



HAL
open science

Sign-changing stationary solutions and blowup for the two power nonlinear heat equation in a ball

Byrame Ben Slimene

► **To cite this version:**

Byrame Ben Slimene. Sign-changing stationary solutions and blowup for the two power nonlinear heat equation in a ball. *Journal of Mathematical Analysis and Applications*, 2017, 454 (2), pp.1067-1084. 10.1016/j.jmaa.2017.05.009 . hal-03868403

HAL Id: hal-03868403

<https://hal.science/hal-03868403>

Submitted on 23 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SIGN-CHANGING STATIONARY SOLUTIONS AND BLOWUP FOR THE TWO POWER NONLINEAR HEAT EQUATION IN A BALL

BYRAME BEN SLIMENE

Université de Tunis El Manar, Faculté des Sciences de Tunis,
Département de Mathématiques, Laboratoire équations aux Dérivées Partielles
LR03ES04, 2092 Tunis, Tunisie.

and

Université Paris 13, Sorbonne Paris Cité, CNRS UMR 7539
Laboratoire Analyse, Géométrie et Applications,
99, Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.
e-mail: byramebenslimene@yahoo.fr

ABSTRACT. Consider the nonlinear heat equation

$$u_t = \Delta u + |u|^{p-1}u - |u|^{q-1}u, \quad (0.1)$$

where $t \geq 0$ and $x \in \Omega$, the unit ball of \mathbb{R}^N , $N \geq 3$, with Dirichlet boundary conditions. Let h be a radially symmetric, sign-changing stationary solution of (0.1). We prove that the solution of (0.1) with initial value λh blows up in finite time if $|\lambda - 1| > 0$ is sufficiently small and if $1 < q < p < p_s = \frac{N+2}{N-2}$ and p sufficiently close to p_s . This proves that the set of initial data for which the solution is global is not star-shaped around 0.

1. INTRODUCTION

This paper studies finite-time blowup of sign-changing, regular solutions of the initial value problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u - |u|^{q-1}u, \\ u|_{\partial\Omega} = 0. \end{cases} \quad (1.1)$$

Here, $u = u(t, x) \in \mathbb{R}$, $t \geq 0$, $x \in \Omega$, and

$$\Omega = B_1, \quad (1.2)$$

2000 *Mathematics Subject Classification.* : 35K91, 35B35, 35B44, 35J91.

Key words and phrases. Semilinear heat equation; finite-time blow-up; sign-changing stationary solutions; linearized operator; asymptotic behavior.

is the (open) unit ball of \mathbb{R}^N ,

$$N \geq 3. \quad (1.3)$$

Furthermore, we consider

$$1 < q < p < p_S, \quad (1.4)$$

where

$$p_S = \frac{N+2}{N-2}. \quad (1.5)$$

It is well known that the initial value problem (1.1) is locally well-posed in $C_0(\Omega)$, where $C_0(\Omega)$ is the Banach space of continuous functions on $\bar{\Omega}$ that vanish on $\partial\Omega$, with the sup norm. More precisely, given $u_0 \in C_0(\Omega)$, there exists a maximal time $0 < T_{u_0} \leq \infty$ and a unique function $u \in C([0, T_{u_0}), C_0(\Omega)) \cap C((0, T_{u_0}), C^2(\bar{\Omega})) \cap C^1((0, T_{u_0}), C_0(\Omega))$ which is a classical solution of (1.1) on $(0, T_{u_0})$ and such that $u(0) = u_0$. Furthermore if $T_{u_0} < \infty$, then $\lim_{t \uparrow T_{u_0}} \|u(t)\|_\infty = \infty$, and we say that u blows up in finite time. In addition, if $v \in C([0, T), C_0(\Omega)) \cap C((0, T), C^2(\bar{\Omega})) \cap C^1((0, T), C_0(\Omega))$ is a supersolution of (1.1), i.e $v_t - \Delta v \geq |v|^{p-1}v - |v|^{q-1}v$, $v|_{\partial\Omega} \geq 0$ and $v(0) \geq u_0$, then $v(t) \geq u(t)$ as long as both u and v are defined. The notion of subsolution is defined with reversed inequalities, yielding the analogous conclusion. See, for example Proposition 52.6 in [17].

We define the set \mathcal{G} by

$$\mathcal{G} = \{u_0 \in C_0(\Omega), T_{u_0} = \infty\}.$$

It is interesting to study the geometrical properties of the set \mathcal{G} . First of all we note that every solution h of

$$\begin{cases} -\Delta h = |h|^{p-1}h - |h|^{q-1}h, \\ h|_{\partial\Omega} = 0, \end{cases} \quad (1.6)$$

is a stationary, hence global, solution of (1.1), whose initial value is of course $u_0 = h$, and so is in \mathcal{G} . Since the nonlinearity $|s|^{p-1}s - |s|^{q-1}s$ satisfies the properties of [3, Theorem 1.1, p. 15], it follows that the set \mathcal{G} is not convex. As $u(t) = 0$ is a solution of (1.1) one can ask if \mathcal{G} has the weaker property of being star-shaped around 0. The aim of this paper is to prove that \mathcal{G} is not star-shaped.

This result is already well-known in the case of a single power nonlinearity

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u|_{\partial\Omega} = 0. \end{cases} \quad (1.7)$$

In particular, it is proved in [2] that if h is a radially symmetric, sign-changing stationary solution of the problem (1.7), with $\Omega = B_1$, then the solution of (1.7) with initial value λh blows up in finite time if $|\lambda - 1| > 0$ is sufficiently small and if p is subcritical and sufficiently close to $p_S = \frac{N+2}{N-2}$. More precisely, there exists $1 < \underline{p} < p_S = \frac{N+2}{N-2}$ such that if $\underline{p} < p < p_S$

and if $h \in C_0(\Omega)$ is a radially symmetric, sign-changing stationary solution of (1.7), then there exists $\varepsilon > 0$ such that if $0 < |\lambda - 1| < \varepsilon$, then the classical solution of (1.7) with the initial condition $u(0) = \lambda h$ blows up in finite time. In particular, \mathcal{G} , for the problem (1.7), is not star-shaped.

The fact that h changes sign is fundamental in this affirmation. In fact in the case where $h > 0$ it follows from the comparison principle of the heat equation that if $0 < \lambda \leq 1$, then the solution is global and if $\lambda > 1$, then u blows up in finite time. For an elementary proof of the case $\lambda > 1$, see Theorem 17.8 in [17]. We remark, as was done in [2], that if h changes sign, then h and λh are not comparable if $\lambda \neq 1$.

In addition to the result in [2], it is known that \mathcal{G} for the problem (1.7) is not star-shaped in several other circumstances :

- $N = 3$, $\Omega = B_1$ and $p > 1$ sufficiently near to 1, see [4] ;
- $N \geq 3$, Ω is a general domain and $p < p_S$ sufficiently near to p_S or $p = p_S$, see [14, 15] ;
- $N = 2$, $\Omega = B_1$ or Ω is a general domain and p sufficiently large, see [8, 9].

See [5, 6, 7, 10, 11, 12, 16] for other properties of the set \mathcal{G} for the problem (1.7).

