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Helicity in dispersive continuum mechanics

S. L. Gavriluk* and H. Gouin*

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Abstract

New conservation laws are obtained for the equations of dispersive continuum mechanics describing Euler–van der Waals–Korteweg’s fluids, fluids containing small gas bubbles, and long–free surface gravity waves (Serre–Green–Naghdi equations). The corresponding mathematical models are characterized by the dependence of the energy on spatial and temporal derivatives of the density. In particular, analogs of helicity invariants are deduced.

Keywords: Dispersive fluid mechanics; capillary fluids; bubbly fluids; Serre–Green–Naghdi equations

1 Introduction

The aim of the paper is to describe the analogs of helicity integral for dispersive systems of equations coming from the Hamilton principle of stationary action. The helicity integral was found for classical barotropic fluids in seminal papers by J. J. Moreau [1] and H. K. Moffatt [2]. This integral is a consequence of symmetry due to the relabeling of the particles for the governing equations [3, 4, 5]. Further important studies of such an invariant and related Ertel’s type invariants for the incompressible Euler equations, magnetohydrodynamics etc. can be found in [6, 7, 8, 9, 10, 11, 12, 13]. In this paper, we present helicity type integrals for dispersive media such as fluids endowed with capillarity (called also Euler–van der Waals–Korteweg’s fluids [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]), and a class of fluids with internal inertia (such as bubbly fluids and long free–surface gravity waves [5, 24, 25, 26, 27, 28, 29, 30, 31]).

We introduce the following notations. For any vectors \mathbf{a} and \mathbf{b} , the scalar product $\mathbf{a} \cdot \mathbf{b}$ is written $\mathbf{a}^T \mathbf{b}$ (line vector or covector \mathbf{a}^T is multiplied by column vector \mathbf{b}), where superscript T denotes the transposition. The tensor product $\mathbf{a} \otimes \mathbf{b}$ is also written $\mathbf{a} \mathbf{b}^T$, the product of column vector \mathbf{a} by line vector \mathbf{b}^T .

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The product of matrix \mathbf{A} by vector \mathbf{a} is denoted by $\mathbf{A}\mathbf{a}$. Notation $\mathbf{b}^T \mathbf{A}$ means the covector \mathbf{c}^T defined as $\mathbf{c}^T = (\mathbf{A}^T \mathbf{b})^T$.

Tensor \mathbf{I} denotes the identity transformation.

The gradient of scalar function $f(\mathbf{x})$ is defined as $\nabla f = \left(\frac{\partial f}{\partial \mathbf{x}} \right)^T$ associated with linear form $df = \left(\frac{\partial f}{\partial \mathbf{x}} \right) d\mathbf{x}$.

Let $\mathbf{v} = (v^1, \dots, v^n)^T$ be a function of $\mathbf{x} = (x^1, \dots, x^n)^T$. Then $\frac{\partial \mathbf{v}}{\partial \mathbf{x}}$ denotes the linear application defined by the relation $d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} d\mathbf{x}$ and represented in canonical coordinates by the matrix

$$\begin{pmatrix} \frac{\partial v^1(x^1, \dots, x^n)}{\partial x^1} & \dots & \frac{\partial v^1(x^1, \dots, x^n)}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial v^n(x^1, \dots, x^n)}{\partial x^1} & \dots & \frac{\partial v^n(x^1, \dots, x^n)}{\partial x^n} \end{pmatrix}$$

The divergence of second order tensor \mathbf{A} is the covector $\text{div } \mathbf{A}$ such that, for any constant vector \mathbf{a} , $(\text{div } \mathbf{A})\mathbf{a} = \text{div}(\mathbf{A}\mathbf{a})$. It implies that for any vector field \mathbf{v}

$$\text{div}(\mathbf{A}\mathbf{v}) = (\text{div } \mathbf{A})\mathbf{v} + \text{Tr} \left(\mathbf{A} \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right)$$

where Tr is the trace operator.

In three-dimensional case we introduce $\boldsymbol{\omega} = \text{curl } \mathbf{u}$. In the Cartesian basis $\boldsymbol{\omega} = (\omega^1, \omega^2, \omega^3)^T$ is defined as

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix}$$

If $f(\mathbf{A})$ is any scalar function of the matrix \mathbf{A} , its differential is

$$df(\mathbf{A}) = \frac{\partial f}{\partial A_j^i} dA_j^i \equiv \text{Tr} \left(\frac{\partial f}{\partial \mathbf{A}} d\mathbf{A} \right) \equiv \nabla_{\mathbf{A}} f : d\mathbf{A}$$

where the repeated indexes mean summation. We use the notation

$$\nabla_{\mathbf{A}} f = \left(\frac{\partial f}{\partial \mathbf{A}} \right)^T \quad \text{with} \quad \left(\frac{\partial f}{\partial \mathbf{A}} \right)_i^j = \frac{\partial f}{\partial A_j^i}$$

where A_j^i are components of \mathbf{A} with i being the line index and j being the column index of \mathbf{A} .

2 Motion of a continuous medium

The motion of a continuous medium is represented by a diffeomorphism φ of a three-dimensional reference configuration \mathcal{D}_0 into the physical space \mathcal{D}_t . Let $\mathbf{X} = (X^1, X^2, X^3)^T$ be the Lagrangian coordinates, and $\mathbf{x} = (x^1, x^2, x^3)^T$ be the Eulerian coordinates. The motion is represented as

$$\mathbf{x} = \varphi(t, \mathbf{X}),$$

where t denotes the time. At given time t , the transformation φ possesses an inverse and has continuous derivatives up to the second order. The deformation gradient is :

$$\mathbf{F} = \frac{\partial \varphi(t, \mathbf{X})}{\partial \mathbf{X}} \equiv \frac{\partial \mathbf{x}}{\partial \mathbf{X}}.$$

Considering \mathbf{F} as a function of the Eulerian coordinates and time (t, \mathbf{x}) , we obtain for \mathbf{F} the evolution equation

$$\frac{D\mathbf{F}}{Dt} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{F} \quad \text{with} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}^T \nabla, \quad (2.1)$$

where $\mathbf{u}(t, \mathbf{x}) = \left. \frac{\partial \varphi(t, \mathbf{X})}{\partial t} \right|_{\mathbf{X}=\varphi^{-1}(t, \mathbf{x})}$ is the velocity field.

