Algebraic norm and capitulation of p-class groups in ramified cyclic p-extensions

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ALGEBRAIC NORMS AND CAPITULATION OF p-CLASS GROUPS IN RAMIFIED CYCLIC p-EXTENSIONS

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Abstract. We examine the phenomenon of capitulation of the p-class group $\mathscr{H}_K$ of a number field K in totally ramified cyclic p-extensions L/K of degree p^N, taking the simplest abelian p-extensions $L \subset K(\mu_\ell)$, with primes $\ell \equiv 1 \pmod{2p^N}$. Using an elementary property of the algebraic norm $\mathcal{N}_{L/K}$, we show that the kernel of capitulation is in relation with the "complexity" of the structure of $\mathscr{H}_L$ measured via its exponent $p^{e(L)}$ and the length $m(L)$ of the usual filtration $\{\mathscr{H}_L^i\}_{i\geq 0}$ associated to $\mathscr{H}_L$ as $\mathbb{Z}_p[\mathrm{Gal}(L/K)]$-module. We prove that a *sufficient condition* of capitulation is given by $e(L) \in [1, N - s(L)]$ if $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$ for $s(L) \in [0, N-1]$ (Theorem 1.1 (i)); this improves the case of "stability" $\#\mathscr{H}_L = \#\mathscr{H}_K$ (i.e., $m(L) = 1$, $s(L) = 0$, $e(L) = e(K)$) (Theorem 1.1 (ii)). Numerical examples (with PARI programs) showing most often capitulation of $\mathscr{H}_K$ in L, are given over cubic fields with $p = 2$ and real quadratic fields with $p = 3$. Some conjectures on the existence of non-zero densities of such ℓ's are proposed (Conjectures 1.2, 2.4).

Contents

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1. Introduction

1.1. **Statement of the main result.** Let L/K be any cyclic p-extension of number fields ($p \geq 2$ prime), of degree p^N, $N \geq 1$, of Galois group G, and let $\mathscr{H}_K$, $\mathscr{H}_L$ be the p-class groups of K, L, respectively. We assume in all the sequel that L/K is totally ramified.

Let $\mathbf{J}_{L/K} : \mathscr{H}_K \to \mathscr{H}_L^G$ be the transfer map (or extension of classes) and let $\mathbf{N}_{L/K} : \mathscr{H}_L \to \mathscr{H}_K$ be the arithmetic norm induced by $\mathbf{N}_{L/K}(\mathfrak{P}) := \mathfrak{p}^f$, for all prime ideal $\mathfrak{P}$ of L and the prime ideal $\mathfrak{p}$ of K under $\mathfrak{P}$ with residue degree f.

We know that $\mathbf{N}_{L/K}(\mathscr{H}_L)$ is the subgroup of $\mathscr{H}_K$ which corresponds to $\mathrm{Gal}(H_K^{\mathrm{nr}}/L \cap H_K^{\mathrm{nr}})$ by the Artin map of class field theory, where H_K^{nr} is the p-Hilbert class field of K in the ordinary sense; thus, by the assumption of total ramification, $\mathbf{N}_{L/K}(\mathscr{H}_L) = \mathscr{H}_K$.

Let $\mathrm{rk}_p(A) := \dim_{\mathbb{F}_p}(A/A^p)$ be the p-rank of any finite abelian group A.

We can state:

Theorem 1.1. *Let L/K be a totally ramified cyclic p-extension of number fields, of degree p^N, $N \geq 1$, of Galois group $G =: \langle\sigma\rangle$:*

(i) Let $p^{e(L)}$ be the exponent of $\mathscr{H}_L$ and let $m(L)$ be the minimal integer such that $(\sigma - 1)^{m(L)}$ annihilates $\mathscr{H}_L$. Then, a sufficient condition for the complete capitulation of $\mathscr{H}_K$ in L, is that $e(L) \in [1, N - s(L)]$ if $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$ for $s(L) \in [0, N - 1]$.

A sufficient condition for a partial capitulation of $\mathscr{H}_K$ in L, is that there exists $h' \in \mathscr{H}_L$ such that $\mathbf{N}_{L/K}(h') \neq 1$, h' being of order p^e and annihilated by $(\sigma - 1)^m$ such that $e \in [1, N - s]$ if $m \in [p^s, p^{s+1} - 1]$ for $s \in [0, N - 1]$.

(ii) Let K_n, $n \in [0, N]$, be the subfield of L of degree p^n over K and set $G_n := \mathrm{Gal}(K_n/K)$. Then $\#\mathscr{H}_{K_1} = \#\mathscr{H}_K$ implies the following properties:

- *$\mathscr{H}_{K_n} = \mathscr{H}_{K_n}^{G_n}$ and $\mathscr{H}_{K_n} \overset{\mathbf{N}_{K_n/K}}{\simeq} \mathscr{H}_K$, for all $n \in [0, N]$.*
- *For any $e \leq N$, the subgroup $\mathscr{H}_K[p^e]$ of $\mathscr{H}_K$, of classes annihilated by p^e, capitulates in K_e; whence $\mathscr{H}_K$ capitulates in $K_{e(K)}$ if $e(K) \leq N$.*

(iii) If $\mathrm{rk}_p(\mathscr{H}_{K_1}) = \mathrm{rk}_p(\mathscr{H}_K)$, then $\mathrm{rk}_p(\mathscr{H}_{K_n}) = \mathrm{rk}_p(\mathscr{H}_K)$ for all $n \in [0, N]$.

The claim (i) will be given by Corollary 2.12 to Theorem 2.9. Then points (ii), (iii) come from [Gras2022[a], Theorem 3.1 & Section 6, § (b)] generalizing [Fuku1994, KrSch1995, Band2007, LOXZ2022, MiYa2021]; it corresponds to the case $m(K_n) = 1$ (from $\mathscr{H}_{K_n} = \mathscr{H}_{K_n}^{G_n}$), whence $s(K_n) = 0$ and $e(K) \leq N$ (from $e(L) = e(K)$ due to the isomorphisms $\mathscr{H}_{K_n} \overset{\mathbf{N}_{K_n/K}}{\simeq} \mathscr{H}_K$ given by the arithmetic norms); this particular case is called the p-class groups stability in the tower $L = \bigcup_{n=0}^N K_n$.

In practice, the knowledge of $m(L)$ determines the unique $s(L) \geq 0$ such that $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$ and one must check if $s(L) \in [0, N - 1]$; this condition is most often fulfilled, except possibly for very small N's, a necessary

condition of capitulation being $N \geq e(K)$. Once this holds, one must have $e(L) \leq N - s(L)$ to get the capitulation.

The case of the p-ranks for the case of $\mathbb{Z}_p$-extensions was given by Fukuda [Fuku1994], then found again by Bandini [Band2007]. It holds for any cyclic totally ramified p-extension as we have explained in [Gras2022^a, § 6 (b)].

Of course, properties of stability may occur from a layer K_{n_0} as, more generally, for any reasoning about capitulation.

Theorem 1.1 expresses that, if the "complexity" of $\mathscr{H}_L$ is not too important, then $\mathscr{H}_K$ capitulates in L. Conversely, if one knows that capitulation is impossible (e.g., framework of abelian imaginary fields and minus p-class groups $\mathscr{H}^-$), then the complexity of the $\mathscr{H}^-_{K_n}$'s is much more important compared to that of $\mathscr{H}^-_K$ and strictly increases with n (see Theorem 3.2).

1.2. **History and aims of the paper.** The problem of capitulation [1] of $\mathscr{H}_K$ in L (measured by the capitulation kernel $\mathrm{Ker}(\mathbf{J}_{L/K} : \mathscr{H}_K \to \mathscr{H}_L)$) has been studied in a very large number of publications, as, precisely in the purpose of the factorization problem for Dedekind rings, exposed in [Mart2011]. It is impossible to give a complete bibliography, but one may cite, among many other contributions subsequent to the historical works of Hilbert–Scholz–Taussky: [Kisil1970, Tera1971, Bond1981, HeSc1982, Schm1985, GrJa1985, Jaul1986, Jaul1988, Iwas1989, Suzu1991, Maire1996, Gras1997, Maire1998, Kuri1999, Thie2000, GrWe2000, KoMo2000, GrWe2003, JauMi2006, Gonz2006, Vali2008, Bosc2009, Bem2012, Mayer2014, AZT2016, AZTM2016, Bisw2016, AZT2017, Jaul2019^b, Jaul2022], in which the reader will find more history and references.

Except some Iwasawa's theory results on capitulation, most of these papers are related to the Artin–Furtwängler theorem and its generalizations on capitulation in the Hilbert class field H_K^{nr} (or to the Hilbert Theorem 94 in cyclic subextensions), which is not our purpose since, on the contrary, we will study totally ramified cyclic p-extensions L/K and more precisely the simplest tamely ramified p-extensions $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$ prime, $[L : K] = p^N$, which, surprisingly, are often capitulation fields of $\mathscr{H}_K$ when K is totally real or in some other cases when K is not totally imaginary.

Many classical articles give cohomological expressions of the capitulation in terms of global units as the fact that, in the non-ramified case, the capitulation kernel $\mathrm{Ker}(\mathbf{J}_{L/K})$ is isomorphic to a subgroup of $\mathrm{H}^1(G, \mathbf{E}_L)$, where $\mathbf{E}_L$ is the group of units of L (see, e.g., [Jaul1986, Chap. III, § 1] and [Jaul1988, Bem2012] for more comments and references).

In the same spirit, using sets of places S, T and tamely ramified Galois extensions, the result of Maire [Maire1996, Théorème 4.1] gives, in our context $L \subset K(\mu_\ell)$, injective maps $\mathscr{H}_{L,(\ell)}/\mathbf{J}_{L/K}(\mathscr{H}_{K,(\ell)}) \hookrightarrow \mathrm{H}^2(G, \mathscr{E}_{L,(\ell)})$, where $\mathscr{E} := \mathbf{E} \otimes \mathbb{Z}_p$ for groups of units and where $\mathscr{H}_{K,(\ell)}$, $\mathscr{H}_{L,(\ell)}$ are ray-class groups modulo (ℓ) and $\mathscr{E}_{L,(\ell)}$ the group of units congruent to 1 modulo (ℓ).

But the aspect "global units" is more difficult since the behavior of the unit groups in L/K is less known compared to that of p-class groups, even if there are some links; indeed, we have the following classical exact sequence obtained from the map which associates with the invariant class of the ideal $\mathfrak{a}$, a unit $\varepsilon := \mathbf{N}_{L/K}(\alpha)$ where $\mathfrak{a}^{\sigma-1} =: (\alpha)$:

$$1 \to \mathbf{J}_{L/K}(\mathscr{H}_K) \cdot \mathscr{H}_L^{\mathrm{ram}} \to \mathscr{H}_L^G \to \mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times)/\mathbf{N}_{L/K}(\mathscr{E}_L) \to 1, \quad (1.1)$$

[1] I recently learned (from a Lemmermeyer text) that the word *capitulation* was coined by Arnold Scholz. It is possible that this term may be considered as incongruous; a solution is to consider that a non principal ideal $\mathfrak{a}$ is a troublemaker with respect to elementary arithmetic, in which case, the terminology is perfectly understandable. Conjectures about ideal capitulations have perhaps a moral significance.

where $\mathscr{H}_L^{\rm ram} \subseteq \mathscr{H}_L^G$ is generated by means of classes of the ramified prime ideals of L; in the right term, if $\mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times)$ depends on easier local considerations, $\mathbf{N}_{L/K}(\mathscr{E}_L)$ is in general unknown.

On the contrary, $\mathbf{J}_{L/K}(\mathscr{H}_K)$, $\mathscr{H}_L^{\rm ram}$, are subgroups of $\mathscr{H}_L^G$ and the order of this group is known from the Chevalley–Herbrand formula [Chev1933, pp. 4002–405] (see [Gras1978] for isotopic components of $\mathscr{H}_L^G$ in the abelian semi-simple case, [Jaul1986, Chapitre III, p. 167] with ramification and decomposition, [Lemm2013] in the spirit of Jaulent's work); then general higher fixed points formulas [Gras1973, Gras2017^a, LiYu2020] allow the algorithmic computation of $\mathscr{H}_L$ from a natural filtration (see Subsection 2.1.1).

In our previous paper [Gras2022^a], giving extensive numerical computations with [PARI], we have proposed a conjecture, whose main consequence should be an obvious and immediate proof of the real abelian Main Conjecture "$\mathscr{H}_{K,\varphi} = (\mathscr{E}_{K,\varphi} : \mathscr{F}_{K,\varphi})$" in terms of indices of Leopoldt's cyclotomic units and φ-components using p-adic characters φ of K, in the semi-simple case [2].

We can improve this conjecture taking into account the new criterion using algebraic norms:

Conjecture 1.2. *(i) Let K be any totally real number field and let $\mathscr{H}_K$ be its p-class group, of exponent $p^{e(K)}$.*

- *There exist infinitely many primes $\ell \equiv 1 \pmod{2p^N}$, $N \geq e(K)$, such that $\mathscr{H}_K$ capitulates in $K(\mu_\ell)$.*
- *There exist infinitely many primes $\ell \equiv 1 \pmod{2p^N}$, N large enough, such that the capitulation of $\mathscr{H}_K$ in $L \subset K(\mu_\ell)$, of degree p^N over K, is due to the property: $e(L) \in [1, N - s(L)]$ if $m(L) \in [p^{s(L)}, p^{s(L)+1} - 1]$ for $s(L) \in [0, N-1]$.*
- *The case of stability (i.e., $m(L) = 1$, $s(L) = 0$, $e(L) = e(K)$) occurs for infinitely many ℓ's.*

(ii) Let $\ell \equiv 1 \pmod{2p^N}$, $N \geq 1$, be a fixed prime number and let $\mathscr{K}_{d,N}$ be the family of totally real number fields K, of degree $d \geq 2$, whose p-class group $\mathscr{H}_K \neq 1$ is of order smaller than p^N.

- *There exist infinitely many $K \in \mathscr{K}_{d,N}$ such that $\mathscr{H}_K$ capitulates in $K(\mu_\ell)$.*
- *There exist infinitely many $K \in \mathscr{K}_{d,N}$ such that $\#\mathscr{H}_{K_1} = \#\mathscr{H}_K$ (stability).*

Remark 1.3. This restriction to the family of p-extensions L/K, $L \subset K(\mu_\ell)$, is another point of view compared to the case of abelian capitulations obtained in [Gras1997], [Kuri1999], [Bosc2009], [Jaul2019^b, Jaul2022]. Indeed, all techniques in these papers need to built a finite set of abelian p-extensions L_k of $\mathbb{Q}$, ramified at various primes, requiring many local arithmetic conditions existing from Chebotarev theorem, whose compositum with K gives a capitulation field of $\mathscr{H}_K$; the method must apply to any abelian field K (of suitable signature), of arbitrary increasing degree, obtained in an iterative process giving, for instance, that the maximal real subfield of $\mathbb{Q}\big(\bigcup_{f>0} \mu_f\big)$ is principal (see in [Bosc2009] the most general statements).

Remark 1.4. Let's introduce the p-ray class group $\mathscr{H}_{K,(\ell)}$ and the p-ray class field $H_K^{\ell{\rm ram}}$ modulo (ℓ) (non-complexified), where ℓ is any prime number

[2] The complete statement being the following [Gras2022^b, Section 1.4, then Theorem 4.6]: Assume that $K/\mathbb{Q}$ is a real cyclic extension, of prime-to-p degree. Let $\ell \equiv 1 \pmod{2p^N}$ be a prime totally inert in $K/\mathbb{Q}$ and let $L \subset K(\mu_\ell)$ be the subfield of degree p^N over K. Then, if $\mathscr{H}_K$ capitulates in L, the "Main Conjecture" on the p-adic components $\mathscr{H}_{K,\varphi}$, of $\mathscr{H}_K$, holds (i.e., $\#\mathscr{H}_{K,\varphi} = (\mathscr{E}_{K,\varphi} : \mathscr{F}_{K,\varphi})$ for all irreducible p-adic character φ of K).

distinct from p, unramified in K, and let $\mathscr{E}_{K,(\ell)} := \mathbf{E}_{K,(\ell)} \otimes \mathbb{Z}_p$ be the subgroup of of units congruent to 1 modulo ℓ. We then have the exact sequence:

$$1 \to \mathscr{E}_K/\mathscr{E}_{K,(\ell)} \longrightarrow \bigoplus_{\mathfrak{l} \mid \ell} \mathbb{F}_{\mathfrak{l}}^{\times} \otimes \mathbb{Z}_p \longrightarrow \mathscr{H}_{K,(\ell)} \longrightarrow \mathscr{H}_K \to 1,$$

where $\mathbb{F}_{\mathfrak{l}}$ is the residue field at the prime ideal $\mathfrak{l}$. This leads to the well-known formula:

$$\#\mathscr{H}_{K,(\ell)} = \#\mathscr{H}_K \times \frac{\prod_{\mathfrak{l} \mid \ell} (\ell^{f_{\mathfrak{l}}} - 1)_p}{(\mathscr{E}_K : \mathscr{E}_{K,(\ell)})},$$

where $f_{\mathfrak{l}}$ is the residue degree of $\mathfrak{l}$ in $K/\mathbb{Q}$ and $(\ )_p$ the p-part of an integer.

For instance from [Gras2005, Theorem II.5.8.3], generalizing the principal ideal theorem in the tame case, $\mathscr{H}_{K,(\ell)}$ capitulates in $H_K^{\ell\mathrm{ram}}$, as $\mathscr{H}_K$ for trivial reason. This enforces the conjecture by analogy with the fact that capitulation of $\mathscr{H}_K$ in H_K^{nr} may occur in various sub-extensions, a problem extensively studied as recalled with many bibliographic items given in Section 1.2.

Assume, to simplify, that $K/\mathbb{Q}$ is cyclic of prime-to-p degree d and that $\ell \equiv 1 \pmod{2\,p^N}$ as usual; let f be the common residue degree and r the number of prime ideals above ℓ; since the real field $L \subset K(\mu_\ell)$ is a subfield of $H_K^{\ell\mathrm{ram}}$, one gets:

$$[H_K^{\ell\mathrm{ram}} : LH_K^{\mathrm{nr}}] = \frac{p^{N\,(r-1)}}{(\mathscr{E}_K : \mathscr{E}_{K,(\ell)})}.$$

If ℓ is totally inert in K, then $H_K^{\ell\mathrm{ram}} = LH_K^{\mathrm{nr}}$. If ℓ totally splits, then $(\mathscr{E}_K : \mathscr{E}_{K,(\ell)})$ may be any divisor of $p^{N\,(d-1)}$.

Returning to a class of p-power order, represented by an ideal $\mathfrak{a}$ of K which is principal in L, then $(\mathfrak{a})_L =: (\alpha)$, $\alpha \in L^{\times}$, and there exists a unit u in $H_K^{\ell\mathrm{ram}}$ such that $\alpha \cdot u \equiv 1 \pmod{\ell}$. Of course, since $\mathfrak{a}$ is always principal in H_K^{nr}, the above equality holds replacing $\alpha \in L^{\times}$ by $\alpha \in H_K^{\mathrm{nr}\,\times}$. The difference of nature between these two kinds of capitulation is that L is a straightforward well-known abelian extension of $\mathbb{Q}$, while H_K^{nr} and $H_K^{\ell\mathrm{ram}}$ are not.

2. Complexity of $\mathscr{H}_L$ versus capitulation of $\mathscr{H}_K$

Let L/K be a totally ramified cyclic p-extension of degree p^N, $N \geq 1$, of Galois group $G =: \langle \sigma \rangle$. Let:

$$\nu_{L/K} := \sum_{i=0}^{p^N-1} \sigma^i,$$

be the algebraic norm in L/K. From the law of decomposition in L/K of an unramified prime ideal $\mathfrak{q}$ of K, of residue degree f, we get for $\mathfrak{Q} \mid \mathfrak{q}$ in L and using the fact that the decomposition group D of $\mathfrak{Q}$ is of order f, $(\mathfrak{q})_L = \prod_{\overline{\sigma} \in G/D} \mathfrak{Q}^{\overline{\sigma}}$, thus:

$$\nu_{L/K}(\mathfrak{Q}) = \prod_{i=0}^{p^N-1} \mathfrak{Q}^{\sigma^i} = (\mathfrak{q})_L^f = (\mathfrak{q}^f)_L = (\mathbf{N}_{L/K}(\mathfrak{Q}))_L;$$

whence:

$$\nu_{L/K}(\mathscr{H}_L) = \mathbf{J}_{L/K} \circ \mathbf{N}_{L/K}(\mathscr{H}_L) = \mathbf{J}_{L/K}(\mathscr{H}_K). \tag{2.1}$$

Thus, $\mathscr{H}_K$ capitulates in L if and only if:

$$\nu_{L/K}(\mathscr{H}_L) = 1.$$

A partial capitulation occurs as soon as $\#\nu_{L/K}(\mathscr{H}_L) < \#\mathscr{H}_K$.

So, the action of the algebraic norm characterizes the capitulation (complete or incomplete) and it is clear that the result mainly depends on the $\mathbb{Z}_p[G]$ structure of $\mathscr{H}_L$ which is expressed by means of the canonical associated filtration $\{\mathscr{H}_L^i\}_{i \geq 0}$ that we are going to recall.

2.1. **Filtration of $\mathscr{H}_L$ in the totally ramified case.** Let L/K be a cyclic p-extension of degree p^N, $N \geq 1$, and Galois group $G = \langle\sigma\rangle$. To avoid technical writings, we assume that any prime ideal $\mathfrak{l}$ of K, ramified in L/K, is totally ramified, and that there are $r \geq 1$ such ramified prime ideals.

2.1.1. *The higher rank Chevalley–Herbrand formulas.* The generalizations of the Chevalley–Herbrand formula to the corresponding filtration $\{\mathscr{H}_L^i\}_{i\geq 0}$ gives rise to the following expressions, where:

$$\mathscr{H}_L^0 = 1, \quad \mathscr{H}_L^1 := \mathscr{H}_L^G, \quad \mathscr{H}_L^{i+1}/\mathscr{H}_L^i := (\mathscr{H}_L/\mathscr{H}_L^i)^G,\ i \geq 0,$$

up to $i = m(L) = \min\{m \geq 0,\ \mathscr{H}_L^{(\sigma-1)^m} = 1\}$, for which $\mathscr{H}_L^{m(L)} = \mathscr{H}_L$. We put $\mathscr{H}_L^i = \mathscr{H}_L$ for all $i \geq m(L)$.

Denote by $\mathscr{I}_L^i$ a $\mathbb{Z}_p[G]$-module of ideals of L, of finite type, generating $\mathscr{H}_L^i$, with $\mathscr{I}_L^0 = 1$, $\mathscr{I}_L^{i+1} \supseteq \mathscr{I}_L^i$, for all $i \geq 0$; the $\mathscr{I}_L^i$ are defined up to the group of principal ideals of L, thus $\mathbf{N}_{L/K}(\mathscr{I}_L^i)$ is defined up to $\big(\mathbf{N}_{L/K}(L^\times)\big)$.

This filtration has the following properties [Gras2017^a, Theorem 3.6]:

$$\text{(2.2)}\qquad \begin{cases} (i) \quad \#\mathscr{H}_L^1 = \#\mathscr{H}_K \times \dfrac{p^{N(r-1)}}{(\mathscr{E}_K : \mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times))}, \\ (ii) \quad \#(\mathscr{H}_L^{i+1}/\mathscr{H}_L^i) = \dfrac{\#\mathscr{H}_K}{\#\mathbf{N}_{L/K}(\mathscr{H}_L^i)} \times \dfrac{p^{N(r-1)}}{(\mathbf{\Lambda}_K^i : \mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times))}, \\ \qquad\qquad \mathbf{\Lambda}_K^i := \{x \in K^\times,\ (x) \in \mathbf{N}_{L/K}(\mathscr{I}_L^i)\} \otimes \mathbb{Z}_p, \\ (iii) \quad \mathscr{H}_L^i = \{h \in \mathscr{H}_L,\ h^{(\sigma-1)^i} = 1\},\ \text{ for all } i \geq 0, \\ (iv) \quad \#(\mathscr{H}_L^{i+1}/\mathscr{H}_L^i) \leq \#\mathscr{H}_L^1,\ \text{ for all } i \geq 0, \\ (v) \quad \#\mathscr{H}_L = \prod_{i=0}^{m(L)-1} \#(\mathscr{H}_L^{i+1}/\mathscr{H}_L^i) \leq (\#\mathscr{H}_L^1)^{m(L)}. \end{cases}$$

The $\mathbf{\Lambda}_K^i$'s are subgroups of $K^\times$ containing $\mathscr{E}_K$, with $\mathbf{\Lambda}_K^0 = \mathscr{E}_K$. In particular, any $x \in \mathbf{\Lambda}_K^i$ is local norm in L/K at all the non-ramified places. So, for any $(x) = \mathbf{N}_{L/K}(\mathfrak{A})$, $\mathfrak{A} \in \mathscr{I}_L^i$, which is also local norm at the ramified places, then $x = \mathbf{N}_{L/K}(y)$, $y \in L^\times$ (Hasse's norm theorem) and there exists an ideal $\mathfrak{B}$ of L such that $\mathfrak{A} = (y)\mathfrak{B}^{\sigma-1}$; this constitutes an algorithm by addition of the $\mathfrak{B}$'s to $\mathscr{I}_L^i$ to get $\mathscr{I}_L^{i+1}$ then $\mathscr{H}_L^{i+1}$.[3] Since $\mathbf{\Lambda}_K^0 = \mathscr{E}_K$ is a $\mathbb{Z}_p$-module of finite type, this algorithm allows to construct $\mathbf{\Lambda}_K^i$ of finite type for all i, with $\mathbf{\Lambda}_K^i \subseteq \mathbf{\Lambda}_K^{i+1}$ (indeed, $\mathbf{N}_{L/K}(\mathscr{I}_L^i)$ is of finite type, there is a finite number of relations of principality between the generators and $\mathbf{\Lambda}_K^i/\mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times)$ is of exponent $\leq p^N$).

