Helicity in dispersive fluid mechanics

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Abstract

By dispersive models of fluid mechanics we are referring to the Euler-Lagrange equations for the constrained Hamilton action functional where the internal energy depends on high order derivatives of unknowns. The mass conservation law is considered as a constraint. The corresponding Euler-Lagrange equations include, in particular, the van der Waals–Korteweg model of capillary fluids, the model of fluids containing small gas bubbles and the model describing long free-surface gravity waves. We obtain new conservation laws generalizing the helicity conservation for classical barotropic fluids.

Keywords: Dispersive fluid mechanics; capillary fluids; bubbly fluids; long free-surface gravity waves

1 Introduction

The helicity integral was first discovered for classical barotropic fluids in seminal papers by J. J. Moreau [1] and H. K. Moffatt [2]. As the Kelvin circulation theorem, conservation of helicity results from the particle re-labeling symmetry [3, 4, 5, 6]. Further important studies of such invariants as well as Ertel's type invariants for the incompressible and compressible Euler equations, magnetohydrodynamics etc. can be found in the literature [7, 8, 9, 10, 11, 12, 13, 14].

In this paper, we derive helicity integrals for dispersive media such as fluids endowed with capillarity (called also Euler–van der Waals–Korteweg's fluids, and a class of fluids with internal inertia (such as bubbly fluids and long free-surface gravity waves).

From a mathematical point of view, we concentrate here on two classes of dispersive equations which are Euler–Lagrange equations for Hamilton's action with a Lagrangian depending not only on the thermodynamic variables but also on their first spatial and time derivatives.

The first class of the models (the Lagrangian depends on the density gradient) is often called *the second gradient models*, and the corresponding dispersive

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terms are called *capillary type dispersion terms*. Indeed, the density is linked directly to the determinant of the deformation gradient, so the density gradient thus linked to second derivatives of the deformation. Historically, such models appeared in the description of regions with non-uniform density and described the surface tension effects due to the van der Waals forces between fluid molecules. The dependence on the density gradient describes in some sense the micro-structure of such a non-homogeneous continuum. Such class of models thus describes a continuum with a characteristic micro-scale length [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

For the second class of models, the Lagrangian depends on the material derivative of the density (the dependence on the material derivative, and not just on the partial time derivative, is important to guarantee the Galilean invariance of the governing equations). This type of dispersion is often called *inertia type dispersion*. Such class of models contains thus a characteristic micro-scale time [5, 26, 27, 28, 29, 30, 31, 32, 33].

These two classes of models can obviously be unified in a general setting [34]. However, for specific applications (Euler–van der Waals–Korteweg fluids, fluids containing small gas bubbles, etc.) the governing equations describe dispersive effects separately: they take into account either the micro-scale structure or a micro-inertia structure. We will therefore study these two classes of models separately.

We introduce the following notations. For any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the scalar product $\boldsymbol{a}\cdot\boldsymbol{b}$ is written $\boldsymbol{a}^T\boldsymbol{b}$ (line vector or covector $\boldsymbol{a}^T$ is multiplied by column vector $\boldsymbol{b}$), where superscript ^{T} denotes the transposition. The tensor product $\boldsymbol{a}\otimes\boldsymbol{b}$ is also written $\boldsymbol{a}\,\boldsymbol{b}^T$.

The product of matrix $\boldsymbol{A}$ by vector $\boldsymbol{a}$ is denoted by $\boldsymbol{A}\,\boldsymbol{a}$. Notation $\boldsymbol{b}^T\boldsymbol{A}$ means the covector $\boldsymbol{c}^T$ defined as $\boldsymbol{c}^T = (\boldsymbol{A}^T\boldsymbol{b})^T$. The mixed product of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ is denoted $\boldsymbol{a}^T(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \det(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ where the sign '$\times$' means the vector product. Tensor $\boldsymbol{I}$ denotes the identity transformation.

The gradient of scalar function $f(\boldsymbol{x})$ is defined by $\nabla f = \left(\dfrac{\partial f}{\partial\boldsymbol{x}}\right)^T$ associated with linear form $df = \left(\dfrac{\partial f}{\partial\boldsymbol{x}}\right)d\boldsymbol{x}$.

Let $\boldsymbol{v} = \left(v^1,...,v^n\right)^T$ be a function of $\boldsymbol{x} = \left(x^1,...,x^n\right)^T$. Then $\dfrac{\partial\boldsymbol{v}}{\partial\boldsymbol{x}}$ denotes the linear application defined by the relation $d\boldsymbol{v} = \dfrac{\partial\boldsymbol{v}}{\partial\boldsymbol{x}}\,d\boldsymbol{x}$ and represented by the matrix

$$\begin{pmatrix} \dfrac{\partial v^1(x^1,...,x^n)}{\partial x^1} & ... & \dfrac{\partial v^1(x^1,...,x^n)}{\partial x^n} \\ \vdots & & \vdots \\ \dfrac{\partial v^n(x^1,...,x^n)}{\partial x^1} & ... & \dfrac{\partial v^n(x^1,...,x^n)}{\partial x^n} \end{pmatrix}.$$

The divergence of second order tensor $\boldsymbol{A}$ is the covector $\operatorname{div} \boldsymbol{A}$ such that, for any constant vector $\boldsymbol{a}$, $(\operatorname{div} \boldsymbol{A})\, \boldsymbol{a} = \operatorname{div}(\boldsymbol{A}\ \boldsymbol{a})$. It implies that for any vector field $\boldsymbol{v}$

$$\operatorname{div}(\boldsymbol{A}\, \boldsymbol{v}) = (\operatorname{div}\ \boldsymbol{A})\, \boldsymbol{v} + \operatorname{Tr}\left(\boldsymbol{A}\, \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right),$$

where Tr is the trace operator.

