



**HAL**  
open science

# Algebraic norm and capitulation of p-class groups in ramified cyclic p-extensions

Georges Gras

► **To cite this version:**

Georges Gras. Algebraic norm and capitulation of p-class groups in ramified cyclic p-extensions. 2023. hal-03865383v3

**HAL Id: hal-03865383**

**<https://hal.science/hal-03865383v3>**

Preprint submitted on 5 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ALGEBRAIC NORMS AND CAPITULATION OF $p$ -CLASS GROUPS IN RAMIFIED CYCLIC $p$ -EXTENSIONS

GEORGES GRAS

ABSTRACT. We examine the phenomenon of capitulation of the  $p$ -class group  $\mathcal{H}_K$  of a real number field  $K$  in totally ramified cyclic  $p$ -extensions  $L/K$  of degree  $p^N$ . Using a property of the algebraic norm  $\nu_{L/K}$ , we show that the kernel of capitulation is in relation with the “complexity” of  $\mathcal{H}_L$  measured via its exponent  $p^{e(L)}$  and the length  $m(L)$  of its Galois filtration  $\{\mathcal{H}_L^i\}_{i \geq 0}$ . We prove that a *sufficient condition* of capitulation is given by  $e(L) \in [1, N - s(L)]$  if  $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$  for  $s(L) \in [0, N - 1]$  (Theorem 1.1); this improves the case of “stability”  $\#\mathcal{H}_L = \#\mathcal{H}_K$  (i.e.,  $m(L) = 1$ ,  $s(L) = 0$ ,  $e(L) = e(K)$ ) (Theorem 1.2). Numerical examples (with PARI programs), showing most often capitulation of  $\mathcal{H}_K$  in  $L$ , are given, taking the simplest ones,  $L \subset K(\mu_\ell)$ , with primes  $\ell \equiv 1 \pmod{2p^N}$ , over cubic fields with  $p = 2$  and real quadratic fields with  $p = 3$ . Some conjectures on the existence of non-zero densities of such  $\ell$ 's are proposed (Conjectures 1.4, 2.4). Capitulation property of other arithmetic invariants is examined.

## CONTENTS

1. Introduction	2
1.1. Statement of the main results	2
1.2. The different aspects of capitulation and aims of the paper	3
2. Complexity of $\mathcal{H}_L$ versus capitulation of $\mathcal{H}_K$	5
2.1. Filtration of $\mathcal{H}_L$ in the totally ramified case	5
2.2. Classes, units and capitulation kernel	8
2.3. Decomposition of the algebraic norm $\nu_{L/K} \in \mathbb{Z}[G]$	12
2.4. Illustrations using the decomposition of $\nu_{L/K}$	15
3. Arithmetic invariants that do not capitulate	18
3.1. Injective transfers and arithmetic consequences	18
3.2. $p$ -class groups of imaginary quadratic field	19
3.3. Torsion groups of abelian $p$ -ramified $p$ -extensions	20
3.4. Remarks on $\text{rk}_p(\mathcal{X}_n)$ -ranks	22
4. Capitulation in $\mathbb{Z}_p$ -extensions	23
4.1. Known results under the assumption $\mu = 0$	23
4.2. Case of the cyclotomic $\mathbb{Z}_p$ -extension of $K$	24
5. Isotopic components and capitulation	26
6. Tables for cubic fields and $p = 2$	30
6.1. Case $\ell = 17$	30
6.2. Case $\ell = 17$ and $f \in \{9283, 7687, 44857\}$	34
6.3. Case $\ell = 97$	35
6.4. Capitulations for $K$ fixed and $\ell$ varying	37
7. Tables for Kummer fields and $p = 2$	39
7.1. Pure cubic fields, $\ell = 17$	40
7.2. Case of $K = \mathbb{Q}(\sqrt[3]{174})$ , $\ell \equiv 1 \pmod{16}$	42
7.3. Pure quintic fields, $L \subset K(\mu_{17})$	43

*Date:* February 5, 2023.

*2020 Mathematics Subject Classification.* 11R29, 11R18, 11R37, 12Y05.

*Key words and phrases.* Capitulation of  $p$ -class groups, cyclotomic extensions of prime conductor, class field theory, Chevalley–Herbrand formula, PARI programs.

8. Tables for real quadratic fields and $p = 3$	43
8.1. Examples with $\mathcal{H}_K \simeq \mathbb{Z}/3\mathbb{Z}$	44
8.2. Examples with $\mathcal{H}_K \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	45
9. Tables for the logarithmic class group	45
9.1. Examples with real quadratic fields and $p = 3$	46
9.2. Examples with cyclic cubic fields and $p = 2$	47
9.3. Capitulation of $\mathcal{H}_K^{\text{lg}}$ in $K^{\text{cy}}$ for $K$ quadratic	48
9.4. Conclusion about $\mathcal{H}_K^{\text{lg}}$	49
10. Conclusions and prospects	49
References	51

## 1. INTRODUCTION

**1.1. Statement of the main results.** Let  $L/K$  be any cyclic  $p$ -extension of number fields, of degree  $p^N$ ,  $N \geq 1$ , of Galois group  $G =: \langle \sigma \rangle$ , and let  $\mathcal{H}_K, \mathcal{H}_L$  be the  $p$ -class groups of  $K, L$ , respectively. For a finite group  $A$ , let  $\text{rk}_p(A) := \dim_{\mathbb{F}_p}(A/A^p)$  be the  $p$ -rank of  $A$ .

We assume, in all the sequel, that  $L/K$  is *totally ramified*.

Let  $\mathbf{J}_{L/K} : \mathcal{H}_K \rightarrow \mathcal{H}_L^G$  be the transfer map (or extension of classes) and let  $\mathbf{N}_{L/K} : \mathcal{H}_L \rightarrow \mathcal{H}_K$  be the arithmetic norm induced by  $\mathbf{N}_{L/K}(\mathfrak{P}) := \mathfrak{p}^f$ , for all prime ideal  $\mathfrak{P}$  of  $L$  and the prime ideal  $\mathfrak{p}$  of  $K$  under  $\mathfrak{P}$  with residue degree  $f$ .

We know that  $\mathbf{N}_{L/K}(\mathcal{H}_L)$  is the subgroup of  $\mathcal{H}_K$  which corresponds, by the Artin map of class field theory, to  $\text{Gal}(H_K^{\text{nr}}/L \cap H_K^{\text{nr}})$  where  $H_K^{\text{nr}}$  is the  $p$ -Hilbert class field of  $K$ ; thus, by assumption of total ramification,  $\mathbf{N}_{L/K}(\mathcal{H}_L) = \mathcal{H}_K$ .

Let  $\nu_{L/K} := \sum_{i=0}^{p^N-1} \sigma^i$ , be the algebraic norm in  $L/K$ . It is immediate to see that  $\nu_{L/K} = \mathbf{J}_{L/K} \circ \mathbf{N}_{L/K}$ , whence  $\nu_{L/K}(\mathcal{H}_L) = \mathbf{J}_{L/K}(\mathcal{H}_K)$  giving the key principle that  $\mathcal{H}_K$  capitulates in  $L$  if and only if  $\nu_{L/K}(\mathcal{H}_L) = 1$ .

We can state, from using this criterion:

**Theorem 1.1.** *Let  $L/K$  be any totally ramified cyclic  $p$ -extension of number fields, of degree  $p^N$ ,  $N \geq 1$ , of Galois group  $G =: \langle \sigma \rangle$ . Let  $p^{e(L)}$  be the exponent of  $\mathcal{H}_L$  and let  $m(L)$  be the minimal integer such that  $(\sigma - 1)^{m(L)}$  annihilates  $\mathcal{H}_L$ .*

(i) *A sufficient condition for the capitulation of  $\mathcal{H}_K$  in  $L$ , is  $e(L) \in [1, N - s(L)]$  if  $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$  for  $s(L) \in [0, N - 1]$ .*

(ii) *A class  $h \neq 1$  of  $\mathcal{H}_K$  capitulates in  $L$  if, in the writing  $h =: \mathbf{N}_{L/K}(h')$ ,  $h'$  is of order  $p^e$  and annihilated by  $(\sigma - 1)^m$  such that  $e \in [1, N - s]$  if  $m \in [p^s, p^{s+1} - 1]$  for  $s \in [0, N - 1]$ .*

**Theorem 1.2.** *Let  $L/K$  be any totally ramified cyclic  $p$ -extension of number fields, of degree  $p^N$ ,  $N \geq 1$ , of Galois group  $G =: \langle \sigma \rangle$ :*

(i) *Let  $K_n$ ,  $n \in [0, N]$ , be the subfield of  $L$  of degree  $p^n$  over  $K$  and set  $G_n := \text{Gal}(K_n/K)$ . Then  $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$  implies the following properties:*

- $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n} \simeq^{\mathbf{N}_{K_n/K}} \mathcal{H}_K$ , for all  $n \in [0, N]$ .

- For any  $e \leq N$ , the subgroup  $\mathcal{H}_K[p^e]$  of  $\mathcal{H}_K$ , of classes annihilated by  $p^e$ , capitulates in  $K_e$ ; whence  $\mathcal{H}_K$  capitulates in  $K_{e(K)}$  if  $e(K) \leq N$ .

(ii) *If  $\text{rk}_p(\mathcal{H}_{K_1}) = \text{rk}_p(\mathcal{H}_K)$ , then  $\text{rk}_p(\mathcal{H}_{K_n}) = \text{rk}_p(\mathcal{H}_K)$ , for all  $n \in [0, N]$ .*

**Remark 1.3.** *Properties (i), (ii) of stability in Theorem 1.2 may occur only from some layer  $K_{n_0}$ , giving  $\#\mathcal{H}_{K_{n_0+1}} = \#\mathcal{H}_{K_{n_0}}$ . Indeed, we will see that the minimal level  $n_0$  (if any) may depend on the structure of  $\mathcal{H}_K$  when the exponent and the  $p$ -rank of  $\mathcal{H}_K$  are large. To get the statement of stability in the first layer, it suffices to replace the base field  $K$  by  $K' = K_{n_0}$ ,  $L/K$  by  $L/K'$  and  $\mathcal{H}_K$  by  $\mathcal{H}_{K'}$ ; then, when*

$N' := N - n_0$  is large enough, the capitulation of  $\mathcal{H}_{K'}$  implies, a fortiori, that of  $\mathcal{H}_K$ . See a detailed example of this phenomenon in §6.4.2.

The proof of Theorem 1.1 will be given by Corollary 2.12 to Theorem 2.9. Then Theorem 1.2 comes from our previous work [46, Theorem 3.1 & Section 6, §(b)] generalizing [27, 68, 8, 72, 83]; it corresponds, in the statement of Theorem 1.1, to the case  $m(K_n) = 1$  (from  $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n}$ , whence  $s(K_n) = 0$  and the condition  $e(K) \leq N$  (since  $e(L) = e(K)$  from the isomorphisms  $\mathcal{H}_{K_n} \xrightarrow{N_{K_n/K}} \mathcal{H}_K$  given by the arithmetic norms); this case is called the  $p$ -class groups stability in the tower  $L = \bigcup_{n=0}^N K_n$ .

In practice, the knowledge of  $m(L)$  determines the unique  $s(L) \geq 0$  such that  $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$  and one must check if  $s(L) \in [0, N - 1]$ ; once this holds, one must have  $e(L) \leq N - s(L)$  to get the capitulation. See the definition of the filtration in §2.2 and its computation with PARI [88] in §2.1.3.<sup>1</sup>

The claim on the  $p$ -ranks for the case of  $\mathbb{Z}_p$ -extensions was given by Fukuda [27], then found again by Bandini [8]. It holds for any cyclic totally ramified  $p$ -extension as we have explained in [46, §6(b)].

Theorems 1.1, 1.2 express that, if the “complexity” of  $\mathcal{H}_L$  is not too important, then, in a not very intuitive way,  $\mathcal{H}_K$  capitulates in  $L$ . Conversely, if one knows that capitulation is impossible (e.g., minus  $p$ -class groups in extensions of CM-fields or some invariants attached to abelian  $p$ -ramification theory), then the complexity of these invariants strictly increases with  $n$  (see Theorem 3.2).

**1.2. The different aspects of capitulation and aims of the paper.** The general problem of capitulation<sup>2</sup> of  $\mathcal{H}_K$  in  $L$  has been studied in a very large number of publications, first in the purpose of the factorization problem for Dedekind rings as exposed in Martin [77]. It is impossible to give a complete bibliography, but one may cite, among many other contributions, subsequent to the pioneering works of Hilbert–Scholz–Tausky (see Kisilevsky <https://doi.org/10.2140/PJM.1997.181.219> for prehistory): [64, 98, 15, 52, 91, 30, 55, 56, 54, 94, 74, 35, 75, 69, 99, 50, 66, 51, 62, 29, 100, 16, 12, 79, 4, 6, 14, 5, 60, 61], in which the reader will find more history and references.

Except some Iwasawa’s theory results on capitulation [53, 54], most of these papers are related to the Artin–Furtwängler theorem and its generalizations on capitulation in the Hilbert class field  $H_K^{\text{pr}}$  (or to the Hilbert Theorem 94 in cyclic sub-extensions of  $H_K^{\text{pr}}/K$  as given in Miyake [82], Suzuki [94]), which is not our purpose since, on the contrary, we will study *totally ramified cyclic  $p$ -extensions*  $L/K$  and more precisely the simplest tamely ramified  $p$ -extensions  $L \subset K(\mu_\ell)$ ,  $\ell \equiv 1 \pmod{2p^N}$  prime,  $[L : K] = p^N$ , which, surprisingly, are often capitulation fields of  $\mathcal{H}_K$  when  $K$  is totally real or for non totally imaginary base fields.

Many classical articles give cohomological expressions of the capitulation in terms of global units as the fact that, in the non-ramified case,  $\text{Ker}(\mathbf{J}_{L/K})$  is isomorphic to a subgroup of  $H^1(G, \mathbf{E}_L)$ , where  $\mathbf{E}_L$  is the group of units of  $L$  and  $G = \text{Gal}(L/K)$  (see, e.g., Jaulent [55, Chap. III, §1], [56], then Bembom [12] for more comments and references).

Similarly, using sets of places  $S$ ,  $T$  and tamely ramified Galois extensions, Maire [74, Théorème 4.1] describes injective maps  $\mathcal{H}_{L,(\ell)}/\mathbf{J}_{L/K}(\mathcal{H}_{K,(\ell)}) \hookrightarrow H^2(G, \mathbf{E}_{L,(\ell)})$ ,

<sup>1</sup>PARI programs may be copy and past, from any pdf file. We only give excerpts of the very numerous numerical results which can be found again from running these programs. They excerpts give an overview suggesting the high frequency of capitulations in the simplest cyclic  $p$ -extensions  $L/K$ ,  $L \subset K(\mu_\ell)$ ,  $\ell \equiv 1 \pmod{2p^N}$ . Detailed examples may be found in Subsections 2.2, 2.4.

<sup>2</sup>I recently learned (from a Lemmermeyer text) that the word *capitulation* was coined by Arnold Scholz. It is possible that this term may be received as incongruous; a solution is to consider that a non principal ideal  $\mathfrak{a}$  is a troublemaker with respect to elementary arithmetic, in which case, the terminology is perfectly understandable.

in the context  $L \subset K(\mu_\ell)$ , where  $\mathcal{H}_{K,(\ell)}$ ,  $\mathcal{H}_{L,(\ell)}$  are the ray-class groups modulo  $(\ell)$  and  $\mathbf{E}_{L,(\ell)}$  is the group of units congruent to 1 modulo  $(\ell)$ .

But the aspect “global units” is more difficult since the behavior of the unit groups in  $L/K$  is less known compared to that of  $p$ -class groups, even if there are some links; indeed, we have the following classical “exact sequence of capitulation” obtained from the map which associates, with the invariant class of the ideal  $\mathfrak{a}$ , a unit  $\varepsilon := \mathbf{N}_{L/K}(\alpha)$  from the relation  $\mathfrak{a}^{\sigma-1} =: (\alpha)$ ,  $\alpha \in L^\times$ :

$$(1.1) \quad 1 \rightarrow \mathbf{J}_{L/K}(\mathcal{H}_K) \cdot \mathcal{H}_L^{\text{ram}} \rightarrow \mathcal{H}_L^G \rightarrow \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times) / \mathbf{N}_{L/K}(\mathbf{E}_L) \rightarrow 1,$$

where  $\mathcal{H}_L^{\text{ram}}$  is generated by the classes of the ramified prime ideals of  $L$ ; in the right term, if  $\mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times)$  depends on easier local norm considerations,  $\mathbf{N}_{L/K}(\mathbf{E}_L)$  is in general unknown.

On the contrary,  $\mathbf{J}_{L/K}(\mathcal{H}_K)$ ,  $\mathcal{H}_L^{\text{ram}}$ , are subgroups of  $\mathcal{H}_L^G$  and the order of this group is known from the Chevalley–Herbrand formula [19, pp. 4002–405]. For generalizations of this formula, see Gras [34] for isotopic components of  $\mathcal{H}_L^G$  in the abelian semi-simple case, Jaulent [55, Chapitre III, p.167] with ramification and decomposition, Lemmermeyer [70] in the spirit of the previous work of Jaulent and some papers of González-Avilés [29]; then our general higher fixed points formulas [32, 37] or its idelic translation [71] by Li–Yu, allow the algorithmic computation of  $\mathcal{H}_L$  from its filtration (see Subsection 2.1.1).

In [46], giving extensive numerical PARI computations, we have proposed a conjecture, whose main consequence should be an obvious and immediate proof of the real abelian Main Conjecture “ $\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi})$ ” (where  $\mathcal{E} := \mathbf{E} \otimes \mathbb{Z}_p$  and  $\mathcal{F} := \mathbf{F} \otimes \mathbb{Z}_p$ ) in terms of indices of Leopoldt’s cyclotomic units and  $\varphi$ -components using  $p$ -adic characters  $\varphi$  of  $K$ , in the semi-simple case<sup>3</sup>.

We can improve the scope of this conjecture in various directions taking into account the new criterion of Theorem 1.1 using algebraic norms (see Remark 1.3 about the definition of the level  $n_0$ ):

**Conjecture 1.4.** *Let  $K$  be any real number field and let  $\mathcal{H}_K$  be its  $p$ -class group, of exponent  $p^{e(K)}$ . For any prime number  $\ell \equiv 1 \pmod{2p^N}$ ,  $N \geq e(K)$ , let  $K_n$  be the sub-extension of  $K(\mu_\ell)$ , of degree  $p^n$  over  $K$ ,  $n \in [0, N]$ :*

- *There exist infinitely many  $\ell$ ’s such that  $\mathcal{H}_K$  capitulates in  $K(\mu_\ell)$ .*
- *There exist infinitely many  $\ell$ ’s such that the capitulation of  $\mathcal{H}_K$  in  $K(\mu_\ell)$  is due, for some  $n \in [1, N]$ , to:  $e(K_n) \in [1, n - s(K_n)]$  if  $m(K_n) \in [p^{s(K_n)}, p^{s(K_n)+1} - 1]$  for  $s(K_n) \in [0, n - 1]$ .*
- *For  $N \gg 0$ , the case  $\#\mathcal{H}_{K_{n_0+1}} = \#\mathcal{H}_{K_{n_0}}$  of stability, for some  $n_0 < N$ , occurs for infinitely many  $\ell$ ’s.*

**Remark 1.5.** *This restriction to the family of  $p$ -extensions  $L/K$ ,  $L \subset K(\mu_\ell)$ , is another point of view with respect to the case of abelian capitulations obtained in Gras [35] (1997), Kurihara [69] (1999), Bosca [16] (2009), Jaulent [60, 61] (2019/22). Indeed, all techniques in these papers need to build a finite set of abelian  $p$ -extensions  $L_k$  of  $\mathbb{Q}$ , ramified at various primes, requiring many local arithmetic conditions existing from Chebotarev theorem, whose compositum with  $K$  gives a capitulation field of  $\mathcal{H}_K$ ; the method must apply to any abelian field  $K$  (of suitable signature), of arbitrary increasing degree, obtained in an iterative process giving, for instance, that the maximal real subfield of  $\mathbb{Q}(\bigcup_{f>0} \mu_f)$  is principal (see in [16] the most general statements).*

<sup>3</sup>The complete statement being the following [47, Section 1.4, then Theorem 4.6]: Assume that  $K/\mathbb{Q}$  is a real cyclic extension, of prime-to- $p$  degree. Let  $\ell \equiv 1 \pmod{2p^N}$  be a prime totally inert in  $K/\mathbb{Q}$  and let  $L \subset K(\mu_\ell)$  be the subfield of degree  $p^N$  over  $K$ . Then, if  $\mathcal{H}_K$  capitulates in  $L$ , the “Main Conjecture” on the  $p$ -adic components  $\mathcal{H}_{K,\varphi}$ , of  $\mathcal{H}_K$ , holds (i.e.,  $\#\mathcal{H}_{K,\varphi} = (\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi})$  for all irreducible  $p$ -adic character  $\varphi$  of  $K$ ).

2. COMPLEXITY OF  $\mathcal{H}_L$  VERSUS CAPITULATION OF  $\mathcal{H}_K$ 

Let  $L/K$  be a totally ramified cyclic  $p$ -extension of degree  $p^N$ ,  $N \geq 1$ , of Galois group  $G = \langle \sigma \rangle$ . Let  $\nu_{L/K} := \sum_{i=0}^{p^N-1} \sigma^i$ , be the algebraic norm in  $L/K$ . From the law of decomposition of an unramified prime ideal  $\mathfrak{q}$  of  $K$ , we get, for  $\mathfrak{Q} \mid \mathfrak{q}$  in  $L$  and the decomposition group  $D$  of  $\mathfrak{Q}$  (of order  $f$ ),  $(\mathfrak{q})_L = \prod_{\bar{\sigma} \in G/D} \mathfrak{Q}^{\bar{\sigma}}$ , thus  $\nu_{L/K}(\mathfrak{Q}) = \prod_{i=0}^{p^N-1} \mathfrak{Q}^{\sigma^i} = (\mathfrak{q})_L^f = (\mathfrak{q}^f)_L = (\mathbf{N}_{L/K}(\mathfrak{Q}))_L$ ; whence the relation:

$$(2.1) \quad \nu_{L/K}(\mathcal{H}_L) = \mathbf{J}_{L/K} \circ \mathbf{N}_{L/K}(\mathcal{H}_L) = \mathbf{J}_{L/K}(\mathcal{H}_K).$$

Thus,  $\mathcal{H}_K$  capitulates in  $L$  if and only if  $\nu_{L/K}(\mathcal{H}_L) = 1$ . So, the action of the algebraic norm characterizes the capitulation (complete or incomplete) and it is clear that the result mainly depends on the  $\mathbb{Z}_p[G]$  structure of  $\mathcal{H}_L$  which is expressed by means of the canonical associated filtration  $\{\mathcal{H}_L^i\}_{i \geq 0}$  that we are going to recall, from [37], improved english translation of <https://doi.org/10.2969/jmsj/04630467>.

**2.1. Filtration of  $\mathcal{H}_L$  in the totally ramified case.** Let  $L/K$  be a cyclic  $p$ -extension of degree  $p^N$ ,  $N \geq 1$ , and Galois group  $G = \langle \sigma \rangle$ . To avoid technical writings, assume that any prime ideal  $\mathfrak{l}$  of  $K$ , ramified in  $L/K$ , is totally ramified, and that there are  $r \geq 1$  such prime ideals. We do not assume  $L/\mathbb{Q}$  Galois.

**2.1.1. Filtration and higher Chevalley–Herbrand formulas.** The generalizations of the Chevalley–Herbrand formula is based on the corresponding filtration  $\{\mathcal{H}_L^i\}_{i \geq 0}$  defined as follows:

$$\mathcal{H}_L^0 = 1, \quad \mathcal{H}_L^1 := \mathcal{H}_L^G, \quad \mathcal{H}_L^{i+1}/\mathcal{H}_L^i := (\mathcal{H}_L/\mathcal{H}_L^i)^G, \quad i \geq 0,$$

up to  $i = m(L) := \min\{m \geq 0, \mathcal{H}_L^{(\sigma-1)^m} = 1\}$ , for which  $\mathcal{H}_L^{m(L)} = \mathcal{H}_L$ .

Denote by  $\mathcal{I}_L^i$  a  $\mathbb{Z}[G]$ -module of ideals of  $L$ , of finite type, generating  $\mathcal{H}_L^i$ , with  $\mathcal{I}_L^0 = 1$ ,  $\mathcal{I}_L^{i+1} \supseteq \mathcal{I}_L^i$ ,  $\forall i \geq 0$ ;  $\mathcal{I}_L^i$  is defined up to the group of principal ideals of  $L$ , thus  $\mathbf{N}_{L/K}(\mathcal{I}_L^i)$  is defined up to  $(\mathbf{N}_{L/K}(L^\times))$ . Note that the above invariants are constant for  $i \geq m(L)$  and that  $m(K) = 1$  for  $\mathcal{H}_K \neq 1$  (otherwise  $m(K) = 0$ ).

This filtration has, for all  $i \geq 0$ , the following properties [37, Theorem 3.6]:

$$(2.2) \quad \left\{ \begin{array}{l} (i) \quad \#\mathcal{H}_L^1 = \#\mathcal{H}_K \times \frac{p^{N(r-1)}}{(\mathbf{E}_K : \mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times))}, \\ (ii) \quad \#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) = \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{L/K}(\mathcal{H}_L^i)} \times \frac{p^{N(r-1)}}{(\mathbf{\Lambda}_K^i : \mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times))}, \\ \quad \quad \quad \mathbf{\Lambda}_K^i := \{x \in K^\times, (x) \in \mathbf{N}_{L/K}(\mathcal{I}_L^i)\}, \\ (iii) \quad \mathcal{H}_L^i = \{h \in \mathcal{H}_L, h^{(\sigma-1)^i} = 1\}, \\ (iv) \quad \#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) \leq \#\mathcal{H}_L^1, \\ (v) \quad \#\mathcal{H}_L = \prod_{i=0}^{m(L)-1} \#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) \leq (\#\mathcal{H}_L^1)^{m(L)}. \end{array} \right.$$

The  $\mathbf{\Lambda}_K^i$ 's are subgroups of  $K^\times$  containing  $\mathbf{E}_K$ , with  $\mathbf{\Lambda}_K^0 = \mathbf{E}_K$  in (i). In particular, any  $x \in \mathbf{\Lambda}_K^i$  is local norm in  $L/K$  at all the non-ramified places. So, for any  $(x) = \mathbf{N}_{L/K}(\mathfrak{A})$ ,  $\mathfrak{A} \in \mathcal{I}_L^i$ , which is also local norm at the ramified places, then  $x = \mathbf{N}_{L/K}(y)$ ,  $y \in L^\times$  (Hasse's norm theorem) and there exists an ideal  $\mathfrak{B}$  of  $L$  such that  $\mathfrak{A} = (y)\mathfrak{B}^{\sigma-1}$ ; this constitutes an algorithm by addition of the  $\mathfrak{B}$ 's to  $\mathcal{I}_L^i$  to get  $\mathcal{I}_L^{i+1}$  then  $\mathcal{H}_L^{i+1}$ .<sup>4</sup> Since  $\mathbf{\Lambda}_K^0 = \mathbf{E}_K$  is a  $\mathbb{Z}$ -module of finite type, this algorithm allows to construct  $\mathbf{\Lambda}_K^i$  of finite type for all  $i$ , with  $\mathbf{\Lambda}_K^i \subseteq \mathbf{\Lambda}_K^{i+1}$  (indeed,  $\mathbf{N}_{L/K}(\mathcal{I}_L^i)$  is of finite type, there is a finite number of relations of principality between the generators and  $\mathbf{\Lambda}_K^i/\mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times)$  is annihilated by  $p^N$ ).

<sup>4</sup>For explicit class field theory, Hasse's norm theorem, norm residue symbols, product formula, see, e.g., [36, Theorem II.6.2, Definition II.3.1.2, Theorems II.3.1.3, 3.4.1].



The  $i$ -sequence  $\#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i)$  is decreasing, from  $\#\mathcal{H}_L^1$  up to 1, because of the injective maps  $\mathcal{H}_L^{i+1}/\mathcal{H}_L^i \hookrightarrow \mathcal{H}_L^i/\mathcal{H}_L^{i-1} \hookrightarrow \dots \hookrightarrow \mathcal{H}_L^1$ , due to the action of  $\sigma - 1$ , giving the inequality in (v).

The first (resp. second) factor in (ii) is called the class (resp. norm) factor.

2.1.2. *Properties of the class and norm factors.* Since ramified places  $v$  of  $K$  are assumed to be totally ramified in  $L/K$ , their inertia groups  $I_v(L/K)$  in  $L/K$  are isomorphic to  $G$ . Let  $\omega_{L/K}$  be the map which associates with  $x \in \mathbf{\Lambda}_K^i$  the family of Hasse's norm symbols  $(\frac{x, L/K}{v}) \in I_v(L/K)$ . Since  $x$  is local norm at the unramified places,  $\omega_{L/K}(\mathbf{\Lambda}_K^i)$  is contained (product formula) in:

$$\Omega_{L/K} := \{(\tau_v)_v \in \bigoplus_v I_v(L/K), \prod_v \tau_v = 1\} \simeq G^{r-1};$$

then  $\#\omega_{L/K}(\mathbf{\Lambda}_K^i) = (\mathbf{\Lambda}_K^i : \mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times))$  divides  $p^{N(r-1)}$ .

Let  $K_n \subseteq L$  be of degree  $p^n$  over  $K$  and set  $G_n := \text{Gal}(K_n/K) =: \langle \sigma_n \rangle$ ,  $n \in [0, N]$ . All of the above applies to the  $K_n$ 's; denote by  $\mathbf{\Lambda}_K^i(n) \subset K^\times$  the invariants corresponding to  $K_n/K$  instead of  $L/K$ ; so  $\mathbf{\Lambda}_K^i(n) = \{x \in K^\times, (x) \in \mathbf{N}_{K_n/K}(\mathcal{I}_{K_n}^i)\}$ , where  $\mathcal{I}_{K_n}^i$  represents  $\mathcal{H}_{K_n}^i$ ; so  $\mathbf{\Lambda}_K^0(n) = \mathbf{E}_K$ . For  $n = 0$  and  $i \geq 1$ ,  $\mathcal{I}_K^i$  generates  $\mathcal{H}_K$  and  $\mathbf{\Lambda}_K^i(0) = \{x \in K^\times, (x) \in \mathcal{I}_K^i\}$  contains  $\mathbf{\Lambda}_K^0(0) = \mathbf{E}_K$  and is given by relations between elements of  $\mathcal{I}_K^i$ ; we have  $\mathcal{I}_K^0 = 1$ .

**Lemma 2.1.** *For any  $i$  fixed we may assume  $\mathbf{\Lambda}_K^i(n+1) \subseteq \mathbf{\Lambda}_K^i(n)$ ,  $\forall n \in [0, N-1]$ .*

*Proof.* We have  $\mathbf{N}_{K_{n+1}/K_n}(\mathcal{H}_{K_{n+1}}^i) \subseteq \mathcal{H}_{K_n}^i$ ; so, for any ideal  $\mathfrak{A}_{n+1} \in \mathcal{I}_{K_{n+1}}^i$ , one may write  $\mathbf{N}_{K_{n+1}/K_n}(\mathfrak{A}_{n+1}) = (\alpha_n)\mathfrak{A}_n$ , where  $\alpha_n \in K_n^\times$  and  $\mathfrak{A}_n \in \mathcal{I}_{K_n}^i$ , in which case *modifying* the definition of  $\mathcal{I}_{K_n}^i$  modulo principal ideals of  $K_n$ , one may assume  $\mathbf{N}_{K_{n+1}/K_n}(\mathcal{I}_{K_{n+1}}^i) \subseteq \mathcal{I}_{K_n}^i$  whence  $\mathbf{N}_{K_{n+1}/K}(\mathcal{I}_{K_{n+1}}^i) \subseteq \mathbf{N}_{K_n/K}(\mathcal{I}_{K_n}^i)$ ; this modifies  $\mathbf{\Lambda}_K^i(n)$  modulo  $\mathbf{N}_{K_n/K}(K_n^\times)$  which does not modify  $\#\omega_{K_n/K}(\mathbf{\Lambda}_K^i(n))$ . Using the process from the top, we obtain  $\mathbf{\Lambda}_K^i(N) \subseteq \mathbf{\Lambda}_K^i(N-1) \subseteq \dots \subseteq \mathbf{\Lambda}_K^i(1) \subseteq \mathbf{\Lambda}_K^i(0)$ . For  $i = 0$ ,  $\mathbf{\Lambda}_K^0(n) = \mathbf{E}_K$ , for all  $n \geq 0$   $\square$

**Lemma 2.2.** *For  $i \geq 0$  fixed, the integers  $\#(\mathcal{H}_{K_n}^{i+1}/\mathcal{H}_{K_n}^i)$  define an increasing  $n$ -sequence from  $\#(\mathcal{H}_K^{i+1}/\mathcal{H}_K^i) = 1$  up to  $\#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i)$ ; the  $\#\mathcal{H}_{K_n}^i$ 's define an increasing  $n$ -sequence from  $\#\mathcal{H}_K^i = \mathcal{H}_K$  up to  $\#\mathcal{H}_L^i$ .*

*Proof.* Consider, for  $i \geq 1$  fixed and  $n \geq 0$ , the two factors of the  $n$ -sequence:

$$\#(\mathcal{H}_{K_n}^{i+1}/\mathcal{H}_{K_n}^i) = \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}^i)} \times \frac{p^{n(r-1)}}{\#\omega_{K_n/K}(\mathbf{\Lambda}_K^i(n))}.$$

As  $\mathbf{N}_{K_{n+1}/K}(\mathcal{H}_{K_{n+1}}^i) \subseteq \mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}^i)$ ,  $p^{c_{K_n}^i} := \frac{\#\mathcal{H}_K}{\#\mathbf{N}_{K_n/K}(\mathcal{H}_{K_n}^i)}$  defines an increasing  $n$ -sequence from 1 up to  $p^{c_L^i} \mid \#\mathcal{H}_K$ . The factor  $p^{\rho_{K_n}^i} := \frac{p^{n(r-1)}}{\#\omega_{K_n/K}(\mathbf{\Lambda}_K^i(n))}$  defines an increasing  $n$ -sequence from 1 up to  $p^{\rho_L^i}$  since, from Lemma 2.1:

$$p^{\rho_{K_{n+1}}^i - \rho_{K_n}^i} = p^{r-1} \frac{\#\omega_{K_n/K}(\mathbf{\Lambda}_K^i(n))}{\#\omega_{K_{n+1}/K}(\mathbf{\Lambda}_K^i(n+1))} \geq p^{r-1} \frac{\#\omega_{K_n/K}(\mathbf{\Lambda}_K^i(n))}{\#\omega_{K_{n+1}/K}(\mathbf{\Lambda}_K^i(n))};$$

so, in the restriction  $\Omega_{K_{n+1}/K} \rightarrow \Omega_{K_n/K}$  (whose kernel is of order  $p^{r-1}$  due to the total ramification of each place), the image of  $\omega_{K_{n+1}/K}(\mathbf{\Lambda}_K^i(n))$  is  $\omega_{K_n/K}(\mathbf{\Lambda}_K^i(n))$  because of the properties of Hasse's symbols, whence  $p^{\rho_{K_{n+1}}^i - \rho_{K_n}^i} \geq 1$  and the result for the  $n$ -sequence  $p^{\rho_{K_n}^i}$ , with maximal value  $p^{\rho_L^i}$ . For  $i = 0$ ,  $\mathcal{H}_{K_n}^0 = 1$  for all  $n \geq 0$ . The first claim of the lemma holds for the  $n$ -sequence  $\#(\mathcal{H}_{K_n}^{i+1}/\mathcal{H}_{K_n}^i)$ ; for  $n = N$ , one gets the formula  $\#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i) = p^{c_L^i} \cdot p^{\rho_L^i}$ .

