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Grothendieck and differential equations.

Yves André

1. Introduction

There is an important, original and varied part of Grothendieck's work devoted to differential equations, a part which had a long posterity and which we would like to delineate.

Scattered in several texts and less documented, it is not so well-known as other parts of his work. To begin with, the expression "differential equation" does not even seem to appear under his pen...

This paradoxical fact deserves a moment's thought. Dealing with mysteries and differential equations, a starting point soon comes to mind, namely:

6accdae13eff7i3l9n4o4qrr4s8t12ux.

This is Newton's answer (1677[1]) to Leibniz' query about his methods of invention: begging the question or rather hiding the answer in a mysterious sentence containing six times the letter a, twice the letter c, one d, one diphthong ae, etc. Scholars deciphered the anagram as follows:

Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa

- note that there are 9 t's, not 8: Newton's or the decipherers' mistake?

Anyway, this is often grossly translated from latin to english as

it is useful to solve differential equations.

The immense development of the theory of differential equations and its applications after Newton gives fully right to this saying, namely to the importance of solving differential equations or studying their solutions.

Nevertheless, some three centuries later, Grothendieck took an opposite view, which could be phrased as follows:

it is also useful not *to solve differential equations, but to study their structure.*

Solutions versus structure: this fundamental tension is reminiscent of its counterpart in algebraic geometry, and of the way Grothendieck overcame it, thanks to the functorial viewpoint: generalizing the notion of point (a solution of algebraic equations) and viewing a variety as the object which represents the functor of its (generalized) points.

But Grothendieck never developed a similar viewpoint for differential equations instead of algebraic equations[2].

[1] "The foundations of these operations is evident enough, in fact; but because I cannot proceed with the explanation of it now, I have preferred to conceal it thus: 6accdae13eff7i3l9n4o4qrr4s8t12ux. On this foundation I have also tried to simplify the theories which concern the squaring of curves, and I have arrived at certain general Theorems".

[2] A differential-algebraic geometry has since been developed in various guises, but it is not based on the functorial viewpoint.

2. Differential operators.

In the context of a linear differential equation

$$L\,y = 0,$$

the two viewpoints in tension are summarized by the emphasis on the study of the differential operator L, or of the solutions y.

It is more or less equivalent to consider a linear differential system

$$\partial \vec{y} = A\vec{y},$$

where A is a matrix, and this leads to the more intrinsic notion of *differential modules* or *integrable connections*: namely, a map $T_S \overset{\nabla}{\to} End_k M$ satisfying Leibniz's rule and commuting with the Lie bracket. Such an action of T_S extends to its "enveloping algebra" $\mathcal{D}_S$, which endows M with a structure of $\mathcal{D}_S$-module.

Grothendieck's contribution [13] stems from his recasting differential calculus in terms of infinitesimal neighborhoods of the diagonal in S^2. By duality, he defines a ring of differential operators $Diff_S = \cup Diff_S^{\leq n}$, and gives a very simple inductive characterization: $L \in Diff_S^n$ if and only if $L \in End_k \mathcal{O}_S$ and for any section f of $\mathcal{O}_S$, $[L, f] \in Diff_S^{n-1}$. In particular, he recovers in this way the tangent bundle and Ehresmann's jets[3], *cf.* [2, II].

One has a map $\mathcal{D}_S \to Diff_S$ which is an isomorphism in characteristic 0, but which is neither injective nor surjective in characteristic $p > 0$.

In char. p and in one variable, $Diff_S$ corresponds to Schmidt's iterated divided derivations - whence two languages, and a further one: infinitely Frobenius-divisible modules. The lively (albeit a little exotic) theory of $Diff_S$-modules (Gieseker, Esnault...) makes full use of triple viewpoint.

Grothendieck and his school (Berthelot...) showed that the "good version" of $\mathcal{D}_S$ in char. p, which restores the link with integrable connections, consists in endowing the ideals of infinitesimal neighborhoods of the diagonal with *divided powers* [3].

In the meantime, inspired by the theory of formal groups and de Rham cohomology (see below), he initiated his *crystalline theory* [15], first announced in his famous letter to Tate (1966):

> un cristal possède deux propriétés caractéristiques: la rigidité, et la faculté de croître dans un voisinage approprié.

In a first version, those appropriate neighborhoods were locally nilpotent closed immersions of open subsets of the base S. In this setting, a crystal in modules amounts to a quasi-coherent module M over S together with an isomorphism $\chi : p_1^* M \cong p_2^* M$ (where p_1, p_2 are the projections of the formal completion of the diagonal of S^2 to S), satisfying a cocyle condition; and this corresponds exactly to a $Diff_S$-module.

