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SIGN-CHANGING STATIONARY SOLUTIONS AND BLOWUP FOR THE TWO POWER NONLINEAR HEAT EQUATION IN A BALL

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Abstract. Consider the nonlinear heat equation

\[ u_t = \Delta u + |u|^{p-1}u - |u|^{q-1}u, \]  

(0.1)

where \( t \geq 0 \) and \( x \in \Omega \), the unit ball of \( \mathbb{R}^N \), \( N \geq 3 \), with Dirichlet boundary conditions. Let \( h \) be a radially symmetric, sign-changing stationary solution of (0.1). We prove that the solution of (0.1) with initial value \( \lambda h \) blows up in finite time if \( |\lambda - 1| > 0 \) is sufficiently small and if \( 1 < q < p < p_S = \frac{N+2}{N-2} \) and \( p \) sufficiently close to \( p_S \). This proves that the set of initial data for which the solution is global is not star-shaped around 0.

1. Introduction

This paper studies finite-time blowup of sign-changing, regular solutions of the initial value problem

\[
\begin{cases}
  u_t = \Delta u + |u|^{p-1}u - |u|^{q-1}u, \\
  u|_{\partial \Omega} = 0.
\end{cases}
\]  

(1.1)

Here, \( u = u(t,x) \in \mathbb{R} \), \( t \geq 0 \), \( x \in \Omega \), and

\[ \Omega = B_1, \]  

(1.2)

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is the (open) unit ball of \( \mathbb{R}^N \),

\[ N \geq 3. \] (1.3)

Furthermore, we consider

\[ 1 < q < p < p_S, \] (1.4)

where

\[ p_S = \frac{N + 2}{N - 2}. \] (1.5)

It is well known that the initial value problem (1.1) is locally well-posed in \( C_0(\Omega) \), where \( C_0(\Omega) \) is the Banach space of continuous functions on \( \overline{\Omega} \) that vanish on \( \partial \Omega \), with the sup norm. More precisely, given \( u_0 \in C_0(\Omega) \), there exists a maximal time \( 0 < T_{u_0} \leq \infty \) and a unique function \( u \in C([0, T_{u_0}), C_0(\Omega)) \cap C((0, T_{u_0}), C^2(\Omega)) \cap C^1((0, T_{u_0}), C_0(\Omega)) \) which is a classical solution of (1.1) on \((0, T_{u_0})\) and such that \( u(0) = u_0 \). Furthermore if \( T_{u_0} < 1 \), then \( \lim_{t \uparrow T_{u_0}} \|u(t)\| = \infty \), and we say that \( u \) blows up in finite time. In addition, if \( v \in C([0, T), C_0(\Omega)) \cap C((0, T), C^2(\Omega)) \cap C^1((0, T), C_0(\Omega)) \) is a supersolution of (1.1), i.e. \( v_t - \Delta v \geq |v|^{p-1}v - |v|^{q-1}v, v|_{\partial \Omega} \geq 0 \) and \( v(0) \geq u_0 \), then \( v(t) \geq u(t) \) as long as both \( u \) and \( v \) are defined. The notion of subsolution is defined with reversed inequalities, yielding the analogous conclusion. See, for example Proposition 52.6 in [17].

We define the set \( G \) by

\[ G = \{ u_0 \in C_0(\Omega), T_{u_0} = \infty \} . \]

It is interesting to study the geometrical properties of the set \( G \). First of all we note that every solution \( h \) of

\[ \begin{align*}
-\Delta h &= |h|^{p-1}h - |h|^{q-1}h, \\
h|_{\partial \Omega} &= 0,
\end{align*} \] (1.6)

is a stationary, hence global, solution of (1.1), whose initial value is of course \( u_0 = h \), and so is in \( G \). Since the nonlinearity \( |s|^{p-1}s - |s|^{q-1}s \) satisfies the properties of [3, Theorem 1.1, p. 15], it follows that the set \( G \) is not convex. As \( u(t) = 0 \) is a solution of (1.1) one can ask if \( G \) has the weaker property of being star-shaped around 0. The aim of this paper is to prove that \( G \) is not star-shaped.

This result is already well-known in the case of a single power nonlinearity

\[ \begin{align*}
 u_t &= \Delta u + |u|^{p-1}u, \\
 u|_{\partial \Omega} &= 0.
\end{align*} \] (1.7)

In particular, it is proved in [2] that if \( h \) is a radially symmetric, sign-changing stationary solution of the problem (1.7), with \( \Omega = B_1 \), then the solution of (1.7) with initial value \( \lambda h \) blows up in finite time if \( |\lambda - 1| > 0 \) is sufficiently small and if \( p \) is subcritical and sufficiently close to \( p_S = \frac{N + 2}{N - 2} \). More precisely, there exists \( 1 < p < p_S = \frac{N + 2}{N - 2} \) such that if \( \frac{N}{2} < p < p_S \)
and if \( h \in C_0(\Omega) \) is a radially symmetric, sign-changing stationary solution of (1.7), then there exists \( \varepsilon > 0 \) such that if \( 0 < |\lambda - 1| < \varepsilon \), then the classical solution of (1.7) with the initial condition \( u(0) = \lambda h \) blows up in finite time. In particular, \( G \), for the problem (1.7), is not star-shaped.

The fact that \( h \) changes sign is fundamental in this affirmation. In fact in the case where \( h > 0 \) it follows from the comparison principle of the heat equation that if \( 0 < \lambda \leq 1 \), then the solution is global and if \( \lambda > 1 \), then \( u \) blows up in finite time. For an elementary proof of the case \( \lambda > 1 \), see Theorem 17.8 in [17]. We remark, as was done in [2], that if \( h \) changes sign, then \( h \) and \( \lambda h \) are not comparable if \( \lambda \neq 1 \).

In addition to the result in [2], it is known that \( G \) for the problem (1.7) is not star-shaped in several other circumstances:

- \( N = 3 \), \( \Omega = B_1 \) and \( p > 1 \) sufficiently near to 1, see [4];
- \( N \geq 3 \), \( \Omega \) is a general domain and \( p < p_S \) sufficiently near to \( p_S \) or \( p = p_S \), see [14, 15];
- \( N = 2 \), \( \Omega = B_1 \) or \( \Omega \) is a general domain and \( p \) sufficiently large, see [8, 9].

See [5, 6, 7, 10, 11, 12, 16] for other properties of the set \( G \) for the problem (1.7).

We now turn to problem (1.1), and we recall the following explosion criterion, see [2, Proposition B.1, p. 447].

