Duality Between Lagrangians and Rockafellians

Michel de Lara

Duality Between Lagrangians and Rockafellians

Michel De Lara
CERMICS, École des Ponts, Marne-la-Vallée, France

March 2, 2023

Abstract

In his monograph *Conjugate Duality and Optimization*, Rockafellar puts forward a "perturbation + duality" method to obtain a dual problem for an original minimization problem. First, one embeds the minimization problem into a family of perturbed problems (thus giving a so-called perturbation function); the perturbation of the original function to be minimized has recently been called a Rockafellian. Second, when the perturbation variable belongs to a primal vector space paired, by a bilinear form, with a dual vector space, one builds a Lagrangian from a Rockafellian; one also obtains a so-called dual function (and a dual problem). The method has been extended from Fenchel duality to generalized convexity: when the perturbation belongs to a primal set paired, by a coupling function, with a dual set, one also builds a Rockafellian from a Lagrangian. Following these paths, we highlight a duality between Lagrangians and Rockafellians. Where the material mentioned above mostly focuses on moving from Rockafellian to Lagrangian, we treat them equally and display formulas that go both ways. We propose a definition of Lagrangian-Rockafellian couples. We characterize these latter as dual functions, with respect to a coupling, and also in terms of generalized convex functions. The duality between perturbation and dual functions is not as clear cut.

1 Introduction

In [10], Rockafellar developed in a systematic way the "perturbation + duality" method (see also [5, 4, 3]). First, one embeds a minimization problem into a family of perturbed problems, thus giving a so-called perturbation function; the bivariate perturbation of the original function to be minimized has recently been called a *Rockafellian* [13, Chapter 5.A]. Second, when the perturbation variable belongs to a primal vector space paired, by a bilinear form, with a dual vector space, Rockafellar showed how one can build a *Lagrangian* from a Rockafellian; one also obtains a so-called dual function (and a dual problem).

A large part of the theory has been developed in the convex bifunction case [10, Sect. 29], that is, when the Rockafellian is jointly convex in both variables, namely decision and perturbation, or at least when the decision set is a linear space (see [10, Sect. 30], [11, Chap. 11.H-I],

[13, Chapter 5]). This has much to do with obtaining strong duality results, for which convex analysis offers powerful tools. In the nonconvex case, it seems that the extension to generalized convexity has begun with [1] that makes use of couplings and conjugacies (see also [16], [6, Sect. 3] and references therein).

In this paper, we simply want to stress a duality between Lagrangians and Rockafellians, in the classic Fenchel bilinear pairing case and also in the more general coupling case. Our contribution is modest and formal. It is modest in that most of the material can be traced back to [10] and then [1] — and that we follow and slightly extend their paths — and also as we do not focus on strong duality. It is formal in that we treat Lagrangians and Rockafellians in a symmetric fashion, and highlight a duality between them (Theorem 4). So, there is no real novelty in the paper, but for a symmetric examination of two-way relationships between Lagrangians and Rockafellians when convexity is not assumed.

The paper is organized as follows. In Sect. 2, we consider the classic Fenchel bilinear pairing case. We revisit some of the results in [10] and try to reformulate the "perturbation + duality" method with as little convexity as possible in the assumptions. We recall and sketch how one can build a Lagrangian from a Rockafellian, and we highlight a converse construction. In Sect. 3, we recall how one can build a Lagrangian from a Rockafellian, in the case where the perturbation belongs to a primal set paired, by a coupling function, with a dual set. We also propose a converse construction, from Lagrangian to Rockafellian. Finally, we propose a notion of Lagrangian-Rockafellian couple, and we formally express duality between Lagrangians and Rockafellians.

2 Lagrangians and Rockafellians: bilinear pairing case

In §2.1, we provide background on the classic Fenchel conjugacy. In §2.2, we sketch, in one table, how one can build a Lagrangian from a Rockafellian as developed in [10]. In §2.3, we sketch, in one table, how one can build a Rockafellian from a Lagrangian.