We now turn to problem (1.1), and we recall the following explosion criterion, see [2, Proposition B.1, p. 447].

Proposition 1.1 ([2, Proposition B.1, p. 447]). *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $g(0) = 0$,*

$$s^2 g'(s) \geq (1 + \epsilon) s g(s), \quad (1.8)$$

and

$$|g(s)| \leq C(1 + |s|^\beta), \quad (1.9)$$

for all $s \in \mathbb{R}$, where $\epsilon > 0$ and $1 \leq \beta < \frac{N+2}{N-2}$. Let $\psi \in C_0(\Omega)$ be a solution of the equation

$$\begin{cases} -\Delta \psi = g(\psi), \\ \psi|_{\partial\Omega} = 0. \end{cases} \quad (1.10)$$

Let $u_0 \in C_0(\Omega)$ and let $u \in C([0, T_{u_0}); C_0(\Omega))$ be the maximal solution of

$$\begin{cases} u_t = \Delta u + g(u), \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1.11)$$

with the initial condition $u(0) = u_0$. If $\psi^+ \neq 0$ and $u_0 \geq \psi$, $u_0 \neq \psi$, then u blows up in finite time. Similarly if $\psi^- \neq 0$ and $u_0 \leq \psi$, $u_0 \neq \psi$, then u blows up in finite time.

Remark 1.2. Note that if $1 < q < p < p_S$, then $g(s) = |s|^{p-1}s - |s|^{q-1}s$ satisfies (1.8) with $\epsilon = q - 1$ and (1.9) with C sufficiently large and $\beta = p$.

It is immediate that if h is a positive solution of (1.6) with $1 < q < p < p_S$, and if u is the solution of (1.1) with initial value $u(0) = \lambda h$, then for $0 < \lambda \leq 1$, u is global (by the comparison principle) and if $\lambda > 1$, then u blows up in finite time (by Proposition 1.1).

The question remains as to whether or not the result in [2], cited above, concerning sign-changing solutions to (1.7) also carries over to sign-changing solutions of (1.1).

The point of view in this paper is to fix a value of q with

$$1 < q < p_S, \quad (1.12)$$

and then consider all p with

$$q < p < p_S. \quad (1.13)$$

In fact we will ultimately consider what happens as $p \rightarrow p_S$. The main purpose of this paper is to establish the following result.

Theorem 1. Assume (1.2)-(1.3). Given $1 < q < p_S = \frac{N+2}{N-2}$. It follows that there exists $1 < q < \underline{p} < p_S$ with the following property. If $\underline{p} < p < p_S$ and if $h_p \in C_0(\Omega)$ is a radially symmetric stationary solution of (1.1) which takes both positive and negative values, then there exist $0 < \underline{\lambda} < 1 < \bar{\lambda}$ such that if $\underline{\lambda} < \lambda < \bar{\lambda}$ and $\lambda \neq 1$, then the classical solution of (1.1) with the initial condition $u(0) = \lambda h$ blows up in finite time.

The first observation is that there does exist a radially symmetric, sign-changing stationary solution of (1.1), since the nonlinearity $|s|^{p-1}s - |s|^{q-1}s$ satisfies the hypothesis of [13, Theorem 2, p. 376]. More precisely, if we consider the problem:

$$\begin{cases} h'' + \frac{N-1}{r}h' + |h|^{p-1}h - |h|^{q-1}h = 0, \\ h(0) = a > 0, \quad h'(0) = 0. \end{cases} \quad (1.14)$$

It is well-known by [13] that (1.14) admits a unique solution $h \in C^2([0, \infty), \mathbb{R})$, which we denote sometimes by $h_p(r, a)$ to emphasize the dependence on a . Recall that we are fixing a value of q satisfying (1.12) and letting p vary in the interval (1.13). Under these conditions, by Theorem 2 in [13] for all integer $m \geq 0$, there exists $a_{p,m}$ such that

- a) $h_p(1, a_{p,m}) = 0$,
- b) $h_p(r, a_{p,m})$ has precisely m zeros in $(0, 1)$.

In particular, $h_p(\cdot, a_{p,m})$, considered as a function on $\Omega = B_1$, is a radially symmetric solution of (1.6) which changes sign precisely m times.

Now, let h_p be any nontrivial solution of (1.6) and consider the linearized operator F_p on $L^2(\Omega)$ defined by

$$\begin{cases} D(F_p) = H^2(\Omega) \cap H_0^1(\Omega), \\ F_p u = -\Delta u - (p|h_p|^{p-1} - q|h_p|^{q-1})u, \quad u \in D(F_p). \end{cases} \quad (1.15)$$

We recall the following result from [3].

Theorem 2 ([3, Corollary 2.5, p. 18]). *Let $h_p \in C_0(\Omega)$ be a sign-changing solution of (1.6). Let φ_p be a positive eigenvector of the self-adjoint operator F_p given by (1.15), corresponding to the first eigenvalue. Suppose that*

$$\int_{\Omega} h_p \varphi_p \neq 0.$$

It follows that there exists $\epsilon > 0$ such that if $0 < |1 - \lambda| < \epsilon$, then the solution of (1.1) with the initial value $u_0 = \lambda h_p$ blows up in finite time.

To prove Theorem 1, it thus suffices to establish the following.

Theorem 3. *Assume (1.2)-(1.3). Given $1 < q < p_S = \frac{N+2}{N-2}$. It follows that there exists $1 < \underline{q} < \underline{p} < p_S$ with the following property. If $\underline{p} < p < p_S$ and if $h_p \in C_0(\Omega)$ is a radially symmetric stationary solution of (1.1) which takes both positive and negative values, then*

$$\int_{\Omega} h_p \varphi_p \neq 0.$$

Where φ_p is a positive eigenvector of the self-adjoint operator F_p given by (1.15), corresponding to the first eigenvalue.

The proof of Theorem 3 is based on rescaling argument. Contrary to the case of single power nonlinearity, a rescaled function v_p defined by (2.4) below in terms of h_p , where h_p is a radially symmetric stationary solution of (1.1) doesn't satisfy the same differential equation satisfied by h_p , which make the situation more difficult. Also, unlike the case of the single power nonlinearity, there exist some solutions $v_p(r)$ of the problem (2.5) below which do not tend to zero as $r \rightarrow \infty$.

The rest of the paper is devoted to proving Theorem 3, which as already noted, implies Theorem 1 when combined with Theorem 2. Our basic approach follows that in [2]. However because of the differences just noted between the single power and the two power cases, many of the arguments in [2] do not immediately apply for the current situation.

Remark 1.3. *The results in this paper are equally valid for*

$$u_t = \Delta u + |u|^{p-1}u - c|u|^{q-1}u,$$

for any $c > 0$. The case where $c < 0$ is not as clear, since in that case, the proof of Proposition 2.1 below is no longer valid.

