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Camell Kachour

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# Globular weak $(n, \infty)$ -Transformations ( $n \in \mathbb{N}$ ) in the sense of Grothendieck

Camell Kachour

June 23, 2021

## Abstract

This article describe globular weak  $(n, \infty)$ -transformations ( $n \in \mathbb{N}$ ) in the sense of Grothendieck, i.e for each  $n \in \mathbb{N}$  we build a coherator  $\Theta_{\mathbb{M}^n}^\infty$  which sets models are globular weak  $(n, \infty)$ -transformations. A natural globular filtration emerges from these coherators.

**Keywords.** higher stacks, higher geometry, higher logic.

**Mathematics Subject Classification (2010).** 18B40, 18C15, 18C20, 18G55, 20L99, 55U35, 55P15.

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## Introduction

We start this article by defining, for each  $n \in \mathbb{N}^*$  a coherator  $\Theta_{\mathbb{M}^n}^\infty$  in the sense of Grothendieck (see [6, 12, 13, 14]), which is a specific sketch [10, 11] such that models of it are globular weak  $(n, \infty)$ -transformations.

Recently John Bourke [4] proved a conjecture of Dimitri Ara [1], which shows that globular weak  $\infty$ -categories in the sense of Batanin [2] and globular weak  $\infty$ -categories in the sense of Grothendieck [6, 13] are equivalent: more

precisely globular weak  $\infty$ -categories in the sense of Batanin are  $\mathbb{B}_C^0$ -algebras where  $\mathbb{B}_C^0$  is a specific globular operad, and globular weak  $\infty$ -categories in the sense of Grothendieck are models of a specific theory  $\Theta_{\mathbb{M}_0^\infty}$ ; it is proved in [3, 15] that the monad associated to any globular operads is strongly cartesian [15] thus for the operad  $\mathbb{B}_C^0$  it leads to a nice theory<sup>1</sup>  $\Theta_{\mathbb{B}_C^0}$  which is Morita equivalent to  $\Theta_{\mathbb{M}_0^\infty}$  [1, 4] in the sense that their category of models are equivalent.

Thus we suspect that models of the coherators  $\Theta_{\mathbb{M}^n}^\infty$  ( $n \in \mathbb{N}^*$ ) build in this article should be also weak  $(n, \infty)$ -transformations in the sense of [7, 8] where they are defined as  $\mathbb{B}_C^n$ -algebras ( $n \in \mathbb{N}^*$ ) for specific globular operads  $\mathbb{B}_C^n$ : similarly to [4], for each integers  $n \geq 1$ , the theory  $\Theta_{\mathbb{M}^n}^\infty$  and the theory  $\Theta_{\mathbb{B}_C^n}$  should be Morita equivalent.

Finally we show that  $\Theta_{\mathbb{M}^2}^\infty$ -models in Sets of dimension 2, i.e globular weak  $(2, \infty)$ -natural transformations of dimension 2, are pseudo-2-natural transformations.

The combinatorics developed in this article lead to an accurate description of the monad  $\mathbb{T}^n$  (for each  $n \in \mathbb{N}^*$ ) defined on the cartesian product of the category of globular sets with itself, which algebras are globular strict  $(n, \infty)$ -transformations; this is similar to the description of the monad  $\mathbb{S}$  on the category of precubical sets as described in [9] which algebras are cubical strict  $\infty$ -categories with connections. We shall give more precisions in a later version of this article. In particular it will show that these monads  $\mathbb{T}^n$  (for each  $n \in \mathbb{N}^*$ ) are cartesian as conjectured in [8]. We shall also use the formalism developed in [5] to show that the coherators  $\Theta_{\mathbb{M}^n}^\infty$  ( $n \in \mathbb{N}^*$ ) lead to monads on the cartesian product of the category of globular sets with itself, which algebras correspond to these globular weak  $(n, \infty)$ -transformations in the sense of Grothendieck.

**Acknowledgement.** In the memory of Alexandre Grothendieck.

## 1 Globular theories $\Theta_{\mathbb{M}^n}$ of $(n, \infty)$ -magmas ( $n \in \mathbb{N}$ ).

### 1.1 The globular $\mathbb{G}_n$ -extensions $\Theta_n$ ( $n \in \mathbb{N}$ ) and their filtration.

The  $n$ -globe categories  $\mathbb{G}_n$  ( $n \in \mathbb{N}$ ) are small categories which objects are the basic arities (or atomic arities) of operations behind the structure of globular higher transformations. Their accurate descriptions (see below) is of first importance for theoretical reasons : their *globular completions*  $\Theta_n$  ( $n \in \mathbb{N}$ ) defined just below, have as objects the arities of operations behind the structure of globular higher transformations, and the important filtration  $\Theta_\bullet$  described below is easily obtain from the filtration  $\mathbb{G}_\bullet$  below, and the universal property of the completion  $\Theta_n$  of each  $\mathbb{G}_n$ .

The 0-globe category is a small category  $\mathbb{G}_0$  which objects are formal symbols  $1(n)$  ( $n \in \mathbb{N}$ ) which arrows are generate by maps  $s_{n-1}^n, t_{n-1}^n$

$$1(0) \begin{array}{c} \xrightarrow{s_0^1} \\ \xrightarrow{t_0^1} \end{array} 1(1) \begin{array}{c} \xrightarrow{s_1^2} \\ \xrightarrow{t_1^2} \end{array} 1(2) \begin{array}{c} \xrightarrow{s_2^3} \\ \xrightarrow{t_2^3} \end{array} 1(3) \begin{array}{c} \xrightarrow{s_3^4} \\ \xrightarrow{t_3^4} \end{array} 1(4) \cdots 1(n-1) \begin{array}{c} \xrightarrow{s_{n-1}^n} \\ \xrightarrow{t_{n-1}^n} \end{array} 1(n) \cdots$$

which follows the *globular equations*  $s_{n-1}^n s_{n-2}^{n-1} = t_{n-1}^n s_{n-2}^{n-1}$  and  $t_{n-1}^n t_{n-2}^{n-1} = s_{n-1}^n t_{n-2}^{n-1}$  for all integers  $n \geq 2$ . The maps  $s_{n-1}^n$  are called *cosources* and the maps  $t_{n-1}^n$  are called *cotargets*. The composition  $s_{n-1}^n s_{n-2}^{n-1} \cdots s_p^{p+1}$  is denoted  $s_p^n$  and the composition  $t_{n-1}^n t_{n-2}^{n-1} \cdots t_p^{p+1}$  is denoted  $t_p^n$ .

**Definition 1** A  $\mathbb{G}_0$ -presheaf i.e an object of  $[\mathbb{G}; \text{Sets}]$  is called a *globular filtration*, whereas a  $\mathbb{G}_0^{\text{op}}$ -presheaf i.e an object of  $[\mathbb{G}^{\text{op}}; \text{Sets}]$  is called a *globular set*. □

Let  $\mathbb{S}^0$  the monad on  $[\mathbb{G}_0^{\text{op}}; \text{Sets}]$  of globular strict  $\infty$ -categories. The category  $(1 \downarrow \mathbb{S}^0(1))$  of elements of the presheaf  $\mathbb{S}^0(1)$  is commonly denoted  $\Theta_0$ , and plays an important role for describing models for theories for globular higher structure (for example the theory  $\Theta_{\mathbb{S}^0}$  which set-models are globular strict  $\infty$ -categories, or the theory  $\Theta_{\mathbb{W}^0}$  which set-models are globular weak  $\infty$ -categories, etc.), those based on monads on  $[\mathbb{G}_0^{\text{op}}; \text{Sets}]$ . Objects of  $\Theta_0$  are globular trees in the sense of [2] and they play the role of arities for operations inside such theories. Actually we can avoid the use of the monad  $\mathbb{S}^0$  to build  $\Theta_0$  by using specific colimits based on the globe category  $\mathbb{G}_0$ : first we consider tables  $t$  of non-negative integers:

<sup>1</sup>By *theory* here we mean a small category equipped with a chosen set of projective cones [10, 11].



We can see that  $\mathbb{G}_1$  has three subcategories denoted respectively by  $\mathbb{G}_{1,0}$ ,  $\mathbb{G}_{1,t}$  and  $\mathbb{G}_{1,1}$ , and which are connected. Also we deliberately use the same notations for the cosources and cotargets maps of these three subcategories because no confusions will occur. The notations that we used for objects of  $\mathbb{G}_1$  shall be suggestive when we will give an accurate description of the monad  $\mathbb{S}^1$  of globular strict  $\infty$ -functors.

