u_{l}(\sigma)|^{p-1}u_{l}(\sigma)) \, d\sigma. \tag{3.28}
\]
The affirmation follows from (3.27)-(3.28). On the other hand we have
\[
\begin{align*}
\|u_{l}(t)\|_{r_1} &= \lambda^{2a_1}\|d_{l}u(\lambda^2 t)\|_{r_1} \\
&= \lambda^{2a_1} \lambda^{-\frac{\lambda^2}{r_1}} \|u(\lambda^2 t)\|_{r_1} \\
&= (\lambda^2)^{-\frac{\lambda}{r_1}} \|u(\lambda^2 t)\|_{r_1} \\

\sup_{t>0} t^{b_1} \|u_{l}(t)\|_{r_1} &= \sup_{\lambda^2 t>0} (\lambda^2)^{b_1} \|u(\lambda^2 t)\|_{r_1} \\
&= \sup_{t>0} t^{b_1} \|u(t)\|_{r_1},
\end{align*}
\]
Similarly \sup_{t>0} t^{b_2} \|v_{l}(t)\|_{r_2} = \sup_{t>0} t^{b_2} \|v(t)\|_{r_2}. It follows so that $\|U_{l}\|_{X} = \|U\|_{X}$. Then by uniqueness in $X_{M}$ $U_{l} = U$ and thus $U$ is self-similar. Let us denote it by $U_{S}$. The fact that $U_{S}(t) \to \Phi$ in $S'(\mathbb{R}^{N})$ as $t \to 0$ follows by statement (c) in Theorem 1. \qed
Proof of Theorem 3. The proof is similar to the one of Theorem 5.1 in [1], we simply indicate that

(i) \( \sup_{t>0} t^{\beta_1+\delta} \| e^{t\Delta} (\varphi_1 - \psi_1) \|_{r_1} < \infty \), for \( 0 < \delta < \frac{N}{2} - \alpha_1 \).

(ii) \( \sup_{t>0} t^{\beta_2+\delta} \| e^{t\Delta} (\varphi_2 - \psi_2) \|_{r_2} < \infty \), for \( 0 < \delta < \frac{N}{2} - \alpha_2 \).

By the formula (2.11), we have that

\[
\sup_{t>0} \left[ t^{\beta_1+\delta} \| u(t) - u_S(t) \|_{r_1}, t^{\beta_2+\delta} \| v(t) - v_S(t) \|_{r_2} \right] \leq CN_\delta(\Phi - \Psi).
\]

That is

\[
\sup_{t>0} \left[ t^{\beta_1+\delta} \| u(t) - u_S(t) \|_{r_1}, t^{\beta_2+\delta} \| v(t) - v_S(t) \|_{r_2} \right] \leq C,
\]

for \( \delta > 0 \) sufficiently small and \( C \) a finite positive constant. This gives (2.15)-(2.16) directly for \( q_1 = r_1 \) and \( q_2 = r_2 \).

We now turn to prove the asymptotic result in the \( L^\infty \)-norm. Write

\[
u(t) - u_S(t) = e^{t\Delta} (u(t/2) - u_S(t/2)) + \
\int_0^t e^{(t-\sigma)\Delta} \left[ |.|^{-\gamma} (|v(\sigma)|^{p-1}v(\sigma) - |v_S(\sigma)|^{p-1}v_S(\sigma)) \right] d\sigma,
\]

\[
v(t) - v_S(t) = e^{t\Delta} (v(t/2) - v_S(t/2)) + \
\int_0^t e^{(t-\sigma)\Delta} \left[ |.|^{-\gamma} (|u(\sigma)|^{q-1}u(\sigma) - |u_S(\sigma)|^{q-1}u_S(\sigma)) \right] d\sigma.
\]