Assuming, by induction on  $i \geq 0$ , that the  $n$ -sequence  $\#\mathcal{H}_{K_n}^i$  is increasing, the property follows for the  $n$ -sequence  $\#\mathcal{H}_{K_n}^{i+1}$ .  $\square$

**Remark 2.3.** *The  $n$ -sequence  $m(K_n)$  is an increasing sequence from 1 (if  $\mathcal{H}_K \neq 1$ , 0 otherwise) up to  $m(L)$ . The  $\#\mathcal{H}_{K_n}$ 's define an increasing  $n$ -sequence from  $\#\mathcal{H}_K$  up to  $\#\mathcal{H}_L$  since  $\#\mathcal{H}_{K_{n+1}} \geq \#\mathcal{H}_{K_{n+1}}^{\text{Gal}(K_{n+1}/K_n)} = \#\mathcal{H}_{K_n} \frac{p^{r-1}}{\omega_{K_{n+1}/K_n}(\mathbf{E}_{K_n})} \geq \#\mathcal{H}_{K_n}$ . The integers  $e(K_n)$  and  $\text{rk}_p(\mathcal{H}_{K_n})$  define increasing  $n$ -sequences.*

The interest of this filtration is that standard probabilities may apply at each level  $n$  to the algorithm computing  $\mathcal{H}_{K_n}^{i+1}$  from  $\mathcal{H}_{K_n}^i$  by means of the factors  $p^{e_{K_n}^i}$  and  $p^{o_{K_n}^i}$ , giving plausible heuristics in the spirit of the works of Koymans-Pagano [67], Smith [92], leading to a considerable generalization of pioneering works as that of Morton [84], Gerth III [28] and many others, the theory over  $\mathbb{Q}$  giving generalizations of the well-known Rédei matrices. Indeed, let  $x \in \mathbf{\Lambda}_K^i(n)$  be such that  $(x) = \mathbf{N}_{K_n/K}(\mathfrak{A})$  (when it holds for  $\mathfrak{A} \in \mathcal{S}_{K_n}^i$ ), and let  $x = \mathbf{N}_{K_n/K}(y)$ , when  $\omega_{K_n/K}(x) = 1$  (depending on Hasse's symbols), so that  $\mathfrak{A} = (y)\mathfrak{B}^{\sigma-1}$  giving  $\mathfrak{B} \in \mathcal{S}_{K_n}^{i+1}$ . So we propose the following conjecture on the algorithmic evolution of the class and norm factors, respectively:

**Conjecture 2.4.** *Let  $L/K$  be any cyclic  $p$ -extension of degree  $p^N$ ,  $N \geq 1$ , of Galois group  $G$ ; we assume that  $L/K$  is ramified at  $r \geq 1$  places of  $K$ , totally ramified in  $L/K$ . Then, the orders of each of the two factors (class and norm) in the  $i$ -sequence  $\#(\mathcal{H}_L^{i+1}/\mathcal{H}_L^i)$ , follow binomial laws as  $i$  increases, based on the following probabilities:*

- Let  $c \in \mathcal{H}_K$ ; the probability, for an ideal  $\mathfrak{C}$  of  $L$ , that the  $p$ -class of the ideal  $\mathbf{N}_{L/K}(\mathfrak{C})$  equals  $c$ , is  $\frac{1}{\#\mathcal{H}_K}$ .
- Let  $\gamma \in G^{r-1}$ ; the probability, for  $x \in K^\times$  local norm at all the non-ramified places, that  $\omega_{L/K}(x) = \gamma$ , is  $\frac{1}{p^{N(r-1)}}$ .

2.1.3. *Program computing the filtrations  $\{\mathcal{H}_{K_n}^i\}_{i \geq 1}$ .* The following program may be used for the calculation of the Galois structure of the  $\mathcal{H}_{K_n}$ 's,  $K_n \subseteq L \subset K(\mu_\ell)$ , whatever the base field  $K$  of prime-to- $p$  degree  $d$ , given, as usual, by means of a monic polynomial of  $\mathbb{Z}[x]$  (the condition  $p \nmid d$  simplifies the computation of a generator  $\sigma_n$  (in  $\mathbf{S}$ ) of  $\text{Gal}(K_n/K)$ ). Let  $r(K_n) := \text{rk}_{\mathbb{Z}}(\mathbf{H}_{K_n})$  (in  $\mathbf{rKn}$ ).

For this, one must indicate the prime  $p$  in  $\mathfrak{p}$ , the number  $\mathbf{Nn}$  of layers  $K_n$  considered, the polynomial  $\mathbf{PK}$  defining  $K$ , a prime  $\mathfrak{ell}$  congruent to 1 modulo  $2p^N$ ,  $\mathbf{N} \geq \mathbf{Nn}$ , and a value  $\mathbf{mKn}$  for computing the  $\mathfrak{h}_j^i := \mathfrak{h}_j^{(\mathbf{S}-1)^i}$  for  $1 \leq i \leq \mathbf{mKn}$  and  $1 \leq j \leq \mathbf{rKn}$ , where the  $\mathfrak{h}_j$ 's are the  $r(K_n)$  generators of the whole class group  $\mathbf{H}_{K_n}$  given by PARI (in  $\mathbf{CKn} = \mathbf{Kn.clgp}$ ), and where  $\mathbf{S}$  is chosen of order  $p^n$  in  $\mathbf{G} = \text{nfgaloisconj}(\mathbf{Kn})$  by testing the orders.<sup>5</sup>

So  $\mathcal{H}_{K_n}^i = \{h \in \mathcal{H}_{K_n}, h^{(\sigma_n-1)^i} = 1\}$ ,  $1 \leq i \leq m(K_n)$  (see (2.2) (iii)). PARI works with independent generators  $\mathfrak{h}_j$  of  $\mathbf{H}_{K_n}$ , of orders  $\mathfrak{e}_j$  (given in  $\mathbf{CKn}[2]$ ); thus, for any data  $[\mathfrak{e}_1, \dots, \mathfrak{e}_{\mathbf{rKn}}]$  given by  $\text{bnfispincipal}(\mathbf{Kn}, \mathbf{Y})[1]$  for an ideal  $\mathbf{Y}$  whose class is  $\mathfrak{h} = \prod_j \mathfrak{h}_j^{\mathfrak{e}_j}$ , the program gives, instead, the list  $\mathbf{E} := [\bar{\mathfrak{e}}_1, \dots, \bar{\mathfrak{e}}_{\mathbf{rKn}}]$  defining the  $p$ -class of  $\mathbf{Y}$  (in  $\mathcal{H}_{K_n}$ ) from  $\bar{\mathfrak{h}} = \prod_j \bar{\mathfrak{h}}_j^{\bar{\mathfrak{e}}_j}$ ,  $\bar{\mathfrak{e}}_j = \text{lift}(\text{Mod}(\mathfrak{e}_j, p^{\mathfrak{n}_j}))$ , where  $p^{\mathfrak{n}_j}$  is the  $p$ -part of the order of  $\mathfrak{h}_j$ ; this does not modify the Galois structure of  $\mathcal{H}_{K_n}$  and the outputs are more readable. So, the ideal  $\mathbf{Y}$  is  $p$ -principal if and only if  $\mathbf{E} = [0, \dots, 0]$ . The outputs are written under the form  $\bar{\mathfrak{h}}_j^{(\sigma-1)^i} = [\bar{\mathfrak{e}}_1, \dots, \bar{\mathfrak{e}}_{\mathbf{rKn}}]$  instead of  $\bar{\mathfrak{h}}_j^{(\sigma-1)^i} = \bar{\mathfrak{h}}_1^{\bar{\mathfrak{e}}_1} \dots \bar{\mathfrak{h}}_{\mathbf{rKn}}^{\bar{\mathfrak{e}}_{\mathbf{rKn}}}$ .

The invariant  $m(K_n)$  is obtained (for  $\mathbf{mKn}$  large enough) for the first  $i$  giving zero matrices in the test of principality of the  $\mathfrak{h}_j^i$ 's.

Below, we take as example the cyclic cubic field of conductor  $f = 703$ , for which, using the structure of  $\mathbf{Z}$ -module,  $\mathbf{Z} = \mathbb{Z}[\exp(\frac{2i\pi}{3})]$ , we have  $\mathcal{H}_K \simeq \mathbf{Z}/2\mathbf{Z}$ ; taking

<sup>5</sup> Warning: for some class group computations, PARI uses random primes in some analytic contexts, so that the generators given by  $\mathbf{Kn.clgp}$  may vary; but the corresponding matrices of exponents are "equivalent". For this observation, run the programs several times.



$\text{ell} = 97$ ,  $mKn = 3$ , the results are given for  $n = 1$  and  $n = 2$  and  $r$  is the number of prime ideals of  $K$  dividing  $\ell$ :

```
PROGRAM COMPUTING THE h_j^[(S-1)^i]:
{p=2;Nn=2;PK=x^3+x^2-234*x-729;ell=97;mKn=3;K=bnfinit(PK,1);CK0=K.clgp;
r=matsize(idealfactor(K,ell))[1];print("p=",p," Nn=",Nn," PK=",PK,
" ell=",ell," mKn=",mKn," CK0=",CK0[2]," r=",r);
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);CKn=Kn.clgp;dn=poldegree(Pn);
print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
\\Search of a generator S of Gal(Kn/K):
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
\\Computation of the image of CKn by (S-1)^i:
for(j=1,rKn,X=CKn[3][j];Y=X;for(i=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpown(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
\\computation in Ehij of the modified exponents of B:
Ehij=List;for(j=1,rKn,c=B[j];w=valuation(CKn[2][j],p);c=lift(Mod(c,p^w));
listput(Ehij,c,j));print("h_",j,"^[","(S-1)^",i,"]=",Ehij);print())}
p=2 Nn=2 f=703 PK=x^3+x^2-234*x-729 ell=97 mKn=3 CK0=[6,2] r=1
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,0,0] h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,0,1,1] h_4^[(S-1)^1]=[0,0,1,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0] h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0] h_4^[(S-1)^3]=[0,0,0,0]
CK2=[12,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,1,1] h_2^[(S-1)^1]=[0,2,1,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,2,0,0] h_2^[(S-1)^2]=[2,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0] h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0] h_4^[(S-1)^3]=[0,0,0,0]
CK3=[12,4,2,2]=[4,4,2,2]
```

This gives  $m(K_1) = 2$ ,  $\mathcal{H}_{K_1}^{\sigma_1^{-1}} = \langle h_1 h_2, h_3 h_4 \rangle$ ,  $\mathcal{H}_{K_1}^{G_1} = \langle h_1 h_2^{-1}, h_3 h_4^{-1} \rangle \simeq \mathbf{Z}/2\mathbf{Z}$ . Then  $m(K_2) = 3$ ,  $\mathcal{H}_{K_2}^{\sigma_2^{-1}} = \langle h_2^2 h_3 h_4, h_2^2 h_3, h_1^2 h_2^2, h_1^2 \rangle = \langle h_1^2, h_2^2, h_3, h_4 \rangle \simeq (\mathbf{Z}/2\mathbf{Z})^2$ , whence  $\mathcal{H}_{K_2}^{G_2} = (\mathcal{H}_{K_2})^2 \simeq \mathbf{Z}/2\mathbf{Z}$ . Then  $\mathcal{H}_{K_2}^{(\sigma_2^{-1})^2} = \langle h_1^2, h_2^2 \rangle \simeq (\mathcal{H}_{K_2})^2 \simeq \mathbf{Z}/2\mathbf{Z}$ ,  $\mathcal{H}_{K_2}^2 = \langle h_1^2, h_2^2, h_3, h_4 \rangle \simeq (\mathbf{Z}/2\mathbf{Z})^2$ . These computations will be pursued to obtain a partial capitulation in  $K_1$  and a complete capitulation in  $K_2$  (stability from  $K_2$ ).

**2.2. Classes, units and capitulation kernel.** In this subsection, we restrict ourselves to the case  $L \subseteq K(\mu_\ell)$ ,  $\ell \equiv 1 \pmod{2p^N}$ , with  $K$  real of prime-to- $p$  degree  $d$ , so that  $L/K$  is totally ramified at all the  $r$  prime ideals  $\mathfrak{l} \mid \ell$  of  $K$ . Chevalley–Herbrand’s formula  $\#\mathcal{H}_L^G = \#\mathcal{H}_K \times \frac{p^{N(r-1)}}{\#(\mathbf{E}_K/\mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times))}$  and exact sequence of capitulation (1.1) lead to the relation:

$$(2.3) \quad \#(\mathbf{J}_{L/K}(\mathcal{H}_K) \mathcal{H}_L^{\text{ram}}) \times \#(\mathbf{E}_K/\mathbf{N}_{L/K}(\mathbf{E}_L)) = \#\mathcal{H}_K \times p^{N(r-1)}.$$

Since  $L_0 \subset \mathbb{Q}(\mu_\ell)$  is  $p$ -principal (indeed,  $\mathcal{H}_{L_0}^{\text{Gal}(L_0/\mathbb{Q})} = 1$ ) and since  $p \nmid d$ , the primes  $\mathfrak{L}_i \mid \mathfrak{l}_i$  of  $L$ ,  $i = 1, \dots, r$ , fulfill a relationship of the form  $\mathfrak{L}_1 \cdots \mathfrak{L}_r = (\alpha_0)_L$ ,  $\alpha_0 \in L_0^\times$ , so that  $\text{rk}_p(\mathcal{H}_L^{\text{ram}}) \leq r - 1$ . Then  $(\mathcal{H}_L^{\text{ram}})^{p^N} = \mathbf{J}_{L/K}(\mathcal{H}_K^{\text{ram}})$ ,  $\mathbf{N}_{L/K}(\mathcal{H}_L^{\text{ram}}) = \mathcal{H}_K^{\text{ram}}$ , where  $\mathcal{H}_K^{\text{ram}} \subseteq \mathcal{H}_K$  is generated by the  $p$ -classes of the  $\mathfrak{l}_i$ ’s. One verifies easily that, in  $L = \bigcup_{n=0}^N K_n$ ,

$$\#\mathcal{H}_{K_n}^{\text{ram}}, \quad \#(\mathbf{E}_K/\mathbf{N}_{K_n/K}(\mathbf{E}_{K_n})), \quad \frac{p^{N(r-1)}}{\#(\mathbf{E}_K/\mathbf{E}_K \cap \mathbf{N}_{L/K}(L^\times))}$$

(see Lemma 2.2 for the last one) define increasing  $n$ -sequences and that  $\#\mathbf{J}_{K_n/K}(\mathcal{H}_K)$  is decreasing. This suggests that for some  $\ell$ ’s, with  $N \gg 0$ , there is capitulation of

$\mathcal{H}_K$  in  $K_{n_0}$  and, from (2.3), relations of the form:

$$\#\mathcal{H}_{K_n}^{\text{ram}} = p^{an+a_0}, \quad \#(\mathbf{E}_K/\mathbf{N}_{K_n/K}(\mathbf{E}_{K_n})) = p^{bn+b_0}, \quad \forall n \geq n_0,$$

with  $a + b = r - 1$  and  $a_0 + b_0 = v_p(\#\mathcal{H}_K)$ .

2.2.1. *Case  $r = 1$ .* The case  $r = 1$  is particular since, whatever  $n$ , all the factors are finite in the relation (2.3) which becomes:

$$\#(\mathbf{J}_{L/K}(\mathcal{H}_K) \mathcal{H}_L^{\text{ram}}) \times \#(\mathbf{E}_K/\mathbf{N}_{L/K}(\mathbf{E}_L)) = \#\mathcal{H}_K,$$

giving, possibly, stationary  $n$ -sequences from some layer  $K_{n_0}$  up to  $L$ . The case  $r = 1$  supposes that  $\ell$  does not split in  $K$  and we can assume, for instance, that  $K/\mathbb{Q}$  is cyclic of prime-to- $p$  degree, in which case  $\mathcal{H}_L^{\text{ram}} = 1$ , whence:

$$\#\mathbf{J}_{L/K}(\mathcal{H}_K) \times \#(\mathbf{E}_K/\mathbf{N}_{L/K}(\mathbf{E}_L)) = \#\mathcal{H}_K,$$

so that complete capitulation in  $L$  is equivalent to:

$$(2.4) \quad \#(\mathbf{E}_K/\mathbf{N}_{L/K}(\mathbf{E}_L)) = \#\mathcal{H}_K,$$

with stationary  $n$ -sequences from some layer  $K_{n_0}$ .

Of course, only orders coincide in (2.4) since structures may be very different; we will consider two examples of this phenomenon. We then compute  $\nu_{K_n/K}(\mathcal{H}_{K_n})$  by means of the  $\nu_{K_n/K}(h_j)$ ,  $1 \leq j \leq r(K_n)$ .

**Remark 2.5.** *An remarkable fact, in a diophantine viewpoint, is that, when the class of  $\mathfrak{a}$  capitulates in some  $K_n$ , the writing of the generator  $\alpha \in K_n^\times$ , of the extended ideal  $(\mathfrak{a})_{K_n}$ , on the  $\mathbb{Q}$ -basis  $\text{Kn.zk}$  of the field  $K_n$ , needs most often oversized coefficients, and increasing with  $n$  (several thousand digits and, often, PARI proves the principality without giving these coefficients). If the reader wishes to verify this fact, it suffices to add the instruction `print(bnfisprincipal(Kn, Y))` giving the whole data for the ideal  $Y$  considered.*

**Example 2.6.** We consider the cubic field of conductor 31923, with  $p = 2$ ,  $\ell = 257$  ( $N = 7$ ).

```
p=2 f=31923 PK=x^3-10641*x+227008 CK0=[6,2,2,2] e11=257 r=1
CK1=[18,6,2,2,2,2]=[2,2,2,2,2,2]
h_1^~[(S-1)^~1]=[0,0,0,0,0,0] h_2^~[(S-1)^~1]=[0,0,0,0,0,0]
h_3^~[(S-1)^~1]=[0,1,0,0,1,0] h_4^~[(S-1)^~1]=[0,0,0,0,1,1]
h_5^~[(S-1)^~1]=[0,0,0,0,0,0] h_6^~[(S-1)^~1]=[0,0,0,0,0,0]
h_1^~[(S-1)^~2]=[0,0,0,0,0,0] h_2^~[(S-1)^~2]=[0,0,0,0,0,0]
h_3^~[(S-1)^~2]=[0,0,0,0,0,0] h_4^~[(S-1)^~2]=[0,0,0,0,0,0]
h_5^~[(S-1)^~2]=[0,0,0,0,0,0] h_6^~[(S-1)^~2]=[0,0,0,0,0,0]
norm in K1/K of the component 1 of CK1: [0,0,0,0,0,0]
norm in K1/K of the component 2 of CK1: [0,0,0,0,0,0]
norm in K1/K of the component 3 of CK1: [0,1,0,0,1,0]
norm in K1/K of the component 4 of CK1: [0,0,0,0,1,1]
norm in K1/K of the component 5 of CK1: [0,0,0,0,0,0]
norm in K1/K of the component 6 of CK1: [0,0,0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=1
CK2=[36,12,2,2,2,2]=[4,4,2,2,2,2]
h_1^~[(S-1)^~1]=[2,2,0,0,0,0] h_2^~[(S-1)^~1]=[2,2,1,1,1,0]
h_3^~[(S-1)^~1]=[0,2,1,1,1,0] h_4^~[(S-1)^~1]=[0,2,0,0,1,1]
h_5^~[(S-1)^~1]=[0,2,1,1,0,1] h_6^~[(S-1)^~1]=[2,0,1,1,0,1]
h_1^~[(S-1)^~2]=[0,0,0,0,0,0] h_2^~[(S-1)^~2]=[0,2,0,0,0,0]
h_3^~[(S-1)^~2]=[0,2,0,0,0,0] h_4^~[(S-1)^~2]=[2,2,0,0,0,0]
h_5^~[(S-1)^~2]=[2,0,0,0,0,0] h_6^~[(S-1)^~2]=[2,0,0,0,0,0]
norm in K2/K of the component 1 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2: [0,0,0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2
```

The data for  $K_2$  give  $\mathcal{H}_{K_2}^{G_2} = \langle h_1 h_5 h_6, h_2 h_4 h_6 \rangle \simeq \mathbf{Z}/4\mathbf{Z}$ , since the two independent generators  $h_1, h_2$ , are of order 4. The exact sequence (1.1) reduces to the isomorphism of  $\mathbf{Z}$ -modules  $\mathcal{H}_{K_2}^{G_2} \simeq \mathbf{E}_K/\mathbf{N}_{K_2/K}(\mathbf{E}_{K_2})$ , which are of order  $\#\mathcal{H}_K = 16$ , but are not isomorphic to  $\mathcal{H}_K$ . We have, from the relation (2.4) and since  $\mathbf{E}_K \simeq \mathbf{Z}$ , the isomorphisms of  $\mathbf{Z}$ -modules  $\mathcal{H}_{K_2}^{G_2} \simeq \mathbf{E}_K/\mathbf{N}_{K_2/K}(\mathbf{E}_{K_2}) \simeq \mathbf{Z}/4\mathbf{Z}$  and  $\mathcal{H}_K \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ .

**Example 2.7.** We consider a quadratic field with  $p = 3, \ell = 19$  inert in  $K$  ( $N = 2$ ). Let  $K = \mathbb{Q}(\sqrt{32009})$  for which  $\mathcal{H}_K \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ . The general Program gives an incomplete capitulation in  $K_1$ , then a complete capitulation in  $K_2$ :

```
PK=x^2-32009 CK0=[3,3] e11=19 r=1
CK1=[9,3]
h_1^[(S-1)^1]=[3,0] h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[9,3]
h_1^[(S-1)^1]=[0,0] h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

The data for  $K_2$  give  $\mathcal{H}_{K_2}^{G_2} = \langle h_1 \rangle \simeq \mathbb{Z}/9\mathbb{Z}$ . Since  $\mathbf{J}_{K_2/K}(\mathcal{H}_K) = 1, \mathcal{H}_{K_2}^{\text{ram}} = 1$  and  $\mathbf{E}_K \simeq \mathbb{Z}$ , the relations (2.3) (2.4) become, in  $K_2/K$ , the isomorphism of cyclic groups  $\mathcal{H}_{K_2}^{G_2} \simeq \mathbf{E}_K/\mathbf{N}_{K_2/K}(\mathbf{E}_{K_2}) \simeq \mathbb{Z}/9\mathbb{Z}$ , while  $\mathcal{H}_K \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ .

2.2.2. *Case  $r > 1$ .* In this case, an heuristic is that there is no obstruction about  $\mathcal{H}_{K_n}^{\text{ram}}, \mathbf{E}_K/\mathbf{N}_{K_n/K}(\mathbf{E}_{K_n})$  as  $\mathbb{Z}_p[G_n]$ -modules of standard  $p$ -ranks, except a bounded exponent which may increase as soon as the orders of the modules increase in (2.3), regarding  $N$ . Under complete capitulation, one gets:

$$(2.5) \quad \#\mathcal{H}_{K_n}^{\text{ram}} \times \#(\mathbf{E}_K/\mathbf{N}_{K_n/K}(\mathbf{E}_{K_n})) = \#\mathcal{H}_K \times p^{n(r-1)}, \quad \forall n \geq n_0.$$

**Example 2.8.** Consider  $K_3$  in the following example with  $p = 2, \ell = 17$  totally split in the cyclic cubic field  $K$  of conductor  $f = 1951$ , and complete capitulation of  $\mathcal{H}_K$  in  $K_1$  (the others layers are computed for checking); we have  $\mathcal{H}_K \simeq \mathbf{Z}/2\mathbf{Z}$  and  $\mathcal{H}_{K_3} \simeq \mathbf{Z}/8\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$ :

```
p=2 f=1951 PK=x^3+x^2-650*x-289 CK0=[2,2] e11=17 r=3
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=2, e(K1)=2
CK2=[4,4,4,4]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[2,0,2,0] h_4^[(S-1)^1]=[0,2,0,2]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K1)=2
CK3=[8,8,4,4]
```

```

h_1^[(S-1)^1]=[0,0,2,0]   h_2^[(S-1)^1]=[2,2,2,0]
h_3^[(S-1)^1]=[2,0,0,0]   h_4^[(S-1)^1]=[6,2,0,2]
h_1^[(S-1)^2]=[4,0,0,0]   h_2^[(S-1)^2]=[0,4,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K3/K of the component 1 of CK3:[0,0,0,0]
norm in K3/K of the component 2 of CK3:[0,0,0,0]
norm in K3/K of the component 3 of CK3:[0,0,0,0]
norm in K3/K of the component 4 of CK3:[0,0,0,0]
Complete capitulation, m(K3)=3, e(K3)=3

```

We compute  $\mathcal{H}_{K_1}^{\text{ram}}$ ,  $\mathcal{H}_{K_2}^{\text{ram}}$  and  $\mathcal{H}_{K_3}^{\text{ram}}$ , adding the following instructions after a program running:

```

For K1:
A=component(idealfactor(Kn,ell),1)[3];bnfisprincipal(Kn,A)[1]=[2,0,0,0]
\\Checking the principality of A^2 with components of a generator:
A2=idealpow(Kn,A,2);bnfisprincipal(Kn,A2)=[[0,0,0,0],[-51,0,-2,2,0,0]]
For K2:
B=component(idealfactor(Kn,ell),1)[3];bnfisprincipal(Kn,B)[1]=[0,0,2,2]
\\Checking the principality of B^2 with components of a generator:
B2=idealpow(Kn,B,2);bnfisprincipal(Kn,B2)=[[0,0,0,0],
[-199260,-90688,102100,13880,4054,-14216,-8292,8559,-6223,-3433,3403,7557]]
For K3:
C=component(idealfactor(Kn,ell),1)[3];bnfisprincipal(Kn,C)[1]=[2,2,0,2]
C2=idealpow(Kn,C,2);bnfisprincipal(Kn,C2)[1]=[4,4,0,0]
\\Checking the principality of C^4 with components of a generator:
C4=idealpow(Kn,C,4);bnfisprincipal(Kn,C4)=[[0,0,0,0],
[-57074733,49681698,-55181004,32125541,42753200,-11450554,20535876,
-4037958, -4486534,-2178833,-1875179,3883122,-1527899,1665071,4332070,
2101150,1108465,-1251165, -1504106,445954,-292536,-677913,157262,-159406]]

```

The previous data and formula (2.5) give:

$$\begin{cases} \mathcal{H}_{K_1} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, & \mathcal{H}_{K_1}^{\sigma_1^{-1}} = \langle h_1^2, h_2^2 \rangle \simeq \mathbf{Z}/2\mathbf{Z}, \\ \mathcal{H}_{K_1}^{G_1} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, & \mathcal{H}_{K_1}^{\text{ram}} \simeq \mathbf{Z}/2\mathbf{Z}, \quad \mathbf{E}_K/\mathbf{N}_{K_1/K}(\mathbf{E}_{K_1}) \simeq \mathbf{Z}/4\mathbf{Z}, \end{cases}$$

since  $\mathbf{E}_K$  is a free  $\mathbf{Z}$ -module of rank 1 and  $\mathbf{E}_K/\mathbf{N}_{K_1/K}(\mathbf{E}_{K_1})$  of order 16.

$$\begin{cases} \mathcal{H}_{K_2} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}, & \mathcal{H}_{K_2}^{\sigma_2^{-1}} = \mathcal{H}_{K_2}^2 \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, \\ \mathcal{H}_{K_2}^{G_2} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, & \mathcal{H}_{K_2}^{\text{ram}} \simeq \mathbf{Z}/2\mathbf{Z}, \quad \mathbf{E}_K/\mathbf{N}_{K_2/K}(\mathbf{E}_{K_2}) \simeq \mathbf{Z}/4\mathbf{Z}, \\ \mathcal{H}_{K_3} \simeq \mathbf{Z}/8\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}, & \mathcal{H}_{K_3}^{\sigma_3^{-1}} = \mathcal{H}_{K_3}^2 \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, \\ \mathcal{H}_{K_3}^{G_3} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, & \mathcal{H}_{K_3}^{\text{ram}} \simeq \mathbf{Z}/4\mathbf{Z}, \quad \mathbf{E}_K/\mathbf{N}_{K_3/K}(\mathbf{E}_{K_3}) \simeq \mathbf{Z}/4\mathbf{Z}, \end{cases}$$

Chevalley–Herbrand’s formula  $\#\mathcal{H}_{K_n}^{G_n} = \#\mathcal{H}_K \times \frac{4^n}{(\mathbf{E}_K : \mathbf{E}_K \cap \mathbf{N}_{K_n/K}(K_n^\times))}$  gives:

$$\mathbf{E}_K/\mathbf{E}_K \cap \mathbf{N}_{K_1/K}(K_1^\times) = 1 \quad \& \quad \mathbf{E}_K/\mathbf{E}_K \cap \mathbf{N}_{K_n/K}(K_n^\times) \simeq \mathbf{Z}/2\mathbf{Z} \quad \text{for } n \in \{2, 3\}.$$

**2.2.3. Conclusion about orders versus structures.** One may ask (for instance in the cubic case with  $p = 2$  to simplify), what happens in the tower  $L = K(\mu_\ell)$  if  $\mathcal{H}_K$  has a large  $p$ -rank and/or a large exponent and if we suppose the capitulation of  $\mathcal{H}_K$  (necessarily in a larger layer  $K_{n_0}$ )? The exact sequence (1.1) looks like:

$$1 \rightarrow \mathbf{Z}/2^{R_n}\mathbf{Z} \rightarrow \mathcal{H}_{K_n}^{G_n} \simeq \mathbf{Z}/2^{x_n}\mathbf{Z} \times \mathbf{Z}/2^{y_n}\mathbf{Z} \rightarrow \mathbf{Z}/2^{E_n}\mathbf{Z} \rightarrow 1, \quad R_n, E_n \geq 0,$$

since  $\mathcal{H}_{K_n}^{\text{ram}} \simeq \mathbf{Z}/2^{R_n}\mathbf{Z}$  and  $\mathbf{E}_K \cap \mathbf{N}_{K_n/K}(K_n^\times)/\mathbf{N}_{K_n/K}(\mathbf{E}_{K_n}) \simeq \mathbf{Z}/2^{E_n}\mathbf{Z}$  are  $\mathbf{Z}$ -monogenic. The Chevalley–Herbrand formula becomes  $4^{x_n+y_n} = 4^H \times 4^{n\rho_n}$ , where  $\#\mathcal{H}_K = 4^H$  ( $\rho_n = 0$  for all  $n$ , if  $r = 1$ , otherwise  $\rho_n$  increases up to some limit  $\rho_N$ ).

So,  $\mathcal{H}_{K_n}^{G_n}$  is at most the product of two  $\mathbf{Z}$ -monogenic components, possibly of large orders since  $x_n + y_n = R_n + E_n = H + n\rho_n$ . In the case  $r = 1$  where  $R_n = \rho_n = 0$ ,  $\mathcal{H}_{K_n}^{G_n} \simeq \mathbf{E}_K/\mathbf{N}_{K_n/K}(\mathbf{E}_{K_n})$  (with orders  $\#\mathcal{H}_K$ ).

The philosophy of such examples is that whatever the structure of  $\mathcal{H}_K$ , there is no obstruction for the relations between *orders and structures* of invariants associated

to the  $\mathcal{H}_{K_n}$ 's by means of the exact sequence (1.1) and the Chevalley–Herbrand formula, invariants whose *algebraic structures* have canonical limitations, especially in terms of  $p$ -ranks (more precisely,  $\mathbf{E}_K/\mathbf{N}_{K_n/K}(\mathbf{E}_{K_n})$  as monogenic  $\mathbb{Z}_p[\text{Gal}(K/\mathbb{Q})]$ -module and  $\mathcal{H}_{K_n}^{\text{ram}}$  of  $p$ -rank bounded by the number of ramified places except one).

In the case of a real abelian base field  $K$ , this is typical of the Main Conjecture philosophy due to the analytic framework giving only orders and not precise structures (see [47]).

**2.3. Decomposition of the algebraic norm  $\nu_{L/K} \in \mathbb{Z}[G]$ .** Let  $G$  be the cyclic group of order  $p^N$  and let  $\sigma$  be a generator of  $G$ . Put  $x := \sigma - 1$ ; then:

$$\nu_{L/K} = \sum_{i=0}^{p^N-1} \sigma^i = \sum_{i=0}^{p^N-1} (x+1)^i = \frac{(x+1)^{p^N} - 1}{x} = \sum_{i=1}^{p^N} \binom{p^N}{i} x^{i-1}.$$

We have the following elementary property which is perhaps known in Iwasawa's theory, but we have not found suitable references; see however Jaulent [55, IV.2 (b)], Washington [102, § 13.3] or Bandini–Caldarola [7, 18] for classical computations in the Iwasawa algebra  $\mathbb{Z}_p[[T]]$ :

**Theorem 2.9.** *The algebraic norm  $\nu_{L/K} = \sum_{i=0}^{p^N-1} \sigma^i \in \mathbb{Z}[G]$  is, for all  $k \in [1, p^N - 1]$ , of the form  $\nu_{L/K} = (\sigma - 1)^k \cdot A_k(\sigma - 1, p) + p^{f(k)} \cdot B_k(\sigma - 1, p)$ ,  $A_k, B_k \in \mathbb{Z}[\sigma - 1, p]$ , where  $f(k) = N - s$  if  $k \in [p^s, p^{s+1} - 1]$  for  $s \in [0, N - 1]$ .*

*Proof.* From:

$$\nu_{L/K} = \binom{p^N}{1} + x \binom{p^N}{2} + \cdots + x^{k-1} \binom{p^N}{k} + x^k \left[ \binom{p^N}{k+1} + x \binom{p^N}{k+2} + \cdots + x^{p^N-1-k} \binom{p^N}{p^N} \right],$$

we deduce that:  $A_k(x, p) = \binom{p^N}{k+1} + x \binom{p^N}{k+2} + \cdots + x^{p^N-1-k} \binom{p^N}{p^N}$ .

The computation of  $B_k(x, p)$  depends on the  $p$ -adic valuations of the  $\binom{p^N}{j}$ ,  $j \in [1, k]$ . To find the maximal factor  $p^{f(k)}$  dividing all the coefficients of the polynomial  $\binom{p^N}{1} + x \binom{p^N}{2} + \cdots + x^{k-1} \binom{p^N}{k}$ , in other words, to find the  $p$ -part of:

$$\gcd \left( \binom{p^N}{1}, \binom{p^N}{2}, \dots, \binom{p^N}{k} \right),$$

we consider  $s \in [0, N - 1]$ .

Let  $v$  be the  $p$ -adic valuation map.

**Lemma 2.10.** *One has  $v\left(\binom{p^N}{p^s}\right) = N - s$ ,  $\forall s \in [0, N - 1]$ .*

*Proof.* We have  $\binom{p^N}{p^s} = \frac{p^N!}{p^s! \cdot (p^N - p^s)!}$ ; then, using the well-known formula:

$$v(m!) = \frac{m - S(m)}{p - 1}, \quad m \geq 1,$$

where  $S(m)$  is the sum of the digits in the writing of  $m$  in base  $p$ , we get:

$$v(p^N!) = \frac{p^N - 1}{p - 1}, \quad v(p^s!) = \frac{p^s - 1}{p - 1}, \quad v((p^N - p^s)!) = \frac{p^N - p^s - (p - 1)(N - s)}{p - 1},$$

since  $p^N - p^s$  may be written  $p^s(p^{N-s} - 1)$  with:

$$p^{N-s} - 1 = 1(p - 1) + p(p - 1) + p^2(p - 1) + \cdots + p^{N-s-2}(p - 1) + p^{N-s-1}(p - 1),$$

giving  $N - s$  times the digit  $p - 1$ . Whence, for all  $s \in [0, N - 1]$ :

$$v\left(\binom{p^N}{p^s}\right) = \frac{1}{p - 1} (p^N - 1 - (p^s - 1) - (p^N - p^s - (p - 1)(N - s))) = N - s. \quad \square$$

**Lemma 2.11.** *For  $k \in [p^s + 1, p^{s+1} - 1]$ ,  $s \in [0, N - 1]$ , we have  $v\left(\binom{p^N}{k}\right) \geq N - s$ .*

*Proof.* Consider  $\binom{p^N}{k} \binom{p^N}{p^s}^{-1}$ ,  $k \in [p^s + 1, p^{s+1} - 1]$  to check that its valuation is non-negative (the interval is empty for  $p = 2$ ,  $s = 0$ , so, for  $p = 2$  we assume implicitly  $s > 0$ ). We have:

$$\begin{aligned} \frac{\binom{p^N}{k}}{\binom{p^N}{p^s}} &= \frac{p^N!}{k!(p^N - k)!} \times \frac{p^s!(p^N - p^s)!}{p^N!} = \frac{p^s!}{k!} \times \frac{(p^N - p^s)!}{(p^N - k)!} \\ &= \frac{1}{(p^s + 1)(p^s + 2) \cdots (p^s + (k - p^s))} \times \frac{(p^N - p^s)!}{(p^N - k)!} \\ &= \frac{(p^N - k + 1)(p^N - k + 2) \cdots (p^N - k + (k - p^s))}{(p^s + 1)(p^s + 2) \cdots (p^s + (k - p^s))}. \end{aligned}$$

Put  $k = p^s + h$ ,  $h \in [1, p^s(p - 1) - 1]$ ; then we can write:

$$\begin{aligned} \frac{\binom{p^N}{k}}{\binom{p^N}{p^s}} &= \frac{[p^N - (p^s + h) + 1][p^N - (p^s + h) + 2] \cdots [p^N - (p^s + h) + h]}{[p^s + 1][p^s + 2] \cdots [p^s + h]} \\ &= \frac{[p^N - (p^s + h) + 1][p^N - (p^s + h) + 2] \cdots [p^N - (p^s + h) + h]}{[p^s + h][p^s + h - 1] \cdots [p^s + h - (h - 1)]} \\ &= \frac{[p^N - (p^s + h) + 1]}{[(p^s + h) - 1]} \frac{[p^N - (p^s + h) + 2]}{[(p^s + h) - 2]} \cdots \frac{[p^N - (p^s + h) + (h - 1)]}{[(p^s + h) - (h - 1)]} \\ &\quad \times \frac{p^N - p^s}{p^s + h}. \end{aligned}$$

We remark that each factor of the form  $\frac{p^N - [(p^s + h) - j]}{[(p^s + h) - j]}$  is a  $p$ -adic unit for  $j \in [1, h - 1]$ ; indeed, one sees that  $(p^s + h) - j \leq p^{s+1} - 2$  with  $s + 1 \leq N$ , whence  $v_p((p^s + h) - j) \leq N - 1$ .

Now, consider the remaining factor  $\frac{p^N - p^s}{p^s + h} = \frac{p^s(p^{N-s} - 1)}{p^s + h} = \frac{p^s}{p^s + h}$ , up to a  $p$ -adic unit since  $s \in [0, N - 1]$ . As  $h \leq p^s(p - 1) - 1$ , one can put  $h = \lambda p^u$ ,  $p \nmid \lambda$ ,  $u \leq s$ ; the case  $u < s$  is obvious and gives a positive valuation; if  $u = s$ , the relation  $h \leq p^s(p - 1) - 1$  implies  $\lambda \leq p - 2$ , thus  $p^s + h = p^s(1 + \lambda)$  with  $1 + \lambda \leq p - 1$  and  $\frac{p^s}{p^s + h}$  is, in this case, a  $p$ -adic unit, whence the lemma.  $\square$

This leads to the expression of  $f(k)$  on  $\bigcup_{s=0}^{N-1} [p^s, p^{s+1} - 1] = [1, p^N - 1]$  and to the proof of the theorem.  $\square$

The following corollary, proving Theorems 1.1, 1.2 and generalizations, is of easy use in practice; we assume, to simplify, that  $L/K$  is totally ramified:

**Corollary 2.12.** *Let  $L/K$  be any totally ramified cyclic  $p$ -extension of degree  $p^N$ ,  $N \geq 1$ , of Galois group  $G = \langle \sigma \rangle$ . Let  $m(L)$  be the minimal integer such that  $(\sigma - 1)^{m(L)}$  annihilates  $\mathcal{H}_L$  and let  $p^{e(L)}$  be the exponent of  $\mathcal{H}_L$ .*

**a)** *Then a sufficient condition of complete capitulation of  $\mathcal{H}_K$  in  $L$  is that  $e(L) \in [1, N - s(L)]$  if  $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$  for  $s(L) \in [0, N - 1]$ .*

*A class  $h \neq 1$  of  $\mathcal{H}_K$  capitulates in  $L$  as soon as  $h =: \mathbf{N}_{L/K}(h')$ , where  $h'$  is of order  $p^e$  and annihilated by  $(\sigma - 1)^m$  such that  $e \in [1, N - s]$  if  $m \in [p^s, p^{s+1} - 1]$  for  $s \in [0, N - 1]$ .*

**b)** *For  $t \geq 1$ , put  $\overline{\mathcal{H}}_L := \mathcal{H}_L / \mathcal{H}_L^{p^t}$  and let  $\overline{m}(L)$ ,  $\overline{e}(L)$ , be the corresponding parameters for which  $\overline{m}(L) \leq m(L)$  and  $\overline{e}(L) \leq e(L)$ ; then  $\overline{\mathcal{H}}_K$  capitulates in  $L$  as soon as  $\overline{e}(L) \in [1, N - \overline{s}(L)]$  if  $\overline{m}(L) \in [p^{\overline{s}(L)}, p^{\overline{s}(L)+1} - 1]$  for  $\overline{s}(L) \in [0, N - 1]$ .*

**c)** *Stability of the  $\overline{\mathcal{H}}_{K_n}$ 's in the tower  $L = \bigcup_{n=0}^N K_n$  holds as soon as  $\#\overline{\mathcal{H}}_{K_1} = \#\overline{\mathcal{H}}_K$  and leads to  $\overline{m}(K_n) = 1$ ,  $\overline{s}(K_n) = 0$ ,  $\overline{e}(K_n) = \overline{e}(K)$  for all  $n$ .*

*Proof.* If  $h' \in \mathcal{H}_L$ ,  $\nu_{L/K}(h') = \mathbf{J}_{L/K}(\mathbf{N}_{L/K}(h')) = (h'^{(\sigma-1)^k})^A \times (h'^{p^f(k)})^B$ ,  $\forall k \in [1, p^N - 1]$ . Thus,  $\mathbf{J}_{L/K}$  is non-injective as soon as  $h$  fulfills the conditions stated



in **(b)** with  $k = m \in [p^s, p^{s+1} - 1]$ ,  $s \in [0, N - 1]$ , and  $f(k) = N - s \geq e$ . For **(a)** on the triviality of  $\nu_{L/K}(\mathcal{H}_L)$ , it suffices that  $m = m(L)$ ,  $s = s(L)$  and  $e = e(L)$  be solution. One obtains Theorem 1.1.

The case **(c)** of quotients is immediate; their capitulation is, a priori, “easier” and means that any ideal  $\mathfrak{a}$  of  $K$  becomes of the form  $(\mathfrak{a})_L = (\alpha) \cdot \mathfrak{A}^{p^t}$ , where  $\mathfrak{A}$  is an ideal of  $L$  and  $\alpha \in L^\times$ . The case  $t = 1$  of stability property gives the stability of the  $p$ -ranks.  $\square$

In other words, if the length  $m(L)$  of the filtration is not too large as well as the exponent  $p^{e(L)}$  of  $\mathcal{H}_L$ , then we obtain  $\nu_{L/K}(h) = 1$  for all  $h \in \mathcal{H}_L$  (or at least for some), whence complete (or partial) capitulation of  $\mathcal{H}_K$  in  $L$ .

Another way to interpret this result is to say that if  $N$  is large enough and if the Galois complexity of the  $p$ -class groups  $\mathcal{H}_{K_n}$  does not increase too much, then  $\mathcal{H}_K$  capitulates in  $L$ . For this, we may introduce the following definition:

**Definition 2.13.** *Let  $L/K$  be a cyclic  $p$ -extension totally ramified of degree  $p^N > 1$  and Galois group  $G =: \langle \sigma \rangle$  (we do not assume that  $L/\mathbb{Q}$  is Galois). Let  $\mathcal{H}_K, \mathcal{H}_L$  be the  $p$ -class groups of  $K, L$ , respectively. Denote by  $p^{e(L)}$  the exponent of  $\mathcal{H}_L$  and by  $m(L)$  the length of the filtration  $\{\mathcal{H}_L^i\}_{i \geq 0}$  (i.e., the least integer  $m(L)$  such that  $(\sigma - 1)^{m(L)}$  annihilates  $\mathcal{H}_L$ ). We will say that  $L/K$  is of smooth complexity when  $e(L) \leq N - s(L)$  if  $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$  for  $s(L) \in [0, N - 1]$ .*

2.3.1. *The Furtwängler property.* This property is the strong equality:

$$\text{Ker}_{\mathcal{H}_L}(\mathbf{N}_{L/K}) = \mathcal{H}_L^{\sigma-1};$$

it is equivalent to say that the genus field of  $L/K$  is  $LH_K^{\text{mf}}$ , then obviously equivalent to  $\#\mathcal{H}_L^G = \#\mathcal{H}_K$ , thus to the triviality of the norm factor in the Chevalley–Herbrand formula (e.g., case  $r = 1$ ). We have discovered some applications in Bembom’s thesis, about the Galois structure of  $\mathcal{H}_L$  via its filtration and the problem of capitulation (see, e.g., [12, §2.6, Theorem 2.6.3; §2.8, Theorem 2.8.9]). Under the Furtwängler property, Nakayama’s Lemma gives immediately:

**Proposition 2.14.** *Let  $L/K$  be a totally ramified cyclic  $p$ -extension such that  $\#\mathcal{H}_L^G = \#\mathcal{H}_K$ . From  $\mathcal{H}_K = \bigoplus_j \langle h_j \rangle$  set  $\mathcal{H}_L' := \bigoplus \langle h'_j \rangle$ ,  $h'_j \in \mathcal{H}_L$  be such that  $h_j = \mathbf{N}_{L/K}(h'_j)$ ,  $\forall i$ . Then  $\mathcal{H}_L$  is generated by  $\mathcal{H}_L'$  as  $\mathbb{Z}_p[\sigma - 1]$ -module and any  $h' \in \mathcal{H}_L$  writes  $h' = \prod_j h'_j \omega_j$ , where  $\omega_j = \sum_{i=0}^{m(L)-1} a_{i,j} (\sigma - 1)^i$ ,  $a_{i,j} \in \mathbb{Z}_p$ .*

2.3.2. *Program giving the decompositions of  $\nu_{L/K}$ .* The following program put  $\nu_{L/K}$  in the form  $P(x, p) = x^k A(x, p) + p^{f(k)} B(x, p)$ ,  $1 \leq k \leq p^N - 1$ , where  $x = \sigma - 1$  and  $p$  are considered as indeterminate variables. This is necessary to have universal expressions for any  $\mathcal{H}_L$  as  $\mathbb{Z}_p[G]$ -module; in other words, we do not reduce modulo  $p$  the coefficients of  $A$  and  $B$ . One must note that, except the cases  $k = 1$ ,  $B = 1$  and  $k = p^N - 1$ ,  $A = 1$ ,  $A$  and  $B$  are not invertible in the group algebra, which allows improvements of the standard reasoning of annihilation with  $((\sigma - 1)^k, p^{f(k)})$ . One must precise the numerical prime number  $p$  in Prime and  $N$  in N:

```
PROGRAM OF DECOMPOSITION OF THE ALGEBRAIC NORM
{Prime=2;N=3;P=0;for(i=1,Prime^N,C=binomial(Prime^N,i);
v=valuation(C,Prime);c=C/Prime^v;P=P+c*p^v*x^(i-1));print("P=",P);
for(k=1,Prime^N-1,B=lift(Mod(P,x^k));w=valuation(B,p);print();
print("P=x^",k,".A+p^",w,".B");print("A=", (P-B)/x^k);print("B=",B/p^w)}}

```

Case p=2, N=1, P=x+p\$

P=x+p

A=1 B=1

Case p=3, N=1, P=x^2+p\*x+p

P=x^1.A+p^1.B

P=x^2.A+p^1.B

A=x+p B=1

A=1 B=x+1

Case  $p=2$ ,  $N=2$ ,  $P=x^3+p^2x^2+3px+p^2$   
 $P=x^1.A+p^2.B$                        $P=x^2.A+p^1.B$                        $P=x^3.A+p^1.B$   
 $A=x^2+p^2x+3p$   $B=1$                        $A=x+p^2$   $B=3x+p$                        $A=1$   $B=p^2+3x+p$

For the next examples, we only write  $P$ .

Case  $p=3$ ,  $N=2$ :  
 $P=x^8+p^2x^7+4p^2x^6+28p^2x^5+14p^2x^4+14p^2x^3+28p^2x^2+4p^2x+p^2$

Case  $p=2$ ,  $N=3$ :  
 $P=x^7+p^3x^6+7p^3x^5+7p^3x^4+35p^3x^3+7p^3x^2+7p^2x+p^3$

Case  $p=2$ ,  $N=4$ :  
 $P=x^{15}+p^4x^{14}+15p^4x^{13}+35p^4x^{12}+455p^4x^{11}+273p^4x^{10}$   
 $+1001p^4x^9+715p^4x^8+6435p^4x^7+715p^4x^6+1001p^4x^5$   
 $+273p^4x^4+455p^4x^3+35p^4x^2+15p^4x+p^4$

Thus, as soon as  $\mathcal{V}_{L/K} = (\sigma - 1)^k \times A_k(\sigma - 1, p) + p^{f(k)} \times B_k(\sigma - 1, p)$ , with  $k \geq m(L)$  and  $f(k) \geq e(L)$ , then  $\mathcal{H}_K$  capitulates in  $L$ . But the reciprocal does not hold and capitulation may occur whatever  $m(L)$  and  $e(L)$ . Indeed, if  $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$  for  $s(L) \in [0, N - 1]$ , with  $e(L) > N - s(L)$  instead of  $e(L) \leq N - s(L)$ , reducing  $P$  modulo the ideal  $(x^{m(L)}, p^{e(L)})$ , the capitulation of  $\mathcal{H}_K$  is equivalent to the annihilation of  $\mathcal{H}_L$  by an explicit polynomial of the form:

$$\sum_{j=1}^{j+h} a_j p^{e_j} x^{m_j}, \quad p \nmid a_j, \quad e_j \in [0, e(L) - 1], \quad m_j \in [0, m(L) - 1],$$

which may hold by accident depending on the numerical data of the filtration; we give such a case in Example 2.15 (ii). We will also give examples where the above property of  $\mathcal{V}_{L/K}$  applies, apart from the obvious case of stability.

**2.4. Illustrations using the decomposition of  $\mathcal{V}_{L/K}$ .** In what follows, the cyclic  $p$ -extensions  $L/K$  are always totally ramified.

**Example 2.15.** Let's begin with an example where the structure of  $\mathcal{H}_{K_n}$  grows sufficiently with  $n$ , giving no capitulation of  $\mathcal{H}_K$  up to  $n = 3$ ; next, another choice of  $\ell$  leads to capitulation of  $\mathcal{H}_K$ , but where Theorem 1.1 does not apply.

(i) We consider the cyclic cubic field  $K$  of conductor  $f = 703$ ,  $p = 2$  and  $\ell = 17$ . Then  $\mathcal{H}_K \simeq \mathbf{Z}/2\mathbf{Z}$  and we get, from Program 6 (several hours for the level  $n = 3$ ):

```
p=2 Nn=3 f=703 PK=x^3+x^2-234*x-729 mKn=3 CK0=[6,2] e11=17 r=1
CK1=[12,4]=[4,4]
h_1^[(S-1)^1]=[0,2] h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
h_1^[(S-1)^3]=[0,0] h_2^[(S-1)^3]=[0,0]
norm in K1/K of the component 1 of CK1:[2,2]
norm in K1/K of the component 2 of CK1:[2,0]
CK2=[24,8]=[8,8]
h_1^[(S-1)^1]=[0,2] h_2^[(S-1)^1]=[6,6]
h_1^[(S-1)^2]=[4,4] h_2^[(S-1)^2]=[4,0]
h_1^[(S-1)^3]=[0,0] h_2^[(S-1)^3]=[0,0]
norm in K2/K of the component 1 of CK2:[4,4]
norm in K2/K of the component 2 of CK2:[4,0]
CK3=[48,16]=[16,16]
h_1^[(S-1)^1]=[8,10] h_2^[(S-1)^1]=[10,14]
h_1^[(S-1)^2]=[4,12] h_2^[(S-1)^2]=[12,8]
norm in K3/K of the component 1 of CK3:[8,8]
norm in K3/K of the component 2 of CK3:[8,0]
```

For  $n = 1$ , we have  $m(K_1) = 2$ ,  $s(K_1) = 1$ ,  $n - s(K_1) = 0$  and  $e(K_1) = 2 > 0$ .

For  $n = 2$ , we have  $m(K_2) = 3$ ,  $s(K_2) = 1$ ,  $n - s(K_2) = 1$  and  $e(K_2) = 3 > 1$ .

For  $n = 3$ , we have  $m(K_3) = 4$ ,  $s(K_3) = 2$ ,  $n - s(K_3) = 1$  and  $e(K_3) = 4 > 1$ .

In this case the complexity (non smooth) increases due to successive exponents 2, 4, 8, 16. Nevertheless  $\mathcal{H}_K = \mathcal{H}_K / \mathcal{H}_K^2$  capitulates in  $K_1$  because  $\bar{m}(K_1) = 1$  since  $h_1^{[(S-1)^1]} = [0, 2]$ ,  $h_2^{[(S-1)^1]} = [2, 2]$  ( $\bar{s}(K_1) = 0$ ), then  $\bar{e}(K_1) = 1 \leq n - \bar{s}(K_1)$ .

(ii) Changing  $\ell = 17$  into  $\ell = 97$  ( $N = 4$ ) gives complete capitulation in  $K_2$  because of a smooth complexity, but higher than a stability in  $K_2/K$  (a stability begins at  $n_0 = 2$ ):

```
p=2 Nn=2 f=703 PK=x^3+x^2-234*x-729 CK0=[6,2] e11=97 r=1
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,0,0] h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,0,1,1] h_4^[(S-1)^1]=[0,0,1,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,0,0]
norm in K1/K of the component 2 of CK1:[1,1,0,0]
norm in K1/K of the component 3 of CK1:[0,0,1,1]
norm in K1/K of the component 4 of CK1:[0,0,1,1]
No capitulation, m(K1)=2, e(K1)=1
CK2=[12,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,1,1] h_2^[(S-1)^1]=[0,2,1,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,2,0,0] h_2^[(S-1)^2]=[2,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0] h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0] h_4^[(S-1)^3]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2
CK3=[12,4,2,2]=[4,4,2,2] (a day of computation)
```

For  $n = 2$ ,  $P = x^3 + 2^2 * x^2 + 3 * 2 * x + 2^2$ ,  $m(K_2) = 3$ ,  $s(K_2) = 1$  and  $e(K_2) = 2 > n - s(K_2) = 1$ ; so the property deduced from the use of  $\mathcal{V}_{K_2/K}$  does not hold. Then, modulo the ideal  $(x^3, 2^2)$ , it follows that  $3 * 2 * x$  must annihilate  $\mathcal{H}_{K_2}$ , which is confirmed by the data since  $h_j^{2(\sigma_2-1)} = 1$ ,  $1 \leq j \leq 4$ . The stability from  $K_2$  gives a smooth complexity of the extension  $L/K$  and confirm the capitulation.

So, Theorem 1.1 gives a sufficient condition of capitulation, not necessary, but some information remains when the condition is not fulfilled.

**Example 2.16.** We consider the cyclic cubic field  $K$  of conductor  $f = 1777$ ,  $p = 2$  and  $\ell = 17$ . Then  $\mathcal{H}_K \simeq \mathbf{Z}/4\mathbf{Z}$  is of exponent 4.

```
p=2 Nn=3 f=1777 PK=x^3+x^2-592*x+724 CK0=[4,4] e11=17 r=3
CK1=[8,8]
h_1^[(S-1)^1]=[0,0] h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation, m(K1)=1, e(K1)=3
CK2=[8,8]
h_1^[(S-1)^1]=[0,0] h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Incomplete capitulation, m(K2)=1, e(K2)=3
CK3=[8,8]
h_1^[(S-1)^1]=[0,0] h_2^[(S-1)^1]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Complete capitulation, m(K3)=1, e(K3)=3
```

In this case, the stability from  $K_1$  implies necessarily the capitulation in  $K_3$  (so the third computation is for checking). Moreover, for all  $n \geq 1$ ,  $\mathcal{H}_{K_n}$  is annihilated by  $\sigma - 1$  and  $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n}$  as expected from Theorem 1.2 (i) and given by Program 2.1.3. Whence  $m(K_n) = 1$ ,  $s(K_n) = 0$ ,  $e(K_n) = 3$  giving  $e(K_n) \leq n - s(K_n)$  only from  $n = 3$ . Note that  $\mathcal{H}_K/\mathcal{H}_K^2$  capitulates in  $K_1$  and that, considering the quotients  $\overline{\mathcal{H}}_{K_n} := \mathcal{H}_{K_n}/\mathcal{H}_{K_n}^4$ , then  $\overline{\mathcal{H}}_K = \mathcal{H}_K/\mathcal{H}_K^4$  capitulates only from  $K_2$ .

**Example 2.17.** (i) We consider the quadratic field  $\mathbb{Q}(\sqrt{142})$  with  $p = 3$  and various primes  $\ell \equiv 1 \pmod{3}$ ,  $\ell \not\equiv 1 \pmod{9}$ , so that  $N = 1$ ,  $L = K_1$ , and the sufficient conditions of Theorem 1.1 are  $e(K_1) = 1 \leq 1 - s(K_1)$ , whence  $s(K_1) = 0$  and  $m(K_1) \in [1, 2]$ :

```
p=3 PK=x^2-142 N=1 CK0=[3] e11=13 r=2
CK1=[3,3]
h_1^[(S-1)^1]=[0,0] h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=1, e(K1)=1

p=3 PK=x^2-142 N=1 CK0=[3] e11=1123 r=2
CK1=[21,3]=[3,3]
h_1^[(S-1)^1]=[1,2] h_2^[(S-1)^1]=[1,2]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=2, e(K1)=1

p=3 PK=x^2-142 N=1 CK0=[3] e11=208057 r=2
CK1=[3,3,3,3]
h_1^[(S-1)^1]=[2,2,0,0] h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,1,1,1] h_4^[(S-1)^1]=[2,1,2,2]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=2, e(K1)=1
```

(ii) For  $p = 5$  and  $N = 1$  the conditions become  $e(K_1) = 1 \leq 1 - s(K_1)$ , whence  $s(K_1) = 0$ , with  $m(K_1) \in [1, 4]$ , which offers more possibilities:

```
p=5 PK=x^2-401 N=1 CK0=[5] e11=1231 r=2
CK1=[5,5]
h_1^[(S-1)^1]=[0,0] h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=1, e(K1)=1

p=5 PK=x^2-401 N=1 CK0=[5] e11=1741 r=1
CK1=[5,5]
h_1^[(S-1)^1]=[3,3] h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=2, e(K1)=1

p=5 PK=x^2-401 CK0=[5] e11=4871 r=1
CK1=[10,10,10,2]=[5,5,5]
h_1^[(S-1)^1]=[4,0,4,0] h_2^[(S-1)^1]=[1,4,0,0] h_3^[(S-1)^1]=[3,4,2,0]
h_1^[(S-1)^2]=[3,1,4,0] h_2^[(S-1)^2]=[3,1,4,0] h_3^[(S-1)^2]=[2,4,1,0]
h_1^[(S-1)^3]=[0,0,0,0] h_2^[(S-1)^3]=[0,0,0,0] h_3^[(S-1)^3]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=3, e(K1)=1
```

**Example 2.18.** We consider the cubic field of conductor  $f = 20887$  with  $p = 2$  and  $\ell = 17$  totally split:

```
p=2 Nn=2 f=20887 PK=x^3+x^2-6962*x-225889 CK0=[4,4,2,2] e11=17 r=3
CK1=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
```

$h_3^{\wedge}[(S-1)^{\wedge}1]=[0,0,0,0]$      $h_4^{\wedge}[(S-1)^{\wedge}1]=[0,0,0,0]$   
 norm in  $K_1/K$  of the component 1 of CK1:  $[2,0,0,0]$   
 norm in  $K_1/K$  of the component 2 of CK1:  $[0,2,0,0]$   
 norm in  $K_1/K$  of the component 3 of CK1:  $[0,0,0,0]$   
 norm in  $K_1/K$  of the component 4 of CK1:  $[0,0,0,0]$   
 Incomplete capitulation,  $m(K_1)=1$ ,  $e(K_1)=3$

We note that  $\mathbf{N}_{K_1/K}(h_j) \neq 1$  for  $j = 3, 4$ , otherwise  $\mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}) = \mathcal{H}_K$  would be of 2-rank 2 (absurd). Since  $m(K_1) = 1$  (all classes are invariant), Theorem 1.1 applies non-trivially for the classes  $h_3, h_4$  of order 2 ( $m = 1$ ,  $s = 0$ ,  $e \in [1, 1]$ , which is indeed the case). Let's give the complete data checking the capitulation of the two classes of  $K$  of order 2; the instruction CK0 = K.clgp gives:

$[64, [4, 4, 2, 2], [[2897, 2889, 2081; 0, 1, 0; 0, 0, 1], [2897, 825, 2889; 0, 1, 0; 0, 0, 1],$   
 $[17, 16, 13; 0, 1, 0; 0, 0, 1], [53, 36, 44; 0, 1, 0; 0, 0, 1]]]$

it describes  $\mathcal{H}_K$  with 4 representative ideals of generating classes; that of order 2 are  $\mathfrak{a}_3 = [17, 16, 13; 0, 1, 0; 0, 0, 1]$ ,  $\mathfrak{a}_4 = [53, 36, 44; 0, 1, 0; 0, 0, 1]$ ; the following 6 large coefficients on the integral basis give integers  $\alpha_i \in L^\times$  with the relations  $(\mathfrak{a}_i)_L = (\alpha_i)$ :

$[[0, 0, 0, 0], [4482450896, -1173749328, 81969609, 69123722, 7646555, 39729395]]$   
 norm in  $K_1/K$  of the component 3 of CK1:  $[0, 0, 0, 0]$   
 $[[0, 0, 0, 0], [-4877380814, 1968946273, -1411818, 102996743, 38571732, 40207952]]$   
 norm in  $K_1/K$  of the component 4 of CK1:  $[0, 0, 0, 0]$

At the level  $n = 2$ , the result is similar, but shows that the classes of order 4 of  $\mathcal{H}_K$  never capitulate:

CK2= $[16, 16, 2, 2]$   
 $h_1^{\wedge}[(S-1)^{\wedge}1]=[8, 0, 0, 0]$      $h_2^{\wedge}[(S-1)^{\wedge}1]=[0, 8, 0, 0]$   
 $h_3^{\wedge}[(S-1)^{\wedge}1]=[0, 0, 0, 0]$      $h_4^{\wedge}[(S-1)^{\wedge}1]=[0, 0, 0, 0]$   
 $h_1^{\wedge}[(S-1)^{\wedge}2]=[0, 0, 0, 0]$      $h_2^{\wedge}[(S-1)^{\wedge}2]=[0, 0, 0, 0]$   
 $h_3^{\wedge}[(S-1)^{\wedge}2]=[0, 0, 0, 0]$      $h_4^{\wedge}[(S-1)^{\wedge}2]=[0, 0, 0, 0]$   
 norm in  $K_2/K$  of the component 1 of CK2:  $[4, 0, 0, 0]$   
 norm in  $K_2/K$  of the component 2 of CK2:  $[0, 4, 0, 0]$   
 norm in  $K_2/K$  of the component 3 of CK2:  $[0, 0, 0, 0]$   
 norm in  $K_2/K$  of the component 4 of CK2:  $[0, 0, 0, 0]$   
 Incomplete capitulation,  $m(K_2)=2$ ,  $e(K_2)=4$

### 3. ARITHMETIC INVARIANTS THAT DO NOT CAPITULATE

The non capitulation of  $p$ -class groups  $\mathcal{H}_K$  in cyclic  $p$ -extensions  $L/K$  implies necessarily, as we have seen, that the structure of  $\mathcal{H}_L$  does not allow the previous use of the algebraic norm (Theorem 1.1) and the complexity is not smooth according to Definition 2.13; that is to say, either  $m(K_n) \geq p^n$  or else  $m(K_n) \in [p^s, p^{s+1} - 1]$  for  $s \in [0, n - 1]$  but in that case,  $e(K_n) > n - s$ . This may be checked by means of a wider framework as follows:

**3.1. Injective transfers and arithmetic consequences.** Let  $p$  be any prime number and let  $\mathcal{K}$  be a family of number fields  $k$  stable by taking subfields.

**Definition 3.1.** Let  $\{\mathcal{X}_k\}_{k \in \mathcal{K}}$  be a family of finite invariants of  $p$ -power order, indexed by the set  $\mathcal{K}$ , fulfilling the following conditions for  $k, k' \in \mathcal{K}$ ,  $k \subseteq k'$ :

- (i) For any Galois extension  $k'/k$ , of Galois group  $G$ ,  $\mathcal{X}_{k'}$  is a  $\mathbb{Z}_p[G]$ -module.
- (ii) There exist norms  $\mathbf{N}_{k'/k} : \mathcal{X}_{k'} \rightarrow \mathcal{X}_k$  and transfers  $\mathbf{J}_{k'/k} : \mathcal{X}_k \rightarrow \mathcal{X}_{k'}$ , such that  $\mathbf{J}_{k'/k} \circ \mathbf{N}_{k'/k} = \mathcal{V}_{k'/k}$ .
- (iii) If  $G = \text{Gal}(k'/k)$  is a cyclic  $p$ -group, we define the associated filtration  $\{\mathcal{X}_{k'}^i\}_{i \geq 0}$  by  $\mathcal{X}_{k'}^{i+1} / \mathcal{X}_{k'}^i := (\mathcal{X}_{k'} / \mathcal{X}_{k'}^i)^G$ ,  $\forall i \geq 0$ .

Thus, for a cyclic  $p$ -extension  $L/K$ ,  $L, K \in \mathcal{K}$ , let  $m(L)$  be the length of the filtration; the condition  $e(L) \in [1, N - s(L)]$  if  $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$  for  $s(L) \in [0, N - 1]$ , of Theorem 1.1, applies in the same way, independently of the fact of being able to calculate the orders of the  $\mathcal{X}_L^i$ 's by means of a suitable algorithm.

**Theorem 3.2.** *Let  $L/K$ ,  $K, L \in \mathcal{K}$ , be a cyclic  $p$ -extension of degree  $p^N$ , let  $K_n$  be the subfield of  $L$  of degree  $p^n$  over  $K$ ,  $n \in [0, N]$  and to simplify, put  $\mathcal{X}_n := \mathcal{X}_{K_n}$ . We assume that, for all  $n \in [0, N-1]$ , the arithmetic norms  $\mathcal{X}_{n+1} \rightarrow \mathcal{X}_n$  are surjective and that the transfer maps  $\mathcal{X}_n \rightarrow \mathcal{X}_{n+1}$  are injective. Let  $p^{e_n}$  be the exponent of  $\mathcal{X}_n$ .*

*Then  $\#\mathcal{X}_{n+h} \geq \#\mathcal{X}_n \cdot \#\mathcal{X}_n[p^h] \geq \#\mathcal{X}_n \cdot p^{\min(e_n, h)}$ , for all  $n \in [0, N]$  and all  $h \in [0, N-n]$ , where  $\mathcal{X}_n[p^h] := \{x \in \mathcal{X}_n, x^{p^h} = 1\}$ .*

*Proof.* Put  $G_n^{n+h} := \text{Gal}(k_{n+h}/k_n)$  and in the same way for  $\mathbf{N}_n^{n+h}$ ,  $\mathbf{J}_n^{n+h}$ . From the exact sequence  $1 \rightarrow \mathbf{J}_n^{n+h} \mathcal{X}_n \rightarrow \mathcal{X}_{n+h} \rightarrow \mathcal{X}_{n+h}/\mathbf{J}_n^{n+h} \mathcal{X}_n \rightarrow 1$ , we get:

$$\begin{aligned} 1 \rightarrow \mathcal{X}_{n+h}^{G_n^{n+h}} / \mathbf{J}_n^{n+h} \mathcal{X}_n &\rightarrow (\mathcal{X}_{n+h} / \mathbf{J}_n^{n+h} \mathcal{X}_n)^{G_n^{n+h}} \\ &\rightarrow \text{H}^1(G_n^{n+h}, \mathbf{J}_n^{n+h} \mathcal{X}_n) \rightarrow \text{H}^1(G_n^{n+h}, \mathcal{X}_{n+h}), \end{aligned}$$

where  $\text{H}^1(G_n^{n+h}, \mathbf{J}_n^{n+h} \mathcal{X}_n) = (\mathbf{J}_n^{n+h} \mathcal{X}_n)[p^h] \simeq \mathcal{X}_n[p^h]$  (injectivity of  $\mathbf{J}_n^{n+h}$ ),

$$\#\text{H}^1(G_n^{n+h}, \mathcal{X}_{n+h}) = \#\text{H}^2(G_n^{n+h}, \mathcal{X}_{n+h}) = \#(\mathcal{X}_{n+h}^{G_n^{n+h}} / \mathbf{J}_n^{n+h} \mathcal{X}_n),$$

since  $\nu_n^{n+h} \mathcal{X}_{n+h} = \mathbf{J}_n^{n+h} \circ \mathbf{N}_n^{n+h} \mathcal{X}_{n+h} = \mathbf{J}_n^{n+h} \mathcal{X}_n$  (surjectivity of  $\mathbf{N}_n^{n+h}$ ), giving an exact sequence of the form:

$$1 \rightarrow A \rightarrow (\mathcal{X}_{n+h} / \mathbf{J}_n^{n+h} \mathcal{X}_n)^{G_n^{n+h}} \rightarrow \mathcal{X}_n[p^h] \rightarrow A', \text{ with } \#A' = \#A.$$

We then obtain the inequality  $\#\mathcal{X}_{n+h} \geq \#\mathcal{X}_n \cdot \#\mathcal{X}_n[p^h]$ , whence the results.  $\square$

**Corollary 3.3.** *The  $n$ -sequence  $\#\mathcal{X}_n$  stabilizes from some  $n_0 \in [0, N-1]$  if and only if  $\mathcal{X}_n = 1$ , for all  $n \in [0, N]$ .*

*In an Iwasawa's theory context with  $\mu = 0$  and  $\lambda > 0$ , let  $p^{e_n}$  (resp.  $r_n$ ) be the exponent (resp. the  $p$ -rank) of  $\mathcal{X}_n$ ,  $n \in [0, N]$ . Then  $r_n$  is a constant  $r$ ,  $\forall n \gg 0$  and  $e_n \rightarrow \infty$  with  $n$ ; in particular, if  $\lambda = \mu = 0$ , then  $\mathcal{X}_n = 1$ , for all  $n \geq 0$ .*

*Proof.* The stability from  $n_0$  means  $\mathcal{X}_{n_0}[p] = 1$ , then  $\mathcal{X}_{n_0} = 1$  and  $\mathcal{X}_n = 1$ , for all  $n \in [n_0, N]$ ; the surjectivity of the norms implies  $\mathcal{X}_n = 1$ , for all  $n \in [0, n_0]$ .

If  $\mu = 0$  in the formula  $\#\mathcal{X}_n = p^{\lambda n + \mu p^{n+\nu}}$ , for  $n \gg 0$ , the relation  $\#\mathcal{X}_{n+1} \geq \#\mathcal{X}_n \cdot \mathcal{X}_n[p]$  implies  $r_n \leq \lambda$ ; since  $r_n$  is increasing (injectivity of the transfers),  $r_n = r$ ,  $\forall n \gg 0$ ; since  $\#\mathcal{X}_n \leq p^{r e_n}$ , one gets  $\lambda n + \nu \leq r e_n$  proving that  $e_n \rightarrow \infty$  with  $n$ . If  $\lambda = \mu = 0$ , then  $\#\mathcal{X}_n$  is constant for  $n \gg 0$ , whence  $\mathcal{X}_n = 1$ ,  $\forall n \geq 0$ .  $\square$

We remark that if  $r_n$  is unbounded, necessarily  $\mu > 0$ . But all of these results are based on strong assumptions avoiding any capitulation. So, Theorem 1.1 does not apply since the complexity of the  $\mathcal{X}_n$ 's crucially increases with  $n$ . We shall give two examples of such families.

**3.2.  $p$ -class groups of imaginary quadratic field.** If  $K$  is an imaginary quadratic field and  $L = L_0 K$ ,  $L_0/\mathbb{Q}$  real cyclic of degree  $p^N$ , we know that there is never capitulation of  $\mathcal{H}_K \neq 1$ . Let's give two numerical examples, and use Program 8 given further:

**Example 3.4.** Consider  $K = \mathbb{Q}(\sqrt{-199})$ ,  $p = 3$ ,  $\ell = 19$ , inert in  $K$ :

```
p=3 Nn=2 PK=x^2+199 CK0=[9] e11=19 r=1
CK1=[513]=[27]
h_1^[(S-1)^1]=[9] h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=2, e(K1)=3
CK2=[749493,19,19]=[81]
h_1^[(S-1)^1]=[45,0,0] h_1^[(S-1)^2]=[0,0,0]
norm in K2/K of the component 1 of CK2:[9]
No capitulation, m(K2)=2, e(K2)=4
```



(i) Case  $n = 1$ . We have  $\mathcal{H}_{K_1} \simeq \mathbb{Z}/3^3\mathbb{Z}$ ,  $m(K_1) = 2$  ( $s(K_1) = 1$ ) and one obtains  $e(K_1) = 3 > n - s(K_1) = 0$ . We get the equality  $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K \cdot \#\mathcal{H}_K[3]$ .

(ii) Case  $n = 2$ . Then  $\mathcal{H}_{K_2} \simeq \mathbb{Z}/3^4\mathbb{Z}$ ,  $m(K_2) = 2$  ( $s(K_2) = 1$ ) and one obtains  $e(K_2) = 4 > n - s(K_2) = 1$ . We have  $\#\mathcal{H}_{K_2} = \#\mathcal{H}_{K_1} \cdot \#\mathcal{H}_{K_1}[3] = \#\mathcal{H}_K \cdot \#\mathcal{H}_K[3^2]$ .

**Example 3.5.** We consider  $K = \mathbb{Q}(\sqrt{-199})$ ,  $\ell = 37$ , inert in  $K$ .

