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Camell Kachour

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Combinatorial approach to the category Θ_0 of cubical pasting diagrams

Camell Kachour

February 22, 2021

Abstract

In these notes we describe models of globular weak (∞, m) -categories ($m \in \mathbb{N}$) in the Grothendieck style, i.e for each $m \in \mathbb{N}$ we define a globular coherator $\Theta_{\mathbb{M}^m}^\infty$ whose set-models are globular weak (∞, m) -categories. Then we describe the combinatorics of the small category Θ_0 whose objects are cubical pasting diagrams and whose morphisms are morphisms of cubical sets. This provides an accurate description of the monad, on the category of cubical sets (without degeneracies and connections), of cubical strict ∞ -categories with connections. We prove that it is a cartesian monad, solving a conjecture in [10]. This puts us in a position to describe the cubical coherator Θ_W^∞ whose set-models are cubical weak ∞ -categories with connections and the cubical coherator $\Theta_{W_0}^\infty$ whose set-models are cubical weak ∞ -groupoids with connections.

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Introduction

Coherators were initiated by Alexander Grothendieck [7] to properly define globular weak ∞ -groupoids. A coherator $\Theta_{\mathbb{M}^0}^\infty$ for globular weak ∞ -groupoids is a theory in the sense of [6] such that $\text{Mod}(\Theta_{\mathbb{M}^0}^\infty)$ is the category of globular weak ∞ -groupoids. These theories generalize those of Lawvere and are powerful yet simple enough to capture many higher structures. For example, a slight modification of the definition of the globular coherator $\Theta_{\mathbb{M}^0}^\infty$ (see [13]) leads to the definition of another globular coherator $\Theta_{\mathbb{M}^1}^\infty$ whose set-models are globular weak ∞ -categories; such models are thus called *Grothendieck's globular weak ∞ -categories*. In [1] it is conjectured that these models are equivalent to Batanin's globular weak ∞ -categories [2], and this conjecture has been proved in [5]. In order to have a feel for the simplicity of this Grothendieck's approach, we first use it to describe globular models of weak (∞, m) -categories ($m \in \mathbb{N}$). Thus for each $m \in \mathbb{N}$ we build a globular coherator $\Theta_{\mathbb{M}^m}^\infty$ whose set-models are globular weak (∞, m) -categories. The author believes these models (for all $m \in \mathbb{N}$) are the simplest in the literature so far (see for example [4, 8]).

The non-trivial part of this article is to describe cubical pasting diagrams. For that we use coordinates of networks which is a formalism close to that of tensors for differential geometry. In our language cubical pasting diagrams are called *rectangular divisors* which are formal finite sums of cubes indexed with coordinates of rectangular shapes. For each rectangular divisor we associate a specific inductive sketch. In fact we shall see that such rectangular divisors form a cubical strict monoidal ∞ -category thus lead to a cubical strict monoidal ∞ -category for their underlying inductive sketches. These sketches are the objects of the cubical Θ_0 . This combinatoric description of cubical pasting diagrams lead us to the monad, on the category of cubical sets (without degeneracies and connections), of cubical strict ∞ -categories with connections. Having then an accurate description, we prove that these monads are cartesian, as conjectured in [10]. See also [11] where, based on this conjecture, we constructed a fundamental cubical weak ∞ -groupoid functor. Also we include here an accurate construction of the cubical coherator Θ_W^∞ whose set-models are cubical weak ∞ -categories with connections and the cubical coherator $\Theta_{W^0}^\infty$ whose set-models are cubical weak ∞ -groupoids with connections. Cubical coherators have also been introduced recently and independently in [3], and it will be interesting to compare that approach with ours.

Here we summarize main achievements of this article :

- In 1.3.5 we build for each $m \in \mathbb{N}$, a globular coherator $\Theta_{\mathbb{M}^m}^\infty$ whose set-models are models of Grothendieck's globular weak (∞, m) -categories.
- In 2.4 we prove that the monad $\mathbb{R} = (R, i, m)$ of cubical reflexive sets is cartesian.
- In 2.5 we prove that the set $\mathbb{C}\text{-Past}$ of cubical pasting diagrams (called rectangular divisors here) is equipped with a structure of cubical strict ∞ -category with connections.
- In 2.5 we prove that the set of sketches associated to cubical pasting diagrams (called rectangular divisors here) is equipped with a structure of cubical strict ∞ -category with connections.
- In 2.7 we prove that the monad $\mathbb{S} = (S, \lambda, \mu)$ acting on $\mathbb{C}\text{Sets}$ which algebras are cubical strict ∞ -categories with connections (described in [9, 10]) is cartesian. A simple consequence appears in 2.7 where we indicate that the other monad $\mathbb{S} = (S, \lambda, \mu)$ acting on $\mathbb{C}\text{Sets}$ which algebras are cubical strict ∞ -categories (without connections) is cartesian.
- In 3 we build the cubical coherator Θ_W^∞ whose models are cubical weak ∞ -categories with connections, and in 4 we build the cubical coherator $\Theta_{W^0}^\infty$ whose models are cubical weak ∞ -groupoids with connections.

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I dedicate this work to my sons, Mohamed-Réda and Ali-Réda.

1 Coherators for globular weak (∞, m) -categories ($m \in \mathbb{N}$)

1.1 Globular magmatic structures

Consider the small category \mathbb{G} with objects $1(n)$ for all $n \in \mathbb{N}$, with morphisms those generated for all $n \in \mathbb{N}$ by the *cosources* $1(n-1) \xrightarrow{s_{n-1}^n} 1(n)$ and the *cotargets* $1(n-1) \xrightarrow{t_{n-1}^n} 1(n)$, which satisfy the following coglobular relations :

- (i) $s_{n-1}^n \circ s_n^{n+1} = t_{n-1}^n \circ s_n^{n+1}$,
- (ii) $s_{n-1}^n \circ t_n^{n+1} = t_{n-1}^n \circ t_n^{n+1}$,

The small category \mathbb{G} is called the *globe category* and we may represent it schematically with its generators :

$$1(0) \begin{array}{c} \xrightarrow{s_0^1} \\ \xrightarrow{t_0^1} \end{array} 1(1) \begin{array}{c} \xrightarrow{s_1^2} \\ \xrightarrow{t_1^2} \end{array} 1(2) \begin{array}{c} \xrightarrow{s_2^3} \\ \xrightarrow{t_2^3} \end{array} 1(3) \begin{array}{c} \xrightarrow{s_3^4} \\ \xrightarrow{t_3^4} \end{array} 1(4) \cdots 1(n-1) \begin{array}{c} \xrightarrow{s_{n-1}^n} \\ \xrightarrow{t_{n-1}^n} \end{array} 1(n) \cdots$$

Definition 1 Globular sets are presheaves on \mathbb{G}^{op} . The category of globular sets is denoted Glob . \square

A globular ∞ -magma M is given by a globular set $\mathbb{G}^{\text{op}} \xrightarrow{M} \text{Sets}$ equipped with operations $M_n \times_{M_p} M_n \xrightarrow{\circ_p^n} M_n$ for all $n \geq 1$ and all $0 \leq p \leq n-1$ such that :

- for $0 \leq p < q < m$, $s_q^m(y \circ_p^m x) = s_q^m(y) \circ_p^q s_q^m(x)$ and $t_q^m(y \circ_p^m x) = t_q^m(y) \circ_p^q t_q^m(x)$
- for $0 \leq q < p < m$, $s_q^m(y \circ_p^m x) = s_q^m(y) = s_q^m(x)$ and $t_q^m(y \circ_p^m x) = t_q^m(y) = t_q^m(x)$
- for $0 \leq p = q < m$, $s_q^m(y \circ_p^m x) = s_q^m(x)$ and $t_q^m(y \circ_p^m x) = t_q^m(x)$

A globular reflexive ∞ -magma is an ∞ -magma equipped with map for reflexivity : $M_n \xrightarrow{1_{n+1}^n} M_{n+1}$, $n \geq 0$ such that :

- $s_k^n(1_n^k(x)) = x = t_k^n(1_n^k(x))$
- $1_n^q(1_q^n(x)) = 1_n^q(x)$

Morphisms between reflexive ∞ -magmas are morphisms of reflexive globular sets between their underlying reflexive globular set structure, i.e for $M \xrightarrow{f} M'$ we have commutative diagrams :

$$\begin{array}{ccc} M_{n+1} & \xrightarrow{f_{n+1}} & M'_{n+1} \\ \uparrow 1_{n+1}^n & & \uparrow 1_{n+1}^n \\ M_n & \xrightarrow{f_n} & M'_n \end{array}$$

which also preserve operations \circ_p^n . The category of reflexive ∞ -magmas is denoted $\infty\text{-Mag}_r$.

An (∞, m) -globular set is a globular set X equipped with j_{n-1}^n -reversors, i.e with maps $X_n \xrightarrow{j_{n-1}^n} X_n$ which satisfy the following equalities :

$$\begin{array}{ccc} X_n & \xrightarrow{j_{n-1}^n} & X_n \\ \searrow s_{n-1}^n & & \swarrow t_{n-1}^n \\ & X_{n-1} & \end{array} \quad \begin{array}{ccc} X_n & \xrightarrow{j_{n-1}^n} & X_n \\ \searrow t_{n-1}^n & & \swarrow s_{n-1}^n \\ & X_{n-1} & \end{array}$$

A morphism of (∞, m) -globular sets is a morphism $X \xrightarrow{f} X'$ of globular sets which satisfy for all $n \geq m$ the following equalities :

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & X'_n \\ j_{n-1}^n \downarrow & & \downarrow j_{n-1}^n \\ X_n & \xrightarrow{f_n} & X'_n \end{array}$$

The category of (∞, m) -globular sets is denoted $(\infty, m)\text{-Glob}$.

A globular reflexive (∞, m) -magma is a globular reflexive ∞ -magma M equipped with a structure of globular (∞, m) -set; a morphism $M \xrightarrow{f} M'$ of globular reflexive (∞, m) -magmas is a morphism of globular reflexive ∞ -magmas which is also a morphism of (∞, m) -sets; the category of globular reflexive (∞, m) -magmas is denoted $(\infty, m)\text{-Mag}_r$.

Remark 1 A globular strict ∞ -category C is given by a globular reflexive ∞ -magma C such that we have the following equalities :

- $x \circ_k^n 1_n^k(s_k^n(x)) = x$ and $1_n^k(t_k^n(x)) \circ_k^n x = x$
- $1_n^q(y \circ_p^q x) = 1_p^q(y) \circ_p^n 1_p^q(x)$
- $x \circ_k^n (y \circ_k^n z) = (x \circ_k^n y) \circ_k^n z$
- $(y' \circ_q^n x') \circ_p^n (y \circ_q^n x) = (y' \circ_p^n y) \circ_q^n (x' \circ_p^n x)$

The category of globular strict ∞ -categories is denoted $\infty\text{-CAT}$. A globular strict (∞, m) -category is given by an (∞, m) -globular set C which is also a globular strict ∞ -category such that if $\alpha \in C_n$ ($n \geq m$) then $\alpha \circ_{n-1}^n j_{n-1}^n(\alpha) = 1_n^{n-1}(t_{n-1}^n(\alpha))$ and $j_{n-1}^n(\alpha) \circ_{n-1}^n \alpha = 1_n^{n-1}(s_{n-1}^n(\alpha))$. This n -cell $j_{n-1}^n(\alpha)$ of C_n is called a \circ_{n-1}^n -inverse of α and it is straightforward to see that such \circ_{n-1}^n -inverse is uniquely defined. The category of globular strict (∞, m) -categories is defined as the full subcategory of $\infty\text{-CAT}$ which objects are globular strict (∞, m) -categories and is denoted $(\infty, m)\text{-CAT}$. \square

1.2 Globular Theories

1.2.1 Globular extensions

A globular tree t is given by a table of non-negative integers :

$$\begin{pmatrix} i_1 & & i_2 & & i_3 & \cdot & \cdot & \cdot & i_{k-1} & & i_k \\ & & i'_1 & & i'_2 & \cdot & \cdot & \cdot & \cdot & & i'_{k-1} \end{pmatrix}$$

where $k \geq 1$, $i_l > i'_l < i_{l+1}$ and $1 \leq l \leq k-1$.

Let \mathcal{C} a category and let $\mathbb{G} \xrightarrow{F} \mathcal{C}$ a functor. We denote $F(1(n)) = D^n$ and we shall keep the same notations for the image of cosources : $F(s_{i_l}^{i_l}) = s_{i_l}^{i_l}$, and for the image of cotargets : $F(t_{i'_l}^{i'_l}) = t_{i'_l}^{i'_l}$, because no risk of confusion will occur. In this case $\mathbb{G} \xrightarrow{F} \mathcal{C}$ is called a globular extension if for all trees t as just above, the colimit of the following diagram exist in \mathcal{C} :

$$\begin{array}{ccccccc} D^{i_1} & & & & D^{i_2} & & & & D^{i_3} & & \dots & & & & D^{i_{k-1}} & & & & D^{i_k} \\ & \swarrow & & & \swarrow & & & & \swarrow & & \dots & & & & \swarrow & & & & \swarrow \\ t_{i'_1}^{i'_1} & & D^{i'_1} & & s_{i'_1}^{i_2} & & & & t_{i'_2}^{i'_2} & & D^{i'_2} & & & & s_{i'_2}^{i_3} & & & & t_{i'_{k-1}}^{i'_{k-1}} & & D^{i'_{k-1}} \end{array}$$

Remark 2 In [7] Alexander Grothendieck calls these colimits *globular sums*. □

A morphism of globular extensions, also called *globular functor*, is given by a commutative triangle in CAT :

$$\begin{array}{ccc} & & \mathcal{C} \\ & \nearrow F & \downarrow H \\ \mathbb{G} & & \mathcal{C}' \\ & \searrow F' & \end{array}$$

such that the functor H preserves globular sums. The category of globular extensions is denoted $\mathbb{G}\text{-Ext}$. In fact this category has an initial object denoted $\mathbb{G} \xrightarrow{i} \Theta_0$. And the small category Θ_0 can be described as the full subcategory of Glob which objects are globular trees, and its role is central for describing different sketches which set models are globular higher structures. In particular this small category Θ_0 is the basic inductive sketch we shall need to describe coherators which set models are globular weak (∞, m) -categories ($m \in \mathbb{N}$).

1.2.2 Globular theories

A globular theory is given by a globular extension $\mathbb{G} \xrightarrow{F} \mathcal{C}$ such that the unique induced functor \overline{F} which makes commutative the diagram :

$$\begin{array}{ccc} & & \Theta_0 \\ & \nearrow i & \downarrow \overline{F} \\ \mathbb{G} & & \mathcal{C} \\ & \searrow F & \end{array}$$

induces a bijection between objects of Θ_0 and objects of \mathcal{C} . The full subcategory of $\mathbb{G}\text{-Ext}$ which objects are globular theories is denoted $\mathbb{G}\text{-Th}$. Consider an object $\mathbb{G} \xrightarrow{F} \mathcal{C}$, in particular it induces the globular functor $\Theta_0 \xrightarrow{\overline{F}} \mathcal{C}$ as just above, which is a bijection on objects. A set model of (F, \mathcal{C}) or for \mathcal{C} for short, is given by a functor : $\mathcal{C} \xrightarrow{X} \text{Sets}$, such that the functor $X \circ \overline{F}$:

$$\Theta_0 \xrightarrow{\overline{F}} \mathcal{C} \xrightarrow{X} \text{Sets}$$

sends globular sums to globular products¹, thus for all objects t of Θ_0 :

$$\begin{pmatrix} i_1 & & i_2 & & i_3 & \cdot & \cdot & \cdot & i_{k-1} & & i_k \\ & & i'_1 & & i'_2 & \cdot & \cdot & \cdot & \cdot & & i'_{k-1} \end{pmatrix}$$

¹Globular products are just dual to globular sums.

we have

$$\begin{aligned}
X(\overline{F}(t)) &= X \left(\text{colim} \left(\begin{array}{ccc} D^{i_1} & & D^{i_2} \\ & \swarrow_{t_{i'_1}^{i_1}} & \nearrow_{s_{i'_1}^{i_2}} \\ & D^{i'_1} & \end{array} \quad \dots \quad \begin{array}{ccc} D^{i_{k-1}} & & D^{i_k} \\ & \swarrow_{t_{i'_{k-1}}^{i_{k-1}}} & \nearrow_{s_{i'_{k-1}}^{i_k}} \\ & D^{i'_{k-1}} & \end{array} \right) \right) \\
&= X \left((D^{i_1}, t_{i'_1}^{i_1}) \coprod_{D^{i'_1}} (s_{i'_1}^{i_2}, D^{i_2}, t_{i'_2}^{i_3}) \coprod_{D^{i'_2}} \dots \coprod_{D^{i'_{k-1}}} (s_{i'_{k-1}}^{i_k}, D^{i_k}) \right) \\
&\simeq X(D^{i_1}) \times_{X(D^{i'_1})} \dots \times_{X(D^{i'_{k-1}})} X(D^{i_k}) \\
X(\overline{F}(t)) &= X \left(\text{colim} \left(\begin{array}{ccc} D^{i_1} & & D^{i_2} \\ & \swarrow_{t_{i'_1}^{i_1}} & \nearrow_{s_{i'_1}^{i_2}} \\ & D^{i'_1} & \end{array} \quad \dots \quad \begin{array}{ccc} D^{i_{k-1}} & & D^{i_k} \\ & \swarrow_{t_{i'_{k-1}}^{i_{k-1}}} & \nearrow_{s_{i'_{k-1}}^{i_k}} \\ & D^{i'_{k-1}} & \end{array} \right) \right) \\
X \left((D^{i_1}, t_{i'_1}^{i_1}) \coprod_{D^{i'_1}} (s_{i'_1}^{i_2}, D^{i_2}, t_{i'_2}^{i_3}) \coprod_{D^{i'_2}} \dots \coprod_{D^{i'_{k-1}}} (s_{i'_{k-1}}^{i_k}, D^{i_k}) \right) &\simeq X(D^{i_1}) \times_{X(D^{i'_1})} \dots \times_{X(D^{i'_{k-1}})} X(D^{i_k})
\end{aligned}$$

The category of set models of \mathcal{C} is the full subcategory of the category of presheaves $[\mathcal{C}, \text{Sets}]$ which objects are set models of \mathcal{C} , and it is denoted $\text{Mod}(\mathcal{C})$.

1.2.3 Examples of globular theories

Example 1 The theory $\Theta_{\mathbb{M}}$

The forgetful functor U :

$$\begin{array}{c}
\infty\text{-Mag}_r \\
\uparrow \quad \downarrow \\
F \dashv U \\
\downarrow \\
\text{Glob}
\end{array}$$

from the category $\infty\text{-Mag}_r$ of globular reflexive ∞ -magmas to the category Glob of globular sets is right adjoint, which left adjoint is denoted F , and this induce a monad $\mathbb{M} = (M, \eta, \mu)$ on Glob such that we have the equivalence of categories $\infty\text{-Mag}_r \simeq \mathbb{M}\text{-Alg}$ because U is monadic. The full subcategory $\Theta_{\mathbb{M}} \subset \text{Kl}(\mathbb{M})$ of the Kleisli category of \mathbb{M} which objects are trees is called the theory of reflexive globular ∞ -magmas. In fact we have the following equivalences of categories :

$$\infty\text{-Mag}_r \simeq \mathbb{M}\text{-Alg} \simeq \text{Mod}(\Theta_{\mathbb{M}})$$

Example 2 The theories $\Theta_{\mathbb{M}^m}$ ($m \in \mathbb{N}$)

The forgetful functor U^m ($m \in \mathbb{N}$) :

$$\begin{array}{c}
(\infty, m)\text{-Mag}_r \\
\uparrow \quad \downarrow \\
F^m \dashv U^m \\
\downarrow \\
\text{Glob}
\end{array}$$

from the category $(\infty, m)\text{-Mag}_r$ of globular reflexive (∞, m) -magmas to the category Glob of globular sets is right adjoint, which left adjoint is denoted F^m , and this induce a monad $\mathbb{M}^m = (M^m, \eta^m, \mu^m)$ on Glob such that we have the equivalence of categories $(\infty, m)\text{-Mag}_r \simeq \mathbb{M}^m\text{-Alg}$ because U^m is monadic. The full subcategory $\Theta_{\mathbb{M}^m} \subset \text{Kl}(\mathbb{M}^m)$ of the Kleisli category of \mathbb{M}^m which objects are trees is called the theory of reflexive globular (∞, m) -magmas. In fact we have the following equivalences of categories :

$$(\infty, m)\text{-Mag}_r \simeq \mathbb{M}^m\text{-Alg} \simeq \text{Mod}(\Theta_{\mathbb{M}^m})$$

1.3 Globular coherators

1.3.1 Admissibility

Let $\mathbb{G} \xrightarrow{F} \mathcal{C}$ be a globular theory, i.e an object of $\mathbb{G}\text{-Th}$; two arrows : $D^n \xrightarrow[f]{g} t$ in \mathcal{C} are parallels if $fs_{n-1}^n = gt_{n-1}^n$ and $ft_{n-1}^n = gs_{n-1}^n$:

$$\begin{array}{ccc} D^n & \xrightarrow[f]{g} & t \\ \uparrow s_{n-1}^n & & \uparrow t_{n-1}^n \\ D^{n-1} & & \end{array}$$

Consider a couple (f, g) of parallels arrows in \mathcal{C} as just above. We say that it is admissible or algebraic if they don't belong to the image of the globular functor \overline{F} :

$$\begin{array}{ccc} & & \Theta_0 \\ & \nearrow i & \downarrow \overline{F} \\ \mathbb{G} & & \mathcal{C} \\ & \searrow F & \end{array}$$

Consider a couple (f, g) of arrows of \mathcal{C} which is admissible as just above; a lifting of (f, g) is given by an arrow h :

$$\begin{array}{ccc} D^{n+1} & & \\ \uparrow s_n^{n+1} & \nearrow h & \\ D^n & \xrightarrow[f]{g} & t \\ \uparrow t_n^{n+1} & & \end{array}$$

such that $hs_n^{n+1} = f$ and $ht_n^{n+1} = g$

1.3.2 Batanin-Grothendieck Sequences

We now define the Batanin-Grothendieck sequence² associated to a globular theory $\mathbb{G} \xrightarrow{F} \mathcal{C}$. We build it by the following induction :

- If $n = 0$ we start with the couple (\mathcal{C}, E) where E denotes the set of admissible pairs of arrows of \mathcal{C} ; we shall write $(\mathcal{C}_0, E_0) = (\mathcal{C}, E)$ this first step.
- If $n = 1$ we consider then the couple (\mathcal{C}_1, E_1) where \mathcal{C}_1 is obtained by formally adding in $\mathcal{C}_0 = \mathcal{C}$ the liftings of all elements $(f, g) \in E_0 = E$, and E_1 is the set of admissible couples of arrows in \mathcal{C}_1 which are not elements of the set E_0 ;
- If for $n \geq 2$ the couple (\mathcal{C}_n, E_n) is well defined then \mathcal{C}_{n+1} is obtained by formally adding in \mathcal{C}_n the liftings of all elements of E_n , and E_{n+1} is the set of couples of arrows of \mathcal{C}_{n+1} which are not elements of E_n

we give a slightly different but equivalent induction to build the Batanin-Grothendieck sequence for such globular theory $\mathbb{G} \xrightarrow{F} \mathcal{C}$:

- If $n = 0$ we start with the couple (\mathcal{C}, E) where E is the set of couple of arrow which are admissible of \mathcal{C} ; we denote $E = E_0 = E'_0 = E'_0 \setminus \emptyset$ (we shall see soon the reason of these notations), and $\mathcal{C}_0 = \mathcal{C}$;
- If $n = 1$ we consider the couple (\mathcal{C}_1, E_1) where \mathcal{C}_1 is obtained by formally adding in \mathcal{C}_0 all liftings of the elements $(f, g) \in E_0$, E'_1 is the set of all pairs of arrows which are admissible in \mathcal{C}_1 , and $E_1 = E'_1 \setminus E_0$; remark that $E_0 = E'_0 \subset E'_1$;
- If $n = 2$ we consider the couple (\mathcal{C}_2, E_2) where \mathcal{C}_2 is obtained by formally adding in \mathcal{C}_1 all liftings of the elements $(f, g) \in E_1$, E'_2 is the set of all pairs of arrows which are admissible in \mathcal{C}_2 , and $E_2 = E'_2 \setminus E'_1$;
- For $n \geq 3$ we suppose that the couple (\mathcal{C}_n, E_n) is well defined with $E_n = E'_n \setminus E'_{n-1}$, then \mathcal{C}_{n+1} is obtained by formally adding in \mathcal{C}_n all liftings of the elements $(f, g) \in E_n$, E'_{n+1} is the set of all pairs of arrows which are admissible in \mathcal{C}_{n+1} , and $E_{n+1} = E'_{n+1} \setminus E'_n$;

²Coherators associated to such sequence are called of *Batanin-Leinster type* by some authors.