The i-sequence $\#(\mathscr{H}_L^{i+1}/\mathscr{H}_L^i)$, $0 \leq i \leq m(L)$, is decreasing, from $\#\mathscr{H}_L^1$ up to 1, because of the injective maps $\mathscr{H}_L^{i+1}/\mathscr{H}_L^i \hookrightarrow \mathscr{H}_L^i/\mathscr{H}_L^{i-1} \hookrightarrow \cdots \hookrightarrow \mathscr{H}_L^1$ due to the action of $\sigma - 1$, giving the inequality in (v).

The first (resp. second) factor in (ii) is called the class (resp. norm) factor.

2.1.2. *Properties of the class and norm factors.* Since the ramified places v (= prime ideals) of K are assumed to be totally ramified in L/K, their inertia groups $I_v(L/K)$ in L/K are isomorphic to G. Let $\omega_{L/K}$ be the map which associates with $x \in \mathbf{\Lambda}_K^i$ the family of Hasse's norm symbols $\big(\frac{x\,,\,L/K}{v}\big) \in I_v(L/K)$. Since x is local norm at the unramified places, $\omega_{L/K}(\mathbf{\Lambda}_K^i)$ is contained in:

$$\Omega_{L/K} := \big\{(\tau_v)_v \in \textstyle\bigoplus_v I_v(L/K),\ \prod_v \tau_v = 1\big\} \simeq G^{r-1}$$

(product formula); then $\mathrm{Ker}(\omega_{L/K}) = \mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times)$.

[3] For explicit class field theory, Hasse's norm theorem, norm residue symbols, product formula, see, e.g., [Gras2005, Theorem II.6.2, Definition II.3.1.2, Theorems II.3.1.3, 3.4.1].

It follows that $\#\omega_{L/K}(\mathbf{\Lambda}^i_K) = (\mathbf{\Lambda}^i_K : \mathbf{\Lambda}^i_K \cap \mathbf{N}_{L/K}(L^\times))$ divides $p^{N(r-1)}$.

Denote by K_n, $0 \le n \le N$, the subfields of L of degree p^n over K and let $G_n := \mathrm{Gal}(K_n/K) =: \langle\sigma_n\rangle$. All the previous definitions and formulas apply to the K_n's; we will denote by $\mathbf{\Lambda}^i_K(n)$ the invariant corresponding to K_n/K instead of L/K; we have $\mathbf{\Lambda}^i_K(n) = \{x \in K^\times,\ (x) \in \mathbf{N}_{K_n/K}(\mathscr{I}^i_{K_n})\}$. So $\mathbf{\Lambda}^i_K = \mathbf{\Lambda}^i_K(N)$ in the previous notations.

Note that $m(K) = 1$ for $\mathscr{H}_K \ne 1$, that $\mathscr{I}^i_K(0)$ generates $\mathscr{H}_K$ and that $\mathbf{\Lambda}^i_K(0)$ contains $\mathbf{\Lambda}^0_K(0) = \mathscr{E}_K$ and is given by relations between elements of $\mathscr{I}^i_K(0)$.

Lemma 2.1. *For any i fixed, we may assume that $\mathbf{\Lambda}^i_K(n+1) \subseteq \mathbf{\Lambda}^i_K(n)$, for all $n \in [0, N-1]$.*

Proof. For all $n \in [0, N-1]$, we have the following diagram, where the norm $\mathbf{N}_{K_{n+1}/K_n}$, on $\mathscr{H}_{K_{n+1}}$ and $(\mathscr{H}_{K_{n+1}})^{(\sigma_{n+1}-1)^i}$, is surjective (total ramification), but its restriction to $\mathscr{H}^i_{K_{n+1}}$ may be non injective nor surjective:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{H}^i_{K_{n+1}} & \longrightarrow & \mathscr{H}_{K_{n+1}} & \xrightarrow{(\sigma_{n+1}-1)^i} & (\mathscr{H}_{K_{n+1}})^{(\sigma_{n+1}-1)^i} & \longrightarrow & 1 \\
 & & \downarrow & & \mathbf{N}_{K_{n+1}/K_n}\downarrow & & \mathbf{N}_{K_{n+1}/K_n}\downarrow & & \\
1 & \longrightarrow & \mathscr{H}^i_{K_n} & \longrightarrow & \mathscr{H}_{K_n} & \xrightarrow{(\sigma_n-1)^i} & (\mathscr{H}_{K_n})^{(\sigma_n-1)^i} & \longrightarrow & 1\,.
\end{array}$$

We have $\mathbf{N}_{K_{n+1}/K_n}(\mathscr{H}^i_{K_{n+1}}) \subseteq \mathscr{H}^i_{K_n}$; so, for any ideal $\mathfrak{A}_{n+1} \in \mathscr{I}^i_{K_{n+1}}$, one may write $\mathbf{N}_{K_{n+1}/K_n}(\mathfrak{A}_{n+1}) = (\alpha_n)\,\mathfrak{A}_n$, where $\alpha_n \in K_n^\times$ and $\mathfrak{A}_n \in \mathscr{I}^i_{K_n}$, in which case *modifying* the definition of $\mathscr{I}^i_{K_n}$ modulo principal ideals of K_n, one may assume $\mathbf{N}_{K_{n+1}/K_n}(\mathscr{I}^i_{K_{n+1}}) \subseteq \mathscr{I}^i_{K_n}$ whence $\mathbf{N}_{K_{n+1}/K}(\mathscr{I}^i_{K_{n+1}}) \subseteq \mathbf{N}_{K_n/K}(\mathscr{I}^i_{K_n})$; this modifies $\mathbf{\Lambda}^i_K(n)$ modulo $\mathbf{N}_{K_n/K}(K_n^\times)$ which does not modify $\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))$ in the formula giving $\#\big(\mathscr{H}^{i+1}_{K_n}/\mathscr{H}^i_{K_n}\big)$. Using the process from the top, we obtain $\mathbf{\Lambda}^i_K(N) \subseteq \mathbf{\Lambda}^i_K(N-1) \subseteq \cdots \subseteq \mathbf{\Lambda}^i_K(1) \subseteq \mathbf{\Lambda}^i_K(0)$. □

Lemma 2.2. *For i fixed, the integers $\#\big(\mathscr{H}^{i+1}_{K_n}/\mathscr{H}^i_{K_n}\big)$ define an increasing n-sequence from $\#\big(\mathscr{H}^{i+1}_K/\mathscr{H}^i_K\big) = 1$ up to $\#\big(\mathscr{H}^{i+1}_L/\mathscr{H}^i_L\big)$; the $\#\mathscr{H}^i_{K_n}$'s define an increasing n-sequence from $\#\mathscr{H}^i_K = \mathscr{H}_K$ up to $\#\mathscr{H}^i_L$.*

Proof. Consider, for $i \ge 0$ fixed, the two factors of the finite n-sequence (for $n = 0$, the two factors are trivial):

$$\#\big(\mathscr{H}^{i+1}_{K_n}/\mathscr{H}^i_{K_n}\big) = \frac{\#\mathscr{H}_K}{\#\mathbf{N}_{K_n/K}(\mathscr{H}^i_{K_n})} \times \frac{p^{n(r-1)}}{\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))}.$$

As $\mathbf{N}_{K_{n+1}/K}(\mathscr{H}^i_{K_{n+1}}) \subseteq \mathbf{N}_{K_n/K}(\mathscr{H}^i_{K_n})$, $p^{c^i_{K_n}} := \dfrac{\#\mathscr{H}_K}{\#\mathbf{N}_{K_n/K}(\mathscr{H}^i_{K_n})}$ defines an increasing n-sequence from 1 up to a value $p^{c^i_L} \mid \#\mathscr{H}_K$. The norm factor:

$$p^{\rho^i_{K_n}} := \frac{p^{n(r-1)}}{\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))}$$

defines an increasing n-sequence from 1 up to $p^{\rho^i_L}$ since, from Lemma 2.1:

$$p^{\rho^i_{K_{n+1}} - \rho^i_{K_n}} = p^{r-1}\frac{\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))}{\#\omega_{K_{n+1}/K}(\mathbf{\Lambda}^i_K(n+1))} \ge p^{r-1}\frac{\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))}{\#\omega_{K_{n+1}/K}(\mathbf{\Lambda}^i_K(n))};$$

then, in the restriction $\Omega_{K_{n+1}/K} \twoheadrightarrow \Omega_{K_n/K}$ (whose kernel is of order p^{r-1} because of the total ramification of each place), the image of $\omega_{K_{n+1}/K}(\mathbf{\Lambda}^i_K(n))$ is $\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))$ because of the properties of Hasse's symbols, whence:

$$p^{\rho^i_{K_{n+1}} - \rho^i_{K_n}} \ge 1$$

and the result for the n-sequence $p^{\rho^i_{K_n}}$, with maximal value $p^{\rho^i_L}$. The first claim of the lemma holds for the n-sequence $\#\big(\mathscr{H}^{i+1}_{K_n}/\mathscr{H}^i_{K_n}\big)$; for $n=N$, one gets the formula $\#\big(\mathscr{H}^{i+1}_{L}/\mathscr{H}^i_{L}\big)=p^{c^i_L}\cdot p^{\rho^i_L}$.

Assuming, by induction, that the n-sequence $\#\mathscr{H}^i_{K_n}$ is increasing, the property follows for the n-sequence $\#\mathscr{H}^{i+1}_{K_n}$. □

Remarks 2.3. (i) The i-sequences $p^{c^i_L}$, $p^{\rho^i_L}$, $\#(\mathscr{H}^{i+1}_{L}/\mathscr{H}^i_{L})=p^{c^i_L}\cdot p^{\rho^i_L}$ are decreasing up to a divisor of $\#\mathscr{H}_K$, $p^{N(r-1)}$, $\#\mathscr{H}_K\cdot p^{N(r-1)}$, respectively.

(ii) The n-sequence $m(K_n)$ is an increasing sequence from 1 up to $m(L)$. Then the $\#\mathscr{H}_{K_n}$'s define an increasing n-sequence from $\#\mathscr{H}_K$ up to $\#\mathscr{H}_L$ since we have:

$$\#\mathscr{H}_{K_{n+1}}\geq \#\big(\mathscr{H}^{\mathrm{Gal}(K_{n+1}/K_n)}_{K_{n+1}}\big)=\#\mathscr{H}_{K_n}\,\frac{p^{r-1}}{\omega_{K_{n+1}/K_n}(\mathscr{E}_{K_n})}\geq \#\mathscr{H}_{K_n}.$$

The integers $e(K_n)$ and $\mathrm{rk}_p(\mathscr{H}_{K_n})$ define increasing n-sequences.

The interest of this filtration is that standard probabilities may apply at each level n to the algorithm computing $\mathscr{H}^{i+1}_{K_n}$ from $\mathscr{H}^i_{K_n}$ by means of the factors $p^{c^i_{K_n}}$ and $p^{\rho^i_{K_n}}$, giving plausible heuristics in the spirit of the works of [KoPa2022, Smith2022] leading to a considerable generalization of pioneering works as that of [Mort1982], [Gerth1986] and many others.

Indeed, let $x\in\mathbf{\Lambda}^i_K$ be such that $(x)=\mathbf{N}_{K_n/K}(\mathfrak{A})$ (when it holds for $\mathfrak{A}\in\mathscr{I}^i_{K_n}$), and let $x=\mathbf{N}_{K_n/K}(y)$, when $\omega_{K_n/K}(x)=1$ (depending on Hasse's symbols), so that $\mathfrak{A}=(y)\mathfrak{B}^{\sigma-1}$ giving $\mathfrak{B}\in\mathscr{I}^{i+1}_{K_n}$.

So we propose the following conjecture on the evolution of the class and norm factors, respectively:

Conjecture 2.4. *Let L/K be a cyclic p-extension of degree p^N, $N\geq 1$, of Galois group G; we assume that L/K is ramified at $r\geq 1$ places of K, totally ramified in L/K. Then, the orders of each of the two factors (class and norm) in the i-sequence $\#(\mathscr{H}^{i+1}_{L}/\mathscr{H}^i_{L})$, follow binomial laws as i increases, based on the following probabilities:*

- *Let $c\in\mathscr{H}_K$; the probability, for an ideal $\mathfrak{C}$ of L, that the p-class of the ideal $\mathbf{N}_{L/K}(\mathfrak{C})$ equals c, is $\frac{1}{\#\mathscr{H}_K}$.*
- *Let $\gamma\in G^{r-1}$; the probability, for $x\in K^\times$ local norm at all the non-ramified places, that $\omega_{L/K}(x)=\gamma$, is $\frac{1}{p^{N(r-1)}}$.*

2.1.3. *Program computing the filtrations $\{\mathscr{H}^i_{K_n}\}_{i\geq 1}$.* The following program may be used for the calculation of the Galois structure of the $\mathscr{H}_{K_n}$'s, $K_n\subseteq L\subset K(\mu_\ell)$, whatever the base field K given, as usual, by means of a monic polynomial of $\mathbb{Z}[x]$ (here, of prime-to-p degree to simplify the computation of a generator S of $\mathrm{Gal}(K_n/K)$).

For this, one must indicate the prime p in p, the number Nn of layers K_n considered, the polynomial PK defining K, a prime ell congruent to 1 modulo $2p^N$, $\mathsf{N}\geq\mathsf{Nn}$, and a value mKn for computing the $\mathsf{h}^{\mathsf{i}}_{\mathsf{j}}:=\mathsf{h}^{(\mathsf{S}-1)^{\mathsf{i}}}_{\mathsf{j}}$ for $1\leq\mathsf{i}\leq\mathsf{mKn}$, where the h_j's are the generators of the whole class group $\mathbf{H}_{K_n}$ given by PARI (in $\mathsf{CKn}=\mathsf{Kn.clgp}$), and where the generator S of G_n is chosen in $\mathsf{G}=\mathsf{nfgaloisconj(Kn)}$ by testing the orders. [4]

[4] The reader must be warned that for class group computations, PARI uses random primes in some analytic contexts, so that the generators given by $\mathsf{Kn.clgp}$ may vary; but the corresponding matrices of exponents are "equivalent". For this observation, run the programs several times.

So $\mathscr{H}_{K_n}^{i} = \mathrm{Ker}(\mathscr{H}_{K_n} \to \mathscr{H}_{K_n}^{(\sigma_n-1)^i})$, $1 \leq i \leq m(K_n)$ (see (2.2) (iii)). The invariant $m(K_n)$ is obtained (assuming mKn large enough) for the first i giving zero matrices in the test of principality of the $\mathsf{h_j^i}$'s. PARI works with independent generators h_i of $\mathbf{H}_{K_n}$, of orders $p^{n_i}t_i$, $p \nmid t_i$ (given in $\mathsf{CKn[2]}$); thus, for any data $[\mathsf{e_1},\dots,\mathsf{e_r}]$ given by $\mathsf{bnfisprincipal(Kn,Y)[1]}$ for an ideal Y whose class is $\mathsf{h_1^{e_1}\cdots h_r^{e_r}}$, the program gives, instead, the list $\mathsf{E} := [\overline{\mathsf{e}}_1,\dots,\overline{\mathsf{e}}_\mathsf{r}]$ defining the p-class of Y (in $\mathscr{H}_{K_n}$) as $\overline{\mathsf{h}}_1^{\overline{\mathsf{e}}_1}\cdots\overline{\mathsf{h}}_\mathsf{r}^{\overline{\mathsf{e}}_\mathsf{r}}$, where $\overline{\mathsf{e}}_\mathsf{i} = \mathsf{lift(Mod(e_i, p^{n_i}))}$; this does not modify the Galois structure of $\mathscr{H}_{K_n}$ and the outputs are more readable. So, the ideal Y is p-principal if and only if $\mathsf{E} = [0,\dots,0]$.

The corresponding outputs are written under the form $\overline{h}_i^{(\sigma-1)^j} = [\overline{e}_1,\dots,\overline{e}_r]$ instead of $\overline{h}_i^{(\sigma-1)^j} = \overline{h}_1^{\overline{e}_1}\cdots\overline{h}_r^{\overline{e}_r}$.

Below, we take as example a cyclic cubic field of conductor $\mathsf{f} = 703$, for which, using the structure of $\mathbf{Z}$-module, where $\mathbf{Z} = \mathbb{Z}_2[\exp(\frac{2i\pi}{3})]$, we have $\mathscr{H}_K \simeq \mathbf{Z}/2\mathbf{Z}$, $\mathsf{ell} = 97$, $\mathsf{mKn} = 3$; the results are given for $n = 1$ and $n = 2$ and r is the number of prime ideals of K dividing ℓ:

```
PROGRAM COMPUTING THE h_i^[(S-1)^j]:
{p=2;Nn=2;PK=x^3+x^2-234*x-729;ell=97;mKn=3;K=bnfinit(PK,1);CK0=K.clgp;
r=matsize(idealfactor(K,ell))[1];print("p=",p," Nn=",Nn," PK=",PK,
" ell=",ell," mKn=",mKn," CK0=",CK0[2]," r=",r);
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);CKn=Kn.clgp;dn=poldegree(Pn);
print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
\\Search of a generator S of Gal(Kn/K):
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
\\Computation of the image of CKn by (S-1)^j:
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
\\computation in Ehij of the modified exponents of B:
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij));print()))}
p=2  Nn=2  f=703  PK=x^3+x^2-234*x-729  ell=97  mKn=3  CK0=[6,2]  r=1
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,0,0]       h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,0,1,1]       h_4^[(S-1)^1]=[0,0,1,1]
h_1^[(S-1)^2]=[0,0,0,0]       h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]       h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0]       h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0]       h_4^[(S-1)^3]=[0,0,0,0]
CK2=[12,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,1,1]       h_2^[(S-1)^1]=[0,2,1,0]
h_3^[(S-1)^1]=[2,2,0,0]       h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,2,0,0]       h_2^[(S-1)^2]=[2,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0]       h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0]       h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0]       h_4^[(S-1)^3]=[0,0,0,0]
```

This gives $m(K_1) = 2$, $\mathscr{H}_{K_1}^{\sigma_1-1} = \langle h_1h_2, h_3h_4\rangle$, $\mathscr{H}_{K_1}^{G_1} = \langle h_1h_2^{-1}, h_3h_4^{-1}\rangle = \langle h_1h_2, h_3h_4\rangle \simeq \mathbf{Z}/2\mathbf{Z}$. Then $m(K_2) = 3$, $\mathscr{H}_{K_2}^{\sigma_2-1} = \langle h_2^2h_3h_4, h_2^2h_3, h_1^2h_2^2, h_1^2\rangle = \langle h_1^2, h_2^2, h_3, h_4\rangle \simeq (\mathbf{Z}/2\mathbf{Z})^2$, whence $\mathscr{H}_{K_2}^{G_2} = (\mathscr{H}_{K_2})^2 \simeq \mathbf{Z}/2\mathbf{Z}$. We have $\mathscr{H}_{K_2}^{(\sigma_2-1)^2} = \langle h_1^2, h_2^2\rangle \simeq (\mathscr{H}_{K_2})^2 \simeq \mathbf{Z}/2\mathbf{Z}$, $\mathscr{H}_{K_2}^{2} = \langle h_1^2, h_2^2, h_3, h_4\rangle \simeq (\mathbf{Z}/2\mathbf{Z})^2$.

These computations will prove later a partial capitulation of $\mathscr{H}_K$ in K_1 and a complete capitulation in K_2.

Remark 2.5. An astonishing fact, in a diophantine viewpoint, is that, when the class of $\mathfrak{a}$ capitulates in some K_n, the writing of the generator $\alpha \in K_n^\times$, of the extended ideal $(\mathfrak{a})_{K_n}$, on the $\mathbb{Q}$-basis $\mathsf{Kn.zk}$ of the field K_n, needs most often oversized coefficients (several thousand digits and, often, PARI proves

the principality without giving these coefficients). If the reader wishes to verify this fact, it suffices to add the instruction $\mathsf{print}(\mathsf{bnfisprincipal}(\mathsf{Kn},\mathsf{Y}))$ giving the whole data for the ideal Y considered.

2.2. **On the structures of the main invariants.** In this subsection, we restrict ourselves to the case $L \subseteq K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$, with K abelian real of prime-to-p degree, so that L/K is totally ramified at all the r prime ideals $\mathfrak{l} \mid \ell$ of K.

The exact sequence (1.1) and the Chevalley–Herbrand formula:

$$\#\mathscr{H}_L^G = \#\mathscr{H}_K \times \frac{p^{N(r-1)}}{\#\big(\mathscr{E}_K/\mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times)\big)}$$

lead to the relation:

$$\#\big(\mathbf{J}_{L/K}(\mathscr{H}_K)\,\mathscr{H}_L^{\rm ram}\big) \times \#\big(\mathscr{E}_K/\mathbf{N}_{L/K}(\mathscr{E}_L)\big) = \#\mathscr{H}_K \times p^{N\,(r-1)}. \tag{2.3}$$

Since $L_0 \subset \mathbb{Q}(\mu_\ell)$, of degree p^N, is p-principal (indeed, $\mathscr{H}_{L_0}^{\mathrm{Gal}(L_0/\mathbb{Q})} = 1$) and since $p \nmid [K:\mathbb{Q}]$, the primes $\mathfrak{L}_i \mid \mathfrak{l}_i$ of L, $i = 1, \ldots, r$, fulfill a dependent relationship of the form $\mathfrak{L}_1 \cdots \mathfrak{L}_r = (\alpha_0)_L$, $\alpha_0 \in L_0^\times$; moreover, $(\mathscr{H}_L^{\rm ram})^{p^N} = \mathbf{J}_{L/K}(\mathscr{H}_K^{\rm ram})$ and $\mathbf{N}_{L/K}(\mathscr{H}_L^{\rm ram}) = \mathscr{H}_K^{\rm ram}$, where $\mathscr{H}_K^{\rm ram} \subseteq \mathscr{H}_K$ is generated by the p-classes of the $\mathfrak{l}_i$'s; this gives some information on the structures.

One verifies easily that, in $L = \bigcup_{n=0}^N K_n$, $\#\mathscr{H}_{K_n}^{\rm ram}$, $\#\big(\mathscr{E}_K/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n})\big)$ and $\frac{p^{N(r-1)}}{\#\big(\mathscr{E}_K/\mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times)\big)}$ (see Lemma 2.2) define increasing n-sequences and that only $\#\mathbf{J}_{K_n/K}(\mathscr{H}_K)$ is decreasing. This suggests that for some ℓ's, with $N \gg 0$, there is capitulation of $\mathscr{H}_K$ and relations of the form:

$$\#\mathscr{H}_{K_n}^{\rm ram} = p^{a\,n+a_0}, \quad \#\big(\mathscr{E}_K/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n})\big) = p^{b\,n+b_0}, \quad n \geq n_0,$$

with $a + b = r - 1$ and $a_0 + b_0 = v_p(\#\mathscr{H}_K)$.

2.2.1. *Case $r = 1$.* The case $r = 1$ is particular since, whatever n, all the factors are finite in the relation (2.3) which becomes for all $n \geq 0$:

$$\#\big(\mathbf{J}_{K_n/K}(\mathscr{H}_K)\,\mathscr{H}_{K_n}^{\rm ram}\big) \times \#\big(\mathscr{E}_K/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n})\big) = \#\mathscr{H}_K,$$

giving, possibly, stationary sequences from some level. The case $r = 1$ supposes that ℓ does not split in K and we can assume, for instance, that $K/\mathbb{Q}$ is cyclic of prime-to-p degree, in which case $\mathscr{H}_{K_n}^{\rm ram} = 1$ for all $n \geq 0$, whence:

$$\#\mathbf{J}_{K_n/K}(\mathscr{H}_K) \times \#\big(\mathscr{E}_K/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n})\big) = \#\mathscr{H}_K,$$

so that complete capitulation in K_{n_0} is equivalent to the stationary relation:

$$\#\big(\mathscr{E}_K/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n})\big) = \#\mathscr{H}_K, \quad n \geq n_0. \tag{2.4}$$

Of course, only orders coincide in (2.4) since structures may be very different; let's consider the following two examples, given in [Gras2022^b, § 5.4] that we have checked again with the forthcoming Programs 6.1 and 7.1:

Example 2.6. We consider a cubic field with $p = 2$, $\ell = 257$ ($N = 7$).

```
p=2  f=31923  PK=x^3-10641*x+227008  CK0=[6,2,2,2]  ell=257  r=1
CK1=[18,6,2,2,2,2]=[2,2,2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0,0,0]
h_3^[(S-1)^1]=[0,1,0,0,1,0]   h_4^[(S-1)^1]=[0,0,0,0,1,1]
h_5^[(S-1)^1]=[0,0,0,0,0,0]   h_6^[(S-1)^1]=[0,0,0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0]   h_6^[(S-1)^2]=[0,0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,1,0,0,1,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0,1,1]
```

```
norm in K1/K of the component 5 of CK1:[0,0,0,0,0,0]
norm in K1/K of the component 6 of CK1:[0,0,0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=1
CK2=[36,12,2,2,2,2]=[4,4,2,2,2,2]
h_1^[(S-1)^1]=[2,2,0,0,0,0]   h_2^[(S-1)^1]=[2,2,1,1,1,0]
h_3^[(S-1)^1]=[0,2,1,1,1,0]   h_4^[(S-1)^1]=[0,2,0,0,1,1]
h_5^[(S-1)^1]=[0,2,1,1,0,1]   h_6^[(S-1)^1]=[2,0,1,1,0,1]
h_1^[(S-1)^2]=[0,0,0,0,0,0]   h_2^[(S-1)^2]=[0,2,0,0,0,0]
h_3^[(S-1)^2]=[0,2,0,0,0,0]   h_4^[(S-1)^2]=[2,2,0,0,0,0]
h_5^[(S-1)^2]=[2,0,0,0,0,0]   h_6^[(S-1)^2]=[2,0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2
```

The data for K_2 give:

$$\mathscr{H}_{K_2}^{G_2} = \langle h_1h_5h_6,\, h_2h_4h_6\rangle \simeq \mathbf{Z}/4\mathbf{Z},$$

since the two independent generators h_1, h_2, are of order 4.