3 Preliminary results

Lemma 1 *The following identities are satisfied:*

$$\operatorname{div} \left(\frac{\mathbf{F}}{\det \mathbf{F}} \right) = 0, \quad (3.2)$$

$$\frac{D}{Dt} \left(\frac{\mathbf{F}}{\det \mathbf{F}} \right) = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} - (\operatorname{div} \mathbf{u}) \mathbf{I} \right) \frac{\mathbf{F}}{\det \mathbf{F}}. \quad (3.3)$$

Proof. We first derive the relation between the unit normal vector \mathbf{n} on the boundary S_t of a material volume \mathcal{D}_t and the unit normal vector \mathbf{n}_0 on the boundary S_0 of an initial volume \mathcal{D}_0 . Consider a curvilinear coordinate system on S_0 . Let $d_{i0}\mathbf{X}$, $i = 1, 2$, be the infinitesimal tangent vectors along the coordinate lines on S_0 , and $d_i\mathbf{x}$, $i = 1, 2$, be infinitesimal tangent vectors along the corresponding material lines at time t on S_t . Then, for any vector field \mathbf{a}

$$\begin{aligned} \det(\mathbf{a}, d_1\mathbf{x}, d_2\mathbf{x}) &= \det(\mathbf{F}\mathbf{F}^{-1}\mathbf{a}, \mathbf{F}d_{10}\mathbf{X}, \mathbf{F}d_{20}\mathbf{X}) = \\ &= (\det \mathbf{F}) \det(\mathbf{F}^{-1}\mathbf{a}, d_{10}\mathbf{X}, d_{20}\mathbf{X}). \end{aligned}$$

Or

$$\mathbf{a}^T (d_1\mathbf{x} \wedge d_2\mathbf{x}) = (\det \mathbf{F}) (\mathbf{F}^{-1}\mathbf{a})^T (d_{10}\mathbf{X} \wedge d_{20}\mathbf{X}),$$

which is equivalent to

$$\frac{1}{\det \mathbf{F}} \mathbf{a}^T (d_1 \mathbf{x} \wedge d_2 \mathbf{x}) = \mathbf{a}^T (\mathbf{F}^{-1})^T (d_{10} \mathbf{X} \wedge d_{20} \mathbf{X}).$$

The relation being valid for any vector \mathbf{a} , one obtains

$$\frac{\mathbf{n}^T \mathbf{F}}{\det \mathbf{F}} = \mathbf{n}_0^T. \quad (3.4)$$

If \mathbf{L}_0 is a vector field such that

$$\operatorname{div}_0 \mathbf{L}_0 = 0,$$

where div_0 denotes the divergence of \mathbf{L}_0 in the Lagrangian variables, the vector field \mathbf{L} such that

$$\mathbf{L} = \frac{\mathbf{F}}{\det \mathbf{F}} \mathbf{L}_0,$$

verifies the equation

$$\operatorname{div} \mathbf{L} = 0.$$

In particular the relation (3.4) implies (3.2). From (2.1) we obtain

$$\frac{D}{Dt} \left(\frac{\mathbf{F}}{\det \mathbf{F}} \right) = \frac{1}{\det \mathbf{F}} \frac{D\mathbf{F}}{Dt} - \frac{1}{(\det \mathbf{F})^2} \frac{D(\det \mathbf{F})}{Dt} \mathbf{F}$$

and Euler's formula for determinants

$$\frac{D(\det \mathbf{F})}{Dt} = (\det \mathbf{F}) \operatorname{Tr} \left(\mathbf{F}^{-1} \frac{D\mathbf{F}}{Dt} \right) = (\det \mathbf{F}) \operatorname{Tr} \left(\mathbf{F}^{-1} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{F} \right) = (\det \mathbf{F}) \operatorname{div} \mathbf{u}$$

implies (3.3). ■

Let us note that an other direct proof of (3.2) of Lemma 1 can also be done using a geometrical interpretation of this identity.

Lemma 2 Let $\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial X^i} = \frac{\partial \varphi(t, \mathbf{X})}{\partial X^i}$ (with lower indexes $i = 1, 2, 3$) and expressed in Eulerian coordinates, and $\mathbf{e}^i = \nabla X^i(t, \mathbf{x})$ (with upper indexes $i = 1, 2, 3$) be the corresponding cobasis (dual basis). Then

$$\frac{\mathbf{e}_k}{\det \mathbf{F}} = \mathbf{e}^i \wedge \mathbf{e}^j,$$

where $\{i, j, k\}$ forms an even permutation of $\{1, 2, 3\}$. Moreover,

$$\operatorname{div} \left(\frac{\mathbf{e}_k}{\det \mathbf{F}} \right) = 0.$$

Proof. The proof of the first formula comes from the identity

$$\delta_j^i = \frac{\partial X^i}{\partial X^j} = \frac{\partial X^i}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial X^j} = \mathbf{e}^{iT} \mathbf{e}_j,$$

where δ_j^i is the Kronecker symbol. For the second formula one has :

$$\begin{aligned} \operatorname{div} \left(\frac{\mathbf{e}_k}{\det \mathbf{F}} \right) &= \operatorname{div} (\mathbf{e}^i \wedge \mathbf{e}^j) \\ &= \mathbf{e}^{jT} \operatorname{curl} \mathbf{e}^i - \mathbf{e}^{iT} \operatorname{curl} \mathbf{e}^j = 0. \end{aligned}$$

■

As an example illustrating Lemma 2, consider the stationary 2D velocity field

$$\mathbf{u} = (u^1, u^2)^T = (x^2, -x^1)^T. \quad (3.5)$$

The vorticity vector of motion associated with (3.5) is $\boldsymbol{\omega} = (0, 0, -2)^T$. The pathlines are circles which coincide with streamlines. The initial materials lines $X^i = \text{const}$ moving along the velocity field (3.5) are also straight lines at each time instant and rotate with a constant angular velocity (see Fig. 1).