2.1 The bilinear pairing case

We consider two real vector spaces $\mathcal{U}$ and $\mathcal{V}$ paired, in the sense of convex analysis, by a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$. The classic Fenchel conjugacy $\star$ is defined, for any functions

$f : \mathcal{U} \to \overline{\mathbb{R}}$ and $g : \mathcal{V} \to \overline{\mathbb{R}}$, by[1]

$$f^\star(v) = \sup_{u\in\mathcal{U}} \Big(\langle u,\, v\rangle + \big(-f(u)\big) \Big) \;,\;\; \forall v \in \mathcal{V} \;, \tag{1a}$$

$$g^{\star'}(u) = \sup_{v\in\mathcal{V}} \Big(\langle u,\, v\rangle + \big(-g(v)\big) \Big) \;,\;\; \forall u \in \mathcal{U} \;, \tag{1b}$$

$$f^{\star\star'}(u) = \sup_{v\in\mathcal{V}} \Big(\langle u,\, v\rangle + \big(-f^\star(v)\big) \Big) \;,\;\; \forall u \in \mathcal{U} \;. \tag{1c}$$

$$g^{\star'\star}(v) = \sup_{u\in\mathcal{U}} \Big(\langle u,\, v\rangle + \big(-g^{\star'}(u)\big) \Big) \;,\;\; \forall v \in \mathcal{V} \;, \tag{1d}$$

A function $f : \mathcal{U} \to \overline{\mathbb{R}}$ is said to be $\star$-convex if $f^{\star\star'} = f$ and a function $g : \mathcal{V} \to \overline{\mathbb{R}}$ is said to be $\star'$-convex if $g^{\star'\star} = g$. In both cases, this is equivalent to the function being either one of the two constant functions $-\infty$ and $+\infty$ or proper convex lsc (lower semi continuous) [12, Corollary 12.2.1].

2.2 From Rockafellians to Lagrangians: bilinear pairing

We sketch below the "perturbation + duality" method developed in [10]. More precisely, we follow [10, Sect. 4], but with less structure. Indeed, in [10, Sect. 4] it is supposed that the decision set $\mathcal{X}$ below is a linear space, in duality with another linear space. However, for this §2.2, and for the whole paper, we do not require any assumption on the decision set $\mathcal{X}$.

Original minimization problem. Suppose given a nonempty set $\mathcal{X}$, a function $h : \mathcal{X} \to \overline{\mathbb{R}}$ and consider the *original minimization problem* $\inf_{x\in\mathcal{X}} h(x)$.

Perturbation scheme, Rockafellian. Let be given a vector space $\mathcal{U}$ and a function $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ such that $h(x) = R(x, 0)$, $\forall x \in \mathcal{X}$. The variables in $\mathcal{U}$ are called perturbations, and the function R is called a *Rockafellian* [13, Chapter 5] (a *dualizing parameterization* in [11, Definition 11.45], but with the requirement that $R(x, u)$ be convex lsc in the perturbation variable u).

Perturbation function. The original minimization problem is now embedded in a family given by the *perturbation function* [12, p. 295] (called *min-value function* in [13, p. 264]) $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$ defined by $\varphi(u) = \inf_{x\in\mathcal{X}} R(x, u)$, $\forall u \in \mathcal{U}$. The original minimization problem corresponds to $\varphi(0) = \inf_{x\in\mathcal{X}} R(x, 0) = \inf_{x\in\mathcal{X}} h(x)$.

Dual problem, dual function, Lagrangian. Suppose that there exists a vector space $\mathcal{V}$ such that $\mathcal{U}$ and $\mathcal{V}$ are paired by a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$. Then, we obtain a *dual problem* by $\sup_{v\in\mathcal{V}} \big(-\varphi^\star(v)\big) = \varphi^{\star\star'}(0) \leq \varphi(0) = \inf_{x\in\mathcal{X}} h(x)$. In the dual problem appears the so-called *dual function* $\psi = -\varphi^\star : \mathcal{V} \to \overline{\mathbb{R}}$, also given by

[1] In convex analysis, one does not use the notation $^{\star'}$, but simply *. We use $^{\star'}$ to be consistent with the notation (2c) for general conjugacies. Also the $+\big(-\cdots\big)$ expression is here to stress the proximity with the $\dot{+}\,\big(-\cdots\big)$ expression in Equations (2).

