Globular weak (n, ∞) -Transformations $(n \in \mathbb{N})$ in the sense of Grothendieck

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Abstract

This article describe globular weak (n, ∞) -transformations $(n \in \mathbb{N})$ in the sense of Grothendieck, i.e for each $n \in \mathbb{N}$ we build a coherator $\Theta_{\mathbb{M}^n}^{\infty}$ which sets models are globular weak (n, ∞) -transformations. A natural globular filtration emerges from these coherators.

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Introduction

We start this article by defining, for each $n \in \mathbb{N}^*$ a coherator $\Theta_{\mathbb{M}^n}^{\infty}$ in the sense of Grothendieck (see [6, 12, 13, 14]), which is a specific sketch [10, 11] such that models of it are globular weak (n, ∞) -transformations.

Recently John Bourke [4] proved a conjecture of Dimitri Ara [1], which shows that globular weak ∞ -categories in the sense of Batanin [2] and globular weak ∞ -categories in the sense of Grothendieck [6, 13] are equivalent: more

precisely globular weak ∞ -categories in the sense of Batanin are \mathbb{B}^0_C -algebras where \mathbb{B}^0_C is a specific globular operad, and globular weak ∞ -categories in the sense of Grothendieck are models of a specific theory $\Theta_{\mathbb{M}^0}^{\infty}$; it is proved in [3, 15] that the monad associated to any globular operads is strongly cartesian [15] thus for the operad \mathbb{B}^0_C it leads to a nice theory $\Theta_{\mathbb{B}^0_C}$ which is Morita equivalent to $\Theta_{\mathbb{M}^0}^{\infty}$ [1, 4] in the sense that their category of models are equivalent.

Thus we suspect that models of the coherators $\Theta_{\mathbb{M}^n}^{\infty}$ $(n \in \mathbb{N}^*)$ build in this article should be also weak (n, ∞) -transformations in the sense of [7, 8] where they are defined as \mathbb{B}_C^n -algebras $(n \in \mathbb{N}^*)$ for specific globular operads \mathbb{B}_C^n : similarly to [4], for each integers $n \geq 1$, the theory $\Theta_{\mathbb{M}^n}$ and the theory $\Theta_{\mathbb{B}_C^n}$ should be Morita equivalent.

Finally we show that $\Theta_{\mathbb{M}^2}^{\infty}$ -models in Sets of dimension 2, i.e globular weak $(2, \infty)$ -natural transformations of dimension 2, are pseudo-2-natural transformations.

The combinatorics developed in this article lead to an accurate description of the monad \mathbb{T}^n (for each $n \in \mathbb{N}^*$) defined on the cartesian product of the category of globular sets with itself, which algebras are globular strict (n, ∞) -transformations; this is similar to the description of the monad \mathbb{S} on the category of precubical sets as described in [9] which algebras are cubical strict ∞ -categories with connections. We shall give more precisions in a later version of this article. In particular it will show that these monads \mathbb{T}^n (for each $n \in \mathbb{N}^*$) are cartesian as conjectured in [8]. We shall also use the formalism developed in [5] to show that the coherators $\Theta_{\mathbb{M}^n}^{\infty}$ $(n \in \mathbb{N}^*)$ lead to monads on the cartesian product of the category of globular sets with itself, which algebras correspond to these globular weak (n, ∞) -transformations in the sense of Grothendieck.

Acknowledgement. In the memory of Alexandre Grothendieck.

1 Globular theories $\Theta_{\mathbb{M}^n}$ of (n, ∞) -magmas $(n \in \mathbb{N})$.

1.1 The globular \mathbb{G}_n -extensions Θ_n $(n \in \mathbb{N})$ and their filtration.

The *n*-globe categories \mathbb{G}_n $(n \in \mathbb{N})$ are small categories which objects are the basic arities (or atomic arities) of operations behind the structure of globular higher transformations. Their accurate descriptions (see below) is of first importance for theoretical reasons: their globular completions Θ_n $(n \in \mathbb{N})$ defined just below, have as objects the arities of operations behind the structure of globular higher transformations, and the important filtration Θ_{\bullet} described below is easily obtain from the filtration \mathbb{G}_{\bullet} below, and the universal property of the completion Θ_n of each \mathbb{G}_n .

The 0-globe category is a small category \mathbb{G}_0 which objects are formal symbols 1(n) $(n \in \mathbb{N})$ which arrows are generate by maps s_{n-1}^n, t_{n-1}^n

$$1(0) \xrightarrow{\begin{array}{c} s_0^1 \\ \longrightarrow \end{array}} 1(1) \xrightarrow{\begin{array}{c} s_1^2 \\ \longrightarrow \end{array}} 1(2) \xrightarrow{\begin{array}{c} s_2^3 \\ \longrightarrow \end{array}} 1(3) \xrightarrow{\begin{array}{c} s_3^4 \\ \longrightarrow \end{array}} 1(4) \cdots 1(n-1) \xrightarrow{\begin{array}{c} s_{n-1}^n \\ \longrightarrow \end{array}} 1(n) \cdots$$

which follows the globular equations $s_{n-1}^n s_{n-2}^{n-1} = t_{n-1}^n s_{n-2}^{n-1}$ and $t_{n-1}^n t_{n-2}^{n-1} = s_{n-1}^n t_{n-2}^{n-1}$ for all integers $n \geq 2$. The maps s_{n-1}^n are called *cosourses* and the maps t_{n-1}^n are called *cotargets*. The composition $s_{n-1}^n s_{n-2}^{n-1} \cdots s_p^{p+1}$ is denoted s_p^n and the composition $t_{n-1}^n t_{n-2}^{n-1} \cdots t_p^{p+1}$ is denoted t_p^n .

Definition 1 A \mathbb{G}_0 -presheaf i.e an object of $[\mathbb{G}; \mathbb{S}ets]$ is called a *globular filtration*, whereas a \mathbb{G}_0^{op} -presheaf i.e an object of $[\mathbb{G}^{op}; \mathbb{S}ets]$ is called a *globular set*.

Let \mathbb{S}^0 the monad on $[\mathbb{G}_0^{op}; \mathbb{S}ets]$ of globular strict ∞ -categories. The category $(1 \downarrow \mathbb{S}^0(1))$ of elements of the presheaf $\mathbb{S}^0(1)$ is commonly denoted Θ_0 , and plays an important role for describing models for theories for globular higher structure (for example the theory $\Theta_{\mathbb{S}^0}$ which set-models are globular strict ∞ -categories, or the theory $\Theta_{\mathbb{W}^0}$ which set-models are globular weak ∞ -categories, etc.), those based on monads on $[\mathbb{G}_0^{op}; \mathbb{S}ets]$. Objects of Θ_0 are globular trees in the sense of [2] and they play the role of arities for operations inside such theories. Actually we can avoid the use of the monad \mathbb{S}^0 to build Θ_0 by using specific colimits based on the globe category \mathbb{G}_0 : first we consider tables t of non-negative integers:

¹By theory here we mean a small category equipped with a chosen set of projective cones [10, 11].

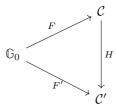
$$\begin{pmatrix} i_1 & i_2 & i_3 & \cdots & i_{k-1} & i_k \\ & i'_1 & & i'_2 & \cdots & \cdots & & i'_{k-1} \end{pmatrix}$$

where $k \ge 1$, $i_l > i'_l < i_{l+1}$ and $1 \le l \le k-1$.

Let C a category and let $\mathbb{G}_0 \xrightarrow{F} C$ a functor. We denote $F(1(n)) = D^n$ and we shall keep the same notations for the image of cosources: $F(s_{i_{l'}}^{i_l}) = s_{i_{l'}}^{i_l}$, and for the image of cotargets: $F(t_{i_{l'}}^{i_l}) = t_{i_{l'}}^{i_l}$, because no risk of confusion will occur. In this case $\mathbb{G}_0 \xrightarrow{F} C$ is called a globular \mathbb{G}_0 -extension if for all tables t as just above, the colimit of the following diagram:



exists in \mathcal{C} . In [6] Alexander Grothendieck calls these colimits globular sums. We prefer call these colimits globular \mathbb{G}_0 -sums. A morphism of globular \mathbb{G}_0 -extensions, also called globular \mathbb{G}_0 -functor, is given by a commutative triangle in $\mathbb{C}AT$:



such that the functor H preserves globular \mathbb{G}_0 -sums. The category of globular \mathbb{G}_0 -extensions is denoted \mathbb{G}_0 - $\mathbb{E}xt$. An initial object of \mathbb{G}_0 - $\mathbb{E}xt$: $\mathbb{G}_0 \longrightarrow \Theta_0$, gives the small category Θ_0 . Tables as above are another formulation of trees in the sense of [2], thus such tables are preferably called *globular* \mathbb{G}_0 -trees. This small category Θ_0 can also be described as the full subcategory of \mathbb{G} lob which objects are globular \mathbb{G}_0 -trees.