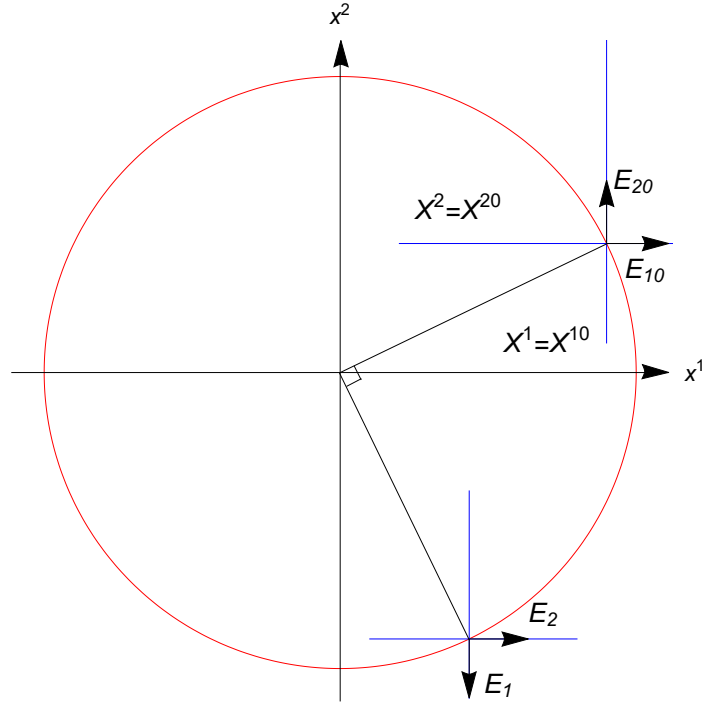


Figure 1: The stream lines (circles) of the velocity field (3.5) are shown in the (x^1, x^2) plane. The initial material lines $X^i = \text{const}$ having tangent vectors \mathbf{E}_{i0} , $i = 1, 2$ represent at each time instant straight lines rotating with a constant angular velocity. The material lines positions at time $t = \pi/2$ and their tangent vectors \mathbf{E}_i , $i = 1, 2$ are shown.

Remark 3 Let $\mathbf{E}_k = \frac{\mathbf{e}_k}{\det \mathbf{F}}$ be the columns of the matrix $\frac{\mathbf{F}}{\det \mathbf{F}}$:

$$\frac{\mathbf{F}}{\det \mathbf{F}} = [\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3].$$

The equations (3.2) and (3.3) yield Helmholtz's equation for the vorticity :

$$\frac{\partial \mathbf{E}_i}{\partial t} + \text{curl}(\mathbf{E}_i \wedge \mathbf{u}) = 0, \quad \text{div} \mathbf{E}_i = 0, \quad i = 1, 2, 3. \quad (3.6)$$

The equations (3.6) can be written as the Lie derivative of a two-form along the velocity field \mathbf{u} (see Appendix A) :

$$d_{L_2} \mathbf{E}_i \equiv \frac{D\mathbf{E}_i}{Dt} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{E}_i + (\text{div} \mathbf{u}) \mathbf{E}_i = 0.$$

Lemma 4 For any motion, scalar field G , vector fields \mathbf{K} and \mathbf{L} of \mathcal{D}_t

$$\mathbf{x} \in \mathcal{D}_t \longrightarrow G(\mathbf{x}) \in \mathbb{R}, \quad \mathbf{x} \in \mathcal{D}_t \longrightarrow \mathbf{K}(\mathbf{x}) \in T_{\mathbf{x}}(\mathcal{D}_t), \quad \mathbf{x} \in \mathcal{D}_t \longrightarrow \mathbf{L}(\mathbf{x}) \in T_{\mathbf{x}}(\mathcal{D}_t),$$

such that $\text{div} \mathbf{L} = 0$, we have the identity

$$\begin{aligned} & \frac{\partial}{\partial t} (\mathbf{K}^T \mathbf{L}) + \text{div} \left\{ \left(\mathbf{u} \mathbf{K}^T + \left(G - \frac{|\mathbf{u}|^2}{2} \right) \mathbf{I} \right) \mathbf{L} \right\} \equiv \\ & \mathbf{L}^T \left(\frac{D\mathbf{K}}{Dt} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{K} + \nabla \left(G - \frac{|\mathbf{u}|^2}{2} \right) \right) + \\ & \mathbf{K}^T \left(\frac{\partial \mathbf{L}}{\partial t} + \frac{\partial \mathbf{L}}{\partial \mathbf{x}} \mathbf{u} + \mathbf{L} \text{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{L} \right). \end{aligned}$$

Proof. Let us denote

$$A = \frac{\partial}{\partial t} (\mathbf{K}^T \mathbf{L}) + \text{div} \left\{ \left(\mathbf{u} \mathbf{K}^T + \left(G - \frac{1}{2} \mathbf{u}^T \mathbf{u} \right) \mathbf{I} \right) \mathbf{L} \right\}.$$