**Proposition 1.1** ([2, Proposition B.1, p. 447]). Let \( \Omega \subset \mathbb{R}^N \) be a smooth bounded domain. Let \( g \in C^1(\mathbb{R}, \mathbb{R}) \) satisfy \( g(0) = 0 \),

\[
s^2 g'(s) \geq (1 + \varepsilon)sg(s),
\]

and

\[
|g(s)| \leq C(1 + |s|^\beta),
\]

for all \( s \in \mathbb{R} \), where \( \varepsilon > 0 \) and \( 1 \leq \beta < \frac{N+2}{N-2} \). Let \( \psi \in C_0(\Omega) \) be a solution of the equation

\[
\begin{aligned}
-\Delta \psi &= g(\psi), \\
\psi_{|\partial\Omega} &= 0.
\end{aligned}
\]

Let \( u_0 \in C_0(\Omega) \) and let \( u \in C([0,T_{u_0});C_0(\Omega)) \) be the maximal solution of

\[
\begin{aligned}
u_t &= \Delta u + g(u), \\
u_{|\partial\Omega} &= 0,
\end{aligned}
\]

with the initial condition \( u(0) = u_0 \). If \( \psi^+ \neq 0 \) and \( u_0 \geq \psi \), \( u_0 \neq \psi \), then \( u \) blows up in finite time. Similarly if \( \psi^- \neq 0 \) and \( u_0 \leq \psi \), \( u_0 \neq \psi \), then \( u \) blows up in finite time.
Remark 1.2. Note that if $1 < q < p < p_S$, then $g(s) = |s|^{p-1}s - |s|^{q-1}s$ satisfies (1.8) with $\epsilon = q - 1$ and (1.9) with $C$ sufficiently large and $\beta = p$.

It is immediate that if $h$ is a positive solution of (1.6) with $1 < q < p < p_S$, and if $u$ is the solution of (1.1) with initial value $u(0) = \lambda h$, then for $0 < \lambda \leq 1$, $u$ is global (by the comparison principle) and if $\lambda > 1$, then $u$ blows up in finite time (by Proposition 1.1).

The question remains as to whether or not the result in [2], cited above, concerning sign-changing solutions to (1.7) also carries over to sign-changing solutions of (1.1).

The point of view in this paper is to fix a value of $q$ with

$$1 < q < p_S,$$  \hspace{1cm} (1.12)

and then consider all $p$ with

$$q < p < p_S.$$ \hspace{1cm} (1.13)

In fact we will ultimately consider what happens as $p \to p_S$. The main purpose of this paper is to establish the following result.

Theorem 1. Assume (1.2)-(1.3). Given $1 < q < p_S = \frac{N+2}{N-2}$. It follows that there exists $1 < q < p < p_S$ with the following property. If $p < p < p_S$ and if $h_p \in C_0(\Omega)$ is a radially symmetric stationary solution of (1.1) which takes both positive and negative values, then there exist $0 < \lambda < 1 < \lambda$ such that if $\lambda < \lambda < \lambda$ and $\lambda \neq 1$, then the classical solution of (1.1) with the initial condition $u(0) = \lambda h$ blows up in finite time.

The first observation is that there does exists a radially symmetric, sign-changing stationary solution of (1.1), since the nonlinearity $|s|^{p-1}s - |s|^{q-1}s$ satisfies the hypothesis of [13, Theorem 2, p. 376]. More precisely, if we consider the problem:

$$\left\{ \begin{array}{l} h'' + \frac{N-1}{r} h' + |h|^{p-1}h - |h|^{q-1}h = 0, \\
    h(0) = a > 0, \quad h'(0) = 0. \end{array} \right.$$  \hspace{1cm} (1.14)

It is well-known by [13] that (1.14) admits a unique solution $h \in C^2([0, \infty), \mathbb{R})$, which we denote sometimes by $h_p(r, a)$ to emphasize the dependence on $a$. Recall that we are fixing a value of $q$ satisfying (1.12) and letting $p$ vary in the interval (1.13). Under these conditions, by Theorem 2 in [13] for all integer $m \geq 0$, there exists $a_{p,m}$ such that

a) $h_p(1, a_{p,m}) = 0,$

b) $h_p(r, a_{p,m})$ has precisely $m$ zeros in $(0, 1)$.

In particular, $h_p(\cdot, a_{p,m})$, considered as a function on $\Omega = B_1$, is a radially symmetric solution of (1.6) which changes sign precisely $m$ times.
Now, let $h_p$ be any nontrivial solution of (1.6) and consider the linearized operator $F_p$ on $L^2(\Omega)$ defined by

$$
\begin{align*}
D(F_p) &= H^2(\Omega) \cap H^1_0(\Omega), \\
F_p u &= -\Delta u - (p|h_p|^{p-1} - q|h_p|^{q-1}) u, \quad u \in D(F_p).
\end{align*}

(1.15)
$$

We recall the following result from [3].

**Theorem 2** ([3, Corollary 2.5, p. 18]). Let $h_p \in C_0(\Omega)$ be a sign-changing solution of (1.6). Let $\varphi_p$ be a positive eigenvector of the self-adjoint operator $F_p$ given by (1.15), corresponding to the first eigenvalue. Suppose that

$$\int_\Omega h_p \varphi_p \neq 0.$$

It follows that there exists $\epsilon > 0$ such that if $0 < |1 - \lambda| < \epsilon$, then the solution of (1.1) with the initial value $u_0 = \lambda h_p$ blows up in finite time.

To prove Theorem 1, it thus suffices to establish the following.

**Theorem 3.** Assume (1.2)-(1.3). Given $1 < q < p_S = \frac{N+2}{N-2}$. It follows that there exists $1 < q < p < p_S$ with the following property. If $p < p < p_S$ and if $h_p \in C_0(\Omega)$ is a radially symmetric stationary solution of (1.1) which takes both positive and negative values, then

$$\int_\Omega h_p \varphi_p \neq 0.$$

Where $\varphi_p$ is a positive eigenvector of the self-adjoint operator $F_p$ given by (1.15), corresponding to the first eigenvalue.

The proof of Theorem 3 is based on rescaling argument. Contrary to the case of single power nonlinearity, a rescaled function $v_p$ defined by (2.4) below in terms of $h_p$, where $h_p$ is a radially symmetric stationary solution of (1.1) doesn’t satisfy the same differential equation satisfied by $h_p$, which make the situation more difficult. Also, unlike the case of the single power nonlinearity, there exist some solutions $v_p(r)$ of the problem (2.5) below which do not tend to zero as $r \to \infty$.

The rest of the paper is devoted to proving Theorem 3, which as already noted, implies Theorem 1 when combined with Theorem 2. Our basic approach follows that in [2]. However because of the differences just noted between the single power and the two power cases, many of the arguments in [2] do not immediately apply for the current situation.