Now we are going to extend this Θ_0 for globular strict ∞ -functors: this analogue shall be denoted Θ_1 . But also globular strict ∞ -natural transformations have their own " Θ " that we denote Θ_2 . Let us put some terminology: globular strict $(0,\infty)$ -transformations are globular strict ∞ -categories; globular strict $(1,\infty)$ -transformations are globular strict ∞ -natural transformations; globular strict $(3,\infty)$ -transformations are globular strict ∞ -natural transformations; globular strict ∞ -natural transformations globular strict ∞ -natural transformations is globular strict ∞ -modifications, and so on. For all integers $n \in \mathbb{N}$ the definition of globular strict (n,∞) -transformations is in [8].

Let us first describe the small categories \mathbb{G}_1 , \mathbb{G}_2 and \mathbb{G}_3 which shall be used to get respectively Θ_1 , Θ_2 and Θ_3 :

The 1-globe category is a small category \mathbb{G}_1 which objects are formal symbols 1(n), $f^n(1(n))$, 2(n) $(n \in \mathbb{N})$, which arrows are generate by maps s^n_{n-1}, t^n_{n-1} which follows the globular equations: $s^n_{n-1} s^{n-1}_{n-2} = t^n_{n-1} s^{n-1}_{n-2}$ and $t^n_{n-1} t^{n-1}_{n-2} = s^n_{n-1} t^{n-1}_{n-2}$ for all integers $n \geq 2$:

$$1(0) \xrightarrow{\begin{array}{c} s_0^1 \\ \longrightarrow \end{array}} 1(1) \xrightarrow{\begin{array}{c} s_1^2 \\ \longrightarrow \end{array}} 1(2) \xrightarrow{\begin{array}{c} s_2^3 \\ \longrightarrow \end{array}} 1(3) \xrightarrow{\begin{array}{c} s_3^4 \\ \longrightarrow \end{array}} 1(4) \cdots 1(n-1) \xrightarrow{\begin{array}{c} s_{n-1}^n \\ \longrightarrow \end{array}} 1(n) \cdots$$

$$f^{0}(1(0)) \xrightarrow[t_{0}^{1}]{} f^{1}(1(1)) \xrightarrow[t_{1}^{2}]{} f^{2}(1(2)) \xrightarrow[t_{2}^{3}]{} f^{3}(1(3)) \xrightarrow[t_{3}^{4}]{} f^{4}(1(4)) \cdots f^{n-1}(1(n-1)) \xrightarrow[t_{n-1}^{n}]{} f^{n}(1(n)) \cdots$$

$$2(0) \xrightarrow{\begin{array}{c} s_0^1 \\ \longrightarrow \end{array}} 2(1) \xrightarrow{\begin{array}{c} s_1^2 \\ \longrightarrow \end{array}} 2(2) \xrightarrow{\begin{array}{c} s_2^3 \\ \longrightarrow \end{array}} 2(3) \xrightarrow{\begin{array}{c} s_3^4 \\ \longrightarrow \end{array}} 2(4) \cdots 2(n-1) \xrightarrow{\begin{array}{c} s_{n-1}^n \\ \longrightarrow \end{array}} 2(n) \cdots$$

which follow the globular equations as just above.

We can see that \mathbb{G}_1 has three subcategories denoted respectively by $\mathbb{G}_{1,0}$, $\mathbb{G}_{1,t}$ and $\mathbb{G}_{1,1}$, and which are connected. Also we deliberately use the same notations for the cosources and cotargets maps of these three subcategories because no confusions will occur. The notations that we used for objects of \mathbb{G}_1 shall be suggestive when we will give an accurate description of the monad \mathbb{S}^1 of globular strict ∞ -functors.

The 2-globe category is a small category \mathbb{G}_2 which objects are formal symbols 1(n), $f^n(1(n))$, ξ_1 , $g^n(1(n))$, 2(n) $(n \in \mathbb{N})$, which arrows are generate by maps s^n_{n-1}, t^n_{n-1} which follows the globular equations: $s^n_{n-1}s^{n-1}_{n-2} = t^n_{n-1}s^{n-1}_{n-2}$ and $t^n_{n-1}t^{n-1}_{n-2} = s^n_{n-1}t^{n-1}_{n-2}$ for all integers $n \ge 2$:

$$1(0) \xrightarrow{\begin{array}{c} s_0^1 \\ \longrightarrow \end{array}} 1(1) \xrightarrow{\begin{array}{c} s_1^2 \\ \longrightarrow \end{array}} 1(2) \xrightarrow{\begin{array}{c} s_2^3 \\ \longrightarrow \end{array}} 1(3) \xrightarrow{\begin{array}{c} s_3^4 \\ \longrightarrow \end{array}} 1(4) \cdots 1(n-1) \xrightarrow{\begin{array}{c} s_{n-1}^n \\ \longrightarrow \end{array}} 1(n) \cdots$$

$$f^{0}(1(0)) \xrightarrow{\xrightarrow{s_{0}^{1}}} f^{1}(1(1)) \xrightarrow{\xrightarrow{s_{1}^{2}}} f^{2}(1(2)) \xrightarrow{\xrightarrow{s_{2}^{3}}} f^{3}(1(3)) \xrightarrow{\xrightarrow{s_{3}^{4}}} f^{4}(1(4)) \cdots f^{n-1}(1(n-1)) \xrightarrow{\xrightarrow{s_{n-1}^{n}}} f^{n}(1(n)) \cdots$$

$$g^{0}(1(0)) \xrightarrow[t_{0}]{s_{0}} g^{1}(1(1)) \xrightarrow[t_{1}]{s_{1}^{2}} g^{2}(1(2)) \xrightarrow[t_{2}]{s_{2}^{3}} g^{3}(1(3)) \xrightarrow[t_{3}]{s_{3}^{4}} g^{4}(1(4)) \cdots g^{n-1}(1(n-1)) \xrightarrow[t_{n-1}]{s_{n-1}^{n}} g^{n}(1(n)) \cdots$$

$$2(0) \xrightarrow{\begin{array}{c} s_0^1 \\ \longrightarrow \end{array}} 2(1) \xrightarrow{\begin{array}{c} s_1^2 \\ \longrightarrow \end{array}} 2(2) \xrightarrow{\begin{array}{c} s_2^3 \\ \longrightarrow \end{array}} 2(3) \xrightarrow{\begin{array}{c} s_3^4 \\ \longrightarrow \end{array}} 2(4) \cdots 2(n-1) \xrightarrow{\begin{array}{c} s_{n-1}^n \\ \longrightarrow \end{array}} 2(n) \cdots$$

The 3-globe category is a small category \mathbb{G}_3 which objects are formal symbols 1(n), $f^n(1(n))$, $\alpha_1(1(0))$, $\xi_2(1(0))$, $\beta_1(1(0))$, $g^n(1(n))$, 2(n) $(n \in \mathbb{N})$, which arrows are generate by maps s^n_{n-1}, t^n_{n-1} which follows the globular equations: $s^n_{n-1}s^{n-1}_{n-2} = t^n_{n-1}s^{n-1}_{n-2}$ and $t^n_{n-1}t^{n-1}_{n-2} = s^n_{n-1}t^{n-1}_{n-2}$ for all integers $n \geq 2$:

$$1(0) \xrightarrow{\stackrel{s_0^1}{\longrightarrow}} 1(1) \xrightarrow{\stackrel{s_1^2}{\longrightarrow}} 1(2) \xrightarrow{\stackrel{s_2^3}{\longrightarrow}} 1(2) \xrightarrow{\stackrel{s_2^3}{\longrightarrow}} 1(3) \xrightarrow{\stackrel{s_3^4}{\longrightarrow}} 1(4) \cdots 1(n-1) \xrightarrow{\stackrel{s_{n-1}^n}{\longrightarrow}} 1(n) \cdots$$

$$f^{0}(1(0)) \xrightarrow{\begin{array}{c} s_{0}^{1} \\ t_{0}^{1} \\ \end{array}} f^{1}(1(1)) \xrightarrow{\begin{array}{c} s_{1}^{2} \\ t_{1}^{2} \\ \end{array}} f^{2}(1(2)) \xrightarrow{\begin{array}{c} s_{2}^{3} \\ t_{2}^{3} \\ \end{array}} f^{3}(1(3)) \xrightarrow{\begin{array}{c} s_{3}^{4} \\ t_{3}^{4} \\ \end{array}} f^{4}(1(4)) \cdots f^{n-1}(1(n-1)) \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ t_{n-1}^{n} \\ \end{array}} f^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{0}^{1} \\ t_{0}^{1} \\ \end{array}} f^{n}(1(0)) \xrightarrow{\begin{array}{c} s_{1}^{2} \\ t_{1}^{2} \\ \end{array}} g^{2}(1(2)) \xrightarrow{\begin{array}{c} s_{2}^{3} \\ t_{2}^{3} \\ \end{array}} g^{3}(1(3)) \xrightarrow{\begin{array}{c} s_{3}^{4} \\ t_{3}^{4} \\ \end{array}} g^{4}(1(4)) \cdots g^{n-1}(1(n-1)) \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ t_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ t_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ t_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \cdots \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\ \end{array}} g^{n}(1(n)) \xrightarrow{\begin{array}{c} s_{n-1}^{n} \\$$

$$2(0) \xrightarrow{\stackrel{s_0^1}{\longrightarrow}} 2(1) \xrightarrow{\stackrel{s_1^2}{\longrightarrow}} 2(2) \xrightarrow{\stackrel{s_2^3}{\longrightarrow}} 2(3) \xrightarrow{\stackrel{s_3^4}{\longrightarrow}} 2(4) \cdots 2(n-1) \xrightarrow{\stackrel{s_{n-1}^n}{\longrightarrow}} 2(n) \cdots$$