Let \( T > 0 \) be an arbitrary real number. By using the smoothing properties of the heat semi-group with \( (s_1, s_2) = (r_1, \infty) \) and the estimate (2.2) with \( (q_1, q_2) = (\infty, \infty) \), it follows that

\[
t^{\alpha_1+\delta} \| u(t) - u_S(t) \|_\infty \leq t^{\alpha_1+\delta} \| e^{\frac{1}{2}\Delta} (u(t/2) - u_S(t/2)) \|_\infty + |a| t^{\alpha_1+\delta} \times \
\int_\frac{T}{2}^t \| e^{(t-\sigma)\Delta} \left[ |.|^{-\gamma} (|v(\sigma)|^{p-1}v(\sigma) - |v_S(\sigma)|^{p-1}v_S(\sigma)) \right] \|_\infty d\sigma 
\leq C t^{\beta_1+\delta} \| u(t/2) - u_S(t/2) \|_{r_1} + |a| Ct^{\alpha_1+\delta} \times \
\int_\frac{T}{2}^t (t - \sigma)^{-\frac{\gamma}{2}} \left( \| v(\sigma) \|_{\infty}^{p-1} + \| v_S(\sigma) \|_{\infty}^{p-1} \right) \| v(\sigma) - v_S(\sigma) \|_\infty d\sigma.
\]
Using (2.11) to estimate the first term and the fact that $\|u(t)\|_{\infty} \leq Ct^{-\alpha_2}$, $\|v(t)\|_{\infty} \leq Ct^{-\alpha_2}$ to estimate the last term, we get
\[
 t^{\alpha_1+\delta} \|u(t) - u_S(t)\|_{\infty} \leq C(\delta) + |a|C \times \left\{ \int_0^1 (1 - \sigma)^{-\frac{3}{2}} \sigma^{-\alpha_2p-\delta} d\sigma \right\} \sup_{t \in (0,T)} \left( t^{\alpha_2+\delta} \|v(t) - v_S(t)\|_{\infty} \right). 
\]
Which leads to
\[
 t^{\alpha_1+\delta} \|u(t) - u_S(t)\|_{\infty} \leq C(\delta) + C \sup_{t \in (0,T)} \left[ t^{\alpha_1+\delta} \|u(t) - u_S(t)\|_{\infty}, t^{\alpha_2+\delta} \|v(t) - v_S(t)\|_{\infty} \right]. 
\]
Similarly we have
\[
 t^{\alpha_2+\delta} \|v(t) - v_S(t)\|_{\infty} \leq C(\delta) + C \sup_{t \in (0,T)} \left[ t^{\alpha_1+\delta} \|u(t) - u_S(t)\|_{\infty}, t^{\alpha_2+\delta} \|v(t) - v_S(t)\|_{\infty} \right]. 
\]
Using (3.29) and (3.30) we obtain
\[
 \sup_{t \in (0,T)} \left[ t^{\alpha_1+\delta} \|u(t) - u_S(t)\|_{\infty}, t^{\alpha_2+\delta} \|v(t) - v_S(t)\|_{\infty} \right] \leq C'(\delta). 
\]
Since the constant $C'(\delta)$ does not depend on $T > 0$, one can take the supremum over $(0, \infty)$. This prove (2.15)-(2.16) for $r_1 = \infty$ and $r_2 = \infty$. Using the interpolation inequality
\[
 \|u(t) - u_S(t)\|_{q_1} \leq \|u(t) - u_S(t)\|_{r_1}^{\mu_1} \|u(t) - u_S(t)\|_{\infty}^{1-\mu_1}, 
\]
where
\[
 \frac{1}{q_1} = \frac{\mu_1}{r_1} + \frac{1 - \mu_1}{\infty} = \frac{\mu_1}{r_1}. 
\]
We get
\[
 \|u(t) - u_S(t)\|_{q_1} \leq \|u(t) - u_S(t)\|_{r_1}^{\mu_1} \|u(t) - u_S(t)\|_{\infty}^{1-\mu_1} \leq Ct^{\mu_1[-\beta_1(r_1)-\delta] + (1-\mu_1)[-\beta_1(\infty)-\delta]} = Ct^{-\beta_1(q_1)-\delta}. 
\]
We have also
\[
 \|v(t) - v_S(t)\|_{q_2} \leq Ct^{-\beta_2(q_2)-\delta}. 
\]
Hence the general results (2.15)-(2.16). The estimate (2.17)-(2.18) follows by a simple dilation argument. We prove just the first estimate (2.17), the proof of the second estimate...
is similar. We have
\[ \|u(t) - u_S(t)\|_{q_1} = \|u(t, \cdot) - t^{-\alpha_1} u_S(1, \frac{\cdot}{\sqrt{t}})\|_{q_1} \]
\[ = \|d_{\frac{1}{\sqrt{t}}} u(t, \cdot\sqrt{t}) - t^{-\alpha_1} d_{\frac{1}{\sqrt{t}}} u_S(1, \cdot)\|_{q_1} \]
\[ = \|d_{\frac{1}{\sqrt{t}}} [u(t, \cdot\sqrt{t}) - t^{-\alpha_1} u_S(1, \cdot)]\|_{q_1} \]
\[ = \left( \frac{1}{\sqrt{t}} \right)^{\frac{q}{p}} \|u(t, \cdot\sqrt{t}) - t^{-\alpha_1} u_S(1, \cdot)\|_{q_1}. \]