**Remark 1.3.** The results in this paper are equally valid for

$$u_t = \Delta u + |u|^{p-1}u - c|u|^{q-1}u,$$
for any $c > 0$. The case where $c < 0$ is not as clear, since in that case, the proof of Proposition 2.1 below is no longer valid.

2. Stationary solutions

The proof of Theorem 3 exploits strongly the radial symmetry of the stationary solutions. By abuse of notation we will use the same letter, for example $h$, to denote a radially symmetric function $h : \mathbb{R}^N \to \mathbb{R}$, and the corresponding function $h : [0, \infty) \to \mathbb{R}$ such that, $h(x) = h(|x|), \forall x \in \mathbb{R}^N$. Throughout this paper, we will use this convention without further comment.

Any radially symmetric solution $h_p \in C_0(\Omega)$ of (1.6) satisfies the ODE

$$\begin{cases}
h'' + \frac{N-1}{r} h' + |h|^{p-1} h - |h|^q h = 0, \\
h_p'(0) = h_p(1) = 0.
\end{cases} \tag{2.1}$$

Since $h_p \neq 0$, it follows by uniqueness for the ODE (2.1) that $h_p(0) \neq 0$. Therefore, since if $u$ satisfies (1.1) then $-u$ satisfies the same problem, it suffice to prove Theorem 3 under the additional assumption

$$h_p(0) > 0. \tag{2.2}$$

In the rest of this paper we set

$$h_p(0) = a_p > 0.$$ 

Clearly $h_p(r) = h_p(r, a_p)$, where $h_p(\cdot, a_p)$ is the solution of (1.14) with $\alpha = a_p$. We let $\lambda_p > 0$ be such that

$$\lambda_p^{\frac{2}{p+1}} = a_p, \tag{2.3}$$

also we define

$$v_p(r) = \lambda_p^{\frac{2}{p+1}} h_p \left( \frac{r}{\lambda_p}, \lambda_p^{\frac{2}{p+1}} \right). \tag{2.4}$$

A simple calculation shows that $v_p$ satisfies

$$\begin{cases}
v'' + \frac{N-1}{r} v' + |v|^{p-1} v - \lambda_p^{2(p-q)} |v|^{q-1} v = 0, \\
v(0) = 1, \quad v'(0) = 0.
\end{cases} \tag{2.5}$$

As such, $v_p$ may be considered as a function $[0, \infty) \to \mathbb{R}$. It is known by [13, Lemma 1, p. 371] that $a_p \geq 1$. In fact, if $0 < a_p \leq \left( \frac{p+1}{q+1} \right)^{\frac{p+1}{q+1}}$ then $h_p(r, a_p) > 0$ for all $r > 0$. Thus,

$$\lambda_p \geq 1. \tag{2.6}$$

We have also

$$v_p(\lambda_p) = 0. \tag{2.7}$$
Proposition 2.1. Let \( \lambda_p \) defined in (2.3), then
\[
\lambda_p \longrightarrow \infty. \tag{2.8}
\]

Proof. Suppose to the contrary that \( \lambda_p \not\rightarrow \infty \) as \( p \rightarrow p_S \). It follows that there exists a subsequence \( (p_k) \) such that \( p_k \longrightarrow \infty \) and
\[
\lambda_{p_k} \longrightarrow \overline{\lambda}, \tag{2.9}
\]
where \( 1 \leq \overline{\lambda} < \infty \), by (2.6). By continuous dependence it follows that
\[
\nu_{p_k} \longrightarrow \overline{\nu}, \tag{2.10}
\]
uniformly on all compact intervals \([0, M] \subset [0, \infty)\), where \( \overline{\nu} \) satisfies
\[
\begin{cases}
\overline{\nu}'' + \frac{N-1}{2} \overline{\nu}' + |\overline{\nu}|^{p_S-1} \overline{\nu} - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)} |\overline{\nu}|^{q-1} \overline{\nu} = 0, \\
\overline{\nu}(0) = 1, \quad \overline{\nu}'(0) = 0.
\end{cases} \tag{2.11}
\]
It follows from (2.7), (2.9) and (2.10) that
\[
\nu(\overline{\lambda}) = 0. \tag{2.12}
\]
And so \( \overline{\nu} \) satisfies the equation
\[
\begin{cases}
-\Delta \overline{\nu} = |\overline{\nu}|^{p_S-1} \overline{\nu} - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)} |\overline{\nu}|^{q-1} \overline{\nu}, \\
\overline{\nu}|_{\partial B(0, \overline{\lambda})} = 0.
\end{cases} \tag{2.13}
\]
If we apply the Pohozaev identity as was done in [1, Remark 1.2, p. 442], and if we set \( g(u) = |u|^{p_S-1}u - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)} |u|^{q-1}u \) and \( G(u) = \frac{|u|^{p_S+1}}{p_S+1} - \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)} |u|^{q+1} \frac{1}{q+1} \), we obtain
\[
\frac{2-N}{2} \int_{B(0, \overline{\lambda})} g(\overline{\nu}) \overline{\nu} + N \int_{B(0, \overline{\lambda})} G(\overline{\nu}) = \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)} \left[ \frac{N-2}{2} - \frac{N}{q+1} \right] \int_{B(0, \overline{\lambda})} |\overline{\nu}|^{q+1}
\]
\[
= \frac{1}{2} \int_{\partial B(0, \overline{\lambda})} (x, \nu) \left( \frac{\partial \overline{\nu}}{\partial \nu} \right)^2 \geq 0. \tag{2.14}
\]
From (2.14), one can conclude that
\[
0 \leq \left( \frac{N-2}{2} - \frac{N}{q+1} \right) \overline{\lambda}^{-\frac{2}{p_S-1}(p_S-q)} ||\overline{\nu}||_{L^{q+1}(B(0, \overline{\lambda}))}^2. \tag{2.15}
\]
Since \( q < p_S \) inequality (2.15) is possible only if \( \overline{\nu} = 0 \), which contradicts \( \overline{\nu}(0) = 1 \). \( \square \)
Let now \( w_p \) be the solution of
\[
\begin{cases}
  w'' + \frac{N-1}{r}w' + |w|^{p-1}w = 0, \\
  w(0) = 1, \quad w'(0) = 0.
\end{cases}
\]
(2.16)

It is well-known and easy to verify that \( w_{PS} \) given by
\[
w_{PS}(r) = \left(1 + \frac{1}{N(N-2)^2}r^2\right)^{-\frac{N-2}{2}}
\]
is the solution of (2.16) with \( p = p_S \).