```
p=3 Nn=2 PK=x^2+199 CK0=[9] ell=37 r=1
CK1=[54,6,3]=[27,3,3]
h_1^[(S-1)^1]=[21,1,0] h_2^[(S-1)^1]=[18,0,1] h_3^[(S-1)^1]=[18,0,0]
h_1^[(S-1)^2]=[0,0,1] h_2^[(S-1)^2]=[18,0,0] h_3^[(S-1)^2]=[0,0,0]
h_1^[(S-1)^3]=[18,0,0] h_2^[(S-1)^3]=[0,0,0] h_3^[(S-1)^3]=[0,0,0]
h_1^[(S-1)^4]=[0,0,0] h_2^[(S-1)^4]=[0,0,0] h_3^[(S-1)^4]=[0,0,0]
norm in K1/K of the component 1 of CK1:[12,0,1]
norm in K1/K of the component 2 of CK1:[18,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
No capitulation, m(K1)=4, e(K1)=3
CK2=[42442542,18,9]=[81,9,9]
No capitulation, m(K2)=4, e(K2)=4
```

(i) Case  $n = 1$ . In this case,  $\mathcal{H}_{K_1} \simeq \mathbb{Z}/3^3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ ; the above data shows that  $m(K_1) = 4$  for  $e(K_1) = 3$ , and a more complex structure, since  $s(K_1) = 1$ , but  $e(K_1) = 3 > n - s(K_1) = 0$ . Here,  $\#\mathcal{H}_{K_1} > \#\mathcal{H}_K \cdot \#\mathcal{H}_K[3]$ .

(ii) Case  $n = 2$ . Then  $\mathcal{H}_{K_2} \simeq \mathbb{Z}/3^4\mathbb{Z} \times \mathbb{Z}/3^2\mathbb{Z} \times \mathbb{Z}/3^2\mathbb{Z}$ ,  $m(K_2) \geq 4$ ,  $s(K_2) \geq 1$  and  $e(K_2) = 4 > n - s(K_2)$ . One has  $\#\mathcal{H}_{K_2} = \#\mathcal{H}_{K_1} \cdot \#\mathcal{H}_{K_1}[3]$ .

**3.3. Torsion groups of abelian  $p$ -ramified  $p$ -extensions.** We will evoke the case of the torsion group  $\mathcal{T}_K$  of the Galois group of the maximal abelian  $p$ -ramified pro- $p$ -extension of a number field  $K$ ; then, under Leopoldt's conjecture, *the transfer map is always injective, whatever the extensions of number fields  $L/K$  considered* (see, e.g., for more information and explicit results, our Crelle's papers (1982), Nguyen Quang Do [86] (1986), Jaulent [55] (1986), Movahhedi [85] (1988), all written in french, collected in our book [36, Chapter IV]).

This has some consequences because of the formula:

$$(3.1) \quad \#\mathcal{T}_K = \#\widetilde{\mathcal{H}}_K \cdot \#\mathcal{R}_K \cdot \#\mathcal{W}_K,$$

where  $\mathcal{W}_K$  is a canonical invariant built on the groups of (local and global) roots of unity of  $p$ -power order of  $K$ ,  $\mathcal{R}_K$  is the normalized  $p$ -adic regulator and  $\widetilde{\mathcal{H}}_K$  a subgroup of  $\mathcal{H}_K$  (see, for instance [36, Theorem IV.2.1], [39, Diagram § 3 and § 5], [43, Section 2]). For the Bertrandias–Payan module  $\mathcal{T}_K^{\text{bp}}$  of  $K$  (isomorphic to  $\mathcal{T}_K/\mathcal{W}_K$ , from [13] about the embedding problem), the transfers  $\mathbf{J}_{L/K}$  are injective, except few special cases discussed in Gras–Jaulent–Nguyen Quang Do [31].

Thus, as we have seen, in any cyclic  $p$ -extension  $L/K$  of Galois group  $G$ , the complexity of the invariants  $\mathcal{T}_{K_n} \neq 1$  is never smooth and is increasing with  $n$ .

**3.3.1. Quadratic fields.** We use our program [40, Corollary 2.2, Program I, § 3.2] computing the group structure of the  $\mathcal{T}_{K_n}$ 's for quadratic fields  $K = \mathbb{Q}(\sqrt{sm})$ ,  $s = \pm 1$ ,  $p = 2$ ,  $K_n \subseteq L \subset K(\mu_\ell)$  (so the number  $r_2 + 1$  of independent  $\mathbb{Z}_p$ -extensions is 1 for  $s = 1$  and  $2^n + 1$  for  $s = -1$ ). One must chose an arbitrary constant  $E$ , “assuming”  $E > e_n + 1$ , to be controlled a posteriori; but taking  $E$  very large does not essentially modify the computer calculation time:

```
MAIN PROGRAM COMPUTING THE STRUCTURE OF TKn (quadratic fields).
Set s=1 (resp. s=-1) for real (resp. imaginary) quadratic fields:
{s=1;p=2;ell=257;Nn=4;E=16;for(m=2,150,if(core(m)!=m,next);
PK=x^2-s*m;print();print("p=",p," ell=",ell," PK=",PK);
for(n=0,Nn,r2=1;if(s==-1,r2=r2+2^n);Qn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);Knmod=bnrinit(Kn,p^E);
CKnmod=Knmod.cyc;TKn=List;d=matsize(CKnmod)[2];for(j=1,d-r2,c=CKnmod[d-j+1];
w=valuation(c,p);if(w>0,listinsert(TKn,p^w,1));print("TK",n,"=",TKn))}
```



```

TK1=[16]
TK2=[32,8,4]
TK3=[64,16,4,4,4,4,4]
TK4=[128,32,4,4,4,4,4,4,4,4,4,4,4,4,4]
p=2 e11=257 PK=x^2+7
TK0=[2]
TK1=[8,2,2]
TK2=[16,4,2,2,2,2,2]
TK3=[32,8,2,2,2,2,2,2,2,2,2,2,2,2,2]
TK4=[64,16,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]

```

This leads to the interesting problem of estimating the maximal unramified sub-extension of the compositum  $\widetilde{K}_n$  of the  $\mathbb{Z}_p$ -extensions of the  $K_n$ 's.

**3.3.2. Cyclic cubic fields.** For cyclic cubic fields,  $p = 2$ , we obtain analogous computations by means of the following program:

```

MAIN PROGRAM COMPUTING THE STRUCTURE OF TKn (cyclic cubic fields):
{p=2;e11=257;Nn=3;E=16;bf=7;Bf=10^3;
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);print("p=",p," f=",f,
" PK=",PK," e11=",e11);for(n=0,Nn,Qn=polsubcyclo(e11,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);Knmod=bnrinit(Kn,p^E);
CKnmod=Knmod.cyc;TKn=List;d=matsize(CKnmod)[2];for(j=1,d-1,c=CKnmod[d-j+1];
w=valuation(c,p);if(w>0,listinsert(TKn,p^w,1));print("TK",n,"=",TKn))}}
p=2 e11=257 f=31 PK=x^3+x^2-10*x-8
TK0=[2,2]
TK1=[8,2,2,2,2]
TK2=[16,4,2,2,2,2,2,2,2,2,2]
TK3=[32,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
p=2 e11=257 f=43 PK=x^3+x^2-14*x+8
TK0=[2,2]
TK1=[16,16,8,2,2]
TK2=[32,32,16,4,4,2,2,2,2,2,2]
TK3=[64,64,32,8,8,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
p=2 e11=257 f=171 PK=x^3-57*x-152
TK0=[8,8]
TK1=[16,16,8,2,2]
TK2=[32,32,16,4,2,2,2,2,2,2,2,2]
TK3=[64,64,32,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
p=2 e11=257 f=277 PK=x^3+x^2-92*x+236
TK0=[4,4]
TK1=[8,4,4,4,4]
TK2=[16,8,8,4,4,4,4,4,2,2,2]
TK3=[32,16,16,8,8,8,8,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]

```

To conclude about the spectacular increasing of the  $p$ -ranks, recall that the  $p$ -rank of the Tate–Chafarevich group  $\text{III}_L^2 := \text{Ker} \left[ \text{H}^2(\widetilde{\mathcal{G}}_L, \mathbb{F}_p) \longrightarrow \bigoplus_{v|p} \text{H}^2(\widetilde{\mathcal{G}}_{L_v}, \mathbb{F}_p) \right]$  ( $\widetilde{\mathcal{G}}_L = \text{Galois group of the maximal } p\text{-ramified pro-}p\text{-extension of } L$ ), is that of  $\mathcal{T}_L$ .

**3.4. Remarks on  $\text{rk}_p(\mathcal{X}_n)$ -ranks.** In the framework of Theorem 3.2, about the family  $\{\mathcal{X}_n\}_{n \geq 0}$  in a tower  $L/K$ , let  $\mathcal{Y}_n := \mathcal{X}_n/\mathcal{X}_n^p$ , and assume that the transfer maps  $\mathcal{Y}_n \rightarrow \mathcal{Y}_{n+1}$  are also injective; so,  $\#\mathcal{Y}_{n+1} \geq \#\mathcal{Y}_n \cdot \#\mathcal{Y}_n[p] = (\#\mathcal{Y}_n)^2$ , whence  $\text{rk}_p(\mathcal{X}_{n+1}) \geq 2 \text{rk}_p(\mathcal{X}_n)$ , for all  $n \in [0, N-1]$ . This “doubling” of the  $p$ -ranks does not seem exceptional; for instance, for  $\mathcal{X} = \mathcal{T}$ ,  $K = \mathbb{Q}(\sqrt{105})$ ,  $p = 2$ ,  $\ell = 257$ , computed above:

```

p=2 e11=257 PK=x^2-105 r=2
TK0=[2,2]

```



we still denote  $(\lambda, \nu)$ . Thus,  $\#\mathcal{H}_K = p^\nu$ ,  $\#\mathcal{H}_{K_n} = p^{\lambda n + \nu}$  and, for all  $n \geq 0$ :

$$(4.1) \quad \begin{cases} \mathcal{H}_{K_n} \simeq \text{Ker}(\mathbf{J}_{\tilde{K}/K_n}) \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{n+\alpha_i}\mathbb{Z}, & \alpha_i \geq 0, \text{ Ker}(\mathbf{J}_{\tilde{K}/K_n}) \simeq \mathbb{T} \\ \mathcal{H}_K \simeq \text{Ker}(\mathbf{J}_{\tilde{K}/K}) \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{\alpha_i}\mathbb{Z}, & \alpha_i \geq 0. \end{cases}$$

**Proposition 4.1.** *Under the above choice of the base field  $K$  in the  $\mathbb{Z}_p$ -extension  $\tilde{K}$  and assuming  $\mu = 0$ , the capitulation of  $\mathcal{H}_K$  in  $\tilde{K}$  is equivalent to the isomorphism  $\mathcal{H}_{K_n} \simeq \mathcal{H}_K \oplus \mathbf{J}_{\tilde{K}/K_n}(\mathcal{H}_{K_n}) \simeq \mathcal{H}_K \oplus (\mathbb{Z}/p^n\mathbb{Z})^\lambda$ ,  $\forall n \geq 0$ .*

*Proof.* If  $\mathcal{H}_K$  capitulates in  $\tilde{K}$ , then  $\text{Ker}(\mathbf{J}_{\tilde{K}/K}) = \mathcal{H}_K$ , whence  $\alpha_i = 0$ , for all  $i \in [1, \lambda]$ , from (4.1), and  $\mathcal{H}_{K_n} \simeq \mathcal{H}_K \oplus (\mathbb{Z}/p^n\mathbb{Z})^\lambda$  for all  $n \geq 0$ , since each capitulation kernel  $\text{Ker}(\mathbf{J}_{\tilde{K}/K_n})$  is isomorphic to  $\mathbb{T}$ , thus isomorphic to  $\text{Ker}(\mathbf{J}_{\tilde{K}/K}) = \mathcal{H}_K$ .

Reciprocally, assume that  $\mathcal{H}_{K_n} \simeq \mathcal{H}_K \oplus (\mathbb{Z}/p^n\mathbb{Z})^\lambda$ ,  $\forall n \geq 0$ ; then, from (4.1),  $\mathcal{H}_{K_n} = \text{Ker}(\mathbf{J}_{\tilde{K}/K_n}) \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{n+\alpha_i}\mathbb{Z} \simeq \mathcal{H}_K \oplus (\mathbb{Z}/p^n\mathbb{Z})^\lambda$ . Comparing the structures for  $n$  large enough gives  $\alpha_i = 0$ , for all  $i \in [1, \lambda]$  and  $\text{Ker}(\mathbf{J}_{\tilde{K}/K_n}) \simeq \mathcal{H}_K$ , for all  $n \geq 0$ , whence the capitulation of  $\mathcal{H}_K$  in  $\tilde{K}$ .  $\square$

**4.2. Case of the cyclotomic  $\mathbb{Z}_p$ -extension of  $K$ .** Assume that  $K$  is totally real and let  $K^{\text{cy}} = \bigcup_{n \geq 0} K_n$  be the cyclotomic  $\mathbb{Z}_p$ -extension of  $K$ , assuming the previous choice of the base field  $K$  in  $K^{\text{cy}}$  ( $K$  is still real with same cyclotomic  $\mathbb{Z}_p$ -extension). One may also assume that Greenberg's conjecture [48] ( $\lambda = \mu = 0$  for  $K^{\text{cy}}$ ) is equivalent to the stability of the  $\#\mathcal{H}_{K_n}$ 's from  $K$ , giving capitulations of all the class groups in  $K^{\text{cy}}$  from  $n = 0$ , then  $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{\text{Gn}} \stackrel{\text{N}_{K_n/K}}{\simeq} \mathcal{H}_K$ , for all  $n \geq 0$ ; thus,  $m(K_n) = 1$  ( $s(K_n) = 0$ ) with  $e(K_n) = e(K)$ , which is exactly the limit case of application of Theorem 1.1, for  $n \geq e(K)$ .

In Kraft–Schoof–Pagani [68, 87] such properties of stability are used to check the conjecture by means of analytic formulas.

In Jaulent [58, 59, 60], it is proved that Greenberg's conjecture is equivalent to the capitulation of the logarithmic class group  $\mathcal{H}_K^{\text{lg}}$  in  $K^{\text{cy}}$ ;<sup>6</sup> this may be effective if, by chance, a capitulation occurs in the firsts layers; indeed, this criterion is probably the only one giving an algorithmic test (using Belabas–Jaulent [11], Diaz–Diaz–Jaulent–Pauli–Pohst–Soriano–Gafiuk [23]) from the base field. We will see, Section 9 that  $\mathcal{H}_K^{\text{lg}}$  may capitulate in real towers  $L/K$ .

**4.2.1. Analysis of the Chevalley–Herbrand formula in  $K^{\text{cy}}/K$ .**

**Hypothesis 4.2.** *Taking in the sequel, as totally real base field  $K$ , a suitable layer  $K_{n_0}$  in  $K^{\text{cy}}$ , we may assume the following properties of  $K^{\text{cy}}/K$ :*

- (i)  $p$  is totally ramified in  $K^{\text{cy}}/K$ ;
- (ii)  $\#\mathcal{H}_{K_n} = p^{\lambda n + \mu p^n + \nu}$  for all  $n \geq 0$  with new non-negative invariants of the form  $(\lambda, \mu p^{n_0}, \nu + \lambda n_0)$  from that of  $K$ , still denoted  $(\lambda, \mu, \nu)$ . Thus,  $\#\mathcal{H}_K = p^{\mu + \nu}$ .

Then, as for the ‘‘tame case’’, formulas (2.2) hold with  $r = \#\{\mathfrak{p}, \mathfrak{p} \mid p \text{ in } K\}$  and the filtration still depends on the class and norm factors. However, the norm factors can be interpreted as divisor of the normalized  $p$ -adic regulator of  $K$ , as follows from class field theory (under Leopoldt's conjecture):

**Definitions 4.3.** (i) Consider  $H_{K_n}^{\text{gen}}$ , the genus field of  $K_n$  (i.e., the subfield of the  $p$ -Hilbert class field  $H_{K_n}^{\text{nr}}$ , abelian over  $K$  and maximal, whence the subfield

<sup>6</sup>This invariant, also denoted  $\tilde{\mathcal{C}}_K$  or  $\tilde{\mathcal{T}}_K$ , was defined in [57] and is, in the class field theory viewpoint, isomorphic to  $\text{Gal}(H_K^{\text{lc}}/K^{\text{cy}})$ , where  $H_K^{\text{lc}}$  is the maximal abelian locally cyclotomic pro- $p$ -extension of  $K$ . In the sequel we will denote  $\mathcal{H}_K^{\text{lg}}$  this group since it behaves more like a class group rather than a torsion group  $\mathcal{T}_K$  which never capitulates.

of  $\text{Gal}(H_{K_n}^{\text{nr}}/K)$  fixed by the image of  $\mathcal{H}_{K_n}^{\sigma_n-1}$ , and  $K^{\text{cy}}H_{K_n}^{\text{gen}}$  which is abelian  $p$ -ramified over  $K$ ; then  $K^{\text{cy}}H_{K_n}^{\text{gen}} \subseteq H_K^{\text{pr}}$ , the maximal  $p$ -ramified abelian pro- $p$ -extension of  $K$ . So  $\mathcal{T}_K := \text{Gal}(H_K^{\text{pr}}/K^{\text{cy}})$  is finite under Leopoldt's conjecture.

We define  $H_{K^{\text{cy}}}^{\text{gen}} := \bigcup_n K^{\text{cy}}H_{K_n}^{\text{gen}}$  and put  $\mathcal{G}_K := \text{Gal}(H_{K^{\text{cy}}}^{\text{gen}}/K^{\text{cy}})$ .

We denote by  $K_{n_1}$ ,  $n_1 \geq 0$ , the minimal layer such that  $K^{\text{cy}}H_{K_{n_1}}^{\text{gen}} = H_{K^{\text{cy}}}^{\text{gen}}$  (even with the above Hypothesis 4.2,  $K_{n_1}$  may be distinct from  $K$ ).

(ii) Let  $H_K^{\text{bp}}$  be the Bertrandias–Payan field fixed by  $\mathcal{W}_K \simeq (\bigoplus_{v|p} \mu_{K_v})/\mu_K$ , where  $K_v$  is the  $v$ -completion of  $K$  and  $\mu_k$  the group of  $p$ th-roots of unity of the field  $k$  (local or global); if  $U_v$  is the group of principal units of  $K_v$ , then  $\mu_{K_v} = \text{tor}_{\mathbb{Z}_p}(U_v)$ .

(iii) Let  $\iota_{\mathcal{E}_K}$  be the image of  $\mathcal{E}_K = \mathbf{E}_K \otimes \mathbb{Z}_p$  in  $U_K := \prod_{v|p} U_v$  and let  $I_v(H_K^{\text{pr}}/K^{\text{cy}})$  be the inertia groups of  $v$  in  $H_K^{\text{pr}}/K^{\text{cy}}$ ; then  $I_v(H_K^{\text{pr}}/K^{\text{cy}}) \simeq \text{tor}_{\mathbb{Z}_p}(U_v/\iota_{\mathcal{E}_K} \cap U_v)$  and the subgroup of  $\mathcal{T}_K$  generated by these inertia groups fixes  $H_{K^{\text{cy}}}^{\text{gen}}$ .

(iv) Let  $\mathcal{R}_K^{\text{nr}} := \text{Gal}(H_{K^{\text{cy}}}^{\text{gen}}/K^{\text{cy}}H_K^{\text{nr}})$  &  $\mathcal{R}_K^{\text{ram}} := \text{Gal}(H_K^{\text{bp}}/H_{K^{\text{cy}}}^{\text{gen}})$ , where  $\mathcal{R}_K := \text{Gal}(H_K^{\text{bp}}/K^{\text{cy}}H_K^{\text{nr}})$  is the normalized  $p$ -adic regulator that we define in [39, § 5].

These definitions may be summarized by the following diagram [43, § 2]:

$$\begin{array}{ccccccc}
 & & & \mathcal{T}_K & & & \\
 & \swarrow & & \searrow & & \swarrow & \\
 K^{\text{cy}} & \xrightarrow{\mathcal{H}_K} & K^{\text{cy}}H_K^{\text{nr}} & \xrightarrow{\mathcal{R}_K^{\text{nr}}} & H_{K^{\text{cy}}}^{\text{gen}} & \xrightarrow{\mathcal{R}_K^{\text{ram}}} & H_K^{\text{bp}} & \xrightarrow{\mathcal{W}_K} & H_K^{\text{pr}} \\
 & \searrow & & \swarrow & & \searrow & & & \\
 & & \mathcal{G}_K & & \mathcal{R}_K & & & & 
 \end{array}$$

where  $\#\mathcal{G}_K = \#\mathcal{H}_{K_n}^{G_n}$  for all  $n \geq n_1$ .

Recall, under the above Hypothesis 4.2, some results that we have given in [41], generalizing some particular results of Taya [95, 96, 97], using  $p$ -adic  $L$ -functions:

**Proposition 4.4.** For all  $n \geq 0$ , the norm factor  $\frac{p^{n \cdot (r-1)}}{\omega_{K_n/K}(\mathbf{E}_K)}$  divides  $\#\mathcal{R}_K^{\text{nr}}$  and there is equality for all  $n \geq n_1$ . Whence  $\#\mathcal{H}_{K_n}^{G_n} = \#\mathcal{H}_K \cdot \frac{p^{n \cdot (r-1)}}{\omega_{K_n/K}(\mathbf{E}_K)}$  divides  $\#\mathcal{G}_K = \#\mathcal{H}_K \cdot \#\mathcal{R}_K^{\text{nr}}$ , for all  $n \geq 0$  with equality for all  $n \geq n_1$ .

Recall that  $m(K_n)$  is the length of the filtration for  $K_n$ , with  $m(K_n) = 1$  if  $\mathcal{H}_{K_n} \neq 1$  ( $m(K_n) = 0$  if  $\mathcal{H}_{K_n} = 1$ ):

**Proposition 4.5.** Let  $v_p$  be the  $p$ -adic valuation. Under Hypothesis 4.2, we have:

$$m(K_n) \leq \lambda \cdot n + \mu \cdot p^n + \nu \leq v_p(\#\mathcal{H}_K \cdot \#\mathcal{R}_K^{\text{nr}}) \cdot m(K_n), \quad \forall n \geq 0.$$

If  $\mathcal{G}_K = 1$ , then Greenberg's conjecture holds with  $\mathcal{H}_{K_n} = 1$  for all  $n$ . If  $\mathcal{G}_K \neq 1$ , then Greenberg's conjecture holds if and only if  $m(K_n)$  is bounded regarding  $n$ .

From these recalls and Hypothesis 4.2, we can deduce:

**Theorem 4.6.** Greenberg's conjecture is equivalent to  $m(K_n) = 1$  (resp.  $= 0$ ) for all  $n \geq 0$ , if  $\mathcal{H}_K \neq 1$  (resp.  $= 1$ ), and  $\mathcal{R}_K^{\text{nr}} = 1$ , whence Greenberg's conjecture is equivalent to  $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n} \mathbf{N}_{K_n/K} \simeq \mathcal{H}_K$  for all  $n \geq 0$ .

*Proof.* If  $\lambda = \mu = 0$ , the stability holds from  $n = 0$  and  $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n} \mathbf{N}_{K_n/K} \simeq \mathcal{H}_K$ , for all  $n \geq 0$  (whence  $m(K_n) \in \{0, 1\}$ , depending on  $\#\mathcal{H}_K = 1$  or not). Since  $\#\mathcal{H}_{K_n}^{G_n} = \#\mathcal{G}_K = \#\mathcal{H}_K$ ,  $\forall n \geq n_1$ , it follows that  $\mathcal{R}_K^{\text{nr}} = 1$  (Proposition 4.4). Reciprocally, if  $\mathcal{R}_K^{\text{nr}} = 1$  and  $m(K_n) \in \{0, 1\}$ , then Proposition 4.5 implies  $\lambda = \mu = 0$  if  $\mathcal{H}_K \neq 1$  (or  $\lambda = \mu = \nu = 0$  if  $\mathcal{H}_K = 1$ ).  $\square$

4.2.2. *Particular case  $\mu = 0$ .* We can wonder, due to the Propositions 4.1 and 4.5, if Greenberg's conjecture may be equivalent to  $\nu_{K_n/K}(\mathcal{H}_{K_n}) = 1$ , for all  $n \geq e(K)$ , obtained with the stronger particular conditions  $m(K_n) \in \{0, 1\}$  (i.e.,  $s(K_n) = 0$ ) and  $e(K_n) = e(K)$  for all  $n \geq 0$ .



Indeed, Greenberg's conjecture implies the capitulation of  $\mathcal{H}_K$  and  $m(K_n) = 1$  for all  $n \geq 0$  (Theorem 4.6); if so, one obtains  $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n} \stackrel{N_{K_n/K}}{\simeq} \mathcal{H}_K$ , for all  $n \geq 0$ ; in some sense, the maximal smoothness complexity.

In the practice these phenomenon (if any) holds from an unknown level in  $K^{\text{cy}}$ .

This possibility may be suggested by the following example of the cyclic cubic field of conductor  $f = 2689$ , of 2-class group  $\mathbf{Z}/2\mathbf{Z}$  and its cyclotomic  $\mathbb{Z}_2$ -extension, giving  $\mathcal{H}_{K_1} \simeq \mathbf{Z}/4\mathbf{Z}$ ,  $\mathcal{H}_{K_2} \simeq \mathbf{Z}/8\mathbf{Z}$ , and  $\mathcal{H}_{K_3} \simeq \mathbf{Z}/8\mathbf{Z}$ :

```
p=2 f=2689 PK=x^3+x^2-896*x+5876 CK0=[2,2]
CK1=[28,4]=[4,4]
h_1^[(S-1)^1]=[0,0] h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation, m(K1)=1, e(K1)=2
CK2=[56,8]=[8,8]
h_1^[(S-1)^1]=[0,0] h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
No capitulation, m(K2)=1, e(K2)=3
CK3=[56,8]=[8,8]
h_1^[(S-1)^1]=[0,0] h_2^[(S-1)^1]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Complete capitulation, m(K3)=1, e(K3)=3
```

At any layer,  $m(K_n) = 1$  and capitulations in  $K^{\text{cy}}$  hold for all  $n$ . Stability occurs from  $K_2$  giving a checking of Greenberg's conjecture.

**Remark 4.7.** *It is interesting to check other examples given by Fukuda [27] for real quadratic fields and  $p = 3$  with a simplified program and suitable polynomials defining the layer  $K_1$ ; for all,  $\mathcal{H}_K \simeq \mathbb{Z}/9\mathbb{Z}$  and  $\mathbf{N}_{K_1/K}(\mathcal{H}_{K_1}) = \mathcal{H}_K^3$ :*