The *2-globe category* is a small category  $\mathbb{G}_2$  which objects are formal symbols  $1(n)$ ,  $f^n(1(n))$ ,  $\xi_1$ ,  $g^n(1(n))$ ,  $2(n)$  ( $n \in \mathbb{N}$ ), which arrows are generate by maps  $s_{n-1}^n, t_{n-1}^n$  which follows the globular equations:  $s_{n-1}^n s_{n-2}^{n-1} = t_{n-1}^n s_{n-2}^{n-1}$  and  $t_{n-1}^n t_{n-2}^{n-1} = s_{n-1}^n t_{n-2}^{n-1}$  for all integers  $n \geq 2$ :

$$\begin{array}{cccccccccccc}
1(0) & \xrightarrow{s_0^1} & 1(1) & \xrightarrow{s_1^2} & 1(2) & \xrightarrow{s_2^3} & 1(3) & \xrightarrow{s_3^4} & 1(4) \cdots 1(n-1) & \xrightarrow{s_{n-1}^n} & 1(n) \cdots \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & & \xrightarrow{t_2^3} & & \xrightarrow{t_3^4} & & \xrightarrow{t_{n-1}^n} & \\
f^0(1(0)) & \xrightarrow{s_0^1} & f^1(1(1)) & \xrightarrow{s_1^2} & f^2(1(2)) & \xrightarrow{s_2^3} & f^3(1(3)) & \xrightarrow{s_3^4} & f^4(1(4)) \cdots f^{n-1}(1(n-1)) & \xrightarrow{s_{n-1}^n} & f^n(1(n)) \cdots \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & & \xrightarrow{t_2^3} & & \xrightarrow{t_3^4} & & \xrightarrow{t_{n-1}^n} & \\
& \searrow s_0^1 & & & & & & & & & \\
& & & & & & & & & & \xi_1 \\
& & & & & & & & & & \nearrow t_0^1 \\
g^0(1(0)) & \xrightarrow{s_0^1} & g^1(1(1)) & \xrightarrow{s_1^2} & g^2(1(2)) & \xrightarrow{s_2^3} & g^3(1(3)) & \xrightarrow{s_3^4} & g^4(1(4)) \cdots g^{n-1}(1(n-1)) & \xrightarrow{s_{n-1}^n} & g^n(1(n)) \cdots \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & & \xrightarrow{t_2^3} & & \xrightarrow{t_3^4} & & \xrightarrow{t_{n-1}^n} & \\
2(0) & \xrightarrow{s_0^1} & 2(1) & \xrightarrow{s_1^2} & 2(2) & \xrightarrow{s_2^3} & 2(3) & \xrightarrow{s_3^4} & 2(4) \cdots 2(n-1) & \xrightarrow{s_{n-1}^n} & 2(n) \cdots \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & & \xrightarrow{t_2^3} & & \xrightarrow{t_3^4} & & \xrightarrow{t_{n-1}^n} & 
\end{array}$$

The *3-globe category* is a small category  $\mathbb{G}_3$  which objects are formal symbols  $1(n)$ ,  $f^n(1(n))$ ,  $\alpha_1(1(0))$ ,  $\xi_2(1(0))$ ,  $\beta_1(1(0))$ ,  $g^n(1(n))$ ,  $2(n)$  ( $n \in \mathbb{N}$ ), which arrows are generate by maps  $s_{n-1}^n, t_{n-1}^n$  which follows the globular equations:  $s_{n-1}^n s_{n-2}^{n-1} = t_{n-1}^n s_{n-2}^{n-1}$  and  $t_{n-1}^n t_{n-2}^{n-1} = s_{n-1}^n t_{n-2}^{n-1}$  for all integers  $n \geq 2$ :

$$\begin{array}{cccccccccccc}
1(0) & \xrightarrow{s_0^1} & 1(1) & \xrightarrow{s_1^2} & 1(2) & \xrightarrow{s_2^3} & 1(3) & \xrightarrow{s_3^4} & 1(4) \cdots 1(n-1) & \xrightarrow{s_{n-1}^n} & 1(n) \cdots \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & & \xrightarrow{t_2^3} & & \xrightarrow{t_3^4} & & \xrightarrow{t_{n-1}^n} & \\
f^0(1(0)) & \xrightarrow{s_0^1} & f^1(1(1)) & \xrightarrow{s_1^2} & f^2(1(2)) & \xrightarrow{s_2^3} & f^3(1(3)) & \xrightarrow{s_3^4} & f^4(1(4)) \cdots f^{n-1}(1(n-1)) & \xrightarrow{s_{n-1}^n} & f^n(1(n)) \cdots \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & & \xrightarrow{t_2^3} & & \xrightarrow{t_3^4} & & \xrightarrow{t_{n-1}^n} & \\
& \searrow t_0^1 & & & & & & & & & \\
& & & & & & & & & & \alpha_1(1(0)) \\
& & & & & & & & & & \nearrow s_0^1 \\
& & & & & & & & & & \searrow s_1^2 \\
& & & & & & & & & & \xi_2(1(0)) \\
& & & & & & & & & & \nearrow t_1^2 \\
& & & & & & & & & & \searrow t_0^1 \\
& & & & & & & & & & \beta_1(1(0)) \\
& & & & & & & & & & \nearrow s_0^1 \\
g^0(1(0)) & \xrightarrow{s_0^1} & g^1(1(1)) & \xrightarrow{s_1^2} & g^2(1(2)) & \xrightarrow{s_2^3} & g^3(1(3)) & \xrightarrow{s_3^4} & g^4(1(4)) \cdots g^{n-1}(1(n-1)) & \xrightarrow{s_{n-1}^n} & g^n(1(n)) \cdots \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & & \xrightarrow{t_2^3} & & \xrightarrow{t_3^4} & & \xrightarrow{t_{n-1}^n} & \\
2(0) & \xrightarrow{s_0^1} & 2(1) & \xrightarrow{s_1^2} & 2(2) & \xrightarrow{s_2^3} & 2(3) & \xrightarrow{s_3^4} & 2(4) \cdots 2(n-1) & \xrightarrow{s_{n-1}^n} & 2(n) \cdots \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & & \xrightarrow{t_2^3} & & \xrightarrow{t_3^4} & & \xrightarrow{t_{n-1}^n} & 
\end{array}$$



$$\begin{array}{ccccccc}
D^{a_{i_1}} & & D^{a_{i_2}} & & \dots & & D^{a_{i_{k-1}}} & & D^{a_{i_k}} \\
\swarrow & & \swarrow & \searrow & & & \swarrow & \searrow & \swarrow \\
t_{i'_1}^{i_1} & & s_{i'_1}^{i_2} & & & & t_{i'_{k-1}}^{i_{k-1}} & & s_{i'_{k-1}}^{i_k} \\
D^{a_{i'_1}} & & D^{a_{i'_2}} & & \dots & & D^{a_{i'_{k-1}}} & & D^{a_{i'_k}}
\end{array}$$

Following the terminology of Grothendieck, such colimits are called  $\mathbb{G}_n$ -globular sums.

A morphism of globular  $\mathbb{G}_n$ -extensions, also called *globular  $\mathbb{G}_n$ -functor*, is given by a commutative triangle in  $\mathbb{CAT}$ :

$$\begin{array}{ccc}
& & \mathcal{C} \\
& \nearrow F & \downarrow H \\
\mathbb{G}_n & & \mathcal{C}' \\
& \searrow F' &
\end{array}$$

such that the functor  $H$  preserves  $\mathbb{G}_n$ -globular sums. The category of globular  $\mathbb{G}_n$ -extensions is denoted  $\mathbb{G}_n\text{-Ext}$ . In fact this category has an initial object denoted  $\mathbb{G}_n \xrightarrow{i} \Theta_n$ . And the small category  $\Theta_n$  can be described as the full subcategory of  $[\mathbb{G}^{op} + \mathbb{G}^{op}; \text{Sets}]$  which objects are globular  $\mathbb{G}_n$ -trees. In particular this small category  $\Theta_n$  is the basic inductive sketch we shall need to describe coherators which set models are globular weak  $(n, \infty)$ -transformations ( $n \in \mathbb{N}$ ) (see 2.4).