2. STATIONARY SOLUTIONS

The proof of Theorem 3 exploits strongly the radial symmetry of the stationary solutions. By abuse of notation we will use the same letter, for example h , to denote a radially symmetric function $h : \mathbb{R}^N \rightarrow \mathbb{R}$, and the corresponding function $h : [0, \infty) \rightarrow \mathbb{R}$ such that, $h(x) = h(|x|)$, $\forall x \in \mathbb{R}^N$. Throughout this paper, we will use this convention without further comment.

Any radially symmetric solution $h_p \in C_0(\Omega)$ of (1.6) satisfies the ODE

$$\begin{cases} h_p'' + \frac{N-1}{r}h_p' + |h_p|^{p-1}h_p - |h_p|^{q-1}h_p = 0, \\ h_p'(0) = h_p(1) = 0. \end{cases} \quad (2.1)$$

Since $h_p \neq 0$, it follows by uniqueness for the ODE (2.1) that $h_p(0) \neq 0$. Therefore, since if u satisfies (1.1) then $-u$ satisfies the same problem, it suffice to prove Theorem 3 under the additional assumption

$$h_p(0) > 0. \quad (2.2)$$

In the rest of this paper we set

$$h_p(0) = a_p > 0.$$

Clearly $h_p(r) = h_p(r, a_p)$, where $h_p(\cdot, a_p)$ is the solution of (1.14) with $a = a_p$. We let $\lambda_p > 0$ be such that

$$\lambda_p^{\frac{2}{p-1}} = a_p, \quad (2.3)$$

also we define

$$v_p(r) = \lambda_p^{-\frac{2}{p-1}} h_p\left(\frac{r}{\lambda_p}, \lambda_p^{\frac{2}{p-1}}\right). \quad (2.4)$$

A simple calculation shows that v_p satisfies

$$\begin{cases} v'' + \frac{N-1}{r}v' + |v|^{p-1}v - \lambda_p^{-\frac{2}{p-1}(p-q)}|v|^{q-1}v = 0, \\ v(0) = 1, \quad v'(0) = 0. \end{cases} \quad (2.5)$$

As such, v_p may be considered as a function $[0, \infty) \rightarrow \mathbb{R}$. It is known by [13, Lemma 1, p. 371] that $a_p \geq 1$. In fact, if $0 < a_p \leq \left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}$ then $h_p(r, a_p) > 0$ for all $r > 0$. Thus,

$$\lambda_p \geq 1. \quad (2.6)$$

We have also

$$v_p(\lambda_p) = 0. \quad (2.7)$$

Proposition 2.1. *Let λ_p defined in (2.3), then*

$$\lambda_p \xrightarrow{p \rightarrow p_S} \infty. \quad (2.8)$$

Proof. Suppose to the contrary that $\lambda_p \not\rightarrow \infty$ as $p \rightarrow p_S$. It follows that there exists a subsequence (p_k) such that $p_k \xrightarrow{k \rightarrow \infty} p_S$ and

$$\lambda_{p_k} \xrightarrow{k \rightarrow \infty} \bar{\lambda}, \quad (2.9)$$

where $1 \leq \bar{\lambda} < \infty$, by (2.6). By continuous dependence it follows that

$$v_{p_k} \xrightarrow{k \rightarrow \infty} \bar{v}, \quad (2.10)$$

uniformly on all compact intervals $[0, M] \subset [0, \infty)$, where \bar{v} satisfies

$$\begin{cases} \bar{v}'' + \frac{N-1}{r}\bar{v}' + |\bar{v}|^{p_S-1}\bar{v} - \bar{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}|\bar{v}|^{q-1}\bar{v} = 0, \\ \bar{v}(0) = 1, \quad \bar{v}'(0) = 0. \end{cases} \quad (2.11)$$

It follows from (2.7), (2.9) and (2.10) that

$$\bar{v}(\bar{\lambda}) = 0. \quad (2.12)$$

And so \bar{v} satisfies the equation

$$\begin{cases} -\Delta \bar{v} = |\bar{v}|^{p_S-1}\bar{v} - \bar{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}|\bar{v}|^{q-1}\bar{v}, \\ \bar{v}|_{\partial B(0, \bar{\lambda})} = 0. \end{cases} \quad (2.13)$$

If we apply the Pohozaev identity as was done in [1, Remark 1.2, p. 442], and if we set $g(u) = |u|^{p_S-1}u - \bar{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}|u|^{q-1}u$ and $G(u) = \frac{|u|^{p_S+1}}{p_S+1} - \bar{\lambda}^{-\frac{2}{p_S-1}(p_S-q)}\frac{|u|^{q+1}}{q+1}$, we obtain

$$\begin{aligned} \frac{2-N}{2} \int_{B(0, \bar{\lambda})} g(\bar{v})\bar{v} + N \int_{B(0, \bar{\lambda})} G(\bar{v}) &= \bar{\lambda}^{-\frac{2}{p_S-1}(p_S-q)} \left[\frac{N-2}{2} - \frac{N}{q+1} \right] \int_{B(0, \bar{\lambda})} |\bar{v}|^{q+1} \\ &= \frac{1}{2} \int_{\partial B(0, \bar{\lambda})} (x \cdot \nu) \left(\frac{\partial \bar{v}}{\partial \nu} \right)^2 \geq 0. \end{aligned} \quad (2.14)$$

From (2.14), one can conclude that

$$0 \leq \left(\frac{N-2}{2} - \frac{N}{q+1} \right) \bar{\lambda}^{-\frac{2}{p_S-1}(p_S-q)} \|\bar{v}\|_{L^{q+1}(B(0, \bar{\lambda}))}^{q+1}. \quad (2.15)$$

Since $q < p_S$ inequality (2.15) is possible only if $\bar{v} = 0$, which contradicts $\bar{v}(0) = 1$. \square

Let now w_p be the solution of

$$\begin{cases} w'' + \frac{N-1}{r}w' + |w|^{p-1}w = 0, \\ w(0) = 1, \quad w'(0) = 0. \end{cases} \quad (2.16)$$

It is well-known and easy to verify that w_{p_S} given by

$$w_{p_S}(r) = \left(1 + \frac{1}{N(N-2)}r^2\right)^{-\frac{N-2}{2}} \quad (2.17)$$

is the solution of (2.16) with $p = p_S$.

Proposition 2.2. *Let v_p defined by (2.4) and w_{p_S} by (2.17), then*

$$v_p \xrightarrow{p \rightarrow p_S} w_{p_S}, \quad (2.18)$$

uniformly on bounded sets of $[0, \infty)$.