```
{p=3;Lm=List([3137,3719,4409,6809,7226,9998]);for(i=1,6,PK=x^2-Lm[i];
K=bnfinit(PK,1);CK0=K.clgp;P1=polcompositum(PK,polsubcyclo(p^2,p))[1];
K1=bnfinit(P1,1);CK1=K1.clgp;print();G=nfgaloisconj(K1);Id=x;for(k=1,2*p,
Z=G[k];ks=1;while(Z!=Id,Z=nfgaloisapply(K1,G[k],Z);ks=ks+1);if(ks==p,
S=G[k];break));A0=CK1[3][1];A=1;for(t=1,p,As=nfgaloisapply(K1,S,A);
A=idealmul(K1,A0,As));C=bnfisprincipal(K1,A)[1];print("PK=",PK,
" CK0=",CK0[2]," CK1=",CK1[2]," Norm of CK1=",C)}}
PK=x^2-3137 CK1=[9] Norm of CK1=[3] PK=x^2-6809 CK1=[9] Norm of CK1=[3]
PK=x^2-3719 CK1=[9] Norm of CK1=[3] PK=x^2-7226 CK1=[18] Norm of CK1=[3]
PK=x^2-4409 CK1=[9] Norm of CK1=[3] PK=x^2-9998 CK1=[9] Norm of CK1=[3]
```

*The stability from  $K$  implies Greenberg's conjecture and capitulation of  $\mathcal{H}_K$  in  $K_2$ , with an incomplete capitulation in  $K_1$  and  $m(K_n) = 1$  for all  $n$ .*

## 5. ISOTOPIC COMPONENTS AND CAPITULATION

Consider a real cyclic field  $K$  of prime-to- $p$  degree  $d$  and  $L = L_0K$  with  $L_0/\mathbb{Q}$  real cyclic of degree  $p^N$ ,  $N \geq 1$ . Then  $L/\mathbb{Q}$  is cyclic of degree  $dp^N$  with Galois group  $\Gamma = g \oplus G$  where  $g = \text{Gal}(L/L_0)$  and  $G = \text{Gal}(L/K)$ . The field  $L$  is associated to an irreducible rational character  $\chi$ , sum of irreducible  $p$ -adic characters  $\varphi$  of "order" the order  $dp^N$  of any  $\psi \mid \varphi$  of degree 1.

This non semi-simple context is problematic for the definition of isotopic  $p$ -adic components of the form  $\mathcal{H}_{L,\varphi}$  and  $\mathcal{H}_{K_n,\varphi_n}$  for the subfields  $K_n$  of  $L$  with corresponding rational and  $p$ -adic characters  $\chi_n$  and  $\varphi_n \mid \chi_n$ ; this is extensively developed in the english translation [44] of our original paper (1978) in french <https://doi.org/10.5802/pmb.a-10>. So we just recall the definitions and explain how the phenomenon of capitulation gives rise to difficulties about the classical algebraic definition of the literature, compared to the arithmetic one that we have introduced to state the Main Conjecture in the non semi-simple case.

Indeed, classical works deal with an algebraic definition of the  $\varphi$ -components of  $p$ -class groups, which presents an inconsistency regarding analytic formulas; this definition is, for  $\Gamma$  cyclic of order  $dp^N$  and for all  $\varphi \mid \chi$ :

$$\mathcal{H}_{L,\varphi}^{\text{alg}} := \mathcal{H}_L \otimes_{\mathbb{Z}_p[\Gamma]} \mathbb{Z}_p[\mu_{dp^N}],$$

with the  $\mathbb{Z}_p[\mu_{dp^N}]$ -action  $\tau \in \Gamma \mapsto \psi(\tau)$ , with  $\psi \mid \varphi$  of order  $dp^N$  (see Solomon [93, II, § 1] or Greither [49, Definition, p. 451]). Let  $P_\chi$  be the  $dp^N$ th cyclotomic polynomial and let  $\sigma_\chi$  be a generator of  $\text{Gal}(L/\mathbb{Q})$ ; then, let  $P_\varphi \mid P_\chi$  be the corresponding local cyclotomic polynomial associated to the above action  $\tau \in \Gamma \mapsto \psi(\tau)$ , with  $\psi \mid \varphi$ . We have defined the notions of  $\chi$  and  $\varphi$ -objects, then proved in [44, § 3.2.4, Theorem 3.7, Definition 3.11], the following interpretations:

$$\begin{cases} \mathcal{H}_{L,\chi}^{\text{alg}} = \{x \in \mathcal{H}_L, P_\chi(\sigma_\chi) \cdot x = 1\} = \{x \in \mathcal{H}_L, \nu_{L/k}(x) = 1, \forall k \subsetneq L\}, \\ \mathcal{H}_{L,\varphi}^{\text{alg}} = \{x \in \mathcal{H}_L, P_\varphi(\sigma_\chi) \cdot x = 1\}, \end{cases}$$

which gives rise to our corresponding arithmetic definitions:

$$\begin{cases} \mathcal{H}_{L,\chi}^{\text{ar}} := \{x \in \mathcal{H}_L, \mathbf{N}_{L/k}(x) = 1, \forall k \subsetneq L\}, \\ \mathcal{H}_{L,\varphi}^{\text{ar}} := \{x \in \mathcal{H}_L, P_\varphi(\sigma) \cdot x = 1 \quad \& \quad \mathbf{N}_{L/k}(x) = 1, \forall k \subsetneq L\} \end{cases}$$

(in other words,  $\mathcal{H}_{L,\varphi}^{\text{ar}} = \mathcal{H}_{L,\chi}^{\text{ar}} \cap \mathcal{H}_{L,\varphi}^{\text{alg}}$ ).

We then have, since  $L/K$  is totally ramified ([44, Theorem 3.15]):

$$(5.1) \quad \#\mathcal{H}_L^{\text{ar}} = \prod_{\chi \in \mathbf{R}_L} \#\mathcal{H}_{L,\chi}^{\text{ar}},$$

where  $\mathbf{R}_L$  is the set of irreducible rational characters of  $L$ . More precisely,  $\chi$  is of the form  $\chi_0 \chi_n$  for a rational characters  $\chi_0$  of  $K$ , of order a divisor  $\text{od } d$ , and the rational characters  $\chi_n$  of  $L_0$  of order  $p^n$ ,  $n \in [1, N]$ ; then  $\mathcal{H}_{L,\chi}^{\text{ar}} = \mathcal{H}_{K_{\chi_0} L_{0,\chi_n}, \chi}^{\text{ar}}$ , where  $K_{\chi_0} \subseteq K$  (resp.  $L_{0,\chi_n} \subseteq L_0$ ) correspond to  $\chi_0$  (resp.  $\chi_n$ ).

These definitions and results lead to an unexpected semi-simplicity, especially in accordance with analytic formulas, which enforces the Main Conjecture in that case [44, Theorem 4.5]; it writes, for all  $\chi \in \mathbf{R}_L$ :

$$(5.2) \quad \mathcal{H}_{L,\chi}^{\text{ar}} = \bigoplus_{\varphi \mid \chi} \mathcal{H}_{L,\varphi}^{\text{ar}},$$

The  $\mathbb{Z}_p[\Gamma]$ -modules of the form  $\mathcal{H}_{L,\chi}^{\text{ar}}$  (resp.  $\mathcal{H}_{L,\varphi}^{\text{ar}}$ ), annihilated by all the arithmetic norms  $\mathbf{N}_{L/k}$ , are called arithmetic  $\chi$ -objects (resp.  $\varphi$ -objects).

We have  $\mathcal{H}_{L,\varphi}^{\text{ar}} = \mathcal{H}_{L,\varphi}^{\text{alg}}$  as soon as the  $\mathbf{J}_{L/k}$ 's are injective for all  $k \subsetneq L$ , but as we have seen, this does not hold in general when  $K \subseteq k \subsetneq L$  since there is often partial capitulation. One can even say that the classic admitted definition is non canonical and imperfect for real  $p$ -class groups (see Examples below).

**Remark 5.1.** *Let  $\chi$  be the rational character associated to  $L$ . Our Main Conjecture [33] (not yet proven in the non semi-simple case contrary to some claims) requires that the equality of orders of  $\chi$ -objects (see [44, Theorem 7.5 (i)]):  $\#\mathcal{H}_{L,\chi}^{\text{ar}} = \#(\mathcal{E}_L/\mathcal{E}_L^0 \cdot \mathcal{F}_L)$ , be valid for the  $\varphi$ -components, for all  $\varphi \mid \chi$ ; in these formula,  $\mathcal{E}_L^0$  is the subgroup of  $\mathcal{E}_L$  generated by the units of the strict subfields of  $L$  and  $\mathcal{F}_L$  is the group of classical Leopoldt's cyclotomic units; the fact that  $\mathcal{E}_L/\mathcal{E}_L^0 \cdot \mathcal{F}_L$  be a  $\chi$ -object is obvious since  $\mathbf{N}_{L/K}(\mathcal{E}_L) \subseteq \mathcal{E}_L^0$  by definition (see [44, Examples 3.12, 3.13]).*

In the case of cubic fields with  $p = 2$  or in the case of real quadratic fields,  $\chi = \varphi$ , so that the Main Conjecture is trivial, but not the definition of arithmetic  $\varphi$ -objects regarding the algebraic ones. Let's give numerical examples showing the consequences of capitulation for the non-arithmetic definitions:

**Example 5.2.** Consider  $K = \mathbb{Q}(\sqrt{4409})$ ,  $p = 3$ ,  $\ell = 19$  split in  $K$  and  $K_2 \subset K(\mu_\ell)$ . We will see that  $\mathcal{H}_{K_n} \simeq \mathbb{Z}/9\mathbb{Z}$ ,  $\forall n \leq 2$  (stability); Program 8 gives:

p=3 PK=x^2-4409 CK0=[9] e11=19 r=2  
 CK1=[9] CK2=[9]  
 norm in K1/K of CK1:[3] norm in K2/K of CK2:[0]  
 Incomplete capitulation in K1 Complete capitulation in K2

The capitulation is complete in  $K_2$  as expected from Theorem 1.2 (stability from  $K$  giving  $\mathcal{H}_{K_n} = \mathcal{H}_{K_n}^{G_n} \stackrel{N_{K_n/K}}{\simeq} \mathcal{H}_K$ , for  $n \in \{1, 2\}$ ).

We use obvious notations for the characters defining the fields  $K_n$ . Since arithmetic norms are isomorphisms, the above computations prove that:

$$\begin{cases} \nu_{K_1/K}(\mathcal{H}_{K_1}) = (\mathcal{H}_{K_1})^{1+\sigma_1+\sigma_1^2} = (\mathcal{H}_{K_1})^3 \simeq \mathbb{Z}/3\mathbb{Z}, \\ \nu_{K_2/K_1}(\mathcal{H}_{K_2}) = (\mathcal{H}_{K_2})^{1+\sigma_2^3+\sigma_2^6} = (\mathcal{H}_{K_2})^3 \simeq \mathbb{Z}/3\mathbb{Z}, \end{cases}$$

whence:

$$\begin{cases} \mathcal{H}_{\chi_2}^{\text{ar}} = \{x \in \mathcal{H}_{K_2}, \mathbf{N}_{K_2/K_1}(x) = 1\} = 1, \\ \mathcal{H}_{\chi_2}^{\text{alg}} = \{x \in \mathcal{H}_{K_2}, \nu_{K_2/K_1}(x) = 1\} = \mathcal{H}_{K_2}[3] \simeq \mathbb{Z}/3\mathbb{Z}. \\ \mathcal{H}_{\chi_1}^{\text{ar}} = \{x \in \mathcal{H}_{K_1}, \mathbf{N}_{K_1/K}(x) = 1\} = 1, \\ \mathcal{H}_{\chi_1}^{\text{alg}} = \{x \in \mathcal{H}_{K_1}, \nu_{K_1/K}(x) = 1\} = \mathcal{H}_{K_1}[3] \simeq \mathbb{Z}/3\mathbb{Z}. \end{cases}$$

Formula (5.1) (or (5.2) since  $\chi = \varphi$ ), gives the product of orders of the  $\chi$ -components  $\mathcal{H}_{\chi_n}^{\text{ar}}$  (since  $\#\mathcal{H}_{\chi_0}^{\text{ar}} = \#\mathcal{H}_K = 9$ ):

$$\begin{cases} \#\mathcal{H}_{K_2} = \#\mathcal{H}_{\chi_0}^{\text{ar}} \cdot \#\mathcal{H}_{\chi_1}^{\text{ar}} \cdot \#\mathcal{H}_{\chi_2}^{\text{ar}} = 9 \cdot 1 \cdot 1 = 3^2, \\ \#\mathcal{H}_{K_1} = \#\mathcal{H}_{\chi_0}^{\text{ar}} \cdot \#\mathcal{H}_{\chi_1}^{\text{ar}} = 9 \cdot 1 = 3^2, \end{cases}$$

These formulas are not fulfilled in the algebraic sense, because:

$$\begin{cases} \#\mathcal{H}_{\chi_0}^{\text{alg}} \cdot \#\mathcal{H}_{\chi_1}^{\text{alg}} \cdot \#\mathcal{H}_{\chi_2}^{\text{alg}} = 9 \cdot 3 \cdot 3 = 3^4, \\ \#\mathcal{H}_{\chi_0}^{\text{alg}} \cdot \#\mathcal{H}_{\chi_1}^{\text{alg}} = 9 \cdot 3 = 3^3. \end{cases}$$

**Example 5.3.** This example, for the cyclic cubic field of conductor  $f = 1951$  with  $p = 2$ ,  $\ell = 17$  split in  $K$ , is analogous except that capitulation takes place in  $K_1$  without any stability (see complete data in Example 2.8):

p=2 f=1951 PK=x^3+x^2-650\*x-289 CK0=[2,2] e11=17  
 CK1=[4,4,2,2]  
 h\_1^~[(S-1)^1]=[2,0,0,0] h\_2^~[(S-1)^1]=[0,2,0,0]  
 h\_3^~[(S-1)^1]=[0,0,0,0] h\_4^~[(S-1)^1]=[0,0,0,0]  
 CK2=[4,4,4,4]  
 h\_1^~[(S-1)^1]=[2,0,0,0] h\_2^~[(S-1)^1]=[0,2,0,0]  
 h\_3^~[(S-1)^1]=[2,0,2,0] h\_4^~[(S-1)^1]=[0,2,0,2]  
 h\_1^~[(S-1)^2]=[0,0,0,0] h\_2^~[(S-1)^2]=[0,0,0,0]  
 h\_3^~[(S-1)^2]=[0,0,0,0] h\_4^~[(S-1)^2]=[0,0,0,0]  
 CK3=[8,8,4,4]  
 h\_1^~[(S-1)^1]=[0,0,2,0] h\_2^~[(S-1)^1]=[2,2,2,0]  
 h\_3^~[(S-1)^1]=[2,0,0,0] h\_4^~[(S-1)^1]=[6,2,0,2]  
 h\_1^~[(S-1)^2]=[4,0,0,0] h\_2^~[(S-1)^2]=[0,4,0,0]  
 h\_3^~[(S-1)^2]=[0,0,0,0] h\_4^~[(S-1)^2]=[0,0,0,0]

Numerical data of the form  $h_j^{[(S-1)^i]} = [e_1, e_2, e_3, e_4]$  give, with  $\mathbf{Z} = \mathbb{Z}_2[\exp(\frac{2i\pi}{3})]$ :

$$\begin{cases} \nu_{K_3/K_2}(\mathcal{H}_{K_3}) = \mathcal{H}_{K_3}^{1+\sigma_3^4} = \mathcal{H}_{K_3}^2 \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, \\ \nu_{K_2/K_1}(\mathcal{H}_{K_2}) = \mathcal{H}_{K_2}^{1+\sigma_2^2} = \mathcal{H}_{K_2}^2 \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, \\ \nu_{K_1/K}(\mathcal{H}_{K_1}) = \mathcal{H}_{K_1}^{1+\sigma_1} = \mathcal{H}_{K_1}^4 = 1. \end{cases}$$

Whence:

$$\left\{ \begin{array}{l} \mathcal{H}_{\chi_3}^{\text{ar}} = \{x \in \mathcal{H}_{K_3}, \mathbf{N}_{K_3/K_2}(x) = 1\} \simeq \mathbf{Z}/2\mathbf{Z}, \\ \mathcal{H}_{\chi_3}^{\text{alg}} = \{x \in \mathcal{H}_{K_3}, \nu_{K_3/K_2}(x) = 1\} = \mathcal{H}_{K_3}[2] \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, \\ \mathcal{H}_{\chi_2}^{\text{ar}} = \{x \in \mathcal{H}_{K_2}, \mathbf{N}_{K_2/K_1}(x) = 1\} \simeq \mathbf{Z}/2\mathbf{Z}, \\ \mathcal{H}_{\chi_2}^{\text{alg}} = \{x \in \mathcal{H}_{K_2}, \nu_{K_2/K_1}(x) = 1\} = \mathcal{H}_{K_2}[2] \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, \\ \mathcal{H}_{\chi_1}^{\text{ar}} = \{x \in \mathcal{H}_{K_1}, \mathbf{N}_{K_1/K}(x) = 1\} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \text{ or } \mathbf{Z}/4\mathbf{Z}, \\ \mathcal{H}_{\chi_1}^{\text{alg}} = \{x \in \mathcal{H}_{K_1}, \nu_{K_1/K}(x) = 1\} = \mathcal{H}_{K_1} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}. \end{array} \right.$$

Which gives, noting that  $\#(\mathbf{Z}/2^k\mathbf{Z}) = 4^k$ :

$$\left\{ \begin{array}{l} \#\mathcal{H}_{K_3} = \#\mathcal{H}_{\chi_0}^{\text{ar}} \cdot \#\mathcal{H}_{\chi_1}^{\text{ar}} \cdot \#\mathcal{H}_{\chi_2}^{\text{ar}} \cdot \#\mathcal{H}_{\chi_3}^{\text{ar}} = 2^2 \cdot 4^2 \cdot 2^2 \cdot 2^2 = 2^{10}, \\ \#\mathcal{H}_{K_2} = \#\mathcal{H}_{\chi_0}^{\text{ar}} \cdot \#\mathcal{H}_{\chi_1}^{\text{ar}} \cdot \#\mathcal{H}_{\chi_2}^{\text{ar}} = 2^2 \cdot 4^2 \cdot 2^2 = 2^8, \\ \#\mathcal{H}_{K_1} = \#\mathcal{H}_{\chi_0}^{\text{ar}} \cdot \#\mathcal{H}_{\chi_1}^{\text{ar}} = 2^2 \cdot 2^4 = 2^6, \end{array} \right.$$

contrary to:

$$\left\{ \begin{array}{l} \#\mathcal{H}_{\chi_0}^{\text{alg}} \cdot \#\mathcal{H}_{\chi_1}^{\text{alg}} \cdot \#\mathcal{H}_{\chi_2}^{\text{alg}} \cdot \#\mathcal{H}_{\chi_3}^{\text{alg}} = 2^2 \cdot 2^6 \cdot 2^4 \cdot 2^4 = 2^{16}, \\ \#\mathcal{H}_{\chi_0}^{\text{alg}} \cdot \#\mathcal{H}_{\chi_1}^{\text{alg}} \cdot \#\mathcal{H}_{\chi_2}^{\text{alg}} = 2^2 \cdot 2^6 \cdot 2^4 = 2^{12}, \\ \#\mathcal{H}_{\chi_0}^{\text{alg}} \cdot \#\mathcal{H}_{\chi_1}^{\text{alg}} = 2^2 \cdot 2^6 = 2^8. \end{array} \right.$$

We will illustrate, in  $K_1$ , the analytic equality, discussed in Remark 5.1:

$$\#\mathcal{H}_{\chi_1}^{\text{ar}} = \#(\mathcal{E}_{K_1}/\mathcal{E}_{K_1}^0 \cdot \mathcal{F}_{K_1}).$$

Let  $k := \mathbb{Q}(\sqrt{17})$ ,  $\text{Gal}(K_1/k) =: \{1, \tau, \tau^2\}$  and  $G_1 = \{1, \sigma\}$ . Since 17 splits in  $K$  and 1951 splits in  $k$ , the generating cyclotomic unit  $\eta$  of  $K_1$  (of conductor  $17 \cdot 1951$ ) is of norm 1, both in  $K_1/k$  and  $K_1/K$ , so it generates  $\mathcal{F}_{\chi_1}$  that we may write  $\langle \eta^\tau, \eta^{\tau^2} \rangle_{\mathbb{Z}}$  since  $\eta^\sigma = \eta^{-1}$  and  $\eta = \eta^{-\tau-\tau^2}$ . Computing in  $\mathbb{Q}(\mu_{17 \cdot 1951})/K_1$  gives (taking logarithms for convenience):

$$\left\{ \begin{array}{l} \log(\eta^\tau) = +32.072728696925313868267792411432213485, \\ \log(\eta^{\tau^2}) = -27.025540940115152089120603990746892715, \\ \log(\eta) = -5.0471877568101617791471884206853207885. \end{array} \right.$$

The group  $\mathcal{E}_{K_1}$  given by PARI is of the form  $\mathcal{E}_{K_1} = \mathcal{E}_k \oplus \mathcal{E}_K \oplus \langle e_4, e_5 \rangle_{\mathbb{Z}}$ , where  $\mathbf{N}_{K_1/K}(e_4) = \mathbf{N}_{K_1/K}(e_5) = 1$ ,  $\mathbf{N}_{K_1/k}(e_4) = \mathbf{N}_{K_1/k}(e_5) = 1$ ,  $\mathcal{E}_{\chi_1} = \langle e_4, e_5 \rangle_{\mathbb{Z}}$ , with:

$$\left\{ \begin{array}{l} \log(e_4) = -8.0181821742313284670669481028585732345, \\ \log(e_5) = +6.7563852350287880222801509976867231766, \end{array} \right.$$

yielding immediately  $\eta^\tau = e_4^{-4}$ ,  $\eta^{\tau^2} = e_5^{-4}$ . Thus, in this example:

$$\mathcal{E}_{K_1}/\mathcal{E}_{K_1}^0 \cdot \mathcal{F}_{K_1} = \mathcal{E}_{\chi_1}/\mathcal{F}_{\chi_1} = \langle e_4, e_5 \rangle_{\mathbb{Z}} / \langle \eta^\tau, \eta^{\tau^2} \rangle_{\mathbb{Z}} \simeq \mathbf{Z}/4\mathbf{Z},$$

of order 16. So, we have  $\#\mathcal{H}_{\chi_1}^{\text{ar}} = \#(\mathcal{E}_{\chi_1}/\mathcal{F}_{\chi_1})$  (possibly with different structures), which relativizes the interest of algebraic definitions, regarding analytic formulas, since  $\mathcal{H}_{\chi_1}^{\text{alg}} = \mathcal{H}_{K_1} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$  in that example.

**Remark 5.4.** *More generally, Galois cohomology groups are based on algebraic definitions of the norms, so that results strongly depend on capitulation phenomena. For instance, let  $L/K$  be a cyclic  $p$ -extension of Galois group  $G$  such that all the  $r$  prime ideals of  $K$  ramified in  $L/K$ , are totally ramified; thus,*

$$\mathbf{H}^1(G, \mathcal{H}_L) = \text{Ker}_{\mathcal{H}_L}(\nu_{L/K}) / \mathcal{H}_L^{\sigma-1} \quad \& \quad \mathbf{H}^2(G, \mathcal{H}_L) = \mathcal{H}_L^G / \nu_{L/K}(\mathcal{H}_L),$$

are of same order  $\frac{\#\mathcal{H}_L^G}{\#\mathbf{J}_{L/K}(\mathcal{H}_K)} = \frac{\#\mathcal{H}_K}{\#\mathbf{J}_{L/K}(\mathcal{H}_K)} \times \frac{p^{N(r-1)}}{\#\omega_{L/K}(\mathbf{E}_K)}$ . So, if  $\text{Ker}(\mathbf{J}_{L/K}) = 1$ , the order is  $\frac{p^{N(r-1)}}{\#\omega_{L/K}(\mathbf{E}_K)}$ ; if  $\mathcal{H}_K$  capitulates, the order becomes  $\#\mathcal{H}_K \times \frac{p^{N(r-1)}}{\#\omega_{L/K}(\mathbf{E}_K)}$  and any intermediate situation does exist.

## 6. TABLES FOR CUBIC FIELDS AND $p = 2$

We consider various totally ramified cyclic  $p$ -extensions  $L/K$ , where  $K$  is a cyclic cubic field and  $L = K(\mu_\ell)$ ,  $\ell \equiv 1 \pmod{2p^N}$ . The program eliminates the cases of stability  $\#\mathcal{H}_{K_1} = \#\mathcal{H}_K$  since capitulation holds in a suitable layer if  $e(K) \leq N$ . Then  $\text{vHK}$  defines the minimal  $p$ -adic valuation of the  $\#\mathcal{H}_K$ 's to be considered (it may be chosen at will) then  $r \in \{1, 3\}$  is the number of prime ideals above  $\ell$  in  $K$ .

The submodules  $\mathcal{V}_{K_n/K}(\mathcal{H}_{K_n}) = \mathbf{J}_{K_n/K}(\mathcal{H}_{K_n})$  are computed for  $n \leq 2$  and given under the form  $\mathbf{h}_j^{[(S-1)^i]} = [e_1, \dots, e_{rKn}]$ , where  $rKn$  is the  $\mathbb{Z}$ -rank of  $\mathbf{H}_{K_n}$ .

**6.1. Case  $\ell = 17$ .** We give an excerpt of the various cases obtained (all these examples show the randomness of the structures and of the capitulations, complete or incomplete). We indicate if  $\mathcal{H}_K$  capitulates in  $K_3$  (not computed) which holds as soon as  $\#\mathcal{H}_{K_2} = \#\mathcal{H}_{K_1}$  (stability from  $K_1$ ) and  $e(K) \leq 2$  (Theorem 1.2 (i)):

MAIN PROGRAM FOR CYCLIC CUBIC FIELDS

```
{p=2;Nn=2;bf=7;Bf=10^4;vHK=2;ell=17;mKn=2;
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
K=bnfinit(PK,1);r=matsize(idealfactor(K,ell))[1];
\\Testing the order of the p-class group of K compared to vHK:
HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
\\Test for elimination of the stability from K:
if(n==1 & valuation(HKn,p)==valuation(HK,p),break);
if(n==1,print("f=",f," PK=",PK," CK0=",CK0[2]," ell=",ell," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
\\Search of a generator S of Gal(Kn/K):
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
\\Computation of the filtration:
for(j=1,rKn,X=CKn[3][j];Y=X;for(i=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(j=1,rKn,c=B[j];w=valuation(CKn[2][j],p);c=lift(Mod(c,p^w));
listput(Ehij,c,j));print("h_",j,"^[","(S-1)^",i,"]=",Ehij));
\\Computation of the algebraic norms of the rKn generators h_j:
for(j=1,rKn,A0=CKn[3][j];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
\\Reduction modulo suitable p-powers of the exponents:
Enu=List;for(j=1,rKn,c=B[j];w=valuation(CKn[2][j],p);
c=lift(Mod(c,p^w));listput(Enu,c,j));
print("norm in K",n,"/K of the component ",j," of CK",n,"=",Enu))))))}
p=2 f=607 PK=x^3+x^2-202*x-1169 CK0=[2,2] ell=17 r=1
CK1=[2,2,2,2]
h_1^[(S-1)^1]=[1,0,0,1] h_2^[(S-1)^1]=[0,1,1,1]
h_3^[(S-1)^1]=[1,1,1,0] h_4^[(S-1)^1]=[1,0,0,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,0,0,1]
norm in K1/K of the component 2 of CK1:[0,1,1,1]
norm in K1/K of the component 3 of CK1:[1,1,1,0]
```

norm in  $K1/K$  of the component 4 of  $CK1$ : [1,0,0,1]  
 No capitulation,  $m(K1)=2$ ,  $e(K1)=1$   
 $CK2=[2,2,2,2]$   
 $h_1^[(S-1)^1]=[1,0,1,0]$      $h_2^[(S-1)^1]=[0,1,0,1]$   
 $h_3^[(S-1)^1]=[1,0,1,0]$      $h_4^[(S-1)^1]=[0,1,0,1]$   
 $h_1^[(S-1)^2]=[0,0,0,0]$      $h_2^[(S-1)^2]=[0,0,0,0]$   
 $h_3^[(S-1)^2]=[0,0,0,0]$      $h_4^[(S-1)^2]=[0,0,0,0]$   
 norm in  $K2/K$  of the component 1 of  $CK2$ : [0,0,0,0]  
 norm in  $K2/K$  of the component 2 of  $CK2$ : [0,0,0,0]  
 norm in  $K2/K$  of the component 3 of  $CK2$ : [0,0,0,0]  
 norm in  $K2/K$  of the component 4 of  $CK2$ : [0,0,0,0]  
 Complete capitulation,  $m(K2)=2$ ,  $e(K2)=1$

$p=2$   $f=1009$   $PK=x^3+x^2-336*x-1719$   $CK0=[2,2]$   $e_{11}=17$   $r=1$   
 $CK1=[28,4]=[4,4]$   
 $h_1^[(S-1)^1]=[0,2]$      $h_2^[(S-1)^1]=[2,2]$   
 $h_1^[(S-1)^2]=[0,0]$      $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K1/K$  of the component 1 of  $CK1$ : [2,2]  
 norm in  $K1/K$  of the component 2 of  $CK1$ : [2,0]  
 No capitulation,  $m(K1)=2$ ,  $e(K1)=2$   
 $CK2=[28,4]=[4,4]$   
 $h_1^[(S-1)^1]=[0,2]$      $h_2^[(S-1)^1]=[2,2]$   
 $h_1^[(S-1)^2]=[0,0]$      $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K2/K$  of the component 1 of  $CK2$ : [0,0]  
 norm in  $K2/K$  of the component 2 of  $CK2$ : [0,0]  
 Complete capitulation,  $m(K2)=2$ ,  $e(K2)=2$

$p=2$   $f=1789$   $PK=x^3+x^2-596*x-5632$   $CK0=[2,2]$   $e_{11}=17$   $r=1$   
 $CK1=[24,8]=[8,8]$   
 $h_1^[(S-1)^1]=[2,0]$      $h_2^[(S-1)^1]=[0,2]$   
 $h_1^[(S-1)^2]=[4,0]$      $h_2^[(S-1)^2]=[0,4]$   
 norm in  $K1/K$  of the component 1 of  $CK1$ : [4,0]  
 norm in  $K1/K$  of the component 2 of  $CK1$ : [0,4]  
 No capitulation,  $m(K1)=3$ ,  $e(K1)=3$   
 $CK2=[312,8]=[8,8]$   
 $h_1^[(S-1)^1]=[2,0]$      $h_2^[(S-1)^1]=[0,2]$   
 $h_1^[(S-1)^2]=[4,0]$      $h_2^[(S-1)^2]=[0,4]$   
 norm in  $K2/K$  of the component 1 of  $CK2$ : [0,0]  
 norm in  $K2/K$  of the component 2 of  $CK2$ : [0,0]  
 Complete capitulation,  $m(K2)=3$ ,  $e(K2)=3$

$p=2$   $f=2077$   $PK=x^3+x^2-692*x-7231$   $CK0=[6,2]$   $e_{11}=17$   $r=1$   
 $CK1=[6,2,2,2]=[2,2,2,2]$   
 $h_1^[(S-1)^1]=[1,1,1,0]$      $h_2^[(S-1)^1]=[0,0,1,1]$   
 $h_3^[(S-1)^1]=[1,1,0,1]$      $h_4^[(S-1)^1]=[1,1,0,1]$   
 $h_1^[(S-1)^2]=[0,0,0,0]$      $h_2^[(S-1)^2]=[0,0,0,0]$   
 $h_3^[(S-1)^2]=[0,0,0,0]$      $h_4^[(S-1)^2]=[0,0,0,0]$   
 norm in  $K1/K$  of the component 1 of  $CK1$ : [1,1,1,0]  
 norm in  $K1/K$  of the component 2 of  $CK1$ : [0,0,1,1]  
 norm in  $K1/K$  of the component 3 of  $CK1$ : [1,1,0,1]  
 norm in  $K1/K$  of the component 4 of  $CK1$ : [1,1,0,1]  
 No capitulation,  $m(K1)=2$ ,  $e(K1)=1$   
 $CK2=[6,2,2,2,2,2]=[2,2,2,2,2,2]$   
 $h_1^[(S-1)^1]=[1,0,0,0,1,0]$      $h_2^[(S-1)^1]=[0,1,0,0,0,1]$   
 $h_3^[(S-1)^1]=[1,1,0,1,0,1]$      $h_4^[(S-1)^1]=[1,0,0,1,0,0]$   
 $h_5^[(S-1)^1]=[1,0,0,1,0,0]$      $h_6^[(S-1)^1]=[0,0,1,1,0,1]$   
 $h_1^[(S-1)^2]=[0,0,0,1,1,0]$      $h_2^[(S-1)^2]=[0,1,1,1,0,0]$   
 $h_3^[(S-1)^2]=[0,1,1,0,1,0]$      $h_4^[(S-1)^2]=[0,0,0,1,1,0]$   
 $h_5^[(S-1)^2]=[0,0,0,1,1,0]$      $h_6^[(S-1)^2]=[0,1,1,1,0,0]$   
 norm in  $K2/K$  of the component 1 of  $CK2$ : [0,0,0,0,0,0]  
 norm in  $K2/K$  of the component 2 of  $CK2$ : [0,0,0,0,0,0]  
 norm in  $K2/K$  of the component 3 of  $CK2$ : [0,0,0,0,0,0]  
 norm in  $K2/K$  of the component 4 of  $CK2$ : [0,0,0,0,0,0]  
 norm in  $K2/K$  of the component 5 of  $CK2$ : [0,0,0,0,0,0]  
 norm in  $K2/K$  of the component 6 of  $CK2$ : [0,0,0,0,0,0]  
 Complete capitulation,  $m(K2)=3$ ,  $e(K2)=1$



$p=2$   $f=2817$   $PK=x^3-939x+6886$   $CK0=[12,4]$   $e_{11}=17$   $r=1$   
 $CK1=[84,4]=[4,4]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 $h_1^[(S-1)^2]=[0,0]$   $h_2^[(S-1)^2]=[0,0]$   
norm in  $K1/K$  of the component 1 of  $CK1:[2,0]$   
norm in  $K1/K$  of the component 2 of  $CK1:[0,2]$   
Incomplete capitulation,  $m(K1)=1$ ,  $e(K1)=2$   
 $CK2=[84,4]=[4,4]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 $h_1^[(S-1)^2]=[0,0]$   $h_2^[(S-1)^2]=[0,0]$   
norm in  $K2/K$  of the component 1 of  $CK2:[0,0]$   
norm in  $K2/K$  of the component 2 of  $CK2:[0,0]$   
Complete capitulation,  $m(K2)=1$ ,  $e(K2)=2$

$p=2$   $f=3357$   $PK=x^3-1119x+9325$   $CK0=[6,2]$   $e_{11}=17$   $r=3$   
 $CK1=[6,2,2,2]=[2,2,2,2]$   
 $h_1^[(S-1)^1]=[0,0,0,0]$   $h_2^[(S-1)^1]=[0,0,0,0]$   
 $h_3^[(S-1)^1]=[0,0,0,0]$   $h_4^[(S-1)^1]=[0,0,0,0]$   
norm in  $K1/K$  of the component 1 of  $CK1:[0,0,0,0]$   
norm in  $K1/K$  of the component 2 of  $CK1:[0,0,0,0]$   
norm in  $K1/K$  of the component 3 of  $CK1:[0,0,0,0]$   
norm in  $K1/K$  of the component 4 of  $CK1:[0,0,0,0]$   
Complete capitulation,  $m(K1)=1$ ,  $e(K1)=1$   
 $CK2=[12,4,2,2]=[4,4,2,2]$   
 $h_1^[(S-1)^1]=[2,0,0,0]$   $h_2^[(S-1)^1]=[0,0,0,0]$   
 $h_3^[(S-1)^1]=[2,0,0,0]$   $h_4^[(S-1)^1]=[0,2,0,0]$   
 $h_1^[(S-1)^2]=[0,0,0,0]$   $h_2^[(S-1)^2]=[0,0,0,0]$   
 $h_3^[(S-1)^2]=[0,0,0,0]$   $h_4^[(S-1)^2]=[0,0,0,0]$   
norm in  $K2/K$  of the component 1 of  $CK2:[0,0,0,0]$   
norm in  $K2/K$  of the component 2 of  $CK2:[0,0,0,0]$   
norm in  $K2/K$  of the component 3 of  $CK2:[0,0,0,0]$   
norm in  $K2/K$  of the component 4 of  $CK2:[0,0,0,0]$   
Complete capitulation,  $m(K2)=2$ ,  $e(K2)=2$

$p=2$   $f=5479$   $PK=x^3+x^2-1826x+13799$   $CK0=[2,2]$   $e_{11}=17$   $r=1$   
 $CK1=[2,2,2,2]$   
 $h_1^[(S-1)^1]=[1,0,0,1]$   $h_2^[(S-1)^1]=[1,1,1,0]$   
 $h_3^[(S-1)^1]=[0,1,1,1]$   $h_4^[(S-1)^1]=[1,0,0,1]$   
 $h_1^[(S-1)^2]=[0,0,0,0]$   $h_2^[(S-1)^2]=[0,0,0,0]$   
 $h_3^[(S-1)^2]=[0,0,0,0]$   $h_4^[(S-1)^2]=[0,0,0,0]$   
norm in  $K1/K$  of the component 1 of  $CK1:[1,0,0,1]$   
norm in  $K1/K$  of the component 2 of  $CK1:[1,1,1,0]$   
norm in  $K1/K$  of the component 3 of  $CK1:[0,1,1,1]$   
norm in  $K1/K$  of the component 4 of  $CK1:[1,0,0,1]$   
No capitulation,  $m(K1)=2$ ,  $e(K1)=1$   
 $CK2=[4,4,4,4]$   
 $h_1^[(S-1)^1]=[3,1,3,1]$   $h_2^[(S-1)^1]=[3,3,0,3]$   
 $h_3^[(S-1)^1]=[0,2,2,2]$   $h_4^[(S-1)^1]=[0,2,3,0]$   
 $h_1^[(S-1)^2]=[0,2,2,0]$   $h_2^[(S-1)^2]=[2,2,2,0]$   
 $h_3^[(S-1)^2]=[2,2,2,2]$   $h_4^[(S-1)^2]=[2,0,2,0]$   
norm in  $K2/K$  of the component 1 of  $CK2:[0,0,2,0]$   
norm in  $K2/K$  of the component 2 of  $CK2:[2,2,2,2]$   
norm in  $K2/K$  of the component 3 of  $CK2:[0,0,0,0]$   
norm in  $K2/K$  of the component 4 of  $CK2:[2,2,0,2]$   
No capitulation,  $m(K2)=4$ ,  $e(K2)=2$

$p=2$   $f=6247$   $PK=x^3+x^2-2082x-35631$   $CK0=[4,4]$   $e_{11}=17$   $r=1$   
 $CK1=[24,8,2,2]=[8,8,2,2]$   
 $h_1^[(S-1)^1]=[0,2,0,1]$   $h_2^[(S-1)^1]=[6,6,1,0]$   
 $h_3^[(S-1)^1]=[0,4,0,0]$   $h_4^[(S-1)^1]=[4,0,0,0]$   
 $h_1^[(S-1)^2]=[0,4,0,0]$   $h_2^[(S-1)^2]=[4,4,0,0]$   
 $h_3^[(S-1)^2]=[0,0,0,0]$   $h_4^[(S-1)^2]=[0,0,0,0]$   
norm in  $K1/K$  of the component 1 of  $CK1:[2,2,0,1]$   
norm in  $K1/K$  of the component 2 of  $CK1:[6,0,1,0]$   
norm in  $K1/K$  of the component 3 of  $CK1:[0,4,0,0]$   
norm in  $K1/K$  of the component 4 of  $CK1:[4,0,0,0]$

No capitulation,  $m(K_1)=4$ ,  $e(K_1)=3$   
 $CK_2=[24,8,2,2]=[8,8,2,2]$   
 $h_1^[(S-1)^1]=[0,6,1,1]$   $h_2^[(S-1)^1]=[2,6,0,1]$   
 $h_3^[(S-1)^1]=[4,4,0,0]$   $h_4^[(S-1)^1]=[0,4,0,0]$   
 $h_1^[(S-1)^2]=[0,4,0,0]$   $h_2^[(S-1)^2]=[4,4,0,0]$   
 $h_3^[(S-1)^2]=[0,0,0,0]$   $h_4^[(S-1)^2]=[0,0,0,0]$   
 norm in  $K_2/K$  of the component 1 of  $CK_2$ :  $[4,4,0,0]$   
 norm in  $K_2/K$  of the component 2 of  $CK_2$ :  $[4,0,0,0]$   
 norm in  $K_2/K$  of the component 3 of  $CK_2$ :  $[0,0,0,0]$   
 norm in  $K_2/K$  of the component 4 of  $CK_2$ :  $[0,0,0,0]$   
 Incomplete capitulation,  $m(K_2)=3$ ,  $e(K_2)=3$   
 Complete capitulation in  $K_3$  (stability from  $K_1$ )  
 $p=2$   $f=9247$   $PK=x^3+x^2-3082*x-27056$   $CK_0=[12,4]$   $e_{11}=17$   $r=3$   
 $CK_1=[24,8]=[8,8]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 norm in  $K_1/K$  of the component 1 of  $CK_1$ :  $[2,0]$   
 norm in  $K_1/K$  of the component 2 of  $CK_1$ :  $[0,2]$   
 No capitulation,  $m(K_1)=1$ ,  $e(K_1)=3$   
 $CK_2=[48,16]=[16,16]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 norm in  $K_2/K$  of the component 1 of  $CK_2$ :  $[4,0]$   
 norm in  $K_2/K$  of the component 2 of  $CK_2$ :  $[0,4]$   
 No capitulation,  $m(K_2)=1$ ,  $e(K_2)=4$   
 $p=2$   $f=20887$   $PK=x^3+x^2-6962*x-225889$   $CK_0=[4,4,2,2]$   $e_{11}=17$   $r=3$   
 $CK_1=[8,8,2,2]$   
 $h_1^[(S-1)^1]=[0,0,0,0]$   $h_2^[(S-1)^1]=[0,0,0,0]$   
 $h_3^[(S-1)^1]=[0,0,0,0]$   $h_4^[(S-1)^1]=[0,0,0,0]$   
 norm in  $K_1/K$  of the component 1 of  $CK_1$ :  $[2,0,0,0]$   
 norm in  $K_1/K$  of the component 2 of  $CK_1$ :  $[0,2,0,0]$   
 norm in  $K_1/K$  of the component 3 of  $CK_1$ :  $[0,0,0,0]$   
 norm in  $K_1/K$  of the component 4 of  $CK_1$ :  $[0,0,0,0]$   
 Incomplete capitulation,  $m(K_1)=1$ ,  $e(K_1)=3$   
 $CK_2=[16,16,2,2]$   
 $h_1^[(S-1)^1]=[8,0,0,0]$   $h_2^[(S-1)^1]=[0,8,0,0]$   
 $h_3^[(S-1)^1]=[0,0,0,0]$   $h_4^[(S-1)^1]=[0,0,0,0]$   
 $h_1^[(S-1)^2]=[0,0,0,0]$   $h_2^[(S-1)^2]=[0,0,0,0]$   
 $h_3^[(S-1)^2]=[0,0,0,0]$   $h_4^[(S-1)^2]=[0,0,0,0]$   
 norm in  $K_2/K$  of the component 1 of  $CK_2$ :  $[4,0,0,0]$   
 norm in  $K_2/K$  of the component 2 of  $CK_2$ :  $[0,4,0,0]$   
 norm in  $K_2/K$  of the component 3 of  $CK_2$ :  $[0,0,0,0]$   
 norm in  $K_2/K$  of the component 4 of  $CK_2$ :  $[0,0,0,0]$   
 Incomplete capitulation,  $m(K_2)=2$ ,  $e(K_2)=4$   
 $p=2$   $f=48769$   $PK=x^3+x^2-16256*x-7225$   $CK_0=[24,8]$   $e_{11}=17$   $r=3$   
 $CK_1=[48,16]=[16,16]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 norm in  $K_1/K$  of the component 1 of  $CK_1$ :  $[2,0]$   
 norm in  $K_1/K$  of the component 2 of  $CK_1$ :  $[0,2]$   
 No capitulation,  $m(K_1)=1$ ,  $e(K_1)=4$   
 $CK_2=[48,16]=[16,16]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 norm in  $K_2/K$  of the component 1 of  $CK_2$ :  $[4,0]$   
 norm in  $K_2/K$  of the component 2 of  $CK_2$ :  $[0,4]$   
 Incomplete capitulation,  $m(K_2)=1$ ,  $e(K_2)=4$   
 $p=2$   $f=55609$   $PK=x^3+x^2-18536*x-823837$   $CK_0=[4,4,2,2]$   $e_{11}=17$   $r=3$   
 $CK_1=[56,8,2,2]=[8,8,2,2]$   
 $h_1^[(S-1)^1]=[4,0,0,0]$   $h_2^[(S-1)^1]=[0,0,0,0]$   
 $h_3^[(S-1)^1]=[4,0,0,0]$   $h_4^[(S-1)^1]=[0,4,0,0]$   
 $h_1^[(S-1)^2]=[0,0,0,0]$   $h_2^[(S-1)^2]=[0,0,0,0]$   
 $h_3^[(S-1)^2]=[0,0,0,0]$   $h_4^[(S-1)^2]=[0,0,0,0]$   
 norm in  $K_1/K$  of the component 1 of  $CK_1$ :  $[6,0,0,0]$   
 norm in  $K_1/K$  of the component 2 of  $CK_1$ :  $[0,2,0,0]$   
 norm in  $K_1/K$  of the component 3 of  $CK_1$ :  $[4,0,0,0]$   
 norm in  $K_1/K$  of the component 4 of  $CK_1$ :  $[0,4,0,0]$

Incomplete capitulation,  $m(K_1)=2$ ,  $e(K_1)=3$   
 $CK_2=[56,8,2,2]=[8,8,2,2]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}1]=[0,0,0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}1]=[0,0,0,0]$   
 $h_{-3}^{\wedge}[(S-1)^{\wedge}1]=[0,4,0,0]$   $h_{-4}^{\wedge}[(S-1)^{\wedge}1]=[4,4,0,0]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   
 $h_{-3}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   $h_{-4}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   
norm in  $K_2/K$  of the component 1 of  $CK_2$ :  $[4,0,0,0]$   
norm in  $K_2/K$  of the component 2 of  $CK_2$ :  $[0,4,0,0]$   
norm in  $K_2/K$  of the component 3 of  $CK_2$ :  $[0,0,0,0]$   
norm in  $K_2/K$  of the component 4 of  $CK_2$ :  $[0,0,0,0]$   
Complete capitulation,  $m(K_2)=2$ ,  $e(K_2)=3$

**6.2. Case  $\ell = 17$  and  $f \in \{9283, 7687, 44857\}$ .** Let's give some comments on interesting examples given by the program:

$p=2$   $f=9283$   $PK=x^3+x^2-3094*x-5501$   $CK_0=[2,2]$   $e_{11}=17$   $r=1$   
 $CK_1=[48,16]=[16,16]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}1]=[6,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}1]=[0,6]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}2]=[4,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}2]=[0,4]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}3]=[8,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}3]=[0,8]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}4]=[0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}4]=[0,0]$   
norm in  $K_1/K$  of the component 1 of  $CK_1$ :  $[8,0]$   
norm in  $K_1/K$  of the component 2 of  $CK_1$ :  $[0,8]$   
No capitulation,  $m(K_1)=4$ ,  $e(K_1)=4$   
 $CK_2=[48,16]=[16,16]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}1]=[6,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}1]=[0,6]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}2]=[4,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}2]=[0,4]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}3]=[8,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}3]=[0,8]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}4]=[0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}4]=[0,0]$   
norm in  $K_2/K$  of the component 1 of  $CK_2$ :  $[0,0]$   
norm in  $K_2/K$  of the component 2 of  $CK_2$ :  $[0,0]$   
Complete capitulation,  $m(K_2)=4$ ,  $e(K_2)=4$

There is complete capitulation, even if conditions of Theorem 1.1 are not fulfilled for the  $K_n/K$ 's (for  $n = 2$ ,  $m(K_2) = 4$ ,  $s(K_2) = 2$ ,  $e(K_2) = 4$ ,  $n - s(K_2) = 0$ ). Moreover, the exponent of  $\mathcal{H}_{K_1}$  is  $2^4$  giving a larger complexity in  $K_1/K$ , but in  $K_n$ ,  $n \geq 2$ , the exponent is still  $2^4$  (no increasing of the complexity). Some other examples are:

$p=2$   $f=7687$   $PK=x^3+x^2-2562*x-48969$   $CK_0=[2,2,2,2]$   $e_{11}=17$   $r=1$   
 $CK_1=[4,4,2,2]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}1]=[0,2,0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}1]=[0,0,0,0]$   
 $h_{-3}^{\wedge}[(S-1)^{\wedge}1]=[2,0,0,0]$   $h_{-4}^{\wedge}[(S-1)^{\wedge}1]=[0,2,0,0]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   
 $h_{-3}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   $h_{-4}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   
norm in  $K_1/K$  of the component 1 of  $CK_1$ :  $[2,2,0,0]$   
norm in  $K_1/K$  of the component 2 of  $CK_1$ :  $[0,2,0,0]$   
norm in  $K_1/K$  of the component 3 of  $CK_1$ :  $[2,0,0,0]$   
norm in  $K_1/K$  of the component 4 of  $CK_1$ :  $[0,2,0,0]$   
Incomplete capitulation,  $m(K_1)=2$ ,  $e(K_1)=2$   
 $CK_2=[4,4,2,2]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}1]=[2,2,0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}1]=[2,0,0,0]$   
 $h_{-3}^{\wedge}[(S-1)^{\wedge}1]=[2,2,0,0]$   $h_{-4}^{\wedge}[(S-1)^{\wedge}1]=[2,0,0,0]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   
 $h_{-3}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   $h_{-4}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   
norm in  $K_2/K$  of the component 1 of  $CK_2$ :  $[0,0,0,0]$   
norm in  $K_2/K$  of the component 2 of  $CK_2$ :  $[0,0,0,0]$   
norm in  $K_2/K$  of the component 3 of  $CK_2$ :  $[0,0,0,0]$   
norm in  $K_2/K$  of the component 4 of  $CK_2$ :  $[0,0,0,0]$   
Complete capitulation,  $m(K_2)=2$ ,  $e(K_2)=2$

$p=2$   $f=44857$   $PK=x^3+x^2-14952*x-704421$   $CK_0=[6,2,2,2]$   $e_{11}=17$   $r=3$   
 $CK_1=[12,12,2,2]=[4,4,2,2]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}1]=[0,0,0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}1]=[0,0,0,0]$   
 $h_{-3}^{\wedge}[(S-1)^{\wedge}1]=[2,0,0,0]$   $h_{-4}^{\wedge}[(S-1)^{\wedge}1]=[2,2,0,0]$   
 $h_{-1}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$   $h_{-2}^{\wedge}[(S-1)^{\wedge}2]=[0,0,0,0]$

$h_3^[(S-1)^2]=[0,0,0,0]$   $h_4^[(S-1)^2]=[0,0,0,0]$   
 norm in  $K1/K$  of the component 1 of  $CK1:[2,0,0,0]$   
 norm in  $K1/K$  of the component 2 of  $CK1:[0,2,0,0]$   
 norm in  $K1/K$  of the component 3 of  $CK1:[2,0,0,0]$   
 norm in  $K1/K$  of the component 4 of  $CK1:[2,2,0,0]$   
 Incomplete capitulation,  $m(K1)=2$ ,  $e(K1)=2$   
 $CK2=[12,12,2,2]=[4,4,2,2]$   
 $h_1^[(S-1)^1]=[0,0,0,0]$   $h_2^[(S-1)^1]=[2,2,0,0]$   
 $h_3^[(S-1)^1]=[2,2,0,0]$   $h_4^[(S-1)^1]=[0,2,0,0]$   
 $h_1^[(S-1)^2]=[0,0,0,0]$   $h_2^[(S-1)^2]=[0,0,0,0]$   
 $h_3^[(S-1)^2]=[0,0,0,0]$   $h_4^[(S-1)^2]=[0,0,0,0]$   
 norm in  $K2/K$  of the component 1 of  $CK2:[0,0,0,0]$   
 norm in  $K2/K$  of the component 2 of  $CK2:[0,0,0,0]$   
 norm in  $K2/K$  of the component 3 of  $CK2:[0,0,0,0]$   
 norm in  $K2/K$  of the component 4 of  $CK2:[0,0,0,0]$   
 Complete capitulation,  $m(K2)=2$ ,  $e(K2)=2$

these examples suggest that the order of magnitude of the  $p$ -ranks of the  $\mathcal{H}_K$ 's is not an obstruction to a capitulation in such cyclic  $p$ -extensions  $L \subset K(\mu_\ell)$ . In the above cases, the capitulation is obtained by means of a stability in larger layers.

**6.3. Case  $\ell = 97$ .** Similarly, we give a table for  $\ell = 97$  allowing capitulations up to  $K_4$ . One finds much more cases of capitulation (not given in the table since they are very numerous); it seems clearly that a larger value of  $N$  may intervene in the phenomenon of capitulation:

$p=2$   $f=349$   $PK=x^3+x^2-116*x-517$   $CK0=[2,2]$   $e_{11}=97$   $r=1$   
 $CK1=[4,4]$   
 $h_1^[(S-1)^1]=[2,2]$   $h_2^[(S-1)^1]=[2,0]$   
 $h_1^[(S-1)^2]=[0,0]$   $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K1/K$  of the component 1 of  $CK1:[0,2]$   
 norm in  $K1/K$  of the component 2 of  $CK1:[2,2]$   
 Incomplete capitulation,  $m(K1)=2$ ,  $e(K1)=2$   
 $CK2=[4,4]$   
 $h_1^[(S-1)^1]=[2,2]$   $h_2^[(S-1)^1]=[2,0]$   
 $h_1^[(S-1)^2]=[0,0]$   $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K2/K$  of the component 1 of  $CK2:[0,0]$   
 norm in  $K2/K$  of the component 2 of  $CK2:[0,0]$   
 Complete capitulation,  $m(K2)=2$ ,  $e(K2)=2$

$p=2$   $f=607$   $PK=x^3+x^2-202*x-1169$   $CK0=[2,2]$   $e_{11}=97$   $r=1$   
 $CK1=[8,8]$   
 $h_1^[(S-1)^1]=[6,4]$   $h_2^[(S-1)^1]=[4,2]$   
 $h_1^[(S-1)^2]=[4,0]$   $h_2^[(S-1)^2]=[0,4]$   
 norm in  $K1/K$  of the component 1 of  $CK1:[0,4]$   
 norm in  $K1/K$  of the component 2 of  $CK1:[4,4]$   
 Incomplete capitulation,  $m(K1)=3$ ,  $e(K1)=3$   
 $CK2=[104,8]=[8,8]$   
 $h_1^[(S-1)^1]=[6,4]$   $h_2^[(S-1)^1]=[4,2]$   
 $h_1^[(S-1)^2]=[4,0]$   $h_2^[(S-1)^2]=[0,4]$   
 norm in  $K2/K$  of the component 1 of  $CK2:[0,0]$   
 norm in  $K2/K$  of the component 2 of  $CK2:[0,0]$   
 Complete capitulation,  $m(K2)=3$ ,  $e(K2)=3$

$p=2$   $f=1957$   $PK=x^3+x^2-652*x+6016$   $CK0=[6,2]$   $e_{11}=97$   $r=3$   
 $CK1=[12,4]=[4,4]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 $h_1^[(S-1)^2]=[0,0]$   $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K1/K$  of the component 1 of  $CK1:[2,0]$   
 norm in  $K1/K$  of the component 2 of  $CK1:[0,2]$   
 No capitulation,  $m(K1)=1$ ,  $e(K1)=2$   
 $CK2=[24,8]=[8,8]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 $h_1^[(S-1)^2]=[0,0]$   $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K2/K$  of the component 1 of  $CK2:[4,0]$   
 norm in  $K2/K$  of the component 2 of  $CK2:[0,4]$

No capitulation,  $m(K2)=1$ ,  $e(K2)=3$

$p=2$   $f=4639$   $PK=x^3+x^2-1546*x+6529$   $CK0=[2,2]$   $e11=97$   $r=1$

$CK1=[4,4]$

$h_1^-[ (S-1)^1 ]=[2,0]$   $h_2^-[ (S-1)^1 ]=[0,2]$

$h_1^-[ (S-1)^2 ]=[0,0]$   $h_2^-[ (S-1)^2 ]=[0,0]$

norm in  $K1/K$  of the component 1 of  $CK1$ :  $[0,0]$

norm in  $K1/K$  of the component 2 of  $CK1$ :  $[0,0]$

Complete capitulation,  $m(K1)=2$ ,  $e(K1)=2$

$CK2=[4,4]$

$h_1^-[ (S-1)^1 ]=[2,0]$   $h_2^-[ (S-1)^1 ]=[0,2]$

$h_1^-[ (S-1)^2 ]=[0,0]$   $h_2^-[ (S-1)^2 ]=[0,0]$

norm in  $K2/K$  of the component 1 of  $CK2$ :  $[0,0]$

norm in  $K2/K$  of the component 2 of  $CK2$ :  $[0,0]$

Complete capitulation,  $m(K2)=2$ ,  $e(K2)=2$

$p=2$   $f=9391$   $PK=x^3+x^2-3130*x-24347$   $CK0=[2,2]$   $e11=97$   $r=3$

$CK1=[4,4,2,2]$

$h_1^-[ (S-1)^1 ]=[2,0,0,0]$   $h_2^-[ (S-1)^1 ]=[0,2,0,0]$

$h_3^-[ (S-1)^1 ]=[2,0,0,0]$   $h_4^-[ (S-1)^1 ]=[2,2,0,0]$

$h_1^-[ (S-1)^2 ]=[0,0,0,0]$   $h_2^-[ (S-1)^2 ]=[0,0,0,0]$

$h_3^-[ (S-1)^2 ]=[0,0,0,0]$   $h_4^-[ (S-1)^2 ]=[0,0,0,0]$

norm in  $K1/K$  of the component 1 of  $CK1$ :  $[0,0,0,0]$

norm in  $K1/K$  of the component 2 of  $CK1$ :  $[0,0,0,0]$

norm in  $K1/K$  of the component 3 of  $CK1$ :  $[2,0,0,0]$

norm in  $K1/K$  of the component 4 of  $CK1$ :  $[2,2,0,0]$

Incomplete capitulation,  $m(K1)=2$ ,  $e(K1)=2$

$CK2=[4,4,4,4,2,2]$

$h_1^-[ (S-1)^1 ]=[0,0,0,2,0,1]$   $h_2^-[ (S-1)^1 ]=[0,0,2,0,1,0]$

$h_3^-[ (S-1)^1 ]=[2,0,2,0,1,0]$   $h_4^-[ (S-1)^1 ]=[0,0,2,0,0,1]$

$h_5^-[ (S-1)^1 ]=[0,2,2,0,0,0]$   $h_6^-[ (S-1)^1 ]=[2,0,0,2,0,0]$

$h_1^-[ (S-1)^2 ]=[2,0,0,2,0,0]$   $h_2^-[ (S-1)^2 ]=[0,2,2,0,0,0]$

$h_3^-[ (S-1)^2 ]=[0,2,2,0,0,0]$   $h_4^-[ (S-1)^2 ]=[2,0,0,2,0,0]$

$h_5^-[ (S-1)^2 ]=[0,0,0,0,0,0]$   $h_6^-[ (S-1)^2 ]=[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 1 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 2 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 3 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 4 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 5 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 6 of  $CK2$ :  $[0,0,0,0,0,0]$

Complete capitulation,  $m(K2)=3$ ,  $e(K2)=2$

$p=2$   $f=20419$   $PK=x^3+x^2-6806*x-3025$   $CK0=[42,2]$   $e11=97$   $r=3$

$CK1=[84,4,2,2]=[4,4,2,2]$

$h_1^-[ (S-1)^1 ]=[0,0,0,0]$   $h_2^-[ (S-1)^1 ]=[0,0,0,0]$

$h_3^-[ (S-1)^1 ]=[2,0,0,0]$   $h_4^-[ (S-1)^1 ]=[0,2,0,0]$

$h_1^-[ (S-1)^2 ]=[0,0,0,0]$   $h_2^-[ (S-1)^2 ]=[0,0,0,0]$

$h_3^-[ (S-1)^2 ]=[0,0,0,0]$   $h_4^-[ (S-1)^2 ]=[0,0,0,0]$

norm in  $K1/K$  of the component 1 of  $CK1$ :  $[2,0,0,0]$

norm in  $K1/K$  of the component 2 of  $CK1$ :  $[0,2,0,0]$

norm in  $K1/K$  of the component 3 of  $CK1$ :  $[2,0,0,0]$

norm in  $K1/K$  of the component 4 of  $CK1$ :  $[0,2,0,0]$

No capitulation,  $m(K1)=2$ ,  $e(K1)=2$

$CK2=[84,4,4,4,2,2]=[4,4,4,4,2,2]$

$h_1^-[ (S-1)^1 ]=[0,0,2,0,1,0]$   $h_2^-[ (S-1)^1 ]=[0,2,0,2,1,0]$

$h_3^-[ (S-1)^1 ]=[2,2,2,2,0,0]$   $h_4^-[ (S-1)^1 ]=[0,2,2,2,1,1]$

$h_5^-[ (S-1)^1 ]=[2,0,0,2,0,0]$   $h_6^-[ (S-1)^1 ]=[0,0,2,2,0,0]$

$h_1^-[ (S-1)^2 ]=[2,0,0,2,0,0]$   $h_2^-[ (S-1)^2 ]=[2,0,0,2,0,0]$

$h_3^-[ (S-1)^2 ]=[0,0,0,0,0,0]$   $h_4^-[ (S-1)^2 ]=[2,0,2,0,0,0]$

$h_5^-[ (S-1)^2 ]=[0,0,0,0,0,0]$   $h_6^-[ (S-1)^2 ]=[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 1 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 2 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 3 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 4 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 5 of  $CK2$ :  $[0,0,0,0,0,0]$

norm in  $K2/K$  of the component 6 of  $CK2$ :  $[0,0,0,0,0,0]$