A globular  $\mathbb{G}_n$ -theory is given by a globular  $\mathbb{G}_n$ -extension  $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$  such that the unique induced functor  $\overline{F}$  which makes commutative the diagram:

$$\begin{array}{ccc}
& & \Theta_n \\
& \nearrow i & \downarrow \overline{F} \\
\mathbb{G}_n & & \mathcal{C} \\
& \searrow F &
\end{array}$$

induces a bijection between objects of  $\Theta_n$  and objects of  $\mathcal{C}$ . The full subcategory of  $\mathbb{G}_n\text{-Ext}$  which objects are globular  $\mathbb{G}_n$ -theories is denoted  $\mathbb{G}_n\text{-Th}$ . Consider an object  $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$  of  $\mathbb{G}_n\text{-Th}$ , in particular it induces the globular  $\mathbb{G}_n$ -functor  $\Theta_n \xrightarrow{\overline{F}} \mathcal{C}$  as just above, which is a bijection on objects. A  $\text{Sets}$ -model of  $(F, \mathcal{C})$  or for  $\mathcal{C}$  for short, is given by a functor  $\mathcal{C} \xrightarrow{X} \text{Sets}$ , such that the functor  $X \circ \overline{F}$ :

$$\Theta_n \xrightarrow{\overline{F}} \mathcal{C} \xrightarrow{X} \text{Sets}$$

sends globular  $\mathbb{G}_n$ -sums to globular  $\mathbb{G}_n$ -products<sup>2</sup>, thus for all objects  $t$  of  $\Theta_n$ :

$$\left( \begin{array}{ccccccccc}
a_{i_1} & & a_{i_2} & & a_{i_3} & \cdot & \cdot & \cdot & a_{i_{k-1}} & & a_{i_k} \\
& & & & & & & & & & \\
& & a_{i'_1} & & a_{i'_2} & \cdot & \cdot & \cdot & \cdot & & a_{i'_{k-1}}
\end{array} \right)$$

we have:

$$\begin{aligned}
X(\overline{F}(t)) &= X \left( \text{colim} \left( \begin{array}{ccccccc}
D^{a_{i_1}} & & D^{a_{i_2}} & & \dots & & D^{a_{i_{k-1}}} & & D^{a_{i_k}} \\
\swarrow & & \swarrow & \searrow & & & \swarrow & \searrow & \swarrow \\
t_{i'_1}^{i_1} & & s_{i'_1}^{i_2} & & & & t_{i'_{k-1}}^{i_{k-1}} & & s_{i'_{k-1}}^{i_k} \\
D^{a_{i'_1}} & & D^{a_{i'_2}} & & \dots & & D^{a_{i'_{k-1}}} & & D^{a_{i'_k}}
\end{array} \right) \right) \\
&= X \left( \left( D^{a_{i_1}}, t_{i'_1}^{i_1} \right) \prod_{D^{a_{i'_1}}} (s_{i'_1}^{i_2}, D^{a_{i_2}}, t_{i'_2}^{i_2}) \prod_{D^{a_{i'_2}}} \dots \prod_{D^{a_{i'_{k-1}}}} (s_{i'_{k-1}}^{i_k}, D^{a_{i_k}}) \right) \\
&\simeq X(D^{a_{i_1}}) \times_{X(D^{a_{i'_1}})} \dots \times_{X(D^{a_{i'_{k-1}}})} X(D^{a_{i_k}})
\end{aligned}$$

<sup>2</sup>Globular  $\mathbb{G}_n$ -products are just dual to globular  $\mathbb{G}_n$ -sums.

The category of Sets-models of  $\mathcal{C}$  is the full subcategory of the category of presheaves  $[\mathcal{C}, \text{Sets}]$  which objects are Sets-models of  $\mathcal{C}$ , and it is denoted  $\text{Mod}(\mathcal{C})$ .

Now it is easy to see that we get the following globular filtration:

$$\begin{array}{ccccccccccccccc}
\mathbb{G}_0 & \xrightarrow{s_0^1} & \mathbb{G}_1 & \xrightarrow{s_1^2} & \mathbb{G}_2 & \xrightarrow{s_2^3} & \mathbb{G}_3 & \xrightarrow{s_3^4} & \mathbb{G}_4 \cdots \mathbb{G}_{n-1} & \xrightarrow{s_{n-1}^n} & \mathbb{G}_n \cdots \\
\downarrow & \xrightarrow{t_0^1} & \downarrow & \xrightarrow{t_1^2} & \downarrow & \xrightarrow{t_2^3} & \downarrow & \xrightarrow{t_3^4} & \downarrow & \xrightarrow{t_{n-1}^n} & \downarrow \\
\Theta_0 & \xrightarrow{s_0^1} & \Theta_1 & \xrightarrow{s_1^2} & \Theta_2 & \xrightarrow{s_2^3} & \Theta_3 & \xrightarrow{s_3^4} & \Theta_4 \cdots \Theta_{n-1} & \xrightarrow{s_{n-1}^n} & \Theta_n \cdots \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & & \xrightarrow{t_2^3} & & \xrightarrow{t_3^4} & & \xrightarrow{t_{n-1}^n} & 
\end{array}$$

## 1.2 The theories $\Theta_{\mathbb{M}^n}$ for $(n, \infty)$ -magmas $(n \in \mathbb{N})$ and their filtration.

A  $(0, \infty)$ -magma  $M$  is given by a globular set  $\mathbb{G}^{\text{op}} \xrightarrow{M} \text{Sets}$  equipped with operations  $M_n \times_{M_p} M_n \xrightarrow{\circ_p^n} M_n$  for all  $n \geq 1$  and all  $0 \leq p \leq n-1$  such that :

- for  $0 \leq p < q < m$ ,  $s_q^m(y \circ_p^m x) = s_q^m(y) \circ_p^q s_q^m(x)$  and  $t_q^m(y \circ_p^m x) = t_q^m(y) \circ_p^q t_q^m(x)$
- for  $0 \leq q < p < m$ ,  $s_q^m(y \circ_p^m x) = s_q^m(y) = s_q^m(x)$  and  $t_q^m(y \circ_p^m x) = t_q^m(y) = t_q^m(x)$
- for  $0 \leq p = q < m$ ,  $s_q^m(y \circ_p^m x) = s_q^m(x)$  and  $t_q^m(y \circ_p^m x) = t_q^m(x)$

A  $(1, \infty)$ -magma  $M \xrightarrow{F} M'$  between two  $(0, \infty)$ -magma  $M$  and  $M'$  is a morphism of globular sets. If  $N \xrightarrow{G} N'$  is another  $(1, \infty)$ -magma, then a strict morphism between them:  $F \xrightarrow{(h, h')} G$  is given by two morphisms of  $(0, \infty)$ -magma:  $M \xrightarrow{h} N$  and  $M' \xrightarrow{h'} N'$  such that the following diagram is commutative in  $\text{Glob}$ :

$$\begin{array}{ccc}
M & \xrightarrow{F} & M' \\
h \downarrow & & \downarrow h' \\
N & \xrightarrow{G} & N'
\end{array}$$

The category of  $(1, \infty)$ -magmas is denoted  $(1, \infty)\text{-Mag}$ .

A  $(2, \infty)$ -magma  $M \begin{array}{c} \xrightarrow{F} \\ \tau \downarrow \\ \xrightarrow{G} \end{array} M'$  is given by two objects  $M \xrightarrow{F} M'$  and  $M \xrightarrow{G} M'$  of  $(1, \infty)\text{-Mag}$  and

a morphism:  $M(0) \xrightarrow{\tau} M'(1)$  of Sets such that for all  $a \in M(0)$ ,  $s_0^1(\tau(a)) = F(a)$  and  $t_0^1(\tau(a)) = G(a)$ . If

$$\begin{array}{ccc}
N & \xrightarrow{H} & N' \\
\rho \downarrow & & \downarrow \rho \\
N & \xrightarrow{K} & N'
\end{array}$$

is another  $(2, \infty)$ -magma, then a strict morphism between them:  $\tau \xrightarrow{(h, h')} \rho$  is given by two morphisms

of  $(0, \infty)$ -magmas:  $M \xrightarrow{h} N$  and  $M' \xrightarrow{h'} N'$  such that for all  $a \in M(0)$  we have:  $h'(\tau(a)) = \rho(h(a))$ . The category of  $(2, \infty)$ -magmas is denoted  $(2, \infty)\text{-Mag}$ .

For  $n \geq 3$  we define  $(n, \infty)$ -magmas by induction: suppose the categories  $(k, \infty)\text{-Mag}$  of  $(k, \infty)$ -magmas are defined for all  $k \in \llbracket 2, n-1 \rrbracket$ . An  $(n, \infty)$ -magma  $\alpha \xrightarrow{\xi} \beta$  between the  $(n-1, \infty)$ -magmas  $\alpha$  and  $\beta$  is given by a morphism

$M(0) \xrightarrow{\xi} M'(n)$  in Sets such that for all  $a \in M(0)$ ,  $s_0^n(\xi(a)) = F(a)$  and  $t_0^n(\xi(a)) = G(a)$ . A morphism between

two  $(n, \infty)$ -magmas  $\xi \xrightarrow{(h, h')} \xi'$  is given by two morphisms of  $(0, \infty)$ -magmas:  $M \xrightarrow{h} N$  and  $M' \xrightarrow{h'} N'$  such that for all  $a \in M(0)$  we have:  $h'(\xi(a)) = \xi'(h(a))$ . The category of  $(n, \infty)$ -magmas is denoted  $(n, \infty)\text{-Mag}$ .