Proof. By Proposition 2.1 $\lambda_p \rightarrow \infty$ as $p \rightarrow p_S$, and so by continuous dependence we can conclude that

$$v_p \xrightarrow{p \rightarrow p_S} w_{p_S},$$

uniformly on bounded sets of $[0, \infty)$. \square

Proposition 2.3. *Given $1 < q < p_S$ and $0 < \eta < p_S - q$. There exist $M, C > 0$ such that for all $p \in [q + \eta, p_S)$ and $r \geq 0$,*

$$|v_p(r)| \leq M \text{ and } |v_p'(r)| \leq C. \quad (2.19)$$

Proof. Let $1 < q < p_S$ and $0 < \eta < p_S - q$. Note first that by (2.5)

$$\left[\frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1} \right]' = -\frac{N-1}{r}|v_p'(r)|^2, \quad (2.20)$$

so that

$$\begin{aligned} \frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1} &\leq \frac{1}{p+1} - \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)} \\ &\leq \frac{1}{p+1}. \end{aligned} \quad (2.21)$$

Now since λ_p satisfies (2.6), it follows from (2.21) that

$$\frac{1}{p+1}|v_p(r)|^{p+1} - \frac{1}{q+1}|v_p(r)|^{q+1} \leq \frac{1}{p+1}. \quad (2.22)$$

Suppose by contradiction that, there exist $(p_n) \subset [q + \eta, p_S)$ and $(r_n) \subset [0, \infty)$ such that

$$|v_{p_n}(r_n)| \xrightarrow{n \rightarrow \infty} \infty.$$

Since (p_n) is bounded we can suppose that $p_n \rightarrow p_* \in [q + \eta, p_S]$, we apply now inequality (2.22), which we note as

$$|v_p(r)|^{p+1} \left(\frac{1}{p+1} - \frac{1}{q+1} |v_p(r)|^{q-p} \right) \leq \frac{1}{p+1},$$

with $p = p_n$, $r = r_n$. By letting $n \rightarrow \infty$, it follows that

$$\infty \leq \frac{1}{p_* + 1},$$

which is absurd. It follows so that there exists $M > 0$, such that for all $p \in [q + \eta, p_S]$ and $r \geq 0$,

$$|v_p(r)| \leq M. \quad (2.23)$$

We turn now to prove the second assertion. It follows from (2.21), $\lambda_p \geq 1$, (2.23) and $p > q$ that

$$\begin{aligned} \frac{1}{2} v_p'(r)^2 &\leq \frac{1}{p+1} + \frac{1}{q+1} |v_p(r)|^{q+1} \\ &\leq \frac{1}{q+1} + \frac{1}{q+1} M^{q+1}, \quad \forall p \in [q + \eta, p_S], \quad \forall r \geq 0, \end{aligned}$$

so that

$$|v_p'(r)| \leq \sqrt{\frac{2}{q+1}} \sqrt{1 + M^{q+1}}, \quad \forall p \in [q + \eta, p_S], \quad \forall r \geq 0.$$

□

The following lemma is one of the key points which differ from the calculations in [2]. Compare Lemma 3.3 in [2]. Indeed, Lemma 3.3 in [2] cannot be true in the present context since not all solutions v_p of (2.5) tend to 0 as $r \rightarrow \infty$. We do obtain, however, a similar estimate, valid only for $r \leq \lambda_p$.

Lemma 2.4. *Given $1 < q < p_S$ and $0 < \eta < p_S - q$. There exists a constant $\gamma = \gamma(N, q)$ such that*

$$\frac{1}{2} |v_p'(r)|^2 + \frac{1}{p+1} |v_p(r)|^{p+1} \leq \gamma \left[\frac{1}{r+1} + \frac{1}{(r+1)^{\frac{2}{p_S-1}\eta}} \right], \quad (2.24)$$

for all $p \in [q + \eta, p_S]$ and for all $0 \leq r \leq \lambda_p$.

Proof. Fix $1 < q < p_S$ and $0 < \eta < p_S - q$. Let r such that $1 \leq r \leq \lambda_p$ and $p \in [q + \eta, p_S]$. Define now

$$F(r) = \frac{1}{2} v_p'(r)^2 + \frac{1}{p+1} |v_p(r)|^{p+1} - \frac{1}{q+1} \lambda_p^{-\frac{2}{p-1}(p-q)} |v_p(r)|^{q+1} + \frac{1}{r} v_p(r) v_p'(r). \quad (2.25)$$

It follows from (2.20) and (2.5) that

$$\begin{aligned}
F'(r) &= -\frac{N-1}{r}v_p'(r)^2 - \frac{1}{r^2}v_p(r)v_p'(r) + \frac{1}{r}v_p'(r)^2 + \frac{1}{r}v_p(r)v_p''(r) \\
&= -\frac{N-2}{r}v_p'(r)^2 - \frac{1}{r^2}v_p(r)v_p'(r) + \frac{1}{r}v_p(r)v_p''(r) \\
&= -\frac{N-2}{r}v_p'(r)^2 - \frac{1}{r^2}v_p(r)v_p'(r) + \\
&\quad \frac{1}{r}v_p(r) \left[-\frac{N-1}{r}v_p'(r) - |v_p(r)|^{p-1}v_p(r) + \lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q-1}v_p(r) \right].
\end{aligned}$$

From (2.19), (1.3), the fact that $1 \leq r \leq \lambda_p$, $1 < q < p$, Young's inequality (applied twice) and denoting $\alpha := \frac{p-q}{p+1} + \frac{2}{p-1}(p-q)$, one can find the estimate

$$\begin{aligned}
F'(r) + \frac{1}{r}F(r) &= -\frac{2N-5}{2r}v_p'(r)^2 - \frac{p}{(p+1)r}|v_p(r)|^{p+1} - \frac{N-1}{r^2}v_p(r)v_p'(r) \\
&\quad + \lambda_p^{-\frac{2}{p-1}(p-q)} \frac{q}{(q+1)r}|v_p(r)|^{q+1} \\
&\leq -\frac{2N-5}{2r}v_p'(r)^2 - \frac{p}{(p+1)r}|v_p(r)|^{p+1} + \frac{1}{2} \left[\frac{(N-1)^2}{r^3}v_p(r)^2 + \frac{1}{r}v_p'(r)^2 \right] \\
&\quad + \frac{q}{q+1}|v_p(r)|^{q+1}r^{-\frac{q+1}{p+1}}r^{-\alpha} \\
&\leq \frac{(N-1)^2}{2r^3}v_p(r)^2 - \frac{p}{(p+1)r}|v_p(r)|^{p+1} \\
&\quad + \frac{q}{(p+1)r}|v_p(r)|^{p+1} + \frac{q(p-q)}{(q+1)(p+1)}r^{-\alpha\frac{p+1}{p-q}} \\
&\leq \frac{(N-1)^2}{2r^3}M^2 + \frac{q(p-q)}{(q+1)^2}r^{-\alpha\frac{p+1}{p-q}}.
\end{aligned}$$

Now since $\alpha\frac{p+1}{p-q} = 1 + 2\frac{p+1}{p-1} \geq 3$, we obtain that for $1 \leq r \leq \lambda_p$

$$F'(r) + \frac{1}{r}F(r) \leq Ar^{-3}.$$

One can conclude now for all $s \in [1, \lambda_p]$, for all $p \in [q + \eta, p_S)$ that

$$\frac{d}{ds}(sF(s)) = sF'(s) + F(s) \leq As^{-2}. \quad (2.26)$$

Integration of (2.26) on $[1, r]$ gives

$$rF(r) - F(1) \leq A \left(-\frac{1}{r} + 1 \right).$$

We can affirm for $r \in [1, \lambda_p]$ that

$$F(r) \leq B\frac{1}{r}. \quad (2.27)$$

Using also (2.25), (2.27), (2.19), $p \in [q + \eta, p_S)$ and the fact that $1 \leq r \leq \lambda_p$, it follows that

$$\begin{aligned} \frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} &\leq \frac{1}{q+1}\lambda_p^{-\frac{2}{p-1}(p-q)}|v_p(r)|^{q+1} - \frac{1}{r}v_p(r)v_p'(r) + B\frac{1}{r} \\ &\leq \frac{1}{q+1}\frac{1}{r^{\frac{2}{p_S-1}\eta}}M^{q+1} + \frac{M.C}{r} + B\frac{1}{r}. \end{aligned}$$