From

$$\text{div} (\mathbf{u} \mathbf{K}^T \mathbf{L}) = \mathbf{K}^T \mathbf{L} (\text{div} \mathbf{u}) + \mathbf{K}^T \frac{\partial \mathbf{L}}{\partial \mathbf{x}} \mathbf{u} + \mathbf{L}^T \frac{\partial \mathbf{K}}{\partial \mathbf{x}} \mathbf{u}$$

and

$$\text{div} \mathbf{L} = 0,$$

one has

$$A = \mathbf{K}^T \frac{\partial \mathbf{L}}{\partial t} + \mathbf{L}^T \frac{\partial \mathbf{K}}{\partial t} + \mathbf{K}^T \mathbf{L} (\text{div} \mathbf{u}) + \mathbf{K}^T \frac{\partial \mathbf{L}}{\partial \mathbf{x}} \mathbf{u} + \mathbf{L}^T \frac{\partial \mathbf{K}}{\partial \mathbf{x}} \mathbf{u} + \frac{\partial}{\partial \mathbf{x}} \left(G - \frac{|\mathbf{u}|^2}{2} \right) \mathbf{L}.$$

From

$$\frac{\partial \mathbf{K}}{\partial t} = \frac{D\mathbf{K}}{Dt} - \frac{\partial \mathbf{K}}{\partial \mathbf{x}} \mathbf{u},$$

we obtain

$$\begin{aligned} A &= \mathbf{K}^T \frac{\partial \mathbf{L}}{\partial t} + \mathbf{L}^T \frac{D\mathbf{K}}{Dt} - \mathbf{L}^T \frac{\partial \mathbf{K}}{\partial \mathbf{x}} \mathbf{u} + \mathbf{K}^T \mathbf{L} (\operatorname{div} \mathbf{u}) \\ &\quad + \mathbf{K}^T \frac{\partial \mathbf{L}}{\partial \mathbf{x}} \mathbf{u} + \mathbf{L}^T \frac{\partial \mathbf{K}}{\partial \mathbf{x}} \mathbf{u} + \frac{\partial}{\partial \mathbf{x}} \left(G - \frac{|\mathbf{u}|^2}{2} \right) \mathbf{L}. \end{aligned}$$

Finally, from

$$\mathbf{L}^T \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{K} = \mathbf{K}^T \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) \mathbf{L}$$

we get

$$\begin{aligned} A &= \mathbf{L}^T \left(\frac{D\mathbf{K}}{Dt} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{K} + \nabla \left(G - \frac{|\mathbf{u}|^2}{2} \right) \right) \\ &\quad + \mathbf{K}^T \left(\frac{\partial \mathbf{L}}{\partial t} + \frac{\partial \mathbf{L}}{\partial \mathbf{x}} \mathbf{u} + \mathbf{L} (\operatorname{div} \mathbf{u}) - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{L} \right). \end{aligned}$$

■

Remark 5 Consider the vector field \mathbf{K} verifying the equation

$$\frac{D\mathbf{K}}{Dt} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{K} + \nabla \left(G - \frac{|\mathbf{u}|^2}{2} \right) = 0. \quad (3.7)$$

The vector field \mathbf{K} equivalently verifies :

$$d_{L_1} \mathbf{K}^T + \frac{\partial}{\partial \mathbf{x}} \left(G - \frac{|\mathbf{u}|^2}{2} \right) = 0,$$

where

$$d_{L_1} \mathbf{K}^T \equiv \frac{D\mathbf{K}^T}{Dt} + \mathbf{K}^T \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)$$

is the Lie derivative of the covector \mathbf{K}^T along the vector field \mathbf{u} (see Appendix A).

Theorem 6 1°) Consider the divergence-free vector field \mathbf{L} satisfying the Helmholtz equation

$$d_{L_2} \mathbf{L} \equiv \frac{\partial \mathbf{L}}{\partial t} + \frac{\partial \mathbf{L}}{\partial \mathbf{x}} \mathbf{u} + \mathbf{L} \operatorname{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{L} = 0,$$

and the field \mathbf{K} satisfying (3.7). Then,

$$\frac{\partial}{\partial t} \left(\mathbf{K}^T \mathbf{L} \right) + \operatorname{div} \left\{ \left(\mathbf{u} \mathbf{K}^T + \left(G - \frac{|\mathbf{u}|^2}{2} \right) \mathbf{I} \right) \mathbf{L} \right\} = 0. \quad (3.8)$$

2°) Consider a material domain \mathcal{D}_t of boundary S_t . If at $t = 0$ the divergence-free vector field \mathbf{L} is tangent to S_0 , then for any time t the vector field \mathbf{L} is tangent to S_t . Moreover, the quantity

$$\mathcal{H} = \iiint_{\mathcal{D}_t} \mathbf{K}^T \mathbf{L} dD, \quad (3.9)$$

we call generalized helicity, keeps a constant value along the motion.

The conservation law (3.8) is a consequence of Lemma 4. The helicity integral (3.9) immediately comes from (3.8).

In particular, the theorem 6 implies the following compact form of the conservation laws in the case $\mathbf{L} = \mathbf{E}_i$

$$\frac{\partial}{\partial t} \left(\mathbf{K}^T \frac{\mathbf{F}}{\det \mathbf{F}} \right) + \operatorname{div} \left\{ \left(\mathbf{u} \mathbf{K}^T + \left(G - \frac{|\mathbf{u}|^2}{2} \right) \mathbf{I} \right) \frac{\mathbf{F}}{\det \mathbf{F}} \right\} = \mathbf{0}^T. \quad (3.10)$$

The analog of conservation law (3.10) combining the field variables and deformation gradient has already appeared in the modeling of multiphase flows described by systems of quasi-linear hyperbolic equations [32].