The Batanin-Grothendieck sequence of the globular theory $\mathbb{G} \xrightarrow{F} \mathcal{C}$ produces the following filtered diagram $(\mathbb{N}, \leq) \xrightarrow{\mathcal{C}_\bullet} \mathbb{G}\text{-Th}$ in the category $\mathbb{G}\text{-Th}$:

$$\mathcal{C}_0 \xrightarrow{i_1} \mathcal{C}_1 \xrightarrow{i_2} \dots \xrightarrow{i_n} \mathcal{C}_n \longrightarrow \dots$$

1.3.3 Coherators for globular theories

We start with datas of the previous subsection, i.e with the Batanin-Grothendieck sequence $(\mathbb{N}, \leq) \xrightarrow{\mathcal{C}_\bullet} \mathbb{G}\text{-Th}$ for a globular theory $\mathbb{G} \xrightarrow{F} \mathcal{C}$.

Definition 2 The colimit $\mathbb{G} \xrightarrow{F_\infty} \mathcal{C}_\infty$ of the previous filtered diagram \mathcal{C}_\bullet :

$$\begin{array}{ccccccc} \mathcal{C}_0 & \xrightarrow{i_1} & \mathcal{C}_1 & \xrightarrow{i_2} & \dots & \xrightarrow{i_n} & \mathcal{C}_n & \longrightarrow & \dots \\ & & & & & & & \searrow & \\ & & & & & & & & \mathcal{C}_\infty \end{array}$$

is called the globular coherator of the type Batanin-Grothendieck associated to the globular theory $\mathbb{G} \xrightarrow{F} \mathcal{C}$. \square

For shorter terminology we shall say that $\mathbb{G} \xrightarrow{F_\infty} \mathcal{C}_\infty$ is the coherator associated to the globular theory $\mathbb{G} \xrightarrow{F} \mathcal{C}$. It is straightforward to see that the Batanin-Grothendieck construction of coherators associated to globular theory is functorial, and the following functor Φ is called the Batanin-Grothendieck functor :

$$\begin{array}{ccc} \mathbb{G}\text{-Th} & \xrightarrow{\Phi} & \mathbb{G}\text{-Th} \\ \mathcal{C} & \longmapsto & \mathcal{C}^\infty \end{array}$$

1.3.4 The coherator $\Theta_{\mathbb{M}}^\infty$

The coherator associated to the globular theory $\mathbb{G} \xrightarrow{j} \Theta_{\mathbb{M}}$ that we obtaine with the composition :

$$\mathbb{G} \xrightarrow{i} \Theta_0 \hookrightarrow \Theta_{\mathbb{M}}$$

is denoted $\Theta_{\mathbb{M}}^\infty$ and $\text{Mod}(\Theta_{\mathbb{M}}^\infty)$ is the category of globular weak ∞ -categories of Grothendieck.

Remark 3 In 2019 John Bourke has proved [5] the *Ara conjecture* [1] which says that the category of globular weak ∞ -categories of Batanin is equivalent to the category of globular weak ∞ -categories of Grothendieck :

$$\text{Mod}(\Theta_{\mathbb{B}_C^0}) \simeq \text{Mod}(\Theta_{\mathbb{M}}^\infty)$$

where here \mathbb{B}_C^0 denotes the globular operad of Batanin [2] which algebras are his models of globular weak ∞ -categories and $\Theta_{\mathbb{B}_C^0}$ is its associated theory. \square

1.3.5 The coherator $\Theta_{\mathbb{M}^m}^\infty$ ($m \in \mathbb{N}$)

The coherator associated to the globular theory $\mathbb{G} \xrightarrow{j_m} \Theta_{\mathbb{M}^m}$ is denoted $\Theta_{\mathbb{M}^m}^\infty$ and $\text{Mod}(\Theta_{\mathbb{M}^m}^\infty)$ is the category of globular weak (∞, m) -categories of Grothendieck ($m \geq 0$). If $m = 0$, the coherator $\Theta_{\mathbb{M}^0}^\infty$ is the one of globular weak ∞ -groupoids of Grothendieck. We trivially have the following filtration in the category $\mathbb{G}\text{-Th}$:

$$\begin{array}{ccccccc} \dots & \longrightarrow & \Theta_{\mathbb{M}^{m+1}}^\infty & \longrightarrow & \Theta_{\mathbb{M}^m}^\infty & \longrightarrow & \dots & \longrightarrow & \Theta_{\mathbb{M}^0}^\infty \\ & & & & \Theta_{\mathbb{M}}^\infty & & & & \\ & & \swarrow & \downarrow & \searrow & & & & \\ \dots & \longrightarrow & \Theta_{\mathbb{M}^{m+1}}^\infty & \longrightarrow & \Theta_{\mathbb{M}^m}^\infty & \longrightarrow & \dots & \longrightarrow & \Theta_{\mathbb{M}^0}^\infty \end{array}$$

which shows that we have the following inclusion of functors when passing to set models :

$$\begin{array}{ccccccc} \text{Mod}(\Theta_{\mathbb{M}^0}^\infty) & \xrightarrow{i_1} & \text{Mod}(\Theta_{\mathbb{M}^1}^\infty) & \xrightarrow{i_2} & \dots & \xrightarrow{i_n} & \text{Mod}(\Theta_{\mathbb{M}^m}^\infty) & \longrightarrow & \dots \\ & & & & & & & \searrow & \\ & & & & & & & & \text{Mod}(\Theta_{\mathbb{M}}^\infty) \end{array}$$

We finish this section by recalling the *Grothendieck Conjecture for Homotopy Theory* :

Conjecture (Grothendieck's Conjecture for Homotopy Theory) *The category $\text{Mod}(\Theta_{\mathbb{M}^m}^\infty)$ is Quillen equivalent to categories of simplicial models of weak (∞, m) -categories (for all $m \in \mathbb{N}$).*

See for example [4] for such existing simplicial models.

2 Cubical Pasting Diagrams

2.1 Tensorial notation

In this section we introduce tensorial notation and shall see that contraction and dilatation of tensors provide interesting cubical strict ∞ -categories, though trivial. In particular it reveals that tensorial calculus has an intrinsic cubical nature.

- For each $n \in \mathbb{N}$ we shall use a coordinate system \mathcal{Z}_n of n -dimensional networks $i \in \{1, \dots, n\}$ such that each $i \in \{1, \dots, n\}$ is the direction of a n -cube whose coordinates are indexed by this network. The coordinate of a n -cube C in \mathcal{Z}_n is written $dx_{k_1}^1 \otimes \dots \otimes dx_{k_j}^j \otimes \dots \otimes dx_{k_n}^n$ which means that C is located for each direction $j \in \{1, \dots, n\}$ at the *depth* $k_j \in \mathbb{Z}$. When no confusion occur we shall denote $dx_{k_i}^i := dx_{k_1}^1 \otimes \dots \otimes dx_{k_j}^j \otimes \dots \otimes dx_{k_n}^n$.

Remark 4 A coordinate $dx_{k_i}^i$ must be thought up to its translations in the network \mathcal{Z}_n . Indeed it is straightforward to see that two coordinates $dx_{k_i}^i, dx_{k'_i}^i \in \mathcal{Z}_n$ are related by translations. For example any coordinates $dx_{k_i}^i \in \mathcal{Z}_n$ gives the coordinate $dx_1^i := dx_1^1 \otimes \dots \otimes dx_1^j \otimes \dots \otimes dx_1^n$ by translations along all directions $j \in \llbracket 1, n \rrbracket$. \square

Two coordinates $dx_{k_i}^i = dx_{k_1}^1 \otimes \dots \otimes dx_{k_n}^n$ and $dx_{k'_i}^i = dx_{k'_1}^1 \otimes \dots \otimes dx_{k'_n}^n$ are j -adjacent if $k_j = k'_j + 1$ or $k_j = k'_j - 1$.

The j -contraction of the coordinate $dx_{k_1}^1 \otimes \dots \otimes dx_{k_j}^j \otimes \dots \otimes dx_{k_n}^n$ is defined as the coordinate

$$dx_{k_i}^i \setminus j = dx_{k_1}^1 \otimes \dots \otimes \widehat{dx_{k_j}^j} \otimes \dots \otimes dx_{k_n}^n$$

in \mathcal{Z}_{n-1} defined by removing the direction j and re-indexing :

$$dx_{k_i}^i \setminus j := dx_{k_1}^1 \otimes \dots \otimes dx_{k_{j-1}}^{j-1} \otimes dx_{k_{j+1}}^j \otimes \dots \otimes dx_{k_n}^{n-1}$$

Sometimes we use also the notation $a_j(dx_{k_i}^i)$ for $dx_{k_i}^i \setminus j$.

If we apply these contractions p -times then we obtain the following coordinate in \mathcal{Z}_{n-p} :

$$dx_{k_i}^i \setminus (j_1, \dots, j_p)$$

where the order of occurrences of the j 's in (j_1, \dots, j_p) is important just because if σ is an element of the permutation group S_p then the action :

$$\sigma \cdot dx_{k_i}^i \setminus (j_1, \dots, j_p) := dx_{k_i}^i \setminus (j_{\sigma(1)}, \dots, j_{\sigma(p)})$$

does not imply the equality between $dx_{k_i}^i \setminus (j_1, \dots, j_p)$ and $dx_{k_i}^i \setminus (j_{\sigma(1)}, \dots, j_{\sigma(p)})$.

The j -dilatation of the coordinate $dx_{k_1}^1 \otimes \dots \otimes dx_{k_j}^j \otimes \dots \otimes dx_{k_n}^n$ is a coordinate in \mathcal{Z}_{n+1} defined by adding in the direction j the guy dx_1^j and re-indexing :

$$dx_{k_i}^i + j := dx_{k_1}^1 \otimes \dots \otimes dx_{k_{j-1}}^{j-1} \otimes dx_{k_1}^j \otimes dx_{k_j}^{j+1} \dots \otimes dx_{k_n}^{n+1}$$

and if we apply these dilatations p -times then we obtain the following coordinate in \mathcal{Z}_{n+p} :

$$dx_{k_i}^i + (j_1, \dots, j_p)$$

where the order of occurrences of the j 's in (j_1, \dots, j_p) is important.

- A n -configuration is given by a family $C_n = \{dx_{k_i}^i/k_1 \in K_1, \dots, k_n \in K_n, \forall i \in \llbracket 1, n \rrbracket, K_i \subset \mathbb{Z} \text{ is a finite set} \}$ of coordinates $dx_{k_i}^i$ in \mathcal{Z}_n . We can also use the notation $C_n = dx_{k_i}^{i,1} + \dots + dx_{k_i}^{i,l} + \dots + dx_{k_i}^{i,r}$ for this n -configuration, where each $dx_{k_i}^{i,l}$ ($l \in \llbracket 1, r \rrbracket$) is a coordinate in \mathcal{Z}_n and $r = \sharp(K_1 \times \dots \times K_n)$. This last notation shall be useful especially when we shall deal with *divisors* in the section 2.5.

Remark 5 A n -configuration C_n must be thought up to its translations in the network \mathcal{Z}_n . \square

- The j -contraction $a_j(C_n)$ of the n -configuration C_n is given by the following $(n-1)$ -configuration : $a_j(C_n) = a_j(dx_{k_i}^{i,1}) + \dots + a_j(dx_{k_i}^{i,l}) + \dots + a_j(dx_{k_i}^{i,r})$.
- If C_n is an n -configuration then it is straightforward to see that others n -configurations C'_n can be equivalent to it by translations. For example if we write $C_n = dx_{k_i}^{i,1} + \dots + dx_{k_i}^{i,l} + \dots + dx_{k_i}^{i,r}$ then we can translate it along the direction $j \in \llbracket 1, n \rrbracket$ with any integer $k \in \mathbb{Z}$ such that the resulting n -configurations C'_n has all its coordinates with depth ≥ 1 for the direction j .
- Two configurations C_n and C'_n are adjacent if
 - $C_n \cap C'_n = \emptyset$
 - $\exists j \in \llbracket 1, n \rrbracket$ and $dx_{k_i}^i \in C_n, dx_{k'_i}^i \in C'_n$ such that $a_j(dx_{k_i}^i) = a_j(dx_{k'_i}^i)$

If two configurations C_n and C'_n are adjacent then they produce the new configuration $C_n +_j C'_n$:

$$C_n +_j C'_n := C_n \cup C'_n$$

that we call their *pasting* along the direction j .

- If $C_n = dx_{k_i}^{i,1} + \dots + dx_{k_i}^{i,l} + \dots + dx_{k_i}^{i,r}$ is an n -configuration then we can associate to it its j -dilatation $\text{dilat}_j(C_n)$, which is the $(n+1)$ -configuration in \mathcal{Z}_{n+1} given by $\text{dilat}_j(C_n) = C_n +_j := (dx_{k_i}^{i,1} + j) + \dots + (dx_{k_i}^{i,l} + j) + \dots + (dx_{k_i}^{i,r} + j)$.
- A *connected* n -configuration is given by a family $C_n = \{dx_{k_i}^i/k_1 \in K_1, \dots, k_n \in K_n, \forall i \in \llbracket 1, n \rrbracket, K_i \subset \mathbb{N} \text{ is a finite set} \}$ of coordinates $dx_{k_i}^i$ in \mathcal{Z}_n such that we have the following *connexity property* :

$\forall dx_{k_i}^i, dx_{k'_i}^i$ in C_n we have :

$\exists r \in \mathbb{N}, \exists l \in \llbracket 1, r \rrbracket, \exists k_j^l \in \mathbb{N}$ where each coordinate $dx_{k_1^1}^1 \otimes \dots \otimes dx_{k_n^n}^n$ belongs to C_n , such that we have the following *zigzag of contractions* between $dx_{k_i}^i$ and $dx_{k'_i}^i$:

$$a_{i_1}(dx_{k_1^1}^1 \otimes \dots \otimes dx_{k_n^n}^n) = a_{i_1}(dx_{k_2^1}^1 \otimes \dots \otimes dx_{k_n^n}^n)$$

$$a_{i_2}(dx_{k_2^1}^1 \otimes \dots \otimes dx_{k_n^n}^n) = a_{i_2}(dx_{k_3^1}^1 \otimes \dots \otimes dx_{k_n^n}^n)$$

.....

$$a_{i_l}(dx_{k_1^1}^1 \otimes \dots \otimes dx_{k_n^n}^n) = a_{i_l}(dx_{k_{l+1}^1}^1 \otimes \dots \otimes dx_{k_n^n}^n)$$

.....

$$a_{i_r}(dx_{k_1^1}^1 \otimes \dots \otimes dx_{k_n^n}^n) = a_{i_r}(dx_{k_{r+1}^1}^1 \otimes \dots \otimes dx_{k_n^n}^n)$$

where $k_1^1 = k_1, k_n^1 = k_n$ and $k_1^{r+1} = k'_1, k_n^{r+1} = k'_n$.

Then we say that the two coordinates $dx_{k_i}^i$ and $dx_{k'_i}^i$ are connected by the zigzag $(a_{i_1}, a_{i_2}, \dots, a_{i_r})$. Of course two coordinates $dx_{k_i}^i$ and $dx_{k'_i}^i$ may have several equivalent zigzag of contractions.

- It is straightforward to see that if two connected n -configurations C_n and C'_n are adjacent, for example along the direction j , then their pasting $C_n +_j C'_n$ is still a connected n -configuration.
- We can see that each n -configuration in \mathcal{Z}_n is built with subsets in it which are connected n -configurations. Thus any configuration C_n is written as a formal sum

$$C_n = C_n^1 + \dots + C_n^l + \dots + C_n^r$$

such that $r = \sharp\{\text{Connected components of } C_n\}$, and for each $l \in \llbracket 1, r \rrbracket, C_n^l$ denote the connected configurations inside C_n .

- Let $C_n = \{dx_{k_i}^i/k_1 \in K_1, \dots, k_n \in K_n, \forall i \in \llbracket 1, n \rrbracket, K_i \subset \mathbb{Z} \text{ is a finite set} \}$ a connected configuration.

Lemma 1 Its j -dilatation $\text{dilat}_j(C_n) = \{dx_{k_i}^i + j/dx_{k_i}^i \in C_n\}$ is a connected $(n+1)$ -configuration. \square

PROOF We have to prove that if two coordinates $dx_{k_i}^i + j$ and $dx_{k'_i}^i + j$ belong to $\text{dilat}_j(C_n)$ then there is a zigzag of contractions in $\text{dilat}_j(C_n)$ between them. Consider a zigzag $(a_{i_1}, a_{i_2}, \dots, a_{i_r})$ of contractions between the two coordinates $dx_{k_i}^i$ and $dx_{k'_i}^i$ in C_n . If $l \in \llbracket 1, r \rrbracket$ write $i'_l = i_l$ if $i_l < j$ and $i'_l = i_l + 1$ if $i_l \geq j$. Then it is easy to see that $(a_{i'_1}, a_{i'_2}, \dots, a_{i'_r})$ is such zigzag. \blacksquare

The j -dilatation $\text{dilat}_j(C_n)$ of a connected configuration C_n is written also $1_{n+1,j}^n(C_n) = 1_{n+1,j}^{n,\gamma=\pm}(C_n) = \text{dilat}_j(C_n)$ in order to have a smell of the structure of cubical strict ∞ -category that we shall put on connected configurations.

- Let $C_n = \{dx_{k_i}^i/k_1 \in K_1, \dots, k_n \in K_n, \forall i \in \llbracket 1, n \rrbracket, K_i \subset \mathbb{Z} \text{ is a finite set} \}$ a connected configuration.

Its j -presources are given by the sub-configuration $C_n^{j\text{-so}} \subset C_n$ built as follow :

First for each coordinate $dx_{k_i}^i = dx_{k_1}^1 \otimes \dots \otimes dx_{k_j}^j \otimes \dots \otimes dx_{k_n}^n$ in C_n we consider the set $C_n^{j, dx_{k_i}^i} \subset C_n$ of all coordinates $dx_{k'_i}^i = dx_{k'_1}^1 \otimes \dots \otimes dx_{k'_j}^j \otimes \dots \otimes dx_{k'_n}^n$ in C_n such that for all $l \in \llbracket 1, n \rrbracket \setminus j$, $k'_l = k_l$. The set $C_n^{j, dx_{k_i}^i} \subset C_n$ is called a j -partition of C_n . These partitions of C_n form a finite set and they may not be connected. Of course if $dx_{k'_i}^i \in C_n^{j, dx_{k_i}^i}$ then $C_n^{j, dx_{k'_i}^i} = C_n^{j, dx_{k_i}^i}$. Thus when we consider the set $C_n^{j, dx_{k_i}^i}$ it means that we have chosen one coordinate $dx_{k_i}^i$ representing this set and we use $dx_{k_i}^i$ to denote this set.

Let us fix a j -partition $C_n^{j, dx_{k_i}^i}$ of C_n . The integer $\min_j = \min\{k'_j \in \mathbb{Z}/dx_{k'_i}^i \in C_n^{j, dx_{k_i}^i}\}$ provides a specific coordinate $dx_{\min_j}^i \in C_n^{j, dx_{k_i}^i}$. We isolate these coordinates for all j -partitions of C_n . They form the set $C_n^{j\text{-so}} \subset C_n$ of j -presources of C_n .

The j -contraction $\text{contr}_j(C_n^{j\text{-so}})$ of $C_n^{j\text{-so}}$ is given by the following set of coordinates in \mathcal{Z}_{n-1} ::

$$\text{contr}_j(C_n^{j\text{-so}}) = \{dx_{k_i}^i \setminus j / dx_{k_i}^i \in C_n^{j\text{-so}}\}$$

Lemma 2 The j -contraction $\text{contr}_j(C_n^{j\text{-so}})$ of $C_n^{j\text{-so}}$ is a connected $(n-1)$ -configuration. □

Definition 3 If C_n is a connected configuration, its j -source is the connected $(n-1)$ -configuration $\text{contr}_j(C_n^{j\text{-so}})$. We denote it by $\sigma_{n-1,j}^n(C_n)$ □

Its j -pretargets are given by the sub-configuration $C_n^{j\text{-tar}} \subset C_n$ built as follow :

As above we fix a j -partition $C_n^{j, dx_{k_i}^i}$ of C_n . The integer $\max_j = \max\{k'_j \in \mathbb{Z}/dx_{k'_i}^i \in C_n^{j, dx_{k_i}^i}\}$ provides a specific coordinate $dx_{\max_j}^i \in C_n^{j, dx_{k_i}^i}$. We isolate these coordinates for all j -partitions of C_n . They form the set $C_n^{j\text{-tar}} \subset C_n$ of j -pretargets of C_n .

The j -contraction $\text{contr}_j(C_n^{j\text{-tar}})$ of $C_n^{j\text{-tar}}$ is given by the following set of coordinates in \mathcal{Z}_{n-1} ::

$$\text{contr}_j(C_n^{j\text{-tar}}) = \{dx_{k_i}^i \setminus j / dx_{k_i}^i \in C_n^{j\text{-tar}}\}$$

Lemma 3 The j -contraction $\text{contr}_j(C_n^{j\text{-tar}})$ of $C_n^{j\text{-tar}}$ is a connected $(n-1)$ -configuration. □

Definition 4 If C_n is a connected configuration, its j -target is the connected $(n-1)$ -configuration $\text{contr}_j(C_n^{j\text{-tar}})$. We denote it by $\tau_{n-1,j}^n(C_n)$ □

- Let $C_n = \{dx_{k_i}^i/k_1 \in K_1, \dots, k_n \in K_n, \forall i \in \llbracket 1, n \rrbracket, K_i \subset \mathbb{Z} \text{ is a finite set} \}$ a connected configuration. A j -move of C_n is given by a new n -configuration $j\text{-move}(C_n)$ built as follow :

Consider a j -partition $C_n^{j, dx_{k_i}^i}$ of C_n as above. If $dx_{k'_i}^i = dx_{k_1}^1 \otimes \dots \otimes dx_{k'_j}^j \otimes \dots \otimes dx_{k_n}^n$ is in $C_n^{j, dx_{k_i}^i} \subset C_n$ then define the new coordinate $j\text{-trans}_k(dx_{k'_i}^i) = dx_{k_1}^1 \otimes \dots \otimes dx_{k'_j+k}^j \otimes \dots \otimes dx_{k_n}^n$ ($k \in \mathbb{Z}$) as the translation of $dx_{k'_i}^i$ by the integer k along the direction j . Then put

$$j\text{-trans}_k(C_n^{j, dx_{k_i}^i}) = \{j\text{-trans}_k(dx_{k'_i}^i) \in \mathcal{Z}_n / dx_{k'_i}^i \in C_n^{j, dx_{k_i}^i}\}$$

Now suppose that C_n has m j -partitions $C_{n,k}^{j, dx_{k_i}^i}$ ($m \in \mathbb{N}$ and $k \in \llbracket 1, m \rrbracket$). For each j -partitions chose an integer $l_k \in \mathbb{Z}$ ($k \in \llbracket 1, m \rrbracket$). Now for each of these partitions $C_{n,k}^{j, dx_{k_i}^i}$ of C_n , consider their different translations by the integers l_k along the direction j :

$$j\text{-trans}_{l_k}(C_{n,k}^{j, dx_{k_i}^i})$$

The new set of coordinates :

$$j\text{-move}_{l_1, \dots, l_m}(C_n) := \bigcup_{k \in \llbracket 1, m \rrbracket} j\text{-trans}_{l_k}(C_{n,k}^{j, dx_{k_i}^i})$$

is called a j -move of C_n . Such j -move of C_n is denoted $j\text{-move}(C_n)$ when no confusion occur for its underlying translations $l_k \in \mathbb{Z}$. A j -move of C_n may not be connected. Such j -moves are central tools to build compositions $\circ_{n-1,j}^n$ between connected n -configurations.

formalism that we are going to build in order to reach cubical pasting diagrams. In fact we shall see that cubical pasting diagrams are built by using a richer version of this formalism of connected configurations. This richness allows to distinguish well degeneracies which won't be collapsed, but also shall give a more precise view of sources, targets and compositions.