Now for all  $n \geq 1$  we have a monadic forgetful functor  $U^n$ :



$$\begin{array}{c}
(n, \infty)\text{-Mag} \\
\uparrow \downarrow \\
F^n \dashv U^n \\
\downarrow \\
\text{Glob}^2
\end{array}$$

from the category  $(n, \infty)\text{-Mag}$  of globular  $(n, \infty)$ -magmas to the category  $\text{Glob}^2$  of pairs of globular sets, which sends  $(n, \infty)$ -magmas  $\xi$  as above to the pair of globular sets  $(U(M), U(M'))$ . The notation  $U(M)$  means that we have forgotten the underlying structure of  $(0, \infty)$ -magma of  $M$ . We use the equivalence  $\text{Glob}^2 \simeq [\mathbb{G}_0^{op} + \mathbb{G}_0^{op}; \text{Sets}]$  to get a monad  $\mathbb{M}^n = (M^n, \eta^n, \mu^n)$  on  $[\mathbb{G}_0^{op} + \mathbb{G}_0^{op}; \text{Sets}]$ . We have the equivalence of categories  $(n, \infty)\text{-Mag} \simeq \mathbb{M}^n\text{-Alg}$  because  $U^n$  is monadic. The full subcategory  $\Theta_{\mathbb{M}^n} \subset \text{Kl}(\mathbb{M}^n)$  of the Kleisli category of  $\mathbb{M}^n$  which objects are  $\mathbb{G}_n$ -trees is called the theory of globular  $(n, \infty)$ -magmas. This is our basic example of globular  $\mathbb{G}_n$ -theory. In fact we have the following equivalences of categories:

$$(n, \infty)\text{-Mag} \simeq \mathbb{M}^n\text{-Alg} \simeq \text{Mod}(\Theta_{\mathbb{M}^n})$$

Now it is easy to see that we get the following globular filtration:

$$\begin{array}{cccccccccccc}
\mathbb{G}_0 & \xrightarrow{s_0^1} & \mathbb{G}_1 & \xrightarrow{s_1^2} & \mathbb{G}_2 & \xrightarrow{s_2^3} & \mathbb{G}_3 & \xrightarrow{s_3^4} & \mathbb{G}_4 \cdots \mathbb{G}_{n-1} & \xrightarrow{s_{n-1}^n} & \mathbb{G}_n \cdots \\
\downarrow & \xrightarrow{t_0^1} & \downarrow & \xrightarrow{t_1^2} & \downarrow & \xrightarrow{t_2^3} & \downarrow & \xrightarrow{t_3^4} & \downarrow & \xrightarrow{t_{n-1}^n} & \downarrow \\
\Theta_0 & \xrightarrow{s_0^1} & \Theta_1 & \xrightarrow{s_1^2} & \Theta_2 & \xrightarrow{s_2^3} & \Theta_3 & \xrightarrow{s_3^4} & \Theta_4 \cdots \Theta_{n-1} & \xrightarrow{s_{n-1}^n} & \Theta_n \cdots \\
\downarrow & \xrightarrow{t_0^1} & \downarrow & \xrightarrow{t_1^2} & \downarrow & \xrightarrow{t_2^3} & \downarrow & \xrightarrow{t_3^4} & \downarrow & \xrightarrow{t_{n-1}^n} & \downarrow \\
\Theta_{\mathbb{M}^0} & \xrightarrow{s_0^1} & \Theta_{\mathbb{M}^1} & \xrightarrow{s_1^2} & \Theta_{\mathbb{M}^2} & \xrightarrow{s_2^3} & \Theta_{\mathbb{M}^3} & \xrightarrow{s_3^4} & \Theta_{\mathbb{M}^4} \cdots \Theta_{\mathbb{M}^{n-1}} & \xrightarrow{s_{n-1}^n} & \Theta_{\mathbb{M}^n} \cdots \\
\downarrow & \xrightarrow{t_0^1} & \downarrow & \xrightarrow{t_1^2} & \downarrow & \xrightarrow{t_2^3} & \downarrow & \xrightarrow{t_3^4} & \downarrow & \xrightarrow{t_{n-1}^n} & \downarrow
\end{array}$$

## 2 Globular $\mathbb{G}_n$ -coherators ( $n \in \mathbb{N}$ )

### 2.1 Admissibility

For a fixed integer  $n \geq 1$ , let  $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$  be a globular  $\mathbb{G}_n$ -theory, i.e an object of  $\mathbb{G}_n\text{-Th}$ ; here we pay attention to the inclusion:  $\mathbb{G}_0 + \mathbb{G}_0 \hookrightarrow \mathbb{G}_n$ . An object  $c_m$  of  $\mathbb{G}_0 + \mathbb{G}_0$  can be the object  $1(m)$  or the object  $2(m)$  for all  $m \in \mathbb{N}$ ; two arrows:  $D^{c_m} \xrightarrow{f} t$  in  $\mathcal{C}$  are parallels if  $fs_{m-1}^m = gs_{m-1}^m$  and  $ft_{m-1}^m = gt_{m-1}^m$ :

$$\begin{array}{ccc}
D^{c_m} & \xrightarrow{f} & t \\
\uparrow & \xrightarrow{g} & \uparrow \\
D^{c_{m-1}} & & D^{c_{m-1}}
\end{array}$$

Consider a couple  $(f, g)$  of parallels arrows in  $\mathcal{C}$  as just above. We say that it is admissible or algebraic if they don't belong to the image of the globular  $\mathbb{G}_n$ -functor  $\overline{F}$ :

$$\begin{array}{ccc}
& & \Theta_n \\
& \nearrow i & \downarrow \overline{F} \\
\mathbb{G}_n & & \mathcal{C} \\
& \searrow F &
\end{array}$$

Consider a couple  $(f, g)$  of arrows of  $\mathcal{C}$  which is admissible as just above; a lifting of  $(f, g)$  is given by an arrow  $h$  :

$$\begin{array}{ccc}
 & D^{c_{m+1}} & \\
 s_m^{m+1} \uparrow & \uparrow & \nearrow h \\
 & D^{c_m} & \xrightarrow{f} t \\
 & \xrightarrow{g} & \searrow
 \end{array}$$

such that  $hs_m^{m+1} = f$  and  $ht_m^{m+1} = g$

## 2.2 Batanin-Grothendieck Sequences

We now define the Batanin-Grothendieck sequence<sup>3</sup> associated to a globular  $\mathbb{G}_n$ -theory  $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$  . We build it by the following induction:

- If  $m = 0$  we start with the couple  $(\mathcal{C}, E)$  where  $E$  denotes the set of admissible pairs of arrows of  $\mathcal{C}$ ; we shall write  $(\mathcal{C}_0, E_0) = (\mathcal{C}, E)$  this first step.
- If  $m = 1$  we consider then the couple  $(\mathcal{C}_1, E_1)$  where  $\mathcal{C}_1$  is obtained by formally adding in  $\mathcal{C}_0 = \mathcal{C}$  the liftings of all elements  $(f, g) \in E_0 = E$ , and  $E_1$  is the set of admissible couples of arrows in  $\mathcal{C}_1$  which are not elements of the set  $E_0$ ;
- If for  $m \geq 2$  the couple  $(\mathcal{C}_m, E_m)$  is well defined then  $\mathcal{C}_{m+1}$  is obtained by formally adding in  $\mathcal{C}_m$  the liftings of all elements of  $E_m$ , and  $E_{m+1}$  is the set of couples of arrows of  $\mathcal{C}_{m+1}$  which are not elements of  $E_m$

We give a slightly different but equivalent induction to build the Batanin-Grothendieck sequence for such globular theory  $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$  :