Let \mathbb{S}^n the monad on $[\mathbb{G}_0^{\text{op}} + \mathbb{G}_0^{\text{op}}; \mathbb{S}\text{ets}]$ of globular strict (n, ∞) -transformations. The category $(1 + 2 \downarrow \mathbb{S}^n (1 + 2))$ of elements of the presheaf $\mathbb{S}^n (1 + 2)$ is denoted Θ_n , and plays an important role for describing models for theories for globular higher structures like the theory $\Theta_{\mathbb{S}^n}$ which set-models are globular strict (n, ∞) -transformations, or the theory $\Theta_{\mathbb{W}^n}$ which set-models are globular weak (n, ∞) -transformations defined in [8]. Now it is interesting to follow the steps of Grothendieck and to construct Θ_n with adapted colimits, avoiding the use of the monad \mathbb{S}^n . For that we start with the following small category \mathbb{G}_n (for $n \geq 4$; \mathbb{G}_1 , \mathbb{G}_2 , \mathbb{G}_3 were described above), that we call the n-globe category: it is a small category which objects are formal symbols 1(n), $f^n(1(n))$, $\alpha_l(1(0))$, $\xi_n(1(0))$, $\beta_l(1(0))$, $g^n(1(n))$, 2(n) ($n \in \mathbb{N}$, and $1 \leq l \leq n - 1$ if n > 1), which arrows are generate by maps s_{n-1}^n , t_{n-1}^n which follows the globular equations: $s_{n-1}^n s_{n-2}^{n-1} = t_{n-1}^n s_{n-2}^{n-1}$ and $t_{n-1}^n t_{n-2}^{n-1} = s_{n-1}^n t_{n-2}^{n-1}$ for all integers $n \geq 2$:

$$1(0) \xrightarrow{\begin{array}{c} s_0^1 \\ \hline \\ t_0^1 \end{array}} 1(1) \xrightarrow{\begin{array}{c} s_1^2 \\ \hline \\ t_1^2 \end{array}} 1(2) \xrightarrow{\begin{array}{c} s_2^3 \\ \hline \\ t_2^3 \end{array}} 1(3) \xrightarrow{\begin{array}{c} s_3^4 \\ \hline \\ t_3^4 \end{array}} 1(4) \cdots 1(n-1) \xrightarrow{\begin{array}{c} s_{n-1}^n \\ \hline \\ t_{n-1}^n \end{array}} 1(n) \cdots$$

$$f^{0}(1(0)) \xrightarrow{s_{0}^{1}} f^{1}(1(1)) \xrightarrow{s_{1}^{2}} f^{2}(1(2)) \xrightarrow{s_{2}^{3}} f^{3}(1(3)) \xrightarrow{s_{3}^{4}} f^{4}(1(4)) \cdots f^{n-1}(1(n-1)) \xrightarrow{s_{n-1}^{n}} f^{n}(1(n)) \cdots \xrightarrow{t_{n-1}^{n}} f^{n}(1(n)) \cdots \xrightarrow{t_{n-1}^{n}} f^{n}(1(n)) \xrightarrow{s_{0}^{1}} \alpha_{1}(1(0)) \xrightarrow{s_{0}^{1}} \alpha_{2}(1(0)) \xrightarrow{s_{0}^{1}} \alpha_{2}(1(0)) \xrightarrow{s_{n-1}^{k}} \alpha_{2}(1(0)) \xrightarrow{s_{n-1}^{k}} \alpha_{2}(1(0)) \xrightarrow{s_{n-1}^{k}} \beta_{2}(1(0)) \xrightarrow{s_{n-1}^{k}} \beta_{2}(1(0)) \xrightarrow{s_{n-1}^{k}} \beta_{2}(1(0)) \xrightarrow{s_{0}^{1}} g^{1}(1(0)) \xrightarrow{s_{0}^{1}} g^{1}(1(0)) \xrightarrow{s_{0}^{1}} g^{2}(1(2)) \xrightarrow{s_{0}^{2}} g^{3}(1(3)) \xrightarrow{s_{0}^{3}} g^{4}(1(4)) \cdots g^{n-1}(1(n-1)) \xrightarrow{s_{n-1}^{n}} g^{n}(1(n)) \cdots$$

$$2(0) \xrightarrow{\begin{array}{c} s_0^1 \\ \longrightarrow \end{array}} 2(1) \xrightarrow{\begin{array}{c} s_1^2 \\ \longrightarrow \end{array}} 2(2) \xrightarrow{\begin{array}{c} s_2^3 \\ \longrightarrow \end{array}} 2(3) \xrightarrow{\begin{array}{c} s_3^4 \\ \longrightarrow \end{array}} 2(4) \cdots 2(n-1) \xrightarrow{\begin{array}{c} s_{n-1}^n \\ \longrightarrow \end{array}} 2(n) \cdots$$

With the small category \mathbb{G}_n (for all $n \geq 1$) it is possible to associate kind of colimits that we call \mathbb{G}_n -sums (see below), which are exactly objects of the category $((1+2\downarrow\mathbb{S}^n(1+2))$ (up to isomorphisms). These objects are also called \mathbb{G}_n -trees. Objects $a \in \mathbb{G}_n(0)$ have dimensions where notations of the diagrams above indicate these dimensions: $\dim(1(n)) = \dim(2(n)) = n$, but also $\dim(f^n(1(n))) = \dim(g^n(1(n))) = n$, $\dim(\alpha_k(1(0))) = \dim(\beta_k(1(0))) = k$, and also $\dim(\xi_n(1(0))) = n$. Thus objects $a \in \mathbb{G}_n(0)$ shall be denoted with a subscript a_l where $l \in \mathbb{N}$ is the dimension of a_l . Objects of \mathbb{G}_n are equipped with the natural order < provided by arrows of \mathbb{G}_n : for $a, b \in \mathbb{G}_n(0)$ we have a < b if and only if there is an arrow $a \longrightarrow b$ in \mathbb{G}_n . We can extend this order < with the order \le by saying: $a \le b$ if and only if a < b or a = b. Now a \mathbb{G}_n -tree is given by a table:

$$\begin{pmatrix} a_{i_1} & a_{i_2} & a_{i_3} & \cdots & a_{i_{k-1}} & a_{i_k} \\ & a_{i'_1} & a_{i'_2} & \cdots & \cdots & a_{i'_{k-1}} \end{pmatrix}$$

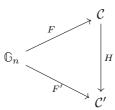
such that $k \geq 1$, and for all $1 \leq l \leq k$, a_{i_l} are objects of \mathbb{G}_n such that $\dim(a_{i_l}) = i_l$, and for all $1 \leq l \leq k-1$ we have $a_{i_l} \geq a_{i'_l} \leq a_{i_{l+1}}$. It is straightforward to see that \mathbb{G}_n -trees are globular sets, and that for n=0 we recover \mathbb{G}_0 -trees.

Let \mathcal{C} a category and let $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$ a functor. We denote $F(a_{i_k}) = D^{a_{i_k}}$ and we shall keep the same notations for the image of cosources: $F(s_{i_{l'}}^{i_l}) = s_{i_{l'}}^{i_l}$, and for the image of cotargets: $F(t_{i_{l'}}^{i_l}) = t_{i_{l'}}^{i_l}$, because no risk of confusion will occur. In this case $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$ is called a globular \mathbb{G}_n -extension if for all \mathbb{G}_n -trees t as just above, the colimit of the following diagram exist in \mathcal{C} :



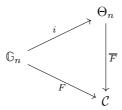
Following the terminology of Grothendieck, such colimits are called \mathbb{G}_n -globular sums.