```

Complete capitulation, m(K2)=3, e(K2)=2
p=2 f=24589 PK=x^3+x^2-8196*x-33696 CK0=[6,2] ell=97 r=3
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,0,0,1] h_2^[(S-1)^1]=[0,1,1,0]
h_3^[(S-1)^1]=[0,1,1,0] h_4^[(S-1)^1]=[1,0,0,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,0,0,1]
norm in K1/K of the component 2 of CK1:[0,1,1,0]
norm in K1/K of the component 3 of CK1:[0,1,1,0]
norm in K1/K of the component 4 of CK1:[1,0,0,1]
No capitulation, m(K1)=2, e(K1)=1
CK2=[6,2,2,2,2,2]=[2,2,2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0,0,0] h_2^[(S-1)^1]=[0,0,0,0,0,0]
h_3^[(S-1)^1]=[1,1,0,1,0,0] h_4^[(S-1)^1]=[1,1,0,0,0,0]
h_5^[(S-1)^1]=[0,1,0,1,1,1] h_6^[(S-1)^1]=[1,1,0,1,1,1]
h_1^[(S-1)^2]=[0,0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0,0]
h_3^[(S-1)^2]=[1,1,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0,0]
h_5^[(S-1)^2]=[0,1,0,0,0,0] h_6^[(S-1)^2]=[0,1,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=1
p=2 f=25171 PK=x^3+x^2-8390*x+273152 CK0=[14,2] ell=97 r=3
CK1=[84,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=2, e(K1)=2
CK2=[84,4,4,4]=[4,4,4,4]
h_1^[(S-1)^1]=[0,0,2,0] h_2^[(S-1)^1]=[0,0,2,0]
h_3^[(S-1)^1]=[0,0,2,0] h_4^[(S-1)^1]=[2,0,2,2]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2

```

**6.4. Capitulations for  $K$  fixed and  $\ell$  varying.** In this subsection, we fix a cyclic cubic field (given via its polynomial  $PK$ ) and consider primes  $\ell \equiv 1 \pmod{2p^N}$  with  $N \geq \text{Nell}$ , and  $\ell \leq \text{Bell}$ ; one must give  $p$ ,  $\text{Nell}$ ,  $\text{Bell}$ :

```

{p=2;Nell=3;Bell=10^3;Nn=2;f=20887;PK=x^3+x^2-6962*x-225889;K=bnfinit(PK,1);
CK0=K.clgp;forprime(ell=1,Bell,N=valuation(ell-1,p)-1;if(N<Nell,next);
r=matsize(idealfactor(K,ell))[1];for(n=1,Nn,Qn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
if(n==1,print();print("f=",f," PK=",PK," CK0=",CK0[2]," ell=",ell," N=",N," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(j=1,rKn,A0=CKn[3][j];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfprincipal(Kn,A)[1];
Enu=List;for(j=1,rKn,c=B[j];w=valuation(CKn[2][j],p);c=lift(Mod(c,p^w));
listput(Enu,c,j));print("norm in K",n,"/K of the component ",j,

```



" of CK",n," : ",Enu))))}

6.4.1. *Cubic field of conductor  $f = 1777$ .* Then  $P_K = x^3 + x^2 - 592x + 724$  and  $\mathcal{H}_K \simeq \mathbf{Z}/4\mathbf{Z}$ ; a complete capitulation in  $K_2$  holds for the following primes  $\ell \equiv 1 \pmod{8}$  (taking  $N_{\ell} = 2$ ):

$\ell \in \{41, 89, 97, 137, 233, 281, 313, 337, 353, 401, 409, 433, 449, 457, 521, 569, 577, 593, 601, 617, 673, 761, 769, 809, 857, 881, 929, 937, 953, 977, 1009, 1049, 1097, 1129, 1153, 1193, 1201, 1217, 1249, 1361, 1409, 1433, 1489, 1553, 1601, 1609, 1657, 1721, 1777, 1801, 2017, 2089, \dots\}$ ;

exceptions are  $\ell \in \{17, 73, 113, 193, 241, 257, 641, 1033, 1289, 1297, 1321, 1481, 1697, 1753, 1873, 1889, 1913, 1993, 2081, \dots\}$ .

6.4.2. *Cubic field of conductor  $f = 20887$ .* We get the more complex structure  $\mathcal{H}_K \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ ; a complete capitulation in  $K_1$  does not hold since  $e(K) = 2$ , but computations in  $K_3$  are out of reach. However, taking  $N_{\ell} = 3$ , the results allow to distinguish between incomplete capitulation in  $K_2$  and possible capitulation in  $K_n$ ,  $n \geq 3$ ; we obtain the following matrices showing always an incomplete capitulation and some stabilities, up to  $\ell = 449$  (the mention  $\text{Im}(J_n) = [a, \dots, z]$  denotes the structure of the image  $\mathbf{J}_{K_n/K}(\mathcal{H}_K)$ , to be compared with  $\text{CK}_0 = [4, 4, 2, 2]$ ):

```
p=2 f=20887 PK=x^3+x^2-6962*x-225889
ell=17 CK0=[4,4,2,2] CK1=[8,8,2,2] CK2=[16,16,2,2] N=3 r=3
norm in K1/K of the component 1 of CK1: [2,0,0,0]
norm in K1/K of the component 2 of CK1: [0,2,0,0]
norm in K1/K of the component 3 of CK1: [0,0,0,0]
norm in K1/K of the component 4 of CK1: [0,0,0,0]
Im(J1)=[4,4]
norm in K2/K of the component 1 of CK2: [4,0,0,0]
norm in K2/K of the component 2 of CK2: [0,4,0,0]
norm in K2/K of the component 3 of CK2: [0,0,0,0]
norm in K2/K of the component 4 of CK2: [0,0,0,0]
Im(J2)=[4,4]
ell=97 CK0=[4,4,2,2] CK1=[8,8,2,2] CK2=[8,8,2,2] N=4 r=3
norm in K2/K of the component 1 of CK2: [4,0,0,0]
norm in K2/K of the component 2 of CK2: [0,4,0,0]
norm in K2/K of the component 3 of CK2: [0,0,0,0]
norm in K2/K of the component 4 of CK2: [0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=353 CK0=[4,4,2,2] CK1=[4,4,4,4,2,2] CK2=[8,8,4,4,2,2] N=4 r=3
norm in K1/K of the component 1 of CK1: [2,2,0,0,0,0]
norm in K1/K of the component 2 of CK1: [2,0,0,0,0,0]
norm in K1/K of the component 3 of CK1: [3,1,2,2,1,1]
norm in K1/K of the component 4 of CK1: [1,2,0,2,1,0]
norm in K1/K of the component 5 of CK1: [2,2,0,0,0,0]
norm in K1/K of the component 6 of CK1: [0,0,0,0,0,0]
Im(J1)=[4,4,2,2], no capitulation (J1 injective)
norm in K2/K of the component 1 of CK2: [0,0,2,2,0,0]
norm in K2/K of the component 2 of CK2: [0,0,0,2,0,0]
norm in K2/K of the component 3 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2: [0,0,0,0,0,0]
Im(J2)=[2,2]
ell=433 CK0=[4,4,2,2] CK1=[8,8,2,2,2,2] CK2=[8,8,4,4,2,2] N=3 r=1
norm in K2/K of the component 1 of CK2: [0,4,2,0,0,0]
norm in K2/K of the component 2 of CK2: [0,4,0,2,0,0]
norm in K2/K of the component 3 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2: [0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2: [0,0,0,0,0,0]
Im(J2)=[2,2]
```

ell=769 CK0=[4,4,2,2] CK1=[4,4,2,2,2,2] CK2=[4,4,4,4,4,4] N=7 r=1  
 norm in K1/K of the component 1 of CK1: [2,0,0,0,0,0]  
 norm in K1/K of the component 2 of CK1: [0,2,1,1,1,0]  
 norm in K1/K of the component 3 of CK1: [0,0,0,1,0,0]  
 norm in K1/K of the component 4 of CK1: [0,0,0,0,0,0]  
 norm in K1/K of the component 5 of CK1: [0,0,0,1,0,0]  
 norm in K1/K of the component 6 of CK1: [0,0,1,0,1,0]  
 Im(J1)=[2,2,2,2]  
 norm in K2/K of the component 1 of CK2: [0,0,0,0,0,0]  
 norm in K2/K of the component 2 of CK2: [0,0,0,0,0,0]  
 norm in K2/K of the component 3 of CK2: [0,0,0,0,2,0]  
 norm in K2/K of the component 4 of CK2: [0,2,0,0,2,2]  
 norm in K2/K of the component 5 of CK2: [0,0,0,0,0,0]  
 norm in K2/K of the component 6 of CK2: [0,0,0,0,0,0]  
 Im(J2)=[2,2]  
 ell=977 CK0=[4,4,2,2] CK1=[8,8,4,4] CK2=[16,16,4,4] N=3 r=3  
 norm in K2/K of the component 1 of CK2: [4,8,0,0]  
 norm in K2/K of the component 2 of CK2: [8,4,0,0]  
 norm in K2/K of the component 3 of CK2: [0,8,0,0]  
 norm in K2/K of the component 4 of CK2: [8,0,0,0]  
 Im(J2)=[4,4]  
 ell=1009 CK0=[4,4,2,2] CK1=[8,8,4,4] CK2=[8,8,4,4] N=3 r=1  
 norm in K2/K of the component 1 of CK2: [4,0,0,0]  
 norm in K2/K of the component 2 of CK2: [0,0,0,0]  
 norm in K2/K of the component 3 of CK2: [4,0,0,0]  
 norm in K2/K of the component 4 of CK2: [4,4,0,0]  
 Im(J2)=[2,2], Capitulation in K3

6.4.3. *Remarks about the towers  $K(\mu_\ell)/K$ ,  $K$  of conductor  $f = 20887$ .* In the above values of  $\ell$ , there is always partial capitulation when  $r = 1$  but never stability from  $K$ ; in other words, if a stability from  $K_{n_0}$  does exist, then  $n_0 \geq 1$  (see Remark 1.3).

This may be explained as follows and many generalizations are possible. For simplicity, put  $\mathbf{N}_{K_1/K} =: \mathbf{N}$ ,  $\mathbf{J}_{K_1/K} =: \mathbf{J}$ ,  $\mathcal{H}_{K_1} =: \mathcal{H}_1$ ,  $\mathbf{E}_{K_1} =: \mathbf{E}_1$ ,  $\mathcal{H}_K =: \mathcal{H}$ ,  $\mathbf{E}_K =: \mathbf{E}$ .

Stability from  $K$  would imply  $\mathcal{H}_1^{G_1} = \mathcal{H}_1 \stackrel{\mathbf{N}}{\simeq} \mathcal{H} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$  (Theorem 1.2), Chevalley–Herbrand formula would imply  $\mathbf{E}/\mathbf{E} \cap \mathbf{N}(K_1^\times) \simeq 1$  or  $\mathbf{Z}/2\mathbf{Z}$  depending on  $r = 1$  or  $r = 3$ , and  $\mathcal{H}_1^{G_1} = \mathcal{H}_1$  would imply  $\mathbf{N}(\mathcal{H}_1^{G_1}) = \mathcal{H}$ .

Let's examine the consequences according to the value of  $r$ :

(i) Case  $r = 3$ . Since  $\mathbf{E}/\mathbf{E}^2 \simeq \mathbf{Z}/2\mathbf{Z}$ , the condition  $\mathbf{E}/\mathbf{E} \cap \mathbf{N}(K_1^\times) \simeq \mathbf{Z}/2\mathbf{Z}$  implies  $\mathbf{E} \cap \mathbf{N}(K_1^\times) = \mathbf{E}^2 = \mathbf{N}(\mathbf{E}_1)$ , whence  $\mathcal{H}_1^{G_1} = \mathbf{J}(\mathcal{H}) \cdot \mathcal{H}_1^{\text{ram}}$ , from exact sequence 1.1, thus  $\mathbf{N}(\mathcal{H}_1^{G_1}) = \mathcal{H}^2 \cdot \mathcal{H}^{\text{ram}} = \mathcal{H}$ ; we deduce from this,  $\mathcal{H} = \mathcal{H}^2 \cdot \mathcal{H}^{\text{ram}} = (\mathcal{H}^2 \cdot \mathcal{H}^{\text{ram}})^2 \cdot \mathcal{H}^{\text{ram}} = \mathcal{H}^4 \cdot \mathcal{H}^{\text{ram}} = \mathcal{H}^{\text{ram}}$  of 2-rank  $\leq 2$  (absurd). The case  $\ell = 353$  gives an example of injective transfer.

(ii) Case  $r = 1$ . The norm factor is trivial,  $\mathcal{H}_1^{\text{ram}} = 1$  and exact sequence 1.1 becomes  $1 \rightarrow \mathbf{J}(\mathcal{H}) \rightarrow \mathcal{H}_1^{G_1} \rightarrow \mathbf{E}/\mathbf{N}(\mathbf{E}_1) \rightarrow 1$  with  $\mathcal{H}_1^{G_1} \stackrel{\mathbf{N}}{\simeq} \mathcal{H}$  and  $\mathbf{E}/\mathbf{N}(\mathbf{E}_1)$  isomorphic to 1,  $\mathbf{Z}/2\mathbf{Z}$  or  $\mathbf{Z}/4\mathbf{Z}$ ; but Theorem 3.2 implies  $\mathbf{J}$  non injective (otherwise we get  $\#\mathcal{H}_1 \geq \#\mathcal{H} \cdot \#\mathcal{H}[2] > \#\mathcal{H}$ ), whence a partial capitulation in  $K_1$  (so,  $\mathbf{J}(\mathcal{H})$  is isomorphic to  $\mathbf{Z}/4\mathbf{Z}$ ,  $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$  or  $\mathbf{Z}/2\mathbf{Z}$ ); but  $\mathbf{N} : \mathcal{H}_1 = \mathcal{H}_1^{G_1} \rightarrow \mathcal{H}$  being an isomorphism, we have  $\mathbf{N}(\mathbf{J}(\mathcal{H})) = \mathcal{H}^2 \simeq \mathbf{Z}/2\mathbf{Z}$ , then  $\mathcal{H}/\mathcal{H}^2 \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \simeq \mathbf{E}/\mathbf{N}(\mathbf{E}_1)$  (absurd).

This gives examples of values of  $n_0 = 1$  for  $\ell = 97$  and 1009.

## 7. TABLES FOR KUMMER FIELDS AND $p = 2$

The purpose is to consider fields of the form  $K = \mathbb{Q}(\sqrt[q]{R})$ ,  $q \geq 3$ , with  $L \subset K(\mu_\ell)$ ,  $\ell \equiv 1 \pmod{2p^N}$ ; although these fields are not totally real, it is known that capitulation may exist in compositum  $L = KL_0$ , with suitable abelian  $p$ -extensions  $L_0/\mathbb{Q}$  (conjectured in [35] (1997), proved by Bosca [16] (2009)).

**7.1. Pure cubic fields,  $\ell = 17$ .** We consider the set of pure cubic fields  $K = \mathbb{Q}(\sqrt[3]{R})$ ; so  $L/\mathbb{Q}$  is not Galois, but, by chance, the instruction  $G = \text{nfgaloisconj}(Kn)$  of PARI computes the group of automorphisms, whence  $\text{Gal}(K_n/K)$  in our case; this simplifies the search of  $S$  of order  $p^n$ .

Taking  $p = 2$ ,  $\ell = 17$ ,  $Nn = 3$  and restricting to fields  $K$  such that  $\#\mathcal{H}_K \geq 4$ , we obtain many capitulations (the program eliminates the cases of stability from  $K$ ):