Then by using inequality (2.15) and relation (2.14), we get (2.17).

Proof of Proposition 2.2. If \( \gamma = 0 \) and \( \rho = 0 \), then (1.9) and (1.10) are verified. Since these are strict inequalities, they must hold for small \( \gamma \geq 0 \) and \( \rho > 0 \). This finishes the proof of the proposition.

Proof of Proposition 2.3. Let \( \alpha_1 \) and \( \alpha_2 \) defined by (1.4) and (1.5) respectively. Under the conditions
\[ q \geq \frac{2 - \rho}{\gamma} + \frac{2}{\gamma}, \]
and
\[ p \geq \frac{2 - \gamma}{\rho} + \frac{2}{\rho}, \]
we have that conditions (1.9) and (1.10) are equivalent to the conditions \( 2\alpha_1 < N \) and \( 2\alpha_2 < N \). Now, since \( q \geq \frac{2-\rho}{N} + \frac{2-\gamma}{N} + 1 \), we see that \( 2\alpha_1 < N \) and since \( p \geq \frac{2-\gamma}{N} + \frac{2-\rho}{N} + 1 \), we obtain that \( 2\alpha_2 < N \). This finishes the proof of the proposition.

4. Stronger uniqueness results

It has been proved in Theorem 1 that for small initial data \( \Phi = (\varphi_1, \varphi_2) \in L^{N_{\alpha_1}}(\mathbb{R}^N) \times L^{N_{\alpha_2}}(\mathbb{R}^N) \) with respect of the norm \( N' \), there exists a solution \( U_\Phi = (u_\Phi, v_\Phi) \) of the integral system (1.2) and uniqueness is guaranteed only among continuous functions \( U : [0, \infty) \to L^{N_{\alpha_1}}(\mathbb{R}^N) \times L^{N_{\alpha_2}}(\mathbb{R}^N) \) which also verify \( \sup_{t>0} \left[ t^{\beta_1} \|u(t)\|_{r_1}, t^{\beta_2} \|v(t)\|_{r_2} \right] \) is sufficiently small. Our aim in this section is to prove that uniqueness is guaranteed for solutions which belong to \( C \left( [0, \infty), L^{N_{\alpha_1}}(\mathbb{R}^N) \right) \times C \left( [0, \infty), L^{N_{\alpha_2}}(\mathbb{R}^N) \right) \cap C \left( (0, \infty), L^{\alpha_1}(\mathbb{R}^N) \right) \times C \left( (0, \infty), L^{\alpha_2}(\mathbb{R}^N) \right) \), which improves the result of uniqueness in Lebesgue spaces given in Theorem 1. We will use arguments of type Brezis Cazenave [2]. We have obtained the following result.
Theorem 4. Let \( N \) be a positive integer. Let \( p, q > 1 \). Let \( 0 < \gamma < \min(N, 2) \) and \( 0 < \rho < \min(N, 2) \). Let \( \alpha_1, \alpha_2 \) defined by (1.4) and (1.5). Suppose that (1.9) and (1.10) are satisfied. Let \( \beta_1, \beta_2 \) be given by (1.6) and (1.7). Let \( r_1 \) and \( r_2 \) be as in Lemma 2.1. Let \( M, R > 0 \) be such that (3.25) is satisfied. Let \( \Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{\gamma}}(\mathbb{R}^N) \times L^{\frac{N}{\rho}}(\mathbb{R}^N) \) satisfying (2.12). Let \( U_\Phi = (u_\Phi, v_\Phi) \in Y_M \) be the solution of the integral system (1.2) with initial data \( \Phi \) constructed by Theorem 1. Then for all \( T > 0 \), there exists a unique solution \( U_{\Phi,T} = U_\Phi \in Y_{M,T} \) of (1.2) with initial data \( \Phi \), where
\[
Y_{M,T} = \left\{ U = (u,v) \in C \left( [0,T) , L^{\frac{N}{\gamma}}(\mathbb{R}^N) \right) \times C \left( [0,T) , L^{\frac{N}{\rho}}(\mathbb{R}^N) \right) \cap C \left( (0,T) , L^{1}(\mathbb{R}^N) \right) \times C \left( (0,T) , L^{2}(\mathbb{R}^N) \right) ; \max_{t \in [0,T]} \sup_{\rho \in (0,T)} \|u(t)\|_{\frac{N}{\gamma}}, \|v(t)\|_{\frac{N}{\rho}}, \sup_{t \in (0,T)} [t^{\beta_1} \|u(t)\|_{r_1}, t^{\beta_2} \|v(t)\|_{r_2}] \leq M \right\}.
\]