A morphism of globular \mathbb{G}_n -extensions, also called *globular* \mathbb{G}_n -functor, is given by a commutative triangle in $\mathbb{C}AT$:



such that the functor H preserves \mathbb{C}_n -globular sums. The category of globular \mathbb{C}_n -extensions is denoted \mathbb{C}_n - \mathbb{E} xt. In fact this category has an initial object denoted $\mathbb{C}_n \xrightarrow{i} \Theta_n$. And the small category Θ_n can be described as the full subcategory of $[\mathbb{C}^{op} + \mathbb{C}^{op}; \operatorname{Sets}]$ which objects are globular \mathbb{C}_n -trees. In particular this small category Θ_n is the basic inductive sketch we shall need to describe coherators which set models are globular weak (n, ∞) -transformations $(n \in \mathbb{N})$ (see 2.4).

A globular \mathbb{G}_n -theory is given by a globular \mathbb{G}_n -extension $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$ such that the unique induced functor \overline{F} which makes commutative the diagram:



induces a bijection between objects of Θ_n and objects of \mathcal{C} . The full subcategory of \mathbb{G}_n -Ext which objects are globular \mathbb{G}_n -theories is denoted \mathbb{G}_n -Th. Consider an object $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$ of \mathbb{G}_n -Th, in particular it induces the globular \mathbb{G}_n -functor $\Theta_n \xrightarrow{\overline{F}} \mathcal{C}$ as just above, which is a bijection on objects. A Sets-model of (F,\mathcal{C}) or for \mathcal{C} for short, is given by a functor : $\mathcal{C} \xrightarrow{X} \mathbb{S}$ ets , such that the functor $X \circ \overline{F}$:

$$\Theta_n \xrightarrow{\overline{F}} \mathcal{C} \xrightarrow{X} \mathbb{S}ets$$

sends globular \mathbb{G}_n -sums to globular \mathbb{G}_n -products², thus for all objects t of Θ_n :

$$\begin{pmatrix} a_{i_1} & a_{i_2} & a_{i_3} & \cdots & a_{i_{k-1}} & a_{i_k} \\ & a_{i'_1} & a_{i'_2} & \cdots & \cdots & a_{i'_{k-1}} \end{pmatrix}$$

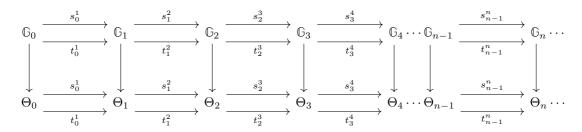
we have:

$$X(\overline{F}(t)) = X \left(\operatorname{colim} \left(\begin{array}{c} D^{a_{i_{1}}} & D^{a_{i_{2}}} & \dots & D^{a_{i_{k-1}}} \\ \sum_{i_{i_{1}}^{i_{1}}} & D^{a_{i_{1}}} & \dots & \sum_{i_{i_{k-1}}^{i_{k-1}}} & D^{a_{i_{k}}} \\ \sum_{i_{i_{k-1}}^{i_{i_{1}}}} & D^{a_{i_{1}}} & \dots & D^{a_{i_{k-1}}} \\ \sum_{i_{i_{k-1}}^{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & \dots & D^{a_{i_{k-1}}} \\ \sum_{i_{k-1}^{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & \dots & D^{a_{i_{k-1}}} \\ \sum_{i_{k-1}^{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & \dots & D^{a_{i_{k-1}}} \\ \sum_{i_{k-1}^{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} \\ \sum_{i_{k-1}^{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} \\ \sum_{i_{k-1}^{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} \\ \sum_{i_{k-1}^{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} \\ \sum_{i_{k-1}^{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} \\ \sum_{i_{k-1}^{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} & D^{a_{i_{1}}} \\ \sum_{i_{k-1}^{i_{1}$$

²Globular \mathbb{G}_n -products are just dual to globular \mathbb{G}_n -sums.

The category of Sets-models of \mathcal{C} is the full subcategory of the category of presheaves $[\mathcal{C}, Sets]$ which objects are Sets-models of \mathcal{C} , and it is denoted $Mod(\mathcal{C})$.

Now it is easy to see that we get the following globular filtration:



1.2 The theories $\Theta_{\mathbb{M}^n}$ for (n, ∞) -magmas $(n \in \mathbb{N})$ and their filtration.

A $(0,\infty)$ -magma M is given by a globular set $\mathbb{G}^{\text{op}} \xrightarrow{M} \mathbb{S}$ ets equipped with operations $M_n \times_{M_p} M_n \xrightarrow{\circ_p^n} M_n$ for all $n \geq 1$ and all $0 \leq p \leq n-1$ such that :

- for $0 \le p < q < m$, $s_q^m(y \circ_p^m x) = s_q^m(y) \circ_p^q s_q^m(x)$ and $t_q^m(y \circ_p^m x) = t_q^m(y) \circ_p^q t_q^m(x)$
- for $0 \le q , <math>s_q^m(y \circ_p^m x) = s_q^m(y) = s_q^m(x)$ and $t_q^m(y \circ_p^m x) = t_q^m(y) = t_q^m(x)$
- for $0 \le p = q < m$, $s_q^m(y \circ_p^m x) = s_q^m(x)$ and $t_q^m(y \circ_p^m x) = t_q^m(x)$

A $(1, \infty)$ -magma $M \xrightarrow{F} M'$ between two $(0, \infty)$ -magma M and M' is a morphism of globular sets. If $N \xrightarrow{G} N'$ is another $(1, \infty)$ -magma, then a strict morphism between them: $F \xrightarrow{(h,h')} G$ is given by two morphisms of $(0, \infty)$ -magma: $M \xrightarrow{h} N$ and $M' \xrightarrow{h'} N'$ such that the following diagram is commutative in Glob:

$$\begin{array}{ccc}
M & \xrightarrow{F} & M' \\
\downarrow h & & \downarrow h' \\
N & \xrightarrow{G} & N'
\end{array}$$

The category of $(1, \infty)$ -magmas is denoted $(1, \infty)$ -Mag.

a morphism: $M(0) \xrightarrow{\tau} M'(1)$ of Sets such that for all $a \in M(0)$, $s_0^1(\tau(a)) = F(a)$ and $t_0^1(\tau(a)) = G(a)$. If

$$N \xrightarrow{\rho} N'$$
 is another $(2, \infty)$ -magma, then a strict morphism between them: $\tau \xrightarrow{(h,h')} \rho$ is given by two morphisms

of $(0, \infty)$ -magmas: $M \xrightarrow{h} N$ and $M' \xrightarrow{h'} N'$ such that for all $a \in M(0)$ we have: $h'(\tau(a)) = \rho(h(a))$. The category of $(2, \infty)$ -magmas is denoted $(2, \infty)$ -Mag.

For $n \geq 3$ we define (n, ∞) -magmas by induction: suppose the categories (k, ∞) -Mag of (k, ∞) -magmas are defined for all $k \in [\![2,n-1]\!]$. An (n,∞) -magmas $\alpha \xrightarrow{\xi} \beta$ between the $(n-1,\infty)$ -magmas α and β is given by a morphism $M(0) \xrightarrow{\xi} M'(n)$ in Sets such that for all $a \in M(0)$, $s_0^n(\xi(a)) = F(a)$ and $t_0^n(\xi(a)) = G(a)$. A morphism between two (n,∞) -magmas $\xi \xrightarrow{(h,h')} \xi'$ is given by two morphisms of $(0,\infty)$ -magmas: $M \xrightarrow{h} N$ and $M' \xrightarrow{h'} N'$ such that for all $a \in M(0)$ we have: $h'(\xi(a)) = \xi'(h(a))$. The category of (n,∞) -magmas is denoted (n,∞) -Mag.