- If  $m = 0$  we start with the couple  $(\mathcal{C}, E)$  where  $E$  is the set of couple of arrow which are admissible of  $\mathcal{C}$ ; we denote  $E = E_0 = E'_0 = E'_0 \setminus \emptyset$  (we shall see soon the reason of these notations), and  $\mathcal{C}_0 = \mathcal{C}$ ;
- If  $m = 1$  we consider the couple  $(\mathcal{C}_1, E_1)$  where  $\mathcal{C}_1$  is obtained by formally adding in  $\mathcal{C}_0$  all liftings of the elements  $(f, g) \in E_0$ ,  $E'_1$  is the set of all pairs of arrows which are admissible in  $\mathcal{C}_1$ , and  $E_1 = E'_1 \setminus E_0$ ; remark that  $E_0 = E'_0 \subset E'_1$ ;
- If  $m = 2$  we consider the couple  $(\mathcal{C}_2, E_2)$  where  $\mathcal{C}_2$  is obtained by formally adding in  $\mathcal{C}_1$  all liftings of the elements  $(f, g) \in E_1$ ,  $E'_2$  is the set of all pairs of arrows which are admissible in  $\mathcal{C}_2$ , and  $E_2 = E'_2 \setminus E'_1$ ;
- For  $m \geq 3$  we suppose that the couple  $(\mathcal{C}_m, E_m)$  is well defined with  $E_m = E'_m \setminus E'_{m-1}$ , then  $\mathcal{C}_{m+1}$  is obtained by formally adding in  $\mathcal{C}_m$  all liftings of the elements  $(f, g) \in E_m$ ,  $E'_{m+1}$  is the set of all pairs of arrows which are admissible in  $\mathcal{C}_{m+1}$ , and  $E_{m+1} = E'_{m+1} \setminus E'_m$ ;

The Batanin-Grothendieck sequence of the globular theory  $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$  produces the following filtered diagram:

$$(\mathbb{N}, \leq) \xrightarrow{\mathcal{C}_\bullet} \mathbb{G}_n\text{-Th} \text{ in the category } \mathbb{G}_n\text{-Th}$$

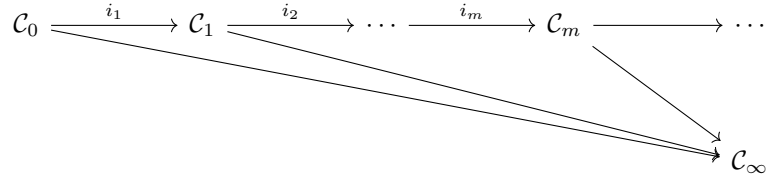
$$\mathcal{C}_0 \xrightarrow{i_1} \mathcal{C}_1 \xrightarrow{i_2} \dots \xrightarrow{i_m} \mathcal{C}_m \longrightarrow \dots$$

## 2.3 $\mathbb{G}_n$ -coherators for globular $\mathbb{G}_n$ -theories ( $n \in \mathbb{N}$ )

Let us fixed an integer  $n \geq 1$ . We start with datas of the previous subsection, i.e with the Batanin-Grothendieck sequence  $(\mathbb{N}, \leq) \xrightarrow{\mathcal{C}_\bullet} \mathbb{G}_n\text{-Th}$  for a globular  $\mathbb{G}_n$ -theory  $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$  :

<sup>3</sup>Coherators associated to such sequence are called of *Batanin-Leinster type* by some authors.

**Definition 2** The colimit  $\mathbb{G}_n \xrightarrow{F_\infty} \mathcal{C}_\infty$  of the previous filtered diagram  $\mathcal{C}_\bullet$  :



is called the globular  $\mathbb{G}_n$ -coherator of the type Batanin-Grothendieck associated to the globular  $\mathbb{G}_n$ -theory  $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$   $\square$

For shorter terminology we shall say that  $\mathbb{G}_n \xrightarrow{F_\infty} \mathcal{C}_\infty$  is the  $\mathbb{G}_n$ -coherator associated to the globular  $\mathbb{G}_n$ -theory  $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$  . It is straightforward to see that the Batanin-Grothendieck construction of  $\mathbb{G}_n$ -coherators associated to globular  $\mathbb{G}_n$ -theory is functorial, and the following functor  $\Phi_n$  is called the Batanin-Grothendieck functor:

$$\begin{array}{ccc} \mathbb{G}_n\text{-Th} & \xrightarrow{\Phi_n} & \mathbb{G}_n\text{-Th} \\ \mathcal{C} & \longmapsto & \mathcal{C}^\infty \end{array}$$

## 2.4 The $\mathbb{G}_n$ -coherators $\Theta_{\mathbb{M}^n}^\infty$ ( $n \in \mathbb{N}$ )

Let us fixed an integer  $n \geq 1$ . The  $\mathbb{G}_n$ -coherator associated to the globular theory  $\mathbb{G}_n \xrightarrow{j} \Theta_{\mathbb{M}^n}$  that we obtain with the composition:

$$\mathbb{G}_n \xrightarrow{i} \Theta_n \hookrightarrow \Theta_{\mathbb{M}^n}$$

is denoted  $\Theta_{\mathbb{M}^n}^\infty$ .

**Definition 3** Globular weak  $(n, \infty)$ -transformations are objects of the category  $\text{Mod}(\Theta_{\mathbb{M}^n}^\infty)$ .  $\square$

Morphisms in  $\Theta_{\mathbb{M}^n}^\infty$  which doesn't belong to the image of  $\Theta_n \hookrightarrow \Theta_{\mathbb{M}^n}^\infty$  are called *algebraic morphisms* of  $\Theta_{\mathbb{M}^n}^\infty$ . Morphisms in  $\Theta_{\mathbb{M}^n}^\infty$  of the form:  $1(p) \xrightarrow{K} t$  or  $2(p) \xrightarrow{K} t$ , where  $t$  are any  $\mathbb{G}_n$ -trees, are the *p-operations* of  $\Theta_{\mathbb{M}^n}^\infty$ .

**Definition 4** If  $\tau \in \text{Mod}(\Theta_{\mathbb{M}^n}^\infty)$  and  $p \geq 1$  is an integer, then  $\dim(\tau) = p$  if for all  $q > p$ , any  $q$ -operations  $K$  in  $\Theta_{\mathbb{M}^n}^\infty$  has its source and target which are equalize by  $\tau$ . Thus if we have:

$$\begin{array}{ccc} 1(q) & & \\ \uparrow s_{q-1}^q & \uparrow t_{q-1}^q & \searrow K \\ 1(q-1) & \xrightarrow{K \circ s_{q-1}^q} & t \\ & \xrightarrow{K \circ t_{q-1}^q} & \end{array}$$

or:

$$\begin{array}{ccc} 2(q) & & \\ \uparrow s_{q-1}^q & \uparrow t_{q-1}^q & \searrow K \\ 2(q-1) & \xrightarrow{K \circ s_{q-1}^q} & t \\ & \xrightarrow{K \circ t_{q-1}^q} & \end{array},$$

then  $\tau(K \circ s_{q-1}^q) = \tau(K \circ t_{q-1}^q)$ .  $\square$

**Remark 1** In 2019 John Bourke has proved [4] the *Ara conjecture* [1] which says that the category of globular weak  $\infty$ -categories of Batanin is equivalent to the category of globular weak  $\infty$ -categories of Grothendieck:

$$\text{Mod}(\Theta_{\mathbb{B}_C^0}) \simeq \text{Mod}(\Theta_{\mathbb{M}}^\infty)$$

where here  $\mathbb{B}_C^0$  denotes the globular operad of Batanin [2] which algebras are his models of globular weak  $\infty$ -categories and  $\Theta_{\mathbb{B}_C^0}$  is its associated theory. In the same vein, for all integers  $n \geq 1$ , the category of globular weak  $(n, \infty)$ -transformations defined in [8] should be equivalent to the category of globular weak  $(n, \infty)$ -transformations with this  $\mathbb{G}_n$ -coherator  $\Theta_{\mathbb{M}^n}^\infty$ :

$$\text{Mod}(\Theta_{\mathbb{B}_C^n}) \simeq \text{Mod}(\Theta_{\mathbb{M}^n}^\infty)$$

where  $\mathbb{B}_C^n$  denotes the globular operad [8] which algebras are models of globular weak  $(n, \infty)$ -transformations and  $\Theta_{\mathbb{B}_C^n}$  is its associated theory. □