The exact sequence (1.1) reduces to the isomorphism of $\mathbf{Z}$-modules $\mathscr{H}_{K_2}^{G_2} \simeq \mathscr{E}_K/\mathbf{N}_{K_2/K}(\mathscr{E}_{K_2})$, which are of order $\#\mathscr{H}_K = 16$, but are not isomorphic to $\mathscr{H}_K$. We have, from the relation (2.4) and since $\mathscr{E}_K \simeq \mathbf{Z}\otimes\mathbb{Z}_3$, the isomorphisms of $\mathbf{Z}$-modules:

$$\mathscr{H}_{K_2}^{G_2} \simeq \mathscr{E}_K/\mathbf{N}_{K_2/K}(\mathscr{E}_{K_2}) \simeq \mathbf{Z}/4\mathbf{Z}, \quad \mathscr{H}_K \simeq \mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}.$$

Example 2.7. We consider a quadratic field with $p = 3$, $\ell = 19$ ($N = 2$). Let $K = \mathbb{Q}(\sqrt{32009})$ for which $\mathscr{H}_K \simeq \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$. The general Program gives an incomplete capitulation in K_1, then a complete capitulation in K_2:

```
PK=x^2-32009  CK0=[3,3]  ell=19  r=1
CK1=[9,3]
h_1^[(S-1)^1]=[3,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[9,3]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

The data for K_2 give:

$$\mathscr{H}_{K_2}^{G_2} = \langle h_1\rangle \simeq \mathbb{Z}/9\mathbb{Z}.$$

Since $\mathbf{J}_{K_2/K}(\mathscr{H}_K) = 1$, $\mathscr{H}_{K_2}^{\mathrm{ram}} = 1$ and $\mathscr{E}_K \simeq \mathbb{Z}_3$, the relation (2.4) becomes, in K_2/K, the isomorphism of cyclic groups:

$$\mathscr{H}_{K_2}^{G_2} \simeq \mathscr{E}_K/\mathbf{N}_{K_2/K}(\mathscr{E}_{K_2}) \simeq \mathbb{Z}/9\mathbb{Z},$$

while $\mathscr{H}_K \simeq \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$.

2.2.2. *Case $r > 1$.* In that case, an heuristic is that there is no obstruction about $\mathscr{H}_{K_n}^{\mathrm{ram}}$, $\mathscr{E}_K/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n})$ as $\mathbb{Z}_p[G_n]$-modules of standard p-ranks, except a bounded exponent which may increase as soon as the orders of the modules increase in (2.3), regarding N. Under complete capitulation, one gets:

$$\#\mathscr{H}_{K_n}^{\mathrm{ram}} \times \#\big(\mathscr{E}_K/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n})\big) = \#\mathscr{H}_K \times p^{n\,(r-1)}, \quad n \geq n_0. \tag{2.5}$$

Example 2.8. Consider K_3 in the following example with $p = 2$, $\ell = 17$ totally split in the cubic field K of conductor 1951, and complete capitulation of $\mathscr{H}_K$ in K_1; we have $\mathscr{H}_K \simeq \mathbf{Z}/2\mathbf{Z}$ and $\mathscr{H}_{K_3} \simeq \mathbf{Z}/8\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$:

```
p=2  f=1951  PK=x^3+x^2-650*x-289  CK0=[2,2]  ell=17  r=3
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0]   h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=2, e(K1)=2
CK2=[4,4,4,4]
h_1^[(S-1)^1]=[2,0,0,0]   h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[2,0,2,0]   h_4^[(S-1)^1]=[0,2,0,2]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K1)=2
CK3=[8,8,4,4]
h_1^[(S-1)^1]=[0,0,2,0]   h_2^[(S-1)^1]=[2,2,2,0]
h_3^[(S-1)^1]=[2,0,0,0]   h_4^[(S-1)^1]=[6,2,0,2]
h_1^[(S-1)^2]=[4,0,0,0]   h_2^[(S-1)^2]=[0,4,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K3/K of the component 1 of CK3:[0,0,0,0]
norm in K3/K of the component 2 of CK3:[0,0,0,0]
norm in K3/K of the component 3 of CK3:[0,0,0,0]
norm in K3/K of the component 4 of CK3:[0,0,0,0]
Complete capitulation, m(K3)=3, e(K3)=3
```

We compute $\mathscr{H}_{K_1}^{\mathrm{ram}}$, $\mathscr{H}_{K_2}^{\mathrm{ram}}$ and $\mathscr{H}_{K_3}^{\mathrm{ram}}$, adding the following instructions after a program execution:

```
For K1:
A=component(idealfactor(Kn,ell),1)[3]
bnfisprincipal(Kn,A)[1]=[2,0,0,0]
\\Checking the principality of A^2 with components of a generator:
A2=idealpow(Kn,A,2)
bnfisprincipal(Kn,A2)=[[0,0,0,0]~,[-51,0,-2,2,0,0]~]
For K2:
B=component(idealfactor(Kn,ell),1)[3]
bnfisprincipal(Kn,B)[1]=[0,0,2,2]
\\Checking the principality of B^2 with components of a generator:
B2=idealpow(Kn,B,2)
bnfisprincipal(Kn,B2)=[[0,0,0,0]~,
[-199260,-90688,102100,13880,4054,-14216,-8292,8559,-6223,-3433,3403,7557]~]
For K3:
C=component(idealfactor(Kn,ell),1)[3]
bnfisprincipal(Kn,C)[1]=[2,2,0,2]
C2=idealpow(Kn,C,2)
bnfisprincipal(Kn,C2)[1]=[4,4,0,0]
\\Checking the principality of C^4 with components of a generator:
C4=idealpow(Kn,C,4)
bnfisprincipal(Kn,C4)=[[0,0,0,0]~,
[-57074733,49681698,-55181004,32125541,42753200,-11450554,20535876,
-4037958, -4486534,-2178833,-1875179,3883122,-1527899,1665071,4332070,
2101150,1108465,-1251165, -1504106,445954,-292536,-677913,157262,-159406]~]
```

The previous data and formula (2.5) give:

$$\begin{cases} \mathscr{H}_{K_1} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z},\ \mathscr{H}_{K_1}^{\sigma-1} = \langle h_1^2, h_2^2 \rangle, \\ \mathscr{H}_{K_1}^{G_2} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z},\ \mathscr{H}_{K_1}^{\mathrm{ram}} \simeq \mathbf{Z}/2\mathbf{Z},\ \mathscr{E}_K/\mathbf{N}_{K_1/K}(\mathscr{E}_{K_1}) \simeq \mathbf{Z}/4\mathbf{Z}, \end{cases}$$

since $\mathscr{E}_K$ is a free $\mathbf{Z}$-module of rank 1 and $\mathscr{E}_K/\mathbf{N}_{K_1/K}(\mathscr{E}_{K_1})$ is of order 16.

$$\begin{cases} \mathscr{H}_{K_2} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z},\ \mathscr{H}_{K_2}^{\sigma-1} = \mathscr{H}_{K_2}^2, \\ \mathscr{H}_{K_2}^{G_2} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z},\ \mathscr{H}_{K_2}^{\mathrm{ram}} \simeq \mathbf{Z}/2\mathbf{Z},\ \mathscr{E}_K/\mathbf{N}_{K_2/K}(\mathscr{E}_{K_2}) \simeq \mathbf{Z}/4\mathbf{Z}, \end{cases}$$

$$\begin{cases} \mathscr{H}_{K_3} \simeq \mathbf{Z}/8\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z},\ \mathscr{H}_{K_3}^{\sigma-1} = \mathscr{H}_{K_3}^2, \\ \mathscr{H}_{K_3}^{G_3} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z},\ \mathscr{H}_{K_3}^{\mathrm{ram}} \simeq \mathbf{Z}/4\mathbf{Z},\ \mathscr{E}_K/\mathbf{N}_{K_3/K}(\mathscr{E}_{K_3}) \simeq \mathbf{Z}/4\mathbf{Z}, \end{cases}$$

The Chevalley–Herbrand formulas $\#\mathscr{H}_{K_n}^{G_n} = \#\mathscr{H}_K \times \frac{4^n}{(\mathscr{E}_K : \mathscr{E}_K \cap \mathbf{N}_{K_n/K}(K_n^\times))}$ leads to $\mathscr{E}_K/\mathscr{E}_K \cap \mathbf{N}_{K_1/K}(K_1^\times) = 1$, $\mathscr{E}_K/\mathscr{E}_K \cap \mathbf{N}_{K_2/K}(K_2^\times) \simeq \mathbf{Z}/2\mathbf{Z}$, and $\mathscr{E}_K/\mathscr{E}_K \cap \mathbf{N}_{K_3/K}(K_3^\times) \simeq \mathbf{Z}/2\mathbf{Z}$.

2.2.3. *Conclusion about orders versus structures.* One may ask (for instance in the cubic case with $p = 2$ to simplify): what happens in the tower $L = K(\mu_\ell)$ if $\mathscr{H}_K$ has a large p-rank and a large exponent and if we suppose the capitulation of $\mathscr{H}_K$ (necessarily in a large layer K_{n_0}) ?

The exact sequence (1.1) looks like:

$$1 \to \mathbf{Z}/2^{R_n}\mathbf{Z} \longrightarrow \mathscr{H}_{K_n}^{G_n} \simeq \mathbf{Z}/2^{x_n}\mathbf{Z} \times \mathbf{Z}/2^{y_n}\mathbf{Z} \longrightarrow \mathbf{Z}/2^{E_n}\mathbf{Z} \to 1,$$

since $\mathscr{H}_{K_n}^{\mathrm{ram}} \simeq \mathbf{Z}/2^{R_n}\mathbf{Z}$ and $\mathscr{E}_K \cap \mathbf{N}_{K_n/K}(K_n^\times)/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n}) \simeq \mathbf{Z}/2^{E_n}\mathbf{Z}$ are $\mathbf{Z}$-monogenic (all exponents being ≥ 0). Then the Chevalley–Herbrand formula looks like $4^{x_n+y_n} = 4^H \times 4^{n\,\rho_n}$, where $\#\mathscr{H}_K = 4^H$ ($\rho_n = 0$ for all n if $r = 1$, otherwise ρ_n increases up to some limit ρ_N).

So, $\mathscr{H}_{K_n}^{G_n}$ is at most the product of two $\mathbf{Z}$-monogenic components, possibly of large orders since $x_n + y_n = R_n + E_n = H + n\rho_n$. In the case $r = 1$ where $R_n = \rho_n = 0$, $\mathscr{H}_{K_n}^{G_n}$ is isomorphic to $\mathscr{E}_K/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n})$.

The philosophy of such examples is that whatever the structure of $\mathscr{H}_K$, there is no obstruction for the relations between *orders and structures* of invariants associated to the $\mathscr{H}_{K_n}$'s by means of the exact sequence (1.1) and the Chevalley–Herbrand formula, invariants whose *algebraic structures* have canonical limitations, especially in terms of p-ranks (more precisely, $\mathscr{E}_K/\mathbf{N}_{K_n/K}(\mathscr{E}_{K_n})$ as monogenic $\mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})]$-module and $\mathscr{H}_{K_n}^{\mathrm{ram}}$ of p-rank bounded by the number of ramified places).

In the case of a real abelian base field K, this is typical of the Main Conjecture philosophy due to the analytic framework giving only orders and not precise structures.

2.3. **Decomposition of the algebraic norm $\nu_{L/K}$.** Put $x := \sigma - 1$; then:

$$\nu_{L/K} = \sum_{i=0}^{p^N-1} \sigma^i = \sum_{i=0}^{p^N-1} (x+1)^i = \frac{(x+1)^{p^N} - 1}{x} = \sum_{i=1}^{p^N} \tbinom{p^N}{i} x^{i-1}.$$

We have the following elementary property which is perhaps known in Iwasawa's theory, but we have not found suitable references; see however [Jaul1986, IV.2 (b)], [Wash1997, § 13.3] or [BaCa2016, Cald2020] for possible informations:

Theorem 2.9. *The prime number $p \geq 2$ and the integer $N \geq 1$ being given, the algebraic norm $\nu_{L/K} = \sum_{i=0}^{p^N-1} \sigma^i$ is, for all $k \in [1, p^N - 1]$, of the form:*

$$\nu_{L/K} = (\sigma - 1)^k \cdot A_k(\sigma - 1, p) + p^{f(k)} \cdot B_k(\sigma - 1, p),$$

$A_k,\ B_k \in \mathbb{Z}[\sigma-1,p]$, *where* $f(k)=N-s$ *if* $k\in[p^s,p^{s+1}-1]$, $s\in[0,N-1]$.

Proof. From:

$$\nu_{L/K}=\tbinom{p^N}{1}+x\tbinom{p^N}{2}+\cdots+x^{k-1}\tbinom{p^N}{k} + x^k\Big[\tbinom{p^N}{k+1}+x\tbinom{p^N}{k+2}+\cdots+x^{p^N-1-k}\tbinom{p^N}{p^N}\Big],$$

we deduce that $A_k(x,p)=\binom{p^N}{k+1}+x\binom{p^N}{k+2}+\cdots+x^{p^N-1-k}\binom{p^N}{p^N}$.

The computation of $B_k(x,p)$ depends on the p-adic valuations of the $\binom{p^N}{j}$, $j\in[1,k]$. To find the maximal factor $p^{f(k)}$ dividing all the coefficients of the polynomial $\binom{p^N}{1}+x\binom{p^N}{2}+\cdots+x^{k-1}\binom{p^N}{k}$, in other words, to find the p-part of $\gcd\Big(\binom{p^N}{1},\binom{p^N}{2},\dots,\binom{p^N}{k}\Big)$, we consider $s\in[0,N-1]$. Let v the p-adic valuation map.

Lemma 2.10. *One has* $v\big(\binom{p^N}{p^s}\big)=N-s$, *for all* $s\in[0,N-1]$.

Proof. We have $\binom{p^N}{p^s}=\frac{p^N!}{p^s!\cdot(p^N-p^s)!}$; then, using the well-known formula:

$$v(m!)=\frac{m-S(m)}{p-1},\ \ m\geq 1,$$

where $S(m)$ is the sum of the digits in the writing of m in base p, we get:

$$v(p^N!)=\frac{p^N-1}{p-1},\quad v(p^s!)=\frac{p^s-1}{p-1},\ \ v((p^N-p^s)!)=\frac{p^N-p^s-(p-1)(N-s)}{p-1},$$

since p^N-p^s may be written $p^s(p^{N-s}-1)$ with:

$$p^{N-s}-1=1(p-1)+p(p-1)+p^2(p-1)+\cdots+p^{N-s-2}(p-1)+p^{N-s-1}(p-1),$$

giving $N-s$ times the digit $p-1$. Whence:

$$v\big(\tbinom{p^N}{p^s}\big)=\frac{1}{p-1}\big(p^N-1-(p^s-1)-(p^N-p^s-(p-1)(N-s))\big)=N-s,$$

for all $s\in[0,N-1]$. □

Lemma 2.11. *For* $k\in[p^s+1,p^{s+1}-1]$, $s\in[0,N-1]$, *we have* $v\big(\binom{p^N}{k}\big)\geq N-s$.

Proof. Consider $\binom{p^N}{k}\binom{p^N}{p^s}^{-1}$, $k\in[p^s+1,p^{s+1}-1]$ to check that its valuation is non-negative (the interval is empty for $p=2$, $s=0$, so, for $p=2$ we assume implicitly $s>0$):

$$\begin{aligned}\frac{\binom{p^N}{k}}{\binom{p^N}{p^s}}&=\frac{p^N!}{k!\,(p^N-k)!}\times\frac{p^s!\,(p^N-p^s)!}{p^N!}\\ &=\frac{p^s!}{k!}\times\frac{(p^N-p^s)!}{(p^N-k)!}\\ &=\frac{1}{(p^s+1)(p^s+2)\cdots(p^s+(k-p^s))}\times\frac{(p^N-p^s)!}{(p^N-k)!}\\ &=\frac{(p^N-k+1)(p^N-k+2)\cdots(p^N-k+(k-p^s))}{(p^s+1)(p^s+2)\cdots(p^s+(k-p^s))}.\end{aligned}$$

Put $k=p^s+h$, $h\in[1,p^s(p-1)-1]$; then we can write:

$$\begin{aligned}\frac{\binom{p^N}{k}}{\binom{p^N}{p^s}}&=\frac{[p^N-(p^s+h)+1][p^N-(p^s+h)+2]\cdots[p^N-(p^s+h)+h]}{[p^s+1][p^s+2]\cdots[p^s+h]}\\&=\frac{[p^N-(p^s+h)+1][p^N-(p^s+h)+2]\cdots[p^N-(p^s+h)+h]}{[p^s+h][p^s+h-1]\cdots[p^s+h-(h-1)]}\\&=\frac{[p^N-(p^s+h)+1]}{[(p^s+h)-1]}\frac{[p^N-(p^s+h)+2]}{[(p^s+h)-2]}\cdots\frac{[p^N-(p^s+h)+(h-1)]}{[(p^s+h)-(h-1)]}\\&\quad\times\frac{p^N-p^s}{p^s+h}.\end{aligned}$$

We remark that each factor of the form $\frac{p^N-[(p^s+h)-j]}{[(p^s+h)-j]}$ is a p-adic unit for $j\in[1,h-1]$; indeed, one sees that $(p^s+h)-j\leq p^{s+1}-2$ with $s+1\leq N$, whence $v_p((p^s+h)-j)\leq N-1$.

Now, consider the remaining factor $\frac{p^N-p^s}{p^s+h}=\frac{p^s(p^{N-s}-1)}{p^s+h}=\frac{p^s}{p^s+h}$, up to a p-adic unit since $s\in[0,N-1]$. As $h\leq p^s(p-1)-1$, one can put $h=\lambda p^u$, $p\nmid\lambda$, $u\leq s$; the case $u<s$ is obvious and gives a positive valuation; if $u=s$, the relation $h\leq p^s(p-1)-1$ implies $\lambda\leq p-2$, thus $p^s+h=p^s(1+\lambda)$ with $1+\lambda\leq p-1$ and $\frac{p^s}{p^s+h}$ is, in this case, a p-adic unit, whence the lemma. □

This leads to the expression of $f(k)$ on $\bigcup_{s=0}^{N-1}[p^s,p^{s+1}-1]=[1,p^N-1]$, to the proof of the theorem. □

The following corollary, proving Theorem 1.1 (i) and some generalizations, is of easy use in practice; we assume, to simplify, that L/K is totally ramified:

Corollary 2.12. *(i) Let $m(L)$ be the minimal integer such that $(\sigma-1)^{m(L)}$ annihilates $\mathscr{H}_L$ and let $p^{e(L)}$ be the exponent of $\mathscr{H}_L$. A sufficient condition for the complete capitulation of $\mathscr{H}_K$ in L, is that $e(L)\in[1,N-s(L)]$ if $m(L)\in[p^{s(L)},p^{s(L)+1}-1]$ for $s(L)\in[0,N-1]$.*

A sufficient condition for a partial capitulation of $\mathscr{H}_K$ in L, is that there exists $h\in\mathscr{H}_L$ such that $\mathbf{N}_{L/K}(h)\neq1$, h being of order p^e and annihilated by $(\sigma-1)^m$, for e,m such that $e\in[1,N-s]$ if $m\in[p^s,p^{s+1}-1]$ for $s\in[0,N-1]$.

(ii) For $t\geq1$, put $\overline{\mathscr{H}}_L:=\mathscr{H}_L/\mathscr{H}_L^{p^t}$ and let $\overline{m}(L)$, $\overline{e}(L)$, be the corresponding invariants for which we have $\overline{m}(L)\leq m(L)$ and $\overline{e}(L)\leq e(L)$; the capitulation of $\overline{\mathscr{H}}_K$ in L holds as soon as $\overline{e}(L)\in[1,N-\overline{s}(L)]$ if $\overline{m}(L)\in[p^{\overline{s}(L)},p^{\overline{s}(L)+1}-1]$ for $\overline{s}(L)\in[0,N-1]$.

(iii) Stability of the $\overline{\mathscr{H}}_{K_n}$'s in the tower $L=\bigcup_{n=0}^{N}K_n$ corresponds to $\overline{m}(K_n)=1$, $\overline{s}(K_n)=0$, $\overline{e}(K_n)=\overline{e}(K)$.

Proof. If $h\in\mathscr{H}_L$, $\nu_{L/K}(h)=\mathbf{J}_{L/K}(\mathbf{N}_{L/K}(h))=\left(h^{(\sigma-1)^k}\right)^A\times\left(h^{p^{f(k)}}\right)^B$, for all $k\in[1,p^N-1]$. Thus, $\mathbf{J}_{L/K}$ is non-injective as soon as h fulfills the conditions stated in the second part of (i) with $k=m\in[p^s,p^{s+1}-1]$, $s\in[0,N-1]$, and $f(k)=N-s\geq e$. For the triviality of $\nu_{L/K}(\mathscr{H}_L)$, it suffices that $m=m(L)$, $s=s(L)$ and $e=e(L)$ be solution. One obtains Theorem 1.1 (i).

The case (ii) of quotients is immediate; their capitulation is "easier" and means that any ideal $\mathfrak{a}$ of K becomes of the form $(\mathfrak{a})_L=(\alpha)\cdot\mathfrak{A}^{p^t}$, where $\mathfrak{A}$ is an ideal of L and $\alpha\in L^\times$. The case $t=1$ of the stability property (iii) gives the stability of the p-ranks. □

In other words, if the length $m(L)$ of the filtration is not too large as well as the exponent $p^{e(L)}$ of $\mathscr{H}_L$, then we obtain $\nu_{L/K}(h)=1$ for all $h\in\mathscr{H}_L$ (or only for some ones), whence complete (or partial) capitulation of $\mathscr{H}_K$ in L.

Another way to interpret this result is to say that if N is large enough and if the Galois complexity of the p-class groups $\mathscr{H}_{K_n}$, for the layers K_n, does not increase too much, then $\mathscr{H}_K$ capitulates in L.

For this, we introduce the following definition:

Definition 2.13. Let L/K be any cyclic p-extension totally ramified of degree p^N, $N \geq 1$, and let σ be a generator of $\mathrm{Gal}(L/K)$. Let $\mathscr{H}_K$ and $\mathscr{H}_L$ be the p-class groups of K and L, respectively. We denote by $p^{e(L)}$ the exponent of $\mathscr{H}_L$ and by $m(L)$ the length of the filtration $\{\mathscr{H}_L^i\}_{i\geq 0}$ (i.e., the least integer $m(L)$ such that $(\sigma-1)^{m(L)}$ annihilates $\mathscr{H}_L$, cf. § 2.1.1).

Then we say that L/K is of *smooth complexity* if the conditions $e(L) \leq N - s(L)$ if $m(L) \in [p^{s(L)}, p^{s(L)+1}-1]$ for $s(L) \in [0, N-1]$, are fulfilled.

2.3.1. *The Furtwängler property.* This property is the strong equality:

$$\mathrm{Ker}_{\mathscr{H}_L}(\mathbf{N}_{L/K}) = \mathscr{H}_L^{\sigma-1};$$

it is obviously equivalent (in our context $L \subset K(\mu_\ell)$) to $\#\mathscr{H}_L^G = \#\mathscr{H}_K$, thus to the triviality of the norm factor in the Chevalley–Herbrand formula. This is for instance the case when ℓ is non-split in K. We have discovered some applications in Bembom's thesis, about the Galois structure of $\mathscr{H}_L$ via its filtration and the problem of capitulation (see for instance [Bem2012, § 2.6, Theorem 2.6.3; § 2.8, Theorem 2.8.9]).

Under the Furtwängler property, Nakayama's Lemma gives immediately:

Proposition 2.14. *Let L/K be a totally ramified cyclic p-extension such that $\#\mathscr{H}_L^G = \#\mathscr{H}_K$. Set $\mathscr{H}_K = \bigoplus_{i=1}^{r} \langle h_i \rangle$ and let $\mathscr{H}'_L = \bigoplus_{i=1}^{r} \langle h'_i \rangle$ where the $h'_i \in \mathscr{H}_L$ are such that $h_i = \mathbf{N}_{L/K}(h'_i)$, for all i. Then $\mathscr{H}_L$ is generated by $\mathscr{H}'_L$ as $\mathbb{Z}_p[\sigma-1]$-module. So, any $h' \in \mathscr{H}_L$ may be written $h' = \prod_{i=1}^{r} h_i'^{\omega_i}$, where $\omega_i = \sum_{j=0}^{m(L)-1} a_{i,j}(\sigma-1)^j$, $a_{i,j} \in \mathbb{Z}_p$.*

2.3.2. *Program giving the decompositions of $\nu_{L/K}$.* The following program put $\nu_{L/K}$ in the form $P(x,p) = x^k A(x,p) + p^{f(k)} B(x,p)$, $1 \leq k \leq p^N - 1$, where $x = \sigma - 1$ and p are considered as indeterminate variables. This is necessary to have universal expressions for any $\mathscr{H}_L$ as $\mathbb{Z}_p[G]$-module; in other words, we do not reduce modulo p the coefficients of A and B. One must note that, except the cases $k = 1$, $B = 1$ and $k = p^N - 1$, $A = 1$, the polynomials A and B are not invertible in the group algebra, which allows improvements of the standard reasoning of annihilation with $(k,\ p^{f(k)})$.

One must precise the numerical prime number p in Prime and N in N:

```
PROGRAM OF DECOMPOSITION OF THE ALGEBRAIC NORM
{Prime=2;N=3;P=0;for(i=1,Prime^N,C=binomial(Prime^N,i);
v=valuation(C,Prime);c=C/Prime^v;P=P+c*p^v*x^(i-1));print("P=",P);
for(k=1,Prime^N-1,B=lift(Mod(P,x^k));w=valuation(B,p);print();
print("P=x^",k,".A+p^",w,".B");print("A=",(P-B)/x^k);print("B=",B/p^w))}
```

2.3.3. *Case $p = 2, N = 1$: $P = x + p$.*

```
P=x+p
A=1    B=1
```

2.3.4. *Case $p = 3, N = 1$: $P = x^2 + p * x + p$.*

```
P=x^1.A+p^1.B              P=x^2.A+p^1.B
A=x+p     B=1              A=1     B=x+1
```

2.3.5. *Case $p = 2, N = 2$: $P = x^3 + p^2 * x^2 + 3 * p * x + p^2$.*

```
P=x^1.A+p^2.B            P=x^2.A+p^1.B          P=x^3.A+p^1.B
A=x^2+p^2*x+3*p  B=1     A=x+p^2  B=3*x+p       A=1  B=p*x^2+3*x+p
```

For the next examples, we only write P.

2.3.6. *Case $p = 3, N = 2$:*