**Proposition 2.2.** Let \( v_p \) defined by (2.4) and \( w_{PS} \) by (2.17), then
\[
v_p \underset{p \to p_S}{\longrightarrow} w_{PS},
\]
uniformly on bounded sets of \([0, \infty)\).

**Proof.** By Proposition 2.1 \( \lambda_p \to \infty \) as \( p \to p_S \), and so by continuous dependence we can conclude that
\[
v_p \underset{p \to p_S}{\longrightarrow} w_{PS},
\]
uniformly on bounded sets of \([0, \infty)\). \(\square\)

**Proposition 2.3.** Given \( 1 < q < p_S \) and \( 0 < \eta < p_S - q \). There exist \( M, C > 0 \) such that for all \( p \in [q + \eta, p_S) \) and \( r \geq 0 \),
\[
|v_p(r)| \leq M \quad \text{and} \quad |v'_p(r)| \leq C.
\]
(2.19)

**Proof.** Let \( 1 < q < p_S \) and \( 0 < \eta < p_S - q \). Note first that by (2.5)
\[
\left[\frac{1}{2}v''_p(r) + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p \frac{2}{p+1}(p-q)|v_p(r)|^{q+1}\right]' = -\frac{N-1}{r} |v'_p(r)|^2,
\]
(2.20)
so that
\[
\frac{1}{2}v''_p(r) + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p \frac{2}{p+1}(p-q)|v_p(r)|^{q+1} \leq \frac{1}{p+1} - \frac{1}{q+1}\lambda_p \frac{2}{p+1}(p-q)
\]
\[
\leq \frac{1}{p+1}.
\]
(2.21)

Now since \( \lambda_p \) satisfies (2.6), it follows from (2.21) that
\[
\frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}|v_p(r)|^{q+1} \leq \frac{1}{p+1}.
\]
(2.22)

Suppose by contradiction that, there exist \( (p_n) \subset [q + \eta, p_S) \) and \( (r_n) \subset [0, \infty) \) such that
\[
|v_{p_n}(r_n)| \underset{n \to \infty}{\longrightarrow} \infty.
\]
Since \((p_n)\) is bounded we can suppose that \(p_n \to p_* \in [q + \eta, p_S]\), we apply now inequality (2.22), which we note as
\[
|v_p(r)|^{p+1} \left( \frac{1}{p+1} - \frac{1}{q+1} |v_p(r)|^{q-p} \right) \leq \frac{1}{p+1},
\]
with \(p = p_n, r = r_n\). By letting \(n \to \infty\), it follows that
\[
\infty \leq \frac{1}{p_* + 1},
\]
which is absurd. It follows so that there exists \(M > 0\), such that for all \(p \in [q + \eta, p_S]\) and \(r \geq 0\),
\[
|v_p(r)| \leq M. \tag{2.23}
\]
We turn now to prove the second assertion. It follows from (2.21), \(\lambda_p \geq 1\), (2.23) and \(p > q\) that
\[
\frac{1}{2} v_p'(r)^2 \leq \frac{1}{p + 1} + \frac{1}{q + 1} |v_p(r)|^{q+1} \leq \frac{1}{q + 1} + \frac{1}{q + 1} M^{q+1}, \forall p \in [q + \eta, p_S], \forall r \geq 0,
\]
so that
\[
|v_p'(r)| \leq \sqrt{\frac{2}{q + 1}} \sqrt{1 + M^{q+1}}, \forall p \in [q + \eta, p_S], \forall r \geq 0.
\]
\(\square\)

The following lemma is one of the key points which differ from the calculations in [2]. Compare Lemma 3.3 in [2]. Indeed, Lemma 3.3 in [2] cannot be true in the present context since not all solutions \(v_p\) of (2.5) tend to 0 as \(r \to \infty\). We do obtain, however, a similar estimate, valid only for \(r \leq \lambda_p\).

**Lemma 2.4.** Given \(1 < q < p_S\) and \(0 < \eta < p_S - q\). There exists a constant \(\gamma = \gamma(N, q)\) such that
\[
\frac{1}{2} |v_p'(r)|^2 + \frac{1}{p + 1} |v_p(r)|^{p+1} \leq \gamma \left[ \frac{1}{r + 1} + \frac{1}{(r + 1)^{\frac{2}{p - q - 1}}} \right], \tag{2.24}
\]
for all \(p \in [q + \eta, p_S]\) and for all \(0 \leq r \leq \lambda_p\).

**Proof.** Fix \(1 < q < p_S\) and \(0 < \eta < p_S - q\). Let \(r\) such that \(1 \leq r \leq \lambda_p\) and \(p \in [q + \eta, p_S]\). Define now
\[
F(r) = \frac{1}{2} v_p'(r)^2 + \frac{1}{p + 1} |v_p(r)|^{p+1} - \frac{1}{q + 1} \lambda_p^{\frac{2}{p - q}} |v_p(r)|^{q+1} + \frac{1}{r} v_p(r) v_p'(r). \tag{2.25}
\]
It follows from (2.20) and (2.5) that
\[
F'(r) = -\frac{N-1}{r} v'_p(r)^2 - \frac{1}{r^2} v_p(r) v'_p(r) + \frac{1}{r} v''_p(r) + \frac{1}{r} v_p(r) v''_p(r) \\
= -\frac{N-2}{r} v'_p(r)^2 - \frac{1}{r^2} v_p(r) v'_p(r) + \frac{1}{r} v_p(r) v''_p(r) \\
= -\frac{N-2}{r} v'_p(r)^2 - \frac{1}{r^2} v_p(r) v'_p(r) + \frac{1}{r} v_p(r)
\]
\[
\frac{1}{v_p(r)} \left[ -\frac{N-1}{r} v'_p(r) - |v_p(r)|^{p-1} v_p(r) + \lambda_p \frac{2^{(p-q)}}{r^q} |v_p(r)|^{q-1} v_p(r) \right].
\]

From (2.19), (1.3), the fact that \(1 \leq r = \lambda_p\), \(1 < q < p\), Young’s inequality (applied twice) and denoting \(\alpha := \frac{p-q}{p+1} + \frac{2}{p+1}(p-q)\), one can find the estimate
\[
F'(r) + \frac{1}{r} F(r) = -\frac{2N-5}{2r} v'_p(r)^2 - \frac{p}{(p+1)r} |v_p(r)|^{p+1} - \frac{N-1}{r^2} v_p(r) v'_p(r) \\
+ \lambda_p \frac{2^{(p-q)}}{r^q} |v_p(r)|^{q+1} \\
\leq -\frac{2N-5}{2r} v'_p(r)^2 - \frac{p}{(p+1)r} |v_p(r)|^{p+1} + \frac{1}{2} \left( \frac{(N-1)^2}{r^3} v_p(r)^2 + \frac{1}{r} v'_p(r)^2 \right) \\
+ \frac{q}{q+1} |v_p(r)|^{p+1} - \frac{\alpha}{p+1} r^{-\alpha} \\
\leq \frac{(N-1)^2}{2r^3} v_p(r)^2 - \frac{p}{(p+1)r} |v_p(r)|^{p+1} \\
+ \frac{q}{(p+1)r} |v_p(r)|^{p+1} + \frac{q(p-q)}{(q+1)(p+1)} r^{-\alpha} \\
\leq \frac{(N-1)^2}{2r^3} M^2 + \frac{q(p-q)}{(q+1)^2} r^{-\alpha}.
\]