4 Euler–van der Waals–Korteweg’s fluids

We consider compressible fluids endowed with internal capillarity when the energy per unit volume is a function of ρ and its gradient [14, 15, 16, 17, 18, 19, 20, 21, 22]. These fluids are also called Euler–van der Waals–Korteweg’s fluids. The volume energy of these fluids is :

$$W = W(\rho, \nabla \rho).$$

The governing equations are the Euler-Lagrange equations associated with the Hamilton action

$$a = \int_{t_1}^{t_2} \int_{D_t} \mathcal{L} \, dt \, dD,$$

where the Lagrangian is :

$$\mathcal{L} = \frac{\rho |\mathbf{u}|^2}{2} - W(\rho, \nabla \rho) - \rho V(t, \mathbf{x}).$$

Here $V(t, \mathbf{x})$ is the potential energy of external forces, t_1, t_2 are given time instants, ∇ is the gradient operator with respect to space variables. Quite often, the volume potential is taken in the form $W = W^h(\rho) + \lambda(\rho) |\nabla \rho|^2 / 2$, where $W^h(\rho)$ is the hydrodynamic (homogeneous) part of the energy, and $\lambda(\rho) > 0$ is the internal capillary coefficient. The mass conservation law

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0 \quad (4.11)$$

is a constraint. The momentum equation is the corresponding Euler-Lagrange equation for the constrained action functional :

$$\frac{\partial \rho \mathbf{u}^T}{\partial t} + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + \mathbf{\Pi}) + \rho \frac{\partial V}{\partial \mathbf{x}} = \mathbf{0}^T, \quad \mathbf{\Pi} = P \mathbf{I} + \frac{\partial W}{\partial \left(\frac{\partial \rho}{\partial \mathbf{x}} \right)} \frac{\partial \rho}{\partial \mathbf{x}}$$

where a *pressure* term P is defined as

$$P = \rho \frac{\delta W}{\delta \rho} - W \quad \text{with} \quad \frac{\delta W}{\delta \rho} = \frac{\partial W}{\partial \rho} - \operatorname{div} \left(\frac{\partial W}{\partial \nabla \rho} \right).$$

The term $\delta W/\delta\rho$ is the variational derivative of W with respect to ρ . The energy conservation law is

$$\frac{\partial e}{\partial t} + \operatorname{div} \left(e\mathbf{u} + \mathbf{\Pi}\mathbf{u} - \frac{\partial\rho}{\partial t} \frac{\partial W}{\partial \left(\frac{\partial\rho}{\partial\mathbf{x}} \right)} \right) - \rho \frac{\partial V}{\partial t} = 0,$$

with the definition of the total volume energy e

$$e = \frac{\rho|\mathbf{u}|^2}{2} + W + \rho V.$$

Due to the mass conservation law (4.11), the momentum equation can be written

$$\frac{D\mathbf{u}}{Dt} + \nabla \left(\frac{\delta W}{\delta\rho} + V \right) = \mathbf{0}.$$

We use the notation

$$H = \frac{\delta W}{\delta\rho} + V$$

for the total specific enthalpy. Obviously, the momentum equation can be written in the form of (3.7) with $\mathbf{K} = \mathbf{u}$:

$$\frac{D\mathbf{u}}{Dt} + \left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}} \right)^T \mathbf{u} + \nabla \left(H - \frac{|\mathbf{u}|^2}{2} \right) = \mathbf{0}. \quad (4.12)$$

We denote the vorticity of the capillary fluid by

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}.$$

It satisfies the Helmholtz equation :

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \frac{\partial\boldsymbol{\omega}}{\partial\mathbf{x}} \mathbf{u} + \boldsymbol{\omega} \operatorname{div} \mathbf{u} - \frac{\partial\mathbf{u}}{\partial\mathbf{x}} \boldsymbol{\omega} = 0.$$

In the case of potential flows, i.e., when $\boldsymbol{\omega} = 0$, the Bernoulli conservation law is satisfied

$$\frac{\partial\mathbf{u}}{\partial t} + \nabla \left(H + \frac{|\mathbf{u}|^2}{2} \right) = 0.$$

Theorem 7 1°) *Equations of capillary fluids (4.12) admit the following conservation law*

$$\frac{\partial}{\partial t} (\mathbf{u}^T \boldsymbol{\omega}) + \operatorname{div} \left\{ (\mathbf{u}^T \boldsymbol{\omega}) \mathbf{u} + \left(H - \frac{|\mathbf{u}|^2}{2} \right) \boldsymbol{\omega} \right\} = 0$$

2°) *Consider a material domain D_t of boundary S_t . If at $t = 0$ the vector field $\boldsymbol{\omega}$ is tangent to S_0 , then for any time t the vector field $\boldsymbol{\omega}$ is tangent to S_t . Moreover, the quantity (called helicity)*

$$\mathcal{H} = \iiint_{D_t} \mathbf{u}^T \boldsymbol{\omega} dD$$

keeps a constant value along the motion. The results remain true if we replace $\boldsymbol{\omega}$ by vectors \mathbf{E}_i , $i = 1, 2, 3$.

The proof of the theorem follows immediately from theorem 6 with $\mathbf{K} = \mathbf{u}$ and $\mathbf{L} = \boldsymbol{\omega}$ or $\mathbf{L} = \mathbf{E}_i$, $i = 1, 2, 3$.

5 Fluids with internal inertia

We focus on a class of dispersive models with *internal inertia* including a non-linear one-velocity model of bubbly fluids with incompressible liquid phase at small volume concentration of gas bubbles (Iordansky–Kogarko–van Wijngaarden model) [24, 25, 26], and Serre–Green–Naghdi (SGN) equations describing long surface gravity waves [5, 29, 30]. These models can be obtained as Euler–Lagrange equations of the Hamilton action [18, 19, 28]

$$a = \int_{t_1}^{t_2} \int_{D_t} \mathcal{L} dt dD,$$

with the Lagrangian

$$\mathcal{L} = \frac{\rho |\mathbf{u}|^2}{2} - W \left(\rho, \frac{D\rho}{Dt} \right) - \rho V(t, \mathbf{x}).$$