In three-dimensional case we introduce vorticity $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$. In the Cartesian basis $\boldsymbol{\omega} = (\omega^1, \omega^2, \omega^3)^T$ is defined by

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix}.$$

2 Fluid motion

A fluid motion is represented by a diffeomorphism $\boldsymbol{\varphi}$ of a three-dimensional reference configuration $\mathcal{D}_0$ into the physical space $\mathcal{D}_t$. Let $\boldsymbol{X} = (X^1, X^2, X^3)^T$ be the Lagrangian coordinates, and $\boldsymbol{x} = (x^1, x^2, x^3)^T$ be the Eulerian coordinates. Fluid motion is

$$\boldsymbol{x} = \boldsymbol{\varphi}\,(t, \boldsymbol{X})\,,$$

where t denotes the time. At a given time t, the transformation $\boldsymbol{\varphi}$ possesses an inverse and has continuous derivatives up to the second order. The deformation gradient is

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\varphi}(t, \boldsymbol{X})}{\partial \boldsymbol{X}} \equiv \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}.$$

Considering $\boldsymbol{F}$ as a function of the Eulerian coordinates, we obtain the evolution equation

$$\frac{D\boldsymbol{F}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \boldsymbol{F} \quad \text{with} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}^T \nabla, \tag{2.1}$$

where $\boldsymbol{u}(t, \boldsymbol{x}) = \left.\dfrac{\partial \boldsymbol{\varphi}(t, \boldsymbol{X})}{\partial t}\right|_{\boldsymbol{X} = \boldsymbol{\varphi}^{-1}(t, \boldsymbol{x})}$ is the particle velocity.

3 Basic lemmas

Lemma 1 *The following identities are satisfied:*

$$\operatorname{div}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\right) = 0, \tag{3.2}$$

$$\frac{D}{Dt}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\right) = \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - (\operatorname{div} \boldsymbol{u})\, \boldsymbol{I}\right) \frac{\boldsymbol{F}}{\det \boldsymbol{F}}. \tag{3.3}$$

Proof. We consider a vector field $\boldsymbol{v}_0$ on $\mathcal{D}_0$ of boundary S_0 and their images $\mathcal{D}_t$ of boundary S_t. We obtain

$$\iiint_{\mathcal{D}_t} \operatorname{div}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\,\boldsymbol{v}_0\right) dD = \iiint_{\mathcal{D}_0} (\det \boldsymbol{F}) \operatorname{div}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\,\boldsymbol{v}_0\right) dD_0.$$

Let $\boldsymbol{X}\,(u_{10}, u_{20})$ be the parametrization of the surface S_0 in the reference space, and $\boldsymbol{x}\,(u_1, u_2)$ the parametrization of its image, S_t, in the actual space.
We denote by $d_{10}\boldsymbol{X} = \dfrac{\partial \boldsymbol{X}}{\partial u_{10}}\, du_{10}$, $d_{20}\boldsymbol{X} = \dfrac{\partial \boldsymbol{X}}{\partial u_{20}}\, du_{20}$ the infinitesimal tangent vectors along the coordinate lines on S_0, and $d_1\boldsymbol{x} = \dfrac{\partial \boldsymbol{x}}{\partial u_1}\, du_1$, $d_2\boldsymbol{x} = \dfrac{\partial \boldsymbol{x}}{\partial u_2}\, du_2$ the infinitesimal tangent vectors along the corresponding material lines on S_t.
For any constant vector $\boldsymbol{a}$,

$$\boldsymbol{a}^T (d_1\boldsymbol{x} \times d_2\boldsymbol{x}) = \det\left(\boldsymbol{a}, d_1\boldsymbol{x}, d_2\boldsymbol{x}\right) = \det\left(\boldsymbol{a}, \boldsymbol{F} d_{10}\boldsymbol{X}, \boldsymbol{F} d_{20}\boldsymbol{X}\right)$$

$$= (\det\boldsymbol{F})\, \det\left(\boldsymbol{F}^{-1}\boldsymbol{a}, d_{10}\boldsymbol{X}, d_{20}\boldsymbol{X}\right) = (\det\boldsymbol{F})\, \left(\boldsymbol{F}^{-1}\boldsymbol{a}\right)^T (d_{10}\boldsymbol{X} \times d_{20}\boldsymbol{X}).$$

This gives us the relationship obtained in [35], p. 77 :

$$d_1\boldsymbol{x} \times d_2\boldsymbol{x} = (\det\boldsymbol{F})\, \boldsymbol{F}^{-T} (d_{10}\boldsymbol{X} \times d_{20}\boldsymbol{X}). \tag{3.4}$$

From Eq. (3.4) and Gauss–Ostrogradsky's formula, we obtain

$$\iiint_{\mathcal{D}_t} \operatorname{div}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\,\boldsymbol{v}_0\right) dD = \iint_{S_t} \det\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\,\boldsymbol{v}_0, d_1\boldsymbol{x}, d_2\boldsymbol{x}\right)$$

$$= \iint_{S_0} \det\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\,\boldsymbol{v}_0, \boldsymbol{F}\, d_{10}\boldsymbol{X}, \boldsymbol{F}\, d_{20}\boldsymbol{X}\right)$$

$$= \iint_{S_0} \det\left(\boldsymbol{v}_0, d_{10}\boldsymbol{X}, d_{20}\boldsymbol{X}\right) = \iiint_{\mathcal{D}_0} \left(\operatorname{div}_0 \boldsymbol{v}_0\right)\, dD_0.$$

Consequently,

$$(\det \boldsymbol{F}) \operatorname{div}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\,\boldsymbol{v}_0\right) = \operatorname{div}_0 \boldsymbol{v}_0.$$

For all constant vector $\boldsymbol{v}_0$ on $\mathcal{D}_0$, we get

$$\operatorname{div}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\,\boldsymbol{v}_0\right) = \operatorname{div}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\right)\,\boldsymbol{v}_0 = 0.$$

This implies (3.2).