Now since \(\alpha \frac{p+1}{p-q} = 1 + 2 \frac{p+1}{p-1} \geq 3\), we obtain that for \(1 \leq r \leq \lambda_p\)
\[
F'(r) + \frac{1}{r} F(r) \leq A r^{-3}.
\]

One can conclude now for all \(s \in [1, \lambda_p]\), for all \(p \in [q, q(p), qS)\) that
\[
\frac{d}{ds} (sF(s)) = sF'(s) + F(s) \leq A s^{-2}.
\]
Integration of (2.26) on \([1, r]\) gives
\[
rF(r) - F(1) \leq A \left( -\frac{1}{r} + 1 \right).
\]

We can affirm for \(r \in [1, \lambda_p]\) that
\[
F(r) \leq B \frac{1}{r}.
\]
Using also (2.25), (2.27), (2.19), \( p \in [q + \eta, p_S) \) and the fact that \( 1 \leq r \leq \lambda_p \), it follows that
\[
\frac{1}{2} v_p'(r)^2 + \frac{1}{p+1} |v_p(r)|^{p+1} \leq \frac{1}{q+1} \lambda_p \frac{2}{r^{q+1}(p-q)} |v_p(r)|^{q+1} - \frac{1}{r} v_p(r) v_p'(r) + B \frac{1}{r} \leq \frac{1}{q+1} \frac{1}{r^{q+1}} M^{q+1} + M C \frac{1}{r} + B \frac{1}{r}.
\]
Finally, using (2.19) one can conclude that there exists \( \gamma > 0 \) such that
\[
\frac{1}{2} v_p'(r)^2 + \frac{1}{p+1} |v_p(r)|^{p+1} \leq \gamma \left[ \frac{1}{r+1} + \frac{1}{(r+1)^{q+1}} \right],
\]
for all \( 0 \leq r \leq \lambda_p \).

We set
\[
\widetilde{v}_p(r) = \begin{cases} v_p(r) & \text{if } 0 \leq r \leq \lambda_p, \\ 0 & \text{if } r > \lambda_p. \end{cases}
\] (2.28)

**Corollary 2.5.** Given \( 1 < q < p_S \) and \( 0 < \eta < p_S - q \). There exists a decreasing function
\( j : [0, \infty) \to [0, \infty) \) satisfying \( j(r) \to 0 \) as \( r \to \infty \) such that
\[
|\widetilde{v}_p(r)| \leq j(r), \quad \forall r \geq 0, \quad \forall p \in [q + \eta, p_S). \quad (2.29)
\]

**Proposition 2.6.** \( \| \widetilde{v}_p - w_{p_S} \|_{L^\infty(\mathbb{R}^N)} \to 0, \) as \( p \to p_S \).

**Proof.** Fix \( 1 < q < p_S, \ 0 < \eta < p_S - q \) and \( R \geq 0 \). Let \( p \in [q + \eta, p_S), \) on the one hand it follows from (2.29) and (2.17) that
\[
|\widetilde{v}_p(r) - w_{p_S}(r)| \leq |\widetilde{v}_p(r)| + w_{p_S}(r) \\
\leq j(r) + w_{p_S}(r) \\
\leq j(R) + w_{p_S}(R), \quad \forall r \geq R.
\]

It follows that
\[
\sup_{r \geq R} |\widetilde{v}_p(r) - w_{p_S}(r)| \leq j(R) + w_{p_S}(R) \to 0, \quad R \to \infty.
\]
Thus, there exists \( R_0 \) such that
\[
\sup_{r \geq R_0} |\widetilde{v}_p(r) - w_{p_S}(r)| \leq \frac{\varepsilon}{2}. \quad (2.30)
\]

On the other hand, since \( \lambda_p \to \infty \) as \( p \to p_S \), by choosing \( p_0 \) sufficiently close to \( p_S \), we can assume that \( R_0 \leq \lambda_p \) for \( p_0 \leq p < p_S \). It follows from (2.18) that there exists \( p_0 \leq \overline{p} < p_S \) such that if \( \overline{p} < p < p_S \) then
\[
\sup_{r \in [0,R_0]} |\widetilde{v}_p(r) - w_{p_S}(r)| = \sup_{r \in [0,R_0]} |v_p(r) - w_{p_S}(r)| \leq \frac{\varepsilon}{2}. \quad (2.31)
\]
One can conclude from (2.30) and (2.31).

3. The linearized operator

We consider now the self-adjoint operator $F_p$ defined on $L^2(\Omega)$ by

\[
\begin{align*}
D(F_p) &= H^2(\Omega) \cap H^1_0(\Omega), \\
F_p u &= -\Delta u - (p|h_p|^{p-1} - q|h_p|^{q-1}) u, \quad \forall u \in D(H_p).
\end{align*}
\]  

(3.1)

We denote by

\[
\theta_p = \theta_p(F_p),
\]

(3.2)

its first eigenvalue and by $\varphi_p$ the corresponding eigenvector, i.e.

\[
F_p \varphi_p = -\Delta \varphi_p - (p|h_p|^{p-1} - q|h_p|^{q-1}) \varphi_p = \theta_p \varphi_p,
\]

(3.3)

where we require

\[
\varphi_p > 0, \quad ||\varphi_p||_{L^2(\Omega)} = 1.
\]

(3.4)

Since $\varphi_p$ is radially symmetric, it satisfies the ODE

\[
\varphi''_p + \frac{N-1}{r} \varphi'_p + (p|h_p|^{p-1} - q|h_p|^{q-1}) \varphi_p + \theta_p \varphi_p = 0.
\]

(3.5)

In order to transform the operator $F_p$ into another operator we introduce $l_p \in \mathbb{R}$ and $\psi_p$, a positive, spherically symmetric function on $\Omega_p$ defined by

\[
\theta_p = \lambda^2_p l_p, \quad \varphi_p(x) = \lambda^\frac{N}{2} \psi_p(\lambda_p x),
\]

(3.6)

where

\[
\Omega_p = B(0, \lambda_p).
\]

(3.7)