Proof. The existence of the unique solution \( U_{\Phi,T} \) of (1.2) with initial data \( \Phi \) follows by a fixed point argument in \( Y_{M,T} \). Let \( U_\Phi \in Y_M \) the solution of (1.2) with initial data \( \Phi \). Owing to the fact that \( U_\Phi \in Y_M \subset Y_{M,T} \) and by uniqueness in \( Y_{M,T} \), we obtain \( U_{\Phi,T} = U_\Phi \). \( \square \)

Lemma 4.2. Let \( N \) be a positive integer. Let \( p, q > 1 \). Let \( 0 < \gamma < \min(N, 2) \) and \( 0 < \rho < \min(N, 2) \). Let \( \alpha_1, \alpha_2 \) defined by (1.4) and (1.5). Suppose that (1.9) and (1.10) are satisfied. Let \( M, R > 0 \) be such that (3.25) is satisfied. Let \( \Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{\gamma}}(\mathbb{R}^N) \times L^{\frac{N}{\rho}}(\mathbb{R}^N) \) satisfying (2.12). Let \( U_\Phi = (u_\Phi, v_\Phi) \) be the solution of the integral system (1.2) with initial data \( \Phi \) constructed by Theorem 1. Let \( (\Phi_\tau) = ((\varphi_{1,\tau}, \varphi_{2,\tau})) \) be a family of functions satisfying (2.12) such that
\[
\Phi_\tau \to \Phi, \text{ in } L^{\frac{N}{\gamma}}(\mathbb{R}^N) \times L^{\frac{N}{\rho}}(\mathbb{R}^N).
\]
Then the family of solutions \( U_{\phi_r} = (u_{\phi_r}, v_{\phi_r}) \) of the integral system (1.2) verify

\[
U_{\phi_r}(t) \longrightarrow U_{\phi}(t), \quad \text{in } L^{N\frac{N}{N-1}}(\mathbb{R}^N) \times L^{N_2\frac{N}{N-2}}(\mathbb{R}^N), \forall t \in [0, \infty).
\]