Finally, from (2.1) we obtain

$$\frac{D}{Dt}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\right) = \frac{1}{\det \boldsymbol{F}}\,\frac{D\boldsymbol{F}}{Dt} - \frac{1}{(\det \boldsymbol{F})^2}\,\frac{D(\det \boldsymbol{F})}{Dt}\,\boldsymbol{F}.$$

The Euler-Jacobi formula

$$\frac{D(\det \boldsymbol{F})}{Dt} = (\det \boldsymbol{F})\,\mathrm{div}\boldsymbol{u}$$

gives (3.3). ■

A simple geometrical interpretation of the identity (3.2) is given below.

Lemma 2 *Let $\boldsymbol{e}_i$ be a local curvilinear basis (with lower indexes $i = 1,2,3$), $\boldsymbol{e}_i = \dfrac{\partial \boldsymbol{x}}{\partial X^i} = \dfrac{\partial \boldsymbol{\varphi}(t,\boldsymbol{X})}{\partial X^i}$, expressed in the Eulerian coordinates, and $\boldsymbol{e}^i = \nabla X^i(t,\boldsymbol{x})$ (with upper indexes $i = 1,2,3$) be the corresponding dual basis (cobasis). Then*

$$\frac{\boldsymbol{e}_k}{\det \boldsymbol{F}} = \boldsymbol{e}^i \times \boldsymbol{e}^j,$$

where $\{i,j,k\}$ is an even permutation of $\{1,2,3\}$. Moreover,

$$\mathrm{div}\left(\frac{\boldsymbol{e}_k}{\det \boldsymbol{F}}\right) = 0.$$

Proof. The proof of the first formula comes from the identity

$$\delta^i_j = \frac{\partial X^i}{\partial X^j} = \frac{\partial X^i}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial X^j} = {\boldsymbol{e}^i}^T \boldsymbol{e}_j,$$

where δ^i_j is the Kronecker symbol. The second formula comes from the identity

$$\mathrm{div}\left(\frac{\boldsymbol{e}_k}{\det \boldsymbol{F}}\right) = \mathrm{div}\left(\boldsymbol{e}^i \times \boldsymbol{e}^j\right)$$

$$= {\boldsymbol{e}^j}^T \mathrm{curl}\,\boldsymbol{e}^i - {\boldsymbol{e}^i}^T \mathrm{curl}\,\boldsymbol{e}^j = 0.$$

■

Remark 3 *Let $\boldsymbol{E}_k = \dfrac{\boldsymbol{e}_k}{\det \boldsymbol{F}}$ be the columns of the matrix $\dfrac{\boldsymbol{F}}{\det \boldsymbol{F}}$:*

$$\frac{\boldsymbol{F}}{\det \boldsymbol{F}} = [\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3].$$

The equations (3.2), (3.3) yield :

$$\frac{\partial \boldsymbol{E}_i}{\partial t} + \mathrm{curl}\,(\boldsymbol{E}_i \times \boldsymbol{u}) = 0, \quad \mathrm{div}\,\boldsymbol{E}_i = 0, \quad i = 1,2,3. \tag{3.5}$$

This is Helmholtz's equation which governs the vorticity in the Euler equations. Equations (3.5) express that the Lie derivative of a two–form along the velocity field $\boldsymbol{u}$ is zero (see Appendix A for details) :

$$d_{L_2}\boldsymbol{E}_i \equiv \frac{D\boldsymbol{E}_i}{Dt} - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{E}_i + (\mathrm{div}\,\boldsymbol{u})\,\boldsymbol{E}_i = 0.$$

Lemma 4 *For any scalar field G and vector fields $\boldsymbol{K}$, $\boldsymbol{L}$ on $\mathcal{D}_t$ such that $\boldsymbol{L}$ is divergence-free* (div $\boldsymbol{L} = 0$)*, we have the identity*

$$\frac{\partial}{\partial t}\left(\boldsymbol{K}^T \boldsymbol{L}\right) + \operatorname{div}\left\{\left(\boldsymbol{u}\,\boldsymbol{K}^T + \left(G - \frac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{I}\right)\boldsymbol{L}\right\} \equiv$$

$$\boldsymbol{L}^T\left(\frac{D\boldsymbol{K}}{Dt} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{K} + \nabla\left(G - \frac{|\boldsymbol{u}|^2}{2}\right)\right) +$$

$$\boldsymbol{K}^T\left(\frac{\partial \boldsymbol{L}}{\partial t} + \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{x}}\,\boldsymbol{u} + \boldsymbol{L}\operatorname{div}\boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\,\boldsymbol{L}\right).$$

Proof. Let us denote

$$A = \frac{\partial}{\partial t}\left(\boldsymbol{K}^T \boldsymbol{L}\right) + \operatorname{div}\left\{\left(\boldsymbol{u}\,\boldsymbol{K}^T + \left(G - \frac{1}{2}\boldsymbol{u}^T\boldsymbol{u}\right)\boldsymbol{I}\right)\boldsymbol{L}\right\}.$$

From

$$\operatorname{div}\left(\boldsymbol{u}\,\boldsymbol{K}^T\boldsymbol{L}\right) = \boldsymbol{K}^T\boldsymbol{L}\,(\operatorname{div}\boldsymbol{u}) + \boldsymbol{K}^T\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{x}}\,\boldsymbol{u} + \boldsymbol{L}^T\frac{\partial \boldsymbol{K}}{\partial \boldsymbol{x}}\,\boldsymbol{u}$$

and

$$\operatorname{div}\boldsymbol{L} = 0, \quad \frac{\partial \boldsymbol{K}}{\partial t} = \frac{D\boldsymbol{K}}{Dt} - \frac{\partial \boldsymbol{K}}{\partial \boldsymbol{x}}\,\boldsymbol{u},$$

we obtain

$$A = \boldsymbol{K}^T\frac{\partial \boldsymbol{L}}{\partial t} + \boldsymbol{L}^T\frac{D\boldsymbol{K}}{Dt} - \boldsymbol{L}^T\frac{\partial \boldsymbol{K}}{\partial \boldsymbol{x}}\,\boldsymbol{u} + \boldsymbol{K}^T\boldsymbol{L}\,(\operatorname{div}\boldsymbol{u})$$

$$+\boldsymbol{K}^T\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{x}}\,\boldsymbol{u} + \boldsymbol{L}^T\frac{\partial \boldsymbol{K}}{\partial \boldsymbol{x}}\,\boldsymbol{u} + \frac{\partial}{\partial \boldsymbol{x}}\left(G - \frac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{L}.$$