Let us emphasize that the novelty does not lie in the construction (indeed, the isomorphism χ is nothing but the classical resolvant of the corresponding differential system), but in the interpretation: here, χ is seen as a descent data in the infinitesimal site.

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[3] this viewpoint as well as Grothendieck's infinitesimal topos were rediscovered later, in the guise of the so-called "synthetic geometry".

In a later version, Grothendieck introduced divided powers in the definition of crystals, initiating a theory which was largely developed by Berthelot and others [3]. Beyond technical aspects, let us note here the novelty which consists in seeing certain p-adic differential systems as *attached to a variety in char. p rather than to a p-adic variety*, which allows a *functorial explanation* of the action of Frobenius on such systems (discovered "experimentally" by Dwork).

Coming back to char. 0, let us recall that the theory of D-modules was mainly developed later under the name "algebraic analysis" in the wake of other traditions: (Sato and) Kashiwara [18], or Bernstein, to quote the founders. It relies on key concepts like the characteristic variety and holonomy, and on a powerful homological viewpoint which integrates the solution viewpoint and goes beyond, putting solutions and cosolutions[4] on equal footing.

One could say that in char. 0, integrable connections and D-modules are *equivalent in theory but not in practice* - the two viewpoints being rather different: the theory of connections restricts itself to the case of $\mathcal{O}_S$-coherent modules, which amounts to rejecting singularities to infinity, whereas the theory of D-modules does not and includes Dirac distributions for instance in its setting.

3. Singularities.

The saying "you never understand a domain so well as through its crises" is especially relevant in the context of differential equations. The study of the behaviour of the solutions of a linear differential equation $Ly = 0$ with meromorphic coefficients, in the neighborhood of a singularity[5] is a rich and subtle theme which takes its roots in the classical works of Fuchs and Poincaré. This will lead us through a long historical detour, at the end of which Grothendieck's viewpoint will reappear.

One has two different ways for zooming at the singularity:

1) one can localize L at a disk D where there is only one singularity (its center), and then pass to the formal completion (i.e. consider the coefficients of L as formal power series).

The problem becomes purely algebraic; it turns out that after finite ramification, L factors as a product of differential operators of order 1 (Levelt).

2) One can restrict L to the punctured disk D^* (i.e. consider the coefficients of L as analytic functions in D^*).

The problem becomes purely topological, and is controlled by the local monodromy.

The favorable situation is the case of a *regular* singularity (Fuchs 1866, Frobenius 1873, *cf.* [12]). In this case, the algebraic theory 1) and the topological theory 2) match: the formal decomposition of L is meromorphic. The condition of regularity is expressed by a simple inequality involving the valuations of the coefficients of L.

But in the case of an *irregular* singularity, some divergent series appear in the algebraic factorization, and the bridge between the algebraic theory and the topological theory becomes much more indirect, going through the theory of asymptotic expansions by zooming to the singularity along sectors, *cf.* [10][21].

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[4] *i.e.* elements of the cokernel of the differential operator.

[5] *i.e.* a pole of the coefficients.

This short account does not follow, admittedly, the incredibly tortuous historical path: in fact, the key point in the irregular case - the so-called *Stokes phenomenon* (1847-58) - predates by two decades the definition of a regular singularity (1866-73).

The theory of differential equations with regular singularities has enjoyed great fortune, due to the fact that the local correspondence with monodromy extends to the global situation. This is the so-called *Riemann-Hilbert correspondence*. "Equations différentielles à points singuliers réguliers" is the title of a famous memoir where Deligne establishes this correspondence between regular integrable connections and local systems on a smooth complex algebraic variety.

The theory got a second wind thanks to the homological viewpoint of the theory of D-modules, where the Riemann-Hilbert correspondence finds its natural extension in the derived setting. Very recently, the theory has been extended to the irregular case, *cf.* [17].

Remark. One finds several different definitions of regularity in the literature, and their equivalence cannot be taken for granted, especially in several variables. Deligne [9] shows an equivalence between an analytic definition (moderate growth of solutions in the neighborhood of a singularity) and various algebraic definitions, notably one by restriction to curves. However, the argument is incorrect - it overlooks the phenomenon of confluence - and Deligne replaced it by a transcendental argument. The question of providing an algebraic proof of the equivalence of the algebraic definitions remained open for some time, until I solved it using the theory of irregularity [2, VI].