Now it is easy to see that we get the following globular filtration:

$$\begin{array}{cccccccccccc}
 \mathbb{G}_0 & \xrightarrow{s_0^1} & \mathbb{G}_1 & \xrightarrow{s_1^2} & \mathbb{G}_2 & \xrightarrow{s_2^3} & \mathbb{G}_3 & \xrightarrow{s_3^4} & \mathbb{G}_4 \cdots \mathbb{G}_{n-1} & \xrightarrow{s_{n-1}^n} & \mathbb{G}_n \cdots \\
 \downarrow & \xrightarrow{t_0^1} & \downarrow & \xrightarrow{t_1^2} & \downarrow & \xrightarrow{t_2^3} & \downarrow & \xrightarrow{t_3^4} & \downarrow & \xrightarrow{t_{n-1}^n} & \downarrow \\
 \Theta_0 & \xrightarrow{s_0^1} & \Theta_1 & \xrightarrow{s_1^2} & \Theta_2 & \xrightarrow{s_2^3} & \Theta_3 & \xrightarrow{s_3^4} & \Theta_4 \cdots \Theta_{n-1} & \xrightarrow{s_{n-1}^n} & \Theta_n \cdots \\
 \downarrow & \xrightarrow{t_0^1} & \downarrow & \xrightarrow{t_1^2} & \downarrow & \xrightarrow{t_2^3} & \downarrow & \xrightarrow{t_3^4} & \downarrow & \xrightarrow{t_{n-1}^n} & \downarrow \\
 \Theta_{\mathbb{M}^0} & \xrightarrow{s_0^1} & \Theta_{\mathbb{M}^1} & \xrightarrow{s_1^2} & \Theta_{\mathbb{M}^2} & \xrightarrow{s_2^3} & \Theta_{\mathbb{M}^3} & \xrightarrow{s_3^4} & \Theta_{\mathbb{M}^4} \cdots \Theta_{\mathbb{M}^{n-1}} & \xrightarrow{s_{n-1}^n} & \Theta_{\mathbb{M}^n} \cdots \\
 \downarrow & \xrightarrow{t_0^1} & \downarrow & \xrightarrow{t_1^2} & \downarrow & \xrightarrow{t_2^3} & \downarrow & \xrightarrow{t_3^4} & \downarrow & \xrightarrow{t_{n-1}^n} & \downarrow \\
 \Theta_{\mathbb{M}^0}^\infty & \xrightarrow{s_0^1} & \Theta_{\mathbb{M}^1}^\infty & \xrightarrow{s_1^2} & \Theta_{\mathbb{M}^2}^\infty & \xrightarrow{s_2^3} & \Theta_{\mathbb{M}^3}^\infty & \xrightarrow{s_3^4} & \Theta_{\mathbb{M}^4}^\infty \cdots \Theta_{\mathbb{M}^{n-1}}^\infty & \xrightarrow{s_{n-1}^n} & \Theta_{\mathbb{M}^n}^\infty \cdots \\
 \downarrow & \xrightarrow{t_0^1} & \downarrow & \xrightarrow{t_1^2} & \downarrow & \xrightarrow{t_2^3} & \downarrow & \xrightarrow{t_3^4} & \downarrow & \xrightarrow{t_{n-1}^n} & \downarrow
 \end{array}$$

## 2.5 Computations in dimensions 2

A natural transformation  $\tau$ :

$$\begin{array}{ccc}
 & F & \\
 X & \begin{array}{c} \curvearrowright \\ \tau \\ \curvearrowleft \end{array} & Z \\
 & H &
 \end{array}$$

where  $X$  and  $Z$  are categories and  $F$  and  $H$  are functors, is given by a class of arrows:

$$\left( F(x) \xrightarrow{\tau(x)} H(x) \right)_{x \in X(0)}$$

in  $Z$  such that if  $x \xrightarrow{a} y$  is an arrow of  $X$ , then we have the following commutative diagrams:

$$\begin{array}{ccc}
 F(x) & \xrightarrow{\tau(x)} & H(x) \\
 \downarrow F(a) & & \downarrow H(a) \\
 F(y) & \xrightarrow{\tau(y)} & H(y)
 \end{array}$$

If instead the datas:

$$\begin{array}{ccc}
& F & \\
X & \begin{array}{c} \curvearrowright \\ \tau \\ \curvearrowleft \end{array} & Z \\
& H & 
\end{array}$$

are given by bicategories  $X$  and  $Z$ , pseudo-2-functors  $F$  and  $H$ , then a class of 1-cells:

$$\left( F(x) \xrightarrow{\tau(x)} H(x) \right)_{x \in X(0)}$$

of the bicategory  $Z$ , and if  $x \xrightarrow{a} y$  is a 1-cell of  $X$ , then we have the following coherence 2-cell  $\omega(a)$ :

$$\begin{array}{ccc}
F(x) & \xrightarrow{\tau(x)} & H(x) \\
F(a) \downarrow & \omega(a) \swarrow & \downarrow H(a) \\
F(y) & \xrightarrow{\tau(y)} & H(y)
\end{array}$$

such that if  $x \xrightarrow{a} y \xrightarrow{b} z$  are 1-cells of  $X$  then we have the following commutative diagram:

$$\begin{array}{ccc}
H_1(b) \circ_0^1 (H_1(a) \circ_0^1 \tau_1(x)) & \xrightarrow{1_{H_1(b)} \circ_0^2 \omega(a)} & H_1(b) \circ_0^1 (\tau_1(y) \circ_0^1 F_1(a)) \\
\downarrow a(H_1(b), H_1(a), \tau_1(x)) & & \downarrow a(H_1(b), \tau_1(y), F_1(a)) \\
(H_1(b) \circ_0^1 H_1(a)) \circ_0^1 \tau_1(x) & & (H_1(b) \circ_0^1 \tau_1(y)) \circ_0^1 F_1(a) \\
\downarrow d_1(a, b) \circ_0^2 1_{\tau_1(x)} & & \downarrow \omega(b) \circ_0 2_0 1_{F_1(a)} \\
H_1(b \circ_0^1 a) \circ_0^1 \tau_1(x) & & (\tau_1(z) \circ_0^1 F_1(b)) \circ_0^1 F_1(a) \\
\downarrow \omega(b \circ_0 1_0 a) & & \downarrow a(\tau_1(z), F_1(b), F_1(a)) \\
\tau_1(z) \circ_0^1 F_1(b \circ_0^1 a) & \xleftarrow{1_{\tau_1(z)} \circ_0^2 d_0(b, a)} & \tau_1(z) \circ_0^1 (F_1(b) \circ_0^1 F_1(a))
\end{array}$$

**Definition 5** Such  $\tau$  described above are called pseudo-2-natural transformations. □

We are going to show that  $\Theta_{\mathbb{M}^2}^\infty$ -models in Sets of dimension 2, i.e globular weak  $(2, \infty)$ -natural transformations of dimension 2, are pseudo-2-natural transformations.

The following diagram of theories:

$$\begin{array}{ccccc}
\Theta_{\mathbb{M}^0}^\infty & \xrightarrow{s_0^1} & \Theta_{\mathbb{M}^1}^\infty & \xrightarrow{s_1^2} & \Theta_{\mathbb{M}^2}^\infty \\
& \xrightarrow{t_0^1} & & \xrightarrow{t_1^2} & \\
\end{array}$$

leads, by passing to Sets-models, to the following diagram in CAT:

$$\begin{array}{ccccc}
\text{Mod}(\Theta_{\mathbb{M}^2}^\infty) & \xrightarrow{\sigma_1^2} & \text{Mod}(\Theta_{\mathbb{M}^1}^\infty) & \xrightarrow{\sigma_0^1} & \text{Mod}(\Theta_{\mathbb{M}^0}^\infty) \\
& \xrightarrow{\tau_1^2} & & \xrightarrow{\tau_0^1} & \\
\end{array}$$

Thus if  $\tau$  is a  $\Theta_{\mathbb{M}^2}^\infty$ -models in Sets then  $F := \sigma_1^2(\tau) \in \text{Mod}(\Theta_{\mathbb{M}^1}^\infty)$  is the domain of  $\tau$  and  $H := \tau_1^2(\tau) \in \text{Mod}(\Theta_{\mathbb{M}^1}^\infty)$  is the codomain of  $\tau$ . Both  $F$  and  $H$  are globular weak  $\infty$ -functors. Also  $X := \sigma_0^2(\tau) \in \text{Mod}(\Theta_{\mathbb{M}^0}^\infty)$  and  $Z := \tau_0^2(\tau) \in \text{Mod}(\Theta_{\mathbb{M}^0}^\infty)$  are the underlying globular weak  $\infty$ -categories of  $\tau$ . Thus for shorter notation we write:

$$\begin{array}{ccc}
& F & \\
X & \begin{array}{c} \curvearrowright \\ \tau \\ \curvearrowleft \end{array} & Z \\
& H & 
\end{array}$$

such globular weak  $(2, \infty)$ -natural transformation.