- If $C_n = \{dx_{k_i}^i/k_1 \in K_1, \dots, k_n \in K_n, \forall i \in \llbracket 1, n \rrbracket, K_i \subset \mathbb{Z} \text{ is a finite set} \}$ is a connected n -configuration then its j -gluing locus is the set $C_n^{j\text{-gluing}}$ of pairs $(dx_{k_i}^i, dx_{k'_i}^i)$ of coordinates in C_n such that $k'_j = k_j + 1$.
- Diagrams of the j -gluing locus $C_n^{j\text{-gluing}}$ of a connected configuration C_n are given by formal diagrams

$$\begin{array}{ccc} dx_{k_i}^i & & dx_{k'_i}^i \\ & \swarrow t_{n-1,j}^n & \nearrow s_{n-1,j}^n \\ & dx_{k_i}^i \setminus j = dx_{k'_i}^i \setminus j & \end{array}$$

- If $C_n = C_n^1 + \dots + C_n^l + \dots + C_n^r$ is an n -configuration such that $r = \#\{\text{Connected components of } C_n\}$ and for each $l \in \llbracket 1, r \rrbracket$, C_n^l denote the connected configurations inside C_n , then its j -gluing locus is the set

$$C_n^{j\text{-gluing}} = \bigcup_{l \in \llbracket 1, r \rrbracket} C_n^{l,j\text{-gluing}}$$

- Diagrams of the j -gluing locus $C_n^{j\text{-gluing}}$ of a configuration $C_n = C_n^1 + \dots + C_n^l + \dots + C_n^r$ are given by the union of diagrams of each j -gluing locus $C_n^{l,j\text{-gluing}}$ of its connected components C_n^l .
- Let $C_n = \{dx_{k_i}^i/k_1 \in K_1, \dots, k_n \in K_n, \forall i \in \llbracket 1, n \rrbracket, K_i \subset \mathbb{Z} \text{ is a finite set} \}$ a connected configuration.

Its j -predomain is given by the sub-configuration $C_n^{j\text{-dom}} \subset C_n$ such that if $dx_{k_i}^i = dx_{k_1}^1 \otimes \dots \otimes dx_{k_j}^j \otimes \dots \otimes dx_{k_n}^n$ belongs to $C_n^{j\text{-dom}}$ then the coordinate $dx_{k_1}^1 \otimes \dots \otimes dx_{k_{j-1}}^{j-1} \otimes \dots \otimes dx_{k_n}^n$ doesn't belong to C_n .

Lemma 4 The j -contraction $\text{contr}_j(C_n^{j\text{-dom}})$ of the j -predomain $C_n^{j\text{-dom}}$ of C_n just above is a connected $(n-1)$ -configuration. \square

PROOF We have to prove that if two coordinates $dx_{k_i}^i \setminus j$ and $dx_{k'_i}^i \setminus j$ belong to $\text{contr}_j(C_n^{j\text{-dom}})$ then there is a zigzag of contractions in $\text{contr}_j(C_n^{j\text{-dom}})$ between them. Consider a zigzag $(a_{i_1}, a_{i_2}, \dots, a_{i_r})$ of contractions between the two coordinates $dx_{k_i}^i$ and $dx_{k'_i}^i$ of $C_n^{j\text{-dom}}$ in C_n . If $l \in \llbracket 1, r \rrbracket$ write $i'_l = i_l$ if $i_l \leq j$ and $i'_l = i_l - 1$ if $i_l > j$. Then it is easy to see that $(a_{i'_1}, a_{i'_2}, \dots, a_{i'_r})$ is such zigzag. \blacksquare

The j -contraction $\text{contr}_j(C_n^{j\text{-dom}})$ of the j -predomain $C_n^{j\text{-dom}}$ of C_n is called the j -domain of C_n and is written $\Sigma_{n-1,j}^n(C_n)$.

The j -precodomain of C_n is given by the sub-configuration $C_n^{j\text{-codom}} \subset C_n$ such that if $dx_{k_i}^i = dx_{k_1}^1 \otimes \dots \otimes dx_{k_j}^j \otimes \dots \otimes dx_{k_n}^n$ belongs to $C_n^{j\text{-codom}}$ then the coordinate $dx_{k_1}^1 \otimes \dots \otimes dx_{k_{j+1}}^j \otimes \dots \otimes dx_{k_n}^n$ doesn't belong to C_n .

Lemma 5 The j -contraction $\text{contr}_j(C_n^{j\text{-codom}})$ of the j -precodomain $C_n^{j\text{-codom}}$ of C_n just above is a connected $(n-1)$ -configuration. \square

The proof is the same. The j -contraction $\text{contr}_j(C_n^{j\text{-codom}})$ of the j -precodomain $C_n^{j\text{-codom}}$ of C_n is called the j -codomain of C_n and is written $\Gamma_{n-1,j}^n(C_n)$.

- Diagrams of the j -predomain $C_n^{j\text{-dom}}$ of a connected configuration C_n are given by the formal arrows

$$dx_{k_i}^i \setminus j \xrightarrow{s_{n-1,j}^n} dx_{k_i}^i$$

where $dx_{k_i}^i$ belongs to $C_n^{j\text{-dom}}$, and diagrams of the j -precodomain $C_n^{j\text{-codom}}$ of a connected configuration C_n are given by the formal arrows

$$dx_{k_i}^i \setminus j \xrightarrow{t_{n-1,j}^n} dx_{k_i}^i$$

where $dx_{k_i}^i$ belongs to $C_n^{j\text{-codom}}$.

- Diagrams of the j -predomain $C_n^{j\text{-dom}}$ of a configuration C_n are given by the union of all diagrams of the j -predomain of its connected components, and diagrams of the j -precodomain $C_n^{j\text{-codom}}$ of a configuration C_n are given by the union of all diagrams of the j -precodomain of its connected components.
- A crucial and straightforward fact is that given a coordinate $dx_{k_i}^i$ in \mathcal{Z}_n , it has a *trivial structure* of n -cubical set where sources and targets are defined by contractions :

$$- s_{n-1,j}^n(dx_{k_i}^i) = t_{n-1,j}^n(dx_{k_i}^i) := dx_{k_i}^i \setminus j,$$

$$- s_{n-p-1,k}^{n-p}(dx_{k_i}^i \setminus (j_1, \dots, j_p)) = t_{n-p-1,k}^{n-p}(dx_{k_i}^i \setminus (j_1, \dots, j_p)) := dx_{k_i}^i \setminus (j_1, \dots, j_p, k)$$

thus different contractions of $dx_{k_i}^i$ are the faces of its underlying trivial n -cubical set.

However this structure of n -cube that $dx_{k_i}^i$ has is too trivial because it does not distinguished sources and targets with the same direction j . And this distinction is crucial because our idea is to label any n -cubical sets A with a coordinate $dx_{k_i}^i$ of \mathcal{Z}_n , such that faces of A must have new coordinates $dx_{k_i}^i \setminus (j_1, \dots, j_p)$ build by contractions and weighted by a notion of sources and targets. In order to correct this default we are going to enriched the coordinates with a notion of *weighted coordinate* or *link*, which are roughly speaking *coordinates equipped with or weighted with* the symbols $\{-, +\}$.

Thus for each coordinate $dx_{k_i}^i$ of the infinite network \mathcal{Z}_n we shall associate an other n -cubical set $\square_{1(n)}^{dx_{k_i}^i}$ called *the box* of $dx_{k_i}^i$ and which formalise better the notion of n -cubical set A labelled by $dx_{k_i}^i$, in the sense that sources and targets of A are then labelled with weighted coordinates, which give the right information of the location of faces of A . Without these weights any p -face of A which is a source in the direction j has the same coordinate (because the trivial structure collapse this source-target information) as the other p -face of A which is a target in the same direction j , and this is counterintuitive : the role of $\square_{1(n)}^{dx_{k_i}^i}$ is to distinguished well coordinates of any faces of any n -cubical set labelled with the coordinate $dx_{k_i}^i$. The next section is devoted to the description of these boxes $\square_{1(n)}^{dx_{k_i}^i}$.

2.2 The basic box $\square_{1(n)}^{dx_{k_i}^i}$ of a coordinate $dx_{k_i}^i$

Given a coordinate $dx_{k_i}^i$ and the elementary n -cube $1(n)$ (which is the unique n -cell of the cubical sketch \mathbb{C}), we associate to it a canonical *free box* $(dx_{k_i}^i) = \square_{1(n)}^{dx_{k_i}^i}$ which is an n -cubical set which faces are congruences of *links*. This n -cubical set $\square_{1(n)}^{dx_{k_i}^i}$ is called *the basic box of the coordinate $dx_{k_i}^i$* . Its sources and its targets are compatible with contractions and obtained by contraction of $dx_{k_i}^i$, and its different degeneracies (classical and connections) are compatible with dilatations and obtained by dilatation of $dx_{k_i}^i$. Its links are seen as terms of a language equipped with the different contractions of $dx_{k_i}^i : dx_{k_i}^i \setminus (j_1, j_2, \dots, j_p)$ plus two symbols $\{-, +\}$ which label these contractions. These symbols $\{-, +\}$ must be interpreted as sources and targets of the different contractions they equipped, and provide a good notion of sources and targets for $\square_{1(n)}^{dx_{k_i}^i}$. These terms are built inductively (see below) and congruences on it use notions of zigzag build with the cubical identities of sources and targets (see below). An other possible description of faces of $\square_{1(n)}^{dx_{k_i}^i}$ is given in the remark below, which looks more natural (it uses the Reverse Polish Notation), but less intuitive for us. Perhaps in the future we would prefer these *RPN notations*.

In this section we will describe only the underlying cubical set of $\square_{1(n)}^{dx_{k_i}^i}$ and degeneracies of it shall be described only in the next section, because they are more subtle and involve *notions of dilated free boxes equipped congruences for degeneracies* (see below). As we wrote in the previous section the role of $\square_{1(n)}^{dx_{k_i}^i}$ can be summarized as follow : if a n -cubical set X is labelled by a coordinate $dx_{k_i}^i$ it means that it is contained in the box $\square_{1(n)}^{dx_{k_i}^i}$ which faces are congruences of *weighted coordinates or links*. The box $\square_{1(n)}^{dx_{k_i}^i}$ and all faces of $\square_{1(n)}^{dx_{k_i}^i}$ have underlying free boxes (see below). But when we consider the free box associated to a face of $\square_{1(n)}^{dx_{k_i}^i}$ we forget that it was "linked" to $\square_{1(n)}^{dx_{k_i}^i}$.

In order to keep the *linked information* of the faces of $\square_{1(n)}^{dx_{k_i}^i}$ we write these links as finite sequences of the form :

$$X = (dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_r), \pm))$$

We can define them by finite decreasing induction :

Definition 5 • For any direction $j \in \llbracket 1, n \rrbracket$, the term $s_{n-1,j}^n(\square_{1(n)}^{dx_{k_i}^i}) = (dx_{k_i}^i, (dx_{k_i}^i \setminus j, -))$ and the term $t_{n-1,j}^n(\square_{1(n)}^{dx_{k_i}^i}) = (dx_{k_i}^i, (dx_{k_i}^i \setminus j, +))$ are 1-links which must be interpreted respectively as the j -source and the j -target of the box $\square_{1(n)}^{dx_{k_i}^i}$.

• If $X = (dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm))$ is an $(n-r)$ -link of the box $\square_{1(n)}^{dx_{k_i}^i}$, then for any direction $j \in \llbracket 1, r \rrbracket$, the terms

$$s_{r-1,j}^r(X) = (dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm), (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}, j), -))$$

$$t_{r-1,j}^r(X) = (dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm), (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}, j), +))$$

are $(n-r-1)$ -links of $\square_{1(n)}^{dx_{k_i}^i}$.

- $(n-r)$ -links of sources-targets of $\square_{1(n)}^{dx_{k_i}^i}$, or $(n-r)$ -links of $\square_{1(n)}^{dx_{k_i}^i}$ for short, are given by such sequences

$$(dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm)).$$

Some notations shall be useful :

$$s_{n_2, j^1}^n := s_{n_2, j_{n_2+1}^1}^{n_2+1} \circ s_{n_2+1, j_{n_2+2}^1}^{n_2+2} \cdots \circ s_{n-2, j_{n-1}^1}^{n-1} \circ s_{n-1, j_n^1}^n$$

where $j^1 = (j_n^1, \dots, j_{n_2+1}^1)$ and $j_n^1 \in \llbracket 1, n \rrbracket, j_{n-1}^1 \in \llbracket 1, n-1 \rrbracket, \dots, j_{n_2+1}^1 \in \llbracket 1, n_2+1 \rrbracket$

$$t_{n_2, j^1}^n := t_{n_2, j_{n_2+1}^1}^{n_2+1} \circ t_{n_2+1, j_{n_2+2}^1}^{n_2+2} \cdots \circ t_{n-2, j_{n-1}^1}^{n-1} \circ t_{n-1, j_n^1}^n$$

where $j^1 = (j_n^1, \dots, j_{n_2+1}^1)$ and $j_n^1 \in \llbracket 1, n \rrbracket, j_{n-1}^1 \in \llbracket 1, n-1 \rrbracket, \dots, j_{n_2+1}^1 \in \llbracket 1, n_2+1 \rrbracket$.

Also for any partition $n_p < n_{p-1} < \dots < n_k < \dots < n_2 < n_1 = n$ with $(p-1)$ intervals $\llbracket n_{k+1}, n_k \rrbracket$ we have 6 different zigzags of sources and targets :

- $s_{n_2, j}^n := s_{n_2, j_{n_2+1}}^{n_2+1} \circ s_{n_2+1, j_{n_2+2}}^{n_2+2} \cdots \circ s_{n-2, j_{n-1}}^{n-1} \circ s_{n-1, j_n}^n$ where $j = (j_n, \dots, j_{n_2+1})$ and $j_n \in \llbracket 1, n \rrbracket, j_{n-1} \in \llbracket 1, n-1 \rrbracket, \dots, j_{n_2+1} \in \llbracket 1, n_2+1 \rrbracket$ called string of sources of type s .
- $t_{n_2, j}^n := t_{n_2, j_{n_2+1}}^{n_2+1} \circ t_{n_2+1, j_{n_2+2}}^{n_2+2} \cdots \circ t_{n-2, j_{n-1}}^{n-1} \circ t_{n-1, j_n}^n$ where $j = (j_n, \dots, j_{n_2+1})$ and $j_n \in \llbracket 1, n \rrbracket, j_{n-1} \in \llbracket 1, n-1 \rrbracket, \dots, j_{n_2+1} \in \llbracket 1, n_2+1 \rrbracket$ called string of targets of type t
- $s_{n_p, j^{p-1}}^{n_{p-1}} \circ t_{n_{p-1}, j^p}^{n_{p-2}} \cdots t_{n_{k+1}, j^k}^{n_k} \circ s_{n_k, j^{k-1}}^{n_{k-1}} \cdots t_{n_3, j^2}^{n_2} \circ s_{n_2, j^1}^n$ called zigzag of sources-targets of type (s, s) .
- $s_{n_p, j^{p-1}}^{n_{p-1}} \circ t_{n_{p-1}, j^p}^{n_{p-2}} \cdots t_{n_{k+1}, j^k}^{n_k} \circ s_{n_k, j^{k-1}}^{n_{k-1}} \cdots s_{n_3, j^2}^{n_2} \circ t_{n_2, j^1}^n$ called zigzag of sources-targets of type (s, t) .
- $t_{n_p, j^{p-1}}^{n_{p-1}} \circ s_{n_{p-1}, j^p}^{n_{p-2}} \cdots t_{n_{k+1}, j^k}^{n_k} \circ s_{n_k, j^{k-1}}^{n_{k-1}} \cdots s_{n_3, j^2}^{n_2} \circ t_{n_2, j^1}^n$ called zigzag of sources-targets of type (t, t) .
- $t_{n_p, j^{p-1}}^{n_{p-1}} \circ s_{n_{p-1}, j^p}^{n_{p-2}} \cdots t_{n_{k+1}, j^k}^{n_k} \circ s_{n_k, j^{k-1}}^{n_{k-1}} \cdots t_{n_3, j^2}^{n_2} \circ s_{n_2, j^1}^n$ called zigzag of sources-targets of type (t, s)

The number of occurrences of the s and of the t in a string or zigzag is called *the size* of the string or zigzag. If X is an r -link of $\square_{1(n)}^{dx_{k_i}^i}$:

$$X = (dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_r), \pm))$$

then it can be written

$$X = z_X(\square_{1(n)}^{dx_{k_i}^i})$$

where z_X denotes its underlying string or zigzag of sources-targets.

All these zigzags or strings build the $(n-n_p)$ faces of any n -cube. Thanks to the cubical identities two different zigzags or strings can be equal. And these equalities build congruences on the sequences defined below, such that equivalence relations of these sequences are the faces of the free box $\square_{1(n)}^{dx_{k_i}^i}$.

More precisely consider two $(n-r)$ -links $X = (dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm))$ and $X' = (dx_{k_i}^i, (dx_{k_i}^i \setminus j'_1, \pm), (dx_{k_i}^i \setminus (j'_1, j'_2), \pm), \dots, (dx_{k_i}^i \setminus (j'_1, j'_2, \dots, j'_{n-r}), \pm))$. Denote by z_X the string or zigzag of sources-targets which gives X , i.e $X = z_X(\square_{1(n)}^{dx_{k_i}^i})$, and $z_{X'}$ the string or zigzag of sources-targets which gives X' , i.e $X' = z_{X'}(\square_{1(n)}^{dx_{k_i}^i})$.

Definition 6 With the above notations, the $(n-r)$ -link X is congruent to the $(n-r)$ -link X' if and only if $z_X = z_{X'}$; in this case it is trivial to see that z_X and $z_{X'}$ have the same size. Then we write $X \equiv X'$. An equivalence classe of $(n-r)$ -link of the free box $\square_{1(n)}^{dx_{k_i}^i}$ are r -faces of $\square_{1(n)}^{dx_{k_i}^i}$. \square

In fact the *terminal element* of the $(n-r)$ -link X :

$$(dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm)$$

gives the precise information of an r -face of $\square_{1(n)}^{dx_{k_i}^i}$ that it can be a source or a target, depending on the sign in $\{-, +\}$: " - " means sources and " + " means target.

Lemma 6 If two $(n-r)$ -links of $\square_{1(n)}^{dx_{k_i}^i}$ are congruents then they have the same terminal element. \square

PROOF The proof is easy and is made by finite decreasing induction :

- We start the induction by proving it with sources and targets of $(dx_{k_i}^i) = \square_{1(n)}^{dx_{k_i}^i}$ (by using the whole cubical identities $ss = ss$, $st = ts$, etc.) and verify that indeed they give the same terminal coordinates : this step shows the magical role of the trivial cubical structure of the coordinates. See the section above.
- We suppose that this is true for two congruent $(n - r)$ -links. When we apply sources and targets of these $(n - r)$ -links then it is straightforward to see that they have the same terminal coordinates. \blacksquare

A face of $\square_{1(n)}^{dx_{k_i}^i}$ is thus an equivalent classe of links of $\square_{1(n)}^{dx_{k_i}^i}$ with the same terminal element.

We can have in mind also that $(dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm)$ is an r -face of $\square_{1(n)}^{dx_{k_i}^i}$ equipped with (or linked by) the link

$$(dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm))$$

Thus when there is no confusion about the prescribed link of a face $(dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm)$ of $\square_{1(n)}^{dx_{k_i}^i}$ we denote this r -face of $\square_{1(n)}^{dx_{k_i}^i}$ just by $(dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm)$ without referring its link in $\square_{1(n)}^{dx_{k_i}^i}$.

The previous lemma allows to build the *free boxes associate to any faces* of $\square_{1(n)}^{dx_{k_i}^i}$:

Definition 7 The *free box* $(dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r})) = \square_{1(r)}^{dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r})}$ of the link $(dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm))$ which represent an r -face of $\square_{1(n)}^{dx_{k_i}^i}$, is the basic box of the coordinate $dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r})$ in \mathcal{Z}_r . \square

When working with this free box $\square_{1(r)}^{dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r})}$, we forget the previous information that it was linked to $\square_{1(n)}^{dx_{k_i}^i}$. Thus the link $(dx_{k_i}^i, (dx_{k_i}^i \setminus j_1, \pm), (dx_{k_i}^i \setminus (j_1, j_2), \pm), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r-1}), \pm))$ which represents a face of $\square_{1(n)}^{dx_{k_i}^i}$, represents also a face of the underlying free box $\square_{1(r)}^{dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r})}$, but with the simpler link $(dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r-1}), \pm))$ when we see it as a face of the free box $\square_{1(r)}^{dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r})}$.

Remark 6 We have others natural notations for links X of $\square_{1(n)}^{dx_{k_i}^i}$ (Reverse Polish Notation, RPN) :

$$X = (dx_{k_i}^i, dx_{k_i}^i \setminus j_1, dx_{k_i}^i \setminus (j_1, j_2), \dots, dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), \pm, \dots, \pm)$$

This presentation allow the following definition of sources and targets of links of $\square_{1(n)}^{dx_{k_i}^i}$ by using underlying free boxes of it :

- $s_{n-1, j}^n(\square_{1(n)}^{dx_{k_i}^i}) = (dx_{k_i}^i, (dx_{k_i}^i \setminus j, -))$
- $s_{r-1, l}^r(X) = (dx_{k_i}^i, dx_{k_i}^i \setminus j_1, dx_{k_i}^i \setminus (j_1, j_2), \dots, dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r-1}), s_{r-1, l}^r(\square_{1(r)}^{dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r})}), \pm, \dots, \pm)$

Thus

$$s_{r-1, l}^r(X) = (dx_{k_i}^i, dx_{k_i}^i \setminus j_1, dx_{k_i}^i \setminus (j_1, j_2), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}, j_{n-(r-1)} = l), -), \pm, \dots, \pm)$$

that we write when removing redondant occurrences of brackets :

$$s_{r-1, j}^r(X) = (dx_{k_i}^i, dx_{k_i}^i \setminus j_1, dx_{k_i}^i \setminus (j_1, j_2), \dots, dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}, j_{n-(r-1)} = l), -, \pm, \dots, \pm)$$

and for targets :

- $t_{n-1, j}^n(\square_{1(n)}^{dx_{k_i}^i}) = (dx_{k_i}^i, (dx_{k_i}^i \setminus j, +))$
- $t_{r-1, l}^r(X) = (dx_{k_i}^i, dx_{k_i}^i \setminus j_1, dx_{k_i}^i \setminus (j_1, j_2), \dots, dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r-1}), t_{r-1, l}^r(\square_{1(r)}^{dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r})}), \pm, \dots, \pm)$ \square

Thus

$$t_{r-1, l}^r(X) = (dx_{k_i}^i, dx_{k_i}^i \setminus j_1, dx_{k_i}^i \setminus (j_1, j_2), \dots, (dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}, j_{n-(r-1)} = l), +), \pm, \dots, \pm)$$

that we write when removing redondant occurrences of brackets :

$$t_{r-1, j}^r(X) = (dx_{k_i}^i, dx_{k_i}^i \setminus j_1, dx_{k_i}^i \setminus (j_1, j_2), \dots, dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}), dx_{k_i}^i \setminus (j_1, j_2, \dots, j_{n-r}, j_{n-(r-1)} = l), +, \pm, \dots, \pm)$$

2.3 Degenerate boxes $(\square_{1(n)}^{dx_{k_i}^i}, \equiv_A)$

We know that the following forgetful functor :

$$[\mathbb{C}_r^{op}, \text{Sets}] \xrightarrow{U} [\mathbb{C}^{op}, \text{Sets}] = \mathbb{C}\text{Sets}$$

which sends cubical sets equipped with degeneracies and connections [9] to cubical sets is right adjoint. Its induced monad \mathbb{R} applied to the terminal object 1 of the category $[\mathbb{C}^{op}, \text{Sets}]$ of cubical sets, gives all kind of degenerates n -cells $A \in \mathbb{R}(1)(n)$ (for all integers $n \in \mathbb{N}$) we need for cubical pasting diagrams. In the next section we shall describe this monad accurately in order to see that it is a cartesian monad.