It follows from (3.5), (2.4) and (3.6) that $\psi_p$ satisfies the equation

\[
\begin{cases}
-\Delta \psi_p - \left[p|v_p|^{p-1} - q\lambda_p^{\frac{2q}{2q+(p-q)}}|v_p|^{q-1}\right] \psi_p = l_p \psi_p & \text{in } \Omega_p, \\
\psi_p = 0 & \text{on } \partial \Omega_p,
\end{cases}
\]

(3.8)

and that

\[
\int_{\Omega} h_p \varphi_p = \lambda^\frac{2}{p-1} \frac{2}{N} \int_{\Omega_p} v_p \psi_p,
\]

(3.9)

and

\[
\psi_p > 0, \quad ||\psi_p||_{L^2(\Omega_p)} = 1.
\]

(3.10)
We have also that $l_p$ is the first eigenvalue associated to the eigenvector $\psi_p$ of the self-adjoint operator $L_p$ defined on $L^2(\Omega_p)$ by

$$
\begin{align*}
D(L_p) &= H^2(\Omega_p) \cap H^1_0(\Omega_p), \\
L_p u &= -\Delta u - \left[ p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p|^{q-1} \right] u, \quad \forall u \in D(L_p).
\end{align*}
$$

(3.11)

Given $0 < p < p_S$, we set

$$
J_p(w) = \int_{\Omega_p} |\nabla w|^2 - \int_{\Omega_p} \left[ p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p|^{q-1} \right] w^2,
$$

(3.12)

for all $w \in H^1_0(\Omega_p)$, so that

$$
l_p = \inf \left\{ J_p(u), u \in H^1_0(\Omega_p), \|u\|_{L^2(\Omega_p)} = 1 \right\}.
$$

(3.13)

Also we define the self-adjoint operator $L_\ast$ on $L^2(\mathbb{R}^N)$ by

$$
\begin{align*}
D(L_\ast) &= H^2(\mathbb{R}^N), \\
L_\ast u &= -\Delta u - p_S w_{p_S}^{p_S-1} u, \quad \forall u \in D(L_\ast),
\end{align*}
$$

(3.14)

where $w_{p_S}$ is given by (2.17). We set

$$
\lambda_\ast = \inf \left\{ J_\ast(u), u \in H^1(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)} = 1 \right\},
$$

(3.15)

where

$$
J_\ast(w) = \int_{\mathbb{R}^N} |\nabla w|^2 - p_S \int_{\mathbb{R}^N} w_{p_S}^{p_S-1} w^2,
$$

(3.16)

for all $w \in H^1(\mathbb{R}^N)$. We recall now the following proposition from [2].

**Proposition 3.1** ([2, Proposition 3.4, p. 439]). If $L_\ast$ is defined by (3.14) and $\lambda_\ast$ is defined by (3.15), then the following properties hold.

(i) $\lambda_\ast < 0$ and $\lambda_\ast$ is an eigenvalue of $L_\ast$.

(ii) There exists a unique eigenvector $\psi_\ast$ of $L_\ast$ corresponding to the eigenvalue $\lambda_\ast$ which is positive, radially decreasing with $\|\psi_\ast\|_{L^2(\mathbb{R}^N)} = 1$.

(iii) If $(u_n)_{n \geq 1} \subset H^1(\mathbb{R}^N)$ is a minimizing sequence of (3.15) and $u_n \geq 0$, then $u_n \to \psi_\ast$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$.

We set

$$
\tilde{\psi}_p(x) = \begin{cases} 
\psi_p(x) & \text{if } 0 \leq |x| < \lambda_p, \\
0 & \text{if } |x| \geq \lambda_p,
\end{cases}
$$

(3.17)

for all $1 < p < p_S$, so that

$$
\tilde{\psi}_p \in H^1(\mathbb{R}^N), \|\tilde{\psi}_p\|_{L^2(\mathbb{R}^N)} = 1, \quad \tilde{\psi}_p \geq 0.
$$

(3.18)
Lemma 3.2. Let \( \psi \in H^1(\mathbb{R}^N) \) such that \( \|\psi\|_{L^2(\mathbb{R}^N)} = 1 \). Consider a smooth radial cut-off function \( \eta : \mathbb{R}^N \to [0,1] \) such that \( \eta(r) = 1 \) for \( r \leq \frac{1}{2} \) and \( \eta(r) = 0 \) for \( r \geq 1 \). Set
\[
k_\lambda(r) = \eta \left( \frac{r}{\lambda} \right) \psi(r),
\]
and
\[
u_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{L^2(\mathbb{R}^N)}}.
\]
Then \( u_\lambda \in H^1_0(\Omega_\lambda) \) and
\[
\|u_\lambda - \psi\|_{H^1(\mathbb{R}^N)} \to 0.
\]
Where \( \Omega_\lambda = B(0, \lambda) \).

Proof. This follows by standard arguments, using the observation that \( \|k_\lambda\|_{L^2(\mathbb{R}^N)} \to 1 \).

Lemma 3.3. Let \( l_p \) defined by (3.6), then
\[l_p \to \lambda_* \quad \text{as} \quad p \to p_S.
\]

Proof. We first use \( \tilde{\psi}_p \) as a test function in (3.15). It follows from (3.18) that
\[
\lambda_* \leq J_*(\tilde{\psi}_p) = J_p(\tilde{\psi}_p) + \int_{\mathbb{R}^N} \left[ p|\tilde{\psi}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}}(p-q)|\tilde{\psi}_p|^{q-1} - pS\lambda_p^{p-1} \right] \tilde{\psi}_p^2.
\]
\[
= l_p + \int_{\mathbb{R}^N} \left[ p|\tilde{\psi}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}}(p-q)|\tilde{\psi}_p|^{q-1} - pS\lambda_p^{p-1} \right] \tilde{\psi}_p^2
\]
\[
= l_p + \int_{\mathbb{R}^N} \left[ p|\tilde{\psi}_p|^{p-1} - pS\lambda_p^{p-1} \right] \tilde{\psi}_p^2 - q\lambda_p^{-\frac{2}{p-1}}(p-q) \int_{\mathbb{R}^N} |\tilde{\psi}_p|^{q-1} \tilde{\psi}_p^2.
\]
It follows from (3.22), Proposition 2.3 and (3.18) that
\[
\lambda_* - l_p \leq \left\| p|\tilde{\psi}_p|^{p-1} - pS\lambda_p^{p-1} \right\|_{L^\infty(\mathbb{R}^N)} + qM^{-1}\lambda_p^{-\frac{2}{p-1}}(p-q), \quad \forall p \in [q + \eta, p_S).
\]
One can conclude now by applying Proposition 2.6 and Proposition 2.1 that
\[
\limsup_{p \to p_S}(\lambda_* - l_p) \leq 0.
\]