Finally, from

$$\boldsymbol{L}^T\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T\boldsymbol{K} = \boldsymbol{K}^T\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)\boldsymbol{L}$$

we get

$$A = \boldsymbol{L}^T\left(\frac{D\boldsymbol{K}}{Dt} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{K} + \nabla\left(G - \frac{|\boldsymbol{u}|^2}{2}\right)\right)$$

$$+\boldsymbol{K}^T\left(\frac{\partial \boldsymbol{L}}{\partial t} + \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{x}}\,\boldsymbol{u} + \boldsymbol{L}\,(\operatorname{div}\boldsymbol{u}) - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\,\boldsymbol{L}\right).$$

■

Remark 5 *The equation*

$$\frac{D\boldsymbol{K}}{Dt} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{K} + \nabla\left(G - \frac{|\boldsymbol{u}|^2}{2}\right) = 0, \tag{3.6}$$

it equivalent to

$$d_{L_1}\boldsymbol{K}^T+\frac{\partial}{\partial \boldsymbol{x}}\left(G-\frac{|\boldsymbol{u}|^2}{2}\right)=0$$

where $d_{L_1}\boldsymbol{K}^T \equiv \dfrac{D\boldsymbol{K}^T}{Dt}+\boldsymbol{K}^T\left(\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)$ *is the Lie derivative of covector* $\boldsymbol{K}^T$ *along the vector field* $\mathbf{u}$ *(see Appendix A for details).*

Theorem 6 1^o) *Consider a divergence–free field* $\boldsymbol{L}$ *satisfying the Helmholtz equation*

$$d_{L_2}\boldsymbol{L}\equiv\frac{\partial \boldsymbol{L}}{\partial t}+\frac{\partial \boldsymbol{L}}{\partial \boldsymbol{x}}\boldsymbol{u}+\boldsymbol{L}\,\mathrm{div}\,\boldsymbol{u}-\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{L}=0,$$

and the field $\boldsymbol{K}$ *satisfying* (3.6). *Then,*

$$\frac{\partial}{\partial t}\left(\boldsymbol{K}^T\,\boldsymbol{L}\right)+\mathrm{div}\left\{\left(\boldsymbol{u}\,\boldsymbol{K}^T+\left(G-\frac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{I}\right)\boldsymbol{L}\right\}=0. \tag{3.7}$$

2^o) *Consider a material domain* $\mathcal{D}_t$ *of boundary* S_t. *If at* $t=0$ *the divergence-free field* $\boldsymbol{L}$ *is tangent to* S_0*, then at all time* $\boldsymbol{L}$ *is tangent to* S_t.
Moreover, the quantity

$$\mathcal{H}=\iiint_{D_t}\boldsymbol{K}^T\boldsymbol{L}\,dD, \tag{3.8}$$

we call generalized helicity, keeps a constant value along the motion.

The conservation law (3.7) is a consequence of Lemma 4. The helicity integral (3.8) immediately comes from (3.7).
In particular, the theorem 6 implies the following compact form of the conservation laws in the case $\boldsymbol{L}=\boldsymbol{E}_i,\ i=1,2,3$

$$\frac{\partial}{\partial t}\left(\boldsymbol{K}^T\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\right)+\mathrm{div}\left\{\left(\boldsymbol{u}\,\boldsymbol{K}^T+\left(G-\frac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{I}\right)\frac{\boldsymbol{F}}{\det\boldsymbol{F}}\right\}=\boldsymbol{0}^T. \tag{3.9}$$

The analog of conservation law (3.9) has already been found in the modeling of multiphase flows [36].

4 Euler–van der Waals–Korteweg's fluids

We consider compressible fluids endowed with internal capillarity when the energy per unit volume is a function of density ρ and $\nabla\rho$ [15, 16, 17, 18, 19, 34, 20, 23, 24]. These fluids are also called Euler–van der Waals–Korteweg's fluids. For barotropic fluids, the volume energy is in the form

$$W=W(\rho,\nabla\rho).$$

The governing equations are the Euler-Lagrange equations associated with the Hamilton action

$$a=\int_{t_1}^{t_2}\int_{D_t}\mathcal{L}\,dt\,dD,$$

where $t_1,\ t_2$ are given time instants. The Lagrangian is

$$\mathcal{L} = \frac{\rho|\boldsymbol{u}|^2}{2} - W(\rho, \nabla\rho) - \rho\, V(t, \boldsymbol{x}). \tag{4.10}$$

Here $V(t, \boldsymbol{x})$ is the potential of external forces. The mass conservation law

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\, \boldsymbol{u}) = 0 \tag{4.11}$$

is a constraint. The momentum equation is the Euler-Lagrange equation for (4.10)

$$\frac{\partial\rho\, \boldsymbol{u}^T}{\partial t} + \operatorname{div}(\rho\, \boldsymbol{u}\otimes\boldsymbol{u} + \boldsymbol{\Pi}) + \rho\,\frac{\partial V}{\partial\boldsymbol{x}} = \boldsymbol{0}^T, \quad \boldsymbol{\Pi} = P\,\boldsymbol{I} + \frac{\partial W}{\partial\nabla\rho}\otimes\nabla\rho$$

where pressure term P is defined as

$$P = \rho\frac{\delta W}{\delta\rho} - W \quad \text{with} \quad \frac{\delta W}{\delta\rho} = \frac{\partial W}{\partial\rho} - \operatorname{div}\left(\frac{\partial W}{\partial\nabla\rho}\right).$$