$\psi(v) = \inf_{x \in \mathcal{X}} L(x, v)$, $\forall v \in \mathcal{V}$, where the *Lagrangian* $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ is defined by[2] $L(x, v) = \inf_{u \in \mathcal{U}} \big\{ R(x, u) - \langle u,\, v \rangle \big\}$, $\forall (x, v) \in \mathcal{X} \times \mathcal{V}$.

Thus, in the classical setting of [10], one can build a Lagrangian from a Rockafellian as summarized and sketched in Table 1. We do not provide a proof of the properties stated in Table 1, as these properties are well-known [10] (and as their proof will be a specialization of the proof of Proposition 1 in §3.2 in the special case of the Fenchel bilinear pairing).

For any bivariate functions $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ and $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$, we denote, for all $x \in \mathcal{X}$, by $L(x, \cdot) : \mathcal{V} \to \overline{\mathbb{R}}$ and $R(x, \cdot) : \mathcal{U} \to \overline{\mathbb{R}}$ the partial functions obtained by "freezing" x. In [12, Sect. 29], Rockafellar uses the vocable of *bifunction* and the notation $L(x, \cdot) = (Lx)$, $R(x, \cdot) = (Rx)$.

sets	optimization set $\mathcal{X}$	primal space $\mathcal{U}$	pairing $\mathcal{U} \overset{\langle \cdot, \cdot \rangle}{\leftrightarrow} \mathcal{V}$	dual space $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$\langle u,\, v \rangle$ $\in \mathbb{R}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition				$L(x, v) =$ $\inf_{u \in \mathcal{U}} \big\{ R(x, u) - \langle u,\, v \rangle \big\}$
property				$-L(x, \cdot) = \big(R(x, \cdot)\big)^\star$
property				$-L(x, \cdot)$ is $\star'$-convex (hence $L(x, \cdot)$ is concave usc)
univariate functions		perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x \in \mathcal{X}} R(x, u)$		$\psi(v) = \inf_{x \in \mathcal{X}} L(x, v)$
property				$-\psi = \varphi^\star$

Table 1: From Rockafellians to Lagrangians: bilinear pairing

2.3 From Lagrangians to Rockafellians: bilinear pairing

Conversely, one can build a Rockafellian from a Lagrangian as sketched in Table 2. This seems to be less classical than the opposite construction recalled in §2.2, although the formula $R(x, u) = \sup_{v \in \mathcal{V}} \big\{ L(x, v) + \langle u,\, v \rangle \big\}$ appears in [11, Equation 11(17)], as a consequence of [11, Equation 11(16)] and of the requirement that $R(x, u)$ be convex lsc in the perturbation variable u. Note also that [12, Chapter 33] establishes a relationship between the class of concave-convex saddle functions (Lagrangians of convex programs being a subclass) and the class of convex Rockafellians via such formula (see especially [12, Corollary 33.12]).

[2] The expression of the Lagrangian is given in [10, Equation (4.2)] but with $+\langle u,\, v \rangle$ instead of $-\langle u,\, v \rangle$. The expression we adopt is the one in [11, Equation 11(16)], but without requiring that $R(x, u)$ be convex lsc in the perturbation variable u.

The properties stated in Table 2 will be a consequence of the results of Proposition 2 in §3.3, in the special case of the Fenchel bilinear pairing.

sets	optimization set $\mathcal{X}$	primal space $\mathcal{U}$	pairing $\mathcal{U} \overset{\langle \cdot, \cdot \rangle}{\leftrightarrow} \mathcal{V}$	dual space $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$\langle u,\ v \rangle$ $\in \mathbb{R}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition		$R(x,u) =$ $\sup_{v \in \mathcal{V}} \big\{ L(x,v) + \langle u,\ v \rangle \big\}$		
property		$R(x,\cdot) = \big(-L(x,\cdot)\big)^{\star'}$		
property		$R(x,\cdot)$ is $\star$-convex (hence $R(x,\cdot)$ is convex lsc)		
univariate functions		perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x \in \mathcal{X}} R(x,u)$		$\psi(v) = \inf_{x \in \mathcal{X}} L(x,v)$
property		$\varphi \geq (-\psi)^{\star'}$		

Table 2: From Lagrangians to Rockafellians: bilinear pairing

3 Lagrangians and Rockafellians: general coupling case

In §3.1, we provide background on couplings and Fenchel-Moreau conjugacies. In §3.2, we recall and sketch, in one table, how one can build a Lagrangian from a Rockafellian [1]. In §3.3, we sketch, in one table, how one can build a Rockafellian from a Lagrangian. Finally, in §3.4, we propose a notion of Lagrangian-Rockafellian couple, and we formally express duality between Lagrangians and Rockafellians.