```

MAIN PROGRAM FOR PURE CUBIC FIELDS:
{p=2;Nn=3;vHK=2;ell=17;mKn=2;for(R=2,10^4,PK=x^3-R;
if(polisirreducible(PK)==0,next);
K=bnfinit(PK,1);r=matsize(idealfactor(K,ell))[1];
\\Testing the order of the p-class group of K compared to vHK:
HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;
\\Test for elimination of the stability from K:
if(n==1 & valuation(HKn,p)==valuation(HK,p),break);
if(n==1,print();print("PK=",PK," CK0=",CK0[2]," ell=",ell," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,p^n,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(j=1,rKn,X=CKn[3][j];Y=X;for(i=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpows(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(j=1,rKn,c=B[j];w=valuation(CKn[2][j],p);c=lift(Mod(c,p^w));
listput(Ehij,c,j));print("h_",j,"^[","(S-1)^",i,"]=",Ehij));
for(j=1,rKn,A0=CKn[3][j];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(j=1,rKn,c=B[j];w=valuation(CKn[2][j],p);c=lift(Mod(c,p^w));
listput(Enu,c,j));print("norm in K",n,"/K of the component ",j,
" of CK",n,":",Enu))}}

p=2 PK=x^3-43 CK0=[12] ell=17 r=2
CK1=[12,6]=[4,2]
h_1^[(S-1)^1]=[0,1] h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,1]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[12,12]=[4,4]
h_1^[(S-1)^1]=[0,1] h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,2]
norm in K2/K of the component 2 of CK2:[0,0]
Incomplete capitulation, m(K2)=2, e(K2)=2
CK3=[12,12]=[4,4]
h_1^[(S-1)^1]=[0,3] h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Complete capitulation (stability from K2), m(K3)=2, e(K3)=2

p=2 PK=x^3-113 CK0=[2,2] ell=17 r=2
CK1=[6,2,2]=[2,2,2]
h_1^[(S-1)^1]=[0,1,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0] h_2^[(S-1)^2]=[0,0,0] h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,1,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=1
CK2=[6,2,2,2,2]=[2,2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,1,0] h_2^[(S-1)^1]=[0,1,1,1,0]
h_3^[(S-1)^1]=[0,1,1,1,0] h_4^[(S-1)^1]=[1,1,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0]
h_1^[(S-1)^2]=[1,1,0,0,0] h_2^[(S-1)^2]=[1,1,0,0,0]
h_3^[(S-1)^2]=[1,1,0,0,0] h_4^[(S-1)^2]=[0,1,1,0,0]

```

$h_5^[(S-1)^2]=[0,0,0,0,0]$   
 norm in  $K2/K$  of the component 1 of  $CK2:[0,1,1,0,0]$   
 norm in  $K2/K$  of the component 2 of  $CK2:[0,1,1,0,0]$   
 norm in  $K2/K$  of the component 3 of  $CK2:[0,1,1,0,0]$   
 norm in  $K2/K$  of the component 4 of  $CK2:[0,0,0,0,0]$   
 norm in  $K2/K$  of the component 5 of  $CK2:[0,0,0,0,0]$   
 Incomplete capitulation,  $m(K2)=3$ ,  $e(K2)=1$   
 $CK3=[12,2,2,2,2]=[4,2,2,2,2]$   
 $h_1^[(S-1)^1]=[2,1,0,0,1]$   $h_2^[(S-1)^1]=[0,0,1,1,0]$   
 $h_3^[(S-1)^1]=[0,1,0,0,0]$   $h_4^[(S-1)^1]=[2,0,1,1,1]$   
 $h_5^[(S-1)^1]=[0,1,0,0,1]$   
 $h_1^[(S-1)^2]=[0,1,1,1,1]$   $h_2^[(S-1)^2]=[2,1,1,1,1]$   
 $h_3^[(S-1)^2]=[0,0,1,1,0]$   $h_4^[(S-1)^2]=[2,0,1,1,0]$   
 $h_5^[(S-1)^2]=[0,1,1,1,1]$   
 norm in  $K3/K$  of the component 1 of  $CK3:[0,0,0,0,0]$   
 norm in  $K3/K$  of the component 2 of  $CK3:[0,0,0,0,0]$   
 norm in  $K3/K$  of the component 3 of  $CK3:[0,0,0,0,0]$   
 norm in  $K3/K$  of the component 4 of  $CK3:[0,0,0,0,0]$   
 norm in  $K3/K$  of the component 5 of  $CK3:[0,0,0,0,0]$   
 Complete capitulation,  $m(K3)=4$ ,  $e(K3)=2$   
 $p=2$   $PK=x^3-122$   $CK0=[12]$   $e11=17$   $r=2$   
 $CK1=[12,4]=[4,4]$   
 $h_1^[(S-1)^1]=[0,1]$   $h_2^[(S-1)^1]=[0,2]$   
 $h_1^[(S-1)^2]=[0,2]$   $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K1/K$  of the component 1 of  $CK1:[2,1]$   
 norm in  $K1/K$  of the component 2 of  $CK1:[0,0]$   
 Incomplete capitulation,  $m(K1)=3$ ,  $e(K1)=2$   
 $CK2=[12,4]=[4,4]$   
 $h_1^[(S-1)^1]=[0,3]$   $h_2^[(S-1)^1]=[0,2]$   
 $h_1^[(S-1)^2]=[0,2]$   $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K2/K$  of the component 1 of  $CK2:[0,2]$   
 norm in  $K2/K$  of the component 2 of  $CK2:[0,0]$   
 Incomplete capitulation,  $m(K2)=3$ ,  $e(K2)=2$   
 $CK3=[12,4]=[4,4]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[3,2]$   
 $h_1^[(S-1)^2]=[0,0]$   $h_2^[(S-1)^2]=[2,0]$   
 norm in  $K3/K$  of the component 1 of  $CK3:[0,0]$   
 norm in  $K3/K$  of the component 2 of  $CK3:[0,0]$   
 Complete capitulation (stability from  $K1$ ),  $m(K3)=3$ ,  $e(K3)=2$   
 $p=2$   $PK=x^3-141$   $CK0=[4,2]$   $e11=17$   $r=2$   
 $CK1=[8,2]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 norm in  $K1/K$  of the component 1 of  $CK1:[2,0]$   
 norm in  $K1/K$  of the component 2 of  $CK1:[0,0]$   
 Incomplete capitulation,  $m(K1)=1$ ,  $e(K1)=3$   
 $CK2=[16,2]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 norm in  $K2/K$  of the component 1 of  $CK2:[4,0]$   
 norm in  $K2/K$  of the component 2 of  $CK2:[0,0]$   
 Incomplete capitulation,  $m(K2)=1$ ,  $e(K2)=4$   
 $CK3=[288,18]=[32,2]$   
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[0,0]$   
 norm in  $K3/K$  of the component 1 of  $CK3:[8,0]$   
 norm in  $K3/K$  of the component 2 of  $CK3:[0,0]$   
 Incomplete capitulation,  $m(K3)=1$ ,  $e(K3)=5$   
 $p=2$   $PK=x^3-174$   $CK0=[6,2]$   $e11=17$   $r=2$   
 $CK1=[12,6,2]=[4,2,2]$   
 $h_1^[(S-1)^1]=[2,1,0]$   $h_2^[(S-1)^1]=[0,0,0]$   $h_3^[(S-1)^1]=[0,0,0]$   
 $h_1^[(S-1)^2]=[0,0,0]$   $h_2^[(S-1)^2]=[0,0,0]$   $h_3^[(S-1)^2]=[0,0,0]$   
 norm in  $K1/K$  of the component 1 of  $CK1:[0,1,0]$   
 norm in  $K1/K$  of the component 2 of  $CK1:[0,0,0]$   
 norm in  $K1/K$  of the component 3 of  $CK1:[0,0,0]$   
 Incomplete capitulation,  $m(K1)=2$ ,  $e(K1)=2$

CK2=[2040,12,2,2]=[8,4,2,2]  
 $h_1^[(S-1)^1]=[4,0,1,1]$   $h_2^[(S-1)^1]=[0,0,0,1]$   
 $h_3^[(S-1)^1]=[0,2,0,1]$   $h_4^[(S-1)^1]=[0,2,0,0]$   
 $h_1^[(S-1)^2]=[0,0,0,1]$   $h_2^[(S-1)^2]=[0,2,0,0]$   
 $h_3^[(S-1)^2]=[0,2,0,0]$   $h_4^[(S-1)^2]=[0,0,0,0]$   
 norm in  $K_2/K$  of the component 1 of CK2: [4,2,0,0]  
 norm in  $K_2/K$  of the component 2 of CK2: [0,0,0,0]  
 norm in  $K_2/K$  of the component 3 of CK2: [0,0,0,0]  
 norm in  $K_2/K$  of the component 4 of CK2: [0,0,0,0]  
 Incomplete capitulation,  $m(K_2)=4$ ,  $e(K_2)=3$   
 CK3=[4080,12,2,2]=[16,4,2,2]

Unfortunately, the computations for  $n = 3$ , in this last example, take too much time. We are going to examine separately this field.

**7.2. Case of  $K = \mathbb{Q}(\sqrt[3]{174})$ ,  $\ell \equiv 1 \pmod{16}$ .** Varying  $\ell$ , we find many capitulations in the layer  $K_2$ :

$p=2$  PK= $x^3-174$  CK0=[6,2]  $e_{11}=193$   $r=1$   
 CK1=[12,6]=[4,2]  
 $h_1^[(S-1)^1]=[2,0]$   $h_2^[(S-1)^1]=[2,0]$   
 $h_1^[(S-1)^2]=[0,0]$   $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K_1/K$  of the component 1 of CK1: [0,0]  
 norm in  $K_1/K$  of the component 2 of CK1: [2,0]  
 Incomplete capitulation,  $m(K_1)=2$ ,  $e(K_1)=2$   
 CK2=[12,6]=[4,2]  
 $h_1^[(S-1)^1]=[0,0]$   $h_2^[(S-1)^1]=[2,0]$   
 $h_1^[(S-1)^2]=[0,0]$   $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K_2/K$  of the component 1 of CK2: [0,0]  
 norm in  $K_2/K$  of the component 2 of CK2: [0,0]  
 Complete capitulation,  $m(K_2)=2$ ,  $e(K_2)=2$

$p=2$  PK= $x^3-174$  CK0=[6,2]  $e_{11}=353$   $r=2$   
 CK1=[48,6]=[16,2]  
 $h_1^[(S-1)^1]=[6,0]$   $h_2^[(S-1)^1]=[8,0]$   
 $h_1^[(S-1)^2]=[4,0]$   $h_2^[(S-1)^2]=[0,0]$   
 norm in  $K_1/K$  of the component 1 of CK1: [8,0]  
 norm in  $K_1/K$  of the component 2 of CK1: [8,0]  
 Incomplete capitulation,  $m(K_1)=4$ ,  $e(K_1)=4$   
 CK2=[48,6,3,3]=[16,2]  
 $h_1^[(S-1)^1]=[6,0,0,0]$   $h_2^[(S-1)^1]=[8,0,0,0]$   
 $h_1^[(S-1)^2]=[4,0,0,0]$   $h_2^[(S-1)^2]=[0,0,0,0]$   
 norm in  $K_2/K$  of the component 1 of CK2: [0,0,0,0]  
 norm in  $K_2/K$  of the component 2 of CK2: [0,0,0,0]  
 Complete capitulation,  $m(K_2)=4$ ,  $e(K_2)=4$

$p=2$  PK= $x^3-174$  CK0=[6,2]  $e_{11}=577$   $r=1$   
 CK1=[84,6,2]=[4,2,2]  
 $h_1^[(S-1)^1]=[0,1,1]$   $h_2^[(S-1)^1]=[2,1,1]$   $h_3^[(S-1)^1]=[2,1,1]$   
 $h_1^[(S-1)^2]=[0,0,0]$   $h_2^[(S-1)^2]=[0,0,0]$   $h_3^[(S-1)^2]=[0,0,0]$   
 norm in  $K_1/K$  of the component 1 of CK1: [2,1,1]  
 norm in  $K_1/K$  of the component 2 of CK1: [2,1,1]  
 norm in  $K_1/K$  of the component 3 of CK1: [2,1,1]  
 Incomplete capitulation,  $m(K_1)=2$ ,  $e(K_1)=2$   
 CK2=[168,6,6,3]=[8,2,2]  
 $h_1^[(S-1)^1]=[2,0,0,0]$   $h_2^[(S-1)^1]=[4,0,0,0]$   $h_3^[(S-1)^1]=[4,1,0,0]$   
 $h_1^[(S-1)^2]=[4,0,0,0]$   $h_2^[(S-1)^2]=[0,0,0,0]$   $h_3^[(S-1)^2]=[4,0,0,0]$   
 $h_1^[(S-1)^3]=[0,0,0,0]$   $h_2^[(S-1)^3]=[0,0,0,0]$   $h_3^[(S-1)^3]=[0,0,0,0]$   
 norm in  $K_2/K$  of the component 1 of CK2: [0,0,0,0]  
 norm in  $K_2/K$  of the component 2 of CK2: [0,0,0,0]  
 norm in  $K_2/K$  of the component 3 of CK2: [0,0,0,0]  
 Complete capitulation,  $m(K_2)=3$ ,  $e(K_2)=3$

The last case  $\ell = 577$  shows that the complexity of the  $\mathcal{H}_{K_n}$ 's is increasing, but nevertheless leads to complete capitulation in  $K_2$ ; so conditions of Theorem 1.1 are

not necessary for capitulation. Indeed, we obtain the following information on the structure of the  $\mathcal{H}_{K_n}$ 's for  $K = \mathbb{Q}(\sqrt[3]{174})$  and  $\ell = 577$ :

In  $K_1$ ,  $m(K_1) = 2$ ,  $s(K_1) = 1$  but  $e(K_1) = 2$ . In  $K_2$ ,  $m(K_2) = 3$ ,  $s(K_2) = 1$ ,  $e(K_2) = 3$ ; thus  $\mathcal{V}_{K_2/K}$  acts like  $4(\sigma - 1)^2 + 6(\sigma - 1) + 4$ . The above data show that this reduces to the annihilation by  $A = 6(\sigma - 1) + 4$ ; indeed,  $h_1^A = h_1^{12}h_1^4 = 1$ ,  $h_j^A = 1$  for the other generators.

**7.3. Pure quintic fields,  $L \subset K(\mu_{17})$ .** Replacing, in the program, the polynomial  $PK = x^3 - R$  by  $PK = x^5 - R$  (or with any odd degree), we get analogous results:

```
p=2 PK=x^5-13 CK0=[4] ell=17 r=2
CK1=[40]=[8]
h_1^[(S-1)^1]=[0] h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[2]
No capitulation, m(K1)=1, e(K1)=3
CK2=[40]=[8]
h_1^[(S-1)^1]=[0] h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[4]
Incomplete capitulation, m(K2)=1, e(K2)=3
Complete capitulation in K3 (stability from K1)

p=2 PK=x^5-122 CK0=[10,2] ell=17 r=2
CK1=[10,2,2]=[2,2,2]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[1,1,0]
h_1^[(S-1)^2]=[0,0,0] h_2^[(S-1)^2]=[0,0,0] h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[1,1,0]
Incomplete capitulation, m(K1)=2, e(K1)=1
CK2=[20,2,2]=[4,2,2]
h_1^[(S-1)^1]=[2,0,1] h_2^[(S-1)^1]=[2,0,1] h_3^[(S-1)^1]=[2,0,0]
h_1^[(S-1)^2]=[2,0,0] h_2^[(S-1)^2]=[2,0,0] h_3^[(S-1)^2]=[0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2
```

## 8. TABLES FOR REAL QUADRATIC FIELDS AND $p = 3$

We consider cyclic  $p$ -extensions  $L/K$ , where  $K = \mathbb{Q}(\sqrt{m})$ ,  $L \subset K(\mu_\ell)$ ,  $\ell \equiv 1 \pmod{2p^N}$ . Results are given for  $p = 3$ ,  $\ell = 19$  ( $N = 2$ ), 109 ( $N = 3$ ) and 163 ( $N = 4$ ), except cases of stability in  $K_1/K$ . The images  $\mathbf{J}_{K_n/K}(\mathcal{H}_K)$  are computed for  $n = 1$  and  $n = 2$ ;  $r \in \{1, 2\}$  is the number of prime ideals above  $\ell$  in  $K$ :

```
MAIN PROGRAM FOR REAL QUADRATIC FIELDS
{p=3;Nn=2;bm=2;Bm=10^8;vHK=1;ell=109;mKn=2;
for(m=bm,Bm,if(core(m)!=m,next);PK=x^2-m;K=bnfinit(PK,1);
\\Testing the order of the p-class group of K compared to vHK:
HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;r=(kronecker(m,ell)+3)/2;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
\\Test for elimination of the stability from K:
if(n==1 & valuation(HKn,p)==valuation(HK,p),break);
if(n==1,print();print("PK=",PK," CK0=",CK0[2]," ell=",ell," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(j=1,rKn,X=CKn[3][j];Y=X;for(i=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(j=1,rKn,c=B[j];w=valuation(CKn[2][j],p);c=lift(Mod(c,p^w));
listput(Ehij,c,j));print("h_",j,"^[","(S-1)^",i,"]=",Ehij));
for(j=1,rKn,A0=CKn[3][j];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(j=1,rKn,c=B[j];w=valuation(CKn[2][j],p);c=lift(Mod(c,p^w));
```



```
listput(Enu,c,j));print("norm in K",n,"/K of the component ",j,
" of CK",n," :",Enu))}}}
```

8.1. **Examples with  $\mathcal{H}_K \simeq \mathbb{Z}/3\mathbb{Z}$ .** This simplest case gives both capitulations or non-capitulations in  $K_2$ :

```
PK=x^2-142 CK0=[3] ell=19 r=2
CK1=[9]
h_1^[(S-1)^1]=[0] h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=1, e(K1)=2
CK2=[9]
h_1^[(S-1)^1]=[0] h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=1, e(K2)=2
PK=x^2-359 CK0=[3] ell=19 r=2
CK1=[9]
h_1^[(S-1)^1]=[0] h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=1, e(K1)=2
CK2=[27]
h_1^[(S-1)^1]=[0] h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[9]
No capitulation, m(K2)=1, e(K2)=3
p=3 PK=x^2-142 CK0=[3] ell=109 r=1
CK1=[18,2]=[9]
h_1^[(S-1)^1]=[3,0] h_1^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
No capitulation, m(K1)=2, e(K1)=2
CK2=[54,2]=[27]
h_1^[(S-1)^1]=[24,0] h_1^[(S-1)^2]=[9,0]
norm in K2/K of the component 1 of CK2:[9,0]
No capitulation, m(K2)=3, e(K2)=3
p=3 PK=x^2-223 CK0=[3] ell=109 r=2
CK1=[9]
h_1^[(S-1)^1]=[6] h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=2, e(K1)=2
CK2=[9]
h_1^[(S-1)^1]=[3] h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=2, e(K2)=2
p=3 PK=x^2-254 CK0=[3] ell=109 r=2
CK1=[3,3]
h_1^[(S-1)^1]=[2,2] h_2^[(S-1)^1]=[1,1]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=2, e(K1)=1
CK2=[3,3]
h_1^[(S-1)^1]=[1,1] h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0] h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=1
p=3 PK=x^2-79 CK0=[3] ell=163 r=1
CK1=[18,2]=[9]
h_1^[(S-1)^1]=[3,0] h_1^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation, m(K1)=2, e(K1)=2
CK2=[18,2]=[9]
h_1^[(S-1)^1]=[3,0] h_1^[(S-1)^2]=[0,0]
```

norm in  $K_2/K$  of the component 1 of  $CK_2$ : [0,0]  
norm in  $K_2/K$  of the component 2 of  $CK_2$ : [0,0]  
Complete capitulation,  $m(K_2)=2$ ,  $e(K_2)=2$   
 $p=3$   $PK=x^2-254$   $CK_0=[3]$   $e_{11}=163$   $r=2$   
 $CK_1=[18,2]=[9]$   
 $h_{-1}[(S-1)^1]=[0,0]$   $h_{-1}[(S-1)^2]=[0,0]$   
norm in  $K_1/K$  of the component 1 of  $CK_1$ : [3,0]  
norm in  $K_1/K$  of the component 2 of  $CK_1$ : [0,0]  
No capitulation,  $m(K_1)=1$ ,  $e(K_1)=2$   
 $CK_2=[18,2]=[9]$   
 $h_{-1}[(S-1)^1]=[0,0]$   $h_{-1}[(S-1)^2]=[0,0]$   
norm in  $K_2/K$  of the component 1 of  $CK_2$ : [0,0]  
norm in  $K_2/K$  of the component 2 of  $CK_2$ : [0,0]  
Complete capitulation,  $m(K_2)=1$ ,  $e(K_2)=2$

**8.2. Examples with  $\mathcal{H}_K \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ .** Due to a very large calculation time for degrees  $[K_2 : \mathbb{Q}] = 18$ , we have only some results showing that, as for the case of cubic fields and  $p = 2$  (degrees  $[K_2 : \mathbb{Q}] = 12$ ), capitulation may occur in  $K_2$ :

$p=3$   $PK=x^2-23659$   $CK_0=[6,3]$   $e_{11}=19$   $r=2$   
 $CK_1=[18,3]=[9,3]$   
 $h_{-1}[(S-1)^1]=[0,0]$   $h_{-2}[(S-1)^1]=[3,0]$   
 $h_{-1}[(S-1)^2]=[0,0]$   $h_{-2}[(S-1)^2]=[0,0]$   
norm in  $K_1/K$  of the component 1 of  $CK_1$ : [3,0]  
norm in  $K_1/K$  of the component 2 of  $CK_1$ : [0,0]  
Incomplete capitulation,  $m(K_1)=2$ ,  $e(K_1)=2$   
 $CK_2=[18,3]=[9,3]$   
 $h_{-1}[(S-1)^1]=[0,0]$   $h_{-2}[(S-1)^1]=[3,0]$   
 $h_{-1}[(S-1)^2]=[0,0]$   $h_{-2}[(S-1)^2]=[0,0]$   
norm in  $K_2/K$  of the component 1 of  $CK_2$ : [0,0]  
norm in  $K_2/K$  of the component 2 of  $CK_2$ : [0,0]  
Complete capitulation,  $m(K_2)=2$ ,  $e(K_2)=2$

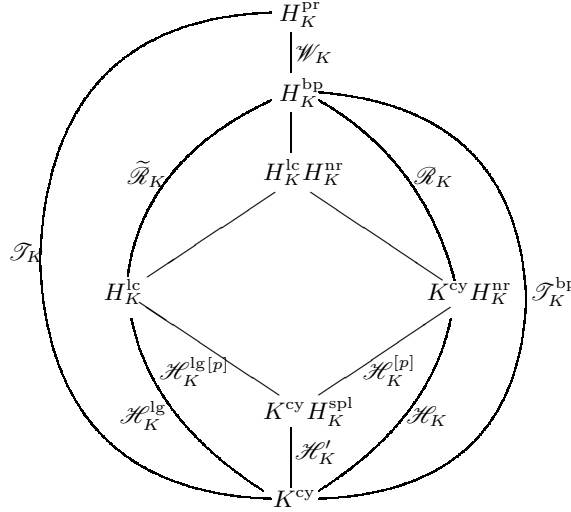
$p=3$   $PK=x^2-23659$   $CK_0=[6,3]$   $e_{11}=37$   $r=2$   
 $CK_1=[18,3,3]=[9,3,3]$   
 $h_{-1}[(S-1)^1]=[0,0,0]$   $h_{-2}[(S-1)^1]=[6,0,1]$   $h_{-3}[(S-1)^1]=[6,0,0]$   
 $h_{-1}[(S-1)^2]=[0,0,0]$   $h_{-2}[(S-1)^2]=[6,0,0]$   $h_{-3}[(S-1)^2]=[0,0,0]$   
norm in  $K_1/K$  of the component 1 of  $CK_1$ : [0,0,0]  
norm in  $K_1/K$  of the component 2 of  $CK_1$ : [3,0,0]  
norm in  $K_1/K$  of the component 3 of  $CK_1$ : [3,0,0]  
Incomplete capitulation,  $m(K_1)=3$ ,  $e(K_1)=2$   
 $CK_2=[18,3,3]=[9,3,3]$   
 $h_{-1}[(S-1)^1]=[0,0,0]$   $h_{-2}[(S-1)^1]=[3,1,1]$   $h_{-3}[(S-1)^1]=[0,2,2]$   
 $h_{-1}[(S-1)^2]=[0,0,0]$   $h_{-2}[(S-1)^2]=[3,0,0]$   $h_{-3}[(S-1)^2]=[6,0,0]$   
norm in  $K_2/K$  of the component 1 of  $CK_2$ : [0,0,0]  
norm in  $K_2/K$  of the component 2 of  $CK_2$ : [0,0,0]  
norm in  $K_2/K$  of the component 3 of  $CK_2$ : [0,0,0]  
Complete capitulation,  $m(K_2)=3$ ,  $e(K_2)=2$

## 9. TABLES FOR THE LOGARITHMIC CLASS GROUP

Questions of capitulation, in various  $p$ -extensions, of other arithmetic invariants, are at the origin of many papers (see, e.g., in a chronological order [74, 63, 66, 62, 17, 100, 58, 31, 59, 60, 61, 46] and their references); they are related to generalized  $p$ -class groups with conditions of ramification and decomposition, to wild kernels of  $\mathbf{K}$ -theory, to torsion groups in  $p$ -ramification theory, to Tate–Chafarevich groups, Bertrandias–Payan modules, logarithmic class groups.

The same techniques, using the algebraic norm, may be applied; the results essentially depend on the properties of the associated filtration (that we do not know for most of the above invariants), whence on the variation of the complexity in the  $p$ -extension  $L/K$  considered.

We shall focus on transfers of the logarithmic class groups  $\mathcal{H}_K^{\text{lg}}$  in some totally ramified cyclic  $p$ -extensions. Recall that  $H_K^{\text{lc}}$  is the maximal abelian locally cyclotomic pro- $p$ -extension of  $K$  and  $K^{\text{cy}}$  its cyclotomic  $\mathbb{Z}_p$ -extension. The tame places totally split in  $H_K^{\text{pr}}/K^{\text{cy}}$ , so that  $H_K^{\text{lc}}$  is the subfield of  $H_K^{\text{pr}}$  fixed by the decomposition groups of the  $p$ -places. In Jaulent [59] one finds the following diagram of the main invariants, showing in particular that  $\mathcal{H}_K$  and  $\mathcal{H}_K^{\text{lg}}$  are isomorphic to quotients of  $\mathcal{T}_K$  and that  $\text{Gal}(H_K^{\text{bp}}/K^{\text{cy}}H_K^{\text{nr}})$ ,  $\text{Gal}(H_K^{\text{bp}}/H_K^{\text{lc}})$  are suitable regulators of units (compare with the diagram in Definitions 4.3):



In this diagram,  $\mathcal{H}_K^{\text{lg}[p]}$  (resp.  $\mathcal{H}_K^{[p]}$ ) is the subgroup, of the logarithmic class group  $\mathcal{H}_K^{\text{lg}}$  (resp. of the  $p$ -class group  $\mathcal{H}_K$ ), generated by the classes of the primes dividing  $p$ . So  $\mathcal{H}_K^l$  is the quotient  $\mathcal{H}_K/\mathcal{H}_K^{[p]}$  where  $H_K^{\text{spl}}$  is the splitting field of  $p$  in  $H_K^{\text{nr}}$ , hence the subfield fixed by the image of  $\mathcal{H}_K^{[p]}$ , noting that in our case,  $H_K^{\text{nr}} \cap K^{\text{cy}} = K$ .

For  $L \subset K(\mu_\ell)$ ,  $\ell \equiv 1 \pmod{2p^N}$ , Theorem 1.1 applies to  $\mathcal{H}_K^{\text{lg}}$  and  $\mathcal{H}_L^{\text{lg}}$ , computable using Diaz y Diaz–Jaulent–Pauli–Pohst–Soriano–Gafiuk [23] or, with the instruction `bnflog(K, p)`, Belabas–Jaulent [11]. Indeed, since  $L/K$  is tamely and totally ramified at  $\ell$ , then  $L^{\text{cy}} = LK^{\text{cy}}$ , thus  $H_K^{\text{lc}}$  is linearly disjoint from  $L^{\text{cy}}$  and the norm  $\mathbf{N}_{L/K} : \mathcal{H}_L^{\text{lg}} = \text{Gal}(H_L^{\text{lc}}/L^{\text{cy}}) \rightarrow \mathcal{H}_K^{\text{lg}} = \text{Gal}(H_K^{\text{lc}}/K^{\text{cy}})$  is surjective.

**Remark 9.1.** *For logarithmic class groups in totally tamely ramified cyclic  $p$ -extensions (in the classical sense), the theory of stability does exist, essentially because the arithmetic norms are surjective allowing the criterion of capitulation with  $\mathcal{V}_{L/K} = \mathbf{J}_{L/K} \circ \mathbf{N}_{L/K}$ , but in numerical applications, our extensions  $L/K$  may be partially locally cyclotomic, say in  $K_{n_0}/K$ , which gives some logarithmic non-ramification (see [57, Théorème 1.4]). For instance, if  $K$  is a cyclic cubic field,  $p = 2$  and  $\ell = 17$ , then  $p$  splits in  $K_1 = K(\sqrt{17})$  and is inert in  $L/K_1$ ; for  $K$  quadratic real,  $p = 3$ ,  $\ell = 109$ ,  $p$  is totally inert in  $L/K$  leading to the classic reasoning.*

In the following program we restrict ourselves to *totally logarithmically ramified cyclic  $p$ -extensions*. Since the invariants  $m(K_n)$  are unknown, we can only prove capitulation from stabilities at some layer, but very probably, many capitulation hold in the forthcoming computations with no stability, especially if  $N$  is large; we give excerpts of the results.

**9.1. Examples with real quadratic fields and  $p = 3$ .** We give examples with  $K = \mathbb{Q}(\sqrt{m})$  given,  $p = 3$ . There are many stabilities, allowing to conclude the capitulation. Recall that PARI gives the data  $[\mathcal{H}_{K_n}^{\text{lg}}, \mathcal{H}_{K_n}^{\text{lg}[p]}, \mathcal{H}_{K_n}^l]$  in this order. One may vary  $\mathbf{p}$ ,  $\mathbf{Nn}$ ,  $\mathbf{Nell}$ ,  $\mathbf{m}$ :

```

{p=3;Nn=2;Nell=2;m=67;PK=x^2-m;K=bnfinit(PK,1);ClogK0=bnflog(K,p);
forprime(e11=1,10^3,N=valuation(e11-1,p);if(N<Nell,next);
a=znormer(Mod(p,e11));if(valuation(a,p)!=N,next);
r=(kronecker(m,e11)+3)/2;for(n=1,Nn,Qn=polsubcyclo(e11,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);if(n==1,print());
print("PK=",PK," ClogK0=",ClogK0," e11=",e11," N=",N," r=",r));
ClogKn= bnflog(Kn,p);print("ClogK",n,"=",ClogKn))}
PK=x^2-67 ClogK0=[[3],[3],[ ]]
e11=19 N=2 r=1 e11=397 N=2 r=2
ClogK1=[[9],[9],[ ]] ClogK1=[[9,3],[9],[3]]
ClogK2=[[9],[9],[ ]] ClogK2=[[27,9],[27],[9]]

e11=37 N=2 r=2 e11=487 N=5 r=2
ClogK1=[[9],[9],[ ]] ClogK1=[[9],[9],[ ]]
ClogK2=[[27],[27],[ ]] ClogK2=[[27],[27],[ ]]

e11=109 N=3 r=1 e11=631 N=2 r=2
ClogK1=[[9],[9],[ ]] ClogK1=[[9,3],[9],[3]]
ClogK2=[[9],[9],[ ]] ClogK2=[[9,9],[9],[9]]

e11=163 N=4 r=1 e11=811 N=4 r=2
ClogK1=[[9],[9],[ ]] ClogK1=[[9],[9],[ ]]
ClogK2=[[9],[9],[ ]] ClogK2=[[27],[27],[ ]]

```

Fixing  $\ell = 109$  and varying  $K = \mathbb{Q}(\sqrt{m})$ , we get for  $r = 1$ :

```

PK=x^2-67 ClogK0=[[3],[3],[ ]] r=1 PK=x^2-473 ClogK0=[[3],[ ],[3]] r=1
ClogK1=[[9],[9],[ ]] ClogK1=[[3],[ ],[3]]
ClogK2=[[9],[9],[ ]] ClogK2=[[3],[ ],[3]]
Complete capitulation in K2 Complete capitulation in K1
PK=x^2-321 ClogK0=[[3],[ ],[3]] r=1 PK=x^2-610 ClogK0=[[3],[3],[ ]] r=1
ClogK1=[[9],[ ],[9]] ClogK1=[[9],[9],[ ]]
ClogK2=[[9],[ ],[9]] ClogK2=[[27],[27],[ ]]
Complete capitulation in K2 No conclusion

```

The last example ( $m = 610$ ) is the only one of non stability in the interval considered. Its 3-class group is trivial, then  $\mathcal{H}_K^1 = 1$  and  $\mathcal{H}_{K_n}^{\text{lg}} = \mathcal{H}_{K_n}^{\text{lg}[3]}$ .

The following list deals with the case  $r = 2$ . If  $\mathcal{H}_{K_n}^{\text{lg}[p]} = 1$ , we can conclude, using the general Program 8 for  $p$ -class groups.

```

PK=x^2-106 ClogK0=[[3],[3],[ ]] r=2 PK=x^2-659 ClogK0=[[3],[ ],[3]] r=2
ClogK1=[[3,3],[3],[3]] ClogK1=[[9],[ ],[9]]
ClogK2=[[9,3],[9],[3]] ClogK2=[[9],[ ],[9]]
No conclusion Complete capitulation in K2
PK=x^2-238 ClogK0=[[3],[3],[ ]] r=2 PK=x^2-679 ClogK0=[[3],[3],[ ]] r=2
ClogK1=[[9,3],[9],[3]] ClogK1=[[9],[9],[ ]]
ClogK2=[[27,9],[27],[9]] ClogK2=[[27],[27],[ ]]
No conclusion No conclusion
PK=x^2-253 ClogK0=[[3],[3],[ ]] r=2 PK=x^2-727 ClogK0=[[3],[3],[ ]] r=2
ClogK1=[[9],[9],[ ]] ClogK1=[[9],[9],[ ]]
ClogK2=[[27],[27],[ ]] ClogK2=[[27],[27],[ ]]
No conclusion No conclusion
PK=x^2-254 ClogK0=[[3],[ ],[3]] r=2 PK=x^2-785 ClogK0=[[3],[ ],[3]] r=2
ClogK1=[[3,3],[ ],[3,3]] ClogK1=[[9],[ ],[9]]
ClogK2=[[3,3],[ ],[3,3]] ClogK2=[[27],[ ],[27]]
Complete capitulation in K2 No conclusion
PK=x^2-326 ClogK0=[[3],[ ],[3]] r=2 PK=x^2-790 ClogK0=[[3],[3],[ ]] r=2
ClogK1=[[3],[ ],[3]] ClogK1=[[3,3],[3],[3]]
ClogK2=[[3],[ ],[3]] ClogK2=[[9,3],[3],[9]]
Complete capitulation in K1 No conclusion

```

**9.2. Examples with cyclic cubic fields and  $p = 2$ .** With an analogous program,  $\ell = 17$  ( $N = 3$ ), one obtains the following results, taking into account Remark 9.1:

```

PK=x^3+x^2-54*x-169
ClogK0=[[2,2],[],[2,2]] r=3
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[2,2],[],[2,2]]
ClogK3=[[2,2],[],[2,2]]
Complete capitulation in K1

PK=x^3+x^2-182*x-81
ClogK0=[[2,2],[],[2,2]] r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[2,2],[],[2,2]]
ClogK3=[[2,2],[],[2,2]]
Complete capitulation in K1

PK=x^3-201*x+1072
ClogK0=[[2,2],[2,2],[]] r=1
ClogK1=[[2,2],[2,2],[]]
ClogK2=[[4,4],[4,4],[]]
ClogK3=[[8,8],[8,8],[]]
No conclusion (fake stability in K1/K)

PK=x^3+x^2-202*x-1169
ClogK0=[[2,2],[],[2,2]] r=1
ClogK1=[[2,2,2,2],[],[2,2,2,2]]
ClogK2=[[2,2,2,2],[],[2,2,2,2]]
ClogK3=[[2,2,2,2],[],[2,2,2,2]]
Complete capitulation in K2

PK=x^3+x^2-234*x-729
ClogK0=[[2,2],[],[2,2]] r=1
ClogK1=[[4,4],[],[4,4]]
ClogK2=[[8,8],[],[8,8]]
ClogK3=[[16,16],[],[16,16]]
No conclusion, probably no capitulation

PK=x^3-291*x-1358
ClogK0=[[2,2],[2,2],[]] r=1
ClogK1=[[4,4],[4,4],[]]
ClogK2=[[4,4],[4,4],[]]
Complete capitulation in K2

PK=x^3+x^2-336*x-1719
ClogK0=[[2,2],[],[2,2]] r=1
ClogK1=[[4,4],[],[4,4]]
ClogK2=[[4,4],[],[4,4]]
Complete capitulation in K2

PK=x^3+x^2-340*x+416
ClogK0=[[2,2],[2,2],[]] r=1
ClogK1=[[2,2],[2,2],[]]
ClogK2=[[2,2,2,2],[2,2,2,2],[]]
No conclusion (fake stability in K1/K)

PK=x^3+x^2-386*x+1760
ClogK0=[[4,4],[4,4],[]] r=1
ClogK1=[[4,4],[4,4],[]]
ClogK2=[[8,8],[8,8],[]]
No conclusion (fake stability in K1/K)

PK=x^3+x^2-486*x+2864
ClogK0=[[2,2],[2,2],[]] r=3
ClogK1=[[2,2],[2,2],[]]
ClogK2=[[4,4,2,2],[4,4],[2,2]]
No conclusion (fake stability in K1/K)

PK=x^3+x^2-650*x-289
ClogK0=[[2,2],[],[2,2]] r=3
ClogK1=[[4,4,2,2],[],[4,4,2,2]]
ClogK2=[[4,4,4,4],[],[4,4,4,4]]
No conclusion

PK=x^3+x^2-692*x-7231
ClogK0=[[2,2],[],[2,2]] r=1
ClogK1=[[2,2,2,2],[],[2,2,2,2]]
ClogK2=[[2,2,2,2,2],[],[2,2,2,2,2]]
No conclusion

```

To verify the capitulations, as for the  $p$ -class groups, it would be interesting to have available the logarithmic instructions replacing `K.clgp`, once the field  $K$  is given as usual with  $K = \text{bnfinit}(PK)$  and the logarithmic class group by `bnflog(K, p)`, then an instruction replacing `bnfisprincipal(K, A)` for an ideal  $A$  in the logarithmic sense.

**9.3. Capitulation of  $\mathcal{H}_K^{\text{lg}}$  in  $K^{\text{cy}}$  for  $K$  quadratic.** The program eliminates cases of stability from  $K$ . The capitulation is obtained in  $K_3/K$  for almost cases; we give an excerpt of the results for  $n \leq 3$  with  $m$  up to 1250. When the exponents increase, no conclusion is possible up to  $K_3$  (e.g., successive exponents 4, 8, 16, 32):

```

{p=2;Nn=3;bm=2;Bm=10^4;for(m=bm,Bm,if(core(m)!=m,next);if(Mod(m,8)==2,next);
PK=x^2-m;K=bnfinit(PK,1);r=matsize(idealfactor(K,2))[1];ClogK0=bnflog(K,p);
if(ClogK0==[[],[],[]],next);for(n=1,Nn,Qn=x;for(i=1,n,Qn=Qn^2-2);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);if(n==1,print());
print("PK=",PK," ClogK0=",ClogK0," r=",r));ClogKn= bnflog(Kn,p);
if(n==1 & ClogKn==ClogK0,break);print("ClogK",n,"=",ClogKn))}

PK=x^2-113 ClogK0=[[2],[2],[]] r=2
ClogK1=[[2],[],[2]]
ClogK2=[[2],[],[2]]
ClogK3=[[2],[],[2]]

PK=x^2-119 ClogK0=[[2],[],[2]] r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[2,2],[],[2,2]]
ClogK3=[[2,2],[],[2,2]]

PK=x^2-161 ClogK0=[[4],[4],[]] r=2
ClogK1=[[4],[2],[2]]

PK=x^2-799 ClogK0=[[4],[],[4]] r=1
ClogK1=[[4,2],[],[4,2]]
ClogK2=[[8,2,2],[],[8,2,2]]
ClogK3=[[8,2,2],[],[8,2,2]]

PK=x^2-805 ClogK0=[[2],[],[2]] r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[8,4],[],[8,4]]
ClogK3=[[8,4],[],[8,4]]