The term $\delta W/\delta\rho$ is the variational derivative of W with respect to ρ. The energy conservation law is

$$\frac{\partial e}{\partial t} + \operatorname{div}\left(e\boldsymbol{u} + \boldsymbol{\Pi}\boldsymbol{u} - \frac{\partial\rho}{\partial t}\frac{\partial W}{\partial\nabla\rho}\right) - \rho\,\frac{\partial V}{\partial t} = 0,$$

with the definition of the total volume energy e

$$e = \frac{\rho|\boldsymbol{u}|^2}{2} + W + \rho\, V.$$

Due to the mass conservation law (4.11), the momentum equation can be written

$$\frac{D\boldsymbol{u}}{Dt} + \nabla\left(\frac{\delta W}{\delta\rho} + V\right) = \boldsymbol{0}.$$

We use further the notation

$$H = \frac{\delta W}{\delta\rho} + V$$

for the total generalized specific enthalpy. Obviously, the momentum equation can be written in the form of Eq. (3.6) with $\boldsymbol{K} = \boldsymbol{u}$

$$\frac{D\boldsymbol{u}}{Dt} + \left(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}}\right)^T\boldsymbol{u} + \nabla\left(H - \frac{|\boldsymbol{u}|^2}{2}\right) = \boldsymbol{0}. \tag{4.12}$$

We denote the vorticity of the capillary fluid by

$$\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{u}.$$

The vorticity satisfies the Helmholtz equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \frac{\partial\boldsymbol{\omega}}{\partial\boldsymbol{x}}\boldsymbol{u} + \boldsymbol{\omega}\operatorname{div}\boldsymbol{u} - \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}}\boldsymbol{\omega} = 0.$$

Applying the theorem 6, we obtain

Theorem 7 1^o) *Equations of capillary fluids* (4.12) *admit the following conservation law*

$$\frac{\partial}{\partial t}\left(\boldsymbol{u}^T\boldsymbol{\omega}\right)+\operatorname{div}\left\{\left(\boldsymbol{u}^T\boldsymbol{\omega}\right)\boldsymbol{u}+\left(H-\frac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{\omega}\right\}=0$$

2^o) *Consider a material domain D_t of boundary S_t. If at $t=0$ the vector field $\boldsymbol{\omega}$ is tangent to S_0, then for any time t the vector field $\boldsymbol{\omega}$ is tangent to S_t and the helicity*

$$\mathcal{H}=\iiint_{D_t}\boldsymbol{u}^T\boldsymbol{\omega}\,dD$$

keeps a constant value along the motion.
The results remain true if we replace $\boldsymbol{\omega}$ by vectors $\boldsymbol{E}_i$, $i=1,2,3$.

The proof of the theorem follows immediately from theorem 6 with $\boldsymbol{K}=\boldsymbol{u}$ and $\boldsymbol{L}=\boldsymbol{\omega}$ or $\boldsymbol{L}=\boldsymbol{E}_i$, $i=1,2,3$.

5 Fluids with internal inertia

We consider dispersive models with *internal inertia* including a non-linear one-velocity model of bubbly fluids with incompressible liquid phase at small volume concentration of gas bubbles (Iordansky–Kogarko–van Wijngarden's model) [26, 27, 28], and Serre–Green–Naghdi's equations (SGN eqations) describing long free–surface gravity waves [5, 31, 32].

As usually, the mass conservation law (4.11) is a constraint. The momentum equation is the Euler-Lagrange equation for the constrained action functional [19, 34, 30]

$$a=\int_{t_1}^{t_2}\int_{D_t}\mathcal{L}\,dt\,dD,$$

where the Lagrangian is

$$\mathcal{L}=\frac{\rho|\boldsymbol{u}|^2}{2}-W\left(\rho,\frac{D\rho}{Dt}\right)-\rho\,V(t,\boldsymbol{x}).$$

Here $W\left(\rho,\dfrac{D\rho}{Dt}\right)$ is a potential depending not only on the density, but also on the material time derivative of the density. We call *internal inertia effect* such a dependence on the material time derivative.
The conservative form of the momentum equation is the Euler–Lagrange equation

$$\frac{\partial\rho\,\boldsymbol{u}^T}{\partial t}+\operatorname{div}\left(\rho\,\boldsymbol{u}\otimes\boldsymbol{u}+p\,\boldsymbol{I}\right)+\rho\frac{\partial V}{\partial\boldsymbol{x}}=\boldsymbol{0}^T,\tag{5.13}$$

where pressure p is defined as

$$p = \rho \frac{\delta W}{\delta \rho} - W$$

with

$$\frac{\delta W}{\delta \rho} = \frac{\partial W}{\partial \rho} - \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial\left(\frac{D\rho}{Dt}\right)}\right) - \operatorname{div}\left(\frac{\partial W}{\partial\left(\frac{D\rho}{Dt}\right)}\,\boldsymbol{u}\right). \tag{5.14}$$

The energy conservation law is

$$\frac{\partial e}{\partial t} + \operatorname{div}\left((e+p)\,\boldsymbol{u}\right) - \rho\,\frac{\partial V}{\partial t} = 0,$$

with the total volume energy e

$$e = \frac{\rho|\boldsymbol{u}|^2}{2} + E + \rho V \quad \text{where} \quad E = W - \frac{D\rho}{Dt}\left(\frac{\partial W}{\partial\left(\frac{D\rho}{Dt}\right)}\right).$$