**Proposition 1** *With the notation above, if  $\tau$  is a  $\Theta_{\mathbb{M}^2}^\infty$ -models in Sets of dimension 2, then it has a structure of pseudo-2-natural transformation.* □

PROOF We need to exhibit a commutative diagram:

$$\begin{array}{ccc}
H_1(b) \circ_0^1 (H_1(a) \circ_0^1 \tau_1(x)) & \xrightarrow{1_{H_1(b)} \circ_0^2 \omega(a)} & H_1(b) \circ_0^1 (\tau_1(y) \circ_0^1 F_1(a)) \\
\downarrow a(H_1(b), H_1(a), \tau_1(x)) & & \downarrow a(H_1(b), \tau_1(y), F_1(a)) \\
(H_1(b) \circ_0^1 H_1(a)) \circ_0^1 \tau_1(x) & & (H_1(b) \circ_0^1 \tau_1(y)) \circ_0^1 F_1(a) \\
\downarrow d_1(a, b) \circ_0^2 1_{\tau_1(x)} & & \downarrow \omega(b) \circ_0 2_{F_1(a)} \\
H_1(b \circ_0^1 a) \circ_0^1 \tau_1(x) & & (\tau_1(z) \circ_0^1 F_1(b)) \circ_0^1 F_1(a) \\
\downarrow \omega(b \circ_0 1_a) & & \downarrow a(\tau_1(z), F_1(b), F_1(a)) \\
\tau_1(z) \circ_0^1 F_1(b \circ_0^1 a) & \xleftarrow{1_{\tau_1(z)} \circ_0^2 d_0(b, a)} & \tau_1(z) \circ_0^1 (F_1(b) \circ_0^1 F_1(a))
\end{array}$$

for 1-cells  $x \xrightarrow{a} y \xrightarrow{b} z$  of  $X$ , where are involved the coherence 2-cells:

$$a \circ_0^1 (b \circ_0^1 c) \xrightarrow{a(a, b, c)} (a \circ_0^1 b) \circ_0^1 c \quad H_1(a) \circ_0^1 \tau(x) \xrightarrow{\omega(a)} \tau(y) \circ_0^1 F_1(a)$$

and:

$$F_1(b) \circ_0^1 F_1(a) \xrightarrow{d_0(b, a)} F_1(b \circ_0^1 a) \quad H_1(b) \circ_0^1 H_1(a) \xrightarrow{d_1(b, a)} H_1(b \circ_0^1 a)$$

In fact this diagram is the realization in Sets, through the presheaf  $\tau$ , of some conglomerate of operations living in  $\Theta_{\mathbb{P}^2}^\infty$ . We are going to describe each operations which underlies expressions in this diagram, and this shall lead to the conglomerate of operations leading, by passing to Sets-models, to this diagram.

In order to organize our computations we write:

$$\begin{array}{ccc}
A & \xrightarrow{(AB)} & B \\
\downarrow (AG) & & \downarrow (BC) \\
G & & C \\
\downarrow (GH) & & \downarrow (CD) \\
H & & D \\
\downarrow (HF) & & \downarrow (DE) \\
F & \xleftarrow{(EF)} & E
\end{array}$$

- Operation for  $(AB)$ :

The operations:

$$h_1(1(1)) \star_0^1 \tau(1(0)) \xrightarrow{H_1 \star_0^1 \tau} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1), \quad \tau(1(0)) \star_0^1 f_1(1(1)) \xrightarrow{\tau \star_0^1 F_1} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1),$$

are such that  $h_1(1(1)) \star_0^1 \tau(1(0)) = \tau(1(0)) \star_0^1 f_1(1(1))$ , also they are parallels, thus lead to the operation  $\omega$ :

$$\begin{array}{ccc}
2(2) & & \\
\uparrow s_1^2 & \nearrow \omega & \\
2(1) & \xrightarrow{(H_1 \star_0^1 \tau) \circ \nu_0^1} & h_1(1(1)) \star_0^1 \tau(1(0)) \\
& \xrightarrow{(\tau \star_0^1 F_1) \circ \nu_0^1} & \\
& & \uparrow t_1^2
\end{array}$$

also we have the operation  $[H_1; H_1]$ :

$$\begin{array}{ccc}
2(2) & & \\
\uparrow s_1^2 & \uparrow t_1^2 & \searrow [H_1; H_1] \\
2(1) & \xrightarrow{H_1} & h_1(1(1)) \\
& \xrightarrow{H_1} & 
\end{array}$$

thus this leads to the operation:

$$2(2) \xrightarrow{\nu_0^2} 2(2) \star_0^2 2(2) \xrightarrow{[H_1; H_1] \star_0^2 \omega} h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0))$$

which gives the operation  $(AB)$ :

$$2(2) \xrightarrow{([H_1; H_1] \star_0^2 \omega) \circ \nu_0^2} h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0))$$

- Operation for  $(BC)$ :

The coherence for associativity is builds as follow:

$$\begin{array}{ccc}
2(2) & & \\
\uparrow s_1^2 & \uparrow t_1^2 & \searrow a \\
2(1) & \xrightarrow{(\nu_0^1 \star_0^1 1_{2(1)}) \circ \nu_0^1} & 2(1) \star_0^1 2(1) \star_0^1 2(1) \\
& \xrightarrow{(1_{2(1)} \star_0^1 \nu_0^1) \circ \nu_0^1} & 
\end{array}$$

thus by precomposing it with:

$$2(1) \star_0^1 2(1) \star_0^1 2(1) \xrightarrow{H_1 \star_0^1 \tau \star_0^1 F_1} h_1(1(1)) \star_0^1 \tau(1(0)) \star_0^1 f_1(1(1))$$

we get the operation  $(BC)$ :

$$2(2) \xrightarrow{(H_1 \star_0^1 \tau \star_0^1 F_1) \circ a} h_1(1(1)) \star_0^1 \tau(1(0)) \star_0^1 f_1(1(1))$$

- Operation for  $(CD)$ : this is similar to the operation  $(AB)$  and we obtain:

$$2(2) \xrightarrow{(\omega \star_0^2 [F_1; F_1]) \circ \nu_0^2} \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

- Operation  $(DE)$ : this is similar to the operation  $(BC)$  and we obtain:

$$2(2) \xrightarrow{(\tau \star_0^1 F_1 \star_0^1 F_1) \circ a} \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

- Operation  $(EF)$ :

the operations:

$$f_1(1(1)) \star_0^1 f_1(1(1)) \xrightarrow{F_1 \star_0^1 F_1} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1), \quad f_1(1(1)) \star_0^1 f_1(1(1)) \xrightarrow{f_1(\mu_0^1)} f_1(1(1)) \xrightarrow{F_1} 2(1),$$

are parallels, thus lead to the operation  $d_0$ :

$$\begin{array}{ccc}
2(2) & & \\
\uparrow s_1^2 & \uparrow t_1^2 & \searrow d_0 \\
2(1) & \xrightarrow{(F_1 \star_0^1 F_1) \circ \nu_0^1} & 2(1) \star_0^1 2(1) \star_0^1 2(1) \\
& \xrightarrow{f_1(\mu_0^1) \circ \nu_0^1} &
\end{array}$$

Also we have the lifting  $[\tau; \tau]$ :

$$\begin{array}{ccc}
2(2) & & \\
\uparrow s_1^2 & \uparrow t_1^2 & \searrow [\tau; \tau] \\
2(1) & \xrightarrow{\tau} & \tau(1(0)) \\
& \xrightarrow{\tau} &
\end{array}$$

thus this leads to:

$$2(2) \xrightarrow{\nu_0^2} 2(2) \star_0^2 2(2) \xrightarrow{[\tau; \tau] \star_0^2 d_0} \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

thus to the operation  $(EF)$ :

$$2(2) \xrightarrow{([\tau; \tau] \star_0^2 d_0) \circ \nu_0^2} \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

- Operation  $(AG)$ : this is similar to the operations  $(BC)$  and  $(DE)$  and we obtain:

$$2(2) \xrightarrow{(H_1 \star_0^1 H_1 \star_0^1 \tau) \circ a} h_1(1(1)) \star_0^1 f_1(1(1)) \star_0^1 \tau(1(0))$$

- Operation  $(GH)$ : this is similar to the operation  $(EF)$  and we obtain:

$$2(2) \xrightarrow{(d_1 \star_0^2 [\tau; \tau]) \circ \nu_0^2} h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0))$$

- Operation  $(HF)$ :

we use the operations:

$$h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0)) \xrightarrow{(H_1 \circ h_1(\mu_0^1)) \star_0^1 \tau} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1)$$

and

$$\tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1)) \xrightarrow{\tau \star_0^1 (F_1 \circ f_1(\mu_0^1))} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1)$$

also we have the equality  $h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0)) = \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$ , and these operations are parallels, thus we get the lifting  $\omega(\mu_0^1)$ :

$$\begin{array}{ccc}
2(2) & & \\
\uparrow s_1^2 & \uparrow t_1^2 & \searrow \omega(\mu_0^1) \\
2(1) & \xrightarrow{((H_1 \circ h_1(\mu_0^1)) \star_0^1 \tau) \circ \nu_0^1} & \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1)) \\
& \xrightarrow{(\tau \star_0^1 (F_1 \circ f_1(\mu_0^1))) \circ \nu_0^1} &
\end{array}$$