Next, we would like to use \( \psi_* \) as a test function in (3.13), but \( \psi_* \not\in H^1_0(\Omega_p) \). Thus, we need to approximate \( \psi_* \) by a sequence in \( H^1_0(\Omega_p) \). Consider a smooth radial cut-off function \( \eta : \mathbb{R}^N \to [0,1] \) such that \( \eta(r) = 1 \) for \( r \leq \frac{1}{2} \) and \( \eta(r) = 0 \) for \( r \geq 1 \). Setting
\[
k_p(r) = \eta \left( \frac{r}{\lambda_p} \right) \psi_*(r),
\]
and
\[
u_p = \frac{n_p}{\|n_p\|_{L^2(\mathbb{R}^N)}}.
\]
and
\[ u_p = \frac{k_p}{\|k_p\|_{L^2(\mathbb{R}^N)}}. \tag{3.26} \]
it follows from Lemma 3.2 (since \( \lambda \to \infty \) as \( p \to p_S \)) that
\[ \|u_p - \psi_\ast\|_{H^1(\mathbb{R}^N)} \underset{p \to p_S}{\longrightarrow} 0. \tag{3.27} \]
Moreover, \( u_p \in H^1_0(\Omega_p) \), so that
\[ l_p \leq J_p(u_p) = \lambda_\ast - J_\ast(\psi_\ast) + J_\ast(u_p) - J_\ast(u_p) + J_\ast(u_p). \tag{3.28} \]
On the one hand we have by Proposition 2.6 and Proposition 2.1 that
\[
|J_\ast(u_p) - J_p(u_p)| = \left| \int_{\mathbb{R}^N} \left[ p|\bar{v}_p|^{p-1} - q \lambda_p - \frac{2}{p} (p-q) \bar{v}_p \right] u_p^2 \right| \\
\leq \int_{\mathbb{R}^N} \left| p|\bar{v}_p|^{p-1} - p_S u_p^{p_S-1} \right| u_p^2 + q \lambda_p - \frac{2}{p} (p-q) \int_{\mathbb{R}^N} |\bar{v}_p|^{q-1} u_p^2 \\
\leq \left| p|\bar{v}_p|^{p-1} - p_S u_p^{p_S-1} \right|_{L^{\infty}(\mathbb{R}^N)} + q \lambda_p^{q-1} \lambda_p^{-\frac{2}{p} (p-q)} \underset{p \to p_S}{\longrightarrow} 0. \tag{3.29} \]
On the other hand, using the fact that \( |J_\ast(\psi_\ast) - J_\ast(u_p)| \leq \left| \|\nabla u_p\|^2_{L^2(\mathbb{R}^N)} - \|\nabla \psi_\ast\|^2_{L^2(\mathbb{R}^N)} \right| + p_S \int_{\mathbb{R}^N} |u_p^2 - \psi_\ast^2| \), it easily follows from (3.27) and the dominated convergence theorem that
\[ |J_\ast(\psi_\ast) - J_\ast(u_p)| \underset{p \to p_S}{\longrightarrow} 0. \tag{3.30} \]
We deduce from (3.29) and (3.30) that
\[ -\liminf_{p \to p_S} (\lambda_\ast - l_p) = \limsup_{p \to p_S} (l_p - \lambda_\ast) \leq 0. \tag{3.31} \]
We can confirm so by (3.24) and (3.31) that
\[ \liminf_{p \to p_S} (\lambda_\ast - l_p) = \limsup_{p \to p_S} (\lambda_\ast - l_p) = \lim_{p \to p_S} (\lambda_\ast - l_p) = 0. \]
The result follows now. \( \square \)

**Lemma 3.4.** Given \( q \in (1, p_S) \) and \( \eta \in (0, p_S - q) \). There exists \( C > 0 \) such that
\[ |\bar{v}_p(r) + |\psi_\ast(r)| \leq C \frac{1}{r^{-\eta}} \leq C, \tag{3.32} \]
for all \( r \geq 1 \) and \( q + \eta \leq p < p_S \).
Proof. Fix $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. We affirm first that

$$l_p < 0 \quad \text{for all} \quad p \in (q, p_S).$$

(3.33)

In fact, since $l_p$ satisfies (3.6), it suffice to prove that $\theta_p < 0$. We have on the one hand since $p > q$

$$\left( \int_{\Omega_p} h_p^2 \right) \theta_p \leq \int_{\Omega_p} |\nabla h_p|^2 - \int_{\Omega_p} (p|h_p|^{p-1} - q|h_p|^{q-1}) h_p^2 \leq \int_{\Omega_p} |\nabla h_p|^2 - q \int_{\Omega_p} (|h_p|^{p+1} - |h_p|^{q+1}).$$

(3.34)

On the other hand since $h_p$ satisfies (2.1) it follows that

$$\int_{\Omega_p} |\nabla h_p|^2 = \int_{\Omega_p} (|h_p|^{p+1} - |h_p|^{q+1}).$$

(3.35)

It follows from (3.34) and (3.35) since $q > 1$ that $\theta_p < 0$.

We complete now our proof. Since $l_p < 0$, we deduce from (3.13) and Proposition 2.3 that

$$||\nabla \tilde{\psi}_p||_{L^2(\mathbb{R}^N)} \leq p_S (M^{q-1} + M^{p_S-1}), \quad \forall p \in [q + \eta, p_S].$$

(3.36)

By (3.18), (3.36) and Strauss’ radial lemma [18] that

$$|\tilde{\psi}_p(r)| \leq c \sqrt{1 + p_S (M^{q-1} + M^{p_S-1})} \frac{1}{r^{\frac{N-1}{2}}}.$$  

(3.37)

for all $r \geq 1$.

A similar argument applies to $\psi_*$ which completes the proof.

\[\square\]

**Lemma 3.5.** Given $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. There exist $R, C > 0, \theta > 0$ and $q + \eta \leq p_0 < p_S$ such that

$$|\tilde{\psi}_p(r)| + |\psi_*(r)| \leq Ce^{-\theta r},$$

(3.38)

for all $r \geq R$ and $p_0 \leq p < p_S$.