The momentum and conservation laws are consequences of the invariance of Lagrangian under space and time shifts. The momentum equation can also be written as

$$\frac{D\boldsymbol{u}}{Dt} + \frac{\nabla p}{\rho} + \nabla V = \mathbf{0}.$$

One introduces the vector field $\boldsymbol{K}$ defined as (see [29])

$$\boldsymbol{K} = \boldsymbol{u} + \frac{\nabla \sigma}{\rho} \qquad \text{where} \qquad \sigma = -\rho\left(\frac{\partial W}{\partial\left(\frac{D\rho}{Dt}\right)}\right). \tag{5.15}$$

Considering the volume internal energy $E(\rho,\tau)$ as a partial Legendre transform of $W(\rho,\dot\rho)$

$$E(\rho,\tau) = W - \frac{D\rho}{Dt}\left(\frac{\partial W}{\partial\left(\frac{D\rho}{Dt}\right)}\right) = W + \frac{D\rho}{Dt}\,\tau \quad \text{where} \quad \tau = -\left(\frac{\partial W}{\partial\left(\frac{D\rho}{Dt}\right)}\right),$$

and defining $\tilde{E}(\rho,\sigma)$ as $\tilde{E}(\rho,\sigma) = E(\rho,\sigma/\rho)$, the momentum equation becomes [29]

$$\frac{D\boldsymbol{K}}{Dt} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{K} + \nabla\left(\tilde{E}_\rho + V - \frac{|\boldsymbol{u}|^2}{2}\right) = 0, \tag{5.16}$$

with $\tilde{E}_\rho = \dfrac{\partial \tilde{E}}{\partial \rho}(\rho,\sigma)$. Equation (5.16) is in the form (3.6).

The notion of generalized vorticity for bubbly fluids was introduced in [29]. This generalized vorticity is defined by

$$\boldsymbol{\Omega} = \operatorname{curl} \boldsymbol{K}.$$

It satisfies the Helmholtz equation

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \operatorname{curl}\left(\boldsymbol{\Omega} \times \boldsymbol{u}\right) = \boldsymbol{0}.$$

Thus, we obtain

Theorem 8 1^o) *The equations of fluids with internal inertia* (5.13) *admit the conservation law*

$$\frac{\partial}{\partial t}\left(\boldsymbol{K}^T \boldsymbol{\Omega}\right) + \operatorname{div}\left\{\left(\boldsymbol{K}^T \boldsymbol{\Omega}\right)\boldsymbol{u} + \left(\tilde{E}_\rho + V - \frac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{\Omega}\right\} = 0$$

2^o) *Consider a material domain D_t of boundary S_t. If at $t = 0$ the vector field $\boldsymbol{\Omega}$ is tangent to S_0, then for any time t the vector field $\boldsymbol{\Omega}$ is tangent to S_t and the helicity*

$$\mathcal{H} = \iiint_{D_t} \boldsymbol{K}^T \boldsymbol{\Omega}\, dD$$

keeps a constant value along the motion.
The results remain true if we replace $\boldsymbol{\Omega}$ by the vectors $\boldsymbol{E}_i$, $i = 1, 2, 3$.

The proof is an immediate application of Theorem 6.

Now, we consider applications to the Serre–Green–Naghdi equations. The SGN equations were derived in [5, 31, 32]. The equations represent a two–dimensional asymptotic reduction of full free–surface Euler equations for long gravity waves. A mathematical justification of this model is presented in [33, 37]. Its numerical study can be found in [38, 39, 40, 41]. The corresponding potential W is given in [5, 19, 29, 30, 42]

$$W(h, \dot{h}) = \frac{g\, h^2}{2} - \frac{h}{6}\left(\frac{Dh}{Dt}\right)^2 \quad \text{with} \quad \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + \nabla h \cdot \boldsymbol{u},$$

where g is the constant gravity acceleration, h is the water depth and $\boldsymbol{u}$ is the depth averaged horizontal velocity. For the SGN equations, we have only to replace ρ by the water depth h in potential $W\left(\rho, \dfrac{D\rho}{Dt}\right)$. The water depth h and depth averaged velocity $\boldsymbol{u}$ are functions of time t and of the horizontal coordinates $\boldsymbol{x} = (x^1, x^2)^T$. The vector $\boldsymbol{K}$ defined by (5.15) can be written in the form

$$\boldsymbol{K} = \boldsymbol{u} - \frac{1}{h}\nabla\left(h\,\frac{\partial W}{\partial\left(\dfrac{Dh}{Dt}\right)}\right) = \boldsymbol{u} - \frac{1}{3h}\nabla\left(h^2\,\frac{Dh}{Dt}\right) = \boldsymbol{u} + \frac{1}{3h}\nabla\left(h^3 \operatorname{div}\boldsymbol{u}\right).$$

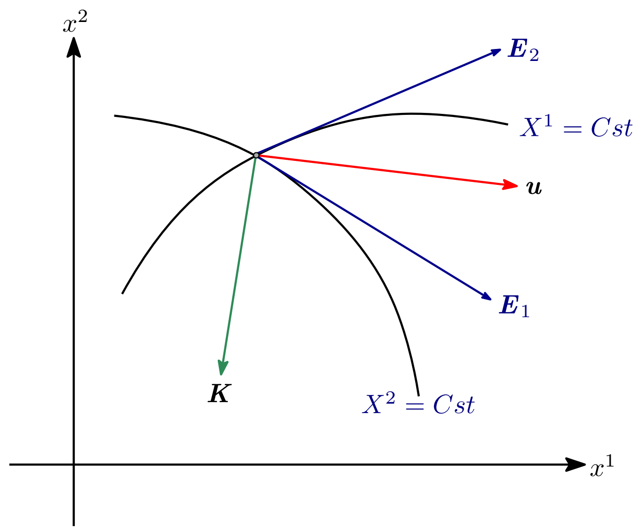

Figure 1: The material curves $X^1(t, x^1, x^2) = X^{10}$ and $X^2(t, x^1, x^2) = X^{20}$ corresponding to coordinate lines $X^1 = X^{10} = Cst$ and $X^2 = X^{20} = Cst$ are drawn. At any point, these curves are tangent to the vectors $\boldsymbol{E}_2$ and $\boldsymbol{E}_1$. *A priori*, vectors $\boldsymbol{u}$ and $\boldsymbol{K}$ are not collinear.