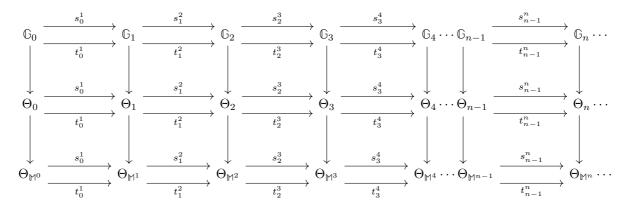
Now for all $n \geq 1$ we have a monadic forgetful functor U^n :

$$(n,\infty)$$
-Mag F^n U^n U^n

from the category (n, ∞) -Mag of globular (n, ∞) -magmas to the category $\mathbb{G}lob^2$ of pairs of globular sets, which sends (n, ∞) -magmas ξ as above to the pair of globular sets (U(M), U(M')). The notation U(M) means that we have forgotten the underlying structure of $(0, \infty)$ -magma of M. We use the equivalence $\mathbb{G}lob^2 \simeq [\mathbb{G}_0^{op} + \mathbb{G}_0^{op}; \mathbb{S}ets]$ to get a monad $\mathbb{M}^n = (M^n, \eta^n, \mu^n)$ on $[\mathbb{G}_0^{op} + \mathbb{G}_0^{op}; \mathbb{S}ets]$. We have the equivalence of categories (n, ∞) -Mag $\cong \mathbb{M}^n$ -Alg because U^n is monadic. The full subcategory $\Theta_{\mathbb{M}^n} \subset \mathbb{K}l(\mathbb{M}^n)$ of the Kleisli category of \mathbb{M}^n which objects are \mathbb{G}_n -trees is called the theory of globular (n, ∞) -magmas. This is our basic example of globular \mathbb{G}_n -theory. In fact we have the following equivalences of categories:

$$(n,\infty)$$
-Mag $\simeq \mathbb{M}^n$ -Alg $\simeq \mathbb{M}$ od $(\Theta_{\mathbb{M}^n})$

Now it is easy to see that we get the following globular filtration:



2 Globular \mathbb{G}_n -coherators $(n \in \mathbb{N})$

2.1 Admissibility

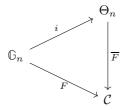
For a fixed integer $n \geq 1$, let $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$ be a globular \mathbb{G}_n -theory, i.e an object of \mathbb{G}_n -Th; here we pay attention to the inclusion: $\mathbb{G}_0 + \mathbb{G}_0 \hookrightarrow \mathbb{G}_n$. An object c_m of $\mathbb{G}_0 + \mathbb{G}_0$ can be the object 1(m) or the object 2(m) for all $m \in \mathbb{N}$; two arrows: $D^{c_m} \xrightarrow{g} t$ in \mathcal{C} are parallels if $fs_{m-1}^m = gs_{m-1}^m$ and $ft_{m-1}^m = gt_{m-1}^m$:

$$D^{c_m} \xrightarrow{f} t$$

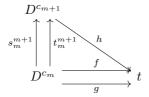
$$s_{m-1}^m \downarrow t_{m-1}^m$$

$$D^{c_{m-1}}$$

Consider a couple (f,g) of parallels arrows in \mathcal{C} as just above. We say that it is admissible or algebraic if they don't belong to the image of the globular \mathbb{G}_n -functor \overline{F} :



Consider a couple (f,g) of arrows of \mathcal{C} which is admissible as just above; a lifting of (f,g) is given by an arrow h:



such that $hs_m^{m+1} = f$ and $ht_m^{m+1} = g$

2.2 Batanin-Grothendieck Sequences

We now define the Batanin-Grothendieck sequence³ associated to a globular \mathbb{G}_n -theory $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$. We build it by the following induction:

- If m = 0 we start with the couple (C, E) where E denotes the set of admissible pairs of arrows of C; we shall write $(C_0, E_0) = (C, E)$ this first step.
- If m = 1 we consider then the couple (C_1, E_1) where C_1 is obtained by formally adding in $C_0 = C$ the liftings of all elements $(f, g) \in E_0 = E$, and E_1 is the set of admissible couples of arrows in C_1 which are not elements of the set E_0 ;
- If for $m \ge 2$ the couple (C_m, E_m) is well defined then C_{m+1} is obtained by formally adding in C_m the liftings of all elements of E_m , and E_{m+1} is the set of couples of arrows of C_{m+1} which are not elements of E_m

We give a slightly different but equivalent induction to build the Batanin-Grothendieck sequence for such globular theory $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$:

- If m = 0 we start with the couple (C, E) where E is the set of couple of arrow which are admissible of C; we denote $E = E_0 = E'_0 = E'_0 \setminus \emptyset$ (we shall see soon the reason of these notations), and $C_0 = C$;
- If m = 1 we consider the couple (C_1, E_1) where C_1 is obtained by formally adding in C_0 all liftings of the elements $(f,g) \in E_0$, E'_1 is the set of all pairs of arrows which are admissible in C_1 , and $E_1 = E'_1 \setminus E_0$; remark that $E_0 = E'_0 \subset E'_1$;
- If m = 2 we consider the couple (C_2, E_2) where C_2 is obtained by formally adding in C_1 all liftings of the elements $(f, g) \in E_1, E'_2$ is the set of all pairs of arrows which are admissible in C_2 , and $E_2 = E'_2 \setminus E'_1$;
- For $m \geq 3$ we suppose that the couple (C_m, E_m) is well defined with $E_m = E'_m \setminus E'_{m-1}$, then C_{m+1} is obtained by formally adding in C_m all liftings of the elements $(f,g) \in E_m$, E'_{m+1} is the set of all pairs of arrows which are admissible in C_{m+1} , and $E_{m+1} = E'_{m+1} \setminus E'_m$;

The Batanin-Grothendieck sequence of the globular theory $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$ produces the following filtered diagram:

$$(\mathbb{N}, \leq) \xrightarrow{\mathcal{C}_{\bullet}} \mathbb{G}_n$$
-Th in the category \mathbb{G}_n -Th:

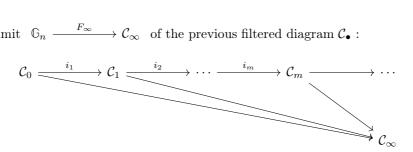
$$C_0 \xrightarrow{i_1} C_1 \xrightarrow{i_2} \cdots \xrightarrow{i_m} C_m \xrightarrow{} \cdots$$

2.3 \mathbb{G}_n -coherators for globular \mathbb{G}_n -theories $(n \in \mathbb{N})$

Let us fixed an integer $n \geq 1$. We start with datas of the previous subsection, i.e with the Batanin-Grothendieck sequence $(\mathbb{N}, \leq) \xrightarrow{C_{\bullet}} \mathbb{G}_n$ -Th for a globular \mathbb{G}_n -theory $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$:

³Coherators associated to such sequence are called *of Batanin-Leinster type* by some authors.

Definition 2 The colimit $\mathbb{G}_n \xrightarrow{F_{\infty}} \mathcal{C}_{\infty}$ of the previous filtered diagram \mathcal{C}_{\bullet} :



is called the globular \mathbb{G}_n -coherator of the type Batanin-Grothendieck associated to the globular \mathbb{G}_n -theory $\mathbb{G}_n \xrightarrow{F} \mathcal{C}_{\square}$

For shorter terminology we shall say that $\mathbb{G}_n \xrightarrow{F_{\infty}} \mathcal{C}_{\infty}$ is the \mathbb{G}_n -coherator associated to the globular \mathbb{G}_n theory $\mathbb{G}_n \xrightarrow{F} \mathcal{C}$. It is straightforward to see that the Batanin-Grothendieck construction of \mathbb{G}_n -coherators associated to globular \mathbb{G}_n -theory is functorial, and the following functor Φ_n is called the Batanin-Grothendieck functor:

$$\mathbb{G}_n\text{-}\mathbb{T}\mathrm{h} \xrightarrow{\Phi_n} \mathbb{G}_n\text{-}\mathbb{T}\mathrm{h}$$

$$\mathcal{C} \longmapsto \mathcal{C}^{\infty}$$

The \mathbb{G}_n -coherators $\Theta_{\mathbb{M}^n}^{\infty}$ $(n \in \mathbb{N})$ 2.4

Let us fixed an integer $n \geq 1$. The \mathbb{G}_n -coherator associated to the globular theory $\mathbb{G}_n \xrightarrow{j} \Theta_{\mathbb{M}^n}$ that we obtain with the composition:

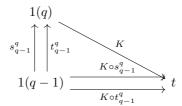
$$\mathbb{G}_n \xrightarrow{i} \Theta_n \hookrightarrow \Theta_{\mathbb{M}^n}$$

is denoted $\Theta_{\mathbb{M}^n}^{\infty}$.

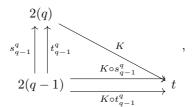
Definition 3 Globular weak (n, ∞) -transformations are objects of the category $Mod(\Theta_{\mathbb{M}^n}^{\infty})$.

Morphisms in $\Theta_{\mathbb{M}^n}^{\infty}$ which doesn't belong to the image of $\Theta_n \subseteq \Theta_{\mathbb{M}^n}^{\infty}$ are called algebraic morphisms of $\Theta_{\mathbb{M}^n}^{\infty}$. Morphisms in $\Theta_{\mathbb{M}^n}^{\infty}$ of the form: $1(p) \xrightarrow{K} t$ or $2(p) \xrightarrow{K} t$, where t are any \mathbb{G}_n -trees, are the p-operations of $\Theta_{\mathbb{M}^n}^{\infty}$.