Finally, using (2.19) one can conclude that there exists $\gamma > 0$ such that

$$\frac{1}{2}v_p'(r)^2 + \frac{1}{p+1}|v_p(r)|^{p+1} \leq \gamma \left[\frac{1}{r+1} + \frac{1}{(r+1)^{\frac{2}{p_S-1}\eta}} \right],$$

for all $0 \leq r \leq \lambda_p$. \square

We set

$$\tilde{v}_p(r) = \begin{cases} v_p(r) & \text{if } 0 \leq r \leq \lambda_p, \\ 0 & \text{if } r > \lambda_p. \end{cases} \quad (2.28)$$

Corollary 2.5. *Given $1 < q < p_S$ and $0 < \eta < p_S - q$. There exists a decreasing function $j : [0, \infty) \rightarrow [0, \infty)$ satisfying $j(r) \xrightarrow{r \rightarrow \infty} 0$ such that*

$$|\tilde{v}_p(r)| \leq j(r), \quad \forall r \geq 0, \quad \forall p \in [q + \eta, p_S). \quad (2.29)$$

Proposition 2.6. $\|\tilde{v}_p - w_{p_S}\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$, as $p \rightarrow p_S$.

Proof. Fix $1 < q < p_S$, $0 < \eta < p_S - q$ and $R \geq 0$. Let $p \in [q + \eta, p_S)$, on the one hand it follows from (2.29) and (2.17) that

$$\begin{aligned} |\tilde{v}_p(r) - w_{p_S}(r)| &\leq |\tilde{v}_p(r)| + w_{p_S}(r) \\ &\leq j(r) + w_{p_S}(r) \\ &\leq j(R) + w_{p_S}(R), \quad \forall r \geq R. \end{aligned}$$

It follows that

$$\sup_{r \geq R} |\tilde{v}_p(r) - w_{p_S}(r)| \leq j(R) + w_{p_S}(R) \xrightarrow{R \rightarrow \infty} 0.$$

Thus, there exists $R_0 \geq 0$ such that

$$\sup_{r \geq R_0} |\tilde{v}_p(r) - w_{p_S}(r)| \leq \frac{\varepsilon}{2}. \quad (2.30)$$

On the other hand, since $\lambda_p \xrightarrow{p \rightarrow p_S} \infty$, by choosing p_0 sufficiently close to p_S , we can assume that $R_0 \leq \lambda_p$ for $p_0 \leq p < p_S$. It follows from (2.18) that there exists $p_0 \leq \bar{p} < p_S$ such that if $\bar{p} < p < p_S$ then

$$\sup_{r \in [0, R_0]} |\tilde{v}_p(r) - w_{p_S}(r)| = \sup_{r \in [0, R_0]} |v_p(r) - w_{p_S}(r)| \leq \frac{\varepsilon}{2}. \quad (2.31)$$

One can conclude from (2.30) and (2.31). \square

3. THE LINEARIZED OPERATOR

We consider now the self-adjoint operator F_p defined on $L^2(\Omega)$ by

$$\begin{cases} D(F_p) = H^2(\Omega) \cap H_0^1(\Omega), \\ F_p u = -\Delta u - (p|h_p|^{p-1} - q|h_p|^{q-1})u, \quad \forall u \in D(F_p). \end{cases} \quad (3.1)$$

We denote by

$$\theta_p = \theta_p(F_p), \quad (3.2)$$

its first eigenvalue and by φ_p the corresponding eigenvector, i.e.

$$F_p \varphi_p = -\Delta \varphi_p - (p|h_p|^{p-1} - q|h_p|^{q-1})\varphi_p = \theta_p \varphi_p, \quad (3.3)$$

where we require

$$\varphi_p > 0, \quad \|\varphi_p\|_{L^2(\Omega)} = 1. \quad (3.4)$$

Since φ_p is radially symmetric, it satisfies the ODE

$$\varphi_p'' + \frac{N-1}{r} \varphi_p' + (p|h_p|^{p-1} - q|h_p|^{q-1})\varphi_p + \theta_p \varphi_p = 0. \quad (3.5)$$

In order to transform the operator F_p into another operator we introduce $l_p \in \mathbb{R}$ and ψ_p , a positive, spherically symmetric function on Ω_p defined by

$$\theta_p = \lambda_p^2 l_p, \quad \varphi_p(x) = \lambda_p^{\frac{N}{2}} \psi_p(\lambda_p x), \quad (3.6)$$

where

$$\Omega_p = B(0, \lambda_p). \quad (3.7)$$

It follows from (3.5), (2.4) and (3.6) that ψ_p satisfies the equation

$$\begin{cases} -\Delta \psi_p - \left[p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)} |v_p|^{q-1} \right] \psi_p = l_p \psi_p & \text{in } \Omega_p, \\ \psi_p = 0 & \text{on } \partial\Omega_p, \end{cases} \quad (3.8)$$

and that

$$\int_{\Omega} h_p \varphi_p = \lambda_p^{\frac{2}{p-1} - \frac{N}{2}} \int_{\Omega_p} v_p \psi_p, \quad (3.9)$$

and

$$\psi_p > 0, \quad \|\psi_p\|_{L^2(\Omega_p)} = 1. \quad (3.10)$$

We have also that l_p is the first eigenvalue associated to the eigenvector ψ_p of the self-adjoint operator L_p defined on $L^2(\Omega_p)$ by

$$\begin{cases} D(L_p) = H^2(\Omega_p) \cap H_0^1(\Omega_p), \\ L_p u = -\Delta u - \left[p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)} |v_p|^{q-1} \right] u, \quad \forall u \in D(L_p). \end{cases} \quad (3.11)$$

Given $0 < p < p_S$, we set

$$J_p(w) = \int_{\Omega_p} |\nabla w|^2 - \int_{\Omega_p} \left[p|v_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)} |v_p|^{q-1} \right] w^2, \quad (3.12)$$

for all $w \in H_0^1(\Omega_p)$, so that

$$l_p = \inf \{ J_p(u), u \in H_0^1(\Omega_p), \|u\|_{L^2(\Omega_p)} = 1 \}. \quad (3.13)$$

Also we define the self-adjoint operator L_* on $L^2(\mathbb{R}^N)$ by

$$\begin{cases} D(L_*) = H^2(\mathbb{R}^N), \\ L_* u = -\Delta u - p_S w_{p_S}^{p_S-1} u, \quad \forall u \in D(L_*), \end{cases} \quad (3.14)$$

where w_{p_S} is given by (2.17). We set

$$\lambda_* = \inf \left\{ J_*(u), u \in H^1(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)} = 1 \right\}, \quad (3.15)$$

where

$$J_*(w) = \int_{\mathbb{R}^N} |\nabla w|^2 - p_S \int_{\mathbb{R}^N} w_{p_S}^{p_S-1} w^2, \quad (3.16)$$

for all $w \in H^1(\mathbb{R}^N)$. We recall now the following proposition from [2].