- We have the equalities:

$$\begin{aligned} h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0)) &= h_1(1(1)) \star_0^1 \tau(1(0)) \star_0^1 f_1(1(1)) \\ &= \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1)) \end{aligned}$$

which shows that operations  $(AB)$ ,  $(BC)$ ,  $(CD)$ ,  $(DE)$ ,  $(EF)$ ,  $(AG)$ ,  $(GH)$  and  $(HF)$  have the same arities, say:

$$\tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

- With the operations  $(AB)$ ,  $(BC)$ ,  $(CD)$ ,  $(DE)$  and  $(EF)$  we have the following morphism in  $\Theta_{\mathbb{M}_2}^\infty$ :

$$\begin{array}{c} 2(2) \star_1^2 2(2) \star_1^2 2(2) \star_1^2 2(2) \star_1^2 2(2) \\ \downarrow \left( ([H_1; H_1] \star_0^2 \omega) \circ \nu_0^2 \right) \star_1^2 \left( (H_1 \star_0^1 \tau \star_0^1 F_1) \circ a \right) \star_1^2 \left( (\omega \star_0^2 [F_1; F_1]) \circ \nu_0^2 \right) \star_1^2 \left( (\tau \star_0^1 F_1 \star_0^1 F_1) \circ a \right) \star_1^2 \left( ([\tau; \tau] \star_0^2 d_0) \circ \nu_0^2 \right) \\ \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1)) \end{array}$$

From it we get the following operation  $(ABCDEF)$  in  $\Theta_{\mathbb{M}_2}^\infty$ :

$$\begin{array}{c} 2(2) \\ \downarrow \nu_1^2 \\ 2(2) \star_1^2 2(2) \\ \downarrow 1_{2(2)} \star_1^2 \nu_1^2 \\ 2(2) \star_1^2 2(2) \star_1^2 2(2) \\ \downarrow 1_{2(2)} \star_1^2 1_{2(2)} \star_1^2 \nu_1^2 \\ 2(2) \star_1^2 2(2) \star_1^2 2(2) \star_1^2 2(2) \\ \downarrow 1_{2(2)} \star_1^2 1_{2(2)} \star_1^2 1_{2(2)} \star_1^2 \nu_1^2 \\ 2(2) \star_1^2 2(2) \star_1^2 2(2) \star_1^2 2(2) \star_1^2 2(2) \\ \downarrow \left( ([H_1; H_1] \star_0^2 \omega) \circ \nu_0^2 \right) \star_1^2 \left( (H_1 \star_0^1 \tau \star_0^1 F_1) \circ a \right) \star_1^2 \left( (\omega \star_0^2 [F_1; F_1]) \circ \nu_0^2 \right) \star_1^2 \left( (\tau \star_0^1 F_1 \star_0^1 F_1) \circ a \right) \star_1^2 \left( ([\tau; \tau] \star_0^2 d_0) \circ \nu_0^2 \right) \\ \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1)) \end{array}$$

- With the operations  $(AG)$ ,  $(GH)$  and  $(HF)$  we have the following morphism in  $\Theta_{\mathbb{M}_2}^\infty$ :

$$\begin{array}{c} 2(2) \star_1^2 2(2) \star_1^2 2(2) \\ \downarrow \left( (H_1 \star_0^1 H_1 \star_0^1 \tau) \circ a \right) \star_1^2 \left( (d_1 \star_0^2 [\tau; \tau]) \circ \nu_0^2 \right) \star_1^2 \left( \omega(\mu_0^1) \right) \\ \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1)) \end{array}$$

and from it we get the following operation  $(AGHF)$  in  $\Theta_{\mathbb{M}_2}^\infty$ :

$$\begin{array}{c}
2(2) \\
\downarrow \nu_1^2 \\
2(2) \star_1^2 2(2) \\
\downarrow 1_{2(2)} \star_1^2 \nu_1^2 \\
2(2) \star_1^2 2(2) \star_1^2 2(2) \\
\downarrow ((H_1 \star_0^1 H_1 \star_0^1 \tau) \circ a) \star_1^2 ((d_1 \star_0^2 [\tau; \tau]) \circ \nu_0^2) \star_1^2 (\omega(\mu_0^1)) \\
\tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))
\end{array}$$

The operations  $(ABCDEF)$  and  $(AGHF)$  are parallels, thus lead to the lifting:

$$\begin{array}{ccc}
2(3) & & \\
\uparrow s_2^3 & \nearrow [(ABCDEF); (AGHF)] & \\
\uparrow t_2^3 & & \\
2(2) & \xrightarrow{(ABCDEF)} & \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1)) \\
& \xrightarrow{(AGHF)} &
\end{array}$$

which is a 3-operation in  $\Theta_{\mathbb{M}^2}^\infty$ . But  $\tau$  has dimension 2, thus the Sets-realization of the operations  $(ABCDEF)$  and  $(AGHF)$  agree, which gives the commutative diagram for pseudo-2-natural transformations.  $\blacksquare$

### 3 The globular weak $\infty$ -category of globular weak $\infty$ -categories in the sense of Grothendieck

Thanks to the result in [4] it is possible to use the formalism of globular operads in [2], at least for the globular free contractible operad  $\mathbb{B}_C^0$  which algebras are globular weak  $\infty$ -categories of Batanin. The main result in [4] solves a conjecture in [1] which basically claimed the equivalence of categories:

$$\text{Mod}(\Theta_{\mathbb{M}^0}^\infty) \simeq \mathbb{B}_C^0\text{-Alg}$$

that is, the approach of globular weak  $\infty$ -categories of Batanin and Grothendieck are equivalent.

In this article we built the following coglobular object in  $\text{Cat}$ :

$$\Theta_{\mathbb{M}^0}^\infty \begin{array}{c} \xrightarrow{s_1^0} \\ \xrightarrow{t_0^1} \end{array} \Theta_{\mathbb{M}^1}^\infty \begin{array}{c} \xrightarrow{s_1^2} \\ \xrightarrow{t_1^2} \end{array} \Theta_{\mathbb{M}^2}^\infty \begin{array}{c} \xrightarrow{s_2^3} \\ \xrightarrow{t_2^3} \end{array} \Theta_{\mathbb{M}^3}^\infty \begin{array}{c} \xrightarrow{s_3^4} \\ \xrightarrow{t_3^4} \end{array} \Theta_{\mathbb{M}^4}^\infty \cdots \Theta_{\mathbb{M}^{n-1}}^\infty \begin{array}{c} \xrightarrow{s_{n-1}^n} \\ \xrightarrow{t_{n-1}^n} \end{array} \Theta_{\mathbb{M}^n}^\infty \cdots$$

If this coglobular object  $\Theta_{\mathbb{M}^\bullet}^\infty$  is a  $\mathbb{B}_C^0$ -coalgebra which means that we have a coaction:

$$\mathbb{B}_C^0 \longrightarrow \text{Coend}(\Theta_{\mathbb{M}^\bullet}^\infty)$$

where  $\text{Coend}(\Theta_{\mathbb{M}^\bullet}^\infty)$  is the coendomorphism operad associated to  $\Theta_{\mathbb{M}^\bullet}^\infty$  (this exists because  $\text{Cat}$  has all small colimits), then this shows that globular weak  $(n, \infty)$ -transformations, for all  $n \in \mathbb{N}$ , by coherators, organize in a globular weak  $\infty$ -categories in the sense of Batanin, thus in the sense of Grothendieck. Thus the  $\mathbb{B}_C^0$ -coalgebraicity of  $\Theta_{\mathbb{M}^\bullet}^\infty$  means that globular weak  $\infty$ -categories of Grothendieck organize in a globular weak  $\infty$ -categories of Grothendieck. This fact is a necessary step for an accurate approach of globular weak  $\infty$ -stacks as wished by Alexandre Grothendieck in [6].

A way to prove this coalgebraicity is to show that the operad  $\text{Coend}(\Theta_{\mathbb{M}^\bullet}^\infty)$  is contractible and equipped with a composition system. We shall give a complete syntactical proof of this coalgebraicity in a future paper.

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Laboratoire de Mathématiques d’Orsay, UMR 8628  
Université de Paris-Saclay and CNRS  
Bâtiment 307, Faculté des Sciences d’Orsay  
94015 ORSAY Cedex, FRANCE  
[camell.kachour@universite-paris-saclay.fr](mailto:camell.kachour@universite-paris-saclay.fr)