```
P=x^8+p^2*x^7+4*p^2*x^6+28*p*x^5+14*p^2*x^4+14*p^2*x^3+28*p*x^2+4*p^2*x+p^2
```

2.3.7. *Case $p = 2, N = 3$:*

```
P=x^7+p^3*x^6+7*p^2*x^5+7*p^3*x^4+35*p*x^3+7*p^3*x^2+7*p^2*x+p^3
```

2.3.8. *Case $p = 2, N = 4$:*

```
P=x^15+p^4*x^14+15*p^3*x^13+35*p^4*x^12+455*p^2*x^11+273*p^4*x^10
  +1001*p^3*x^9+715*p^4*x^8+6435*p*x^7+715*p^4*x^6+1001*p^3*x^5
  +273*p^4*x^4+455*p^2*x^3+35*p^4*x^2+15*p^3*x+p^4
```

Thus, as soon as $\nu_{L/K} = (\sigma-1)^k \times A_k(\sigma-1,p) + p^{f(k)} \times B_k(\sigma-1,p)$, with $k \geq m(L)$ and $f(k) \geq e(L)$, then $\mathscr{H}_K$ capitulates in L. But the reciprocal does not hold and capitulation may occur whatever $m(L)$ and $e(L)$. Indeed, if $m(L) \in [p^{s(L)}, p^{s(L)+1}-1]$ for $s(L) \in [0, N-1]$, with $e(L) > N - s(L)$, reducing P modulo the ideal $\left(x^{m(L)}, p^{e(L)}\right)$, the capitulation of $\mathscr{H}_K$ is equivalent to the annihilation of $\mathscr{H}_L$ by an explicit polynomial of the form:

$$\sum_{j=1}^{j+h} a_j\, p^{e_j} x^{m_j}, \quad p \nmid a_j,\ e_j \in [1, e(L)-1],\ m_j \in [0, m(L)-1],$$

which may hold by accident depending on the numerical data of the filtration; we give such a case in Example 2.15 (ii). We will also give examples where the above property of $\nu_{L/K}$ applies, apart from the obvious case of stability.

2.4. **Illustrations using the decomposition of $\nu_{L/K}$.** In what follows, the cyclic p-extensions L/K are always totally ramified.

Example 2.15. Let's begin with an example where the structure of $\mathscr{H}_{K_n}$ grows sufficiently with n, giving no capitulation up to $n = 3$; next, another choice of ℓ leads to capitulation, but where Theorem 1.1 (i) does not apply.

(i) We consider the cyclic cubic field K of conductor $f = 703$, $p = 2$ and $\ell = 17$. Then $\mathscr{H}_K \simeq \mathbf{Z}/2\mathbf{Z}$ and we get, from the Program 6.1 (with several hours for the level $n = 3$):

```
p=2  Nn=3  f=703  PK=x^3+x^2-234*x-729  mKn=3  CK0=[6,2]  ell=17  r=1
CK1=[12,4]=[4,4]
h_1^[(S-1)^1]=[0,2]      h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]      h_2^[(S-1)^2]=[0,0]
h_1^[(S-1)^3]=[0,0]      h_2^[(S-1)^3]=[0,0]
norm in K1/K of the component 1 of CK1:[2,2]
norm in K1/K of the component 2 of CK1:[2,0]
No capitulation, m(K1)=2, e(K1)=2
CK2=[24,8]=[8,8]
h_1^[(S-1)^1]=[0,2]      h_2^[(S-1)^1]=[6,6]
h_1^[(S-1)^2]=[4,4]      h_2^[(S-1)^2]=[4,0]
h_1^[(S-1)^3]=[0,0]      h_2^[(S-1)^3]=[0,0]
norm in K2/K of the component 1 of CK2:[4,4]
norm in K2/K of the component 2 of CK2:[4,0]
No capitulation, m(K2)=3, e(K2)=3
CK3=[48,16]=[16,16]
h_1^[(S-1)^1]=[8,10]     h_2^[(S-1)^1]=[10,14]
h_1^[(S-1)^2]=[4,12]     h_2^[(S-1)^2]=[12,8]
norm in K3/K of the component 1 of CK3:[8,8]
norm in K3/K of the component 2 of CK3:[8,0]
No capitulation, m(K3)=4, e(K3)=4
```

For $n = 1$, we have $m(K_1) = 2$, $s(K_1) = 1$, $n - s(K_1) = 0$ and $e(K_1) = 2 > 0$.

For $n = 2$, we have $m(K_2) = 3$, $s(K_2) = 1$, $n - s(K_2) = 1$ and $e(K_2) = 3 > 1$.

For $n = 3$, we have $m(K_3) = 4$, $s(K_3) = 2$, $n - s(K_3) = 1$ and $e(K_3) = 4 > 1$.

In this case the complexity increases due to successive exponents $1, 2, 3, 4$.

Nevertheless $\overline{\mathscr{H}}_K = \mathscr{H}_K/\mathscr{H}_K^2$ capitulates in K_1 because $\overline{m}(K_1) = 1$ since $\mathsf{h}_1^{[(\mathsf{S}-1)^1]} = [0,2]$ and $\mathsf{h}_2^{[(\mathsf{S}-1)^1]} = [2,2]$ ($\overline{s}(K_1) = 0$), then $\overline{e}(K_1) = 1 \leq n - \overline{s}(K_1)$.

(ii) Changing $\ell = 17$ into $\ell = 97$ gives complete capitulation in K_2 because of a smooth complexity, but higher than a stability:

```
p=2  Nn=2  f=703  PK=x^3+x^2-234*x-729  CK0=[6,2]  ell=97  r=1
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,0,0]      h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,0,1,1]      h_4^[(S-1)^1]=[0,0,1,1]
h_1^[(S-1)^2]=[0,0,0,0]      h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]      h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,0,0]
norm in K1/K of the component 2 of CK1:[1,1,0,0]
norm in K1/K of the component 3 of CK1:[0,0,1,1]
norm in K1/K of the component 4 of CK1:[0,0,1,1]
No capitulation, m(K1)=2, e(K1)=1
CK2=[12,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,1,1]      h_2^[(S-1)^1]=[0,2,1,0]
h_3^[(S-1)^1]=[2,2,0,0]      h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,2,0,0]      h_2^[(S-1)^2]=[2,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0]      h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0]      h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0]      h_4^[(S-1)^3]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2
```

For $n = 2$, $\mathsf{P} = \mathsf{x}^3 + 2^2 * \mathsf{x}^2 + 3 * 2 * \mathsf{x} + 2^2$, $m(K_2) = 3$, $s(K_2) = 1$ and $e(K_2) = 2 > n - s(K_2) = 1$; so the property deduced from the use of $\nu_{K_2/K}$ does not hold. Then, modulo the ideal $(x^3, 2^2)$, it follows that $3 * 2 * \mathsf{x}$ must annihilate $\mathscr{H}_{K_2}$, which is confirmed by the data since $h_i^{2(\sigma_2-1)} = 1$ for all i.

So, Theorem 1.1 (i) gives a sufficient condition of capitulation, not necessary, but some information remains when the condition is not fulfilled.

Example 2.16. We consider the cyclic cubic field K of conductor 1777, $p = 2$ and $\ell = 17$. Then $\mathscr{H}_K \simeq \mathbf{Z}/4\mathbf{Z}$ is of exponent 4.

```
p=2  Nn=3  f=1777  PK=x^3+x^2-592*x+724  CK0=[4,4]  ell=17  r=3
CK1=[8,8]
h_1^[(S-1)^1]=[0,0]     h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation, m(K1)=1, e(K1)=3
CK2=[8,8]
h_1^[(S-1)^1]=[0,0]     h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Incomplete capitulation, m(K2)=1, e(K2)=3
CK3=[8,8]
h_1^[(S-1)^1]=[0,0]     h_2^[(S-1)^1]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Complete capitulation, m(K3)=1, e(K3)=3
```

In this case, the stability from K_1 implies necessarily the capitulation in K_3 (so the third computation is for checking). Moreover, for all $n \geq 1$, $\mathscr{H}_{K_n}$ is annihilated by $\sigma - 1$ and $\mathscr{H}_{K_n} = \mathscr{H}_{K_n}^{G_n}$ as expected from Theorem 1.1 (ii) and given by Program 2.1.3. Whence $m(K_n) = 1$, $s(K_n) = 0$, $e(K_n) = 3$ giving $e(K_n) \leq n - s(K_n)$ only from $n = 3$. Note that $\mathscr{H}_K/\mathscr{H}_K^2$ capitulates in K_1 and that, considering the $\overline{\mathscr{H}}_{K_n} := \mathscr{H}_{K_n}/\mathscr{H}_{K_n}^4$, then $\overline{\mathscr{H}}_K = \mathscr{H}_K/\mathscr{H}_K^4$ capitulates only in K_2.

Example 2.17. (i) We consider the quadratic field $\mathbb{Q}(\sqrt{142})$ with $p = 3$ and various primes $\ell \equiv 1 \pmod 3$, $\ell \not\equiv 1 \pmod 9$, so that $N = 1$, $L = K_1$, and the sufficient conditions of Theorem 1.1 (i) are $e(K_1) = 1 \leq 1 - s(K_1)$, whence $s(K_1) = 0$ and $m(K_1) \in [1, 2]$:

```
p=3  PK=x^2-142  N=1  CK0=[3]  ell=13  r=2
CK1=[3,3]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=1, e(K1)=1
p=3  PK=x^2-142  N=1  CK0=[3]  ell=1123  r=2
CK1=[21,3]=[3,3]
h_1^[(S-1)^1]=[1,2]   h_2^[(S-1)^1]=[1,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=2, e(K1)=1
p=3  PK=x^2-142  N=1  CK0=[3]  ell=208057  r=2
CK1=[3,3,3,3]
h_1^[(S-1)^1]=[2,2,0,0]   h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,1,1,1]   h_4^[(S-1)^1]=[2,1,2,2]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=2, e(K1)=1
```

(ii) For $p = 5$ and $N = 1$ the conditions become $e(K_1) = 1 \leq 1 - s(K_1)$, whence $s(K_1) = 0$, with $m(K_1) \in [1, 4]$, which offers more possibilities:

```
p=5  PK=x^2-401  N=1  CK0=[5]  ell=1231  r=2
CK1=[5,5]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=1, e(K1)=1
p=5  PK=x^2-401  N=1  CK0=[5]  ell=1741  r=1
CK1=[5,5]
h_1^[(S-1)^1]=[3,3]    h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=2, e(K1)=1
p=5  PK=x^2-401  CK0=[5]  ell=4871  r=1
CK1=[10,10,10,2]=[5,5,5]
h_1^[(S-1)^1]=[4,0,4,0]  h_2^[(S-1)^1]=[1,4,0,0]  h_3^[(S-1)^1]=[3,4,2,0]
h_1^[(S-1)^2]=[3,1,4,0]  h_2^[(S-1)^2]=[3,1,4,0]  h_3^[(S-1)^2]=[2,4,1,0]
h_1^[(S-1)^3]=[0,0,0,0]  h_2^[(S-1)^3]=[0,0,0,0]  h_3^[(S-1)^3]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=3, e(K1)=1
```

Example 2.18. We consider the cubic field of conductor $f = 20887$ with $p = 2$ and $\ell = 17$ totally split:

```
p=2  Nn=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]  ell=17  r=3
CK1=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
```

```
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=1, e(K1)=3
```

We note that $\mathbf{N}_{K_1/K}(h_i) \neq 1$ for $i = 3, 4$, otherwise $\mathbf{N}_{K_1/K}(\mathscr{H}_{K_1}) = \mathscr{H}_K$ would be of 2-rank 2 instead of 4 (absurd). Since $m(K_1) = 1$ (all classes are invariant), Theorem 1.1 (i) applies non-trivially for the classes h_3, h_4 of order 2 ($m = 1$, $s = 0$, $e \in [1, 1]$, which is indeed the case).

Let's give the complete data checking the capitulation of the two classes of K of order 2; the instruction $\mathsf{CK0} = \mathsf{K.clgp}$ gives:

```
[64,[4,4,2,2],[[2897,2889,2081;0,1,0; 0,0,1],[2897,825,2889;0,1,0;0,0,1],
[17,16,13;0,1,0;0,0,1],[53,36,44;0,1,0; 0,0,1]]]
```

it describes $\mathscr{H}_K$ with 4 representative ideals of generating classes; that of order 2 are $\mathfrak{a}_3 = [17, 16, 13; 0, 1, 0; 0, 0, 1]$, $\mathfrak{a}_4 = [53, 36, 44; 0, 1, 0; 0, 0, 1]$; the following 6 large coefficients (on the integral basis computed by PARI) give integers α_i of L with the relations $(\mathfrak{a}_i)_L = (\alpha_i)$:

```
[[0,0,0,0]~,[4482450896,-1173749328,81969609,69123722,7646555,39729395]~]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
[[0,0,0,0]~,[-4877380814,1968946273,-1411818,102996743,38571732,40207952]~]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
```

At the level $n = 2$, the result is similar, but shows that the classes of order 4 of $\mathscr{H}_K$ never capitulate:

```
CK2=[16,16,2,2]
h_1^[(S-1)^1]=[8,0,0,0] h_2^[(S-1)^1]=[0,8,0,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=2, e(K2)=4
```

3. Arithmetic invariants that do not capitulate

The non capitulation of p-class groups $\mathscr{H}_K$ in cyclic p-extensions L/K implies necessarily, as we have seen, that the structure of $\mathscr{H}_L$ does not allow the previous use of the algebraic norm (Theorem 1.1 (i)) and the complexity is not smooth according to the Definition 2.13; that is to say, either $m(K_n) \geq p^n$ or $m(K_n) \in [p^s, p^{s+1} - 1]$ for $s \in [0, n-1]$ but in that case, $e(K_n) > n - s$. This may be checked by means a more general framework.

3.1. **General case of injective transfers $\mathbf{J}_{L/K}$.** Let p be any prime number and let $\mathscr{K}$ a family of number fields stable by taking subfields, extensions, compositum (e.g., family of totally real number fields).

Definition 3.1. Assume given a family $\mathscr{X}_k$ of finite invariants of p-power order, indexed by the set of number fields $\mathscr{K}$, fulfilling the following conditions for $k, k' \in \mathscr{K}$:

(i) For any Galois extension k'/k, of Galois group G, $\mathscr{X}_{k'}$ is a $\mathbb{Z}_p[G]$-module.

(ii) There exist surjective arithmetic norms $\mathbf{N}_{k'/k} : \mathscr{X}_{k'} \to \mathscr{X}_k$ and transfer maps $\mathbf{J}_{k'/k} : \mathscr{X}_k \to \mathscr{X}_{k'}$, such that $\mathbf{J}_{k'/k} \circ \mathbf{N}_{k'/k} = \nu_{k'/k}$ for all k, k', $k \subseteq k'$.

(iii) If G is a cyclic p-group, we define the associated filtration $\{\mathscr{X}^i_{k'}\}_{i \geq 0}$ defined by $\mathscr{X}^{i+1}_{k'}/\mathscr{X}^i_{k'} := (\mathscr{X}_{k'}/\mathscr{X}^i_{k'})^G$, for all $i \geq 0$.

Thus, for a cyclic p-extension L/K, $L, K \in \mathscr{K}$, let $m(L)$ be the length of the filtration; the condition $e(L) \in [1, N-s(L)]$ if $m(L) \in [p^{s(L)}, p^{s(L)+1}-1]$ for $s(L) \in [0, N-1]$, of Theorem 1.1 (i), applies in the same way, independently of the fact of being able to calculate the orders of the $\mathscr{X}_L^i$'s by means of a suitable algorithm moving from $\mathscr{X}_L^i$ to $\mathscr{X}_L^{i+1}$.

Theorem 3.2. *Let L/K, $K, L \in \mathscr{K}$, be a cyclic p-extension of degree p^N, $N \geq 1$, and let K_n be the subfield of L of degree p^n over K, $n \in [0, N]$. We assume that, for all $n \in [0, N-1]$, the arithmetic norms $\mathbf{N}_{K_{n+1}/K_n}$ are surjective and that the transfer maps $\mathbf{J}_{K_{n+1}/K_n}$ are injective.*
Then $\#\mathscr{X}_{n+h} \geq \#\mathscr{X}_n \cdot \#\mathscr{X}_n[p^h]$, for all $n \in [0, N]$ and all $h \in [0, N-n]$, where $\mathscr{X}_n[p^h] := \{x \in \mathscr{X}_n,\ \ x^{p^h} = 1\}$.
In particular, if for $n + h \leq N]$, p^h annihilates $\mathscr{X}_n$, then $\#\mathscr{X}_{n+h} \geq (\#\mathscr{X}_n)^2$.

Proof. Put $G_n^{n+h} := \mathrm{Gal}(k_{n+h}/k_n)$ and in the same way for $\mathbf{N}_n^{n+h}$, $\mathbf{J}_n^{n+h}$. From the exact sequence:

$$1 \to \mathbf{J}_n^{n+h}\mathscr{X}_n \to \mathscr{X}_{n+h} \to \mathscr{X}_{n+h}/\mathbf{J}_n^{n+h}\mathscr{X}_n \to 1,$$

we get:

$$1 \to \mathscr{X}_{n+h}^{G_n^{n+h}}/\mathbf{J}_n^{n+h}\mathscr{X}_n \to (\mathscr{X}_{n+h}/\mathbf{J}_n^{n+h}\mathscr{X}_n)^{G_n^{n+h}} \to \mathrm{H}^1(G_n^{n+h}, \mathbf{J}_n^{n+h}\mathscr{X}_n) \to \mathrm{H}^1(G_n^{n+h}, \mathscr{X}_{n+h}),$$

where $\mathrm{H}^1(G_n^{n+h}, \mathbf{J}_n^{n+h}\mathscr{X}_n) = (\mathbf{J}_n^{n+h}\mathscr{X}_n)[p^h] \simeq \mathscr{X}_n[p^h]$ (injectivity of $\mathbf{J}_n^{n+h}$),

$$\#\mathrm{H}^1(G_n^{n+h}, \mathscr{X}_{n+h}) = \#\mathrm{H}^2(G_n^{n+h}, \mathscr{X}_{n+h}) = \#(\mathscr{X}_{n+h}^{G_n^{n+h}}/\mathbf{J}_n^{n+h}\mathscr{X}_n),$$

since $\nu_n^{n+h}\mathscr{X}_{n+h} = \mathbf{J}_n^{n+h} \circ \mathbf{N}_n^{n+h}\mathscr{X}_{n+h} = \mathbf{J}_n^{n+h}\mathscr{X}_n$ (surjectivity of $\mathbf{N}_n^{n+h}$), giving an exact sequence of the form:

$$1 \to A \to (\mathscr{X}_{n+h}/\mathbf{J}_n^{n+h}\mathscr{X}_n)^{G_n^{n+h}} \to \mathscr{X}_n[p^h] \to A', \text{ with } \#A' = \#A.$$

We then obtain the inequality $\#\mathscr{X}_{n+h} \geq \#\mathscr{X}_n \cdot \#\mathscr{X}_n[p^h]$. □

Corollary 3.3. *Let p^{e_n} be the exponent of $\mathscr{X}_{K_n}$ and let r_n be its p-rank, $n \in [0, N]$. The n-sequence $\#\mathscr{X}_{K_n}$ stabilizes from some $n_0 \in [0, N-1]$ (i.e., $\#\mathscr{X}_{K_{n_0+1}} = \#\mathscr{X}_{K_{n_0}}$) if and only if $\mathscr{X}_{K_n} = 1$ for all $n \in [0, N]$.*
In an Iwasawa's theory context with $\mu = 0$ and $\lambda > 0$, then r_n is a constant r for all $n \gg 0$ and $e_n \to \infty$ with n; in particular, if $\lambda = \mu = 0$, then $\mathscr{X}_{K_n} = 1$ for all $n \geq 0$.

Proof. The first claim is obvious. The stability from n_0 means $\mathscr{X}_{K_{n_0}}[p] = 1$, whence $\mathscr{X}_{K_{n_0}} = 1$, then $\mathscr{X}_{K_n} = 1$ for all $n \geq 0$ (injectivity of $\mathbf{J}_{K_n/K}$).

If $\mu = 0$ in the formula $\#\mathscr{X}_n = p^{\lambda n + \mu p^n + \nu}$ for all n large enough, the relation $\#\mathscr{X}_{n+1} \geq \#\mathscr{X}_n \cdot \mathscr{X}_n[p]$ implies $r_n \leq \lambda$; since the p-rank is increasing, $r_n = r$ for all $n \gg 0$; then one may write $\#\mathscr{H}_{K_n} \geq p^{r\,e_n}$, whence $\lambda n + \nu \leq r e_n$ proving that $e_n \to \infty$ with n. If $\lambda = \mu = 0$, then $\#\mathscr{X}_n$ is constant for n large enough, whence $\mathscr{X}_n = 1$ for all $n \geq 0$. □

We remark that if the p-rank r_n is unbounded, necessarily $\mu > 0$.

We must notice, in the practice, that Theorem 1.1 (i) does not apply since the complexity of the $\mathscr{X}_{K_n}$ crucially increases with n.

3.2. **Case of imaginary quadratic field and p-class groups.** If K is an imaginary quadratic field and $L = L_0K$, $L_0/\mathbb{Q}$ real cyclic of degree p^N, we know that there is never capitulation of $\mathscr{H}_K \neq 1$. In the following example, for $p = 3$, $n = 1$, 2, we will examine the structure of $\mathscr{H}_{K_n}$.

Example 3.4. Consider $K = \mathbb{Q}(\sqrt{-199})$, $p = 3$ and $\ell = 19$, inert in K, and use a modified version of Program 7.1 given further:

```
{p=3;Nn=2;m=199;ell=19;mKn=2;PK=x^2+m;K=bnfinit(PK,1);
CK0=K.clgp;r=(kronecker(-m,ell)+3)/2;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
dn=poldegree(Pn);Kn=bnfinit(Pn,1);if(n==1,print();
print("PK=",PK," CK0=",CK0[2]," ell=",ell," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))}
p=3  Nn=2  PK=x^2+199  CK0=[9]  ell=19  r=1
CK1=[513]=[27]
h_1^[(S-1)^1]=[9]       h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=2, e(K1)=3
CK2=[749493,19,19]=[81]
h_1^[(S-1)^1]=[45,0,0]  h_1^[(S-1)^2]=[0,0,0]
norm in K2/K of the component 1 of CK2:[9]
No capitulation, m(K2)=2, e(K2)=4
```

(i) Case $n = 1$. We have $\mathscr{H}_K \simeq \mathbb{Z}/3^2\mathbb{Z}$ and $\mathscr{H}_{K_1} \simeq \mathbb{Z}/3^3\mathbb{Z}$; then $m(K_1) = 2$ ($s(K_1) = 1$) and one obtains $e(K_1) = 3 > n - s(K_1) = 0$. We get the equality $\#\mathscr{H}_{K_1} = \#\mathscr{H}_K \cdot \#\mathscr{H}_K[3]$.

(ii) Case $n = 2$. Then $\mathscr{H}_{K_2} \simeq \mathbb{Z}/3^4\mathbb{Z}$ and $m(K_2) = 2$ ($s(K_2) = 1$) and $e(K_2) = 4 > n - s(K_2) = 1$. Here, $\#\mathscr{H}_{K_2} > \#\mathscr{H}_{K_1} \cdot \#\mathscr{H}_{K_1}[3]$, but $\#\mathscr{H}_{K_2} = \#\mathscr{H}_K \cdot \#\mathscr{H}_K[3^2]$.

Example 3.5. We consider $K = \mathbb{Q}(\sqrt{-199})$, $\ell = 37$, inert in K.