Definition 4 If $\tau \in Mod(\Theta_{M^n}^{\infty})$ and $p \geq 1$ is an integer, then $\dim(\tau) = p$ if for all q > p, any q-operations K in $\Theta_{M^n}^{\infty}$ has its source and target which are equalize by τ . Thus if we have:



or:



then $\tau(K \circ s_{q-1}^q) = \tau(K \circ t_{q-1}^q).$

Remark 1 In 2019 John Bourke has proved [4] the Ara conjecture [1] which says that the category of globular weak ∞ -categories of Batanin is equivalent to the category of globular weak ∞ -categories of Grothendieck:

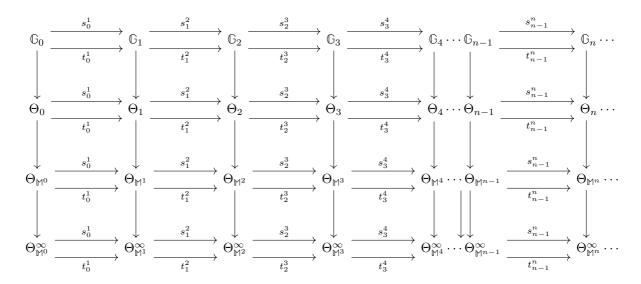
$$\operatorname{\mathbb{M}od}(\Theta_{\mathbb{B}^0_G}) \simeq \operatorname{\mathbb{M}od}(\Theta^\infty_{\mathbb{M}})$$

where here \mathbb{B}^0_C denotes the globular operad of Batanin [2] which algebras are his models of globular weak ∞ -categories and $\Theta_{\mathbb{B}^0_C}$ is its associated theory. In the same veine, for all integers $n \geq 1$, the category of globular weak (n, ∞) -transformations defined in [8] should be equivalent to the category of globular weak (n, ∞) -transformations with this \mathbb{G}_n -coherator $\Theta_{\mathbb{M}^m}^{\infty}$:

$$\operatorname{Mod}(\Theta_{\mathbb{B}^n_C}) \simeq \operatorname{Mod}(\Theta^{\infty}_{\mathbb{M}^n})$$

where \mathbb{B}^n_C denotes the globular operad [8] which algebras are models of globular weak (n, ∞) -transformations and $\Theta_{\mathbb{B}^n_C}$ is its associated theory.

Now it is easy to see that we get the following globular filtration:



2.5 Computations in dimensions 2

A natural transformation τ :

$$X \xrightarrow{\tau} Z$$

where X and Z are categories and F and H are functors, is given by a class of arrows:

$$\left(F(x) \xrightarrow{\tau(x)} H(x) \right)_{x \in X(0)}$$

in Z such that if $x \xrightarrow{a} y$ is an arrow of X, then we have the following commutative diagrams:

$$F(x) \xrightarrow{\tau(x)} H(x)$$

$$F(a) \downarrow \qquad \qquad \downarrow^{H(a)}$$

$$F(y) \xrightarrow{\tau(y)} H(y)$$

If instead the datas:

$$X$$
 T
 H
 Z

are given by bicategories X and Z, pseudo-2-functors F and H, then a class of 1-cells:

$$\left(F(x) \xrightarrow{\tau(x)} H(x) \right)_{x \in X(0)}$$

of the bicategory Z, and if $x \xrightarrow{a} y$ is a 1-cell of X, then we have the following coherence 2-cell $\omega(a)$:

$$F(x) \xrightarrow{\tau(x)} H(x)$$

$$F(a) \downarrow \qquad \qquad \downarrow^{H(a)}$$

$$F(y) \xrightarrow{\tau(y)} H(y)$$

such that if $x \xrightarrow{a} y \xrightarrow{b} z$ are 1-cells of X then we have the following commutative diagram:

$$H_{1}(b) \circ_{0}^{1} (H_{1}(a) \circ_{0}^{1} \tau_{1}(x)) \xrightarrow{1_{H_{1}(b)} \circ_{0}^{2} \omega(a)} H_{1}(b) \circ_{0}^{1} (\tau_{1}(y) \circ_{0}^{1} F_{1}(a))$$

$$a(H_{1}(b), H_{1}(a), \tau_{1}(x)) \downarrow \qquad \qquad \downarrow a(H_{1}(b), \tau_{1}(y), F_{1}(a))$$

$$(H_{1}(b) \circ_{0}^{1} H_{1}(a)) \circ_{0}^{1} \tau_{1}(x) \qquad \qquad (H_{1}(b) \circ_{0}^{1} \tau_{1}(y)) \circ_{0}^{1} F_{1}(a)$$

$$d_{1}(a,b) \circ_{0}^{2} 1_{\tau_{1}(x)} \downarrow \qquad \qquad \downarrow \omega(b) \circ_{2} 1_{F_{1}(a)}$$

$$H_{1}(b \circ_{0}^{1} a) \circ_{0}^{1} \tau_{1}(x) \qquad \qquad (\tau_{1}(z) \circ_{0}^{1} F_{1}(b)) \circ_{0}^{1} F_{1}(a)$$

$$\omega(b \circ 1_{0} a) \downarrow \qquad \qquad \downarrow a(\tau_{1}(z), F_{1}(b), F_{1}(a))$$

$$\tau_{1}(z) \circ_{0}^{1} F_{1}(b \circ_{0}^{1} a) \xleftarrow{1_{\tau_{1}(z) \circ_{0}^{2} d_{0}(b,a)}} \tau_{1}(z) \circ_{0}^{1} (F_{1}(b) \circ_{0}^{1} F_{1}(a))$$

Definition 5 Such τ described above are called pseudo-2-natural transformations.

We are going to show that $\Theta_{\mathbb{M}^2}^{\infty}$ -models in Sets of dimension 2, i.e globular weak $(2, \infty)$ -natural transformations of dimension 2, are pseudo-2-natural transformations.

The following diagram of theories:

$$\Theta_{\mathbb{M}^0}^{\infty} \xrightarrow{\begin{array}{c} s_0^1 \\ \\ \end{array}} \Theta_{\mathbb{M}^1}^{\infty} \xrightarrow{\begin{array}{c} s_1^2 \\ \\ \end{array}} \Theta_{\mathbb{M}^2}^{\infty}$$

leads, by passing to Sets-models, to the following diagram in CAT:

$$\mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^{2}}) \xrightarrow{\sigma^{2}_{1}} \mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^{1}}) \xrightarrow{\sigma^{1}_{0}} \mathbb{M}\mathrm{od}(\Theta^{\infty}_{\mathbb{M}^{0}})$$

Thus if τ is a $\Theta_{\mathbb{M}^2}^{\infty}$ -models in Sets then $F := \sigma_1^2(\tau) \in \mathbb{M}\mathrm{od}(\Theta_{\mathbb{M}^1}^{\infty})$ is the domain of τ and $H := \tau_1^2(\tau) \in \mathbb{M}\mathrm{od}(\Theta_{\mathbb{M}^1}^{\infty})$ is the codomain of τ . Both F and H are globular weak ∞ -functors. Also $X := \sigma_0^2(\tau) \in \mathbb{M}\mathrm{od}(\Theta_{\mathbb{M}^0}^{\infty})$ and $Z := \tau_0^2(\tau) \in \mathbb{M}\mathrm{od}(\Theta_{\mathbb{M}^0}^{\infty})$ are the underlying globular weak ∞ -categories of τ . Thus for shorter notation we write:

$$X$$
 T
 H
 Z

such globular weak $(2, \infty)$ -natural transformation.

Proposition 1 With the notation above, if τ is a $\Theta_{\mathbb{M}^2}^{\infty}$ -models in Sets of dimension 2, then it has a structure of pseudo-2-natural transformation.