The physical meaning of $\boldsymbol{K}$ is quite interesting : $\boldsymbol{K}$ is the fluid velocity tangent to the free surface [43, 44]. Since $\sigma = \dfrac{h^2}{3}\dfrac{Dh}{Dt}$, the energy expressed in terms of h and σ is

$$\tilde{E}(h,\sigma) = W - \frac{Dh}{Dt}\left(\frac{\partial W}{\partial\left(\dfrac{Dh}{Dt}\right)}\right) = \frac{g\,h^2}{2} + \frac{9\,\sigma^2}{6h^3}$$

Hence

$$\tilde{E}_h(h,\sigma) = g\,h - \frac{9\,\sigma^2}{2\,h^4} = g\,h - \frac{1}{2}\left(\frac{Dh}{Dt}\right)^2$$

and the equation for $\boldsymbol{K}$ is

$$\frac{D\boldsymbol{K}}{Dt} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{K} + \nabla\left(g\,h - \frac{1}{2}\left(\frac{Dh}{Dt}\right)^2 - \frac{|\boldsymbol{u}|^2}{2}\right) = \boldsymbol{0}.$$

If theorem 8 is applied with the vorticity vector $\boldsymbol{\Omega}$, the result is trivial because $\boldsymbol{\Omega}$ and $\boldsymbol{K}$ are orthogonal. However, the result is not trivial if we use, instead of vector $\boldsymbol{\Omega}$, the column vectors $\boldsymbol{E}_i$, $i = 1, 2$, of the matrix $\boldsymbol{F}/\det \boldsymbol{F}$

$$\frac{\partial}{\partial t}\left(\boldsymbol{K}^T\boldsymbol{E}_i\right) + \operatorname{div}\left(\boldsymbol{u}\left(\boldsymbol{K}^T\boldsymbol{E}_i\right) + \left(g\,h - \frac{1}{2}\left(\frac{Dh}{Dt}\right)^2 - \frac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{E}_i\right) = 0.$$

This equation can be interpreted as conservation laws for the covariant components of $\boldsymbol{K}$ in the basis $\boldsymbol{E}_i$, $i = 1, 2$ (see Fig 1).

6 Conclusion and discussion

We have found new conservation laws for the equations of dispersive fluids where the energy depends on the density and gradient of density (capillary fluids), or depends on the density and material time derivative of the density (fluids with internal inertia). These conservation laws keep constant generalized helicity.

We have also found new integrals representing the variation of the components of velocity $\boldsymbol{u}$ (for capillary fluids) or of vector $\boldsymbol{K}$ (for fluids with internal inertia) in a time–dependent curvilinear basis $\boldsymbol{E}_i$, $i = 1, 2, 3$. These divergence–free basis vectors are tangent at each time to material curves along which coordinate X^i varies and other coordinates X^j, $j \neq i$ are fixed.

Other invariants can be found in the case when the potential W depends on additional scalar transported by the fluid such as the specific entropy η. Indeed, η satisfies the equation

$$\frac{D\eta}{Dt} = 0. \tag{6.17}$$

The analog of Ertel's invariant is

$$\frac{D}{Dt}\left(\frac{\partial \eta}{\partial \boldsymbol{x}}\frac{\boldsymbol{L}}{\rho}\right) = 0$$

where $\boldsymbol{L}$ is either the scaled basis vectors $\boldsymbol{E}_i$, $i = 1, 2, 3$ or the *generalized* vorticity vector. Indeed, even if the equation for the vorticity contains an extra term related to the entropy gradient, it vanishes by combining the Helmholtz equation for the *generalized* vorticity with the Lie derivative of covector $\dfrac{\partial \eta}{\partial \boldsymbol{x}}$ coming from (6.17) (see Appendix A for details)

$$d_{L_1}\left(\frac{\partial \eta}{\partial \boldsymbol{x}}\right) \equiv \frac{D}{Dt}\left(\frac{\partial \eta}{\partial \boldsymbol{x}}\right) + \frac{\partial \eta}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} = 0.$$

In the same way, higher order conservation laws can be found as it was already done for the classical dispersionless equations (Euler equations of compressible and incompressible fluids, equations of magnetohydrodynamics, two-fluid plasma equations etc.) [7, 9, 10, 11]. However, the physical and geometrical meaning of these conservation laws still remains to be done.

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A Lie derivatives of one-form and two-form fields

In the literature, one generally uses the notation $\mathcal{L}_{\boldsymbol{v}}$ where $\boldsymbol{v}$ is a vector field. However, the analytical expression of the Lie derivative is different for scalars, one–forms, two–forms and so on. This is why we denote d_{L_1} for the Lie derivative of a one–form and d_{L_2} for the Lie derivative of a two–form [45].