Now we are going to define the notion of zigzag of degeneracies in order to capture *the depth of a degenerate n -cell A* in $\mathbb{R}(1)(n)$ which is the greatest integer r such that r -faces of A are of the form $1(r)$. We begin with the notations :

$$1_{n,i^1}^{n_2} := 1_{n,i_1^1}^{n-1} \circ 1_{n-1,i_2^1}^{n-2} \circ \dots \circ 1_{n-k+1,i_k^1}^{n-k} \circ \dots \circ 1_{n_2+1,i_{n-n_2}^1}^{n_2} \quad \text{where } i^1 = (i_1^1, \dots, i_k^1, \dots, i_{n-n_2}^1), k \in \llbracket 1, n-n_2 \rrbracket \text{ and}$$

$$i_1^1 \in \llbracket 1, n \rrbracket, \dots, i_k^1 \in \llbracket 1, n-k+1 \rrbracket \dots i_{n-n_2}^1 \in \llbracket 1, n_2+1 \rrbracket.$$

$$1_{n,j^1}^{n_2,\gamma} := 1_{n,j_1^1}^{n-1,\gamma} \circ 1_{n-1,j_2^1}^{n-2,\gamma} \circ \dots \circ 1_{n-k+1,j_k^1}^{n-k,\gamma} \circ \dots \circ 1_{n_2+1,j_{n-n_2}^1}^{n_2,\gamma} \quad \text{where } j^1 = (j_1^1, \dots, j_k^1, \dots, j_{n-n_2}^1), k \in \llbracket 1, n-n_2 \rrbracket \text{ and}$$

$$j_1^1 \in \llbracket 1, n-1 \rrbracket, \dots, j_k^1 \in \llbracket 1, n-k \rrbracket \dots j_{n-n_2}^1 \in \llbracket 1, n_2 \rrbracket.$$

Also for any partition $n_p < n_{p-1} < \dots < n_k < \dots < n_2 < n_1 = n$ with $(p-1)$ intervals $\llbracket n_{k+1}, n_k \rrbracket$ we have 6 different zigzags of reflexivities and connections :

- $1_{n,i^1}^{n_2} := 1_{n,i_1^1}^{n-1} \circ 1_{n-1,i_2^1}^{n-2} \circ \dots \circ 1_{n-k+1,i_k^1}^{n-k} \circ \dots \circ 1_{n_2+1,i_{n-n_2}^1}^{n_2}$ where $i = (i_1, \dots, i_k, \dots, i_{n-n_2})$, $k \in \llbracket 1, n-n_2 \rrbracket$ called strings of degeneracies of type 1.
- $1_{n,j^1}^{n_2,\gamma} := 1_{n,j_1^1}^{n-1,\gamma} \circ 1_{n-1,j_2^1}^{n-2,\gamma} \circ \dots \circ 1_{n-k+1,j_k^1}^{n-k,\gamma} \circ \dots \circ 1_{n_2+1,j_{n-n_2}^1}^{n_2,\gamma}$ where $j = (j_1, \dots, j_k, \dots, j_{n-n_2})$, $k \in \llbracket 1, n-n_2 \rrbracket$ called strings of degeneracies of type γ .
- $1_{n,i^1}^{n_2} \circ 1_{n_2,i^2}^{n_3,\gamma} \circ \dots \circ 1_{n_{k-1},i^k}^{n_k,\gamma} \circ 1_{n_k,i^{k+1}}^{n_{k+1}} \circ \dots \circ 1_{n_{p-2},i^{p-2}}^{n_{p-1},\gamma} \circ 1_{n_{p-1},i^{p-1}}^{n_p}$ called zigzags of degeneracies of type $(1, 1)$.
- $1_{n,i^1}^{n_2,\gamma} \circ 1_{n_2,i^2}^{n_3} \circ \dots \circ 1_{n_{k-1},i^k}^{n_k,\gamma} \circ 1_{n_k,i^{k+1}}^{n_{k+1}} \circ \dots \circ 1_{n_{p-2},i^{p-2}}^{n_{p-1},\gamma} \circ 1_{n_{p-1},i^{p-1}}^{n_p}$ called zigzags of degeneracies of type $(\gamma, 1)$.
- $1_{n,i^1}^{n_2,\gamma} \circ 1_{n_2,i^2}^{n_3} \circ \dots \circ 1_{n_{k-1},i^k}^{n_k,\gamma} \circ 1_{n_k,i^{k+1}}^{n_{k+1}} \circ \dots \circ 1_{n_{p-2},i^{p-2}}^{n_{p-1}} \circ 1_{n_{p-1},i^{p-1}}^{n_p,\gamma}$ called zigzags of degeneracies of type (γ, γ) .
- $1_{n,i^1}^{n_2} \circ 1_{n_2,i^2}^{n_3,\gamma} \circ \dots \circ 1_{n_{k-1},i^k}^{n_k,\gamma} \circ 1_{n_k,i^{k+1}}^{n_{k+1}} \circ \dots \circ 1_{n_{p-2},i^{p-2}}^{n_{p-1}} \circ 1_{n_{p-1},i^{p-1}}^{n_p,\gamma}$ called zigzags of degeneracies of type $(1, \gamma)$.

The number of occurrences of the operations $1_{r+1,i}^r$, $1_{r+1,i}^{r,\gamma}$ in such zigzags or such strings are respectively called *the size of a zigzag* or *the size of a string*.

Definition 8 Consider an n -cell $A \in \mathbb{R}(1)(n)$ which is not equal to $1(n)$. Thus it is a degenerate n -cell and is build with zigzag or string of degeneracies as described just above. The *depth* of A is the integer $p \in \mathbb{N}$ such that A is equal to a zigzag of size $n-p$ or a string of size $n-p$ of degeneracies of the p -cell $1(p)$ of the cubical site, i.e A is written $a_A(1(p))$ where a_A denotes its underlying string or zigzag of degeneracies and a_A has size $n-p$. \square

Remark 7 Thanks to the axioms of degeneracies the degenerate n -cell A has zigzags or strings of degeneracies with different shapes and which are equals. \square

Suppose A is a degenerate n -cell in $\mathbb{R}(1)(n)$ with depth $p < n$. Zigzags or strings of sources-targets of A with sizes which are less or equal to $(n-p)$ are the one which build a congruence \equiv_A on faces of the basic n -box $\square_{1(n)}^{dx_{k_i}^i}$, and this congruence is defined as follow : if $p < q \leq n$, two q -faces x and y of $\square_{1(n)}^{dx_{k_i}^i}$ are A -congruent : $x \equiv_A y$ if and only if any strings or zigzags of sources-targets z_x of x (i.e z_x is the underlying string or the underlying zigzag of sources-targets of any link of $\square_{1(n)}^{dx_{k_i}^i}$ which gives the q -face x (any two such links are equivalent)) and any strings or zigzags of sources-targets z_y (i.e z_y is the underlying string or the underlying zigzag of sources-targets of any link of $\square_{1(n)}^{dx_{k_i}^i}$ which gives the q -face y (any two such links are equivalent)) of y , equalize A i.e are such that $z_x(A) = z_y(A)$.

The quotient $\square_{1(n)}^{dx_{k_i}^i} / \equiv_A$ is a boxe with coordinate $dx_{k_i}^i$ such that it sources and targets agree with those of A . We denote it with the bracket notation $(\square_{1(n)}^{dx_{k_i}^i}, \equiv_A)$.

Definition 9 • Sources and targets of degenerate boxes :

$$s_{n-1,j}^n((\square_{1(n)}^{dx_{k_i}^i}, \equiv A)) := (\square_{1(n-1)}^{dx_{k_i}^i \setminus j}, \equiv_{s_{n-1,j}^n(A)})$$

and

$$t_{n-1,j}^n((\square_{1(n)}^{dx_{k_i}^i}, \equiv A)) := (\square_{1(n-1)}^{dx_{k_i}^i \setminus j}, \equiv_{t_{n-1,j}^n(A)})$$

• Degeneracies of degenerate boxes :

$$1_{n+1,j}^n((\square_{1(n)}^{dx_{k_i}^i}, \equiv A)) := (\square_{1(n+1)}^{dx_{k_i}^i + j}, \equiv_{1_{n+1,j}^n(A)})$$

and

$$1_{n+1,j}^{n,\gamma}((\square_{1(n)}^{dx_{k_i}^i}, \equiv A)) := (\square_{1(n+1)}^{dx_{k_i}^i + j}, \equiv_{1_{n+1,j}^{n,\gamma}(A)})$$

Definition 10 A *basic divisor* is the expression $Adx_{k_i}^i$ which mean that the n -cell $A \in \mathbb{R}(1)(n)$ has coordinate $dx_{k_i}^i$ and when we write $Adx_{k_i}^i$; we furthermore mean that A is located in its degenerate box $(\square_{1(n)}^{dx_{k_i}^i}, \equiv A)$. \square

We use the following notations for sources and targets of basic divisors :

$$\bullet s_{n-1,j}^n(Adx_{k_i}^i) := s_{n-1,j}^n(A)dx_{k_i}^i \setminus j$$

$$\bullet t_{n-1,j}^n(Adx_{k_i}^i) := t_{n-1,j}^n(A)dx_{k_i}^i \setminus j$$

With it we get two formal inclusions :

$$s_{n-1,j}^n(Adx_{k_i}^i) \xrightarrow{s_{n-1,j}^n} Adx_{k_i}^i \quad t_{n-1,j}^n(Adx_{k_i}^i) \xrightarrow{t_{n-1,j}^n} Adx_{k_i}^i ,$$

We use the following notations for degeneracies of basic divisors :

$$\bullet 1_{n+1,j}^n(Adx_{k_i}^i) := 1_{n+1,j}^n(A)(dx_{k_i}^i + j)$$

$$\bullet 1_{n+1,j}^{n,\gamma}(Adx_{k_i}^i) := 1_{n+1,j}^{n,\gamma}(A)(dx_{k_i}^i + j)$$

Two basic divisors $Adx_{k_i}^i, A'dx_{k'_i}^i$ in X located respectively in the coordinates $dx_{k_i}^i = dx_{k_1}^1 \otimes \cdots \otimes dx_{k_n}^n$ and $dx_{k'_i}^i = dx_{k'_1}^1 \otimes \cdots \otimes dx_{k'_n}^n$ are j -adjacent for a direction $j \in \llbracket 1, n \rrbracket$ if their coordinates are j -adjacent and if $s_{n-1,j}^n(Adx_{k_i}^i) = t_{n-1,j}^n(A'dx_{k'_i}^i)$ if $k_j = k'_j + 1$ or $t_{n-1,j}^n(Adx_{k_i}^i) = s_{n-1,j}^n(A'dx_{k'_i}^i)$ if $k_j = k'_j - 1$. An *isolated basic divisor* in X is a basic divisor $Adx_{k_i}^i$ which is not j -adjacent to any other basic divisor of X for all direction $j \in \llbracket 1, n \rrbracket$.

Also we have the following simple fact :

Proposition 1 Any basic divisor has an underlying structure of cubical set with connections. \square

The set of basic divisors is denoted $\mathbb{B}\text{Div}$ and by the previous proposition it is straightforward that it has an underlying structure of cubical set with connections where its n -cells are the basic n -divisors. Consider the full subcategory $\Theta_{\mathbb{B}\text{Div}} \subset \mathbb{C}\text{Sets}$ which objects are basic divisors. The Yoneda embedding

$$\Theta_{\mathbb{B}\text{Div}} \xrightarrow{Y} \mathbb{C}\text{Sets}$$

$$X \longmapsto \text{hom}_{\mathbb{C}\text{Sets}}(X, -)$$

shall be useful in the next section. However a little comment is necessary here.

2.4 The monad of reflexive cubical sets

As we wrote in the previous section the forgetful functor :

$$[\mathbb{C}_r^{\text{op}}, \text{Sets}] \xrightarrow{U} [\mathbb{C}^{\text{op}}, \text{Sets}]$$

which sends cubical sets equipped with degeneracies and connections [9] to cubical sets is right adjoint and its induced monad is written $\mathbb{R} = (R, i, m)$ where $1_{\mathbb{C}\text{Sets}} \xrightarrow{i} R$ is its unit and $R^2 \xrightarrow{m} R$ is its multiplication.

If $C \in \mathbb{C}\text{Sets}$ is a cubical set, then we put :

$$R(C) := \bigcup_{X \in \mathbb{B}\text{Div}} \text{hom}_{\mathbb{C}\text{Sets}}(Y(X), C)$$

The multiplication m of the monad \mathbb{R} is very simple : it is obtained with the concatenation of two strings of degeneracies, or one string of degeneracies with one zigzag of degeneracies, or with two zigzags of degeneracies. The unit i of the monad \mathbb{R} sends n -cells c to the decorated box $cdx_{k_i}^i$.

Let us be more precise : the multiplication $R^2(C) \xrightarrow{m} R(C)$ is defined as follow : the cubical set $R^2(C)$ is defined by the formula :

$$R^2(C) = \bigcup_{X \in \mathbb{B}\text{Div}} \text{hom}_{\mathbb{C}\text{Sets}}(Y(X), R(C)) = \bigcup_{X' \in \mathbb{B}\text{Div}} \text{hom}_{\mathbb{C}\text{Sets}}(Y(X'), C)$$

thus an n -cell x of $R^2(C)$ is an expression of the form : $z(z'(c))$ where c is a p -cell of C , $p \leq n$ (for the case $p = n$ it means that x is non-degenerate and equal to c), z' is a string or a zigzag of degeneracies which when apply to c gives a degenerate q -cell $z'(c)$ of $R(C)$ ($p < q \leq n$), and where z is a string or a zigzag of degeneracies which degenerates again $z'(c)$. The multiplication m sends $z(z'(c)) \in R^2(C)$ to $(z + z')(c) \in R(C)$ where here $z + z'$ is just the concatenation of z and z' .

Proposition 2 *The monad $\mathbb{R} = (R, i, m)$ of cubical reflexive sets with connections is cartesian* □

PROOF The definition of the endofunctor R shows that it preserves fiber products.

We are going to prove that the multiplication m is cartesian, i.e we are going to prove that if $C \in \mathbb{C}\text{Sets}$ is a cubical set then the commutative diagram :

$$\begin{array}{ccc} R^2(C) & \xrightarrow{R^2(!)} & R^2(1) \\ m(C) \downarrow & & \downarrow m(1) \\ R(C) & \xrightarrow{R(!)} & R(1) \end{array}$$

is a cartesian square. Consider the commutative diagram in $\mathbb{C}\text{Sets}$:

$$\begin{array}{ccc} C' & \xrightarrow{g} & R^2(1) \\ f \downarrow & & \downarrow m(1) \\ R(C) & \xrightarrow{R(!)} & R(1) \end{array}$$

Thus if x is an n -cell of C' then $f(x) = z(c)$ where $c \in C(q)$ ($q \leq n$) and $R^2(!)(f(x)) = R^2(!)(z(c)) = z(1(q))$, and $g(x) = z''(z'(1(p)))$, thus $m(1)(g(x)) = m(1)(z''(z'(1(p)))) = (z'' + z')(1(p))$, thus the commutativity of the square gives $z = z'' + z'$ and $p = q$

$$\begin{array}{ccccc} C' & & & & \\ & \searrow g & & & \\ & & R^2(C) & \xrightarrow{R^2(!)} & R^2(1) \\ & \searrow f & \downarrow m(C) & & \downarrow m(1) \\ & & R(C) & \xrightarrow{R(!)} & R(1) \end{array}$$

Thus the unique arrow l is defined as follow : $l(x) = z''(z'(c))$, and we can see that $m(C)(z''(z'(c))) = (z'' + z')(c) = z(c) = f(x)$ and that $R^2(!)(z''(z'(c))) = z''(z'(1(p))) = g(x)$.

The cartesianity of the unit

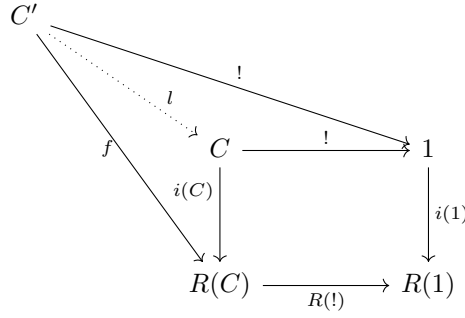
$$C \xrightarrow{i} R(C)$$

is easier and goes as follow : we start with a commutative diagram in $\mathbb{C}\text{Sets}$

$$\begin{array}{ccc} C' & \xrightarrow{!} & 1 \\ f \downarrow & & \downarrow i(1) \\ R(C) & \xrightarrow{R(!)} & R(1) \end{array}$$

Let x be an n -cell of C' , thus we have $f(x) = z(c)$, thus $R(!)(z(c)) = z(1(p))$ and the commutativity gives: $z(1(p)) = i(1)(1(n)) = 1(n)$; which shows that $z = \emptyset$ and $p = n$, thus $f(x) = c$.

It shows that there is a unique map l :



defined by $l(x) = f(x)$. ■

2.5 Rectangular divisors

Definition 11 A n -divisor is a configuration C_n equipped with chosen boxes for each coordinate in it, thus it is an expression :

$$X = A_{(k_1^1, \dots, k_n^1)} dx_{k_1^1}^i + \dots + A_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i + \dots + A_{(k_1^r, \dots, k_n^r)} dx_{k_1^r}^i$$

where $Adx_{k_i}^i$ are basic divisors. □

Remark 8 A n -divisor X must be thought up to its translations in the network \mathcal{Z}_n . Coordinates are used as guides to build their associated sketches 2.6. □

Proposition 3 Any n -divisor has an underlying structure of cubical set. □

PROOF If $X = A_{(k_1^1, \dots, k_n^1)} dx_{k_1^1}^i + \dots + A_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i + \dots + A_{(k_1^r, \dots, k_n^r)} dx_{k_1^r}^i$ is a n -divisor, then put :

- j -sources ($j \in \llbracket 1, n \rrbracket$) are given by :

$$s_{n-1,j}^n(X) = s_{n-1,j}^n(A_{(k_1^1, \dots, k_n^1)} dx_{k_1^1}^i) + \dots + s_{n-1,j}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i) + \dots + s_{n-1,j}^n(A_{(k_1^r, \dots, k_n^r)} dx_{k_1^r}^i)$$

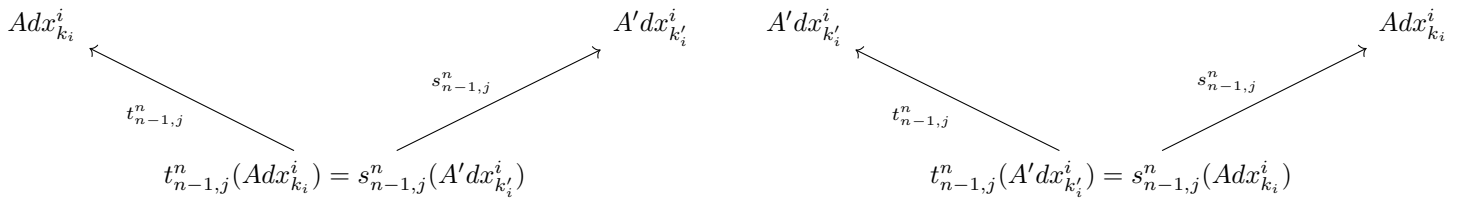
- j -targets ($j \in \llbracket 1, n \rrbracket$) are given by :

$$t_{n-1,j}^n(X) = s_{n-1,j}^n(A_{(k_1^1, \dots, k_n^1)} dx_{k_1^1}^i) + \dots + t_{n-1,j}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i) + \dots + t_{n-1,j}^n(A_{(k_1^r, \dots, k_n^r)} dx_{k_1^r}^i)$$

- If $x \in f_p(X)$ is a p -face of X , then it is an $(n-p)$ -divisor obtained by a zigzag of sources-targets of the n -divisor X .

Some notions attached to n -divisors shall be useful :

- Let $X = A_{(k_1^1, \dots, k_n^1)} dx_{k_1^1}^i + \dots + A_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i + \dots + A_{(k_1^r, \dots, k_n^r)} dx_{k_1^r}^i$ be a divisor. If two basic divisors $Adx_{k_i}^i$, $A'dx_{k'_i}^i$ in X are j -adjacent we get two possible diagrams :



- A j -gluing data ($j \in \llbracket 1, n \rrbracket$) for X is given by a couple $(A_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i, A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_1^{l'}}^i)$ of basic divisors of X which are j -adjacent and are such that

$$t_{n-1,j}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i) = s_{n-1,j}^n(A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_1^{l'}}^i)$$

Such j -gluing data can be written $A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i +_j A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i$ and underlies the diagram of the type

$$\begin{array}{ccc} A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i & & A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i \\ & \swarrow \scriptstyle t_{n-1, j}^n & \searrow \scriptstyle s_{n-1, j}^n \\ & t_{n-1, j}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i) = s_{n-1, j}^n(A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i) & \end{array}$$

called a *basic j -gluing locus* of X . The *j -gluing locus* of X is the set of such basic j -gluing locus. The *gluing locus* of X is the set of all j -gluing locus for all the direction $j \in \llbracket 1, n \rrbracket$.

- A basic divisor $A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i$ of a divisor X has *free j -source* if there is no basic divisor $A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i$ of X such that the couple $(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i, A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i)$ is a j -gluing data. The set of formal arrows :

$$s_{n-1, j}^n(A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i) \xrightarrow{s_{n-1, j}^n} A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i$$

associated to basic divisors in X which have free j -sources is called the *j -free sources locus* of X . The *free sources locus* of X is the set of all j -free sources locus for all the direction $j \in \llbracket 1, n \rrbracket$.

- A basic divisor $A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i$ of a divisor X has *free j -target* if there is no basic divisor $A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i$ of X such that the couple $(A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i, A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i)$ is a j -gluing data. The set of formal arrows :

$$t_{n-1, j}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i) \xrightarrow{t_{n-1, j}^n} A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i$$

associated to basic divisors in X which have free j -targets is called the *j -free targets locus* of X . The *free targets locus* of X is the set of all j -free targets locus for all the direction $j \in \llbracket 1, n \rrbracket$.

Consider a n -divisor $X = A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i + \dots + A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i + \dots + A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i$. We are going to define degeneracies of $X : 1_{n+1, i}^n(X), 1_{n+1, i}^{n, \gamma}(X)$ by using *the axioms of degeneracies-compositions* :

Definition 12 – $1_{n+1, i}^n(a \circ_j^n b) = 1_{n+1, i}^n(a) \circ_{j+1}^{n+1} 1_{n+1, i}^n(b)$ if $1 \leq i \leq j \leq n$

$$1_{n+1, i}^n(a \circ_j^n b) = 1_{n+1, i}^n(a) \circ_j^{n+1} 1_{n+1, i}^n(b) \text{ if } 1 \leq j < i \leq n+1$$

$$- 1_{n+1, i}^{n, \gamma}(a \circ_j^n b) = 1_{n+1, i}^{n, \gamma}(a) \circ_{j+1}^{n+1} 1_{n+1, i}^{n, \gamma}(b) \text{ if } 1 \leq i < j \leq n$$

$$1_{n+1, i}^{n, \gamma}(a \circ_j^n b) = 1_{n+1, i}^{n, \gamma}(a) \circ_j^{n+1} 1_{n+1, i}^{n, \gamma}(b) \text{ if } 1 \leq j < i \leq n$$

– First transport laws : for $1 \leq j \leq n$

$$1_{n+1, j}^{n, +}(a \circ_j^n b) = \begin{bmatrix} 1_{n+1, j}^{n, +}(a) & 1_{n+1, j}^n(a) \\ 1_{n+1, j+1}^n(a) & 1_{n+1, j}^{n, +}(b) \end{bmatrix}$$

– Second transport laws : for $1 \leq j \leq n$

$$1_{n+1, j}^{n, -}(a \circ_j^n b) = \begin{bmatrix} 1_{n+1, j}^{n, -}(a) & 1_{n+1, j+1}^n(a) \\ 1_{n+1, j}^n(a) & 1_{n+1, j}^{n, -}(b) \end{bmatrix}$$

□

Definition 13 Consider a divisor $X = A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i + \dots + A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i + \dots + A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i$. If $Adx_{k_i^l}^i$ is an isolated basic divisor of X then we just define $1_{n+1, i}^n(Adx_{k_i^l}^i)$, $1_{n+1, i}^{n, -}(Adx_{k_i^l}^i)$ and $1_{n+1, i}^{n, +}(Adx_{k_i^l}^i)$ as in 2.3; for the direction $j \in \llbracket 1, n \rrbracket$ consider a j -gluing data $A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i +_j A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i$ of X (see just above).