3.1 The general coupling case

When we manipulate functions with values in $\overline{\mathbb{R}} = [-\infty, +\infty]$, we adopt the Moreau *lower addition* [7, 8] that extends the usual addition with $(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = -\infty$, and the Moreau *upper addition* that extends the usual addition with $(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty$.

For background on coupling and conjugacies, we refer the reader to [15, 14, 6]. We consider two sets $\mathcal{U}$ and $\mathcal{V}$ paired by a *coupling* $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$. Then, one associates *conjugacies* from the set $\overline{\mathbb{R}}^{\mathcal{U}}$ of functions $\mathcal{U} \to \overline{\mathbb{R}}$ to the set $\overline{\mathbb{R}}^{\mathcal{V}}$ of functions $\mathcal{V} \to \overline{\mathbb{R}}$, and from $\overline{\mathbb{R}}^{\mathcal{V}}$ to $\overline{\mathbb{R}}^{\mathcal{U}}$ as follows. The *c-Fenchel-Moreau conjugate* of a function $f : \mathcal{U} \to \overline{\mathbb{R}}$ is the

function $f^c : \mathcal{V} \to \overline{\mathbb{R}}$ defined by

$$f^c(v) = \sup_{u \in \mathcal{U}} \Big(c(u,v) \dot{+} \big(-f(u)\big) \Big) \ , \ \ \forall v \in \mathcal{V} \ . \tag{2a}$$

With the coupling c, we associate the *reverse coupling* c' defined by

$$c' : \mathcal{V} \times \mathcal{U} \to \overline{\mathbb{R}} \ , \ \ c'(v,u) = c(u,v) \ , \ \ \forall (v,u) \in \mathcal{V} \times \mathcal{U} \ . \tag{2b}$$

The *c'-Fenchel-Moreau conjugate* of a function $g : \mathcal{V} \to \overline{\mathbb{R}}$ is the function $g^{c'} : \mathcal{U} \to \overline{\mathbb{R}}$ defined by

$$g^{c'}(u) = \sup_{v \in \mathcal{V}} \Big(c'(v,u) \dot{+} \big(-g(v)\big) \Big) = \sup_{v \in \mathcal{V}} \Big(c(u,v) \dot{+} \big(-g(v)\big) \Big) \ , \ \ \forall u \in \mathcal{U} \ . \tag{2c}$$

The *c-Fenchel-Moreau biconjugate* of a function $f : \mathcal{U} \to \overline{\mathbb{R}}$ is the function $f^{cc'} : \mathcal{U} \to \overline{\mathbb{R}}$ defined by

$$f^{cc'}(u) = \big(f^c\big)^{c'}(u) = \sup_{v \in \mathcal{V}} \Big(c(u,v) \dot{+} \big(-f^c(v)\big) \Big) \ , \ \ \forall u \in \mathcal{U} \ . \tag{2d}$$

The *c'-Fenchel-Moreau biconjugate* of a function $g : \mathcal{V} \to \overline{\mathbb{R}}$ is the function $g^{c'c} : \mathcal{V} \to \overline{\mathbb{R}}$ defined by

$$g^{c'c}(v) = \big(g^{c'}\big)^{c}(v) = \sup_{u \in \mathcal{U}} \Big(c(u,v) \dot{+} \big(-g^{c'}(u)\big) \Big) \ , \ \ \forall v \in \mathcal{V} \ . \tag{2e}$$

A function $f : \mathcal{U} \to \overline{\mathbb{R}}$ is said to be c-convex if $f^{cc'} = f$ (which is equivalent to $f = g^{c'}$ for some $g : \mathcal{V} \to \overline{\mathbb{R}}$). A function $g : \mathcal{V} \to \overline{\mathbb{R}}$ is said to be c'-convex if $g^{c'c} = g$ (which is equivalent to $g = f^c$ for some $f : \mathcal{U} \to \overline{\mathbb{R}}$).