```
p=3  Nn=2  PK=x^2+199  CK0=[9]  ell=37  r=1
CK1=[54,6,3]=[27,3,3]
h_1^[(S-1)^1]=[21,1,0] h_2^[(S-1)^1]=[18,0,1] h_3^[(S-1)^1]=[18,0,0]
h_1^[(S-1)^2]=[0,0,1]  h_2^[(S-1)^2]=[18,0,0] h_3^[(S-1)^2]=[0,0,0]
h_1^[(S-1)^3]=[18,0,0] h_2^[(S-1)^3]=[0,0,0]  h_3^[(S-1)^3]=[0,0,0]
h_1^[(S-1)^4]=[0,0,0]  h_2^[(S-1)^4]=[0,0,0]  h_3^[(S-1)^4]=[0,0,0]
norm in K1/K of the component 1 of CK1:[12,0,1]
norm in K1/K of the component 2 of CK1:[18,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
No capitulation, m(K1)=4, e(K1)=3
CK2=[42442542,18,9]=[81,9,9]
No capitulation, m(K2)=4, e(K2)=4
```

(i) Case $n = 1$. In this case, $\mathscr{H}_{K_1} \simeq \mathbb{Z}/3^3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$; the above data shows that $m(K_1) = 4$ for $e(K_1) = 3$, and a more complex structure, since $s(K_1) = 1$, but $e(K_1) = 3 > n - s(K_1) = 0$.

(ii) Case $n = 2$. Then $\mathscr{H}_{K_2} \simeq \mathbb{Z}/3^4\mathbb{Z} \times \mathbb{Z}/3^2\mathbb{Z} \times \mathbb{Z}/3^2\mathbb{Z}$. So $m(K_2) \geq 4$ (from relation (2.2) (v)), $s(K_2) \geq 1$ with $e(K_2) = 4 > n - s(K_2) = 1$. One has $\#\mathscr{H}_{K_2} = \#\mathscr{H}_{K_1} \cdot \#\mathscr{H}_{K_1}[3]$.

3.3. **Case of torsion groups in abelian p-ramification theory.** We will evoke the case of the torsion group $\mathscr{T}_K$ of the Galois group of the maximal abelian p-ramified pro-p-extension of a number field K; then, under Leopoldt's conjecture, the transfer map is always injective, whatever the extensions of number fields L/K considered. This has some consequences because of the formula:

$$\#\mathscr{T}_K = \#\widetilde{\mathscr{H}_K} \cdot \#\mathscr{R}_K \cdot \#\mathscr{W}_K,$$

where $\mathscr{W}_K$ is a canonical invariant built on the groups of (local and global) roots of unity of p-power order of K, $\mathscr{R}_K$ is the normalized p-adic regulator and $\widetilde{\mathscr{H}_K}$ a sub-group of $\mathscr{H}_K$ (see, for instance [Gras2005, Theorem IV.2.1], [Gras2018, Diagram § 3 and § 5], [Gras2021^a]). Note that for the Bertrandias–Payan module $\mathscr{T}_K^{\rm bp}$ of K (isomorphic to $\mathscr{T}_K/\mathscr{W}_K$) the transfers $\mathbf{J}_{L/K}$ are injective, except few special cases discussed in [GJN2016].

Thus, as we have seen, in any cyclic p-extension L/K of Galois group G, the complexity of the invariants $\mathscr{T}_{K_n}$ will never be smooth and will be increasing with n, regarding that of K.

Example 3.6. We use the program [Gras2019^a, Corollary 2.2, Program I, § 3.2] computing the group structure of the $\mathscr{T}_{K_n}$'s for quadratic fields $K = \mathbb{Q}(\sqrt{m})$, $p = 2$, $K_n \subset K(\mu_\ell)$ (so the number $r_2 + 1$ of independent $\mathbb{Z}_p$-extensions is 1 for $m > 0$ and $2^n + 1$ for $m < 0$). One must chose an arbitrary constant E, "assuming" $E > e_n + 1$, to be controlled a posteriori:

```
MAIN PROGRAM COMPUTING THE STRUCTURE OF TKn (real quadratic fields):
{p=2;ell=257;Nn=4;E=16;for(m=2,150,if(core(m)!=m,next);PK=x^2-m;
print("p=",p," PK=",PK," ell=",ell);for(n=0,Nn,Qn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);Knmod=bnrinit(Kn,p^E);
CKnmod=Knmod.cyc;TKn=List;d=matsize(CKnmod)[2];for(j=1,d-1,c=CKnmod[d-j+1];
w=valuation(c,p);if(w>0,listinsert(TKn,p^w,1)));print("TK",n,"=",TKn)))}
```

In a very simple context ($K = \mathbb{Q}(\sqrt{m})$, $p = 2$, $L \subset K(\mu_{257})$), the complexity of the torsion groups $\mathscr{T}_{K_n}$ is growing dramatically (for the p^k-ranks as well as the exponents) as shown by the following excerpts:

```
p=2  PK=x^2-2  ell=257
TK0=[]
TK1=[8,8]
TK2=[16,16,4,4,2,2]
TK3=[32,32,8,8,4,4,4,2,2,2,2]
TK4=[64,64,16,16,4,4,4,4,4,4,4,4,4,4,2,2,2,2]
p=2  ell=257  PK=x^2-73
TK0=[2]
TK1=[64,8,2,2]
TK2=[128,16,8,4,4,2,2,2]
TK3=[256,32,16,16,8,8,8,4,2,2,2,2,2,2,2,2]
TK4=[512,64,32,32,16,16,16,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
p=2  ell=257  PK=x^2-105
TK0=[2,2]
TK1=[16,8,2,2]
TK2=[32,16,8,4,2,2,2,2]
TK3=[64,32,8,8,8,8,8,2,2,2,2,2,2,2,2,2]
TK4=[128,64,16,16,16,16,16,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
p=2  ell=257  PK=x^2-113
TK0=[4]
TK1=[128,16,4]
TK2=[256,32,8,4,4,4,2]
TK3=[512,64,16,8,8,8,2,2,2,2,2,2,2,2,2]
TK4=[1024,128,32,16,16,16,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
```

In the context $L \subset K(\mu_\ell)$, the analogue of the Chevalley–Herbrand formula gives $\#\mathscr{T}_{K_n}^{G_n} = \#\mathscr{T}_K \cdot p^{r\,n}$, where $r = 1$ or 2 is the number of primes $\mathfrak{l} \mid \ell$ in $K/\mathbb{Q}$ [Gras2005, Theorem IV.3.3, Exercise 3.3.1]; unfortunately, we do not know formulas, similar to that of (2.2), for the orders of the $\mathscr{T}_{K_n}^i$ for $i > 1$.

Consider, for instance, the above case of $m = 113$, for K_4, $\#\mathscr{T}_{K_4} = 2^{59}$, $r = 2$, $\nu_{K_4/K}(\mathscr{T}_{K_4}) \simeq \mathbb{Z}/4\mathbb{Z}$, with the associated polynomial:

```
P=x^15+p^4*x^14+15*p^3*x^13+35*p^4*x^12+455*p^2*x^11+273*p^4*x^10
  +1001*p^3*x^9+715*p^4*x^8+6435*p*x^7+715*p^4*x^6+1001*p^3*x^5
  +273*p^4*x^4+455*p^2*x^3+35*p^4*x^2+15*p^3*x+p^4
```

Since $\#(\mathscr{T}_{K_4}^{i+1}/\mathscr{T}_{K_4}^{i}) \leq \#\mathscr{T}_{K_4}^{G_4} = 2^{10}$, this gives $m(K_4) \geq 10$ ($s(K_4) \geq 3$). These computations show that $m(K_4)$ and $e(K_4) = 10$ are large. Moreover, the conditions of Theorem 1.1 (i), $e(K_4) \leq 4 - s(K_4)$ can not be satisfied.

Taking imaginary quadratic fields does not modify the behavior of the $\mathscr{T}_{K_n}$'s since, for all n, $\mathbf{J}_{K_n/K}$ is still injective and $\mathbf{N}_{K_n/K}$ surjective:

```
MAIN PROGRAM COMPUTING THE STRUCTURE OF TKn (imaginary quadratic fields):
{p=2;ell=257;Nn=3;E=16;for(m=2,150,if(core(m)!=m,next);PK=x^2+m;print("p=",p,
" ell=",ell," PK=",PK);for(n=0,Nn,r2=2^n+1;PKn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,PKn)[1];Kn=bnfinit(Pn,1);Knmod=bnrinit(Kn,p^E);
CKnmod=Knmod.cyc;TKn=List;d=matsize(CKnmod)[2];for(j=1,d-r2,c=CKnmod[d-j+1];
w=valuation(c,p);if(w>0,listinsert(TKn,p^w,1)));print("TK",n,"=",TKn)))}

p=2  PK=x^2+2  ell=257
TK0=[]
TK1=[16,2]
TK2=[16,4,4,4,4]
TK3=[32,8,4,4,4,4,4,4,4]
TK4=[64,16,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4]

p=2  ell=257  PK=x^2+3
TK0=[]
TK1=[16]
TK2=[32,8,4]
TK3=[64,16,4,4,4,4,4]
TK4=[128,32,4,4,4,4,4,4,4,4,4,4,4,4,4]

p=2  ell=257  PK=x^2+7
TK0=[2]
TK1=[8,2,2]
TK2=[16,4,2,2,2,2,2]
TK3=[32,8,2,2,2,2,2,2,2,2,2,2,2,2,2]
TK4=[64,16,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
```

For cyclic cubic fields, $p = 2$, $\ell = 257$, we obtain for instance:

```
MAIN PROGRAM COMPUTING THE STRUCTURE OF TKn (cyclic cubic fields):
{p=2;ell=257;Nn=3;E=16;bf=7;Bf=10^3;
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);print("p=",p," f=",f,
" PK=",PK," ell=",ell);for(n=0,Nn,Qn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);Knmod=bnrinit(Kn,p^E);
CKnmod=Knmod.cyc;TKn=List;d=matsize(CKnmod)[2];for(j=1,d-1,c=CKnmod[d-j+1];
w=valuation(c,p);if(w>0,listinsert(TKn,p^w,1)));print("TK",n,"=",TKn)))))}

p=2  f=31  PK=x^3+x^2-10*x-8  ell=257
TK0=[2,2]
TK1=[8,2,2,2,2]
TK2=[16,4,2,2,2,2,2,2,2,2,2]
TK3=[32,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]

p=2  f=43  PK=x^3+x^2-14*x+8  ell=257
TK0=[2,2]
TK1=[16,16,8,2,2]
TK2=[32,32,16,4,4,4,2,2,2,2,2]
TK3=[64,64,32,8,8,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]

p=2  f=171  PK=x^3-57*x-152  ell=257
TK0=[8,8]
TK1=[16,16,8,2,2]
TK2=[32,32,16,4,2,2,2,2,2,2,2]
TK3=[64,64,32,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]

p=2  f=277  PK=x^3+x^2-92*x+236  ell=257
TK0=[4,4]
TK1=[8,4,4,4,4]
```

```
TK2=[16,8,8,4,4,4,4,4,2,2,2]
TK3=[32,16,16,8,8,8,8,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
```

Remark 3.7. Consider, in the framework of Theorem 3.2, $\mathscr{Y}_n := \mathscr{X}_n/\mathscr{X}_n^p$ for all $n \in [0, N]$, and assume that the transfer maps $\mathscr{Y}_n \to \mathscr{Y}_{n+1}$ are injective; thus, $\#\mathscr{Y}_{n+1} \geq \#\mathscr{Y}_n \cdot \#\mathscr{Y}_n[p] = (\#\mathscr{Y}_n)^2$, whence $\mathrm{rk}_p(\mathscr{X}_{n+1}) \geq 2\,\mathrm{rk}_p(\mathscr{X}_n)$ for all $n \in [0, N-1]$. This doubling of the p-ranks does not seem exceptional; for instance, for $\mathscr{X} = \mathscr{T}$, $K = \mathbb{Q}(\sqrt{105})$, $p = 2$, $\ell = 257$, computed above, we obtain precisely, for $r_n := \mathrm{rk}_p(\mathscr{T}_n)$, $r_0 = 2$, $r_1 = 4$, $r_2 = 8$, $r_3 = 16$, $r_4 = 32$, and in the other similar examples, some irregularities appear, but for all them, ℓ splits in K.

The case $\mathscr{X} = \mathscr{H}$ and K imaginary quadratic may give similar results; for instance $K = \sqrt{-53}$, $p = 3$, $\ell = 109$, leads to:

```
PK=x^2+53  CK0=[6]  ell=109  r=1
CK1=[114,6,6]=[3,3,3]
CK2=[7510662,342,18,3,3]=[9,9,9,3,3]
```

for which $r_0 = 1$, $r_1 = 3$, $r_2 = 5$.

Then, with $K = \sqrt{-5}$, $p = 2$, $\ell = 257$, one obtains:

```
PK=x^2+5  CK0=[2]  ell=257  r=1
CK1=[6,6]=[2,2]
CK2=[174,6,2,2]=[2,2,2,2]
CK3=[19662,6,6,6,2,2,2,2]=[2,2,2,2,2,2,2,2]
CK4=[5353549698,42,6,6,2,2,2,2,2,2,2,2,2,2,2,2]
                                  =[2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
```

for which $r_n = 2^n$ for all $n \leq 4$.

Note that, even if the transfers $\mathscr{X}_n \to \mathscr{X}_{n+1}$ are injective, $\mathscr{Y}_n \to \mathscr{Y}_{n+1}$ may be non injective, meaning that some $x_n \in \mathscr{X}_n$ become pth powers in $\mathscr{X}_{n+1}$, which "explains" that the exponents p^{e_n} increase with n in many of the above numerical examples. So one may hope that any pair (p^{e_n}, r_n), compatible with Galois action, does exist.

This study suggests that classical bounds given by genus theory in L/K, in the form of the genus exact sequence (e.g., [AnJau2000, Théorème 2.2.9], [Jaul1986, Théorème III.2.7]), [Gras2005, Corollary IV.4.5.1], [Maire2018, Theorem 2.2], [Liu2022, Theorem 6.5]), may be much largely exceeded, including totally real base fields K.

4. Capitulation in $\mathbb{Z}_p$-extensions

The problem of capitulations in a $\mathbb{Z}_p$-extension $\widetilde{K} = \bigcup_{n\geq 0} K_n$ of K has a long history from Iwasawa pioneering works showing, for instance, that the capitulation kernels $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n})$ have a bounded order as $n \to \infty$ [Iwas1973, Theorem 10, § 5]. The reader may refer for instance to [GrJa1985, BaCa2016, Cald2020] for classical context of p-class groups and to [KoMo2000, Vali2008] for wild kernels, [Jaul2016, Jaul2019^a] for logarithmic class groups (see § Ê5).

Let $\mathrm{X}_{\widetilde{K}} := \varprojlim_n \mathscr{H}_{K_n}$ (for the arithmetic norms), isomorphic to the Galois group of the maximal unramified abelian pro-p-extension of $\widetilde{K}$ and let $\mathscr{H}_{\widetilde{K}} := \varinjlim_n \mathscr{H}_{K_n}$ (for the transfer maps) be the p-class group of $\widetilde{K}$.

4.1. Survey of known results under the assumption $\mu = 0$.

If $\mu = 0$, in the writing $\#\mathscr{H}_{K_n} = p^{\lambda n + \mu p^n + \nu}$ for $n \gg 0$, the following properties are proved in [GrJa1985, Théorème, p. 214]:

- $\mathrm{X}_{\widetilde{K}} \simeq \mathrm{T} \bigoplus \mathbb{Z}_p^{\lambda}$, where T is a finite p-group,

• $\mathbf{N}_{\widetilde{K}/K_n} : \mathrm{X}_{\widetilde{K}} \to \mathscr{H}_{K_n}$ induces an isomorphism of T onto $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n})$, for all $n \gg 0$,

• $\mathscr{H}_{K_n} \simeq \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \oplus \mathbf{J}_{\widetilde{K}/K_n}(\mathscr{H}_{K_n}) \simeq \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{n+\alpha_i}\mathbb{Z}$, for all $n \gg 0$, with some relative integers α_i.

In [Vali2008, Théorème 3.2.5] is proved analogous results for even groups of the **K**-theory of rings of integers of number fields, after similar results as that of [KoMo2000].

From now on, we take the base field K_{n_0}, n_0 large enough, in such a way that $\widetilde{K}/K_{n_0}$ is totally ramified and such that all the above properties are fulfilled from n_0. By abuse of notation, we write K instead of K_{n_0} and K_n now denotes K_{n_0+n}. So, the $\mathbb{Z}_p$-extension $\widetilde{K}/K$ has new Iwasawa invariants of the form $(\lambda,\, \mu = 0,\, \nu + \lambda n_0)$ that we still denote $(\lambda,\, \nu)$.

Thus, $\#\mathscr{H}_{K_n} = p^{\lambda n+\nu}$ for all $n \geq 0$, $\#\mathscr{H}_K = p^{\nu}$, and:

$$\mathscr{H}_{K_n} \simeq \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{n+\alpha_i}\mathbb{Z}, \text{ with } \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \simeq \mathrm{T}\ ,\ \ \forall n \geq 0. \tag{4.1}$$

In particular, in this new context, $\alpha_i \geq 0$ and:

$$\mathscr{H}_K \simeq \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K}) \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{\alpha_i}\mathbb{Z},\ \ \alpha_i \geq 0. \tag{4.2}$$

Proposition 4.1. *Under the above choice of the base field K in the $\mathbb{Z}_p$-extension $\widetilde{K}$ and assuming $\mu = 0$, the capitulation of $\mathscr{H}_K$ in $\widetilde{K}$ is equivalent to the isomorphism $\mathscr{H}_{K_n} \simeq \mathscr{H}_K \oplus \mathbf{J}_{\widetilde{K}/K_n}(\mathscr{H}_{K_n}) \simeq \mathscr{H}_K \oplus \big(\mathbb{Z}/p^n\mathbb{Z}\big)^{\lambda}$, for all $n \geq 0$.*

Proof. If $\mathscr{H}_K$ capitulates in $\widetilde{K}$, then $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K}) = \mathscr{H}_K$ and, from (4.2), $\alpha_i = 0$ for all $i \in [1, \lambda]$. So $\mathscr{H}_{K_n} \simeq \mathscr{H}_K \oplus \big(\mathbb{Z}/p^n\mathbb{Z}\big)^{\lambda}$, for all $n \geq 0$, since each capitulation kernel $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n})$ is isomorphic to T, whence isomorphic to $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K}) = \mathscr{H}_K$ (isomorphisms given by the arithmetic norms).

Reciprocally, assume that $\mathscr{H}_{K_n} \simeq \mathscr{H}_K \oplus \big(\mathbb{Z}/p^n\mathbb{Z}\big)^{\lambda}$, for all $n \geq 0$; then, from (4.1):

$$\mathscr{H}_{K_n} = \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{n+\alpha_i}\mathbb{Z} \simeq \mathscr{H}_K \oplus \big(\mathbb{Z}/p^n\mathbb{Z}\big)^{\lambda}.$$

Comparing the structures for n large enough gives $\alpha_i = 0$ for all $i \in [1, \lambda]$ and $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \simeq \mathscr{H}_K$ for all n, whence the capitulation of $\mathscr{H}_K$ in $\widetilde{K}$. □

4.2. **Case of the cyclotomic $\mathbb{Z}_p$-extension of K.** Assume that K is totally real and let $K_\infty = \bigcup_{n\geq 0} K_n$ be the cyclotomic $\mathbb{Z}_p$-extension of K, assuming the previous choice of the base field K in K_∞ (it is still real with same cyclotomic $\mathbb{Z}_p$-extension). Greenberg's conjecture [Gree1976] for K_∞ ($\lambda = \mu = 0$) is equivalent to the stability of the $\#\mathscr{H}_{K_n}$'s from K, giving capitulations of all the class groups in K_∞ from $n = 0$, then equalities $\mathscr{H}_{K_n} = \mathscr{H}_K^{G_n}$ for all $n \geq 0$, and the isomorphisms $\mathscr{H}_{K_n} \overset{\mathbf{N}_{K_n/K}}{\simeq} \mathscr{H}_K$, for all $n \geq 0$; thus, $m(K_n) = 1$ ($s(K_n) = 0$) with $e(K_n) = e(K)$, which is exactly the limit case of application of the Theorem 1.1 (i) for $n \geq e(K)$.

In [KrSch1995, Paga2022] such properties of stability are used to check the conjecture by means of analytic formulas.

In [Jaul2016, Jaul2019^a, Jaul2019^b], it is proved that Greenberg's conjecture is equivalent to the capitulation of the logarithmic class group $\mathscr{H}_K^{\log}$ in K_∞; this may be effective if, by chance, a capitulation occurs in the firsts layers

over the base field K; indeed, this criterion is probably the only one giving an algorithmic test (using [BeJa2016, DJPPS2005]) from the base field.

In what follows, we will analyze the generalized Chevalley–Herbrand formula in K_∞/K.

Conventions 4.2. *Taking in the sequel, as totally real base field K, a suitable layer K_{n_0} in K_∞, we may assume the following properties of K_∞/K:*

(i) p is totally ramified in K_∞/K;

(ii) Iwasawa's formula $\#\mathscr{H}_{K_n} = p^{\lambda n+\mu p^n+\nu}$ is valid for all $n \geq 0$ with the new invariants $(\lambda,\ \mu p^{n_0},\ \nu+\lambda n_0)$ that we still denote (λ,μ,ν). Thus, $\#\mathscr{H}_K = p^{\mu p^{n_0}+\nu+\lambda n_0}$ and, except the trivial case $\lambda=\mu=\nu=0$, $\mathscr{H}_K \neq 1$ in this new writing.

Then, as for the "tame casee", formulas (2.2) hold with $r = \#\{\mathfrak{p},\ \mathfrak{p}\,|\,p \text{ in } K\}$ and the filtration still depends on the class and norm factors. However, the norm factors can be interpreted, from generalizations of Taya results, as divisor of the normalized p-adic regulator of K, as follows from class field theory:

Definitions 4.3. (i) Let $H_{K_n}^{\rm gen}$ be the genus field of K_n (i.e., the subfield of the p-Hilbert class field $H_{K_n}^{\rm nr}$, abelian over K and maximal, whence the subfield of $\mathrm{Gal}(H_{K_n}^{\rm nr}/K)$ fixed by the image of $\mathscr{H}_{K_n}^{\sigma_n-1}$), and let $K_\infty H_{K_n}^{\rm gen}$, which is abelian p-ramified over K, whence $K_\infty H_{K_n}^{\rm gen} \subseteq H_K^{\rm pr}$, the maximal p-ramified abelian pro-p-extension of K. So $\mathscr{T}_K := \mathrm{Gal}(H_K^{\rm pr}/K_\infty)$ is finite under Leopoldt's conjecture.

We define $H_{K_\infty}^{\rm gen} := \bigcup_n K_\infty H_{K_n}^{\rm gen}$ and put $\mathscr{G}_K := \mathrm{Gal}(H_{K_\infty}^{\rm gen}/K_\infty)$.

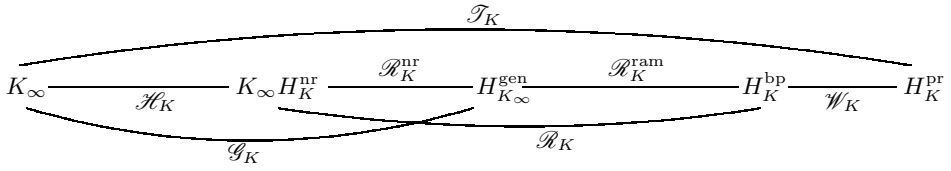
We denote by K_{n_1}, $n_1 \geq 0$, the minimal layer such that $K_\infty H_{K_{n_1}}^{\rm gen} = H_{K_\infty}^{\rm gen}$ (even with the above conventions, K_{n_1} may be distinct from K).

(ii) Let $H_K^{\rm bp}$ be the Bertrandias–Payan field fixed by $\mathscr{W}_K \simeq \big(\oplus_{v|p}\mu_{K_v}\big)/\mu_K$, where K_v is the v-completion of K and μ_k the group of pth-roots of unity of the field k (local or global); if U_v is the group of principal units of K_v, then $\mu_{K_v} = \mathrm{tor}_{\mathbb{Z}_p}(U_v)$.

(iii) Let $\iota\mathscr{E}_K$ be the image of $\mathscr{E}_K$ in $U_K := \prod_{v|p} U_v$ and let $I_v(H_K^{\rm pr}/K_\infty) := \mathrm{tor}_{\mathbb{Z}_p}(U_v/\iota\mathscr{E}_K \cap U_v)$ be the inertia groups of v in $H_K^{\rm pr}/K_\infty$; the subgroup of $\mathscr{T}_K$ generated by these inertia groups fixes $H_{K_\infty}^{\rm gen}$.

(iv) Let $\mathscr{R}_K^{\rm nr} := \mathrm{Gal}(H_{K_\infty}^{\rm gen}/K_\infty H_K^{\rm nr})$ and let $\mathscr{R}_K^{\rm ram} := \mathrm{Gal}(H_K^{\rm bp}/H_{K_\infty}^{\rm gen})$, where $\mathscr{R}_K := \mathrm{Gal}(H_K^{\rm bp}/K_\infty H_K^{\rm nr})$ is the normalized p-adic regulator defined in [Gras2018, Section 5].

These definitions may be summarized by the following diagram [Gras2021^a, Section 2]:



Recall, under the above Conventions 4.2 about the choice of the base field K, some results, given in [Gras2017^b, Gras2019^b] and generalizing some particular results of Taya [Taya1996, Taya1999, Taya2000]:

Proposition 4.4. *For all $n \geq 0$, the norm factor $\dfrac{p^{n\cdot(r-1)}}{\omega_{K_n/K}(\mathscr{E}_K)}$ divides $\#\mathscr{R}_K^{\rm nr}$ and $\#\mathscr{H}_{K_n}^{G_n} = \#\mathscr{H}_K \times \dfrac{p^{n\cdot(r-1)}}{\omega_{K_n/K}(\mathscr{E}_K)}$ divides $\#\mathscr{G}_K = \#\mathscr{H}_K \cdot \#\mathscr{R}_K^{\rm nr}$; then, equality holds for all $n \geq n_1$.*

Thus, the norm factors $\dfrac{p^{n\cdot(r-1)}}{\omega_{K_n/K}(\mathbf{\Lambda}_K^i)}$, associated to the filtration, divide $\#\mathscr{R}_K^{\rm nr}$, which allows computations in the base field K without Hasse's symbols. Recall that $m(K_n)$ is the length of the filtration for K_n and that $m(K)=1$ if $\mathscr{H}_K\neq 1$ ($m(K)=0$ if $\mathscr{H}_K=1$); so, formulas (2.2) apply in general for $L=K_n$, $n\geq 0$:

Proposition 4.5. *Let v_p denotes the p-adic valuation. Under the previous Conventions 4.2 about the base field K, we have for all $n\geq 0$:*

$$m(K_n)\leq \lambda\cdot n+\mu\cdot p^n+\nu\leq v_p(\#\mathscr{H}_K\cdot\#\mathscr{R}_K^{\rm nr})\cdot m(K_n).$$

From these recalls and conventions about the base field K we can deduce (under Leopoldt's conjecture):

Corollary 4.6. *Greenberg's conjecture is equivalent to $m(K_n)=1$ (resp. 0) if $\mathscr{H}_K\neq 1$ (resp. $\mathscr{H}_K=1$) and $\mathscr{R}_K^{\rm nr}=1$, whence $\mathscr{H}_{K_n}=\mathscr{H}_{K_n}^{G_n}$ for all $n\geq 0$.*

Proof. (i) If $\lambda=\mu=0$, there is stability from the level $n=0$ and we know that $\mathscr{H}_{K_n}=\mathscr{H}_{K_n}^{G_n}\overset{\mathbf{N}_{K_n/K}}{\simeq}\mathscr{H}_K$ for all $n\geq 0$ (whence $m(K_n)\in\{0,1\}$ for all n). There exists $n_1\gg 0$ such that $\mathrm{Gal}(K_\infty H_{K_{n_1}}^{\rm gen}/K_\infty)=\mathscr{G}_K$ by definition; thus, since $\#\mathscr{H}_{K_n}^{G_n}=\#\mathscr{H}_K$, for all $n\geq n_1$, it follows that $\mathscr{R}_K^{\rm nr}=1$.

(ii) Reciprocally, if $\mathscr{R}_K^{\rm nr}=1$ and $m(K_n)\in\{0,1\}$, then Proposition 4.5 implies $\lambda=\mu=0$ if $\mathscr{H}_K\neq 1$ (or $\lambda=\mu=\nu=0$ if $\mathscr{H}_K=1$). □

We can wonder, due to Proposition 4.1, if Greenberg's conjecture is equivalent, under Conventions 4.2 and $\mu=0$, to $\nu_{K_n/K}(\mathscr{H}_{K_n})=1$ for all $n\geq e(K)$, obtained with the stronger particular conditions $m(K_n)=1$ (i.e., $s(K_n)=0$) and $e(K_n)=e(K)$.

Indeed, under Greenberg's conjecture, in the non trivial case $\mathscr{H}_K\neq 1$, one has $\mathscr{H}_{K_n}\overset{\mathbf{N}_{K_n/K}}{\simeq}\mathscr{H}_K$, for all $n\geq 0$, which characterizes the stability with $m(K_n)=1$ and $e(K_n)=e(K)$, precisely the kind of annihilation of the $\mathscr{H}_{K_n}$'s by $\nu_{K_n/K}$ with $m(K_n)=1$ (i.e., $s(K_n)=0$) and $e(K_n)=e(K)$.

Reciprocally, if $\mathscr{H}_K$ capitulates because of the conditions $m(K_n)=1$ and $e(K_n)=e(K)$ for all $n\geq e(K)$, then $\lambda=0$.

But, unfortunately, in the practice, these phenomenon (if any) holds from an unknown level.

This possibility may be suggested by the following example of the cyclic cubic field of conductor $f=2689$, of 2-class group $\mathbf{Z}/2\mathbf{Z}$ and its cyclotomic $\mathbb{Z}_2$-extension, giving $\mathscr{H}_{K_1}\simeq\mathbf{Z}/4\mathbf{Z}$, $\mathscr{H}_{K_2}\simeq\mathbf{Z}/8\mathbf{Z}$, and $\mathscr{H}_{K_3}\simeq\mathbf{Z}/8\mathbf{Z}$ (see details of the PARI programs and data in § 6.3):