– If $1 \leq i \leq j \leq n$ or $1 \leq j < i \leq n+1$, then we put :

$$1_{n+1, i}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i +_j A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i) := 1_{n+1, i}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i) + 1_{n+1, i}^n(A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i)$$

– if $1 \leq i < j \leq n$ or $1 \leq j < i \leq n$ the we put :

$$1_{n+1, i}^{n, \gamma}(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i +_j A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i) := 1_{n+1, i}^{n, \gamma}(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i) + 1_{n+1, i}^{n, \gamma}(A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^i)$$

– If $1 \leq j \leq n$ then we put :

$$1_{n+1,j}^{n,+}(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i + A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i) := 1_{n+1,j}^{n,+}(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i) + 1_{n+1,j}^{n,+}(A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i) + 1_{n+1,j}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i) + 1_{n+1,j}^n(A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i)$$

and

$$1_{n+1,j}^{n,-}(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i + A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i) := 1_{n+1,j}^{n,-}(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i) + 1_{n+1,j}^{n,-}(A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i) + 1_{n+1,j}^n(A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i) + 1_{n+1,j}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i)$$

These give the definition of $1_{n+1,i}^n(X)$ for $i \in \llbracket 1, n+1 \rrbracket$ and of $1_{n+1,i}^{n,-}(X)$, $1_{n+1,i}^{n,+}(X)$ for $i \in \llbracket 1, n \rrbracket$ \square

Proposition 4 Any n -divisor has an underlying structure of reflexive cubical set. \square

Definition 14 A *connected divisor* is a divisor with underlying connected configuration such that all its pairs of basic divisors which have adjacent coordinates are adjacents. \square

Now we are going to define specific connected n -divisors, called *rectangular n -divisors*, which are our cubical pasting diagrams. These rectangular n -divisors have another notions of sources and targets that we call the *pasting-sources* and the *pasting-targets*. Their j -sources and j -targets that we have defined above shall be used to build their sketches, but their j -pasting-sources and their j -pasting-targets shall be used to build a structure of cubical ∞ -category on cubical pasting diagrams.

– A rectangular n -configuration is given by a n -configuration $C_n = \{dx_{k_i}^i / k_1 \in K_1, \dots, k_n \in K_n, \forall i \in \llbracket 1, n \rrbracket, K_i \subset \mathbb{N} \text{ is a finite set}\}$ of coordinates $dx_{k_i}^i \in \mathcal{Z}_n$ such that for all $i \in \llbracket 1, n \rrbracket$, the set $K_i \subset \mathbb{N}$ has the following form : $K_i = \{n_1^i, n_1^i + 1, \dots, n_1^i + l^i\}$ where $l^i \in \mathbb{N}$. When the set of integers \mathbb{N} is seen as a category which morphisms are given by its order, then another way to describe rectangular n -configurations is to see them as n -configurations C_n such that all finite subcategories $K_i \subset \mathbb{N}$ are connected. It is evident to see that rectangular n -configurations are specific connected n -configurations.

Definition 15 For all integer $n \in \mathbb{N}$, a *rectangular n -divisor* is a connected divisor X which underlying n -configuration is rectangular. Rectangular divisors are also called *cubical pasting n -diagrams*. \square

The set of rectangular divisors is denoted $\mathbb{C}\text{-Past}$. The full subcategory $\Theta_0 \subset \mathbb{C}\text{Sets}$ which objects are rectangular divisors is called the *cubical Θ_0* and the Yoneda embedding

$$\Theta_0 \xrightarrow{\mathbf{Y}} \mathbb{C}\text{Sets}$$

$$X \longmapsto \text{hom}_{\mathbb{C}\text{Sets}}(X, -)$$

shall be useful when we will describe the monad \mathbb{S} of cubical strict ∞ -categories with connections.

– Consider a rectangular divisors $X = A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i + \dots + A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i + \dots + A_{(k_1^r, \dots, k_n^r)} dx_{k_i^r}^i$ with underlying n -configuration $C_n = \{dx_{k_i}^i / k_1 \in K_1, \dots, k_n \in K_n, \forall i \in \llbracket 1, n \rrbracket, K_i \subset \mathbb{N} \text{ is a finite set}\}$. For each direction $j \in \llbracket 1, n \rrbracket$ the subset $K_j = \{n_1^j, n_1^j + 1, \dots, n_1^j + l^j\} \subset \mathbb{N}$ has a minimum n_j and a maximum $n_1^j + l^j$. The finite set of basic divisors $A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i$ of X which $dx_{k_i^l}^i$ has its j -depth equal to n_j , gives a new rectangular divisor denoted by $\text{pre-}\sigma_{n-1,j}^n(X)$ that we call the *j -pre-source* of X . Also the finite set of basic divisors $A_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i$ of X which $dx_{k_i^{l'}}^i$ has its j -depth equal to $n_j + l^j$, gives a new rectangular n -divisor denoted by $\text{pre-}\tau_{n-1,j}^n(X)$ that we call the *j -pre-target* of X .

The important fact here is : if X' is another rectangular n -divisor such that its j -pre-source $\text{pre-}\sigma_{n-1,j}^n(X')$ is built with basic divisors $A'_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i$ which are j -adjacent to basic divisors $A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i$ in $\text{pre-}\tau_{n-1,j}^n(X)$ and vice-versa : each basic divisor $A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i$ in $\text{pre-}\tau_{n-1,j}^n(X)$ is j -adjacent to a basic divisor $A'_{(k_1^{l'}, \dots, k_n^{l'}} dx_{k_i^{l'}}^i$ in $\text{pre-}\sigma_{n-1,j}^n(X')$ then the sum $X + X'$ is a rectangular n -divisor. Furthermore $s_{n-1,j}^n(\text{pre-}\sigma_{n-1,j}^n(X'))$ and $t_{n-1,j}^n(\text{pre-}\sigma_{n-1,j}^n(X))$ are rectangular $(n-1)$ -divisors and are equal.

Thus we put :

Definition 16 If X is a rectangular divisor, then its *j -pasting source* is :

$$\sigma_{n-1,j}^n(X) := s_{n-1,j}^n(\text{pre-}\sigma_{n-1,j}^n(X))$$

and its *j -pasting target* is :

$$\tau_{n-1,j}^n(X) := t_{n-1,j}^n(\text{pre-}\tau_{n-1,j}^n(X))$$

Proposition 5 $\sigma_{n-1,j}^n(X)$ and $\tau_{n-1,j}^n(X)$ are rectangular $(n-1)$ -divisors called respectively the j -source and the j -target of X . These faces are the one adapted to define a cubical ∞ -categorical structure on cubical pasting diagrams. \square

Proposition 6 If X is a rectangular n -divisor, then $1_{n+1,j}^n(X)$ and $1_{n+1,j}^{n,\gamma}(X)$ are rectangular $(n+1)$ -divisors. \square

PROOF Evident : just see that the dilatation of a rectangular n -configuration \blacksquare

Proposition 7 The set $\mathbb{C}\text{-Past}$ of rectangular divisors is equipped with a structure of cubical strict ∞ -category with connections. \square

PROOF – For all integer $n \in \mathbb{N}$, n -cells of $\mathbb{C}\text{-Past}$ are rectangular n -divisors.

– Consider two rectangular n -divisors X and X' such that $\tau_{n-1,j}^n(X) = \sigma_{n-1,j}^n(X')$. Then we define :

$$X \circ_{n-1,j}^n X' := X + X'$$

– If X is a rectangular n -divisor, then we defined degeneracies $1_{n+1,j}^n(X)$ for $j \in \llbracket 1, n+1 \rrbracket$ and of $1_{n+1,j}^{n,-}(X)$, $1_{n+1,j}^{n,-}(X)$ for $j \in \llbracket 1, n \rrbracket$. \blacksquare

2.6 Cubical inductive sketches

Let $X = A_{(k_1^1, \dots, k_n^1)} dx_{k_1^1}^1 + \dots + A_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^l + \dots + A_{(k_1^r, \dots, k_n^r)} dx_{k_1^r}^r$ be a divisor. In this section we associate to it a sketch \mathcal{E}_X which has a cubical set structure such that n -cells of it are themselves sketches build with cocones called *basic gluing locus* and *free sources-targets locus*. When X is a rectangular divisor, its *rectangular sketch* \mathcal{E}_X has like rectangular divisors, another notions of sources-targets, called again *pasting-sources* and *pasting-targets*. These last notion of sources-targets allow to build a cubical strict ∞ -categorical structure with connections on rectangular sketches.

Definition 17 If X is a divisor, its j -sketch source denoted by $\mathcal{E}_{s_{n-1,j}^{n-1}}(X)$ is the sketch obtained as the union of its j -gluing locus and its j -free sources locus. Its j -sketch target denoted by $\mathcal{E}_{t_{n-1,j}^{n-1}}(X)$ is the sketch obtained as the union of its j -gluing locus and its j -free targets locus. \square

Thus these sketches $\mathcal{E}_{s_{n-1,j}^{n-1}}(X)$ and $\mathcal{E}_{t_{n-1,j}^{n-1}}(X)$ are characterized as follow : we consider the $(n-1)$ -divisors $s_{n-1,j}^n(X)$ and $t_{n-1,j}^n(X)$, and with it we select as above j -gluing locus, j -free sources locus and j -free targets locus of X . This characterization is crucial because just by using j -sources $s_{n-1,j}^n(X)$ and j -targets $t_{n-1,j}^n(X)$ of the divisor X we can identify these sketches $\mathcal{E}_{s_{n-1,j}^{n-1}}(X)$ and $\mathcal{E}_{t_{n-1,j}^{n-1}}(X)$. We easily see that we can do the same construction for not necessary connected n -divisors.

• Let X be a fixed divisor. As we saw it has a canonical structure of n -cubical set, and we denote $f_p(X)$ the set of its p -faces (which are not necessarily connected divisors). If $x \in f_p(X)$ is a p -face of X then we denote $f_q(x)$ the set of q -faces of x . We associate to it the following cubical set \mathcal{E}_X of sketches :

– \mathcal{E}_X has only one n -cell still denoted by X , which is a *punctual sketch*, i.e which base is reduced to the point X see as the unique formal point of this base. We denote this singleton by \mathcal{E}_X^n .

– Consider the following cubical set :

$$\begin{array}{ccccccc}
\begin{array}{c} \xrightarrow{s_{n-1,n}^n} \\ \cdots \cdots \cdots \rightarrow \\ \xrightarrow{s_{n-1,j}^n} \\ \cdots \cdots \cdots \rightarrow \\ \xrightarrow{s_{n-1,1}^n} \\ \xrightarrow{t_{n-1,1}^n} \\ \cdots \cdots \cdots \rightarrow \\ \xrightarrow{t_{n-1,j}^n} \\ \cdots \cdots \cdots \rightarrow \\ \xrightarrow{t_{n-1,n}^n} \end{array} & \xrightarrow{\mathcal{E}_{f_{n-1}(X)}^{n-1}} \cdots \xrightarrow{\mathcal{E}_{f_4(X)}^4} & \begin{array}{c} \xrightarrow{s_{3,4}^4} \\ \xrightarrow{s_{3,3}^4} \\ \xrightarrow{s_{3,2}^4} \\ \xrightarrow{s_{3,1}^4} \\ \xrightarrow{t_{3,1}^4} \\ \xrightarrow{t_{3,2}^4} \\ \xrightarrow{t_{3,3}^4} \\ \xrightarrow{t_{3,4}^4} \end{array} & \xrightarrow{\mathcal{E}_{f_3(X)}^3} & \begin{array}{c} \xrightarrow{s_{2,3}^3} \\ \xrightarrow{s_{2,2}^3} \\ \xrightarrow{s_{2,1}^3} \\ \xrightarrow{t_{2,1}^3} \\ \xrightarrow{t_{2,2}^3} \\ \xrightarrow{t_{2,3}^3} \end{array} & \xrightarrow{\mathcal{E}_{f_2(X)}^2} & \begin{array}{c} \xrightarrow{s_{1,2}^2} \\ \xrightarrow{s_{1,1}^2} \\ \xrightarrow{t_{1,1}^2} \\ \xrightarrow{t_{1,2}^2} \end{array} & \xrightarrow{\mathcal{E}_{f_1(X)}^1} & \xrightarrow{s_0^1} & \mathcal{E}_{f_0(X)}^0
\end{array}$$

where $\mathcal{E}_{f_p(X)}^p$ denotes the sketch which is the set of all sketches \mathcal{E}_x^p associated to p -faces $x \in f_p(X)$ of X . We define it by using a simple finite decreasing induction :

– as we wrote, the set \mathcal{E}_X^n has only the punctual sketch X as element. The maps $s_{n-1,j}^n$ send X to the sketches $\mathcal{E}_{s_{n-1,j}^{n-1}}(X) \subset \mathcal{E}_{f_{n-1}(X)}^{n-1}$ (the j -sketch source of the n -divisor X), and the maps $t_{n-1,j}^n$ send X to the sketches $\mathcal{E}_{t_{n-1,j}^{n-1}}(X) \subset \mathcal{E}_{f_{n-1}(X)}^{n-1}$ (the j -sketch target of the n -divisor X), for all directions $j \in \llbracket 1, n \rrbracket$.

- If $x \in f_p(X)$ is a p -face of X (it is obtained by zigzags of $(n-p)$ -sequences of sources and targets of X) then the maps $s_{p-1,k}^p$ send the sketch \mathcal{E}_x^p to the sketches $\mathcal{E}_{s_{p-1,k}^p(x)}^{p-1} \subset \mathcal{E}_{f_{p-1}(X)}^{p-1}$ ($\mathcal{E}_{s_{p-1,k}^p(x)}^{p-1}$ is the k -sketch source of the $(n-p)$ -divisor x) and the maps $t_{p-1,k}^p$ send the sketch \mathcal{E}_x^p to the sketches $\mathcal{E}_{t_{p-1,k}^p(x)}^{p-1} \subset \mathcal{E}_{f_{p-1}(X)}^{p-1}$, ($\mathcal{E}_{t_{p-1,k}^p(x)}^{p-1}$ is the k -sketch target of the $(n-p)$ -divisor x), for all directions $k \in \llbracket 1, p \rrbracket$, where $s_{p-1,k}^p(x) \in f_{p-1}(x)$ and $t_{p-1,k}^p(x) \in f_{p-1}(x)$ are $(p-1)$ -faces of x .

Remark 9 When we associate a sketch \mathcal{E}_X to a n -divisor X we forget coordinates. Here we can see the crucial use of coordinates : it allows us to have an accurate description of such sketches and these coordinates are here used jus as "guides" for building them. \square

Now we are going to give another description of the sources and targets of \mathcal{E}_X .

- Consider the following $(n-p)$ -divisor :

$$x = A_{(k_1^i, \dots, k_p^i)} dx_{k_i^i}^i + \dots + A_{(k_1^i, \dots, k_p^i)} dx_{k_i^i}^i + \dots + A_{(k_1^i, \dots, k_p^i)} dx_{k_i^i}^i$$

which is a p -face of X . The j -gluing locus d of x are the following cocones d of the sketches $\mathcal{E}_{s_{p-1,j}^p(x)}^{p-1}$ and $\mathcal{E}_{t_{p-1,j}^p(x)}^{p-1}$:

$$\begin{array}{ccc} A_{(k_1^i, \dots, k_j^i, \dots, k_p^i)} dx_{k_i^i}^i & & A_{(k_1^{i'}, \dots, k_j^{i'}, \dots, k_p^{i'})} dx_{k_i^{i'}}^i \\ & \swarrow t_{p-1,j}^p & \searrow s_{p-1,j}^p \\ & t_{p-1,j}^p(A_{(k_1^i, \dots, k_j^i, \dots, k_p^i)} dx_{k_i^i}^i) = s_{p-1,j}^p(A_{(k_1^{i'}, \dots, k_j^{i'}, \dots, k_p^{i'})} dx_{k_i^{i'}}^i) & \end{array}$$

where $A_{(k_1^i, \dots, k_j^i, \dots, k_p^i)} dx_{k_i^i}^i$ and $A_{(k_1^{i'}, \dots, k_j^{i'}, \dots, k_p^{i'})} dx_{k_i^{i'}}^i$ are basic divisors of x which are j -adjacent and thus are such that :

$$(k_1^{i'}, \dots, k_j^{i'}, \dots, k_p^{i'}) = (k_1^i, \dots, k_j^i + 1, \dots, k_p^i)$$

and where $t_{p-1,j}^p(A_{(k_1^i, \dots, k_j^i, \dots, k_p^i)} dx_{k_i^i}^i)$ is itself a basic divisor of the divisor $t_{p-1,j}^p(x)$ and $s_{p-1,j}^p(A_{(k_1^{i'}, \dots, k_j^{i'}, \dots, k_p^{i'})} dx_{k_i^{i'}}^i)$ is a basic divisor of the divisor $s_{p-1,j}^p(x)$. With this notation we can also describe morphisms of sketches :

$$\begin{array}{ccc} \begin{array}{c} \xrightarrow{s_{p-2,p-1}^{p-1}} \\ \dots \xrightarrow{s_{p-2,k}^{p-1}} \\ \dots \xrightarrow{s_{p-2,1}^{p-1}} \\ \mathcal{E}_{s_{p-1,j}^p(x)}^{p-1} \end{array} & \xrightarrow{\quad} & \begin{array}{c} \xrightarrow{s_{p-2,p-1}^{p-1}} \\ \dots \xrightarrow{s_{p-2,k}^{p-1}} \\ \dots \xrightarrow{s_{p-2,1}^{p-1}} \\ \mathcal{E}_{f_{p-2}(s_{p-1,j}^p(x))}^{p-2} \end{array} \\ \begin{array}{c} \xrightarrow{t_{p-2,k}^{p-1}} \\ \dots \xrightarrow{t_{p-2,1}^{p-1}} \\ \mathcal{E}_{t_{p-1,j}^p(x)}^{p-1} \end{array} & \xrightarrow{\quad} & \begin{array}{c} \xrightarrow{t_{p-2,k}^{p-1}} \\ \dots \xrightarrow{t_{p-2,1}^{p-1}} \\ \mathcal{E}_{f_{p-2}(t_{p-1,j}^p(x))}^{p-2} \end{array} \\ \xrightarrow{t_{p-2,p-1}^{p-1}} & & \xrightarrow{t_{p-2,p-1}^{p-1}} \end{array}$$

on the j -gluing locus of x ; the description of these morphisms of sketches on the j -free sources locus of x and on the j -free targets locus of x is straightforward when we define their actions only on the j -gluing locus.

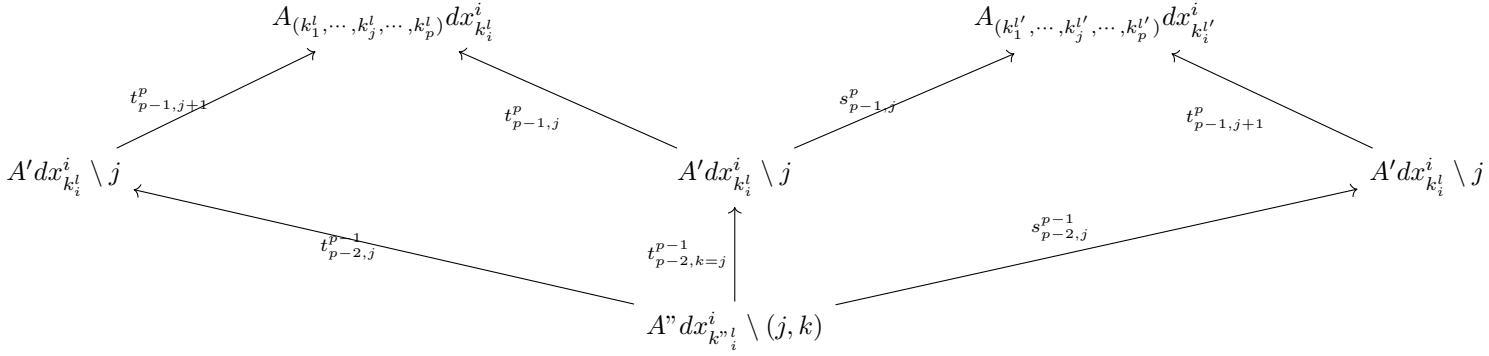
We describe these morphisms of sketches by defining cocones $s_{p-2,k}^{p-1}(d)$ and $t_{p-2,k}^{p-1}(d)$ as precomposition of the j -gluing data d just above

$$\begin{array}{ccc} A_{(k_1^i, \dots, k_j^i, \dots, k_p^i)} dx_{k_i^i}^i & & A_{(k_1^{i'}, \dots, k_j^{i'}, \dots, k_p^{i'})} dx_{k_i^{i'}}^i \\ & \swarrow t_{p-1,j}^p & \searrow s_{p-1,j}^p \\ & A' dx_{k_i^i}^i \setminus j & \end{array}$$

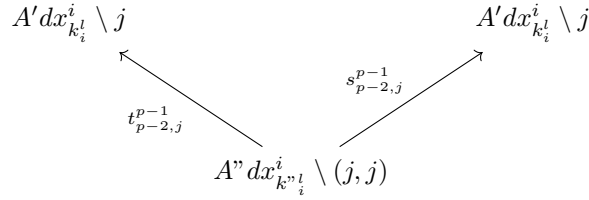
where we denoted $A' = t_{p-1,j}^p(A_{(k_1^i, \dots, k_j^i, \dots, k_p^i)} dx_{k_i^i}^i) = s_{p-1,j}^p(A_{(k_1^{i'}, \dots, k_j^{i'}, \dots, k_p^{i'})} dx_{k_i^{i'}}^i)$ and $A''_s = s_{p-2,k}^{p-1}(A')$, $A''_t = t_{p-2,k}^{p-1}(A')$, which is precomposed with the maps

$$A''_s dx_{k_i^i}^i \setminus (j, k) \xrightarrow{s_{p-2,k}^{p-1}} A' dx_{k_i^i}^i \setminus j \quad A''_t dx_{k_i^i}^i \setminus (j, k) \xrightarrow{t_{p-2,k}^{p-1}} A' dx_{k_i^i}^i \setminus j$$