For Kashiwara, regularity corresponds to reduceness of the characteristic variety; the fact that this matches with Deligne's conditions is not at all obvious and was proved only recently [6].

Let us now turn to the *p-adic case*: it is in this situation that Grothendieck's crystalline viewpoint played a decisive role, in the way of conceiving what a singularity really is.

The solutions y of a p-adic differential equation $Ly = 0$ usually do not converge up to the next singularity (for instnce $y = e^t$, solution of $\frac{dy}{dt} - y = 0$, has radius of convergence $p^{-1/(p-1)}$). Dwork, the founder of the theory, stressed the importance of radii of convergence of solutions, and introduced the notion of overconvergence and of Frobenius structure, which allowed him to select a good category of differential equations (which turned out to correspond to overconvergent F-isocrystals in Berthelot's setting).

Following Grothendieck's viewpoint, one should consider overconvergent p-adic differential equations as "living" in char. p, which leads to consider singularities no longer as points, but as open discs. Overconvergence allows to enter a little into those singular discs. The analog of studying a complex-analytic L by restriction to the punctured disc (with center the singularity) becomes, in the p-adic case, restricting to *a thin annulus at the inner boundary of the singular disc.*

This viewpoint, advocated by Crew, progressively led to an essentially complete understanding of p-adic differential equations in the framework of Berkovich geometry (Christol-Mebkhout, Baldassarri, Kedlaya, Pulita-Poineau... *cf.* [20][8]).

4. Gauss-Manin connection.

This is Grothendieck's central contribution to differential equations.

In his letter to Atiyah (1963, augmented and published as [14]), Grothendieck defines *algebraic De Rham cohomology* of a smooth algebraic variety in char. 0 (hypercohomology of the algebraic De Rham complex), and shows that this provides a reasonable theory: it gives the usual Betti numbers.

This short but incredibly rich letter goes beyond. For instance, Grothendieck observes that the comparison isomorphism between the transcendental Betti cohomology and his algebraic De Rham cohomology gives rises to some classical constants (the periods), which were studied in transcendental number theory; in a cryptic footnote, he alludes to a general transcendence conjecture about them without stating it (the now famous *Grothendieck period conjecture*).

In the relative case, i.e. when the periods depends on a parameter, it had been known since the XIX century that they are solutions of a linear differential equation with algebraic coefficients: the *Picard-Fuchs equation*, *cf.* [12].

The prototype is Gauss hypergeometric equation (1812)

$$t(1-t)\frac{d^2y}{dt^2} + (c-(a+b+1)t)\frac{dy}{dt} - ab\,y = 0$$

satisfied by $\int_1^\infty z^{a-c}(1-z)^{c-b-1}(1-tz)^{-a}dz$.

The problem posed by Grothendieck is to give an *algebraic construction* of the connection of which the periods are solutions. In the case of a family of curves (Fuchs 1871), Manin had just provided such an algebraic connection.

The problem to construct an algebraic connection on algebraic relative de Rham cohomology amounts to *understanding algebraically "integration in the fiber"*. This problem seems so fundamental in Grothendieck's view that he does not hesitate to change the established terminology and replace "Picard-Fuchs" by "Gauss-Manin" in order to underline the change of viewpoint, bracketing a century work on Picard-Fuchs equations without having yet any result to propose!

Soon after, however, he had such a construction. Katz and Oda proposed a more general one, with coefficients in an arbitrary integrable connection, *cf.* [2, VII]. This was the beginning of *the yoga of De Rham coefficients*, in Grothendieck's terms, which reached maturity with the formalism of the six operations in Kashiwara's theory of holonomic D-modules.

In the p-adic case, Berthelot came to a synthesis between the crystalline and the D-module viewpoints: the theory of $D^\dagger$-modules [4]. However, the yoga of p-adic De Rham coefficients started to work only after combination with the transcendental study of p-adic singularities (Caro).

Remark. By the way, is Grothendieck-Katz-Oda's Gauss-Manin connection the same thing as the direct image in the sense of D-module theory?

During the last quarter of the XXc., the question has been relegated in the folklore, the answer often taken for granted and not worth being written down... until Dimca, Maaref, Sabbah, Saito (2000) took the pain to write down two proofs [11]; the simplified proof given in [6] is still far from being simple.