Here $W \left(\rho, \frac{D\rho}{Dt} \right)$ represents a potential depending not only on the density, but also on the material time derivative of the density. We call *internal inertia effect* such a dependence on the material time derivative. As usually, the mass conservation law (4.11) is a constraint. The conservative form of the momentum equation which is the corresponding Euler–Lagrange equation

$$\frac{\partial \rho \mathbf{u}^T}{\partial t} + \operatorname{div} (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}) + \rho \frac{\partial V}{\partial \mathbf{x}} = \mathbf{0}^T,$$

where the pressure p defined as

$$p = \rho \frac{\delta W}{\delta \rho} - W \quad \text{with} \quad \frac{\delta W}{\delta \rho} = \frac{\partial W}{\partial \rho} - \frac{\partial}{\partial t} \left(\frac{\partial W}{\partial \left(\frac{D\rho}{Dt} \right)} \right) - \operatorname{div} \left(\frac{\partial W}{\partial \left(\frac{D\rho}{Dt} \right)} \mathbf{u} \right). \quad (5.13)$$

The equation (5.13) is a generalized Rayleigh–Lamb equation describing oscillations of spherical bubbles in a fluid (see [27, 28] for details). The energy conservation law is

$$\frac{\partial e}{\partial t} + \operatorname{div} ((e + p) \mathbf{u}) - \rho \frac{\partial V}{\partial t} = 0,$$

with the definition of the total volume energy e

$$e = \frac{\rho|\mathbf{u}|^2}{2} + E + \rho V \quad \text{where} \quad E = W - \frac{D\rho}{Dt} \left(\frac{\partial W}{\partial \left(\frac{D\rho}{Dt} \right)} \right).$$

These laws are consequences of the invariance of Lagrangian under space and time shifts. The momentum equation can also be written as

$$\frac{D\mathbf{u}}{Dt} + \frac{\nabla p}{\rho} + \nabla V = \mathbf{0}.$$

One introduces the vector field \mathbf{K} [27] :

$$\mathbf{K} = \mathbf{u} + \frac{\nabla \sigma}{\rho} \quad \text{where} \quad \sigma = -\rho \left(\frac{\partial W}{\partial \left(\frac{D\rho}{Dt} \right)} \right). \quad (5.14)$$

With a special choice of the scalar function G , vector \mathbf{K} satisfies the same equation (3.7). Indeed, considering the volume internal energy $E(\rho, \tau)$ as a partial Legendre transform of $W(\rho, \dot{\rho})$, one has

$$E(\rho, \tau) = W - \frac{D\rho}{Dt} \left(\frac{\partial W}{\partial \left(\frac{D\rho}{Dt} \right)} \right) = W + \frac{D\rho}{Dt} \tau \quad \text{where} \quad \tau = - \left(\frac{\partial W}{\partial \left(\frac{D\rho}{Dt} \right)} \right).$$

Instead of $E(\rho, \tau)$ we define $\tilde{E}(\rho, \sigma) = E(\rho, \sigma/\rho)$. The momentum equation becomes [27]

$$\frac{D\mathbf{K}}{Dt} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{K} + \nabla \left(\tilde{E}_\rho + V - \frac{|\mathbf{u}|^2}{2} \right) = 0, \quad (5.15)$$

with $\tilde{E}_\rho = \frac{\partial \tilde{E}}{\partial \rho}(\rho, \sigma)$. This is exactly the form (3.7).

The notion of generalized vorticity for bubbly fluids was introduced in [27]. Let us define

$$\boldsymbol{\Omega} = \text{curl } \mathbf{K}.$$

Then the generalized vorticity satisfies the Helmholtz equation :

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \text{curl}(\boldsymbol{\Omega} \wedge \mathbf{u}) = \mathbf{0}.$$

In the case of vanishing vorticity (the case of potential flows), one obtains the Bernoulli conservation law in the form

$$\frac{\partial \mathbf{K}}{\partial t} + \nabla \left(\mathbf{K}^T \mathbf{u} + \tilde{E}_\rho + V - \frac{|\mathbf{u}|^2}{2} \right) = 0,$$

Finally, the following theorem can be formulated :

Theorem 8 1^o) *The equations of fluids with internal inertia (5) admit the conservation law*

$$\frac{\partial}{\partial t} (\mathbf{K}^T \boldsymbol{\Omega}) + \operatorname{div} \left\{ (\mathbf{K}^T \boldsymbol{\Omega}) \mathbf{u} + \left(\tilde{E}_\rho + V - \frac{|\mathbf{u}|^2}{2} \right) \boldsymbol{\Omega} \right\} = 0$$

2^o) *Consider a material domain \mathcal{D}_t of boundary S_t . If at $t = 0$ the vector field $\boldsymbol{\Omega}$ is tangent to S_0 , then for any time t the vector field $\boldsymbol{\Omega}$ is tangent to S_t . Moreover, the quantity*

$$\mathcal{H} = \iiint_{\mathcal{D}_t} \mathbf{K}^T \boldsymbol{\Omega} dD$$

we call generalized helicity, keeps a constant value along the motion. The results remain true if we replace $\boldsymbol{\Omega}$ by the vectors \mathbf{E}_i , $i = 1, 2, 3$.

The proof is an immediate application of Theorem 6.