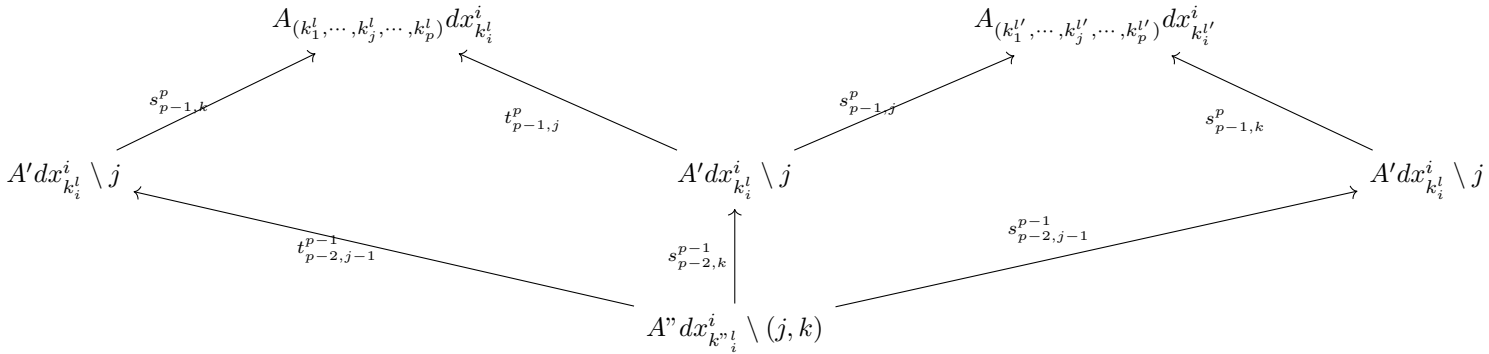
And we obtain $t_{p-2,k}^{p-1}(d)$ by using the diagram :



where we denote $A'' dx_{k_i}^i \setminus (j, k) = t_{p-2,k}^{p-1}(A' dx_{k_i}^i \setminus j)$, and thus the morphism of sketches $t_{p-2,k}^{p-1}$ sends d to the following diagram $t_{p-2,k}^{p-1}(d)$ of the sketches $\mathcal{E}_{f_{p-2}(s_{p-1,j}^p(x))}^{p-2}$ and $\mathcal{E}_{f_{p-2}(t_{p-1,j}^p(x))}^{p-2}$:



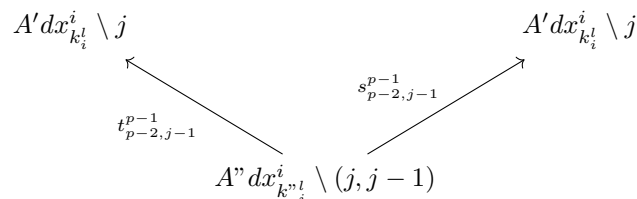
- When $k < j$ then we obtain $s_{p-2,k}^{p-1}(d)$ by using the diagram :



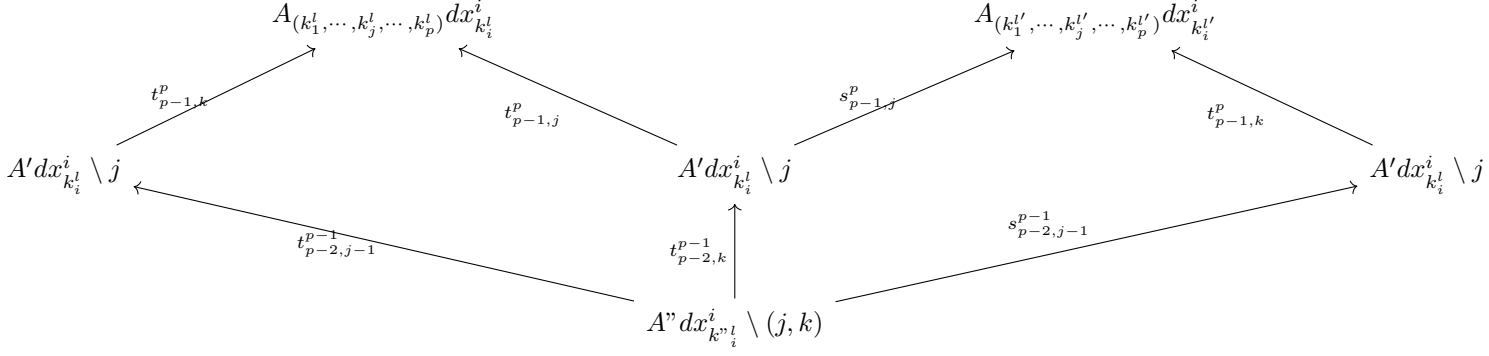
where we denote $A'' dx_{k_i}^i \setminus (j, k) = s_{p-2,k}^{p-1}(A' dx_{k_i}^i \setminus j)$

Remark 11 Of course we have also : $t_{p-2,j-1}^{p-1}(A' dx_{k_i}^i \setminus j) = s_{p-2,k}^{p-1}(A' dx_{k_i}^i \setminus j) = s_{p-2,j-1}^{p-1}(A' dx_{k_i}^i \setminus j)$ but in $t_{p-2,j-1}^{p-1}(A' dx_{k_i}^i \setminus j) = t_{p-2,j-1}^{p-1}(A' dx_{k_i}^i \setminus (j, j-1))$ and $s_{p-2,j-1}^{p-1}(A' dx_{k_i}^i \setminus j) = s_{p-2,j-1}^{p-1}(A' dx_{k_i}^i \setminus (j, j-1))$ the basic divisors A'' , $t_{p-2,j-1}^{p-1}(A')$ and $s_{p-2,j-1}^{p-1}(A')$ are not necessarily equals. \square

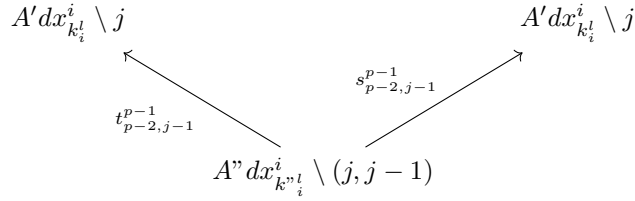
and thus the morphism of sketches $s_{p-2,k}^{p-1}$ sends d to the following diagram $s_{p-2,k}^{p-1}(d)$ of the sketches $\mathcal{E}_{f_{p-2}(s_{p-1,j}^p(x))}^{p-2}$ and $\mathcal{E}_{f_{p-2}(t_{p-1,j}^p(x))}^{p-2}$:



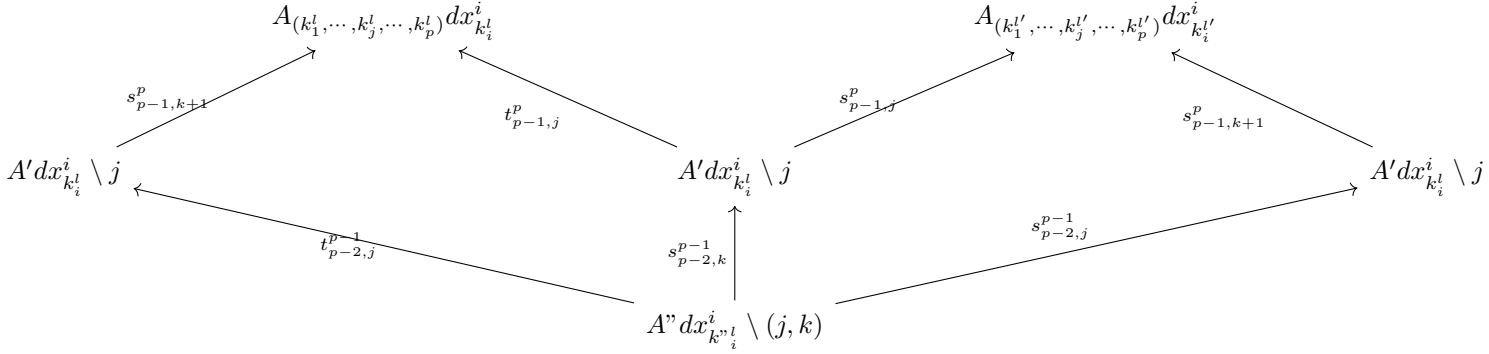
And we obtain $t_{p-2,k}^{p-1}(d)$ by using the diagram :



where we denote $A'' dx_{k_i^{l'}}^i \setminus (j, k) = t_{p-2, k}^{p-1}(A' dx_{k_i^l}^i \setminus j)$, and thus the morphism of sketches $t_{p-2, k}^{p-1}$ sends d to the following diagram $t_{p-2, k}^{p-1}(d)$ of the sketches $\mathcal{E}_{f_{p-2}(s_{p-1, j}^p(x))}^{p-2}$ and $\mathcal{E}_{f_{p-2}(t_{p-1, j}^p(x))}^{p-2}$:



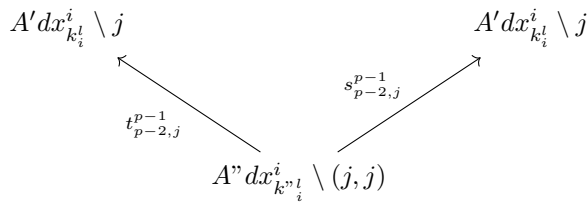
- When $k > j$ then we obtain $s_{p-2, k}^{p-1}(d)$ by using the diagram :



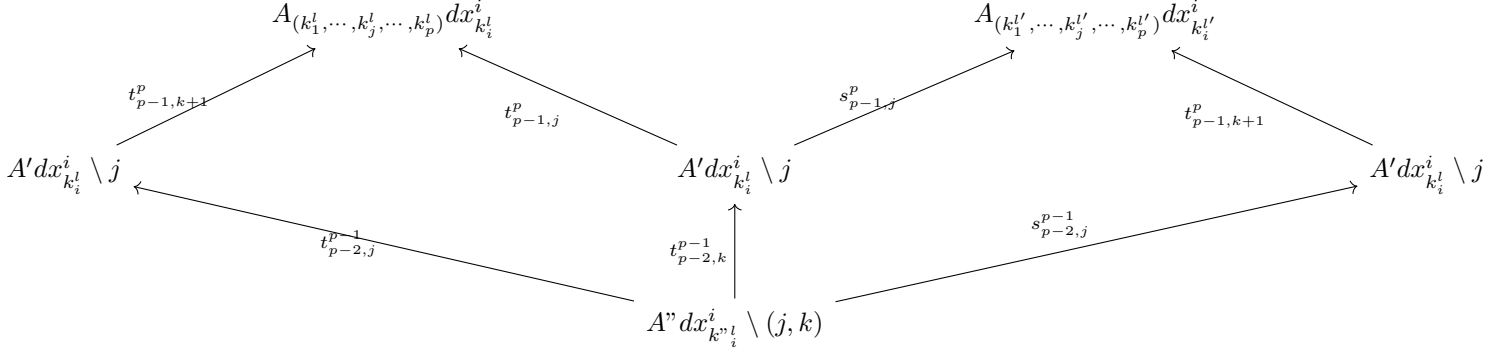
where we denote $A'' dx_{k_i^{l'}}^i \setminus (j, k) = s_{p-2, k}^{p-1}(A' dx_{k_i^l}^i \setminus j)$.

Remark 12 Of course we have also : $t_{p-2, j}^{p-1}(A' dx_{k_i^l}^i \setminus j) = s_{p-2, k}^{p-1}(A' dx_{k_i^l}^i \setminus j) = s_{p-2, j}^{p-1}(A' dx_{k_i^l}^i \setminus j)$ but in $t_{p-2, j}^{p-1}(A' dx_{k_i^l}^i \setminus j) = t_{p-2, j}^{p-1}(A') dx_{k_i^l}^i \setminus (j, j)$ and $s_{p-2, j}^{p-1}(A' dx_{k_i^l}^i \setminus j) = s_{p-2, j}^{p-1}(A') dx_{k_i^l}^i \setminus (j, j)$ the basic divisors A'' , $t_{p-2, j}^{p-1}(A')$ and $s_{p-2, j}^{p-1}(A')$ are not necessarily equals. \square

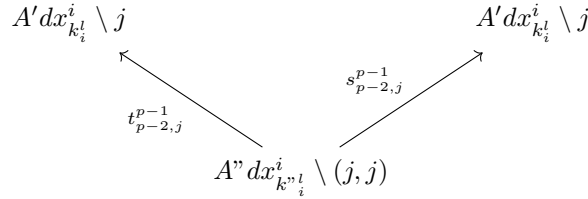
and thus the morphism of sketches $s_{p-2, k}^{p-1}$ sends d to the following diagram $s_{p-2, k}^{p-1}(d)$ of the sketches $\mathcal{E}_{f_{p-2}(s_{p-1, j}^p(x))}^{p-2}$ and $\mathcal{E}_{f_{p-2}(t_{p-1, j}^p(x))}^{p-2}$:



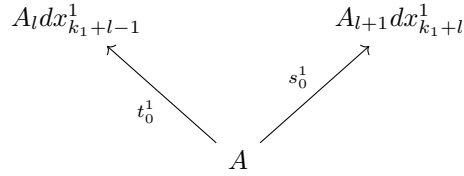
And we obtain $t_{p-2, k}^{p-1}(d)$ by using the diagram :



where we denote $A'' dx_{k_i}^i \setminus (j, k) = t_{p-2,k}^{p-1}(A' dx_{k_i}^i \setminus j)$, and thus the morphism of sketches $t_{p-2,k}^{p-1}$ sends d to the following diagram $t_{p-2,k}^{p-1}(d)$ of the sketches $\mathcal{E}_{f_{p-2}(s_{p-1,j}^p(x))}^{p-2}$ and $\mathcal{E}_{f_{p-2}(t_{p-1,j}^p(x))}^{p-2}$:

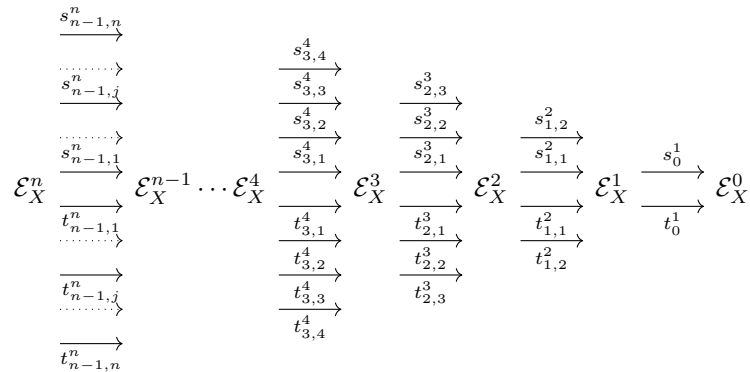


- The 1-faces $x \in f^1(X)$ of X are all of the form $x = A_1 dx_{k_1}^1 + A_2 dx_{k_1+1}^1 + \dots + A_l dx_{k_1+l-1}^1 + A_{l+1} dx_{k_1+l}^1 \dots + A_r dx_{k_1+r-1}^1$ where any basic divisor $A_l dx_{k_1+l-1}^1$ of x can be $1(1) dx_{k_1+l-1}^1$ or $1_1^0(1(0)) dx_{k_1+l-1}^1$ and the sketch $\mathcal{E}_{f_0(X)}^0$ is a set of diagrams of the form :



where A denotes the unique 0-cell $1(0)$ of the cubical site \mathbb{C} .

- It is interesting to notice that the sketch \mathcal{E}_X can be seen also as a n -cubical object in the category Sketch of sketches :



where we put : $\mathcal{E}_X^p := \bigcup_{x \in f_p(X)} \mathcal{E}_x^p$.

- **Definition 18** If \mathcal{E}_X is the sketch associated to a divisor X then it has a straightforward structure of cubical set given by different faces of X , and also it has a straightforward structure of reflexive cubical set given by :

$$1_{n+1,j}^n(\mathcal{E}_X) := \mathcal{E}_{1_{n+1,j}^n(X)} \text{ for } j \in \llbracket 1, n+1 \rrbracket, \text{ and } 1_{n+1,j}^{n,\gamma}(\mathcal{E}_X) := \mathcal{E}_{1_{n+1,j}^{n,\gamma}(X)} \text{ for } j \in \llbracket 1, n \rrbracket$$

- **Definition 19** A *rectangular n -sketch* are the one of the form \mathcal{E}_X where X is a rectangular n -divisor. □

Because rectangular divisors have two notions of sources and targets :

- the j -sources and j -targets ($j \in \mathbb{N}$ is a direction) which are useful to build associated sketches,
- the j -pasting sources and j -pasting targets which are useful to put a cubical strict ∞ -categorical structure with connections on rectangular divisors

their associated rectangular sketches inherit also two notions of sources-targets :

Definition 20 If X is a rectangular n -divisor and \mathcal{E}_X is its associated sketch, then we define :

$$\sigma_{n-1,j}^n(\mathcal{E}_X) := \mathcal{E}_{\sigma_{n-1,j}^n(X)} \text{ and } \tau_{n-1,j}^n(\mathcal{E}_X) := \mathcal{E}_{\tau_{n-1,j}^n(X)}$$

Proposition 8 *The set of sketches associated to rectangular divisors is equipped with a structure of cubical strict ∞ -category with connections that we denote by $\mathbb{C}\text{-}\mathbb{P}'\text{ast}$* □

PROOF – For all integer $n \in \mathbb{N}$, n -cells of $\mathbb{C}\text{-}\mathbb{P}'\text{ast}$ are sketches \mathcal{E}_X where X is a rectangular n -divisor.

- Consider two sketches \mathcal{E}_X and $\mathcal{E}_{X'}$ such that $\tau_{n-1,j}^n(\mathcal{E}_X) = \sigma_{n-1,j}^n(\mathcal{E}_{X'})$. Then we define :

$$\mathcal{E}_X \circ_{n-1,j}^n \mathcal{E}_{X'} := \mathcal{E}_{X \circ_{n-1,j}^n X'}$$

- If X is a rectangular n -divisor and \mathcal{E}_X is its associated sketch, then we defined above degeneracies $1_{n+1,j}^n(\mathcal{E}_X)$ for $j \in \llbracket 1, n+1 \rrbracket$ and $1_{n+1,j}^{n,-}$, $1_{n+1,j}^{n,+}$ for $j \in \llbracket 1, n \rrbracket$. ■

We thus have another description of the cubical Θ_0 which is the full subcategory $\Theta_0 \subset \mathbb{C}\text{Sets}$ which objects are rectangular sketches.

2.7 The monad of cubical strict ∞ -categories with connections

Consider a rectangular n -divisor $X = A_{(k_1^i, \dots, k_n^i)} dx_{k_i^i}^i + \dots + A_{(k_1^i, \dots, k_n^i)} dx_{k_i^i}^i + \dots + A_{(k_1^r, \dots, k_n^r)} dx_{k_i^r(X)}^i$ and a cubical set $C \in \mathbb{C}\text{Sets}$.

A *decoration* of X by cells of C is given by a *C -decorated rectangular n -divisor* :

$$\langle X, C \rangle = c_{(k_1^1, \dots, k_n^1)} dx_{k_i^1}^1 + \dots + c_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^l + \dots + c_{(k_1^r, \dots, k_n^r)} dx_{k_i^r(X)}^r$$

i.e a filling of X with cells $c_{(k_1^l, \dots, k_n^l)}$ of $\mathbb{R}(C)$ (i.e we substitute the $1(q)$'s in each basic divisors of X which are formally degenerate or not with the q -cells of C) such that for all directions $j \in \llbracket 1, n \rrbracket$ if $(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^l, A_{(k_1^{l'}, \dots, k_n^{l'}}) dx_{k_i^{l'}}^{l'})$ are j -gluing datas for X , i.e are such that $t_{n-1,j}^n(A_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^l) = s_{n-1,j}^n(A_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^{l'})$ then :

$$t_{n-1,j}^n(c_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^l) = s_{n-1,j}^n(c_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_i^{l'}}^{l'})$$

The set of decorations of X by cells of C is denoted $\text{Decor}(X, C)$. The colimit $\text{colim}\mathcal{E}_{\langle X, C \rangle}$ is taken in Sets and gives an n -cell of the free cubical strict ∞ -category $S(C)$ with connections.

We have another description of these n -cells $\text{colim}\mathcal{E}_{\langle X, C \rangle}$ by using gluing of representables. But first let us define what are *cubical sums* : consider a functor $\mathbb{C} \xrightarrow{F} \mathcal{C}$ where we denote $F(1(n)) = I^n$ and $F(s_{n-1,j}^n) = \mathbf{s}_{\mathbf{n}-1, \mathbf{j}}$, and $F(t_{n-1,j}^n) = \mathbf{t}_{\mathbf{n}-1, \mathbf{j}}$. The *F -decorated rectangular n -divisor* :

$$\langle X, F \rangle = c_{(k_1^1, \dots, k_n^1)} dx_{k_i^1}^1 + \dots + c_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^l + \dots + c_{(k_1^r, \dots, k_n^r)} dx_{k_i^r(X)}^r$$

is a filling of X by the n -cells I^n of \mathcal{C} in the sense that in each occurrence of the $1(n)$'s in the degenerate boxes of X , we substitute $1(n)$ by I^n . Here this is important to notice that the expressions $c_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^l$ that we obtain are formal degenerate terms build with the objects I^n of \mathcal{C} . Also there is only a unique F -decoration $\langle X, F \rangle$ of F for each rectangular n -divisor X .

We associate to $\langle X, F \rangle$ the sketch $\mathcal{E}_{\langle X, F \rangle}$ as in 2.6, and in fact this is just the realization of the sketch \mathcal{E}_X by F , i.e all formal cosources-cotargets $s_{p-1,j}^p, t_{p-1,j}^p$ of \mathcal{E}_X are sent to $\mathbf{s}_{\mathbf{p}-1, \mathbf{j}}, \mathbf{t}_{\mathbf{p}-1, \mathbf{j}}$ by F , and all formal degenerate terms $c_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^l$ in $\langle X, F \rangle$ must be well realized in \mathcal{C} . At this point this is interesting to notice that such realizations are possible in any category \mathcal{C} of presheaves, because in any category of presheaves we can build degenerate terms (like the one of the left adjoint of the forgetful functor U in 2.3).

If the colimit $\text{colim}\mathcal{E}_{\langle X, F \rangle}$ exists in \mathcal{C} then we say that it has the cubical sum associated to the F -decoration $\langle X, F \rangle$, or it has *X -cubical sum* for short. If the X -cubical sum exists in \mathcal{C} for all such decorations $\langle X, F \rangle$ and for all rectangular n -divisor X then we say that F is a *cubical extension* or has all *cubical sums*. A morphism of cubical extensions is given by a commutative triangle

$$\begin{array}{ccc}
& & \mathcal{C} \\
& \nearrow F & \\
\mathcal{C} & & \\
& \searrow F' & \\
& & \mathcal{C}'
\end{array}$$

such that the functor f preserves cubical sums, i.e we have $f(\operatorname{colim}\mathcal{E}_{\langle X, F \rangle}) = \operatorname{colim}\mathcal{E}_{\langle X, F' \rangle}$ for each F -decoration $\langle X, F \rangle$ of each rectangular divisor X . We denote by $\mathbb{C}\text{-Ext}$ the category of cubical extensions. It is easy to see that the functor i :

$$\mathbb{C} \xrightarrow{i} \Theta_0$$

which sends each objects $1(n)$ of \mathbb{C} to the basic box $1(n)dx_{k_i}^i$ is an initial object of $\mathbb{C}\text{-Ext}$, such that the unique map is given by the functor

$$\begin{array}{ccc}
& & \Theta_0 \\
& \nearrow i & \\
\mathbb{C} & & \\
& \searrow F & \\
& & \mathcal{C}
\end{array}
\quad \begin{array}{c} \\ \\ \\ \\ \operatorname{colim}\mathcal{E}_{\langle -, F \rangle} \\ \\ \\ \end{array}$$

where Θ_0 is here seen as the category of rectangular n -divisors (for each $n \in \mathbb{N}$) 2.5, and thus Θ_0 inherits a universal property. The Yoneda embedding :

$$\mathbb{C} \xrightarrow{Y} \mathbb{C}\text{Sets}$$

$$1(n) \longmapsto \operatorname{hom}_{\mathbb{C}\text{Sets}}(1(n), -)$$

is such cubical extension because $\mathbb{C}\text{Sets}$ has all small colimits. The full image $Y(\mathbb{C})$ of Y is a cubical set builds with representables of the pre-cubical site \mathbb{C} , and we put $Y(1(n)) = \operatorname{hom}_{\mathbb{C}\text{Sets}}(1(n), -)$ for all integer $n \in \mathbb{N}$, and $Y(s_{n-1,j}^n) = \mathbf{s}_{n-1,j}^n$, and $Y(t_{n-1,j}^n) = \mathbf{t}_{n-1,j}^n$. A decoration $\langle X, Y(\mathbb{C}) \rangle$ of X by cells of $Y(\mathbb{C})$ has here the same meaning as a Y -decorated rectangular n -divisor $\langle X, Y \rangle$, i.e it is given by the $Y(\mathbb{C})$ -decorated rectangular n -divisor

$$\langle X, Y(\mathbb{C}) \rangle = \langle X, Y \rangle = c_{(k_1^1, \dots, k_n^1)} dx_{k_i^1}^i + \dots + c_{(k_1^l, \dots, k_n^l)} dx_{k_i^l}^i + \dots + c_{(k_1^r, \dots, k_n^r)} dx_{k_i^r}^i$$

i.e we substitute each $1(p)$'s of X with the representable $\operatorname{hom}_{\mathbb{C}\text{Sets}}(1(p), -)$ for $p \leq n$.