```
p=2  f=2689  PK=x^3+x^2-896*x+5876  CK0=[2,2]
CK1=[28,4]=[4,4]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation,  m(K1)=1, e(K1)=2
CK2=[56,8]=[8,8]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
No capitulation, m(K2)=1, e(K2)=3
CK3=[56,8]=[8,8]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
```

```
Complete capitulation, m(K3)=1, e(K3)=3
```

At any layer, $\sigma_n - 1$ annihilates $\mathscr{H}_{K_n}$ (i.e., $m(K_n) = 1$) and capitulations in K_∞ hold for all n. Stability occurs from K_2 giving a checking of Greenberg's conjecture.

Remark 4.7. It is interesting to check the numerical examples given by Fukuda [Fuku1994] for real quadratic fields and $p = 3$ (stability from K) with the simplified usual program and suitable polynomials defining the layers of the cyclotomic $\mathbb{Z}_3$-extension; here we limit the computations at the first level ($\mathsf{n} = 1$) and use the fact that the $\mathscr{H}_{K_n}$'s are cyclic:

```
{p=3;n=1;Lm=List([3137,3719,4409,6809,7226,9998]);for(i=1,6,PK=x^2-Lm[i];
K=bnfinit(PK,1);CK0=K.clgp;Pn=polcompositum(PK,polsubcyclo(p^(n+1),p^n))[1];
Kn=bnfinit(Pn,1);CKn=Kn.clgp;print();G=nfgaloisconj(Kn);Id=x;for(k=1,2*p^n,
Z=G[k];ks=1;while(Z!=Id,Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,
S=G[k];break));A0=CKn[3][1];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));C=bnfisprincipal(Kn,A)[1];print("PK=",PK,
" CK0=",CK0[2]," CK",n,"=",CKn[2]," norm in K",n,"/K of CK",n,":",C))}
```

```
PK=x^2-3137   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
PK=x^2-3719   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
PK=x^2-4409   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
PK=x^2-6809   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
PK=x^2-7226   CK0=[18]   CK1=[18]   norm in K1/K of CK1:[3]~
PK=x^2-9998   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
```

The stability from K implies Greenberg's conjecture and capitulation of $\mathscr{H}_K$ in K_2, with an incomplete capitulation in K_1 and $m(K_n) = 1$ for all n.

5. Capitulation of the logarithmic class group

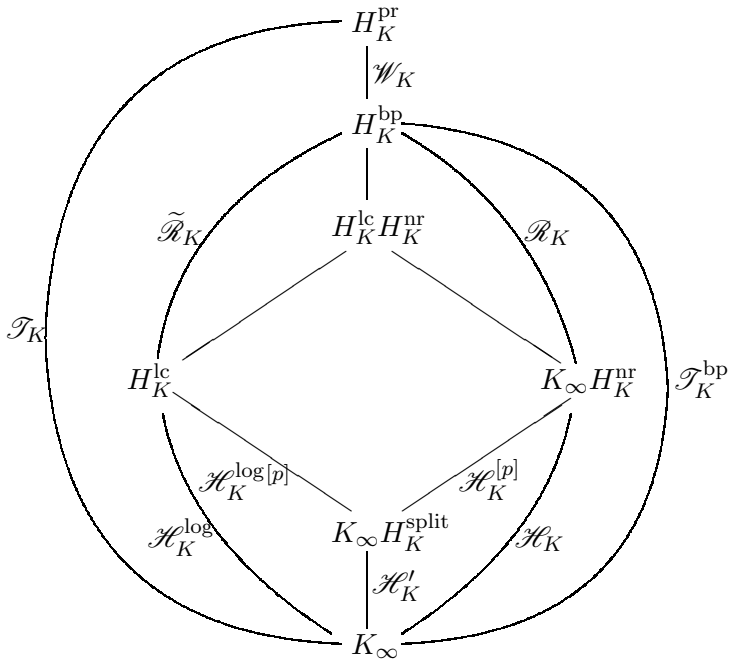
Questions of capitulation, in various p-extensions, of other arithmetic invariants, are at the origin of many papers, (see, e.g., [Maire1996, KoMo2000, JauMi2006, Brig2007, Vali2008, Jaul2016, GJN2016, Jaul2019^a, Jaul2019^b, KhPr2000, Jaul2022, Gras2022^a] and their references); they are related to generalized p-class groups with conditions of ramification and decomposition, to wild kernels of K-theory, then, in a nearby setting, to torsion groups in p-ramification theory, to Tate–Chafarevich groups, Bertrandias–Payan modules about the embedding problem and logarithmic class groups.

The same techniques, using the algebraic norm, may be applied; the results essentially depend on the properties of the associated filtration, whence on the variation of the complexity in the p-extension L/K considered.

We shall focus on transfers of the logarithmic class groups in some totally ramified cyclic p-extensions. This invariant, usually denoted $\widetilde{\mathscr{C}\ell}_K$ or $\widetilde{\mathscr{T}}_K$, was defined in [Jaul1994] and is, in the class field theory viewpoint, isomorphic to $\mathrm{Gal}(H_K^{\mathrm{lc}}/K^{\mathrm{cyc}})$, where H_K^{lc} is the maximal abelian locally cyclotomic pro-p-extension of K and $K^{\mathrm{cyc}} = K_\infty$ its cyclotomic $\mathbb{Z}_p$-extension.

It is well known that tame places totally split in $H_K^{\mathrm{pr}}/K^{\mathrm{cyc}}$, so that H_K^{lc} is the subfield of H_K^{pr} fixed by the decomposition groups of the p-places. In the sequel we will denote $\mathscr{H}_K^{\log}$ this group since it behaves more like a class group rather than a torsion group $\mathscr{T}_K$ which never capitulates.

In [Jaul2019^a] one finds the following diagram of the main invariants, showing in particular that $\mathscr{H}_K$ and $\mathscr{H}_K^{\log}$ are quotients of $\mathscr{T}_K$ where the groups $\mathrm{Gal}(H_K^{\mathrm{bp}}/K_\infty H_K^{\mathrm{nr}})$ and $\mathrm{Gal}(H_K^{\mathrm{bp}}/H_K^{\mathrm{lc}})$ are suitable regulators of units:



In this diagram, $\mathscr{H}_K^{\log[p]}$ (resp. $\mathscr{H}_K^{[p]}$) is the subgroup, of the logarithmic class group $\mathscr{H}_K^{\log}$ (resp. of the p-class group $\mathscr{H}_K$), generated by the classes of the primes dividing p. So $\mathscr{H}'_K$ is the quotient $\mathscr{H}_K/\mathscr{H}_K^{[p]}$ and $H_K^{\rm split}$ is the splitting field of p in $H_K^{\rm nr}$, hence the subfield fixed by the image of $\mathscr{H}_K^{[p]}$, noting that in our case, $H_K^{\rm nr}\cap K^{\rm cyc}=K$ (compare with the diagram in Definitions 4.3).

These invariants are connected by means of the exact sequences:

$$1\to\mathscr{H}_K^{\log[p]}\longrightarrow\mathscr{H}_K^{\log}\longrightarrow\mathscr{H}'_K\to 1,\quad 1\to\mathscr{H}_K^{[p]}\longrightarrow\mathscr{H}_K\longrightarrow\mathscr{H}'_K\to 1.$$

In our context (K totally real, $L\subset K(\mu_\ell)$, $\ell\equiv 1\,(\mathrm{mod}\ 2p^N)$), Theorem 1.1 applies to the logarithmic class groups $\mathscr{H}_K^{\log}$ and $\mathscr{H}_L^{\log}$, computable using [BeJa2016, DJPPS2005]. Indeed, since L/K is tamely and totally ramified at ℓ, then $L^{\rm cyc}=LK^{\rm cyc}$, thus $H_K^{\rm lc}$ is linearly disjoint from $L^{\rm cyc}$ and the norm $\mathbf{N}_{L/K}:\mathscr{H}_L^{\log}=\mathrm{Gal}(H_L^{\rm lc}/L^{\rm cyc})\to\mathscr{H}_K^{\log}=\mathrm{Gal}(H_K^{\rm lc}/K^{\rm cyc})$ is surjective.

Remark 5.1. For logarithmic class groups in totally ramified (in the classical sense) cyclic p-extensions, the theory of stability does exist, essentially because the arithmetic norms are *surjective* allowing the criterion of capitulation with $\nu_{L/K}=\mathbf{J}_{L/K}\circ\mathbf{N}_{L/K}$, but in numerical applications, our extensions L/K may be partially locally cyclotomic, say in K_{n_0}/K, which gives some logarithmic non-ramification (see [Jaul1994, Théorème 1.4]). For instance, if K is a cyclic cubic field, $p=2$ and $\ell=17$, then p splits in $K_1=K(\sqrt{17})$ and is inert in L/K_1; for K quadratic real, $p=3$, $\ell=109$, p is totally inert in L/K leading to the classic reasoning in *totally logarithmically ramified cyclic p-extensions.*

Nevertheless, one can illustrate the problem of capitulation in the tricky case, from the base field K_1 (instead of K), or by using Theorem 1.1 (i) with a direct proof as follows:

Proposition 5.2. *Let $L\subset K(\mu_\ell)$, $\ell\equiv 1\ (\mathrm{mod}\ p^N)$, as usual. Assume that for some $n_0\geq 0$ and $h\geq 1$, $\mathscr{H}_{K_{n_0+h}}^{\log}\simeq\mathscr{H}_{K_{n_0}}^{\log}$ with $e(K_{n_0})\in[1,n_0+h]$. Then $\mathscr{H}_K^{\log}$ capitulates in K_{n_0+h}.*

Proof. Assume that $\mathscr{H}^{\log}_{K_{n_0+h}} \simeq \mathscr{H}^{\log}_{K_{n_0}}$ and let $G_0 = \mathrm{Gal}(K_{n_0+h}/K_{n_0}) =: \langle\sigma_0\rangle$; necessarily the isomorphism is given by the norm $\mathbf{N}_{K_{n_0+h}/K_{n_0}}$ which is surjective; thus $\mathbf{N}_{K_{n_0+h}/K_{n_0}}((\mathscr{H}^{\log}_{K_{n_0+h}})^{\sigma_0-1}) \simeq 1$ giving $\mathscr{H}^{\log}_{K_{n_0+h}} = (\mathscr{H}^{\log}_{K_{n_0+h}})^{G_0}$, whence $m(K_{n_0+h}) = 1$, $s(K_{n_0+h}) = 0$, which yields the capitulation of $\mathscr{H}^{\log}_{K_{n_0}}$ in $\mathscr{H}^{\log}_{K_{n_0+h}}$ and, a fortiori, that of $\mathscr{H}^{\log}_{K}$. □

5.1. **Examples with real quadratic fields and $p = 3$.** We give examples for quadratic field $K = \mathbb{Q}(\sqrt{m})$, $p = 3$, $\ell = 109$. There are many stabilities, allowing to conclude the capitulation. Recall that PARI gives the data $\big[\mathscr{H}^{\log}_{K_n}, \mathscr{H}^{\log[p]}_{K_n}, \mathscr{H}'_{K_n}\big]$ in this order.

```
{p=3;Nn=2;bm=2;Bm=10^4;ell=109;for(m=bm,Bm,if(core(m)!=m,next);
PK=x^2-m;K=bnfinit(PK,1);ClogK0=bnflog(K,p);if(ClogK0==[[],[],[]],next);
r=(kronecker(m,ell)+3)/2;for(n=1,Nn,Qn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);if(n==1,print();
print("PK=",PK," ClogK0=",ClogK0," ell=",ell," r=",r));
ClogKn= bnflog(Kn,p);print("ClogK",n,"=",ClogKn)))}
```

In the case $r = 1$, we obtain examples of the following form:

```
PK=x^2 - 67 ClogK0=[[3],[3],[]] ell=109 r=1
ClogK1=[[9],[9],[]]
ClogK2=[[9],[9],[]]
Complete capitulation in K2
PK=x^2 - 321 ClogK0=[[3],[],[3]] ell=109 r=1
ClogK1=[[9],[],[9]]
ClogK2=[[9],[],[9]]
Complete capitulation in K2
PK=x^2 - 454 ClogK0=[[3],[3],[]] ell=109 r=1
ClogK1=[[3],[3],[]]
ClogK2=[[3],[3],[]]
Complete capitulation in K1
PK=x^2 - 473 ClogK0=[[3],[],[3]] ell=109 r=1
ClogK1=[[3],[],[3]]
ClogK2=[[3],[],[3]]
Complete capitulation in K1
PK=x^2 - 610 ClogK0=[[3],[3],[]] ell=109 r=1
ClogK1=[[9],[9],[]]
ClogK2=[[27],[27],[]]
No conclusion
```

The last example ($m = 610$) is the only one of non stability in the interval considered. Its 3-class group is trivial, then $\mathscr{H}'_K = 1$ and $\mathscr{H}^{\log}_{K_n} = \mathscr{H}^{\log[3]}_{K_n}$.

The following list deals with the case $r = 2$. If $\mathscr{H}^{\log[p]}_{K_n} = 1$, we can conclude, using the general Program 7.1 for p-class groups.

```
PK=x^2-106  ClogK0=[[3],[3],[]]  ell=109  r=2
ClogK1=[[3,3],[3],[3]]
ClogK2=[[9,3],[9],[3]]
No conclusion
PK=x^2-238  ClogK0=[[3],[3],[]]  ell=109  r=2
ClogK1=[[9,3],[9],[3]]
ClogK2=[[27,9],[27],[9]]
No conclusion
PK=x^2-253  ClogK0=[[3],[3],[]]  ell=109  r=2
ClogK1=[[9],[9],[]]
ClogK2=[[27],[27],[]]
No conclusion
PK=x^2-254  ClogK0=[[3],[],[3]]  ell=109  r=2
ClogK1=[[3,3],[],[3,3]]
ClogK2=[[3,3],[],[3,3]]
```

```
Complete capitulation in K2
PK=x^2-326  ClogK0=[[3],[],[3]]  ell=109  r=2
ClogK1=[[3],[],[3]]
ClogK2=[[3],[],[3]]
Complete capitulation in K1
PK=x^2-443  ClogK0=[[3],[],[3]]  ell=109  r=2
ClogK1=[[9],[],[9]]
ClogK2=[[9],[],[9]]
Complete capitulation in K2
PK=x^2-659  ClogK0=[[3],[],[3]]  ell=109  r=2
ClogK1=[[9],[],[9]]
ClogK2=[[9],[],[9]]
Complete capitulation in K2
PK=x^2-679  ClogK0=[[3],[3],[]]  ell=109  r=2
ClogK1=[[9],[9],[]]
ClogK2=[[27],[27],[]]
No conclusion
PK=x^2-727  ClogK0=[[3],[3],[]]  ell=109  r=2
ClogK1=[[9],[9],[]]
ClogK2=[[27],[27],[]]
No conclusion
PK=x^2-785  ClogK0=[[3],[],[3]]  ell=109  r=2
ClogK1=[[9],[],[9]]
ClogK2=[[27],[],[27]]
No conclusion
PK=x^2-790 ClogK0=[[3],[3],[]] ell=109 r=2
ClogK1=[[3,3],[3],[3]]
ClogK2=[[9,3],[3],[9]]
No conclusion
```

5.2. **Examples with cyclic cubic fields and $p = 2$.** With an analogous program, one obtains the following results, taking into account Remark 5.1:

```
PK=x^3 + x^2 - 54*x - 169  ClogK0=[[2,2],[],[2,2]]  ell=17  r=3
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[2,2],[],[2,2]]
ClogK3=[[2,2],[],[2,2]]
Complete capitulation in K1
PK=x^3 + x^2 - 182*x - 81  ClogK0=[[2,2],[],[2,2]]  ell=17  r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[2,2],[],[2,2]]
ClogK3=[[2,2],[],[2,2]]
Complete capitulation in K1
PK=x^3 - 201*x + 1072  ClogK0=[[2,2],[2,2],[]]  ell=17  r=1
ClogK1=[[2,2],[2,2],[]]
ClogK2=[[4,4],[4,4],[]]
ClogK3=[[8,8],[8,8],[]]
No conclusion (fake stability in K1/K)
PK=x^3 + x^2 - 202*x - 1169  ClogK0=[[2,2],[],[2,2]]  ell=17  r=1
ClogK1=[[2,2,2,2],[],[2,2,2,2]]
ClogK2=[[2,2,2,2],[],[2,2,2,2]]
ClogK3=[[2,2,2,2],[],[2,2,2,2]]
Complete capitulation in K2
PK=x^3 + x^2 - 234*x - 729  ClogK0=[[2,2],[],[2,2]]  ell=17  r=1
ClogK1=[[4,4],[],[4,4]]
ClogK2=[[8,8],[],[8,8]]
ClogK3=[[16,16],[],[16,16]]
No conclusion, probably no capitulation
(...)
PK=x^3 - 291*x - 1358  ClogK0=[[2,2],[2,2],[]]  ell=17  r=1
ClogK1=[[4,4],[4,4],[]]
ClogK2=[[4,4],[4,4],[]]
```

```
Complete capitulation in K2

PK=x^3 + x^2 - 336*x - 1719  ClogK0=[[2,2],[],[2,2]]  ell=17  r=1
ClogK1=[[4,4],[],[4,4]]
ClogK2=[[4,4],[],[4,4]]
Complete capitulation in K2

PK=x^3 + x^2 - 340*x + 416  ClogK0=[[2,2],[2,2],[]]  ell=17  r=1
ClogK1=[[2,2],[2,2],[]]
ClogK2=[[2,2,2,2],[2,2,2,2],[]]
No conclusion (fake stability in K1/K)

PK=x^3 + x^2 - 386*x + 1760  ClogK0=[[4,4],[4,4],[]]  ell=17  r=1
ClogK1=[[4,4],[4,4],[]]
ClogK2=[[8,8],[8,8],[]]
No conclusion (fake stability in K1/K)

PK=x^3 + x^2 - 486*x + 2864  ClogK0=[[2,2],[2,2],[]]  ell=17  r=3
ClogK1=[[2,2],[2,2],[]]
ClogK2=[[4,4,2,2],[4,4],[2,2]]
No conclusion (fake stability in K1/K)

PK=x^3 + x^2 - 650*x - 289  ClogK0=[[2,2],[],[2,2]]  ell=17  r=3
ClogK1=[[4,4,2,2],[],[4,4,2,2]]
ClogK2=[[4,4,4,4],[],[4,4,4,4]]
No conclusion

PK=x^3 + x^2 - 692*x - 7231  ClogK0=[[2,2],[],[2,2]]  ell=17  r=1
ClogK1=[[2,2,2,2],[],[2,2,2,2]]
ClogK2=[[2,2,2,2,2,2],[],[2,2,2,2,2,2]]
No conclusion
```

To verify the capitulations as for the p-class groups, it would be interesting to have available the logarithmic instructions replacing K.clgp, once the field K is given as usual with K = bnfinit(PK) and the logarithmic class group by bnflog(K, p), then an instruction replacing bnfisprincipal(K, A) for an ideal A in the logarithmic sense.

5.3. **Capitulation in the $\mathbb{Z}_2$-extension of $\mathbb{Q}(\sqrt{m})$.** The program eliminates trivial cases (as the stability from K) and only considers totally ramified cyclotomic $\mathbb{Z}_2$-extensions of K. The capitulation is obtained in K_3/K for almost cases; we give an excerpt of the results and the case where no conclusion is possible for $n \leq 3$ with m up to 1250:

```
{p=2;Nn=3;bm=2;Bm=10^4;for(m=bm,Bm,if(core(m)!=m,next);if(Mod(m,8)==2,next);
PK=x^2-m;K=bnfinit(PK,1);r=matsize(idealfactor(K,2))[1];ClogK0=bnflog(K,p);
if(ClogK0==[[],[],[]],next);for(n=1,Nn,Qn=x;for(i=1,n,Qn=Qn^2-2);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);if(n==1,print();
print("PK=",PK," ClogK0=",ClogK0," r=",r));ClogKn= bnflog(Kn,p);
if(n==1 & ClogKn==ClogK0,break);print("ClogK",n,"=",ClogKn)))}

PK=x^2-113  ClogK0=[[2],[2],[]]  r=2
ClogK1=[[2],[],[2]]
ClogK2=[[2],[],[2]]
ClogK3=[[2],[],[2]]

PK=x^2-119  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[2,2],[],[2,2]]
ClogK3=[[2,2],[],[2,2]]

PK=x^2-161  ClogK0=[[4],[4],[]]  r=2
ClogK1=[[4],[2],[2]]
ClogK2=[[4],[],[4]]
ClogK3=[[4],[],[4]]

PK=x^2-221  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[4],[],[4]]
ClogK2=[[4],[],[4]]
ClogK3=[[4],[],[4]]
```

```
PK=x^2-255  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[4,2],[],[4,2]]
ClogK3=[[4,2],[],[4,2]]

PK=x^2-323  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[8],[],[8]]
ClogK2=[[8],[],[8]]
ClogK3=[[8],[],[8]]

PK=x^2-357  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[4,2,2],[],[4,2,2]]
ClogK3=[[4,2,2],[],[4,2,2]]

PK=x^2-527  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[2,2,2],[],[2,2,2]]
ClogK3=[[2,2,2],[],[2,2,2]]

PK=x^2-627  ClogK0=[[2,2],[],[2,2]]  r=1
ClogK1=[[4,4],[],[4,4]]
ClogK2=[[8,8],[],[8,8]]
ClogK3=[[16,8],[],[16,8]]
No conclusion up to K3

PK=x^2-791  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[2,2,2,2],[],[2,2,2,2]]
ClogK3=[[2,2,2,2],[],[2,2,2,2]]

PK=x^2-799  ClogK0=[[4],[],[4]]  r=1
ClogK1=[[4,2],[],[4,2]]
ClogK2=[[8,2,2],[],[8,2,2]]
ClogK3=[[8,2,2],[],[8,2,2]]

PK=x^2-805  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[8,4],[],[8,4]]
ClogK3=[[8,4],[],[8,4]]

PK=x^2-1023  ClogK0=[[4],[],[4]]  r=1
ClogK1=[[8],[],[8]]
ClogK2=[[16],[],[16]]
ClogK3=[[32],[],[32]]
No conclusion up to K3

PK=x^2-1067  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[8],[],[8]]
ClogK2=[[16],[],[16]]
ClogK3=[[32],[],[32]]
No conclusion up to K3

PK=x^2-1217  ClogK0=[[8],[8],[]]  r=2
ClogK1=[[16,2],[8],[4]]
ClogK2=[[16,2],[4],[4,2]]
ClogK3=[[16,2],[2],[8,2]]

PK=x^2-1221  ClogK0=[[4],[],[4]]  r=1
ClogK1=[[8],[],[8]]
ClogK2=[[16],[],[16]]
ClogK3=[[16],[],[16]]

PK=x^2-1243  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[2,2],[],[2,2]]
ClogK2=[[8,4],[],[8,4]]
ClogK3=[[8,8],[],[8,8]]
No conclusion up to K3

PK=x^2-1245  ClogK0=[[2],[],[2]]  r=1
ClogK1=[[8],[],[8]]
ClogK2=[[16],[],[16]]
ClogK3=[[32],[],[32]]
No conclusion up to K3
```

```
PK=x^2-1249  ClogK0=[[4],[4],[]]  r=2
ClogK1=[[8],[4],[2]]
ClogK2=[[8],[2],[4]]
ClogK3=[[8],[],[8]]
PK=x^2-1254  ClogK0=[[2,2],[],[2,2]]  r=1
ClogK1=[[4,4],[],[4,4]]
ClogK2=[[8,8],[],[8,8]]
ClogK3=[[16,8],[],[16,8]]
No conclusion up to K3
```

5.4. **Conclusion.** As a conclusion, one can say, from the above examples, that the logarithmic class group of a totally real field K may capitulate in the simplest cyclic p-extensions L/K, $L \subset K(\mu_\ell)$, as for p-class groups; this was not so obvious, but in [Jaul2019[b]] is proved the existence (as for p-class groups) of abelian extensions $L_0/\mathbb{Q}$ such that $L = L_0K$ is a capitulation field for $\mathscr{H}_K^{\log}$ (some more general conditions of signature may be assumed for K).

Clearly, for imaginary quadratic fields, the fact that, probably, $\mathscr{H}_K^{\log}$ never capitulates in L seems plausible, because of a systematic non-smooth increasing complexity (p-rank and/or exponent) as shown by the following excerpt:

```
PK=x^2 + 14   ClogK0=[[3],[3],[]]   ell=109   r=1
ClogK1=[[9],[9],[]]
ClogK2=[[27],[27],[]]
PK=x^2 + 41   ClogK0=[[27],[27],[]]   ell=109   r=1
ClogK1=[[81],[81],[]]
ClogK2=[[243],[243],[]]
PK=x^2 + 74   ClogK0=[[9],[9],[]]   ell=109   r=2
ClogK1=[[27,3],[27],[3]]
ClogK2=[[81,9],[81],[9]]
PK=x^2 + 107   ClogK0=[[9],[9],[]]   ell=109   r=1
ClogK1=[[27,9,3],[27],[9,3]]
ClogK2=[[81,27,9],[81],[27,9]]
```

6. Tables for cubic fields and $p = 2$

We consider various totally ramified cyclic p-extensions L/K, where K is a cyclic cubic field and $L = KL_0$ with $L_0/\mathbb{Q}$ cyclic, especially $L_0 \subset \mathbb{Q}(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$.

6.1. **Cyclic cubic fields, $L \subset K(\mu_\ell)$, $\ell \in \{17, 97\}$.** In that examples, L_0 is the real subfield of degree 8 of $\mathbb{Q}(\mu_\ell)$. The program eliminates the cases of stability $\#\mathscr{H}_{K_1} = \#\mathscr{H}_K$ since complete capitulation holds in a suitable layer if $e(K) \leq 3$. The number vHK defines the minimal p-adic valuation of the $\#\mathscr{H}_K$'s to be considered; it may be chosen at will. The submodules $\nu_{K_n/K}(\mathscr{H}_L) = \mathbf{J}_{K_n/K}(\mathscr{H}_K)$ are computed for $n \leq 2$. The number $r \in \{1, 3\}$ is the number of prime ideals above ℓ in K.

6.1.1. *Case $\ell = 17$.* We give an excerpt of the various cases obtained (all these examples show the randomness of the structures and of the capitulations, complete or incomplete). We indicate if $\mathscr{H}_K$ capitulates in K_3 (not computed) which holds as soon as $\#\mathscr{H}_{K_2} = \#\mathscr{H}_{K_1}$ (stability from K_1) and $e(K) \leq 2$ (Theorem 1.1 (ii)):

```
MAIN PROGRAM FOR CYCLIC CUBIC FIELDS
{p=2;Nn=2;bf=7;Bf=10^4;vHK=2;ell=17;mKn=2;
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
```

```
K=bnfinit(PK,1);r=matsize(idealfactor(K,ell))[1];
\\Testing the order of the p-class group of K compared to vHK:
HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
\\Test for elimination of the stability from K:
if(n==1 & valuation(HKn,p)==valuation(HK,p),break);
if(n==1,print("f=",f," PK=",PK," CK0=",CK0[2]," ell=",ell," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
\\Search of a generator S of Gal(Kn/K):
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
\\Computation of the filtration:
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
\\Computation of the algebraic norms of the rKn generators h_i:
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
\\Reduction modulo suitable p-powers of the exponents:
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);
c=lift(Mod(c,p^w));listput(Enu,c,ii));
print("norm in K",n,"/K of the component ",i," of CK",n,":",Enu))))))}
```

```
p=2  f=607  PK=x^3+x^2-202*x-1169  CK0=[2,2]  ell=17  r=1
CK1=[2,2,2,2]
h_1^[(S-1)^1]=[1,0,0,1]   h_2^[(S-1)^1]=[0,1,1,1]
h_3^[(S-1)^1]=[1,1,1,0]   h_4^[(S-1)^1]=[1,0,0,1]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,0,0,1]
norm in K1/K of the component 2 of CK1:[0,1,1,1]
norm in K1/K of the component 3 of CK1:[1,1,1,0]
norm in K1/K of the component 4 of CK1:[1,0,0,1]
No capitulation, m(K1)=2, e(K1)=1
CK2=[2,2,2,2]
h_1^[(S-1)^1]=[1,0,1,0]   h_2^[(S-1)^1]=[0,1,0,1]
h_3^[(S-1)^1]=[1,0,1,0]   h_4^[(S-1)^1]=[0,1,0,1]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=1
```

```
p=2  f=1009  PK=x^3+x^2-336*x-1719  CK0=[2,2]  ell=17  r=1
CK1=[28,4]=[4,4]
h_1^[(S-1)^1]=[0,2]   h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,2]
norm in K1/K of the component 2 of CK1:[2,0]
No capitulation, m(K1)=2, e(K1)=2
CK2=[28,4]=[4,4]
h_1^[(S-1)^1]=[0,2]   h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=1197  PK=x^3-399*x+2926  CK0=[6,6]  ell=17  r=3
CK1=[12,12]=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
```

```
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation, m(K1)=1, e(K1)=2
CK2=[12,12]=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=1, e(K2)=2
```

```
p=2  f=1789  PK=x^3+x^2-596*x-5632  CK0=[2,2]  ell=17  r=1
CK1=[24,8]=[8,8]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[4,0]
norm in K1/K of the component 2 of CK1:[0,4]
No capitulation, m(K1)=3, e(K1)=3
CK2=[312,8]=[8,8]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=3, e(K2)=3
```

```
p=2  f=2077  PK=x^3+x^2-692*x-7231  CK0=[6,2]  ell=17  r=1
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,1,0] h_2^[(S-1)^1]=[0,0,1,1]
h_3^[(S-1)^1]=[1,1,0,1] h_4^[(S-1)^1]=[1,1,0,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,1,0]
norm in K1/K of the component 2 of CK1:[0,0,1,1]
norm in K1/K of the component 3 of CK1:[1,1,0,1]
norm in K1/K of the component 4 of CK1:[1,1,0,1]
No capitulation, m(K1)=2, e(K1)=1
CK2=[6,2,2,2,2,2]=[2,2,2,2,2,2]
h_1^[(S-1)^1]=[1,0,0,0,1,0] h_2^[(S-1)^1]=[0,1,0,0,0,1]
h_3^[(S-1)^1]=[1,1,0,1,0,1] h_4^[(S-1)^1]=[1,0,0,1,0,0]
h_5^[(S-1)^1]=[1,0,0,1,0,0] h_6^[(S-1)^1]=[0,0,1,1,0,1]
h_1^[(S-1)^2]=[0,0,0,1,1,0] h_2^[(S-1)^2]=[0,1,1,1,0,0]
h_3^[(S-1)^2]=[0,1,1,0,1,0] h_4^[(S-1)^2]=[0,0,0,1,1,0]
h_5^[(S-1)^2]=[0,0,0,1,1,0] h_6^[(S-1)^2]=[0,1,1,1,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=1
```

```
p=2  f=2817  PK=x^3-939*x+6886  CK0=[12,4]  ell=17  r=1
CK1=[84,4]=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
Incomplete capitulation, m(K1)=1, e(K1)=2
CK2=[84,4]=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=1, e(K2)=2
```

```
p=2  f=3357  PK=x^3-1119*x+9325  CK0=[6,2]  ell=17  r=3
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
```

```
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=1, e(K1)=1
CK2=[12,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=3409  PK=x^3+x^2-1136*x-10732  CK0=[6,2]  ell=17  r=3
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,1,0] h_2^[(S-1)^1]=[1,1,0,1]
h_3^[(S-1)^1]=[0,0,1,1] h_4^[(S-1)^1]=[0,0,1,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,1,0]
norm in K1/K of the component 2 of CK1:[1,1,0,1]
norm in K1/K of the component 3 of CK1:[0,0,1,1]
norm in K1/K of the component 4 of CK1:[0,0,1,1]
No capitulation, m(K1)=2, e(K1)=1
CK2=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,0,1] h_2^[(S-1)^1]=[1,1,0,1]
h_3^[(S-1)^1]=[0,0,0,1] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=1
```

```
p=2  f=5479  PK=x^3+x^2-1826*x+13799  CK0=[2,2]  ell=17  r=1
CK1=[2,2,2,2]
h_1^[(S-1)^1]=[1,0,0,1] h_2^[(S-1)^1]=[1,1,1,0]
h_3^[(S-1)^1]=[0,1,1,1] h_4^[(S-1)^1]=[1,0,0,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,0,0,1]
norm in K1/K of the component 2 of CK1:[1,1,1,0]
norm in K1/K of the component 3 of CK1:[0,1,1,1]
norm in K1/K of the component 4 of CK1:[1,0,0,1]
No capitulation, m(K1)=2, e(K1)=1
CK2=[4,4,4,4]
h_1^[(S-1)^1]=[3,1,3,1] h_2^[(S-1)^1]=[3,3,0,3]
h_3^[(S-1)^1]=[0,2,2,2] h_4^[(S-1)^1]=[0,2,3,0]
h_1^[(S-1)^2]=[0,2,2,0] h_2^[(S-1)^2]=[2,2,2,0]
h_3^[(S-1)^2]=[2,2,2,2] h_4^[(S-1)^2]=[2,0,2,0]
norm in K2/K of the component 1 of CK2:[0,0,2,0]
norm in K2/K of the component 2 of CK2:[2,2,2,2]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[2,2,0,2]
No capitulation, m(K2)=4, e(K2)=2
```

```
p=2  f=6247  PK=x^3+x^2-2082*x-35631  CK0=[4,4]  ell=17  r=1
CK1=[24,8,2,2]=[8,8,2,2]
h_1^[(S-1)^1]=[0,2,0,1] h_2^[(S-1)^1]=[6,6,1,0]
h_3^[(S-1)^1]=[0,4,0,0] h_4^[(S-1)^1]=[4,0,0,0]
h_1^[(S-1)^2]=[0,4,0,0] h_2^[(S-1)^2]=[4,4,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
```

```
norm in K1/K of the component 1 of CK1:[2,2,0,1]
norm in K1/K of the component 2 of CK1:[6,0,1,0]
norm in K1/K of the component 3 of CK1:[0,4,0,0]
norm in K1/K of the component 4 of CK1:[4,0,0,0]
No capitulation, m(K1)=4, e(K1)=3
CK2=[24,8,2,2]=[8,8,2,2]
h_1^[(S-1)^1]=[0,6,1,1] h_2^[(S-1)^1]=[2,6,0,1]
h_3^[(S-1)^1]=[4,4,0,0] h_4^[(S-1)^1]=[0,4,0,0]
h_1^[(S-1)^2]=[0,4,0,0] h_2^[(S-1)^2]=[4,4,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,4,0,0]
norm in K2/K of the component 2 of CK2:[4,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=3, e(K2)=3
Complete capitulation in K3 (stability from K1)
```

```
p=2  f=7687  PK=x^3+x^2-2562*x-48969  CK0=[2,2,2,2]  ell=17  r=1
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0]
Incomplete capitulation,  m(K1)=2, e(K1)=2
CK2=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0] h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=8629  PK=x^3+x^2-2876*x-50176  CK0=[14,2]  ell=17  r=1
CK1=[56,8]=[8,8]
h_1^[(S-1)^1]=[2,4]   h_2^[(S-1)^1]=[4,6]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[4,4]
norm in K1/K of the component 2 of CK1:[4,0]
No capitulation, m(K1)=3, e(K1)=3
CK2=[112,16]=[16,16]
h_1^[(S-1)^1]=[6,4]   h_2^[(S-1)^1]=[12,10]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[0,8]
norm in K2/K of the component 2 of CK2:[8,8]
No capitulation, m(K2)=4, e(K2)=4
```

```
p=2  f=9247  PK=x^3+x^2-3082*x-27056  CK0=[12,4]  ell=17  r=3
CK1=[24,8]=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation, m(K1)=1, e(K1)=3
CK2=[48,16]=[16,16]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
No capitulation, m(K2)=1, e(K2)=4
```

```
p=2  f=9283  PK=x^3+x^2-3094*x-5501  CK0=[2,2]  ell=17  r=1
CK1=[48,16]=[16,16]
```

```
h_1^[(S-1)^1]=[6,0]   h_2^[(S-1)^1]=[0,6]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[8,0]
norm in K1/K of the component 2 of CK1:[0,8]
No capitulation, m(K1)=4, e(K1)=4
CK2=[48,16]=[16,16]
h_1^[(S-1)^1]=[6,0]   h_2^[(S-1)^1]=[0,6]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=4, e(K2)=4
(...)
p=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]  ell=17  r=3
CK1=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=1, e(K1)=3
CK2=[16,16,2,2]
h_1^[(S-1)^1]=[8,0,0,0] h_2^[(S-1)^1]=[0,8,0,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=2, e(K2)=4
p=2  f=25119  PK=x^3-8373*x+2791  CK0=[12,4]  ell=17  r=1
CK1=[12,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,1] h_2^[(S-1)^1]=[0,0,1,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,1]
norm in K1/K of the component 2 of CK1:[0,2,1,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[24,8,2,2,2,2]=[8,8,2,2,2,2]
h_1^[(S-1)^1]=[6,4,0,0,1,1] h_2^[(S-1)^1]=[4,6,1,0,0,1]
h_3^[(S-1)^1]=[0,4,0,0,0,0] h_4^[(S-1)^1]=[4,4,0,1,1,0]
h_5^[(S-1)^1]=[0,0,0,1,1,0] h_6^[(S-1)^1]=[0,4,1,0,0,0]
h_1^[(S-1)^2]=[4,4,1,1,1,0] h_2^[(S-1)^2]=[0,4,1,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0] h_4^[(S-1)^2]=[4,4,0,0,0,0]
h_5^[(S-1)^2]=[4,4,0,0,0,0] h_6^[(S-1)^2]=[0,4,0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Incomplete capitulation, m(K2)=4, e(K2)=3
p=2  f=29467  PK=x^3+x^2-9822*x-20736  CK0=[84,4]  ell=17  r=3
CK1=[168,8,2,2]=[8,8,2,2]
h_1^[(S-1)^1]=[4,0,1,0] h_2^[(S-1)^1]=[4,4,1,1]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[6,0,1,0]
norm in K1/K of the component 2 of CK1:[4,6,1,1]
```

```
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
No capitulation, m(K1)=2, e(K1)=3
CK2=[336,16,2,2]=[16,16,2,2]
h_1^[(S-1)^1]=[0,8,1,0] h_2^[(S-1)^1]=[8,8,0,1]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
No capitulation, m(K2)=2, e(K2)=4
```

```
p=2  f=33061  PK=x^3+x^2-11020*x-262039  CK0=[6,2,2,2]  ell=17  r=1
CK1=[12,4,2,2,2,2]=[4,4,2,2,2,2]
h_1^[(S-1)^1]=[0,0,1,0,1,1] h_2^[(S-1)^1]=[0,2,1,1,0,1]
h_3^[(S-1)^1]=[2,0,0,0,0,0] h_4^[(S-1)^1]=[0,2,0,0,0,0]
h_5^[(S-1)^1]=[0,2,0,0,0,0] h_6^[(S-1)^1]=[2,2,0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0] h_6^[(S-1)^2]=[0,0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,1,0,1,1]
norm in K1/K of the component 2 of CK1:[0,0,1,1,0,1]
norm in K1/K of the component 3 of CK1:[2,0,0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0,0,0]
norm in K1/K of the component 5 of CK1:[0,2,0,0,0,0]
norm in K1/K of the component 6 of CK1:[2,2,0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[24,8,4,4,2,2]=[8,8,4,4,2,2]
h_1^[(S-1)^1]=[6,4,2,2,0,1] h_2^[(S-1)^1]=[6,0,2,0,1,0]
h_3^[(S-1)^1]=[2,2,0,2,0,0] h_4^[(S-1)^1]=[2,0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,2,2,0,0] h_6^[(S-1)^1]=[0,0,0,2,0,0]
h_1^[(S-1)^2]=[4,4,0,2,0,0] h_2^[(S-1)^2]=[0,4,2,2,0,0]
h_3^[(S-1)^2]=[4,0,0,0,0,0] h_4^[(S-1)^2]=[4,0,0,0,0,0]
h_5^[(S-1)^2]=[0,4,0,0,0,0] h_6^[(S-1)^2]=[4,0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[4,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[4,4,0,0,0,0]
norm in K2/K of the component 4 of CK2:[4,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Incomplete capitulation, m(K2)=4, e(K2)=3
```

```
p=2  f=37087  PK=x^3+x^2-12362*x-401089  CK0=[2,2,2,2]  ell=17  r=3
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=1, e(K1)=2
CK2=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=1, e(K2)=3
```

```
p=2  f=44857  PK=x^3+x^2-14952*x-704421  CK0=[6,2,2,2]  ell=17  r=3
CK1=[12,12,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[2,2,0,0]
```

```
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[2,2,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[12,12,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[2,2,0,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=48769  PK=x^3+x^2-16256*x-7225  CK0=[24,8]  ell=17  r=3
CK1=[48,16]=[16,16]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation,  m(K1)=1, e(K1)=4
CK2=[48,16]=[16,16]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Incomplete capitulation, m(K2)=1, e(K2)=4
```

```
p=2  f=55609  PK=x^3+x^2-18536*x-823837  CK0=[4,4,2,2]  ell=17  r=3
CK1=[56,8,2,2]=[8,8,2,2]
h_1^[(S-1)^1]=[4,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[4,0,0,0] h_4^[(S-1)^1]=[0,4,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[6,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[4,0,0,0]
norm in K1/K of the component 4 of CK1:[0,4,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=3
CK2=[56,8,2,2]=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,4,0,0] h_4^[(S-1)^1]=[4,4,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=3
```

Let's give some comments on interesting examples found above:

```
p=2  f=9283  PK=x^3+x^2-3094*x-5501  CK0=[2,2]  ell=17  r=1
CK1=[48,16]=[16,16]
h_1^[(S-1)^1]=[6,0]  h_2^[(S-1)^1]=[0,6]
h_1^[(S-1)^2]=[4,0]  h_2^[(S-1)^2]=[0,4]
h_1^[(S-1)^3]=[8,0]  h_2^[(S-1)^3]=[0,8]
h_1^[(S-1)^4]=[0,0]  h_2^[(S-1)^4]=[0,0]
norm in K1/K of the component 1 of CK1:[8,0]
norm in K1/K of the component 2 of CK1:[0,8]
No capitulation, m(K1)=4, e(K1)=4
CK2=[48,16]=[16,16]
h_1^[(S-1)^1]=[6,0]  h_2^[(S-1)^1]=[0,6]
h_1^[(S-1)^2]=[4,0]  h_2^[(S-1)^2]=[0,4]
h_1^[(S-1)^3]=[8,0]  h_2^[(S-1)^3]=[0,8]
```

```
h_1^[(S-1)^4]=[0,0]  h_2^[(S-1)^4]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=4, e(K2)=4
```

There is complete capitulation, even if conditions of Theorem 1.1 (i) are not fulfilled for the K_n/K's (for $n = 2$, $m(K_2) = 4$, $s(K_2) = 2$, $e(K_2) = 4$, $n-s(K_2) = 0$). Moreover, the exponent of $\mathscr{H}_{K_1}$ is 2^4 giving a larger complexity in K_1/K, but in K_n, $n \geq 2$, the exponent is still 2^4 (no increasing of the complexity). Some other examples are:

```
p=2  f=7687  PK=x^3+x^2-2562*x-48969  CK0=[2,2,2,2]  ell=17  r=1
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0] h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
p=2  f=44857  PK=x^3+x^2-14952*x-704421  CK0=[6,2,2,2]  ell=17  r=3
CK1=[12,12,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[2,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[2,2,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[12,12,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[2,2,0,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

they suggest that the size of the p-rank is not an obstruction to a capitulation in such cyclic sub-p-extensions of $K(\mu_\ell)$; here, the 2-ranks are even because of the structure of $\mathbf{Z}$-module of the $\mathscr{H}_{K_n}$'s. In the above cases, the capitulation is obtained by means of a stability in larger layers.

6.1.2. *Case* $\ell = 97$. Similarly, we give a table for $\ell = 97$ allowing capitulations up to K_4. One finds much more cases of capitulation (not in the table below since they are very numerous); it seems clearly that a larger value of N intervenes in the phenomenon of capitulation:

```
p=2  f=349  PK=x^3+x^2-116*x-517  CK0=[2,2]  ell=97  r=1
```

```
CK1=[4,4]
h_1^[(S-1)^1]=[2,2]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,2]
norm in K1/K of the component 2 of CK1:[2,2]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[4,4]
h_1^[(S-1)^1]=[2,2]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=547  PK=x^3+x^2-182*x-81  CK0=[2,2]  ell=97  r=1
CK1=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,1,0]   h_2^[(S-1)^1]=[0,1,0,1]
h_3^[(S-1)^1]=[1,0,1,1]   h_4^[(S-1)^1]=[0,1,0,1]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,1,0]
norm in K1/K of the component 2 of CK1:[0,1,0,1]
norm in K1/K of the component 3 of CK1:[1,0,1,1]
norm in K1/K of the component 4 of CK1:[0,1,0,1]
Incomplete capitulation, m(K1)=2, e(K1)=1
CK2=[2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,1,0,0]   h_4^[(S-1)^1]=[1,1,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=1
```

```
p=2  f=607  PK=x^3+x^2-202*x-1169  CK0=[2,2]  ell=97  r=1
CK1=[8,8]
h_1^[(S-1)^1]=[6,4]   h_2^[(S-1)^1]=[4,2]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[0,4]
norm in K1/K of the component 2 of CK1:[4,4]
Incomplete capitulation, m(K1)=3, e(K1)=3
CK2=[104,8]=[8,8]
h_1^[(S-1)^1]=[6,4]   h_2^[(S-1)^1]=[4,2]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=3, e(K2)=3
```

```
p=2  f=1957  PK=x^3+x^2-652*x+6016  CK0=[6,2]  ell=97  r=3
CK1=[12,4]=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation, m(K1)=1, e(K1)=2
CK2=[24,8]=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
No capitulation, m(K2)=1, e(K2)=3
```

```
p=2  f=4207  PK=x^3+x^2-1402*x+14335  CK0=[6,2]  ell=97  r=3
CK1=[12,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0]   h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[0,2,0,0]   h_4^[(S-1)^1]=[2,2,0,0]
```

```
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,0]
norm in K1/K of the component 2 of CK1:[2,2,0,0]
norm in K1/K of the component 3 of CK1:[0,2,0,0]
norm in K1/K of the component 4 of CK1:[2,2,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=32
CK2=[12,4,4,4]=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,0,0]   h_2^[(S-1)^1]=[2,2,0,0]
h_3^[(S-1)^1]=[0,0,0,2]   h_4^[(S-1)^1]=[0,0,2,2]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=4639  PK=x^3+x^2-1546*x+6529  CK0=[2,2]  ell=97  r=1
CK1=[4,4]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=2, e(K1)=2
CK2=[4,4]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=9391  PK=x^3+x^2-3130*x-24347  CK0=[2,2]  ell=97  r=3
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0]   h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[2,0,0,0]   h_4^[(S-1)^1]=[2,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[2,2,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[4,4,4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,2,0,1]   h_2^[(S-1)^1]=[0,0,2,0,1,0]
h_3^[(S-1)^1]=[2,0,2,0,1,0]   h_4^[(S-1)^1]=[0,0,2,0,0,1]
h_5^[(S-1)^1]=[0,2,2,0,0,0]   h_6^[(S-1)^1]=[2,0,0,2,0,0]
h_1^[(S-1)^2]=[2,0,0,2,0,0]   h_2^[(S-1)^2]=[0,2,2,0,0,0]
h_3^[(S-1)^2]=[0,2,2,0,0,0]   h_4^[(S-1)^2]=[2,0,0,2,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0]   h_6^[(S-1)^2]=[0,0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2
```

```
p=2  f=10513  PK=x^3+x^2-3504*x-80989  CK0=[8,8]  ell=97  r=3
CK1=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
Incomplete capitulation, m(K1)=1, e(K1)=3
CK2=[8,8]
```

```
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Incomplete capitulation, m(K2)=1, e(K2)=3
Complete capitulation in K3 (stability from K)
```

```
p=2  f=11149  PK=x^3+x^2-3716*x+39228  CK0=[2,2]  ell=97  r=3
CK1=[12,4]=[4,4]
h_1^[(S-1)^1]=[2,2]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,2]
norm in K1/K of the component 2 of CK1:[2,2]
No capitulation, m(K1)=2, e(K1)=2
CK2=[12,4]=[4,4]
h_1^[(S-1)^1]=[0,2]   h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=15823  PK=x^3+x^2-5274*x+141821  CK0=[4,4]  ell=97  r=3
CK1=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
Incomplete capitulation, m(K1)=1, e(K1)=2
CK2=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=1, e(K2)=2
```

```
p=2  f=19177  PK=x^3+x^2-6392*x-79549  CK0=[6,2]  ell=97  r=3
CK1=[24,8,2,2]=[8,8,2,2]
h_1^[(S-1)^1]=[6,4,0,1]   h_2^[(S-1)^1]=[0,6,1,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[4,0,0,0]   h_2^[(S-1)^2]=[0,4,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,4,0,1]
norm in K1/K of the component 2 of CK1:[0,0,1,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
No capitulation, m(K1)=3, e(K1)=3
CK2=[336,112,4,4]=[16,16,4,4]
h_1^[(S-1)^1]=[2,4,1,2]   h_2^[(S-1)^1]=[12,6,1,1]
h_3^[(S-1)^1]=[0,8,0,2]   h_4^[(S-1)^1]=[8,0,2,2]
h_1^[(S-1)^2]=[4,8,2,2]   h_2^[(S-1)^2]=[8,12,0,2]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[8,8,2,0]
norm in K2/K of the component 2 of CK2:[8,0,2,2]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
No capitulation, m(K2)=4, e(K2)=2
```

```
p=2  f=20419  PK=x^3+x^2-6806*x-3025  CK0=[42,2]  ell=97  r=3
CK1=[84,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0]   h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0]
```

```
No capitulation, m(K1)=2, e(K1)=2
CK2=[84,4,4,4,2,2]=[4,4,4,4,2,2]
h_1^[(S-1)^1]=[0,0,2,0,1,0]   h_2^[(S-1)^1]=[0,2,0,2,1,0]
h_3^[(S-1)^1]=[2,2,2,2,0,0]   h_4^[(S-1)^1]=[0,2,2,2,1,1]
h_5^[(S-1)^1]=[2,0,0,2,0,0]   h_6^[(S-1)^1]=[0,0,2,2,0,0]
h_1^[(S-1)^2]=[2,0,0,2,0,0]   h_2^[(S-1)^2]=[2,0,0,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0]   h_4^[(S-1)^2]=[2,0,2,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0]   h_6^[(S-1)^2]=[0,0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2

p=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]  ell=97  r=3
CK1=[8,8,2,2]
h_1^[(S-1)^1]=[4,4,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[4,4,0,0]   h_4^[(S-1)^1]=[4,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[6,4,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[4,4,0,0]
norm in K1/K of the component 4 of CK1:[4,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=3
CK2=[8,8,2,2]
h_1^[(S-1)^1]=[0,4,0,0]   h_2^[(S-1)^1]=[4,4,0,0]
h_3^[(S-1)^1]=[4,0,0,0]   h_4^[(S-1)^1]=[4,4,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=2, e(K2)=3
Complete capitulation in K3  (stability from K1)

p=2  f=21931  PK=x^3+x^2-7310*x-3249  CK0=[12,12]  ell=97  r=3
CK1=[24,24]=[8,8]
h_1^[(S-1)^1]=[4,4]   h_2^[(S-1)^1]=[4,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[6,4]
norm in K1/K of the component 2 of CK1:[4,2]
No capitulation, m(K1)=2, e(K1)=3
CK2=[24,24]=[8,8]
h_1^[(S-1)^1]=[0,4]   h_2^[(S-1)^1]=[4,4]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Incomplete capitulation, m(K2)=2, e(K2)=3
Complete capitulation in K3 (stability from K1)

p=2  f=24589  PK=x^3+x^2-8196*x-33696  CK0=[6,2]  ell=97  r=3
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,0,0,1]   h_2^[(S-1)^1]=[0,1,1,0]
h_3^[(S-1)^1]=[0,1,1,0]   h_4^[(S-1)^1]=[1,0,0,1]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,0,0,1]
norm in K1/K of the component 2 of CK1:[0,1,1,0]
norm in K1/K of the component 3 of CK1:[0,1,1,0]
norm in K1/K of the component 4 of CK1:[1,0,0,1]
No capitulation, m(K1)=2, e(K1)=1
CK2=[6,2,2,2,2,2]=[2,2,2,2,2,2]
```

```
h_1^[(S-1)^1]=[0,0,0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0,0,0]
h_3^[(S-1)^1]=[1,1,0,1,0,0]   h_4^[(S-1)^1]=[1,1,0,0,0,0]
h_5^[(S-1)^1]=[0,1,0,1,1,1]   h_6^[(S-1)^1]=[1,1,0,1,1,1]
h_1^[(S-1)^2]=[0,0,0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0,0,0]
h_3^[(S-1)^2]=[1,1,0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0,0,0]
h_5^[(S-1)^2]=[0,1,0,0,0,0]   h_6^[(S-1)^2]=[0,1,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=1
p=2  f=25171  PK=x^3+x^2-8390*x+273152  CK0=[14,2]  ell=97  r=3
CK1=[84,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0]   h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=2, e(K1)=2
CK2=[84,4,4,4]=[4,4,4,4]
h_1^[(S-1)^1]=[0,0,2,0]   h_2^[(S-1)^1]=[0,0,2,0]
h_3^[(S-1)^1]=[0,0,2,0]   h_4^[(S-1)^1]=[2,0,2,2]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

6.2. **Cyclic cubic fields, $L \subset K(\mu_{17\cdot 97})$.** We use the same program with suitable defining polynomials, of degrees 2 and 4, respectively, given in the list $\mathsf{L4} = [x^2 - x - 412, x^4 - x^3 - 618 * x^2 + 1752 * x + 8960]$, so that $L_0/\mathbb{Q}$ be cyclic with total ramification of 17 and 97. Since two primes ramify in K_n/K, the factor norm is in general non trivial, which gives larger 2-class groups with $\mathscr{H}_{L_0} \neq 1$. The number r of ramified primes is equal to 2, 4 or 6 depending on the splitting of 17 and 97; as we have explained, capitulation is favored as soon as the complexity is smooth (in the meaning of Definition 2.13).