5.1 Serre–Green–Naghdi equations

The results of this section can be considered as a physically important application of the general results obtained before. The Serre–Green–Naghdi (SGN) equations were derived in [5, 29, 30]. The equations represent a two–dimensional asymptotic reduction of the full free–surface Euler equations for long gravity waves. A mathematical justification of this model is presented in [31, 33]. Its numerical study can be found in [34, 35, 36, 37]. The corresponding potential W is [5, 18, 27, 28, 38] :

$$W(h, \dot{h}) = \frac{g h^2}{2} - \frac{h}{6} \left(\frac{Dh}{Dt} \right)^2 \quad \text{with} \quad \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + \nabla h \cdot \mathbf{u}.$$

Here g is the constant gravity acceleration, h is the water depth and \mathbf{u} is the depth averaged horizontal velocity. For the SGN equations we have only to replace ρ by the water depth h in the generic potential $W \left(\rho, \frac{D\rho}{Dt} \right)$. Thus, the water depth h and depth averaged velocity \mathbf{u} are functions of time t and of the horizontal coordinates $\mathbf{x} = (x^1, x^2)^T$. The vector \mathbf{K} defined by (5.14) can be written as

$$\mathbf{K} = \mathbf{u} - \frac{1}{h} \nabla \left(h \frac{\partial W}{\partial \left(\frac{Dh}{Dt} \right)} \right) = \mathbf{u} - \frac{1}{3h} \nabla \left(h^2 \frac{Dh}{Dt} \right) = \mathbf{u} + \frac{1}{3h} \nabla (h^3 \operatorname{div} \mathbf{u}).$$

The physical meaning of \mathbf{K} is quite interesting : it is the fluid velocity tangent to the free surface [39, 40]. Since $\sigma = \frac{h^2}{3} \frac{Dh}{Dt}$, the energy expressed in terms of h and σ is

$$\tilde{E}(h, \sigma) = W - \frac{Dh}{Dt} \left(\frac{\partial W}{\partial \left(\frac{Dh}{Dt} \right)} \right) = \frac{g h^2}{2} + \frac{9 \sigma^2}{6 h^3}$$

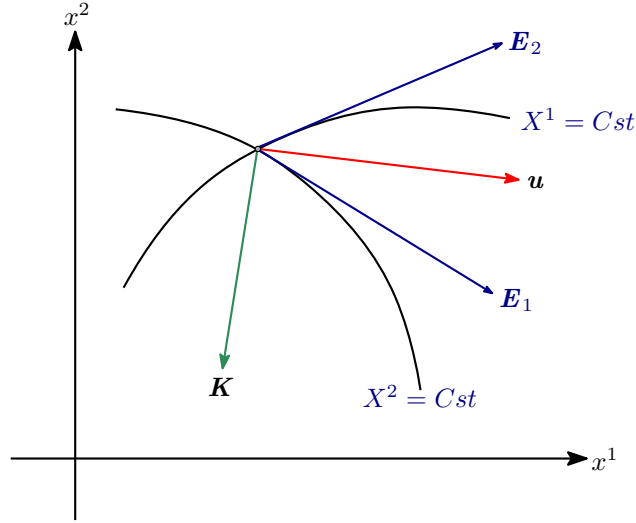


Figure 2: In the (x^1, x^2) – plane associated with Eulerian flow domain for the SGN model, we have shown at a given time t the material curves $X^1(t, x^1, x^2) = X^{10}$ and $X^2(t, x^1, x^2) = X^{20}$ corresponding to coordinate lines $X^1 = X^{10} = Cst$ and $X^2 = X^{20} = Cst$ in the plane of Lagrangian coordinates (X^1, X^2) . At any point, these curves are tangent to the vectors \mathbf{E}_2 and \mathbf{E}_1 . *A priori*, the vectors \mathbf{u} and \mathbf{K} are not collinear.

Hence

$$\tilde{E}_h(h, \sigma) = gh - \frac{9\sigma^2}{2h^4} = gh - \frac{1}{2} \left(\frac{Dh}{Dt} \right)^2$$

and the equation for \mathbf{K} is :

$$\frac{D\mathbf{K}}{Dt} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{K} + \nabla \left(gh - \frac{1}{2} \left(\frac{Dh}{Dt} \right)^2 - \frac{|\mathbf{u}|^2}{2} \right) = \mathbf{0}.$$

If theorem 8 is applied with the vorticity vector $\mathbf{\Omega}$, the result is trivial because $\mathbf{\Omega}$ and \mathbf{K} are orthogonal. However, the result is not trivial if we use, instead of vector $\mathbf{\Omega}$, the column vectors \mathbf{E}_i , $i = 1, 2$, of the matrix $\frac{\mathbf{F}}{\det \mathbf{F}}$:

$$\frac{\partial}{\partial t} (\mathbf{K}^T \mathbf{E}_i) + \text{div} \left(\mathbf{u} (\mathbf{K}^T \mathbf{E}_i) + \left(gh - \frac{1}{2} \left(\frac{Dh}{Dt} \right)^2 - \frac{|\mathbf{u}|^2}{2} \right) \mathbf{E}_i \right) = 0.$$

This equation can be interpreted as conservation laws for the covariant components of \mathbf{K} in the basis \mathbf{E}_i , $i = 1, 2$ (see Fig 2).

6 Conclusion and discussion

We have found new conservation laws for the equations of dispersive media when the energy depends on the density and density gradient (capillary fluids), or when the energy depends on the density and material time derivatives of the density (fluids with internal inertia). In particular, these conservation laws imply the generalized helicity integrals.

We have also found new physically interesting integrals representing the variation of the components of velocity \mathbf{u} (for capillary fluids) or of the vector \mathbf{K} (for fluids with internal inertia) in a time-dependent curvilinear basis \mathbf{E}_i , $i = 1, 2, 3$. These divergence-free basis vectors are tangent at each time to material curves along which the coordinate X^i varies and the other coordinates X^j , $j \neq i$ are fixed. We note that like the vorticity, they satisfy the Helmholtz equation.

We also note that the Schrödinger equation can be written in hydrodynamic form, a special case of the capillary fluid equations [41]. Thus, the new conservation laws obtained for capillary fluids can be used in the quantum mechanical case.