3.2 From Rockafellians to Lagrangians: general coupling

Following [1] — and taking inspiration from §2.2 (see also [6, Sect. 3]) — one can build a Lagrangian from a Rockafellian as sketched in Table 3. The formal statement is given in Proposition 1. The result is not new [6, Sect. 3], but we give a proof for the sake of completeness and symmetry in the exposition.

Proposition 1 (from [1, 6]) *We consider a set $\mathcal{X}$, and two sets $\mathcal{U}$ and $\mathcal{V}$ paired by a coupling $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$. We consider a Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ and we define the* Lagrangian *$L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ by*[3]

$$L(x,v) = \inf_{u \in \mathcal{U}} \Big\{ R(x,u) \dot{+} \big(-c(u,v)\big) \Big\} \ , \ \ \forall (x,v) \in \mathcal{X} \times \mathcal{V} \ , \tag{3}$$

the perturbation function *$\varphi : \mathcal{U} \to \overline{\mathbb{R}}$ and the* dual function *$\psi : \mathcal{V} \to \overline{\mathbb{R}}$ by*

$$\varphi(u) = \inf_{x \in \mathcal{X}} R(x,u) \ , \ \ \forall u \in \mathcal{U} \ \ \textit{and} \ \ \psi(v) = \inf_{x \in \mathcal{X}} L(x,v) \ , \ \ \forall v \in \mathcal{V} \ . \tag{4}$$

Then, we have that $-L(x,\cdot) = \big(R(x,\cdot)\big)^c$ and $-L(x,\cdot)$ is c'-convex, for all $x \in \mathcal{X}$, and that $-\psi = \varphi^c$.

[3] Our definition is close to [6, p. 252].

sets	optimization set $\mathcal{X}$	primal set $\mathcal{U}$	coupling $\mathcal{U} \overset{c}{\leftrightarrow} \mathcal{V}$	dual set $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$c(u,v)$ $\in \overline{\mathbb{R}}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition				$L(x,v) =$ $\inf_{u \in \mathcal{U}} \left\{ R(x,u) \dot{+} \big(-c(u,v)\big) \right\}$
property				$-L(x,\cdot) = \big(R(x,\cdot)\big)^c$
property				$-L(x,\cdot)$ is c'-convex
univariate functions		perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x \in \mathcal{X}} R(x,u)$		$\psi(v) = \inf_{x \in \mathcal{X}} L(x,v)$
property				$-\psi = \varphi^c$

Table 3: From Rockafellians to Lagrangians: general coupling

Proof. The equality $-L(x,\cdot) = \big(R(x,\cdot)\big)^c$, for all $x \in \mathcal{X}$, is a straightforward application of definition (2a) with $f = R(x,\cdot)$ and of $-L(x,v) = \sup_{u \in \mathcal{U}} \left\{ c(u,v) \dot{+} \big(-R(x,u)\big) \right\}$, for all $(x,v) \in \mathcal{X} \times \mathcal{V}$.

By property of conjugacies [6, Proposition 6.1 (ii)], we have that $\big(-L(x,\cdot)\big)^{c'c} = \big(R(x,\cdot)\big)^{cc'c} = \big(R(x,\cdot)\big)^c = -L(x,\cdot)$, for all $x \in \mathcal{X}$. Hence, $-L(x,\cdot)$ is c'-convex, for all $x \in \mathcal{X}$.

Finally, as conjugacies turn an infimum into a supremum, we have that $\varphi^c = \big(\inf_{x \in \mathcal{X}} R(x,\cdot)\big)^c = \sup_{x \in \mathcal{X}} \big(R(x,\cdot)\big)^c = \sup_{x \in \mathcal{X}} \big(-L(x,\cdot)\big) = -\inf_{x \in \mathcal{X}} L(x,\cdot) = -\psi$ by (4). □

The dual problem is $\varphi^{cc'}(u) = (-\psi)^{c'}(u) = \sup_{v \in \mathcal{V}} \big(c(u,v) \dot{+} \psi(v)\big)$. In the Fenchel bilinear pairing case, and in [10], we have that $u = 0$ and $c(0,v) = \langle 0,\ v \rangle = 0$, $\forall v \in \mathcal{V}$. This is generally no longer the case in the general coupling case.