A further remark. Let us come back to the starting point: Gauss hypergeometric equation with parameter $a = 1/2$, $b = 1/2$, $c = 1$, which controls the variation of periods of the Legendre family of elliptic curves. In many courses and articles, one can read the monodromy matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ at 0, $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ at 1, often followed by the "well-known"

fact that they generate the subgroup $\Gamma(2)$ of $GL_2(\mathbf{Z})$ consisting of matrices congruent to id mod.2: "well-known", but wrong![6]

5. Galois aspects.

Algebraic functions satisfy linear differential equations. The idea to generalize generalize Galois theory from algebraic equations to differential equations (maybe foreseen by Galois, and initiated by Liouville) took shape when Picard (1883) defined the differential Galois group attached to a linear differential equation $Ly = 0$: form the extension of the field of coefficients generated by the solutions y and their derivatives; the differential Galois group is the automorphism group of this extension of differential fields. This is an algebraic group which acts linearly on the spaces of solutions.

Galois theory became increasingly efficient with the development of finite group theory (in practice, the Galois correspondence is applied in one direction); similarly, differential Galois theory became useful with the development of linear algebraic group theory, and was one of its main early motivations (Kolchin), *cf.* [21].

Apparently, Grothendieck has never shown any interest for this theory. This is strange if one considers Grothendieck's creation of the theory of the etale fundamental group, and even more if one considers his theory of tannakian categories, which are linear analogues of discrete Galois categories - in fact, Deligne developed a tannakian interpretation of differential Galois theory.

One reason could be that (until recently) differential Galois theory was a theory of differential *fields* only: it turns out that in order to get a Galois theory of differential algebras and schemes, one needs more than the classical tools (groups and torsors), namely the theory of quasi-homogeneous varieties [1].

Another reason could be that Grothendieck did not know that on replacing monodromy groups by monodromy groupoids, one gets rid of the regularity condition: the Zariski closure of the monodromy groupoid is always the differential Galois groupoid (Cartier, Malgrange *cf.* [7, §5])[7].

There are nonetheless two contact points between Grothendieck's theory of the etale fundamental group and differential Galois theory.

1) the situation where the two theories are trivial: if S is proper smooth and if $\pi_1^{et}(S) = 0$, then there is no non-trivial D-module on S (Grothendieck).

2) Grothendieck's *p-curvature conjecture* (a differential analogue of the fact that Galois groups of number fields are generated by Frobenii). This stems from Fuchs's influential question (1875): characterize those differential equations which have a full set of algebraic solutions[8].

Grothendieck's conjecture predicts that this happens when this happens modulo p for almost all p (which translates into the vanishing of p-curvatures). Its analog is known in equal char. 0 (Hrushovsky, myself) and in equal char. p (Esnault-Mehta), but still open in mixed characteristic, *i.e.* in the original setting.

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[6] $\Gamma(2)$ is the product of the monodromy group by $\{\pm 1\}$.

[7] for instance for $y = e^t$, the monodromy group is trivial while the differential Galois group is $\mathbb{C}^*$; but the monodromy groupoid is given by the graph of $(t_1, t_2) \mapsto e^{t_2 - t_1}$ which is Zariski-dense in the differential Galois groupoid $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^*$.

[8] this question lies at the source of the uniformization theorem of algebraic curves (Poincaré-Koebe), *cf.* [12].

6. Conclusion

In this text, I put forward viewpoints rather than results, since many of Grothendieck's contributions to the theme "differential equations" lie in his novel viewpoints: differential operators, infinitesimal and crystalline topoi, algebraic de Rham cohomology and the Gauss-Manin connection...

Even restricted to the domain of algebraic analysis considered in this article, there is a remarkable plurality - sometimes a plethora - of viewpoints on differential equations. The subject often gets advantage of the crossed perspectives; but occasionally, they ignore each other for lack of dictionaries between different languages. Such dictionaries are as rare as valuable: the variations which they allow help keeping fertile a viewpoint

- those fertile viewpoints about which Grothendieck once wrote [16]:

> Le point de vue fécond est celui qui nous révèle, comme autant de parties vivantes d'un même Tout qui les englobe et leur donne un sens, ces questions brûlantes que nul ne sentait, et (comme en réponse peut-être à ces questions) ces notions tellement naturelles que personne n'avait songé à dégager, et ces énoncés enfin qui semblent couler de source, et que personne ne risquait de poser, aussi longtemps que les questions qui les ont suscités, et les notions qui permettent de les formuler, n'étaient pas apparues encore.
>
> Plus encore que ce qu'on appelle les théorèmes-clef en mathématique, ce sont les points de vue féconds qui sont, dans notre art, les plus puissants outils de découverte - ou plutôt, ce ne sont pas des outils, mais ce sont les yeux même du chercheur qui, passionnément, veut connaître la nature des choses mathématiques.

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