**Proof.** By continuous dependance (3.26) in \( Y_M \), it follows that

\[
\max \left\{ \|u_{\phi_r}(t) - u_{\phi}(t)\|_{N\frac{N}{N-1}}, \|v_{\phi_r}(t) - v_{\phi}(t)\|_{N_2\frac{N}{N-2}} \right\} \leq (1 - K)^{-1} \times \max \left\{ \|\varphi_{1, \tau} - \varphi_{1}\|_{N\frac{N}{N-1}}, \|\varphi_{2, \tau} - \varphi_{2}\|_{N_2\frac{N}{N-2}} \right\}, \quad \forall t \in [0, \infty).
\]

By letting \( \tau \to 0 \), we obtain the result. \( \square \)

**Proof of Theorem 4.** Since \( V = (v_1, v_2) \in C \left( [0, \infty), L^{N\frac{N}{N-1}}(\mathbb{R}^N) \right) \times C \left( [0, \infty), L^{N_2\frac{N}{N-2}}(\mathbb{R}^N) \right) \), then there exists \( \varepsilon_1 > 0 \) such that

\[
N'(V(s)) = \max \left\{ \|v_1(s)\|_{N\frac{N}{N-1}}, \|v_2(s)\|_{N_2\frac{N}{N-2}} \right\} < R, \quad \forall s \in [0, \varepsilon_1]. \tag{4.1}
\]

Let us define \( V_r = (v_{1, r}, v_{2, r}) \) by \( V_r(t) = V(t + \tau), \forall \tau \in (0, \frac{\varepsilon_1}{2}], \forall t \in [0, \frac{\varepsilon_1}{2}] \). We have from (4.1) and since \( (t^{\beta_1}\|v_{1, r}(t)\|_{L^2}, t^{\beta_2}\|v_{2, r}(t)\|_{L^2}) \to (0, 0) \) as \( t \to 0 \), \( \forall \tau \in (0, \frac{\varepsilon_1}{2}] \)

(a) \( \max \left\{ \|v_{1, r}(0)\|_{N\frac{N}{N-1}}, \|v_{2, r}(0)\|_{N_2\frac{N}{N-2}} \right\} = \max \left\{ \|v_1(\tau)\|_{N\frac{N}{N-1}}, \|v_2(\tau)\|_{N_2\frac{N}{N-2}} \right\} < R, \forall \tau \in (0, \frac{\varepsilon_1}{2}] \),

(b) \( \sup_{t \in (0, \frac{\varepsilon_1}{2}]} \left\{ \|v_{1, r}(t)\|_{N\frac{N}{N-1}}, \|v_{2, r}(t)\|_{N_2\frac{N}{N-2}} \right\} < R \leq M, \forall \tau \in (0, \frac{\varepsilon_1}{2}] \),

(c) There exists \( 0 < T_r \leq \varepsilon_1 \) such that \( \sup_{t \in (0, \frac{\varepsilon_1}{2}]} \left\{ t^{\beta_1}\|v_{1, r}(t)\|_{L^2}, t^{\beta_2}\|v_{2, r}(t)\|_{L^2} \right\} \leq M, \forall \tau \in (0, \frac{\varepsilon_1}{2}] \).

It follows then that \( V_r \in Y_M, \forall \tau \), using now Lemma 4.1 we deduce that \( V_r(t) = U_{V_r(0)}(t), \forall \tau \in (0, \frac{\varepsilon_1}{2}], \forall t \in [0, \frac{\varepsilon_1}{2}] \), where \( U_{V_r(0)} \) is the solution of the integral system (1.2) with initial data \( V_r(0) \) constructed by Theorem 1. Hence \( V_r(t) = U_{V_r(0)}(t), \forall \tau \in (0, \frac{\varepsilon_1}{2}], \forall t \in [0, \infty) \).