The colimit $\operatorname{colim}\mathcal{E}_{\langle X, Y \rangle}$ (in $\mathbb{C}\text{Sets}$) is in fact a cubical set builds by gluing representables along their sources-targets : more precisely we have for all directions $j \in \llbracket 1, n \rrbracket$, $s_{n-1,j}^n(\operatorname{colim}\mathcal{E}_{\langle X, Y \rangle}) = \operatorname{colim}\mathcal{E}_{\langle s_{n-1,j}^n(X), Y \rangle}$, $t_{n-1,j}^n(\operatorname{colim}\mathcal{E}_{\langle X, Y \rangle}) = \operatorname{colim}\mathcal{E}_{\langle t_{n-1,j}^n(X), Y \rangle}$, and any zigzag of sources-targets of $\operatorname{colim}\mathcal{E}_{\langle X, Y \rangle}$ is equal to the colimit $\operatorname{colim}\mathcal{E}_{\langle x, Y \rangle}$ where x is the face of X obtained by this zigzag. Now we are ready to describe the monad $\mathbb{S} = (S, \lambda, \mu)$ of cubical strict ∞ -categories with connections : as we wrote in [9] the forgetful functor :

$$\infty\text{-CCAT} \xrightarrow{U} [\mathbb{C}^{op}, \text{Sets}]$$

which sends cubical strict ∞ -categories with connections to cubical sets is right adjoint and its induced monad is written $\mathbb{S} = (S, \lambda, \mu)$ where $1_{\mathbb{C}\text{Sets}} \xrightarrow{\lambda} S$ is its unit and $S^2 \xrightarrow{\mu} S$ is its multiplication.

If $C \in \mathbb{C}\text{Sets}$ is a cubical set, then we put :

$$S(C) := \bigcup_{X \in \Theta_0} \operatorname{colim}\mathcal{E}_{\langle X, C \rangle} = \bigcup_{X \in \Theta_0} \operatorname{hom}_{\mathbb{C}\text{Sets}}(\operatorname{colim}\mathcal{E}_{\langle X, Y \rangle}, C)$$

This description shows immediately that S preserved fiber products and $S(C)$ is a cubical strict ∞ -categories with connections. The unit λ of \mathbb{S} is given by the map :

$$C \xrightarrow{\lambda} S(C)$$

$$c \longmapsto c$$

The multiplication μ of \mathbb{S} :

$$S^2(C) \xrightarrow{\mu} S(C)$$

is more subtle and need more work. Consider a decoration of X by cells of $\mathbb{C}\text{-Past}$ i.e we start with a $\mathbb{C}\text{-Past}$ -decorated rectangular n -divisor :

$$\langle X, \mathbb{C}\text{-Past} \rangle = X_{(k_1^1, \dots, k_n^1)} dx_{k_1^1}^i + \dots + X_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i + \dots + X_{(k_1^r, \dots, k_n^r)} dx_{k_1^r}^i$$

i.e a filling of X with cells $X_{(k_1^l, \dots, k_n^l)}$ of $\mathbb{C}\text{-Past}$ such that for all directions $j \in \llbracket 1, n \rrbracket$ if $(A_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i, A_{(k_1^{l'}, \dots, k_n^{l'}}) dx_{k_1^{l'}}^i)$ are j -gluing datas for X , then :

$$\tau_{n-1, j}^n(X_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i) = \sigma_{n-1, j}^n(X_{(k_1^{l'}, \dots, k_n^{l'})} dx_{k_1^{l'}}^i)$$

Here sources $\sigma_{n-1, j}^n$ and targets $\tau_{n-1, j}^n$ are the pasting-sources and the pasting-targets for rectangular divisors. We obtain then a new rectangular n -divisor denoted $X_{(X, \mathbb{C}\text{-Past})}$ that we obtain by reindexing all coordinates of basic divisors inside each rectangular n -divisors $X_{(k_1^l, \dots, k_n^l)}$. An important fact is the sketch $\mathcal{E}_{X_{(X, \mathbb{C}\text{-Past})}}$ is obtained by using the realization of the $s_{n-1, j}^n$ to the $\sigma_{n-1, j}^n$ and the realization of the $t_{n-1, j}^n$ to the $\tau_{n-1, j}^n$: this is here that we see the interplay between sources-targets of a rectangular sketch and pasting-sources and pasting-targets between rectangular sketches. This provides on $\mathbb{C}\text{-Past}$ an operation that we call *the substitution* of $\mathbb{C}\text{-Past}$. The multiplication μ of \mathbb{S} :

$$S^2(C) \xrightarrow{\mu} S(C)$$

is then given by :

$$\bigcup_{X \in \Theta_0} \text{hom}_{\mathbb{C}\text{Sets}}(\text{colim} \mathcal{E}_{\langle X, Y \rangle}, S(C)) \xrightarrow{\mu(C)} \bigcup_{X \in \Theta_0} \text{hom}_{\mathbb{C}\text{Sets}}(\text{colim} \mathcal{E}_{\langle X, Y \rangle}, C)$$

where

$$\bigcup_{X \in \Theta_0} \text{hom}_{\mathbb{C}\text{Sets}}(\text{colim} \mathcal{E}_{\langle X, Y \rangle}, S(C))$$

is given by :

$$\bigcup_{X \in \Theta_0} \text{hom}_{\mathbb{C}\text{Sets}}\left(\text{colim} \mathcal{E}_{\langle X, Y \rangle}, \bigcup_{X \in \Theta_0} \text{hom}_{\mathbb{C}\text{Sets}}(\text{colim} \mathcal{E}_{\langle X, Y \rangle}, C)\right)$$

An element of $S^2(C)$ is described as follow : it underlies a decoration of X by C -decorated rectangular n -divisors $\langle X_{(k_1^l, \dots, k_n^l)}, C \rangle$, and we write such decoration :

$$\langle X, \mathbb{C}\text{-Past}, C \rangle = \langle X_{(k_1^1, \dots, k_n^1)}, C \rangle dx_{k_1^1}^i + \dots + \langle X_{(k_1^l, \dots, k_n^l)}, C \rangle dx_{k_1^l}^i + \dots + \langle X_{(k_1^r, \dots, k_n^r)}, C \rangle dx_{k_1^r}^i$$

which itself underlies the $\mathbb{C}\text{-Past}$ -decorated rectangular n -divisor :

$$\langle X, \mathbb{C}\text{-Past} \rangle = X_{(k_1^1, \dots, k_n^1)} dx_{k_1^1}^i + \dots + X_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i + \dots + X_{(k_1^r, \dots, k_n^r)} dx_{k_1^r}^i$$

In the decoration $\langle X, \mathbb{C}\text{-Past}, C \rangle$ we substitute each $\langle X_{(k_1^l, \dots, k_n^l)}, C \rangle$ by the n -cells $\text{colim} \mathcal{E}_{\langle X_{(k_1^l, \dots, k_n^l)}, C \rangle}$ of $S(C)$, which are also described as maps between the gluing of representables $\text{colim} \mathcal{E}_{\langle X_{(k_1^l, \dots, k_n^l)}, Y \rangle}$ to C . Thus we obtain a decoration of a rectangular n -divisor X by n -cells of $S(C)$ that we write :

$$[X, \mathbb{C}\text{-Past}, C] = \text{colim}(\langle X_{(k_1^1, \dots, k_n^1)}, C \rangle) dx_{k_1^1}^i + \dots + \text{colim}(\langle X_{(k_1^l, \dots, k_n^l)}, C \rangle) dx_{k_1^l}^i + \dots + \text{colim}(\langle X_{(k_1^r, \dots, k_n^r)}, C \rangle) dx_{k_1^r}^i$$

An element of $S^2(C)$ is then given by $\text{colim} \mathcal{E}_{[X, \mathbb{C}\text{-Past}, C]}$. Thus elements of $S^2(C)$ are described by a colimit of colimits of n -cells in $S(C)$. But colimit of colimits of cocones is the colimit of a cocone obtained by gluing all cocones together, and we can already guess that the multiplication $\mu(C)$ uses this simple fact : we use $X_{(X, \mathbb{C}\text{-Past})}$, the substitution associated to the $\mathbb{C}\text{-Past}$ -decorated rectangular n -divisor :

$$\langle X, \mathbb{C}\text{-Past} \rangle = X_{(k_1^1, \dots, k_n^1)} dx_{k_1^1}^i + \dots + X_{(k_1^l, \dots, k_n^l)} dx_{k_1^l}^i + \dots + X_{(k_1^r, \dots, k_n^r)} dx_{k_1^r}^i$$

in order to glue all cocones of the sketches \mathcal{E}_X and $\mathcal{E}_{X_{(k_1^l, \dots, k_n^l)}}$ ($l \in \llbracket 1, r(X) \rrbracket$) and to obtain cocones of $\mathcal{E}_{X_{(X, \mathbb{C}\text{-Past})}}$.

Thus we have the following definition of $\mu(C)$:

$$S^2(C) \xrightarrow{\mu(C)} S(C)$$

$\mu(C)$ sends $\text{colim} \mathcal{E}_{[X, \mathbb{C}\text{-Past}, C]}$ to $\text{colim} \mathcal{E}_{\langle X_{(X, \mathbb{C}\text{-Past})}, C \rangle}$ which is the same thing as to say that it sends $\text{colim} \mathcal{E}_{[X, \mathbb{C}\text{-Past}, C]}$ to a map between the gluing of representables $\text{colim} \mathcal{E}_{\langle X_{(X, \mathbb{C}\text{-Past})}, Y \rangle}$ to C .

Theorem 1 *The monad $\mathbb{S} = (S, \lambda, \mu)$ acting on $\mathbb{C}\text{Sets}$ which algebras are cubical strict ∞ -categories with connections (described in [9, 10]) is cartesian. \square*

PROOF The description of the monad $\mathbb{S} = (S, \lambda, \mu)$ above shows that its underlying endofunctor S does preserve fibred products. We are going to prove that the multiplication μ is cartesian, i.e we are going to prove that if $C \in \mathbb{C}\text{Sets}$ is a cubical set then the commutative diagram :

$$\begin{array}{ccc} S^2(C) & \xrightarrow{S^2(!)} & S^2(1) \\ \mu(C) \downarrow & & \downarrow \mu(1) \\ S(C) & \xrightarrow{S(!)} & S(1) \end{array}$$

is a cartesian square. Consider the commutative diagram in $\mathbb{C}\text{Sets}$:

$$\begin{array}{ccc} C' & \xrightarrow{g} & S^2(1) \\ f \downarrow & & \downarrow \mu(1) \\ S(C) & \xrightarrow{S(!)} & S(1) \end{array}$$

Thus if x is an n -cell of C' then $f(x) = \text{colim}\mathcal{E}_{\langle X', C \rangle}$ where X' is a rectangular n -divisor, and $S(!)(f(x)) = S(!)(\text{colim}\mathcal{E}_{\langle X', C \rangle}) = \text{colim}\mathcal{E}_{\langle X', 1 \rangle}$, and $g(x) = \text{colim}\mathcal{E}_{[X, \mathbb{C}\text{-Past}, 1]}$ thus $\mu(1)(g(x)) = \mu(1)(\text{colim}\mathcal{E}_{[X, \mathbb{C}\text{-Past}, 1]}) = \text{colim}\mathcal{E}_{\langle X_{\langle X, \mathbb{C}\text{-Past} \rangle}, 1 \rangle}$. But the commutativity gives $S(!)(f(x)) = \mu(1)(g(x))$ thus this commutativity gives $X' = X_{\langle X, \mathbb{C}\text{-Past} \rangle}$ and $f(x) = \text{colim}\mathcal{E}_{\langle X_{\langle X, \mathbb{C}\text{-Past} \rangle}, C \rangle}$. It is then easy to see that we get a unique map l :

$$\begin{array}{ccccc} C' & & & & \\ & \searrow & & & \\ & & S^2(C) & \xrightarrow{S^2(!)} & S^2(1) \\ & \searrow f & \downarrow \mu(C) & & \downarrow \mu(1) \\ & & S(C) & \xrightarrow{S(!)} & S(1) \end{array}$$

defined by $l(x) = \text{colim}\mathcal{E}_{[X, \mathbb{C}\text{-Past}, C]}$, and is such that $\mu(C)(l(x)) = \text{colim}\mathcal{E}_{\langle X_{\langle X, \mathbb{C}\text{-Past} \rangle}, C \rangle} = f(x)$ and $S^2(!)(l(x)) = S^2(!)(\text{colim}\mathcal{E}_{[X, \mathbb{C}\text{-Past}, C]}) = \text{colim}\mathcal{E}_{[X, \mathbb{C}\text{-Past}, 1]} = g(x)$. The cartesianity of the unit

$$C \xrightarrow{\lambda} S(C)$$

is easier and goes as follow : we start with a commutative diagram in $\mathbb{C}\text{Sets}$

$$\begin{array}{ccc} C' & \xrightarrow{!} & 1 \\ f \downarrow & & \downarrow \lambda(1) \\ S(C) & \xrightarrow{S(!)} & S(1) \end{array}$$

Let x an n -cell of C' , thus we have $f(x) = \text{colim}\mathcal{E}_{\langle X', C \rangle}$ where X' is a rectangular n -divisor. Thus $S(!)(f(x)) = \text{colim}\mathcal{E}_{\langle X', 1 \rangle}$, and then the commutativity gives $\text{colim}\mathcal{E}_{\langle X', 1 \rangle} = 1$, and this shows that $X' = 1(n)dx_{k_i}^i$ is just the basic n -box without degeneracies. It shows that there is a unique map l :

$$\begin{array}{ccccc} C' & & & & \\ & \searrow & & & \\ & & C & \xrightarrow{!} & 1 \\ & \searrow f & \downarrow \lambda(C) & & \downarrow \lambda(1) \\ & & S(C) & \xrightarrow{S(!)} & S(1) \end{array}$$

defined by $l(x) = \text{colim}\mathcal{E}_{\langle X', C \rangle}$, and such that $\lambda(C)(l(x)) = \text{colim}\mathcal{E}_{\langle X', C \rangle} = f(x)$. ■

With this theorem we solved the conjecture in [10] for the monad of cubical strict ∞ -categories with connections which provides a complete description of the cubical operad \mathbb{B}_C^0 of cubical weak ∞ -categories with connections.

Proposition 9 *The monad $S = (S, \lambda, \mu)$ acting on $\mathbb{C}\text{Sets}$ which algebras are cubical strict ∞ -categories (without connections) is cartesian.* \square

PROOF This is easy, here we just use the previous proof by using only rectangular divisors build with classical degeneracies $1_{n+1,j}^n$ ($n \in \mathbb{N}$ and $j \in \llbracket 1, n+1 \rrbracket$) and their associated rectangular sketches. \blacksquare

With this proposition we can easily use the materials in [10] to build the cubical operad \mathbb{B}_C^0 of cubical weak ∞ -categories without connections. In particular it is interesting to know that \mathbb{B}_C^0 -algebras of dimensions 2 are exactly *double categories of Verity* [14]. The proof of such fact is made just by mimic the proof of Michael Batanin in [2] where he proved that with globular operads, \mathbb{B}_C^0 -algebras of dimensions 2 are exactly bicategories.

3 The cubical coherator Θ_W^∞ of cubical weak ∞ -categories with connections

- A cubical theory is given by a cubical extension (2.7) :

$$\mathbb{C} \xrightarrow{F} \mathcal{C}$$

such that the induced unique functor \bar{F} :

$$\Theta_0 \xrightarrow{\bar{F}} \mathcal{C}$$

is bijective on objects, and thus a cubical theory³ Θ is a small category which objects are identify with rectangular divisors. In particular a chosen initial object in $\mathbb{C}\text{-Ext}$:

$$\mathbb{C} \xrightarrow{i} \Theta_0$$

is a specific cubical theory called the initial cubical theory. The full subcategory of $\mathbb{C}\text{-Ext}$ which objects are cubical theories is denoted $\mathbb{C}\text{-Th}$ and the cubical theory $\mathbb{C} \xrightarrow{i} \Theta_0$ is initial in it. It is interesting to notice that morphisms G in $\mathbb{C}\text{-Th}$:

$$\begin{array}{ccc} & & \Theta \\ & \nearrow F & \downarrow G \\ \mathbb{C} & & \Theta' \\ & \searrow F' & \end{array}$$

induce, thanks to the universality of Θ_0 , the following commutative triangles in the category Cat of small categories :

$$\begin{array}{ccc} & & \Theta \\ & \nearrow \bar{F} & \downarrow G \\ \Theta_0 & & \Theta' \\ & \searrow \bar{F}' & \end{array}$$

and more precisely this is a commutative triangle in the subcategory $\mathbb{C}\text{-Sketch} \subset \text{Sketch}$ of the category of small sketches equipped with the cocones which underly the rectangular sketches \mathcal{E}_X of the section 2.6 where X is a rectangular n -divisor, because all the functors G , \bar{F} and \bar{F}' do preserve these cocones.

- A set-model for the theory Θ or a Θ -model for short is given by a functor :

$$\Theta^{\text{op}} \xrightarrow{G} \text{Sets}$$

which sends cubical sums to cubical products : more precisely by using the diagram :

$$\Theta_0^{\text{op}} \xrightarrow{\bar{F}^{\text{op}}} \Theta^{\text{op}} \xrightarrow{G} \text{Sets}$$

³A cubical extension is written by using the Greek letter Θ when it is a cubical theory.

we require the equality $(G \circ \bar{F}^{op})(\mathcal{E}_X) = \lim \mathcal{E}_{G(X)}^{op}$, where $G(X)$ denotes the rectangular divisor decorated by occurrences of $G(I^p)$ i.e we substitute in each basic divisor of X , $1(p)$ by $G(I^p)$, and the projective sketch $\mathcal{E}_{G(X)}^{op}$ is defined as the opposite sketch of $\mathcal{E}_{G(X)}$.

- A crucial example of cubical theory is the one $\Theta_{\mathbb{M}}$ of cubical reflexive ∞ -magmas. We recall their definition ([9]) : consider a cubical reflexive set

$$(C, (1_{n+1,j}^n)_{n \in \mathbb{N}, j \in \llbracket 1, n+1 \rrbracket}, (1_{n+1,j}^{n,\gamma})_{n \geq 1, j \in \llbracket 1, n \rrbracket})$$

equipped with partial operations $(\circ_j^n)_{n \geq 1, j \in \llbracket 1, n \rrbracket}$ where if $a, b \in C(n)$ then $a \circ_j^n b$ is defined for $j \in \{1, \dots, n\}$ if $s_j^n(b) = t_j^n(a)$. We also require these operations to follow the following axioms of positions :

- (i) For $1 \leq j \leq n$ we have : $s_{n-1,j}^n(a \circ_j^n b) = s_{n-1,j}^n(a)$ and $t_{n-1,j}^n(a \circ_j^n b) = t_{n-1,j}^n(a)$,
- (ii) $s_{n-1,i}^n(a \circ_j^n b) = \begin{cases} s_{n-1,i}^n(a) \circ_{j-1}^{n-1} s_{n-1,i}^n(b) & \text{if } 1 \leq i < j \leq n \\ s_{n-1,i}^n(a) \circ_j^{n-1} s_{n-1,i}^n(b) & \text{if } 1 \leq j < i \leq n \end{cases}$
- (iii) $t_{n-1,i}^n(a \circ_j^n b) = \begin{cases} t_{n-1,i}^n(a) \circ_{j-1}^{n-1} t_{n-1,i}^n(b) & \text{if } 1 \leq i < j \leq n \\ t_{n-1,i}^n(a) \circ_j^{n-1} t_{n-1,i}^n(b) & \text{if } 1 \leq j < i \leq n \end{cases}$

Definition 21 Cubical ∞ -magmas are cubical sets equipped with partial operations like above. A morphism between two cubical ∞ -magmas is a morphism of their underlying cubical sets which respects partial operations $(\circ_j^n)_{n \geq 1, j \in \llbracket 1, n \rrbracket}$. The category of cubical ∞ -magmas is noted $\infty\text{-CMag}$ \square

Definition 22 Cubical reflexive ∞ -magmas are cubical reflexive set equipped a structure of ∞ -magmas. A morphism between two cubical reflexive ∞ -magmas is a morphism of their underlying cubical reflexive sets which respects partial operations $(\circ_j^n)_{n \geq 1, j \in \llbracket 1, n \rrbracket}$. The category of cubical reflexive ∞ -magmas is noted $\infty\text{-CMag}_r$ \square

Now the forgetful functor

$$\infty\text{-CMag}_r \xrightarrow{V} \mathbb{C}\text{Sets}$$

is right adjoint and it induces the monad $\mathbb{M} = (M, \eta, \mu)$ of cubical reflexive ∞ -magmas with its Kleisli category $\mathbb{Kl}(\mathbb{M})$. Denote by $\Theta_{\mathbb{M}}$ the full subcategory of $\mathbb{Kl}(\mathbb{M})$ which objects are objects of Θ_0 . This small category $\Theta_{\mathbb{M}}$ equipped with the canonical inclusion functor $\Theta_0 \xrightarrow{j} \Theta_{\mathbb{M}}$ is an important cubical theory because it is the basic data we need to build the coherator Θ_W which models are cubical weak ∞ -categories with connections.