PROOF We need to exhibit a commutative diagram:

$$\begin{array}{c} H_1(b) \circ_0^1 \left(H_1(a) \circ_0^1 \tau_1(x) \right) \xrightarrow{1_{H_1(b)} \circ_0^2 \omega(a)} H_1(b) \circ_0^1 \left(\tau_1(y) \circ_0^1 F_1(a) \right) \\ a(H_1(b), H_1(a), \tau_1(x)) \Big\downarrow & & & & & & \\ \left(H_1(b) \circ_0^1 H_1(a) \right) \circ_0^1 \tau_1(x) & & & & & \\ \left(H_1(b) \circ_0^1 \tau_1(y) \right) \circ_0^1 F_1(a) & & & & \\ d_1(a,b) \circ_0^2 l_{\tau_1(x)} \Big\downarrow & & & & & \\ d_1(a,b) \circ_0^2 l_{\tau_1(x)} \Big\downarrow & & & & & \\ H_1(b \circ_0^1 a) \circ_0^1 \tau_1(x) & & & & & \\ \left(\tau_1(z) \circ_0^1 F_1(b) \right) \circ_0^1 F_1(a) & & & \\ \omega(b \circ l_0 a) \Big\downarrow & & & & & \\ \omega(b \circ l_0 a) \Big\downarrow & & & & & \\ \tau_1(z) \circ_0^1 F_1(b) \circ_0^1 F_1(a) & & & \\ \end{array}$$

for 1-cells $x \xrightarrow{a} y \xrightarrow{b} z$ of X, where are involved the coherence 2-cells:

$$a \circ_0^1 (b \circ_0^1 c) \xrightarrow{a(a,b,c)} (a \circ_0^1 b) \circ_0^1 c \qquad H_1(a) \circ_0^1 \tau(x) \xrightarrow{\omega(a)} \tau(y) \circ_0^1 F_1(a)$$

and:

$$F_1(b) \circ_0^1 F_1(a) \xrightarrow{d_0(b,a)} F_1(b \circ_0^1 a) \qquad H_1(b) \circ_0^1 H_1(a) \xrightarrow{d_1(b,a)} H_1(b \circ_0^1 a)$$

In fact this diagram is the realization in Sets, through the presheaf τ , of some conglomerate of operations living in $\Theta_{\mathbb{M}^2}^{\infty}$. We are going to describe each operations which underlies expressions in this diagram, and this shall lead to the conglomerate of operations leading, by passing to Sets-models, to this diagram.

In order to organize our computations we write:

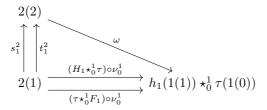
$$\begin{array}{ccc}
A & \xrightarrow{(AB)} & B \\
(AG) \downarrow & & \downarrow (BC) \\
G & & C \\
(GH) \downarrow & & \downarrow (CD) \\
H & & D \\
(HF) \downarrow & & \downarrow (DE) \\
F \longleftarrow & E
\end{array}$$

• Operation for (AB):

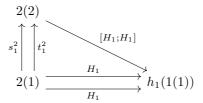
The operations:

$$h_1(1(1)) \star_0^1 \tau(1(0)) \xrightarrow{H_1 \star_0^1 \tau} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1), \qquad \tau(1(0)) \star_0^1 f_1(1(1)) \xrightarrow{\tau \star_0^1 F_1} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1),$$

are such that $h_1(1(1)) \star_0^1 \tau(1(0)) = \tau(1(0)) \star_0^1 f_1(1(1))$, also they are parallels, thus lead to the operation ω :



also we have the operation $[H_1; H_1]$:



thus this leads to the operation:

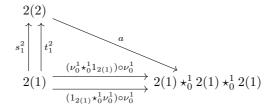
$$2(2) \xrightarrow{\nu_0^2} 2(2) \star_0^2 2(2) \xrightarrow{[H_1; H_1] \star_0^2 \omega} h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0))$$

which gives the operation (AB):

$$2(2) \xrightarrow{([H_1;H_1]\star_0^2\omega)\circ\nu_0^2} h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0))$$

• Operation for (BC):

The coherence for associativity is builds as follow:



thus by precomposing it with:

$$2(1) \star_0^1 2(1) \star_0^1 2(1) \xrightarrow{H_1 \star_0^1 \tau \star_0^1 F_1} h_1(1(1)) \star_0^1 \tau(1(0)) \star_0^1 f_1(1(1))$$

we get the operation (BC):

$$2(2) \xrightarrow{(H_1 \star_0^1 \tau \star_0^1 F_1) \circ a} h_1(1(1)) \star_0^1 \tau(1(0)) \star_0^1 f_1(1(1))$$

• Operation for (CD): this is similar to the operation (AB) and we obtain:

$$2(2) \xrightarrow{(\omega \star_0^2[F_1; F_1]) \circ \nu_0^2} \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

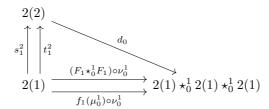
• Operation (DE): this is similar to the operation (BC) and we obtain:

$$2(2) \xrightarrow{(\tau \star_0^1 F_1 \star_0^1 F_1) \circ a} \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

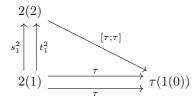
• Operation (EF): the operations:

$$f_1(1(1)) \star_0^1 f_1(1(1)) \xrightarrow{F_1 \star_0^1 F_1} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1), \qquad f_1(1(1)) \star_0^1 f_1(1(1)) \xrightarrow{f_1(\mu_0^1)} f_1(1(1)) \xrightarrow{F_1} 2(1),$$

are parallels, thus lead to the operation d_0 :



Also we have the lifting $[\tau; \tau]$:



thus this leads to:

$$2(2) \xrightarrow{\nu_0^2} 2(2) \star_0^2 2(2) \xrightarrow{[\tau;\tau] \star_0^2 d_0} \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

thus to the operation (EF):

$$2(2) \xrightarrow{([\tau;\tau]\star_0^2 d_0) \circ \nu_0^2} \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

• Operation (AG): this is similar to the operations (BC) and (DE) and we obtain:

$$2(2) \xrightarrow{(H_1 \star_0^1 H_1 \star_0^1 \tau) \circ a} h_1(1(1)) \star_0^1 f_1(1(1)) \star_0^1 \tau(1(0))$$

• Operation (GH): this is similar to the operation (EF) and we obtain:

$$2(2) \xrightarrow{\quad (d_1 \star_0^2 [\tau;\tau]) \circ \nu_0^2} \to h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0))$$

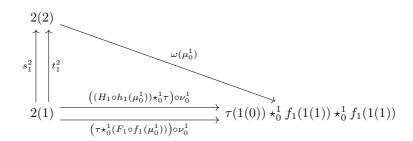
• Operation (*HF*): we use the operations:

$$h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0)) \xrightarrow{(H_1 \circ h_1(\mu_0^1)) \star_0^1 \tau} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1)$$

and

$$\tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1)) \xrightarrow{-\tau \star_0^1(F_1 \circ f_1(\mu_0^1))} 2(1) \star_0^1 2(1) \xrightarrow{\nu_0^1} 2(1)$$

also we have the equality $h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0)) = \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$, and these operations are parallels, thus we get the lifting $\omega(\mu_0^1)$:



• We have the equalities:

$$\begin{aligned} h_1(1(1)) \star_0^1 h_1(1(1)) \star_0^1 \tau(1(0)) &= h_1(1(1)) \star_0^1 \tau(1(0)) \star_0^1 f_1(1(1)) \\ &= \tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1)) \end{aligned}$$

which shows that operations (AB), (BC), (CD), (DE), (EF), (AG), (GH) and (HF) have the same arities, say:

$$\tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))$$

• With the operations (AB), (BC), (CD), (DE) and (EF) we have the following morphism in $\Theta_{\mathbb{M}^2}^{\infty}$:

$$2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2)$$

$$\downarrow \left(([H_{1}; H_{1}] \star_{0}^{2} \omega) \circ \nu_{0}^{2} \right) \star_{1}^{2} \left((H_{1} \star_{0}^{1} \tau \star_{0}^{1} F_{1}) \circ a \right) \star_{1}^{2} \left((\omega \star_{0}^{2} [F_{1}; F_{1}]) \circ \nu_{0}^{2} \right) \star_{1}^{2} \left((\tau \star_{0}^{1} F_{1} \star_{0}^{1} F_{1}) \circ a \right) \star_{1}^{2} \left(([\tau; \tau] \star_{0}^{2} d_{0}) \circ \nu_{0}^{2} \right)$$

$$\tau(1(0)) \star_{0}^{1} f_{1}(1(1)) \star_{0}^{1} f_{1}(1(1))$$

From it we get the following operation (ABCDEF) in $\Theta_{M^2}^{\infty}$:

$$2(2) \downarrow_{\nu_{1}^{2}}$$

$$2(2) \star_{1}^{2} 2(2)$$

$$\downarrow_{1_{2(2)} \star_{1}^{2} \nu_{1}^{2}}$$

$$2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2)$$

$$\downarrow_{1_{2(2)} \star_{1}^{2} 1_{2(2)} \star_{1}^{2} \nu_{1}^{2}}$$

$$2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2)$$

$$\downarrow_{1_{2(2)} \star_{1}^{2} 1_{2(2)} \star_{1}^{2} \nu_{1}^{2}}$$

$$2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2)$$

$$\downarrow_{1_{2(2)} \star_{1}^{2} 1_{2(2)} \star_{1}^{2} 1_{2(2)} \star_{1}^{2} \nu_{1}^{2}}$$