By Lemma 4.2, we obtain \( V(t) \longrightarrow U_{\phi}(t), \) in \( L^{N\frac{N}{N-1}}(\mathbb{R}^N) \times L^{N_2\frac{N}{N-2}}(\mathbb{R}^N), \forall t \in [0, \infty) \). On the other hand \( V_r(t) = V(t + \tau) \longrightarrow V(t), \) in \( L^{N\frac{N}{N-1}}(\mathbb{R}^N) \times L^{N_2\frac{N}{N-2}}(\mathbb{R}^N), \forall t \in [0, \infty), \) (since \( V \) is continuous in \([0, \infty)\)). Finally, we conclude by uniqueness of the limit that \( V(t) = U_{\phi}(t), \forall t \in [0, \infty) \). \( \square \)

Consider now the integral equation

\[
u(t) = e^{t\Delta} \varphi + a \int_0^t e^{(t-s)\Delta} \left( |\cdot|^{-\gamma}|u(s)|^{p-1}u(s) \right) ds, \tag{4.2}
\]
where \( u = u(t, x) \in \mathbb{R}, \ t > 0, \ x \in \mathbb{R}^N, \ a \in \mathbb{R}, \ 0 < \gamma < \min(N, 2) \) and \( p > 1 \). Set
\[
q_c = \frac{N(p - 1)}{2 - \gamma}.
\]
(4.3)
Suppose that
\[
\frac{N(p - 1)}{2 - \gamma} > 1, \ \text{(i.e.} \ q_c > 1). \quad \ (4.4)
\]
By choosing \( \gamma = \rho, \ p = q \) and \( r_1 = r \) in Lemma A.1, using the fact that \( \frac{1}{q_c} - \frac{2}{Np} = \frac{2 + (2 - \gamma)p - \gamma p^2}{Np(p^2 - 1)} \) and the equivalence \( q_c > 1 \) \( \Leftrightarrow \) (A.1), it follows that there exists \( r > q_c \) satisfying
\[
\frac{1}{q_c} - \frac{2}{Np} < \frac{1}{r} - \frac{N - \gamma}{Np} \quad \ (4.5)
\]
Corollary 4.3. Let \( N \) be a positive integer. Suppose that \( p > 1 \). Let \( 0 < \gamma < \min(N, 2) \), let \( q_c \) defined by (4.3). suppose that (4.4) is satisfied. Let \( r > q_c \) satisfying (4.5). Let \( \varphi \in L^p(\mathbb{R}^N) \) sufficiently small. Then there exists a global solution of the integral equation (4.2), which is unique in the class of functions \( u \in C \left([0, \infty), L^p(\mathbb{R}^N)\right) \cap C \left((0, \infty), L^r(\mathbb{R}^N)\right) \).

Proof. Let \( N \) be a positive integer. Suppose that \( p = q > 1 \). Suppose that \( \gamma = \rho \) with \( 0 < \gamma < \min(N, 2) \). Let \( \alpha_1 = \alpha_2 \) defined by (1.4). Suppose that (A.1) is satisfied. Let \( \beta_1, \beta_2 \) be given by (1.6) and (1.7). Let \( r_1 = r_2 \) be as in Lemma 2.1. Let \( M, R > 0 \) be such that (3.25) is satisfied. Let \( \Phi = (\varphi_1, \varphi_2) \in L^{\frac{N}{q_1}}(\mathbb{R}^N) \times L^{\frac{N}{q_2}}(\mathbb{R}^N) \) satisfying (2.12). Let \( U_{\Phi} = (u_{\Phi}, v_{\Phi}) \in \mathcal{Y}_M \) be the solution of the integral system (1.2) with initial data \( \Phi \) constructed by Theorem 1. Let \( V = (v_1, v_2) \in C \left([0, \infty), L^{\frac{N}{q_1}}(\mathbb{R}^N)\right) \times C \left([0, \infty), L^{\frac{N}{q_2}}(\mathbb{R}^N)\right) \cap C \left((0, \infty), L^r(\mathbb{R}^N)\right) \) be a solution of (1.2) with the same initial data \( \Phi \). Then by Theorem 4
\[
V(t) = U_{\Phi}(t), \ \forall t \in [0, \infty).
\]
This finishes the proof. \( \square \)

Remark 4.4. The previous corollary improves the class of uniqueness for the scalar Hardy-Hénon parabolic equations given by Theorem 1.1 (iii)-(b) in [1].