Consider an object $\mathbb{C} \xrightarrow{i} \Theta$ of the category $\mathbb{C}\text{-Th}$ of cubical theories and the unique functor $\Theta_0 \xrightarrow{\bar{i}} \Theta$. A I^n -arrow in Θ is one arrow of it with domain the object I^n (which is by definition equal to $i(1(n))$). A pair (f, g) of I^n -arrows in Θ :

$$I^n \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} X$$

is called

- admissible if it doesn't belong to the image of \bar{i}
- j -admissible (for a direction $j \in \llbracket 1, n \rrbracket$) if it is admissible and it is j -parallel, i.e $f \circ s_{n-1,j}^n = g \circ s_{n-1,j}^n$ and $f \circ t_{n-1,j}^n = g \circ t_{n-1,j}^n$

$$\begin{array}{ccc} I^n & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & X \\ \uparrow s_{n-1,j}^n & & \uparrow t_{n-1,j}^n \\ I^{n-1} & & \end{array}$$

If a pair (f, g) of I^n -arrows : $I^n \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} X$ is admissible, then we define its *liftings* which are all j -lifting arrows $[f, g]_{n+1,j}^n$ for all $j \in \llbracket 1, n+1 \rrbracket$:

$$\begin{array}{ccc}
I^{n+1} & & \\
\uparrow s_{n,i}^{n+1} & \uparrow t_{n,i}^{n+1} & \text{---} [f,g]_{n+1,j}^{n+1} \text{---} \\
I^n & \xrightarrow{f} & X \\
\uparrow s_{n-1,i}^n & \uparrow t_{n-1,i}^n & \text{---} g \text{---} \\
I^{n-1} & &
\end{array}$$

by using an induction. Thus we suppose that such operations $[-, -]_{p+1,k}^p$ ($p \leq n-1$ and $k \in \llbracket 1, p+1 \rrbracket$) exists for all faces of f and g . The definition of $[f, g]_{n+1,j}^n$ goes as follow :

- if $1 \leq i < j \leq n+1$ then $[f, g]_{n+1,j}^n \circ s_{n,i}^{n+1} = [f \circ s_{n-1,i}^n, g \circ s_{n-1,i}^n]_{n,j-1}^{n-1}$ and $[f, g]_{n+1,j}^n \circ t_{n,i}^{n+1} = [f \circ t_{n-1,i}^n, g \circ t_{n-1,i}^n]_{n,j-1}^{n-1}$
- if $1 \leq j \leq n+1$ then $[f, g]_{n+1,j}^n \circ s_{n,j}^{n+1} = f$ and $[f, g]_{n+1,j}^n \circ t_{n,j}^{n+1} = g$
- if $1 \leq j < i \leq n+1$ then $[f, g]_{n+1,j}^n \circ s_{n,i}^{n+1} = [f \circ s_{n-1,i-1}^n, g \circ s_{n-1,i-1}^n]_{n,j}^{n-1}$ and $[f, g]_{n+1,j}^n \circ t_{n,i}^{n+1} = [f \circ t_{n-1,i-1}^n, g \circ t_{n-1,i-1}^n]_{n,j}^{n-1}$

If a pair (f, g) of I^n -arrows : $I^n \xrightarrow{f} X$ is j -admissible (for a fixed direction $j \in \llbracket 1, n \rrbracket$), then we define its $(j, -)$ -lifting arrow $[f, g]_{n+1,j}^{n,-}$ or its $(j, -)$ -lifting for short, the following I^{n+1} -arrow :

$$\begin{array}{ccc}
I^{n+1} & & \\
\uparrow s_{n,i}^{n+1} & \uparrow t_{n,i}^{n+1} & \text{---} [f,g]_{n+1,j}^{n,-} \text{---} \\
I^n & \xrightarrow{f} & X \\
\uparrow s_{n-1,i}^n & \uparrow t_{n-1,i}^n & \text{---} g \text{---} \\
I^{n-1} & &
\end{array}$$

by using an induction. Thus we suppose that such operations $[-, -]_{p+1,k}^{p,-}$ ($p \leq n-1$ and $k \in \llbracket 1, p \rrbracket$) exists for all faces of f and g , but also (see the induction used just below) we have to suppose that the operations $[-, -]_{p+1,k}^p$ ($p \leq n-1$ and $k \in \llbracket 1, p+1 \rrbracket$) defined above exists for such faces. The definition of $[f, g]_{n+1,j}^{n,-}$ goes as follow :

- if $1 \leq i < j \leq n$ then $[f; g]_{n+1,j}^{n,-} \circ s_{n,i}^{n+1} = [f \circ s_{n-1,i}^n; g \circ s_{n-1,i}^n]_{n,j-1}^{n-1,-}$ and $[f; g]_{n+1,j}^{n,-} \circ t_{n,i}^{n+1} = [f \circ t_{n-1,i}^n; g \circ t_{n-1,i}^n]_{n,j-1}^{n-1,-}$
- if $1 \leq j \leq n$ then $[f; g]_{n+1,j}^{n,-} \circ s_{n,j}^{n+1} = f$ and $[f; g]_{n+1,j}^{n,-} \circ s_{n,j+1}^{n+1} = g$, and $[f; g]_{n+1,j}^{n,-} \circ t_{n,j}^{n+1} = [f; g]_{n+1,j}^{n,-} \circ t_{n,j+1}^{n+1} = [f \circ t_{n-1,j}^n, g \circ t_{n-1,j}^n]_{n,j}^{n-1}$
- if $2 \leq j+1 < i \leq n+1$ then $[f; g]_{n+1,j}^{n,-} \circ s_{n,i}^{n+1} = [f \circ s_{n-1,i-1}^n; g \circ s_{n-1,i-1}^n]_{n,j}^{n-1,-}$ and $[f; g]_{n+1,j}^{n,-} \circ s_{n,i}^{n+1} = [f \circ s_{n-1,i-1}^n; g \circ s_{n-1,i-1}^n]_{n,j}^{n-1,-}$

If a pair (f, g) of I^n -arrows : $I^n \xrightarrow{f} X$ is j -admissible (for a fixed direction $j \in \llbracket 1, n \rrbracket$), then we define its $(j, +)$ -lifting arrow $[f, g]_{n+1,j}^{n,+}$ or its $(j, +)$ -lifting for short, the following I^{n+1} -arrow :

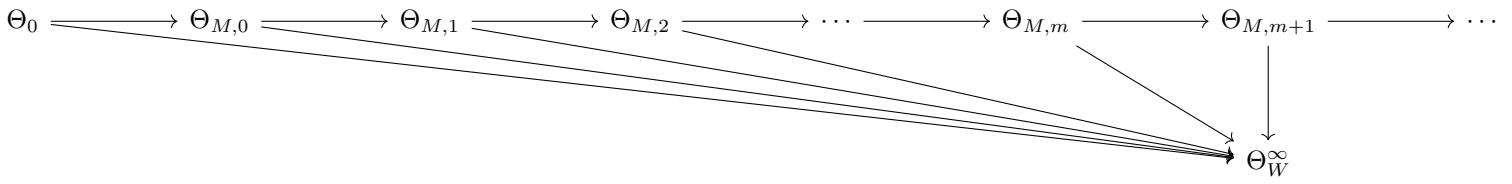
$$\begin{array}{ccc}
I^{n+1} & & \\
\uparrow s_{n,i}^{n+1} & \uparrow t_{n,i}^{n+1} & \text{---} [f,g]_{n+1,j}^{n,+} \text{---} \\
I^n & \xrightarrow{f} & X \\
\uparrow s_{n-1,i}^n & \uparrow t_{n-1,i}^n & \text{---} g \text{---} \\
I^{n-1} & &
\end{array}$$

by using an induction. Thus we suppose that such operations $[-, -]_{p+1, k}^{p, +}$ ($p \leq n-1$ and $k \in \llbracket 1, p \rrbracket$) exists for all faces of f and g , but also (see the induction used just below) we have to suppose that the operations $[-, -]_{p+1, k}^p$ ($p \leq n-1$ and $k \in \llbracket 1, p+1 \rrbracket$) defined above exists for such faces. The definition of $[f, g]_{n+1, j}^{n, +}$ goes as follow :

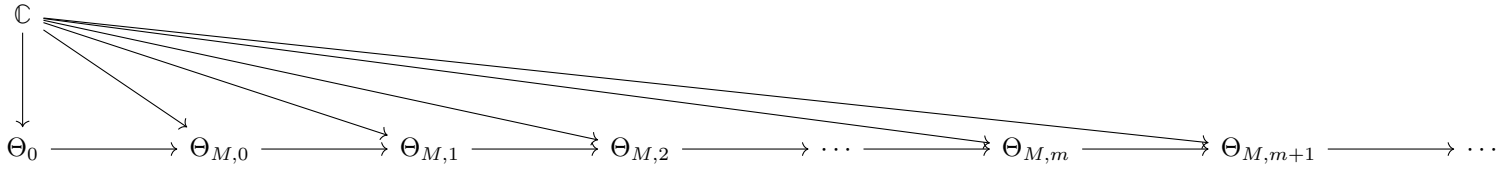
- if $1 \leq i < j \leq n$ then $[f; g]_{n+1, j}^{n, +} \circ s_{n, i}^{n+1} = [f \circ s_{n-1, i}^n; g \circ s_{n-1, i}^n]_{n, j-1}^{n-1, +}$ and $[f; g]_{n+1, j}^{n, +} \circ t_{n, i}^{n+1} = [f \circ t_{n-1, i}^n; g \circ t_{n-1, i}^n]_{n, j-1}^{n-1, +}$
- if $1 \leq j \leq n$ $[f; g]_{n+1, j}^{n, -} \circ s_{n, j}^{n+1} = [f; g]_{n+1, j}^{n, -} \circ s_{n, j+1}^{n+1} = [f \circ s_{n-1, j}^n; g \circ s_{n-1, j}^n]_{n, j}^{n-1}$ and $[f; g]_{n+1, j}^{n, +} \circ t_{n, j}^{n+1} = f$ and $[f; g]_{n+1, j}^{n, +} \circ t_{n, j+1}^{n+1} = g$,
- if $2 \leq j+1 < i \leq n+1$ then $[f; g]_{n+1, j}^{n, +} \circ s_{n, i}^{n+1} = [f \circ s_{n-1, i-1}^n; g \circ s_{n-1, i-1}^n]_{n, j}^{n-1, +}$ and $[f; g]_{n+1, j}^{n, +} \circ t_{n, i}^{n+1} = [f \circ t_{n-1, i-1}^n; g \circ t_{n-1, i-1}^n]_{n, j}^{n-1, +}$

Definition 23 A cubical theory Θ is contractible if for all integer $n \geq 1$, for all pairs (f, g) of I^n -arrows in it which are admissible, have liftings, and for all pairs (f, g) of I^n -arrows in it which are j -admissible ($j \in \llbracket 1, n \rrbracket$), have a $(j, -)$ -lifting and have a $(j, +)$ -lifting $_{\square}$

Now we are going to build a cubical contractible theory Θ_W^∞ which set-models are cubical weak ∞ -categories with connections. This theory Θ_W^∞ is a coerator in the sense of Grothendieck ([13]), i.e it is obtained as a colimit of a diagram \mathcal{D}_{Θ_W} in $\mathbb{C}at$ of cubical theories :



and this diagram \mathcal{D}_{Θ_W} is a sequence in the category $\mathbb{C}\text{-Th}$ of cubical theories :



that we may define inductively :

- We start the induction with $\Theta_{M,0} = \Theta_{\mathbb{M}}$ i.e with the cubical theory of cubical reflexive ∞ -magmas.
- We denote by $E_{M,0}$ the set which is the union of all admissible pairs of I^n -arrows in $\Theta_{\mathbb{M}}$ (for all $n \geq 1$), all j -admissible pairs of I^n -arrows in $\Theta_{\mathbb{M}}$ (for all directions $j \in \llbracket 1, n \rrbracket$ for all $n \geq 1$);
- $\Theta_{M,1}$ is obtained by formally (see just below a precise meaning of "formally") adding in $\Theta_{M,0}$ all kind of liftings of elements of $E_{M,0}$.
- Denote by $E_{M,1}$ the set which is the union of : all admissible pairs of I^n -arrows in $\Theta_{M,1}$ which are not in $E_{M,0}$, and all j -admissible pairs of I^n -arrows in $\Theta_{M,1}$ which are not in $E_{M,0}$.
- $\Theta_{M,2}$ is obtained by formally adding in $\Theta_{M,1}$ all kind of liftings of elements of $E_{M,1}$.
- we suppose that until the integer $m-1$ the sequence :

$$(\Theta_{M,0}, E_{M,0}) \longrightarrow (\Theta_{M,1}, E_{M,1}) \longrightarrow \dots \longrightarrow (\Theta_{M,m-2}, E_{M,m-2}) \longrightarrow (\Theta_{M,m-1}, E_{M,m-1})$$

is well defined. Thus $\Theta_{M,m}$ is obtained by formally adding in $\Theta_{M,m-1}$ all kind of liftings of elements of $E_{M,m-1}$.

- we associate to $\Theta_{M,m}$ the set $E_{M,m}$ which is the union of : all admissible pairs of I^n -arrows in $\Theta_{M,m}$ which are not in $E_{M,m-1}$, all j -admissible pairs of I^n -arrows in $\Theta_{M,m}$ which are not in $E_{M,m-1}$.

An important fact is the cubical theory $\Theta_{M,m}$ obtained by formally adding in $\Theta_{M,m-1}$ all liftings of elements of $E_{M,m-1}$ is universal for this adding. To give a precise meaning of "formally adding" is just an application of the following theorem of Christian Lair⁴ :

Theorem 2 (Lair) *The category Sketch of Sketches is projectively sketchable, that is there a projective sketch $\mathcal{E}_{\text{Sketch}}$ such that the category $\text{Mod}(\mathcal{E}_{\text{Sketch}})$ of set-models of $\mathcal{E}_{\text{Sketch}}$ is equivalent to the category Sketch.* \square

⁴This result was found by Christian Lair, but we were not able to find an exact reference of it.

Also the category \mathbf{Cat} of small categories is also projectively sketchable by a projective sketch $\mathcal{E}_{\mathbf{Cat}}$ and we have an easy morphism of projective sketches : $\mathcal{E}_{\mathbf{Cat}} \xrightarrow{i} \mathcal{E}_{\mathbf{Sketch}}$ which induces a left adjunction F with the functor $\mathbf{Mod}(i) : \mathbf{Sketch} \xrightarrow{F} \mathbf{Cat}$. This construction is called the *free prototype functor*⁵. With these results in hands it is useful to see the cubical theory $\Theta_{M,m}$ obtained by formally adding in $\Theta_{M,m-1}$ all liftings of elements of $E_{M,m-1}$ as the free category (with the free prototype functor) generated by this adding. Thus we start with the object $\Theta_{M,m-1} + E_{M,m-1}$ of \mathbf{Sketch} , where we formally⁶ add all liftings of elements of $E_{M,m-1}$ in the sketch $\Theta_{M,m-1}$, then $\Theta_{M,m}$ is just the free category $F(\Theta_{M,m-1} + E_{M,m-1})$ generated by the free prototype functor.

The colimits Θ_W^∞ in \mathbf{Cat} :

$$\begin{array}{ccccccccccc} \Theta_0 & \rightrightarrows & \Theta_{M,0} & \rightrightarrows & \Theta_{M,1} & \rightrightarrows & \Theta_{M,2} & \longrightarrow & \cdots & \longrightarrow & \Theta_{M,m} & \longrightarrow & \Theta_{M,m+1} & \longrightarrow & \cdots \\ & & & & & & & & & & & & & \downarrow & & \\ & & & & & & & & & & & & & \Theta_W^\infty & & \end{array}$$

is called the *cubical coherator* of cubical weak ∞ -categories with connections. Denote by $\mathbf{Mod}(\Theta_W^\infty)$ the category of Θ_W^∞ -models in \mathbf{Sets} . The category $\mathbf{Mod}(\Theta_W^\infty)$ is a category of models of cubical weak ∞ -categories with connections.

Remark 13 Of course we suspect this category to be equivalent to the category of algebras for the cubical operad built in [10], especially because this is true for the globular geometry [5]. But we prefer to avoid such question here. \square

4 The cubical coherator $\Theta_{W_0}^\infty$ of cubical weak ∞ -groupoids with connections

Cubical $(\infty, 0)$ -sets underly a new sketch (see diagrams below) which we use to define a coherator which models are cubical weak ∞ -groupoids. Here we define cubical version of the formalism developed in [8] for globular $(\infty, 0)$ -sets. This formalism of this cubical world is very similar to its globular analogue. Consider a cubical set $\mathcal{C} = (C_n, s_{n-1,j}^n, t_{n-1,j}^n)_{1 \leq j \leq n}$. If $n \geq 1$ and $1 \leq j \leq n$, then a

(n, j) -reversor on it is given by a map $C_n \xrightarrow{j_j^n} C_n$ such that the following two diagrams commute :

$$\begin{array}{ccc} C_n & \xrightarrow{j_j^n} & C_n \\ s_{n-1,j}^n \searrow & & \swarrow t_{n-1,j}^n \\ & C_{n-1} & \end{array} \quad \begin{array}{ccc} C_n & \xrightarrow{j_j^n} & C_m \\ t_{n-1,j}^n \searrow & & \swarrow s_{n-1,j}^n \\ & C_{n-1} & \end{array}$$

If for each $n > 0$ and for each $1 \leq j \leq n$, there are such (n, j) -reversor j_j^n on \mathcal{C} , then we say that \mathcal{C} is a cubical $(\infty, 0)$ -set. The family of maps $(j_j^n)_{n>0, 1 \leq j \leq n}$ for all $(n \in \mathbb{N}^*)$ is called an $(\infty, 0)$ -structure and in that case we shall say that \mathcal{C} is equipped with the $(\infty, 0)$ -structure $(j_j^n)_{n>0, 1 \leq j \leq n}$. When we speak about such $(\infty, 0)$ -structure $(j_j^n)_{n>0, 1 \leq j \leq n}$ on \mathcal{C} , it means that it is for all integers $n \in \mathbb{N}^*$ such that C_n is non-empty. Seen as cubical $(\infty, 0)$ -set we denote it by $\mathcal{C} = ((C_n, s_{n-1,j}^n, t_{n-1,j}^n)_{1 \leq j \leq n}, (j_j^n)_{n>0, 1 \leq j \leq n})$. If $\mathcal{C}' = ((C'_n, s_{n-1,j}^m, t_{n-1,j}^m)_{1 \leq j \leq n}, (j_j^n)_{n>0, 1 \leq j \leq n})$ is another $(\infty, 0)$ -set, then a morphism of $(\infty, 0)$ -sets

$$\mathcal{C} \xrightarrow{f} \mathcal{C}'$$

is given by a morphism of cubical sets such that for each $n > 0$ and for each $1 \leq j \leq n$ we have the following commutative diagrams

$$\begin{array}{ccc} C_n & \xrightarrow{j_j^n} & C_n \\ f_n \downarrow & & \downarrow f_n \\ C'_n & \xrightarrow{j_j^n} & C'_n \end{array}$$

The category of cubical $(\infty, 0)$ -sets is denoted $(\infty, 0)\text{-CSets}$. A cubical reflexive $(\infty, 0)$ -magma is an object of $\infty\text{-CMag}_r$ such that its underlying cubical set is equipped with an $(\infty, 0)$ -structure. Morphisms between cubical reflexive $(\infty, 0)$ -magmas are those of $\infty\text{-CMag}_r$ which are also morphisms of $(\infty, 0)\text{-CSets}$, i.e they preserve the underlying $(\infty, 0)$ -structures. The category of cubical reflexive $(\infty, 0)$ -magmas is denoted $(\infty, 0)\text{-CMag}_r$.

Now the forgetful functor

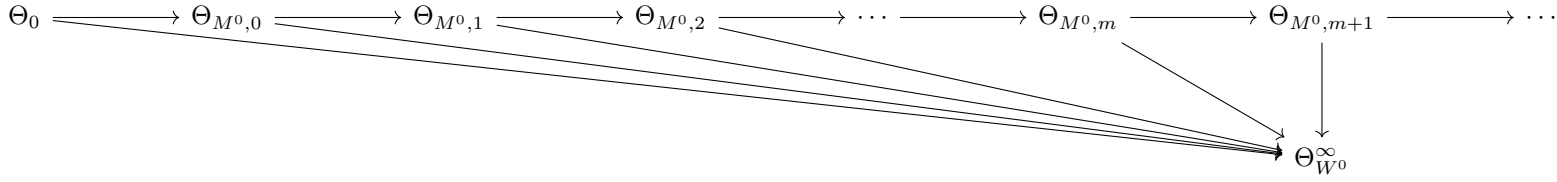
$$(\infty, 0)\text{-CMag}_r \xrightarrow{V} \text{CSets}$$

⁵Private communication with Christian Lair.

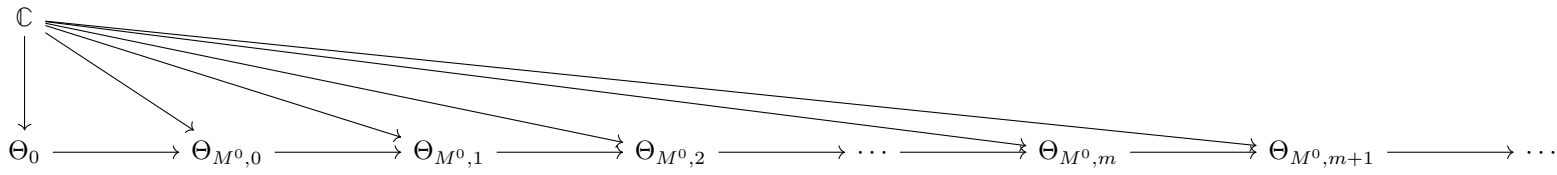
⁶Here "formally" has an accurate logical sense.

is right adjoint and it induces the monad $\mathbb{M}^0 = (M^0, \eta^0, \mu^0)$ of cubical reflexive $(\infty, 0)$ -magmas with its Kleisli category $\mathbb{Kl}(\mathbb{M}^0)$. Denote by $\Theta_{\mathbb{M}^0}$ the full subcategory of $\mathbb{Kl}(\mathbb{M}^0)$ which objects are objects of Θ_0 . This small category $\Theta_{\mathbb{M}^0}$ equipped with the canonical inclusion functor $\Theta_0 \xrightarrow{j^0} \Theta_{\mathbb{M}^0}$ is an important cubical theory because it is the basic data we need to build the coherator $\Theta_{W^0}^\infty$ which models are cubical weak ∞ -groupoids with connections.

This theory $\Theta_{W^0}^\infty$ is a coherator in the sense of Grothendieck ([13]), i.e it is obtained as a colimit of a diagram $\mathcal{D}_{\Theta_{W^0}}$ in $\mathbb{C}at$ of cubical theories :



and this diagram $\mathcal{D}_{\Theta_{W^0}^\infty}$ is a sequence in the category $\mathbb{C}\text{-Th}$ of cubical theories :



that we may define inductively :

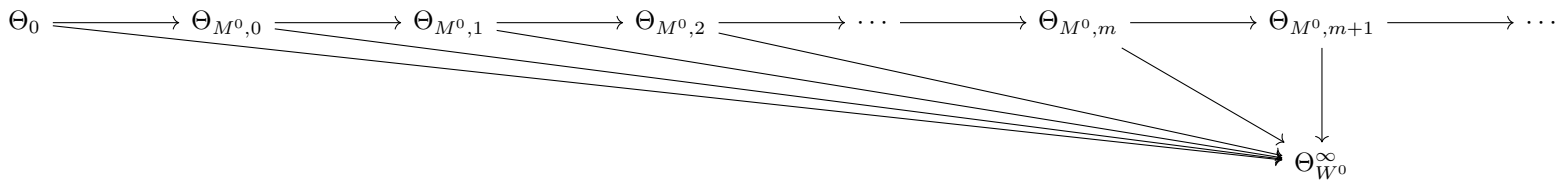
- We start the induction with $\Theta_{M^0,0} = \Theta_{\mathbb{M}^0}$ i.e with the cubical theory of cubical reflexive $(\infty, 0)$ -magmas.
- We denote by $E_{M^0,0}$ the set which is the union of all admissible pairs of I^n -arrows in $\Theta_{\mathbb{M}^0}$ (for all $n \geq 1$), all j -admissible pairs of I^n -arrows in $\Theta_{\mathbb{M}^0}$ (for all directions $j \in \llbracket 1, n \rrbracket$ for all $n \geq 1$);
- $\Theta_{M^0,1}$ is obtained by formally adding in $\Theta_{M^0,0}$ all kind of liftings of elements of $E_{M^0,0}$.
- Denote by $E_{M^0,1}$ the set which is the union of : all admissible pairs of I^n -arrows in $\Theta_{M^0,1}$ which are not in $E_{M^0,0}$, all j -admissible pairs of I^n -arrows in $\Theta_{M^0,1}$ which are not in $E_{M^0,0}$.
- $\Theta_{M^0,2}$ is obtained by formally adding in $\Theta_{M^0,1}$ all kind of liftings of elements of $E_{M^0,1}$.
- we suppose that until the integer $m - 1$ the sequence :

$$(\Theta_{M^0,0}, E_{M^0,0}) \longrightarrow (\Theta_{M^0,1}, E_{M^0,1}) \longrightarrow \dots \longrightarrow (\Theta_{M^0,m-2}, E_{M^0,m-2}) \longrightarrow (\Theta_{M^0,m-1}, E_{M^0,m-1})$$

is well defined. Thus $\Theta_{M^0,m}$ is obtained by formally adding in $\Theta_{M^0,m-1}$ all kind of liftings of elements of $E_{M^0,m-1}$.

- we associate to $\Theta_{M^0,m}$ the set $E_{M^0,m}$ which is the union of : all admissible pairs of I^n -arrows in $\Theta_{M^0,m}$ which are not in $E_{M^0,m-1}$, all j -admissible pairs of I^n -arrows in $\Theta_{M^0,m}$ which are not in $E_{M^0,m-1}$.

The colimits $\Theta_{W^0}^\infty$ in $\mathbb{C}at$:



is called the *cubical coherator* of cubical weak ∞ -groupoids with connections. Denote by $\text{Mod}(\Theta_{W^0}^\infty)$ the category of $\Theta_{W^0}^\infty$ -models in Sets . The category $\text{Mod}(\Theta_{W^0}^\infty)$ is a category of models of cubical weak ∞ -groupoids with connections.

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Laboratoire de Mathématiques d’Orsay, UMR 8628
Université de Paris-Saclay and CNRS
Bâtiment 307, Faculté des Sciences d’Orsay
94015 ORSAY Cedex, FRANCE
camell.kachour@universite-paris-saclay.fr