Proposition 3.1 ([2, Proposition 3.4, p. 439]). *If L_* is defined by (3.14) and λ_* is defined by (3.15), then the following properties hold.*

- (i) $\lambda_* < 0$ and λ_* is an eigenvalue of L_* .
- (ii) There exists a unique eigenvector ψ_* of L_* corresponding to the eigenvalue λ_* which is positive, radially decreasing with $\|\psi_*\|_{L^2(\mathbb{R}^N)} = 1$.
- (iii) If $(u_n)_{n \geq 1} \subset H^1(\mathbb{R}^N)$ is a minimizing sequence of (3.15) and $u_n \geq 0$, then $u_n \rightarrow \psi_*$ in $L^2(\mathbb{R}^N)$ as $n \rightarrow \infty$.

We set

$$\tilde{\psi}_p(x) = \begin{cases} \psi_p(x) & \text{if } 0 \leq |x| < \lambda_p, \\ 0 & \text{if } |x| \geq \lambda_p, \end{cases} \quad (3.17)$$

for all $1 < p < p_S$, so that

$$\tilde{\psi}_p \in H^1(\mathbb{R}^N), \|\tilde{\psi}_p\|_{L^2(\mathbb{R}^N)} = 1, \tilde{\psi}_p \geq 0. \quad (3.18)$$

Lemma 3.2. *Let $\psi \in H^1(\mathbb{R}^N)$ such that $\|\psi\|_{L^2(\mathbb{R}^N)} = 1$. Consider a smooth radial cut-off function $\eta : \mathbb{R}^N \rightarrow [0, 1]$ such that $\eta(r) = 1$ for $r \leq \frac{1}{2}$ and $\eta(r) = 0$ for $r \geq 1$. Set*

$$k_\lambda(r) = \eta\left(\frac{r}{\lambda}\right) \psi(r), \quad (3.19)$$

and

$$u_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{L^2(\mathbb{R}^N)}}. \quad (3.20)$$

Then $u_\lambda \in H_0^1(\Omega_\lambda)$ and

$$\|u_\lambda - \psi\|_{H^1(\mathbb{R}^N)} \xrightarrow{\lambda \rightarrow \infty} 0. \quad (3.21)$$

Where $\Omega_\lambda = B(0, \lambda)$.

Proof. This follows by standard arguments, using the observation that $\|k_\lambda\|_{L^2(\mathbb{R}^N)} \xrightarrow{\lambda \rightarrow \infty} 1$. \square

Lemma 3.3. *Let l_p defined by (3.6), then*

$$l_p \rightarrow \lambda_* \quad \text{as } p \rightarrow p_S.$$

Proof. We first use $\tilde{\psi}_p$ as a test function in (3.15). It follows from (3.18) that

$$\begin{aligned} \lambda_* \leq J_*(\tilde{\psi}_p) &= J_p(\tilde{\psi}_p) + \int_{\mathbb{R}^N} \left[p|\tilde{v}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)} |\tilde{v}_p|^{q-1} - p_S w_{p_S}^{p_S-1} \right] \tilde{\psi}_p^2 \\ &= l_p + \int_{\mathbb{R}^N} \left[p|\tilde{v}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)} |\tilde{v}_p|^{q-1} - p_S w_{p_S}^{p_S-1} \right] \tilde{\psi}_p^2 \\ &= l_p + \int_{\mathbb{R}^N} \left[p|\tilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1} \right] \tilde{\psi}_p^2 - q\lambda_p^{-\frac{2}{p-1}(p-q)} \int_{\mathbb{R}^N} |\tilde{v}_p|^{q-1} \tilde{\psi}_p^2. \end{aligned} \quad (3.22)$$

It follows from (3.22), Proposition 2.3 and (3.18) that

$$\lambda_* - l_p \leq \|p|\tilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1}\|_{L^\infty(\mathbb{R}^N)} + qM^{q-1} \lambda_p^{-\frac{2}{p-1}(p-q)}, \quad \forall p \in [q + \eta, p_S]. \quad (3.23)$$

One can conclude now by applying Proposition 2.6 and Proposition 2.1 that

$$\limsup_{p \rightarrow p_S} (\lambda_* - l_p) \leq 0. \quad (3.24)$$

Next, we would like to use ψ_* as a test function in (3.13), but $\psi_* \notin H_0^1(\Omega_p)$. Thus, we need to approximate ψ_* by a sequence in $H_0^1(\Omega_p)$. Consider a smooth radial cut-off function $\eta : \mathbb{R}^N \rightarrow [0, 1]$ such that $\eta(r) = 1$ for $r \leq \frac{1}{2}$ and $\eta(r) = 0$ for $r \geq 1$. Setting

$$k_p(r) = \eta\left(\frac{r}{\lambda_p}\right) \psi_*(r), \quad (3.25)$$

and

$$u_p = \frac{k_p}{\|k_p\|_{L^2(\mathbb{R}^N)}}. \quad (3.26)$$

it follows from Lemma 3.2 (since $\lambda_p \rightarrow \infty$ as $p \rightarrow p_S$) that

$$\|u_p - \psi_*\|_{H^1(\mathbb{R}^N)} \xrightarrow{p \rightarrow p_S} 0. \quad (3.27)$$

Moreover $u_p \in H_0^1(\Omega_p)$, so that

$$l_p \leq J_p(u_p) = \lambda_* - J_*(\psi_*) + J_*(u_p) - J_*(u_p) + J_p(u_p). \quad (3.28)$$

On the one hand we have by Proposition 2.6 and Proposition 2.1 that

$$\begin{aligned} |J_*(u_p) - J_p(u_p)| &= \left| \int_{\mathbb{R}^N} \left[p|\tilde{v}_p|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\tilde{v}_p|^{q-1} - p_S w_{p_S}^{p_S-1} \right] u_p^2 \right| \\ &\leq \int_{\mathbb{R}^N} |p|\tilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1}| u_p^2 + q\lambda_p^{-\frac{2}{p-1}(p-q)} \int_{\mathbb{R}^N} |\tilde{v}_p|^{q-1} u_p^2 \\ &\leq \|p|\tilde{v}_p|^{p-1} - p_S w_{p_S}^{p_S-1}\|_{L^\infty(\mathbb{R}^N)} + qM^{q-1}\lambda_p^{-\frac{2}{p-1}(p-q)} \xrightarrow{p \rightarrow p_S} 0. \end{aligned} \quad (3.29)$$