We denote by $T_{\boldsymbol{x}}(\mathcal{D}_t)$ and $T_{\boldsymbol{x}}^{\star}(\mathcal{D}_t)$ the tangent and cotangent vector spaces at any point $\boldsymbol{x}$ of a material domain $\mathcal{D}_t$. A vector field $\boldsymbol{J} \in T_{\boldsymbol{x}}(\mathcal{D}_t)$ has the inverse image $\boldsymbol{J}_0 \in T_{\boldsymbol{X}}(\mathcal{D}_0)$ such that

$$\boldsymbol{J}(t,\boldsymbol{x}) = \boldsymbol{F}\,\boldsymbol{J}_0(t,\boldsymbol{X}). \tag{A.18}$$

Consequently, a one-form field $\boldsymbol{C} \in T_{\boldsymbol{x}}^{\star}(\mathcal{D}_t)$ has the inverse image $\boldsymbol{C}_0 \in T_{\boldsymbol{X}}^{\star}(\mathcal{D}_0)$ such that

$$\boldsymbol{C}(t,\boldsymbol{x}) = \boldsymbol{C}_0(t,\boldsymbol{X})\,\boldsymbol{F}^{-1}.$$

The Lie derivative of $\boldsymbol{C}$ is the image of $\partial \boldsymbol{C}_0(t,\boldsymbol{X})/\partial t$ in $T_{\boldsymbol{x}}^{\star}(\mathcal{D}_t)$ and is denoted by d_{L_1}. From the evolution equation (2.1) for $\boldsymbol{F}$, the Lie derivative d_{L_1} of form field $\boldsymbol{C}$ is deduced from the diagram

$$\begin{tikzcd}
\boldsymbol{C} \in T_{\boldsymbol{x}}^{*}(\mathcal{D}_t) \arrow[r, "\boldsymbol{F}"] \arrow[d, "d_{L_1}"'] & \boldsymbol{C}_0 = \boldsymbol{C}\,\boldsymbol{F} \in T_{\boldsymbol{X}}^{*}(\mathcal{D}_o) \arrow[d, "\frac{D}{Dt}"] \\
\dfrac{D\boldsymbol{C}}{Dt} + \boldsymbol{C}\,\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \in T_{\boldsymbol{x}}^{*}(\mathcal{D}_t) & \dfrac{D\boldsymbol{C}}{Dt}\boldsymbol{F} + \boldsymbol{C}\,\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{F} \in T_{\boldsymbol{X}}^{*}(\mathcal{D}_o) \arrow[l, "\boldsymbol{F}^{-1}"']
\end{tikzcd}$$

Consequently, the Lie derivative of $\boldsymbol{C}$ is

$$d_{L_1}\boldsymbol{C} = \frac{D\boldsymbol{C}}{Dt} + \boldsymbol{C}\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \equiv \frac{\partial \boldsymbol{C}}{\partial t} + \boldsymbol{u}^T \left(\frac{\partial \boldsymbol{C}^T}{\partial \boldsymbol{x}}\right)^T + \boldsymbol{C}\,\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}.$$

A two-form field $\boldsymbol{\omega} \in T_{\boldsymbol{x}}^{2\star}(\mathcal{D}_t)$ has the inverse image $\boldsymbol{\omega}_0 \in T_{\boldsymbol{X}}^{2\star}(\mathcal{D}_0)$. There exists an isomorphism $\boldsymbol{\omega}(t,\boldsymbol{x}) \in T_{\boldsymbol{x}}^{2\star}(\mathcal{D}_t) \longrightarrow \boldsymbol{w}(t,\boldsymbol{x}) \in T_{\boldsymbol{x}}(\mathcal{D}_t)$ defined as

$$\forall\ \boldsymbol{J}_1 \text{ and } \boldsymbol{J}_2 \in T_{\boldsymbol{x}}(\mathcal{D}_t), \quad \boldsymbol{\omega}\,(\boldsymbol{J}_1,\boldsymbol{J}_2) = \det(\boldsymbol{w},\boldsymbol{J}_1,\boldsymbol{J}_2).$$

We can define a vector field $\boldsymbol{w}_0$ such that:

$$\det\,(\boldsymbol{w},\boldsymbol{J}_1,\boldsymbol{J}_2) = \det(\boldsymbol{w}_{\boldsymbol{0}},\boldsymbol{J}_{10},\boldsymbol{J}_{20})$$

where $\boldsymbol{J}_{10}$ and $\boldsymbol{J}_{20}$ are the inverse image in $T_{\boldsymbol{X}}(\mathcal{D}_0)$ of $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ which verify (A.18). Similarly to the proof of lemma 1, we obtain

$$\boldsymbol{w}\,(t,\boldsymbol{x}) = \frac{\boldsymbol{F}}{\det \boldsymbol{F}}\,\boldsymbol{w}_0\,(t,\boldsymbol{X}) \qquad \text{or} \qquad \boldsymbol{w}_0\,(t,\boldsymbol{X}) = (\det \boldsymbol{F})\,\boldsymbol{F}^{-1}\boldsymbol{w}\,(t,\boldsymbol{x}).$$

The Lie derivative d_{L_2} of $\boldsymbol{\omega}(t, \boldsymbol{x})$ is isomorphic to the image of $\dfrac{\partial \boldsymbol{w}_0(t, \boldsymbol{X})}{\partial t}$ and we deduce from (2.1)

$$\det\,(d_{L_2}\boldsymbol{w}, \boldsymbol{J}_1, \boldsymbol{J}_2) = \det\left(\frac{D\boldsymbol{w}}{Dt} + \boldsymbol{w}\,\mathrm{div}\,\boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\,\boldsymbol{w}, \boldsymbol{J}_1, \boldsymbol{J}_2\right),$$

or

$$d_{L_2}\boldsymbol{w} = \frac{D\boldsymbol{w}}{Dt} + \boldsymbol{w}\,\mathrm{div}\,\boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\,\boldsymbol{w}.$$

Let us note that $d_{L_1}\boldsymbol{C}$ and $d_{L_2}\boldsymbol{w}$ are zero if and only if $\boldsymbol{C}_0$ and $\boldsymbol{w}_0$ do not depend on time t (they depend only on $\boldsymbol{X}$). A typical example of the two-form field is the vorticity vector. Another important example is the cross product of two one-forms $\dfrac{\partial \alpha}{\partial \boldsymbol{x}} \times \dfrac{\partial \beta}{\partial \boldsymbol{x}}$, where α and β are two scalar fields [45].