3.3 From Lagrangians to Rockafellians: general coupling

Taking inspiration from §2.3, we show one can build a Rockafellian from a Lagrangian as sketched in Table 4. The formal statement is given in Proposition 2, wich seems to be new.

Contrarily to Proposition 1 — where we obtained the equality $-\psi = \varphi^c$ between dual and perturbation functions — we now only obtain the inequality $\varphi \geq -\psi^{c'}$. This is because the definition (5) of a Rockafellian built from a Lagrangian involves a supremum operation, which does not behave, with a conjugacy, as nicely as an infimum operation does.

Proposition 2 *We consider a set $\mathcal{X}$, and two sets $\mathcal{U}$ and $\mathcal{V}$ paired by a coupling $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$. We consider a Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ and we define the* Rockafellian *$R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ by*

$$R(x,u) = \sup_{v \in \mathcal{V}} \left\{ L(x,v) \dot{+} c(u,v) \right\}, \quad \forall (x,u) \in \mathcal{X} \times \mathcal{U}, \tag{5}$$

sets	optimization set $\mathcal{X}$	primal set $\mathcal{U}$	coupling $\mathcal{U} \overset{c}{\leftrightarrow} \mathcal{V}$	dual set $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$c(u,v)$ $\in \overline{\mathbb{R}}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition		$R(x,u) =$ $\sup_{v \in \mathcal{V}} \left\{ L(x,v) \dot{+} c(u,v) \right\}$		
property		$R(x,\cdot) = \left(-L(x,\cdot)\right)^{c'}$		
property		$R(x,\cdot)$ is c-convex		
univariate functions		perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x \in \mathcal{X}} R(x,u)$		$\psi(v) = \inf_{x \in \mathcal{X}} L(x,v)$
property		$\varphi \geq (-\psi)^{c'}$		

Table 4: From Lagrangians to Rockafellians: general coupling

and the perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$ *and the* dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$ *as in* (4).

Then, we have that $R(x,\cdot) = \left(-L(x,\cdot)\right)^{c'}$ *and* $R(x,\cdot)$ *is c-convex, for all* $x \in \mathcal{X}$*, and that* $\varphi \geq -\psi^{c'}$.

Proof. The equality $R(x,\cdot) = \left(-L(x,\cdot)\right)^{c'}$ for all $x \in \mathcal{X}$, is a straightforward application of definition (2c) with $g = -L(x,\cdot)$ and $R(x,\cdot)$ given by (5).

By property of conjugacies [6, Proposition 6.1 (ii)], we have that $\left(R(x,\cdot)\right)^{cc'} = \left(-L(x,\cdot)\right)^{c'cc'} = \left(-L(x,\cdot)\right)^{c'} = R(x,\cdot)$, for all $x \in \mathcal{X}$. Hence, $R(x,\cdot)$ is c-convex, for all $x \in \mathcal{X}$.

Finally, as conjugacies turn an infimum into a supremum, for all $x \in \mathcal{X}$ we have that $\varphi^c = \left(\inf_{x \in \mathcal{X}} R(x,\cdot)\right)^c = \sup_{x \in \mathcal{X}} \left(R(x,\cdot)\right)^c = \sup_{x \in \mathcal{X}} \left(-L(x,\cdot)\right)^{cc'} \leq \sup_{x \in \mathcal{X}} \left(-L(x,\cdot)\right) = -\inf_{x \in \mathcal{X}} L(x,\cdot) = -\psi$ by (4), and where we have used that $\left(-L(x,\cdot)\right)^{cc'} \leq -L(x,\cdot)$. Then, we get that $\varphi^{cc'} \geq (-\psi)^{c'}$, hence that $\varphi \geq \varphi^{cc'} \geq (-\psi)^{c'}$. □

3.4 Duality between Lagrangians and Rockafellians

To formally express duality between Lagrangians and Rockafellians, we take inspiration from [9, Sect. 8], especially the notions of *fonctions sur-duales* and of *fonctions duales* (dual functions), that is, minimal elements in a generalized Fenchel-Young type inequality (see also [12, p. 104-105]). The following definition and theorem are new.