PK=x^2-1023 ClogK0=[[4],[],[4]] r=1
ClogK1=[[8],[],[8]]

```

ClogK2=[[4], [], [4]]	ClogK2=[[16], [], [16]]
ClogK3=[[4], [], [4]]	ClogK3=[[32], [], [32]]
PK=x^2-221 ClogK0=[[2], [], [2]] r=1	PK=x^2-1067 ClogK0=[[2], [], [2]] r=1
ClogK1=[[4], [], [4]]	ClogK1=[[8], [], [8]]
ClogK2=[[4], [], [4]]	ClogK2=[[16], [], [16]]
ClogK3=[[4], [], [4]]	ClogK3=[[32], [], [32]]
PK=x^2-255 ClogK0=[[2], [], [2]] r=1	PK=x^2-1217 ClogK0=[[8], [8], []] r=2
ClogK1=[[2, 2], [], [2, 2]]	ClogK1=[[16, 2], [8], [4]]
ClogK2=[[4, 2], [], [4, 2]]	ClogK2=[[16, 2], [4], [4, 2]]
ClogK3=[[4, 2], [], [4, 2]]	ClogK3=[[16, 2], [2], [8, 2]]
PK=x^2-323 ClogK0=[[2], [], [2]] r=1	PK=x^2-1221 ClogK0=[[4], [], [4]] r=1
ClogK1=[[8], [], [8]]	ClogK1=[[8], [], [8]]
ClogK2=[[8], [], [8]]	ClogK2=[[16], [], [16]]
ClogK3=[[8], [], [8]]	ClogK3=[[16], [], [16]]
PK=x^2-357 ClogK0=[[2], [], [2]] r=1	PK=x^2-1243 ClogK0=[[2], [], [2]] r=1
ClogK1=[[2, 2], [], [2, 2]]	ClogK1=[[2, 2], [], [2, 2]]
ClogK2=[[4, 2, 2], [], [4, 2, 2]]	ClogK2=[[8, 4], [], [8, 4]]
ClogK3=[[4, 2, 2], [], [4, 2, 2]]	ClogK3=[[8, 8], [], [8, 8]]
PK=x^2-627 ClogK0=[[2, 2], [], [2, 2]] r=1	PK=x^2-1249 ClogK0=[[4], [4], []] r=2
ClogK1=[[4, 4], [], [4, 4]]	ClogK1=[[8], [4], [2]]
ClogK2=[[8, 8], [], [8, 8]]	ClogK2=[[8], [2], [4]]
ClogK3=[[16, 8], [], [16, 8]]	ClogK3=[[8], [], [8]]

No conclusion up to K3

**9.4. Conclusion about  $\mathcal{H}_K^{\text{lg}}$ .** As a conclusion, one can say, from the above examples, that the logarithmic class group of a real field  $K$  may capitulate in the simplest cyclic  $p$ -extensions  $L/K$ ,  $L \subset K(\mu_\ell)$ , as for  $p$ -class groups; this was not so obvious, but in Jaulent [60] is proved the existence (as for  $p$ -class groups with Bosca techniques [16]) of abelian extensions  $L_0/\mathbb{Q}$  such that  $L = L_0K$  is a capitulation field for  $\mathcal{H}_K^{\text{lg}}$  (some more general conditions of signature may be assumed for  $K$ ).

Clearly, for imaginary quadratic fields, the fact that, probably,  $\mathcal{H}_K^{\text{lg}}$  never capitulates in  $L$  seems plausible, because of a systematic non-smooth increasing complexity ( $p$ -rank and/or exponent) as shown by the following excerpt:

PK=x^2+14 e11=109 r=1	PK=x^2+74 e11=109 r=2
ClogK0=[[3], [3], []]	ClogK0=[[9], [9], []]
ClogK1=[[9], [9], []]	ClogK1=[[27, 3], [27], [3]]
ClogK2=[[27], [27], []]	ClogK2=[[81, 9], [81], [9]]
PK=x^2+41 e11=109 r=1	PK=x^2+107 e11=109 r=1
ClogK0=[[27], [27], []]	ClogK0=[[9], [9], []]
ClogK1=[[81], [81], []]	ClogK1=[[27, 9, 3], [27], [9, 3]]
ClogK2=[[243], [243], []]	ClogK2=[[81, 27, 9], [81], [27, 9]]

The case of capitulation of  $\mathcal{H}_K^{\text{lg}}$  in the cyclotomic  $\mathbb{Z}_p$ -extension of a totally real number field (equivalent to Greenberg's conjecture) for which no proof does exist, is made very plausible due to a general principle of capitulation of  $\mathcal{H}_K^{\text{lg}}$ .

## 10. CONCLUSIONS AND PROSPECTS

**a)** We have conjectured in Conjecture 1.4 that, varying  $\ell \equiv 1 \pmod{2p^N}$ ,  $N$  large enough, there are infinitely many cases of stability from a suitable layer in  $K(\mu_\ell)$ , yielding capitulation of  $\mathcal{H}_K$  (Theorem 1.2 (i)), which is stronger than the more general capitulation conjecture for infinitely many  $\ell$ 's; this would be coherent with Greenberg's conjecture, equivalent to the stability in  $K^{\text{cy}}$ . In other words, our conjecture may be seen as a "tame version" of Greenberg's one, it being understood that the towers are finite, so that capitulation needs large  $N$ 's.

Furthermore, the particular criterion of Theorem 1.1, using the algebraic norm  $\nu_{L/K}$  by means of the invariants  $m(L)$  and  $e(L)$ , yields capitulation without there necessarily being stability; it shows the link between capitulation and complexity (in the meaning of Definition 2.13), of the filtration of  $\mathcal{H}_L$ , likely to be governed by



natural density results (Conjecture 2.4). It is reasonable to think that, restricting to primes  $\ell \equiv 1 \pmod{2p^N}$  with  $N \rightarrow \infty$ ,  $N - s(L)$  becomes larger than  $e(L)$  taking into account that  $s(L) = \left\lceil \frac{\log(m(L))}{\log(p)} \right\rceil$  is logarithmic regarding  $m(L)$  which basically depends on the algorithm defining  $\mathcal{H}_L^{i+1}$  from  $\mathcal{H}_L^i$ . This would say that, in huge towers, stabilization occurs at some layer with an increasing probability regarding  $N$ , or, at least, a smooth complexity.

But in our computations, we were limited to testing with few values of  $\ell$  (among infinitely many !) and only for the levels  $n \leq 3$ . Similarly, we were limited to small primes  $p$  because of the degrees  $[K_n : \mathbb{Q}] = [K : \mathbb{Q}] p^n$  for PARI calculations.

When capitulation is, on the contrary, structurally impossible (e.g., case of minus  $p$ -class groups of CM-fields or case of torsion groups  $\mathcal{T}_K$  of  $p$ -ramification theory), the complexity of the corresponding invariants necessarily increases in any totally ramified cyclic  $p$ -tower.

**b)** Because capitulation of  $p$ -class groups, in a totally ramified cyclic  $p$ -extension, is in connection with its class group complexity, one may wonder if this has some repercussion (or not) on the very numerous heuristics on repartition of  $p$ -class groups  $\mathcal{H}_k$  when  $\text{Gal}(k/\mathbb{Q})$  is of order divisible by  $p$  (see, e.g., A. Bartel–Johnston–Lenstra [10, 9] dealing with some difficulties about the classical heuristics of Cohen–Lenstra–Martinet–Malle [20, 21, 22, 1], and giving attempts to modify them), or on the numerous probabilistic works and  $\epsilon$ -conjectures, like Ellenberg–Venkatesh [24, Conjecture 1, § 1.2], then [25, 90, 101, 89, 78] and [26] replacing the class groups of abelian extensions fields  $K/k$  by their genus groups  $\mathfrak{g}_{K/k}$  of order essentially given by a “norm factor”  $\frac{\prod_v e_v(K/k)}{(\mathbf{E}_k : \mathbf{E}_k \cap \mathcal{N}_{K/k})}$ , where  $\mathcal{N}_{K/k}$  is the group of local norms in  $K/k$  [36, Theorem IV.4.2]. Then one may cite from [89, Section 5.1]:

*“the truth of the  $\ell$ -torsion Conjecture [=  $p$ -torsion  $\epsilon$ -Conjecture] is implied by the truth of the Cohen–Lenstra–Martinet heuristics on the distribution of class groups.*

showing the similarities of these conjectures, all within a complex analytic frame. Thus the only remaining question is: by means of which parameters of  $k$ ,  $p$ -class group heuristics may be defined (structure of  $\text{Gal}(k/\mathbb{Q})$ , of the prescribed groups  $\mathcal{H}_k$ , signature  $(r_1(k), r_2(k))$ , discriminant  $D_k$ , complex analytic formulas ?); all this been insufficient in our opinion since behavior of  $p$ -class groups in a  $p$ -tower is more  $p$ -adic in nature than archimedean, as suggested in [38], and experiments with extensions  $K(\mu_\ell)/K$ ,  $K$  fixed, show very sensitive results depending on the chosen ramification  $\ell$  and not of the orders of magnitude of the parameters.

In other words, is capitulation a governing principle for complexity, or, on the contrary, is complexity a governing principle for capitulation ? This is a difficult question all the more that numerical examples show that if the tower  $L/K$ , of degree  $dp^N$  with  $N$  large enough, complexity may become as smooth as possible (for instance, stability from some layer, Greenberg’s conjecture in  $K^{\text{cy}}$ , etc.), while the discriminants become oversized in the towers.

The huge discriminant values are going in the right direction for *the  $p$ -ranks  $\epsilon$ -conjectures*, as proved in Klüners–Wang in some  $p$ -extensions [65, Theorems 2.1, 2.3] and us in [42, 45] for arbitrary towers of  $p$ -cyclic extensions. The much more difficult case of the  $p$ -torsion  $\epsilon$ -conjecture  $\#\mathcal{H}_k \ll_{\epsilon, p, [k:\mathbb{Q}]} D_k^\epsilon$  remains plausible even if this depends on the  $p$ -adic complexity given, in cyclic  $p$ -extensions, by the length of the filtration and the exponent of the class group.

**c)** The remarkable circumstance of capitulations in these simplest tamely ramified cyclic  $p$ -extensions  $L/K$ , is certainly a basic principle for many arithmetic properties, as the following ones:

(i) The real abelian Main Conjecture, whose proof becomes trivial in the semi-simple case as soon as  $\ell$  is inert in  $K/\mathbb{Q}$  and  $\mathcal{H}_K$  capitulates in  $L$ , because of the

general relations  $(\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi}) = (\mathbf{N}_{L/K}(\mathcal{E}_{L,\varphi}) : \mathcal{F}_{K,\varphi}) \times \frac{\#\mathcal{H}_{K,\varphi}}{\#\mathbf{J}_{L/K}(\mathcal{H}_{K,\varphi})}$ , implying  $(\mathcal{E}_{K,\varphi} : \mathcal{F}_{K,\varphi}) \geq \#\mathcal{H}_{K,\varphi}$  for all  $\varphi \in \Phi_K$  under capitulation [47, Theorem 4.6].

(ii) In the Iwasawa theory context, we refer for instance to Assim–Mazigh–Oukhaba [3, 80, 81] about generalized Main Conjectures associated to Euler systems built over Stark units replacing Leopoldt’s cyclotomic units, hoping that capitulation phenomena may give new insights in these theories using auxiliary cyclic  $p$ -extensions  $L/K$  in which filtration and exact sequence (1.1) are identical.

(iii) Capitulations prevent to get efficient algebraic definitions of  $p$ -adic isotopic components  $\mathcal{X}_\varphi^{\text{alg}}$  of arithmetic invariants  $\mathcal{X}$  in the non semi-simple case; which suggests to replace algebraic norms by arithmetic ones in the definitions and thus to use instead the arithmetic  $\varphi$ -objects  $\mathcal{X}_\varphi^{\text{ar}}$ .

## REFERENCES

- [1] M. Adam, G. Malle, *A class group heuristic based on the distribution of 1-eigenspaces in matrix groups*, J. Number Theory **149** (2015), 225–235.  
<https://doi.org/10.1016/j.jnt.2014.10.018> 50
- [2] B. Anglès, J-F. Jaulent, *Théorie des genres des corps globaux*, Manuscripta Math. **101**(4) (2000), 513–532.<https://doi.org/10.1007/s002290050229> 23
- [3] J. Assim, Y. Mazigh, H. Oukhaba, *Théorie d’Iwasawa des unités de Stark et groupe de classes*, Int. J. Number Theory **13**(5) (2017), 1165–1190.  
<https://doi.org/10.1142/S1793042117500634> 51
- [4] A. Azizi, A. Zekhnini, M. Taous, *On the strongly ambiguous classes of some bi-quadratic number fields*, Mathematica Bohemica **141**(3) (2016), 363–384.  
<http://eudml.org/doc/286792> 3
- [5] A. Azizi, A. Zekhnini, M. Taous, *Capitulation in the absolutely abelian extensions of some number fields II*, Acta Mathematica Vietnamica **42** (2017), 81–97.  
<https://doi.org/10.1007/s40306-016-0194-8> 3
- [6] A. Azizi, A. Zekhnini, M. Taous, D.C. Mayer, *Principalization of 2-class groups of type (2, 2, 2) of biquadratic fields  $\mathbb{Q}(\sqrt{p_1 p_2 q}, \sqrt{-1})$* , Int. J. Number Theory **11**(4) (2015), 1177–1215.<https://doi.org/10.1142/S1793042115500645> 3
- [7] A. Bandini, F. Caldarola, *Stabilization for Iwasawa modules in  $\mathbb{Z}_p$ -extensions*, Rend. Sem. Mat. Univ. Padova **136** (2016), 137–155.<https://doi.org/10.4171/RSMUP/136-10> 12, 23
- [8] A. Bandini, *A note on  $p$ -ranks of class groups in  $\mathbb{Z}_p$ -extensions*, JP Journal of Algebra, Number Theory and Applications **9**(1) (2007), 95–103.<http://hdl.handle.net/11568/925088> 3
- [9] A. Bartel, H. Johnston, H.W. Lenstra Jr., *Arakelov class groups of random number fields*, arXiv (2022).<https://arxiv.org/abs/2005.11533> 50
- [10] A. Bartel, H.W. Lenstra Jr., *On class groups of random number fields*, Proc. London Math. Soc. **121**(3) (2020), 927–953.<https://doi.org/10.1112/plms.12343> 50
- [11] K. Belabas, J-F. Jaulent, *The logarithmic class group package in PARI/GP*, Publ. Math. Fac. Sc. Besançon (Algèbre et théorie des nombres), PUFC (2016), 5–18.  
<https://doi.org/10.5802/pmb.o-1> 24, 46
- [12] T. Bembom, *The Capitulation Problem in Class Field Theory* (Dissertation), Georg–August–Universität–Göttingen (2012).<http://dx.doi.org/10.53846/goediss-2521> 3, 14
- [13] F. Bertrandias, J-J. Payan,  *$\Gamma$ -extensions et invariants cyclotomiques*, Ann. Sci. Ec. Norm. Sup. 4e série, **5**(4) (1972), 517–543. <https://doi.org/10.24033/asens.1236> 20
- [14] S. Biswas, *Capitulation, unit groups, and the cohomology of  $S$ -idèle classes*, arXiv (2016).  
<https://arxiv.org/abs/1610.00733> 3
- [15] R. Bond, *Some results on the Capitulation Problem*, J. Number Theory **13**(2) (1981), 246–254.[https://doi.org/10.1016/0022-314X\(81\)90007-X](https://doi.org/10.1016/0022-314X(81)90007-X) 3
- [16] S. Bosca, *Principalization of ideals in Abelian extensions of number fields*, Int. J. Number Theory **5**(3) (2009), 527–539.<https://doi.org/10.1142/S1793042109002213> 3, 4, 39, 49
- [17] C. Brighi, *Capitulation des classes logarithmiques et étude de certaines tours de corps de nombres*, Thèse, Université Paul Verlaine–Metz (2007).  
<https://hal.univ-lorraine.fr/tel-01749002> 45
- [18] F. Caldarola, *On the maximal finite Iwasawa submodule in  $\mathbb{Z}_p$ -extensions and capitulation of ideals*, Rendiconti Sem. Mat. Univ. Politecnico Torino **78**(1) (2020), 27–42.  
<http://www.seminariomatematico.polito.it/rendiconti/78-1/Caldarola.pdf> 12, 23
- [19] C. Chevalley, *Sur la théorie du corps de classes dans les corps finis et les corps locaux*, Thèse 155, Jour. of the Faculty of Sciences Tokyo **2** (1933), 365–476.  
[http://archive.numdam.org/item/THESE\\_1934\\_155\\_365\\_0/](http://archive.numdam.org/item/THESE_1934_155_365_0/) 4

- [20] H. Cohen, H. W. Lenstra Jr., *Heuristics on class groups of number fields*. In: Number theory, Noordwijkerhout 1983. Springer, Berlin 1984, 33–62. <https://doi.org/10.1007/BFb0099440> 50
- [21] H. Cohen, J. Martinet, *Étude heuristique des groupes de classes des corps de nombres*, J. reine angew. Math. **404** (1990), 39–76. <https://doi.org/10.1515/crll.1990.404.39> 50
- [22] H. Cohen, J. Martinet, *Heuristics on class groups: some good primes are not too good*, Math. Comp. **63** (1994), 329–334. <https://doi.org/10.2307/2153578> 50
- [23] F. Diaz y Diaz, J-F. Jaulent, S. Pauli, M. Pohst, F. Soriano–Gafiuk, *A New Algorithm for the Computation of Logarithmic  $\ell$ -Class groups of Number Fields*, Experiment. Math. **14**(1) (2005), 65–74. <https://doi.org/10.1080/10586458.2005.10128902> 24, 46
- [24] J.S. Ellenberg, A. Venkatesh, *Reflection principles and bounds for class group torsion*, Int. Math. Res. Not. **2007**(1) (2007). <https://doi.org/10.1093/imrn/rnm002> 50
- [25] J.S. Ellenberg, L.B. Pierce, M.M. Wood, *On  $\ell$ -torsion in class groups of number fields*, Algebra & Number Theory **11**(8) (2017), 1739–1778. <https://doi.org/10.2140/ant.2017.11.1739> 50
- [26] C. Frei, D. Loughran, R. Newton, *Distribution of genus numbers of abelian number fields*, arXiv (2022). <https://doi.org/10.48550/arXiv.2209.10659> 50
- [27] T. Fukuda, *Remarks on  $\mathbb{Z}_p$ -extensions of number fields*, Proc. Japan Acad. Ser. A **70**(8) (1994), 264–266. <https://doi.org/10.3792/pjaa.70.264> 3, 26
- [28] F. Gerth, III, *Densities for certain  $\ell$ -ranks in cyclic fields of degree  $\ell^n$* , Compositio Mathematica, **60**(3) (1986), 295–322. [http://www.numdam.org/item/?id=CM\\_1986\\_\\_60\\_3\\_295\\_0](http://www.numdam.org/item/?id=CM_1986__60_3_295_0) 7
- [29] C.D. González-Avilés, *Capitulation, ambiguous classes and the cohomology of the units*, J. reine angew. Math. **613** (2007), 75–97. <https://doi.org/10.1515/CRELLE.2007.092> 3, 4
- [30] M. Grandet, J-F. Jaulent, *Sur la capitulation dans une  $\mathbb{Z}_\ell$ -extension*, J. reine angew. Math. **362** (1985), 213–217. <http://eudml.org/doc/152777> 3, 23
- [31] G. Gras, J-F. Jaulent, T. Nguyen Quang Do, *Sur le module de Bertrandias–Payan dans une  $p$ -extension – Noyau de capitulation*, pp. 25–44. *Sur la capitulation pour le module de Bertrandias–Payan*, pp. 45–58. *Descente galoisienne et capitulation entre modules de Bertrandias–Payan*, pp. 59–79. Publ. Math. Fac. Sc. Besançon (Algèbre et théorie des nombres), PUFC (2016). [https://pmb.centre-mersenne.org/volume/PMB\\_2016\\_\\_\\_/](https://pmb.centre-mersenne.org/volume/PMB_2016___/) 20, 45
- [32] G. Gras, *Sur les  $\ell$ -classes d'idéaux dans les extensions cycliques relatives de degré premier  $\ell$ , I, II*, Ann. Inst. Fourier (1973), **23**(3), 1–44, **23**(4), 1–48. <http://doi.org/10.5802/aif.471> <http://doi.org/10.5802/aif.480> 4
- [33] G. Gras, *Étude d'invariants relatifs aux groupes des classes des corps abéliens*, Journées Arithmétiques de Caen (1976), Astérisque **41–42** (1977), 19 pp. [http://www.numdam.org/item/AST\\_1977\\_\\_41-42\\_\\_35\\_0/](http://www.numdam.org/item/AST_1977__41-42__35_0/) 27
- [34] G. Gras, *Nombre de  $\varphi$ -classes invariantes. Application aux classes des corps abéliens*, Bulletin Soc. Math. France **106** (1978), 337–364. <https://doi.org/10.24033/bsmf.1876> 4
- [35] G. Gras, *Principalisation d'idéaux par extensions absolument abéliennes*, J. Number Theory **62**(2) (1997), 403–421. <https://doi.org/10.1006/jnth.1997.2068> 3, 4, 39
- [36] G. Gras, *Class Field Theory: from theory to practice*, corr. 2nd ed. Springer Monographs in Mathematics, Springer, xiii+507 pages (2005). 5, 20, 21, 23, 50
- [37] G. Gras, *Invariant generalized ideal classes – Structure theorems for  $p$ -class groups in  $p$ -extensions*, Proc. Indian Acad. Sci. (Math. Sci.) **127**(1) (2017), 1–34. <https://doi.org/10.1007/s12044-016-0324-1> 4, 5
- [38] G. Gras, *Heuristics and conjectures in the direction of a  $p$ -adic Brauer–Siegel theorem*, Math. Comp. **88**(318) (2018/19), 1929–1965. <https://doi.org/10.1090/mcom/3395> 50
- [39] G. Gras, *The  $p$ -adic Kummer–Leopoldt Constant: Normalized  $p$ -adic Regulator*, Int. J. of Number Theory, **14**(2) (2018), 329–337. <https://doi.org/10.1142/S1793042118500203> 20, 25
- [40] G. Gras, *On  $p$ -rationality of number fields. Applications–PARI/GP programs*, Publ. Math. Fac. Sc. Besançon (Algèbre et théorie des nombres) **2**, PUFC (2019), 29–51. <https://doi.org/10.5802/pmb.35> 20
- [41] G. Gras, *Normes d'idéaux dans la tour cyclotomique et conjecture de Greenberg*, Ann. math. du Québec **43** (2019), 249–280. <https://doi.org/10.1007/s40316-018-0108-3> 25
- [42] G. Gras, *Genus theory and  $\varepsilon$ -conjectures on  $p$ -class groups*, Jour. Number Theory **207** (2020), 423–459. <https://doi.org/10.1016/j.jnt.2019.07.008> 50
- [43] G. Gras, *Algorithmic complexity of Greenberg's conjecture*, Arch. Math. **117** (2021), 277–289. <https://doi.org/10.1007/s00013-021-01618-9> 20, 25
- [44] G. Gras, *Application of the notion of  $\varphi$ -object to the study of  $p$ -class groups and  $p$ -ramified torsion groups of abelian extensions*, arXiv (2021). <https://arxiv.org/abs/2112.02865> 26, 27
- [45] G. Gras, *The  $p$ -rank  $\varepsilon$ -conjecture on class groups is true for towers of  $p$ -extensions of a number field*, J. Ramanujan Math. Soc. **37**(2) (2022), 129–138. <http://www.mathjournals.org/jrms/2022-037-002/> 50

- [46] G. Gras, *On the  $\lambda$ -stability of  $p$ -class groups along cyclic  $p$ -towers of a number field*, Int. J. of Number Theory **18**(10) (2022), 2241–2263.  
<https://doi.org/10.1142/S1793042122501147> 3, 4, 45
- [47] G. Gras, *The Chevalley–Herbrand formula and the real abelian conjecture. New criterion using capitulation of the class group*, arXiv (2022).<https://arxiv.org/abs/2207.13911> 4, 12, 51
- [48] R. Greenberg, *On the Iwasawa invariants of totally real number fields*, Amer. J. Math. **98**(1) (1976), 263–284.<https://doi.org/10.2307/2373625> 24
- [49] C. Greither, *Class groups of abelian fields, and the main conjecture*, Ann. Inst. Fourier **42**(3) (1992), 449–499.<https://doi.org/10.5802/aif.1299> 27
- [50] K.W. Gruenberg, A. Weiss, *Capitulation and transfer kernels*, J. Théorie Nombres Bordeaux **12**(1) (2000), 219–226.[http://www.numdam.org/item/JTNB\\_2000\\_\\_12\\_1\\_219\\_0/](http://www.numdam.org/item/JTNB_2000__12_1_219_0/) 3
- [51] K.W. Gruenberg, A. Weiss, *Capitulation and Transfer Triples*, Proc. London Math. Soc. **87**(2) (2003), 273–290.<https://doi.org/10.1112/S0024611503014199> 3
- [52] F-P. Heider, B. Schmithals, *Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen*, J. Reine Angew. Math. **336** (1982), 1–25.  
<https://doi.org/10.1515/crll.1982.336.1> 3
- [53] K. Iwasawa, *On  $\mathbb{Z}_\ell$ -extensions of algebraic number fields*, Ann. Math. **98** (1973), 243–326.  
<https://doi.org/10.2307/1970784> 3, 23
- [54] K. Iwasawa, *A note on capitulation problem for number fields II*, Proc. Japan Acad. Ser. A Math. Sci. **65**(6) (1989), 183–186.<https://doi.org/10.3792/pjaa.65.183> 3
- [55] J-F. Jaulent, *L'arithmétique des  $\ell$ -extensions* (Thèse d'état), Publ. Math. Fac. Sc. Besançon (Algèbre et théorie des nombres) **1**(1), PUFC (1986), 1–357.  
<https://doi.org/10.5802/pmb.a-42> 3, 4, 12, 20, 23
- [56] J-F. Jaulent, *L'état actuel du problème de la capitulation*, Sémin. Théor. Nombres Bordeaux, exposé 17 (1987/1988), 1–33.<https://www.jstor.org/stable/44166467> 3
- [57] J-F. Jaulent, *Classes logarithmiques des corps de nombres*, J. Théorie Nombres Bordeaux **6**(2) (1994), 301–325.[https://jtnb.centre-mersenne.org/article/JTNB\\_1994\\_\\_6\\_2\\_301\\_0.pdf](https://jtnb.centre-mersenne.org/article/JTNB_1994__6_2_301_0.pdf) 24, 46
- [58] J-F. Jaulent, *Classes logarithmiques et capitulation*, Funct. Approx. Comment. Math. **54**(2) (2016), 227–239.<https://doi.org/10.7169/facm/2016.54.2.6> 23, 24, 45
- [59] J-F. Jaulent, *Note sur la conjecture de Greenberg*, J. Ramanujan Math. Soc. **34**(1) (2019) 59–80.<http://www.mathjournals.org/jrms/2019-034-001/> 23, 24, 45, 46
- [60] J-F. Jaulent, *Principalisation abélienne des groupes de classes logarithmiques*, Funct. Approx. Comment. Math. **61**(2) (2019), 257–275.  
<https://doi.org/10.7169/facm/1765> 3, 4, 24, 45, 49
- [61] J-F. Jaulent, *Capitulation abélienne des groupes de classes de rayons*, J. Number Theory **231** (2022), 316–332.<https://doi.org/10.1016/j.jnt.2021.05.003> 3, 4, 45
- [62] J-F. Jaulent, A. Michel, *Approche logarithmique des noyaux étales des corps de nombres*, J. Number Theory **120** (2006), 72–91.<https://doi.org/10.1016/j.jnt.2005.11.011> 3, 45
- [63] C. Khare, D. Prasad, *On the Steinitz module and capitulation of ideals*, Nagoya Math. J. **160** (2000), 1–15.<https://doi.org/10.1017/S0027763000007686> 45
- [64] H. Kisilevsky, *Some results related to Hilbert's theorem 94*, J. Number Theory **2**(2) (1970), 199–206.[https://doi.org/10.1016/0022-314X\(70\)90020-X](https://doi.org/10.1016/0022-314X(70)90020-X) 3
- [65] J. Klüners, J. Wang,  *$\ell$ -torsion bounds of the class group of number fields with an  $\ell$ -group as Galois group*, OWP-2020-11.<https://doi.org/10.14760/OWP-2020-11>  
Proc. Amer. Math. Soc. **150** (2022), 2793–2805.<https://doi.org/10.1090/proc/15882> 23, 50
- [66] M. Kolster, A. Movahhedi, *Galois co-descent for étale wild kernels and capitulation*, Annales Institut Fourier **50**(1) (2000), 35–65.<https://doi.org/10.5802/aif.1746> 3, 23, 45
- [67] P. Koymans, C. Pagano, *On the distribution of  $Cl(K)[\ell^\infty]$  for degree  $\ell$  cyclic fields*, J. Eur. Math. Soc. **24**(4) (2022), 1189–1283.<https://doi.org/10.4171/JEMS/1112> 7
- [68] J.S. Kraft, R. Schoof, *Computing Iwasawa modules of real quadratic number fields*, Compositio Math. **97**(1-2) (1995), 135–155.<http://eudml.org/doc/90370> 3, 24  
Erratum. Compositio Math., **103**(2) (1996), p. 241.  
[http://www.numdam.org/item?id=CM\\_1996\\_\\_103\\_2\\_241\\_0](http://www.numdam.org/item?id=CM_1996__103_2_241_0)
- [69] M. Kurihara, *On the ideal class groups of the maximal real subfields of number fields with all roots of unity*, J. Eur. Math. Soc. **1**(1) (1999), 35–49.  
<https://doi.org/10.1007/PL00011159> 3, 4
- [70] F. Lemmermeyer, *The ambiguous class number formula revisited*, Journal of Ramanujan Math. Soc. **28**(4) (2013), 415–421.<http://www.mathjournals.org/jrms/2013-028-004/> 4
- [71] J. Li, C.F. Yu, *The Chevalley–Gras formula over global fields*, J. Théor. Nombres Bordeaux **32**(2) (2020), 525–543.<https://doi.org/10.5802/jtnb.1133> 4
- [72] J. Li, Y. Ouyang, Y. Xu, S. Zhang,  *$\ell$ -Class groups of fields in Kummer towers*, Publ. Mat. **66**(1) (2022), 235–267.<https://doi.org/10.5565/PUBLMAT6612210> 3



- [73] Y. Liu, *On the  $p$ -rank of class groups of  $p$ -extensions*, arXiv (2022).  
<https://arxiv.org/abs/2212.09888> 23
- [74] C. Maire,  *$T$ - $S$  capitulation*, Publ. Math. Fac. Sc. Besançon (Algèbre et théorie des nombres) **2**, PUFC (1996), 32 pp.<https://doi.org/10.5802/pmb.a-79> 3, 45
- [75] C. Maire, *Une remarque sur la capitulation du groupe des classes au sens restreint*, Publ. Math. Fac. Sc. Besançon (Algèbre et théorie des nombres) **2**, PUFC (1998), 1–10.  
<https://doi.org/10.5802/pmb.a-87> 3
- [76] C. Maire, *Genus theory and governing fields*, New York J. Math. **24** (2018), 1056–1067.  
<http://nyjm.albany.edu/j/2018/24-50.html> 23
- [77] K. Martin, *Non-unique factorization and principalization in number fields*, Proceedings of the American Mathematical Society **139**(9) (2011), 3025–3038.  
<https://doi.org/10.1090/S0002-9939-2011-11053-0> 3
- [78] M. Matchett Wood, *Probability theory for random groups arising in number theory*, arXiv (2023).<https://doi.org/10.48550/arXiv.2301.09687> 50
- [79] D.C. Mayer, *Principalization algorithm via class group structure*, J. Théorie Nombres Bordeaux **26**(2) (2014), 415–464.<https://doi.org/10.5802/jtnb.874> 3
- [80] Y. Mazigh, *Iwasawa theory of Rubin–Stark units and class groups*, manuscripta math. **153** (2017), 403–430.<https://doi.org/10.1007/s00229-016-0889-0> 51
- [81] Y. Mazigh, *Unités de Stark et théorie d’Iwasawa* (Thèse), Université Bourgogne Franche-Comté, Besançon.<https://theses.hal.science/tel-01795150> 51
- [82] K. Miyake, *Algebraic investigations of Hilbert’s Theorem 94, the principal ideal theorem and the capitulation problem*, Expo. Math. **7** (1989), 289–346.<https://doi.org/10.5802/pmb.a-63> 3
- [83] Y. Mizusawa, K. Yamamoto, *On  $p$ -class groups of relative cyclic  $p$ -extensions*, Arch. Math. **117**(3) (2021), 253–260.<https://doi.org/10.1007/s00013-021-01619-8> 3
- [84] P. Morton, *Density results for the 2-class groups of imaginary quadratic fields*, J. reine angew. Math. **332** (1982), 156–187.<https://doi.org/10.1515/crll.1982.332.156> 7
- [85] A. Movahhedi, *Sur les  $p$ -extensions des corps  $p$ -rationnels*, Thèse, Univ. Paris VII, 1988.  
<https://www.theses.fr/1988PA077126> 20
- [86] T. Nguyen Quang Do, *Sur la  $\mathbb{Z}_p$ -torsion de certains modules galoisiens*, Ann. Inst. Fourier, **36**(2) (1986), 27–46. <https://doi.org/10.5802/aif.1045> 20
- [87] L. Pagani, *Greenberg’s conjecture for real quadratic fields and the cyclotomic  $\mathbb{Z}_2$ -extension*, Math. Comp. **91** (2022) 1437–1467.<https://doi.org/10.1090/mcom/3712> 24
- [88] The PARI Group, *PARI/GP, version 2.5.3* (2013), Univ. Bordeaux.  
<http://pari.math.u-bordeaux.fr/download.html> 3
- [89] L.B. Pierce, *Counting problems: class groups, primes, and number fields*, Proceedings of the ICM 2022, arXiv (2022).<https://doi.org/10.48550/arXiv.2206.08351> 50
- [90] L.B. Pierce, C.L. Turnage–Butterbaugh, M.M. Wood, *An effective Chebotarev density theorem for families of number fields, with an application to  $\ell$ -torsion in class groups*, Invent. Math. **219**(2) (2020), 701–778.<https://doi.org/10.1007/s00222-019-00915-z> 50
- [91] B. Schmithals, *Kapitulation der Idealklassen und Einheitenstruktur in Zahlkörpern*, J. reine angew. math. **358** (1985), 43–60.<https://doi.org/10.1515/crll.1985.358.43> 3
- [92] A. Smith, *The distribution of  $\ell^\infty$ -Selmer groups in degree  $\ell$  twist families*, arXiv (2022).  
<https://doi.org/10.48550/arXiv.2207.05674> 7
- [93] D. Solomon, *On the class groups of imaginary abelian fields*, Ann. Inst. Fourier **40**(3) (1990), 467–492.<https://doi.org/10.5802/aif.1221> 27
- [94] H. Suzuki, *A generalization of Hilbert’s Theorem 94*, Nagoya Math. J. **121** (1991), 161–169.  
<https://doi.org/10.1017/S0027763000003445> 3
- [95] H. Taya, *On cyclotomic  $\mathbb{Z}_p$ -extensions of real quadratic fields*, Acta Arithmetica **74**(2) (1996), 107–119.<http://matwbn.icm.edu.pl/ksiazki/aa/aa74/aa7422.pdf> 25
- [96] H. Taya, *On  $p$ -adic zeta functions and  $\mathbb{Z}_p$ -extensions of certain totally real number fields*, Tohoku Math. J. **51**(1) (1999), 21–33.<https://doi.org/10.2748/tmj/1178224850> 25
- [97] H. Taya, *Iwasawa invariants and class numbers of quadratic fields for the prime 3*, Proc. Amer. Math. Soc. **128**(5), 1285–1292.<https://doi.org/10.1090/S0002-9939-99-05177-1> 25
- [98] F. Terada, *A principal ideal theorem in the genus field*, Tohoku Math. J. (2) **23**(4) (1971), 697–718.<https://doi.org/10.2748/tmj/1178242555> 3
- [99] C. Thiébaud, *Sur la capitulation dans les corps de genres d’une extension abélienne d’un corps quadratique imaginaire*, J. Number Theory **85**(1) (2000), 92–107.  
<https://doi.org/10.1006/jnth.2000.2550> 3
- [100] R. Validire, *Capitulation des noyaux sauvages étales*, Thèse, Université de Limoges (24 Juin 2008).<https://tel.archives-ouvertes.fr/tel-00343427/document> 3, 23, 45
- [101] J. Wang, *Pointwise bound for  $\ell$ -torsion in class groups: Elementary abelian extensions*, J. reine angew. Math. **2021**(773), 129–151.<https://doi.org/10.1515/crelle-2020-0034> 50

- [102] L.C. Washington, *Introduction to Cyclotomic Fields*, GTM 83, Springer enlarged second edition 1997. [12](#)

VILLA LA GARDETTE, 4 CHEMIN DE CHÂTEAU GAGNIÈRE, F-38520 LE BOURG D'OISANS  
*Email address:* [g.mn.gras@wanadoo.fr](mailto:g.mn.gras@wanadoo.fr)  
*URL:* <http://orcid.org/0000-0002-1318-4414>