**Proof.** Fix $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. We start first by showing that there exists $R, C > 0, \theta > 0$ and $q + \eta \leq p_0 < p_S$ such that

$$|\tilde{\psi}_p(r)| \leq Ce^{-\theta r},$$

(3.39)

for all $r \geq R$ (with $r \leq \lambda_p$) and $p_0 \leq p < p_S$. It follows from (3.8) that $\psi_p$ satisfies

$$-\psi''_p(r) - \frac{N-1}{r} \psi'_p(r) - \left\{ p|\tilde{\psi}_p(r)|^{p-1} - q\lambda_p \frac{2}{r} (|\tilde{\psi}_p(r)|^{q-1}) + l_p \right\} \psi_p(r) = 0.$$  

(3.40)
for all $0 \leq r < \lambda_p$. We would like to use a method of energy in equation (3.40), but the term $-\left[p|\tilde{u}_p(r)|^{p-1} - q\lambda_p^{\frac{2}{p-1}(p-q)}|\tilde{v}_p(r)|^{q-1}\right] - l_p$ is difficult to handle so we may estimate it. On the one hand, since $|p|\tilde{u}_p(r)|^{p-1} - q\lambda_p^{\frac{2}{p-1}(p-q)}|\tilde{v}_p(r)|^{q-1}| \leq L(r) \to 0$ by Corollary 2.5 and the fact that $p \in [q + \eta, p_S]$, it follows that there exists $R > 0$, such that for all $r \geq R$

$$-\left[p|\tilde{u}_p(r)|^{p-1} - q\lambda_p^{\frac{2}{p-1}(p-q)}|\tilde{v}_p(r)|^{q-1}\right] \geq \frac{\lambda_*}{4}. \quad (3.41)$$

On the other hand, since $-l_p \to -\lambda_*$ as $p \to p_S$ by Lemma 3.3, it follows that there exists $p_0 \in [q + \eta, p_S]$ such that for all $p_0 \leq p < p_S$

$$-l_p \geq -\frac{3}{4}\lambda_* \quad (3.42)$$

Finally one can conclude from (3.41) and (3.42) that there exist $R > 0$ and $q + \eta \leq p_0 < p_S$ such that

$$-\left[p|\tilde{u}_p(r)|^{p-1} - q\lambda_p^{\frac{2}{p-1}(p-q)}|\tilde{v}_p(r)|^{q-1}\right] - l_p \geq -\frac{\lambda_*}{2} > 0, \quad (3.43)$$

for all $p_0 \leq p < p_S$ and all $r \geq R$. By choosing $p_0$ possibly larger, we also may assume that $\lambda_p > R$ for $p_0 \leq p < p_S$. Since $\psi_p \geq 0$, we deduce from (3.40) and (3.43) that

$$\psi_p'' + \frac{N - 1}{r}\psi_p' \geq -\frac{\lambda_*}{2}\psi_p, \quad (3.44)$$

for all $R \leq r \leq \lambda_p$. We now claim that

$$\psi_p'(r) < 0, \quad (3.45)$$

for all $p_0 \leq p < p_S$ and all $R < r < \lambda_p$. We argue by contradiction and suppose that $\psi_p'(r_p) \geq 0$ for some $p_0 \leq p < p_S$ and some $R < r_p < \lambda_p$. Since $\psi_p(\lambda_p) = 0$, there exists $r_p \leq r_p' < \lambda_p$ such that $\psi_p'(r_p') = 0$ and $\psi_p''(r_p') \leq 0$. This is impossible by (3.44) since $\lambda_* < 0$. Multiplying (3.44) by $\psi_p' < 0$, see that

$$\psi_p''\psi_p' + \frac{N - 1}{r}\psi_p'\psi_p' \leq -\frac{\lambda_*}{2}\psi_p\psi_p',$$

which implies

$$\left(\psi_p'^2 + \frac{\lambda_*}{2}\psi_p^2\right)' \leq 0, \quad (3.46)$$

for $R \leq r \leq \lambda_p$. It follows from (3.46) that

$$\left[\psi_p'^2 + \frac{\lambda_*}{2}\psi_p^2(r)\right] \geq \psi_p'\lambda_p^2 \geq 0,$$
for $R < r < \lambda_p$. Since $\psi_p > 0$ and $\psi'_p < 0$, we obtain that $\psi'_p + \frac{\lambda_p}{2} \psi_p \leq 0$ for $R < r < \lambda_p$, so that

$$\psi_p(r) \leq \psi_p(R)e^{\frac{\lambda_p}{2}}R^e^{-\frac{\lambda_p}{2}r}$$

for $R < r < \lambda_p$. By choosing $R \geq 1$ we have $\psi_p(R) \leq C$ by Lemma 3.4. The exponential decay follows. As remarked in [2], the proof for $\psi_*$ is similar. This completes the proof. □

**Lemma 3.6.** $\tilde{\psi}_p$ and $\psi_* \in L^1(\mathbb{R}^N)$. Moreover, $\|\tilde{\psi}_p - \psi_*\|_{L^1(\mathbb{R}^N)} \to 0$ as $p \to p_S$.

**Proof.** The proof is similar to the proof of Lemma 3.7 in [2]. □

**Proof of Theorem 3.** Fix $q \in (1, p_S)$ and $0 < \eta < p_S - q$. Let $h_p \in C_0(\Omega)$ be a radially symmetric, sign-changing stationary solution of (1.1). Let $\varphi_p$ be the positive eigenvector normalized in $L^2(\Omega)$ of the self-adjoint operator $F_p$ given by (3.1), corresponding to the first eigenvalue. We have from Proposition 2.3

$$\left| \int_{\Omega_p} v_p \psi_p - \int_{\mathbb{R}^N} w_{pS} \psi_* \right| = \left| \int_{\mathbb{R}^N} \tilde{v}_p \tilde{\psi}_p - \int_{\mathbb{R}^N} w_{pS} \psi_* \right|$$

$$\leq \left| \int_{\mathbb{R}^N} \tilde{v}_p \left( \tilde{\psi}_p - \psi_* \right) \right| + \left| \int_{\mathbb{R}^N} \left( \tilde{v}_p - w_{pS} \right) \psi_* \right|$$

$$\leq M \left\| \tilde{v}_p - \psi_* \right\|_{L^1(\mathbb{R}^N)} + \left\| \psi_* \right\|_{L^1(\mathbb{R}^N)} \left\| \tilde{v}_p - w_{pS} \right\|_{L^\infty(\mathbb{R}^N)},$$

\forall p \in [q + \eta, p_S]$. It follows so by Lemma 3.6, Proposition 2.6 that

$$\int_{\Omega} v_p \varphi_p \to_{p \to p_S} \int_{\mathbb{R}^N} w_{pS} \psi_* > 0.$$ We can now conclude from (3.9) that there exists $1 < q < \frac{p}{p} < p_S$ such that if $\frac{p}{p} < p < p_S$, then

$$\int_{\Omega} h_p \varphi_p > 0.$$ This finishes the proof. □

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BLOWUP FOR THE TWO POWER NONLINEAR HEAT EQUATION

References