Definition 3 *We consider a set* $\mathcal{X}$*, and two sets* $\mathcal{U}$ *and* $\mathcal{V}$ *paired by a coupling* $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$*. We equip* $\overline{\mathbb{R}}^{\mathcal{X} \times \mathcal{U}} \times \overline{\mathbb{R}}^{\mathcal{X} \times \mathcal{V}}$ *with the natural ordering. We say that the functions* $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$

and $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ *form a* Lagrangian-Rockafellian *couple* (L, R) *if* $(-L, R)$ *is minimal in the inequality*

$$\big(-L(x,v)\big) \dot{+} R(x,u) \geq c(u,v) \ , \ \ \forall (x,u,v) \in \mathcal{X} \times \mathcal{U} \times \mathcal{V} \ . \tag{6}$$

We can now state our main result, which establishes a duality between Lagrangians and Rockafellians, expressed in different equivalent forms: as minimal elements in a generalized Fenchel-Young type inequality; by means of Fenchel-Moreau conjugates dual pairs; and also by means of generalized convex functions (equal to their Fenchel-Moreau biconjugate).

Theorem 4 *We consider a set* $\mathcal{X}$*, two sets* $\mathcal{U}$ *and* $\mathcal{V}$ *paired by a coupling* $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$*, and two functions* $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ *and* $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$*. Then, the following are equivalent.*

1. *The functions* L *and* R *form a Lagrangian-Rockafellian couple* (L, R)*.*

2. *Equality* (3) *and Equality* (5) *hold true, that is,*

$$L(x,v) = \inf_{u \in \mathcal{U}} \Big\{ R(x,u) \dot{+} \big(-c(u,v)\big) \Big\} \ , \ \ \forall (x,v) \in \mathcal{X} \times \mathcal{V} \ , \tag{7a}$$

$$R(x,u) = \sup_{v \in \mathcal{V}} \big\{ L(x,v) \dot{+} c(u,v) \big\} \ , \ \ \forall (x,u) \in \mathcal{X} \times \mathcal{U} \ . \tag{7b}$$

3. *The functions* $-L$ *and* R *are dual functions, with respect to the coupling* c*, in the sense that*

$$-L(x,\cdot) = \big(R(x,\cdot)\big)^c \ \ \textit{and} \ \ R(x,\cdot) = \big(-L(x,\cdot)\big)^{c'} \ , \ \ \forall x \in \mathcal{X} \ . \tag{8}$$

4. *We have that*

$$-L(x,\cdot) = \big(R(x,\cdot)\big)^c \ \ \textit{and} \ \ \big(R(x,\cdot)\big)^{cc'} = R(x,\cdot) \ , \ \ \forall x \in \mathcal{X} \ . \tag{9}$$

5. *We have that*

$$R(x,\cdot) = \big(-L(x,\cdot)\big)^{c'} \ \ \textit{and} \ \ \big(-L(x,\cdot)\big)^{c'c} = -L(x,\cdot) \ , \ \ \forall x \in \mathcal{X} \ . \tag{10}$$

The above equivalences are formal, in that they rather easily follow from definitions (Definition 3, Equation (2) in §3.1) and from basic properties of conjugacies. They simply establish, in different equivalent ways, the duality between Lagrangians and Rockafellians. In particular, in Item 1, Item 2 and Item 3, Lagrangians and Rockafellians are characterized in a balanced fashion, by contrast with Item 4 and Item 5.

Proof.
Item 1 $\iff$ Item 2.

The equivalence follows from Definition 3 and from the following equivalences [7, 8] (see also [2, Appendix A.1]): for any $(x, u, v) \in \mathcal{X} \times \mathcal{U} \times \mathcal{V}$,

$$\big(-L(x,v)\big) \dot{+} R(x,u) \geq c(u,v) \;, \tag{11a}$$
$$\iff L(x,v) \leq R(x,u) \dot{+} \big(-c(u,v)\big) \;, \tag{11b}$$
$$\iff R(x,u) \geq L(x,v) \dot{+} c(u,v) \;. \tag{11c}$$

Item 2 $\iff$ Item 3.

Equations (7) are equivalent to Equations (8) by definitions (2a) and (2c).

Item 3 $\implies$ Item 4.

Equations (8) imply that $-L(x,\cdot) = \big(R(x,\cdot)\big)^c$ and $\big(R(x,\cdot)\big)^{cc'} = \Big(\big(R(x,\cdot)\big)^c\Big)^{c'} = \big(-L(x,\cdot)\big)^{c'} = R(x,\cdot)$, for all $x \in \mathcal{X}$. Hence, we obtain Equations (9).