On the other hand, using the fact that $|J_*(\psi_*) - J_*(u_p)| \leq \left| \|\nabla u_p\|_{L^2(\mathbb{R}^N)}^2 - \|\nabla \psi_*\|_{L^2(\mathbb{R}^N)}^2 \right| + p_S \int_{\mathbb{R}^N} |u_p^2 - \psi_*^2|$, it easily follows from (3.27) and the dominated convergence theorem that

$$|J_*(\psi_*) - J_*(u_p)| \xrightarrow{p \rightarrow p_S} 0. \quad (3.30)$$

We deduce from (3.29) and (3.30) that

$$-\liminf_{p \rightarrow p_S} (\lambda_* - l_p) = \limsup_{p \rightarrow p_S} (l_p - \lambda_*) \leq 0. \quad (3.31)$$

We can confirm so by (3.24) and (3.31) that

$$\liminf_{p \rightarrow p_S} (\lambda_* - l_p) = \limsup_{p \rightarrow p_S} (\lambda_* - l_p) = \lim_{p \rightarrow p_S} (\lambda_* - l_p) = 0.$$

The result follows now. \square

Lemma 3.4. *Given $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. There exists $C > 0$ such that*

$$|\tilde{\psi}_p(r)| + |\psi_*(r)| \leq C \frac{1}{r^{\frac{N-1}{2}}} \leq C, \quad (3.32)$$

for all $r \geq 1$ and $q + \eta \leq p < p_S$.

Proof. Fix $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. We affirm first that

$$l_p < 0 \quad \text{for all } p \in (q, p_S). \quad (3.33)$$

In fact, since l_p satisfies (3.6), it suffice to prove that $\theta_p < 0$. We have on the one hand since $p > q$

$$\begin{aligned} \left(\int_{\Omega_p} h_p^2 \right) \theta_p &\leq \int_{\Omega_p} |\nabla h_p|^2 - \int_{\Omega_p} (p|h_p|^{p-1} - q|h_p|^{q-1})h_p^2 \\ &\leq \int_{\Omega_p} |\nabla h_p|^2 - q \int_{\Omega_p} (|h_p|^{p+1} - |h_p|^{q+1}). \end{aligned} \quad (3.34)$$

On the other hand since h_p satisfies (2.1) it follows that

$$\int_{\Omega_p} |\nabla h_p|^2 = \int_{\Omega_p} (|h_p|^{p+1} - |h_p|^{q+1}). \quad (3.35)$$

It follows from (3.34) and (3.35) since $q > 1$ that $\theta_p < 0$.

We complete now our proof. Since $l_p < 0$, we deduce from (3.13) and Proposition 2.3 that

$$\|\nabla \tilde{\psi}_p\|_{L^2(\mathbb{R}^N)}^2 \leq p_S(M^{q-1} + M^{p_S-1}), \quad \forall p \in [q + \eta, p_S]. \quad (3.36)$$

By (3.18), (3.36) and Strauss' radial lemma [18] that

$$|\tilde{\psi}_p(r)| \leq c\sqrt{1 + p_S(M^{q-1} + M^{p_S-1})} \frac{1}{r^{\frac{N-1}{2}}}, \quad (3.37)$$

for all $r \geq 1$.

A similar argument applies to ψ_* which completes the proof. \square

Lemma 3.5. *Given $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. There exist $R, C > 0, \theta > 0$ and $q + \eta \leq p_0 < p_S$ such that*

$$|\tilde{\psi}_p(r)| + |\psi_*(r)| \leq Ce^{-\theta r}, \quad (3.38)$$

for all $r \geq R$ and $p_0 \leq p < p_S$.

Proof. Fix $q \in (1, p_S)$ and $\eta \in (0, p_S - q)$. We start first by showing that there exists $R, C > 0, \theta > 0$ and $q + \eta \leq p_0 < p_S$ such that

$$|\psi_p(r)| \leq Ce^{-\theta r}, \quad (3.39)$$

for all $r \geq R$ (with $r \leq \lambda_p$) and $p_0 \leq p < p_S$. It follows from (3.8) that ψ_p satisfies

$$-\psi_p''(r) - \frac{N-1}{r}\psi_p'(r) - \left\{ \left[p|\tilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\tilde{v}_p(r)|^{q-1} \right] + l_p \right\} \psi_p(r) = 0, \quad (3.40)$$

for all $0 \leq r < \lambda_p$. We would like to use a method of energy in equation (3.40), but the term $-\left[p|\tilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\tilde{v}_p(r)|^{q-1}\right] - l_p$ is difficult to handle so we may estimate it. On the one hand, since $|p|\tilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\tilde{v}_p(r)|^{q-1}| \leq L(r) \xrightarrow{r \rightarrow \infty} 0$ by Corollary 2.5 and the fact that $p \in [q + \eta, p_S)$, it follows that there exists $R > 0$, such that for all $r \geq R$

$$-\left[p|\tilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\tilde{v}_p(r)|^{q-1}\right] \geq \frac{\lambda_*}{4}. \quad (3.41)$$

On the other hand, since $-l_p \rightarrow -\lambda_*$ as $p \rightarrow p_S$ by Lemma 3.3, it follows that there exists $p_0 \in [q + \eta, p_S)$ such that for all $p_0 \leq p < p_S$

$$-l_p \geq -\frac{3}{4}\lambda_*. \quad (3.42)$$

Finally one can conclude from (3.41) and (3.42) that there exist $R > 0$ and $q + \eta \leq p_0 < p_S$ such that

$$-\left[p|\tilde{v}_p(r)|^{p-1} - q\lambda_p^{-\frac{2}{p-1}(p-q)}|\tilde{v}_p(r)|^{q-1}\right] - l_p \geq -\frac{\lambda_*}{2} > 0, \quad (3.43)$$

for all $p_0 \leq p < p_S$ and all $r \geq R$. By choosing p_0 possibly larger, we also may assume that $\lambda_p > R$ for $p_0 \leq p < p_S$. Since $\psi_p \geq 0$, we deduce from (3.40) and (3.43) that

$$\psi_p'' + \frac{N-1}{r}\psi_p' \geq -\frac{\lambda_*}{2}\psi_p, \quad (3.44)$$

for all $R \leq r \leq \lambda_p$. We now claim that

$$\psi_p'(r) < 0, \quad (3.45)$$

for all $p_0 \leq p < p_S$ and all $R < r < \lambda_p$. We argue by contradiction and suppose that $\psi_p'(r_p) \geq 0$ for some $p_0 \leq p < p_S$ and some $R < r_p < \lambda_p$. Since $\psi_p(\lambda_p) = 0$, there exists $r_p \leq r'_p < \lambda_p$ such that $\psi_p'(r'_p) = 0$ and $\psi_p''(r'_p) \leq 0$. This is impossible by (3.44) since $\lambda_* < 0$. Multiplying (3.44) by $\psi_p' < 0$, see that

$$\psi_p''\psi_p' + \frac{N-1}{r}\psi_p'\psi_p' \leq -\frac{\lambda_*}{2}\psi_p\psi_p',$$

which implies

$$\left(\psi_p'^2 + \frac{\lambda_*}{2}\psi_p^2\right)' \leq 0, \quad (3.46)$$

for $R \leq r \leq \lambda_p$. It follows from (3.46) that

$$\left[\psi_p'^2 + \frac{\lambda_*}{2}\psi_p^2\right](r) \geq \psi_p'(\lambda_p)^2 \geq 0,$$

for $R < r < \lambda_p$. Since $\psi_p > 0$ and $\psi'_p < 0$, we obtain that $\psi'_p + \sqrt{-\frac{\lambda_*}{2}}\psi_p \leq 0$ for $R < r < \lambda_p$, so that

$$\psi_p(r) \leq \psi_p(R)e^{\sqrt{-\frac{\lambda_*}{2}}R}e^{-\sqrt{-\frac{\lambda_*}{2}}r},$$

for $R < r < \lambda_p$. By choosing $R \geq 1$ we have $\psi_p(R) \leq C$ by Lemma 3.4. The exponential decay follows. As remarked in [2], the proof for ψ_* is similar. This completes the proof. \square

Lemma 3.6. $\tilde{\psi}_p$ and $\psi_* \in L^1(\mathbb{R}^N)$. Moreover, $\|\tilde{\psi}_p - \psi_*\|_{L^1(\mathbb{R}^N)} \rightarrow 0$ as $p \rightarrow p_S$.