Remark 4.5. Using the same steps as above we prove that for initial data \( \Phi = (\varphi_1, \varphi_2) \in L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N) \) such that \( \frac{N}{q_1} < q_1 < r_1 \) and \( \frac{N}{q_2} < q_2 < r_2 \), there exists a local solution \( U_{\Phi} = (u_{\Phi}, v_{\Phi}) \) of the integral system (1.2) and uniqueness is guaranteed in the class of solutions which belong to \( C \left([0, T], L^q(\mathbb{R}^N)\right) \times C \left([0, T], L^{q_2}(\mathbb{R}^N)\right) \cap C \left((0, T], L^r(\mathbb{R}^N)\right) \times C \left((0, T], L^{r_2}(\mathbb{R}^N)\right) \), for any fixed \( 0 < T < T_{\max} \), where \( T_{\max} \) is the maximal existence time. This improves the result of uniqueness in Lebesgue spaces given by Theorem 1.1 (iii)-(a) in [1].
Lemma A.1. Let $N$ be a positive integer. Let $p, q > 1$. Let $0 \leq \gamma < \min(N, 2)$ and $0 < \rho < \min(N, 2)$. Let $k$ given by (1.3). Suppose that (1.9) and (1.10) are satisfied. Then there exists a real number $r_1$ satisfying the conditions

(i) $N \frac{pq-1}{(2-\rho)p+(2-\gamma)} < r_1$,
(ii) $N k \frac{pq-1}{(2-\gamma)q+(2-\rho)} < r_1$,
(iii) $N \frac{kp}{N-\rho} < r_1$,
(iv) $N \frac{pq}{N-\rho} < r_1$,
(v) $N \frac{kp}{N-\rho} < r_1$,
(vi) $N \frac{kp}{N-\rho} < r_1$,
(vii) $N \frac{kp}{N-\rho} < r_1$,
(viii) $N \frac{kp}{N-\rho} < r_1$.

Proof. We will treat the cases where $2 + (2 - \rho)p - \gamma pq > 0$ and $2 + (2 - \gamma)q - \rho pq > 0$, the other cases are simple. One can easily see that $r_1$ exists if and only if the left-hand sides of inequalities (i)-(vi) are less than the right-hand sides of inequalities (vii) and (viii). Since $pq - 1 > 0$ we verify easily

(i) $N \frac{pq-1}{(2-\rho)p+(2-\gamma)} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(ii) $N k \frac{pq-1}{(2-\gamma)q+(2-\rho)} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(iii) $N \frac{kp}{N-\rho} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(iv) $N \frac{kp}{N-\rho} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(v) $N \frac{kp}{N-\rho} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(vi) $N \frac{kp}{N-\rho} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(vii) $N \frac{kp}{N-\rho} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$,
(viii) $N \frac{kp}{N-\rho} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$.

Condition $2\alpha_1 < N$ implies that $N \frac{kp}{2+(2-\rho)p-\gamma pq} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$, condition $2\alpha_1 < \frac{p}{q}(N - \rho) \frac{pq-1}{2+(2-\rho)p-\gamma pq}$ implies that $N \frac{kp}{2+(2-\rho)p-\gamma pq} < N k \frac{p(pq-1)}{2+(2-\rho)p-\gamma pq}$, condition $2\alpha_2 < N$ implies that $N \frac{kp}{2+(2-\gamma)q-\rho pq} < N k \frac{p(pq-1)}{2+(2-\gamma)q-\rho pq}$ and finally condition $2\alpha_2 < \frac{q}{p}(N - \gamma) \frac{pq-1}{2+(2-\gamma)q-\rho pq}$ implies that $N \frac{kp}{2+(2-\gamma)q-\rho pq} < N k \frac{p(pq-1)}{2+(2-\gamma)q-\rho pq}$. This finishes the proof of the lemma.

Proof of Lemma 2.1. Owing to relation (2.3) and Lemma A.1, the proof of Lemma 2.1 is simple and can be omitted.
Remark A.2. In the case where $\gamma = \rho$ and $p = q$ it suffice to change the hypothesis (1.9) and (1.10) by the hypothesis

$$2\alpha_1 < N.$$  \hspace{1cm} \text{(A.1)}

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References