Item 4 $\implies$ Item 3.

Equations (9) imply that $-L(x,\cdot) = \big(R(x,\cdot)\big)^c$ and $R(x,\cdot) = \big(R(x,\cdot)\big)^{cc'} = \Big(\big(R(x,\cdot)\big)^c\Big)^{c'} = \big(-L(x,\cdot)\big)^{c'}$, for all $x \in \mathcal{X}$. Hence, we obtain Equations (8).

Item 3 $\iff$ Item 5.

This equivalence is shown by two implications (8) $\implies$ (10) and (10) $\implies$ (8) as above.

□

As a consequence, for a Lagrangian-Rockafellian couple (L, R), one necessarily has that $-L(x,\cdot)$ is c'-convex and $R(x,\cdot)$ is c-convex, for all $x \in \mathcal{X}$.

4 Conclusion

In this paper, we have highlighted a duality between Lagrangians and Rockafellians, as these two functions appear in the "perturbation + duality" method of [10]. We have treated both functions equally, and have provided formulas that go both way: from Rockafellian to Lagrangian (classical); from Lagrangian to Rockafellian (less classical). The setting is the one of so-called abstract or generalized convexity, where the perturbation belongs to a primal set paired, by a coupling function, with a dual set. It encompasses the classical Fenchel duality setting.

We have proposed a notion of Lagrangian-Rockafellian couple — minimal elements in a generalized Fenchel-Young type inequality — and we have formally expressed duality between Lagrangians and Rockafellians by means of Fenchel-Moreau conjugates dual pairs, and also by means of generalized convex functions (equal to their Fenchel-Moreau biconjugate).

Acknowledgements. The Author thanks Johannes Royset and Roger Wets for their comments on a first version of the paper. The Author thanks the Reviewer for her/his comments that helped improve the paper, notably by clarifying the contribution and by pointing suitable sources.

References

[1] E. J. Balder. An extension of duality-stability relations to nonconvex optimization problems. *SIAM Journal on Control and Optimization*, 15(2):329–343, 1977.

[2] J.-P. Chancelier and M. De Lara. Fenchel-Moreau conjugation inequalities with three couplings and application to the stochastic Bellman equation. *Journal of Convex Analysis*, 26(3):945–966, 2019.

[3] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems*. Society for Industrial and Applied Mathematics, 1999.

[4] J. L. Joly and P. J. Laurent. Stability and duality in convex minimization problems. *Revue française d'informatique et de recherche opérationnelle. Série rouge*, 5(R2):3–42, 1971.

[5] P.-J. Laurent. *Approximation et optimisation*. Hermann, Paris, 1972.

[6] J. E. Martínez-Legaz. Generalized convex duality and its economic applications. In S. S. Hadjisavvas N., Komlósi S., editor, *Handbook of Generalized Convexity and Generalized Monotonicity. Nonconvex Optimization and Its Applications*, volume 76, pages 237–292. Springer-Verlag, 2005.

[7] J. J. Moreau. Fonctions à valeurs dans $[-\infty, +\infty]$; notions algébriques. *Faculté des Sciences de Montpellier, Séminaires de Mathématiques*, 1963.

[8] J. J. Moreau. Remarques sur les fonctions à valeurs dans $[-\infty, +\infty]$ définies sur un demi-groupe. *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 257:3107–3109, 1963.

[9] J. J. Moreau. Fonctionnelles convexes. *Séminaire Jean Leray*, 2:1–108, 1966-1967.

[10] R. T. Rockafellar. *Conjugate Duality and Optimization*. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, 1974.

[11] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*. Springer-Verlag, Berlin, 1998.

[12] T. R. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, N.J., 1970.

[13] J. O. Royset and R. J.-B. Wets. *An Optimization Primer*. Springer International Publishing, Cham, 2021.

[14] A. Rubinov. *Abstract convexity and global optimization*, volume 44 of *Nonconvex Optimization and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000.

[15] I. Singer. *Abstract Convex Analysis*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1997.

[16] I. Singer. *Duality for Nonconvex Approximation and Optimization*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2006.