$$2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2)$$

$$\downarrow_{(([H_{1};H_{1}] \star_{0}^{2} \omega) \circ \nu_{0}^{2}) \star_{1}^{2} ((H_{1} \star_{0}^{1} \tau \star_{0}^{1} F_{1}) \circ a) \star_{1}^{2} ((\omega \star_{0}^{2} [F_{1};F_{1}]) \circ \nu_{0}^{2}) \star_{1}^{2} ((\tau \star_{0}^{1} F_{1} \star_{0}^{1} F_{1}) \circ a) \star_{1}^{2} (([\tau;\tau] \star_{0}^{2} d_{0}) \circ \nu_{0}^{2})$$

$$\tau(1(0)) \star_{0}^{1} f_{1}(1(1)) \star_{0}^{1} f_{1}(1(1))$$

• With the operations (AG), (GH) and (HF) we have the following morphism in $\Theta_{\mathbb{M}^2}^{\infty}$:

$$2(2) \star_{1}^{2} 2(2) \star_{1}^{2} 2(2)$$

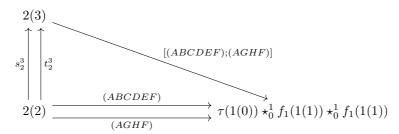
$$\downarrow ((H_{1} \star_{0}^{1} H_{1} \star_{0}^{1} \tau) \circ a) \star_{1}^{2} ((d_{1} \star_{0}^{2} [\tau; \tau]) \circ \nu_{0}^{2}) \star_{1}^{2} (\omega(\mu_{0}^{1}))$$

$$\tau(1(0)) \star_{0}^{1} f_{1}(1(1)) \star_{0}^{1} f_{1}(1(1))$$

and from it we get the following operation (AGHF) in $\Theta_{\mathbb{M}^2}^{\infty}$:

$$\begin{array}{c}
2(2) \\
\downarrow^{\nu_1^2} \\
2(2) \star_1^2 2(2) \\
\downarrow^{1_{2(2)} \star_1^2 \nu_1^2} \\
2(2) \star_1^2 2(2) \star_1^2 2(2) \\
\downarrow^{\left((H_1 \star_0^1 H_1 \star_0^1 \tau) \circ a\right) \star_1^2 \left((d_1 \star_0^2 [\tau; \tau]) \circ \nu_0^2\right) \star_1^2 \left(\omega(\mu_0^1)\right)} \\
\tau(1(0)) \star_0^1 f_1(1(1)) \star_0^1 f_1(1(1))
\end{array}$$

The operations (ABCDEF) and (AGHF) are parallels, thus lead to the lifting:



which is a 3-operation in $\Theta_{\mathbb{M}^2}^{\infty}$. But τ has dimension 2, thus the Sets-realization of the operations (ABCDEF) and (AGHF) agree, which gives the commutative diagram for pseudo-2-natural transformations.

3 The globular weak ∞ -category of globular weak ∞ -categories in the sense of Grothendieck

Thanks to the result in [4] it is possible to use the formalism of globular operads in [2], at least for the globular free contractible operad $\mathbb{B}^0_{\mathbb{C}}$ which algebras are globular weak ∞ -categories of Batanin. The main result in [4] solves a conjecture in [1] which basically claimed the equivalence of categories:

$$\operatorname{Mod}(\Theta^{\infty}_{\mathbb{M}^0}) \simeq \mathbb{B}^0_{\mathbb{C}}$$
-Alg

that is, the approach of globular weak ∞ -categories of Batanin and Grothendieck are equivalent. In this article we built the following coglobular object in \mathbb{C} at:

$$\Theta_{\mathbb{M}^0}^{\infty} \xrightarrow{\begin{array}{c} s_1^{0s_0^1} \\ \longrightarrow \end{array}} \Theta_{\mathbb{M}^1}^{\infty} \xrightarrow{\begin{array}{c} s_1^2 \\ \longrightarrow \end{array}} \Theta_{\mathbb{M}^2}^{\infty} \xrightarrow{\begin{array}{c} s_2^3 \\ \longrightarrow \end{array}} \Theta_{\mathbb{M}^3}^{\infty} \xrightarrow{\begin{array}{c} s_3^4 \\ \longrightarrow \end{array}} \Theta_{\mathbb{M}^4}^{\infty} \cdots \Theta_{\mathbb{M}^{n-1}}^{\infty} \xrightarrow{\begin{array}{c} s_{n-1}^n \\ \longrightarrow \end{array}} \Theta_{\mathbb{M}^n}^{\infty} \cdots$$

If this coglobular object $\Theta_{\mathbb{M}^{\bullet}}^{\infty}$ is a $\mathbb{B}_{\mathbb{C}}^{0}$ -coalgebra which means that we have a coaction:

$$\mathbb{B}^0_{\mathbb{C}} \longrightarrow \mathbb{C}\mathrm{oend}(\Theta^{\infty}_{\mathbb{M}^{\bullet}})$$

where $\mathbb{C}\text{oend}(\Theta_{\mathbb{M}^{\bullet}}^{\infty})$ is the coendomorphism operad associated to $\Theta_{\mathbb{M}^{\bullet}}^{\infty}$ (this exists because \mathbb{C} at has all small colimits), then this shows that globular weak (n, ∞) -transformations, for all $n \in \mathbb{N}$, by coherators, organize in a globular weak ∞ -categories in the sense of Batanin, thus in the sense of Grothendieck. Thus the $\mathbb{B}_{\mathbb{C}}^0$ -coalgebraicity of $\Theta_{\mathbb{M}^{\bullet}}^{\infty}$ means that globular weak ∞ -categories of Grothendieck organize in a globular weak ∞ -categories of Grothendieck. This fact is a necessary step for an accurate approach of globular weak ∞ -stacks as wished by Alexandre Grothendieck in [6].

A way to prove this coalgebraicity is to show that the operad $\mathbb{C}oend(\Theta_{\mathbb{M}^{\bullet}}^{\infty})$ is contractible and equipped with a composition system. We shall give a complete syntactical proof of this coalgebraicity in a future paper.

References

- [1] Dimitri Ara, Sur les ∞-groupoïdes de Grothendieck et une variante ∞-catègorique, Thèse de 3e cycle, Paris 7, Septembre 2010, https://www.i2m.univ-amu.fr/perso/dimitri.ara/files/these.pdf 1, 2, 11, 17
- [2] Michael Batanin, Monoidal globular categories as a natural environment for the theory of weak-n-categories, Advances in Mathematics (1998), volume 136, pages 39–103. 1, 2, 3, 11, 17
- [3] Clèmens Berger, A cellular nerve for higher categories, Advances in Mathematics, Volume 169, 2002, pages 118–175. 2
- [4] John Bourke, Iterated Algebraic Injectivity and the Faithfullness conjecture, https://arxiv.org/pdf/1811.09532.pdf 1, 2, 11, 17
- [5] John Bourke and Richard Garner *Monads and theories*, Advances in Mathematics, Volume 351, 31 July 2019, pages 1024–1071. 2
- [6] Alexandre Grothendieck, *Pursuing stacks*, Manuscript, https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.imj-prg.fr/~georges.maltsiniotis/ps.https://webusers.maltsiniotis/
- [7] Camell Kachour, Aspects of Globular Higher Category Theory, PhD thesis, Macquarie University, 2014 2
- [8] Camell Kachour, Steps toward the weak higher category of weak higher categories in the globular setting, Published in Categories and General Algebraic Structures with Applications, 2015, Volume 4, pages 9–42. 2, 3, 5, 11
- [9] Camell Kachour, Combinatorial approach to the Θ_0 of cubical pasting diagrams, https://arxiv.org/abs/2102.09787 2
- [10] Christian Lair and Laurent Coppey, Leçons de théorie des esquisses, http://www.numdam.org/article/DIA_1984__12__A4_0.pdf 1, 2
- [11] Makkai and Robert Paré, Accessible Categories: The Foundations of Categorical Model Theory, AMS, Contemporary Mathematics, 1989, Volume 104. 1, 2
- . 1

[12] Georges Maltsiniotis, Infini Groupoides d'après Grothendieck, preprint, 2007, https://webusers.imj-prg.fr/~georges.m

- [13] Georges Maltsiniotis, Infini catégories non strictes, une nouvelle définition, preprint, 2007, https://webusers.imj-prg.fr/~georges.maltsiniotis/ps/infctart.pdf . 1
- [14] Georges Maltsiniotis, $Grothendieck \propto -groupoids$ and still another definition of ∞ -categories, preprint, 2010, https://webusers.imj-prg.fr/~georges.maltsiniotis/ps/infgrcat.pdf . 1
- [15] Mark Weber, Familial 2-functors and parametric right adjoints, Theory and Applications of Categories, Volume 18, pages 665–732, 2007. 2

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