Obviously, other invariants can be found in the case where the potential W depends on additional scalar η which moves with a fluid, such as the specific entropy. Indeed, η satisfies the equation

$$\frac{D\eta}{Dt} = 0. \quad (6.16)$$

For example, the analog of Ertel's invariant is

$$\frac{D}{Dt} \left(\frac{\partial \eta}{\partial \mathbf{x}} \mathbf{L} \right) = 0$$

where \mathbf{L} is either the scaled basis vectors \mathbf{E}_i , $i = 1, 2, 3$ or (generalized) vorticity vector. Indeed, even if the equation for the vorticity contains an extra term related to the entropy gradient, it vanishes by combining the Helmholtz equation for the (generalized) vorticity with the Lie derivative of covector $\frac{\partial \eta}{\partial \mathbf{x}}$ coming from (6.16) (see Appendix A)

$$d_{L_1} \left(\frac{\partial \eta}{\partial \mathbf{x}} \right) = \frac{D}{Dt} \left(\frac{\partial \eta}{\partial \mathbf{x}} \right) + \frac{\partial \eta}{\partial \mathbf{x}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = 0,$$

Also, higher order conservation laws can be found as it was already done for the classical dispersionless equations (Euler equations of compressible and incompressible fluids, equations of magnetohydrodynamics, two-fluid plasma equations etc.) [6, 8, 9, 10]. However, the physical and geometrical meaning of these conservation laws still remains to be done.

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A The Lie derivatives of one-form and two-form fields

We denote by $T_{\mathbf{x}}(\mathcal{D}_t)$ and $T_{\mathbf{x}}^*(\mathcal{D}_t)$ the tangent and cotangent vector space at any point \mathbf{x} of a material domain \mathcal{D}_t . A vector field $\mathbf{J} \in T_{\mathbf{x}}(\mathcal{D}_t)$ has the inverse image $\mathbf{J}_0 \in T_{\mathbf{X}}(\mathcal{D}_0)$ such that

$$\mathbf{J}(t, \mathbf{x}) = \mathbf{F} \mathbf{J}_0(t, \mathbf{X}). \quad (\text{A.17})$$

A one-form field $\mathbf{C} \in T_{\mathbf{x}}^*(\mathcal{D}_t)$ has the inverse image $\mathbf{C}_0 \in T_{\mathbf{X}}^*(\mathcal{D}_0)$ such that

$$\mathbf{C}(t, \mathbf{x}) = \mathbf{C}_0(t, \mathbf{X}) \mathbf{F}^{-1} \quad \text{or} \quad \mathbf{C}_0(t, \mathbf{X}) = \mathbf{C}(t, \mathbf{x}) \mathbf{F}.$$

The Lie derivative of \mathbf{C} is denoted by d_{L_1} and is the image of $\frac{\partial \mathbf{C}_0(t, \mathbf{X})}{\partial t}$ in $T_{\mathbf{x}}^*(\mathcal{D}_t)$. From the evolution equation (2.1) for \mathbf{F} , we obtain

$$d_{L_1} \mathbf{C} = \frac{D\mathbf{C}}{Dt} + \mathbf{C} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}.$$

A two-form field $\boldsymbol{\omega} \in T_{\mathbf{x}}^{2*}(\mathcal{D}_t)$ has the inverse image $\boldsymbol{\omega}_0 \in T_{\mathbf{X}}^{2*}(\mathcal{D}_0)$. There exists an isomorphism $\boldsymbol{\omega}(t, \mathbf{x}) \in T_{\mathbf{x}}^{2*}(\mathcal{D}_t) \rightarrow \boldsymbol{w}(t, \mathbf{x}) \in T_{\mathbf{x}}(\mathcal{D}_t)$ defined as

$$\forall \mathbf{J}_1 \text{ and } \mathbf{J}_2 \in T_{\mathbf{x}}(\mathcal{D}_t), \quad \boldsymbol{\omega}(\mathbf{J}_1, \mathbf{J}_2) = \det(\boldsymbol{w}, \mathbf{J}_1, \mathbf{J}_2).$$

We can define a vector field \boldsymbol{w}_0 such that:

$$\det(\boldsymbol{w}, \mathbf{J}_1, \mathbf{J}_2) = \det(\boldsymbol{w}_0, \mathbf{J}_{10}, \mathbf{J}_{20})$$

where \mathbf{J}_{10} and \mathbf{J}_{20} are the inverse image in $T_{\mathbf{X}}(\mathcal{D}_0)$ of \mathbf{J}_1 and \mathbf{J}_2 which verify (A.17). Similarly to the proof of Lemma 1, we obtain

$$\boldsymbol{w}(t, \mathbf{x}) = \frac{\mathbf{F}}{\det \mathbf{F}} \boldsymbol{w}_0(t, \mathbf{X}) \quad \text{or} \quad \boldsymbol{w}_0(t, \mathbf{X}) = (\det \mathbf{F}) \mathbf{F}^{-1} \boldsymbol{w}(t, \mathbf{x}).$$

The Lie derivative d_{L_2} of $\boldsymbol{\omega}(t, \mathbf{x})$ is isomorphic to the image of $\frac{\partial \boldsymbol{w}_0(t, \mathbf{X})}{\partial t}$ and we deduce from (2.1):

$$\det(d_{L_2} \boldsymbol{w}, \mathbf{J}_1, \mathbf{J}_2) = \det\left(\frac{D\boldsymbol{w}}{Dt} + \boldsymbol{w} \operatorname{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \boldsymbol{w}, \mathbf{J}_1, \mathbf{J}_2\right),$$

or

$$d_{L_2} \boldsymbol{w} = \frac{D\boldsymbol{w}}{Dt} + \boldsymbol{w} \operatorname{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \boldsymbol{w}.$$

Let us note that $d_{L_1} \mathbf{C}$ and $d_{L_2} \boldsymbol{w}$ are zero if and only if \mathbf{C}_0 and \boldsymbol{w}_0 do not depend on time t (they depend only on \mathbf{X}). A typical example of the two-form field is the vorticity vector. Another important example is the cross product of two one-forms $\frac{\partial \alpha}{\partial \mathbf{x}} \wedge \frac{\partial \beta}{\partial \mathbf{x}}$, where α and β are two scalar fields.