Proof. The proof is similar to the proof of Lemma 3.7 in [2]. \square

Proof of Theorem 3. Fix $q \in (1, p_S)$ and $0 < \eta < p_S - q$. Let $h_p \in C_0(\Omega)$ be a radially symmetric, sign-changing stationary solution of (1.1). Let φ_p be the positive eigenvector normalized in $L^2(\Omega)$ of the self-adjoint operator F_p given by (3.1), corresponding to the first eigenvalue. We have from Proposition 2.3

$$\begin{aligned} \left| \int_{\Omega_p} v_p \psi_p - \int_{\mathbb{R}^N} w_{p_S} \psi_* \right| &= \left| \int_{\mathbb{R}^N} \tilde{v}_p \tilde{\psi}_p - \int_{\mathbb{R}^N} w_{p_S} \psi_* \right| \\ &\leq \left| \int_{\mathbb{R}^N} \tilde{v}_p (\tilde{\psi}_p - \psi_*) \right| + \left| \int_{\mathbb{R}^N} (\tilde{v}_p - w_{p_S}) \psi_* \right| \\ &\leq M \left\| \tilde{\psi}_p - \psi_* \right\|_{L^1(\mathbb{R}^N)} + \|\psi_*\|_{L^1(\mathbb{R}^N)} \|\tilde{v}_p - w_{p_S}\|_{L^\infty(\mathbb{R}^N)}, \end{aligned}$$

$\forall p \in [q + \eta, p_S)$. It follows so by Lemma 3.6, Proposition 2.6 that

$$\int_{\Omega} v_p \varphi_p \xrightarrow{p \rightarrow p_S} \int_{\mathbb{R}^N} w_{p_S} \psi_* > 0.$$

We can now conclude from (3.9) that there exists $1 < q < \underline{p} < p_S$ such that if $\underline{p} < p < p_S$, then

$$\int_{\Omega} h_p \varphi_p > 0.$$

This finishes the proof. \square

4. ACKNOWLEDGEMENT

This article is part of my doctoral thesis. I am pleased to thank Professors Fred Weissler and Slim Tayachi for their guidance. I am grateful for the opportunity to have spent two months at University Paris 13, supported by the Faculty of Sciences of Tunis, Department of Mathematic, Laboratory of partial differential equations and by the Preparatory Institute of the Nabeul engineering studies.

REFERENCES

- [1] Brezis, H., Nirenberg, L.: *Positive Solutions of Nonlinear Elliptic Equations involving Critical Exponents*. Communications on Pure and Applied Mathematics. **34** (1983), pp. 437-477.
- [2] Cazenave, T., Dickstein, F., Weissler, F. B.: *Sign-changing stationary solutions and blow up for the nonlinear heat equation in a ball*. Math. Ann. **344** (2) (2009), pp. 431-449.
- [3] Cazenave, T., Dickstein, F., Weissler, F. B.: *Structural properties of the set of global solutions of the nonlinear heat equation*. in Current Advances in Nonlinear Analysis and Related Topics 1323, GAKUTO Internat. Ser. Math. Sci. Appl. **32** (2010), pp. 13-23.
- [4] Cazenave, T., Dickstein, F., Weissler, F. B.: *Spectral properties of stationary solutions of the nonlinear heat equation*. Publications Mathématiques. **55** (2011), pp. 185-200.
- [5] Cazenave, T., Dickstein, F., Weissler, F. B.: *On the structure of global solutions of the nonlinear heat equation in a ball*. J. Math. Anal. Appl. **360** (2) (2009), pp. 537-547.
- [6] Cazenave, T., Haraux, A.: *An Introduction to Semilinear Evolution Equations*. Oxford Lecture Series in Mathematics and its Applications vol. 13, xiv+186 pp. Oxford University Press, Oxford (1998). ISBN: 0-19-850277-X.
- [7] Cazenave, T., Lions, P.-L.: *Solutions globales d'équations de la chaleur semi linéaires*. Commun. Partial Differ. Equ. **9** (1984), pp. 955-978.
- [8] De Marchis, F., Ianni, I.: *Blow up of solutions of semilinear heat equations in non radial domains of \mathbb{R}^2* . Discrete Contin. Dyn. Syst. **35** (2015), pp. 891-907.
- [9] Dickstein, F., Pacella, F., Sciunzi, F.: *Sign-changing stationary solutions and blowup for a nonlinear heat equation in dimension two*. J. Evol. Equ. **14**(2014), pp. 617-633.
- [10] Gazzola, F., Weth, T.: *Finite-time blow-up and global solutions for semilinear parabolic equations with initial data at high energy levels*. Differ. Integral Equ. **18**(9) (2005), pp. 961-990.
- [11] Giga, Y.: *A bound for global solutions of semilinear heat equations*. Commun. Math. Phys. **103** (1986), pp. 415-421.
- [12] Lions, P.-L.: *Asymptotic behavior of some nonlinear heat equations*. Nonlinear Phenomena Phys. D **5** (1982), pp. 293-306.
- [13] McLeod, K., Troy, W. C., Weissler, F. B.: *Radial solutions of $\Delta u + f(u) = 0$ with prescribed numbers of zeros*. J. Differ. Equ. **83** (1990), pp. 368-378.
- [14] Ianni, I., Musso, M., Pistoia, A.: *Blow-up for sign-changing solutions of the critical heat equation in domains with a small hole*. Commun. Contemp. Math. **18** (2016), 1550017, 18 pp.
- [15] Marino, V., Pacella, F., Sciunzi, B.: *Blow up of solutions of semilinear heat equations in general domains*. Commun. Contemp. Math. **17** (2015).
- [16] Quittner, P.: *A priori bounds for global solutions of a semilinear parabolic problem*. Acta Math. Univ. Comenian. (N.S.) **68**(2) (1999), pp. 195-203.
- [17] Quittner, P., Souplet, P.: *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*. Birkhäuser Advanced Texts. Birkhäuser Verlag, Basel (2007).
- [18] Strauss, W.A.: *Existence of solitary waves in higher dimensions*. Commun. Math. Phys. **55** (1977), pp. 149-162.