```
{p=2;Nn=2;bf=7;Bf=10^6;vHK=4;mKn=2;L4=List;
L4=[x^2-x-412,x^4-x^3-618*x^2+1752*x+8960];
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);K=bnfinit(PK,1);
HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
r=matsize(idealfactor(K,17))[1]+matsize(idealfactor(K,97))[1];
for(n=1,Nn,Qn=L4[n];Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);
HKn=Kn.no;dn=poldegree(Pn);if(n==1 & valuation(HKn,p)==vHK,break);
if(n==1,print();print("f=",f," PK=",PK," CK0=",CK0[2]," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
```

```
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))))}
```

```
p=2  f=1777  PK=x^3+x^2-592*x+724  CK0=[4,4]  r=6
CK1=[8,8,2]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
No capitulation, m(K1)=1, e(K1)=3
CK2=[16,16,4]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
No capitulation, m(K2)=1, e(K2)=4
```

```
p=2  f=2817  PK=x^3-939*x+6886  CK0=[12,4]  r=2
CK1=[444,4,2,2,2]=[4,4,2,2,2]
h_1^[(S-1)^1]=[2,0,1,0,1] h_2^[(S-1)^1]=[0,2,1,1,0]
h_3^[(S-1)^1]=[0,0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,1,0,1]
norm in K1/K of the component 2 of CK1:[0,0,1,1,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0,0]
norm in K1/K of the component 5 of CK1:[0,0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[888,8,4,2,2]=[8,8,4,2,2]
h_1^[(S-1)^1]=[2,4,2,1,0] h_2^[(S-1)^1]=[0,2,2,0,1]
h_3^[(S-1)^1]=[0,4,0,0,0] h_4^[(S-1)^1]=[4,4,0,0,0]
h_5^[(S-1)^1]=[4,0,0,0,0]
h_1^[(S-1)^2]=[0,4,0,0,0] h_2^[(S-1)^2]=[4,4,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=3
```

```
p=2  f=4297  PK=x^3+x^2-1432*x+20371  CK0=[4,4]  r=2
CK1=[4,4,2]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=1, e(K1)=2
CK2=[4,4,4]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Complete capitulation, m(K2)=1, e(K2)=2
```

```
p=2  f=7687  PK=x^3+x^2-2562*x-48969  CK0=[2,2,2,2]  r=2
CK1=[4,4,2,2,2]
h_1^[(S-1)^1]=[2,2,0,0,0] h_2^[(S-1)^1]=[0,0,0,0,0]
h_3^[(S-1)^1]=[2,2,0,0,0] h_4^[(S-1)^1]=[2,0,0,0,0]
```

```
h_5^[(S-1)^1]=[2,2,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0,0]
norm in K1/K of the component 3 of CK1:[2,2,0,0,0]
norm in K1/K of the component 4 of CK1:[2,0,0,0,0]
norm in K1/K of the component 5 of CK1:[2,2,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[4,4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0,0] h_2^[(S-1)^1]=[2,0,0,0,0]
h_3^[(S-1)^1]=[2,0,2,0,0] h_4^[(S-1)^1]=[0,0,2,0,0]
h_5^[(S-1)^1]=[2,0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2

p=2  f=10513  PK=x^3+x^2-3504*x-80989  CK0=[8,8]  r=4
CK1=[8,8,2]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=1, e(K1)=3
CK2=[8,8,4]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Incomplete capitulation, m(K2)=1, e(K2)=3

p=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]  r=6
CK1=[8,8,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0,0]
norm in K1/K of the component 5 of CK1:[0,0,0,0,0]
Incomplete capitulation, m(K1)=1, e(K1)=3
CK2=[16,16,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0]
Incomplete capitulation, m(K2)=1, e(K2)=4

p=2  f=21931  PK=x^3+x^2-7310*x-3249  CK0=[12,12]  r=6
CK1=[12,12,2,2,2,2,2]=[4,4,2,2,2,2,2]
h_1^[(S-1)^1]=[2,0,0,1,0,0,0] h_2^[(S-1)^1]=[0,2,0,0,1,0,0]
h_3^[(S-1)^1]=[0,0,0,0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0,0,0] h_6^[(S-1)^1]=[0,0,0,0,0,0,0]
h_7^[(S-1)^1]=[0,0,0,0,0,0,0]
```

```
h_1^[(S-1)^2]=[0,0,0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0,0] h_6^[(S-1)^2]=[0,0,0,0,0,0,0]
h_7^[(S-1)^2]=[0,0,0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,1,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0,1,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0,0,0,0]
norm in K1/K of the component 5 of CK1:[0,0,0,0,0,0,0]
norm in K1/K of the component 6 of CK1:[0,0,0,0,0,0,0]
norm in K1/K of the component 7 of CK1:[0,0,0,0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[24,24,4,4,4,2,2]=[8,8,4,4,4,2,2]
h_1^[(S-1)^1]=[6,0,3,0,3,1,1] h_2^[(S-1)^1]=[2,0,2,3,1,1,0]
h_3^[(S-1)^1]=[4,0,0,2,2,0,0] h_4^[(S-1)^1]=[0,0,0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0,0,0] h_6^[(S-1)^1]=[4,0,2,0,2,0,0]
h_7^[(S-1)^1]=[4,0,0,2,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0,0] h_6^[(S-1)^2]=[0,0,0,0,0,0,0]
h_7^[(S-1)^2]=[0,0,0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,2,0,2,0,0]
norm in K2/K of the component 2 of CK2:[4,4,0,2,2,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0,0]
norm in K2/K of the component 7 of CK2:[0,0,0,0,0,0,0]
Incomplete capitulation, m(K2)=2, e(K2)=3
```

6.3. **Cyclic cubic fields, $L = K(\sqrt{2+\sqrt{2}})$.** In that case, L is a subfield of the cyclotomic $\mathbb{Z}_2$-extension of K and this leads to a checking of Greenberg's conjecture, equivalent to the stability from some layer, but capitulation of $\mathscr{H}_K$ may occur before, as shown by some examples of the following list. The program is unchanged, except that the cyclotomic polynomials $\mathsf{Qn} = \mathsf{polsubcyclo}(\mathsf{ell}, \mathsf{p}^{\mathsf{n}})$ are replaced by $\mathsf{Q1} = \mathsf{x}^2 - 2$ and $\mathsf{Q2} = \mathsf{x}^4 - 4 * \mathsf{x}^2 + 2$; we find very often cases of capitulation from stability at some level, noting that "N" is unlimited; so we only write few examples of the various structures obtained:

```
{p=2;Nn=2;bf=7;Bf=10^6;vHK=2;mKn=2;
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
K=bnfinit(PK,1);r=matsize(idealfactor(K,p))[1];HK=K.no;
if(valuation(HK,p)<vHK,next);CK0=K.clgp;for(n=1,Nn,Qn=x;
for(i=1,n,Qn=Qn^2-2);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
if(n==1,print();print("f=",f," PK=",PK," CK0=",CK0[2]," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))))}
```

```
p=2  f=1879  PK=x^3+x^2-626*x-5289  CK0=[2,2]  r=1
CK1=[6,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0]  h_2^[(S-1)^1]=[1,0,0,0]
h_3^[(S-1)^1]=[1,1,1,1]  h_4^[(S-1)^1]=[0,1,1,1]
h_1^[(S-1)^2]=[0,0,0,0]  h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]  h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[1,0,0,0]
norm in K1/K of the component 3 of CK1:[1,1,1,1]
norm in K1/K of the component 4 of CK1:[0,1,1,1]
No capitulation, m(K1)=2, e(K1)=2
CK2=[6,2,2,2,2,2]=[2,2,2,2,2,2]
h_1^[(S-1)^1]=[0,1,0,0,0,1]  h_2^[(S-1)^1]=[0,1,1,0,1,0]
h_3^[(S-1)^1]=[1,1,0,1,0,1]  h_4^[(S-1)^1]=[0,1,0,0,0,1]
h_5^[(S-1)^1]=[1,1,0,0,1,0]  h_6^[(S-1)^1]=[1,1,1,1,1,0]
h_1^[(S-1)^2]=[1,0,0,1,0,0]  h_2^[(S-1)^2]=[0,1,1,1,0,1]
h_3^[(S-1)^2]=[1,0,0,1,0,0]  h_4^[(S-1)^2]=[1,0,0,1,0,0]
h_5^[(S-1)^2]=[1,1,1,0,0,1]  h_6^[(S-1)^2]=[0,1,1,1,0,1]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=1

p=2  f=6381  PK=x^3-2127*x+21979  CK0=[6,2]  r=1
CK1=[24,8]=[8,8]
h_1^[(S-1)^1]=[2,4]  h_2^[(S-1)^1]=[4,6]
h_1^[(S-1)^2]=[4,0]  h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[4,4]
norm in K1/K of the component 2 of CK1:[4,0]
No capitulation, m(K1)=3, e(K1)=3
CK2=[48,16]=[16,16]
h_1^[(S-1)^1]=[10,12]  h_2^[(S-1)^1]=[12,14]
h_1^[(S-1)^2]=[4,0]    h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[8,8]
norm in K2/K of the component 2 of CK2:[8,0]
No capitulation, m(K2)=4, e(K2)=4

p=2  f=11899  PK=x^3+x^2-3966*x+19391  CK0=[6,2]  r=1
CK1=[24,8]=[8,8]
h_1^[(S-1)^1]=[6,4]  h_2^[(S-1)^1]=[4,2]
h_1^[(S-1)^2]=[4,0]  h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[0,4]
norm in K1/K of the component 2 of CK1:[4,4]
No capitulation, m(K1)=3, e(K1)=3
CK2=[24,8]=[8,8]
h_1^[(S-1)^1]=[6,4]  h_2^[(S-1)^1]=[4,2]
h_1^[(S-1)^2]=[4,0]  h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=3, e(K2)=3

p=2  f=13531  PK=x^3+x^2-4510*x-98225  CK0=[6,2]  r=1
CK1=[78,2,2,2]=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,1,1]  h_2^[(S-1)^1]=[1,1,1,1]
h_3^[(S-1)^1]=[1,1,0,0]  h_4^[(S-1)^1]=[1,1,0,0]
h_1^[(S-1)^2]=[0,0,0,0]  h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]  h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,1,1]
norm in K1/K of the component 2 of CK1:[1,1,1,1]
norm in K1/K of the component 3 of CK1:[1,1,0,0]
norm in K1/K of the component 4 of CK1:[1,1,0,0]
No capitulation, m(K1)=2, e(K1)=1
CK2=[156,4,2,2]=[4,4,2,2]
```

```
h_1^[(S-1)^1]=[2,0,1,1]  h_2^[(S-1)^1]=[0,0,1,0]
h_3^[(S-1)^1]=[0,2,0,0]  h_4^[(S-1)^1]=[2,2,0,0]
h_1^[(S-1)^2]=[2,0,0,0]  h_2^[(S-1)^2]=[0,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0]  h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2
```

```
p=2  f=1777  PK=x^3+x^2-592*x+724  CK0=[4,4]  r=3
CK1=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
Incomplete capitulation, m(K1)=1, e(K1)=2
CK2=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=1, e(K2)=2
```

```
p=2  f=7687  PK=x^3+x^2-2562*x-48969  CK0=[2,2,2,2]  r=1
CK1=[12,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0]     h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]     h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]     h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]     h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,0]
norm in K1/K of the component 2 of CK1:[2,2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[24,8,2,2]=[8,8,2,2]
h_1^[(S-1)^1]=[0,6,0,0]   h_2^[(S-1)^1]=[6,2,0,0]
h_3^[(S-1)^1]=[0,4,0,0]   h_4^[(S-1)^1]=[4,0,0,0]
h_1^[(S-1)^2]=[4,4,0,0]   h_2^[(S-1)^2]=[4,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,4,0,0]
norm in K2/K of the component 2 of CK2:[4,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=3, e(K2)=3
```

```
p=2  f=10513  PK=x^3+x^2-3504*x-80989  CK0=[8,8]  r=1
CK1=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0])
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
Incomplete capitulation, m(K1)=1, e(K1)=2
CK2=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Incomplete capitulation, m(K2)=1, e(K2)=3
Complete capitulation in K3 (stability from K)
```

```
p=2  f=16363  PK=x^3+x^2-5454*x-16969  CK0=[2,2,2,2]  r=1
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0]    h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0]    h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,0]
norm in K1/K of the component 2 of CK1:[2,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0]
```

```
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,0,0]    h_2^[(S-1)^1]=[2,2,0,0]
h_3^[(S-1)^1]=[2,2,0,0]    h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=20599  PK=x^3+x^2-6866*x+216671  CK0=[28,4]  r=1
CK1=[28,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,1]    h_2^[(S-1)^1]=[0,2,1,1]
h_3^[(S-1)^1]=[0,0,0,0]    h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,1]
norm in K1/K of the component 2 of CK1:[0,0,1,1]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[56,8,2,2]=[8,8,2,2]
h_1^[(S-1)^1]=[0,6,0,1]    h_2^[(S-1)^1]=[2,6,1,1]
h_3^[(S-1)^1]=[0,0,0,0]    h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[4,4,0,0]    h_2^[(S-1)^2]=[4,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,4,0,0]
norm in K2/K of the component 2 of CK2:[4,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=3, e(K2)=3
```

```
p=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]  r=1
CK1=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0]    h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,4,0,0]    h_4^[(S-1)^1]=[4,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[0,4,0,0]
norm in K1/K of the component 4 of CK1:[4,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=3
CK2=[8,8,2,2]
h_1^[(S-1)^1]=[0,4,0,0]    h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,4,0,0]    h_4^[(S-1)^1]=[4,4,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=2, e(K2)=3
Complete capitulation in K3 (stability from K1)
```

```
p=2  f=31513  PK=x^3+x^2-10504*x-417839  CK0=[28,4]  r=1
CK1=[84,4,2,2]=[4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,1]    h_2^[(S-1)^1]=[0,2,1,1]
h_3^[(S-1)^1]=[0,0,0,0]    h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,1]
norm in K1/K of the component 2 of CK1:[0,0,1,1]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
```

```
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[168,8,2,2,2,2]=[8,8,2,2,2,2]
h_1^[(S-1)^1]=[2,0,1,0,1,1] h_2^[(S-1)^1]=[0,6,0,0,0,1]
h_3^[(S-1)^1]=[0,0,1,0,0,1] h_4^[(S-1)^1]=[4,0,0,1,1,0]
h_5^[(S-1)^1]=[4,4,0,1,1,0] h_6^[(S-1)^1]=[4,4,1,0,0,1]
h_1^[(S-1)^2]=[4,0,0,1,1,0] h_2^[(S-1)^2]=[4,0,1,0,0,1]
h_3^[(S-1)^2]=[4,4,0,0,0,0] h_4^[(S-1)^2]=[0,4,0,0,0,0]
h_5^[(S-1)^2]=[0,4,0,0,0,0] h_6^[(S-1)^2]=[4,4,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,4,0,0,0,0]
norm in K2/K of the component 2 of CK2:[4,4,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Incomplete capitulation, m(K2)=4, e(K2)=3
p=2  f=35353  PK=x^3+x^2-11784*x+45828  CK0=[2,2]  r=3
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0]  h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[0,2,0,0]  h_4^[(S-1)^1]=[2,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0]  h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]  h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,0]
norm in K1/K of the component 2 of CK1:[2,2,0,0]
norm in K1/K of the component 3 of CK1:[0,2,0,0]
norm in K1/K of the component 4 of CK1:[2,2,0,0]
No capitulation, m(K1)=2, e(K1)=2
CK2=[4,4,4,4]
h_1^[(S-1)^1]=[0,2,0,0]  h_2^[(S-1)^1]=[2,2,0,0]
h_3^[(S-1)^1]=[0,2,2,2]  h_4^[(S-1)^1]=[2,2,2,0]
h_1^[(S-1)^2]=[0,0,0,0]  h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]  h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

6.4. **Statistics varying the primes ℓ.** In this subsection, we fix a cyclic cubic field and consider the primes $\ell \equiv 1 \pmod{2p^N}$ for $\mathsf{N} \geq \mathsf{Nell}$:

```
{p=2;Nell=3;Nn=2;f=20887;PK=x^3+x^2-6962*x-225889;K=bnfinit(PK,1);
CK0=K.clgp;forprime(ell=1,10^3,N=valuation(ell-1,p)-1;if(N<Nell,next);
r=matsize(idealfactor(K,ell))[1];for(n=1,Nn,Qn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
if(n==1,print("f=",f," PK=",PK," CK0=",CK0[2]," ell=",ell," N=",N," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))}
```

(i) For the cubic field of conductor $f = 1777$, with $P_K = x^3+x^2-592x+724$ and $\mathscr{H}_K \simeq \mathbf{Z}/4\mathbf{Z}$, a complete capitulation in K_2 holds for the following primes $\ell \equiv 1 \pmod 8$:

$\ell \in \{41,\ 89,\ 97,\ 137,\ 233,\ 281,\ 313,\ 337,\ 353,\ 401,\ 409,\ 433,\ 449,\ 457,\ 521,\ 569,\ 577,\ 593,\ 601,\ 617,\ 673,\ 761,\ 769,\ 809,\ 857,\ 881,\ 929,\ 937,\ 953,\ 977,\ 1009,\ 1049,\ 1097,\ 1129,\ 1153,\ 1193,\ 1201,\ 1217,\ 1249,\ 1361,\ 1409,\ 1433,\ 1489,\ 1553,\ 1601,\ 1609,\ 1657,\ 1721,\ 1777,\ 1801,\ 2017,\ 2089, \ldots\}$;
exceptions are $\ell \in \{17,\ 73,\ 113,\ 193,\ 241,\ 257,\ 641,\ 1033,\ 1289,\ 1297,\ 1321,\ 1481,\ 1697,\ 1753,\ 1873,\ 1889,\ 1913,\ 1993,\ 2081, \ldots\}$.

(ii) For the cubic field of conductor $f = 20887$, with the more complex structure $\mathscr{H}_K \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, a complete capitulation in K_2 does not hold since $e(K) = 2$, but computations in K_3 is out of reach; however, the results for K_2 allow to distinguish between incomplete capitulation in K_2, then possible capitulation in K_n, $n \geq 3$; more precisely, we obtain the following matrices for K_2 showing always an incomplete capitulation and/or a stability, up to $\ell = 449$ (the mention $\mathsf{Im(J2)} = [\mathsf{a}, ..., \mathsf{z}]$ denotes the structure of $\mathbf{J}_{K_2/K}(\mathscr{H}_K)$ to be compared to $\mathsf{CK0} = [4, 4, 2, 2]$):

```
p=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]

ell=17  r=3  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[16,16,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[4,4]
ell=97  r=3  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=113  r=3  CK0=[4,4,2,2] CK1=[8,8,2,2]  CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=193  r=1  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=241  r=1  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=257  r=1  CK0=[4,4,2,2]  CK1=[24,24,2,2]  CK2=[48,48,2,2]
norm in K2/K of the component 1 of CK2:[4,8,0,0]
norm in K2/K of the component 2 of CK2:[8,12,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[4,4]
ell=337  r=1  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=353  r=3  CK0=[4,4,2,2]  CK1=[4,4,4,4,2,2]  CK2=[8,8,4,4,2,2]
norm in K2/K of the component 1 of CK2:[0,0,2,2,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,2,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Im(J2)=[2,2]
ell=401  r=3  CK0=[4,4,2,2]  CK1=[40,8,2,2]  CK2=[40,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
```

```
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=433  r=1  CK0=[4,4,2,2]  CK1=[8,8,2,2,2,2]  CK2=[8,8,4,4,2,2]
norm in K2/K of the component 1 of CK2:[0,4,2,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,2,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Im(J2)=[2,2]
ell=449  r=1  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[104,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=577  r=3  CK0=[4,4,2,2]  CK1=[56,8,2,2]  CK2=[112,16,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[4,4]
ell=593  r=3  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=641  r=3  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[80,16,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[4,4]
ell=673  r=3  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[8,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=769  r=1  CK0=[4,4,2,2]  CK1=[156,4,2,2,2,2]  CK2=[156,4,4,4,4,4]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,2,0]
norm in K2/K of the component 4 of CK2:[0,2,0,0,2,2]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Im(J2)=[2,2]
ell=881  r=3  CK0=[4,4,2,2]  CK1=[8,8,2,2]  CK2=[16,16,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[4,4]
ell=929  r=1  CK0=[4,4,2,2]  CK1=[104,8,2,2]  CK2=[104,8,2,2]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Im(J2)=[2,2], Capitulation in K3
ell=977  r=3  CK0=[4,4,2,2]  CK1=[8,8,4,4]  CK2=[80,16,4,4]
norm in K2/K of the component 1 of CK2:[4,8,0,0]
norm in K2/K of the component 2 of CK2:[8,4,0,0]
```

```
norm in K2/K of the component 3 of CK2:[0,8,0,0]
norm in K2/K of the component 4 of CK2:[8,0,0,0]
Im(J2)=[2,2]
ell=1009  r=1  CK0=[4,4,2,2]  CK1=[56,8,4,4]  CK2=[56,8,4,4]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[4,0,0,0]
norm in K2/K of the component 4 of CK2:[4,4,0,0]
Im(J2)=[2,2]
```

In the above interval of primes ℓ, there is always partial capitulation.

6.5. **Kummer fields, $L \subset K(\mu_{17})$.** The purpose is to consider fields of the form $K = \mathbb{Q}(\sqrt[q]{R})$, $q \geq 3$; although these fields are not totally real, it is known that capitulation may exist in compositum $L = KL_0$, with suitable abelian p-extensions $L_0/\mathbb{Q}$ (conjectured in [Gras1997], proved in [Bosc2009]); so we limit ourselves to the usual $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$.

6.5.1. *Pure cubic fields, $L \subset K(\mu_{17})$.* We consider the set of pure cubic fields $K = \mathbb{Q}(\sqrt[3]{R})$ The extension $L/\mathbb{Q}$ is not Galois, but, by chance, the instruction $\mathsf{G} = \mathsf{nfgaloisconj}(\mathsf{Kn})$ of PARI computes the group of automorphisms, whence $\mathrm{Gal}(K_n/K)$ in our case; this simplifies the search of S of order p^n.

Taking $\mathsf{p} = 2$, $\ell = 17$, $\mathsf{Nn} = 3$ and restricting to fields K such that $\#\mathscr{H}_K \geq 4$, we obtain many capitulations (the program eliminates cases of stability):

```
MAIN PROGRAM FOR PURE CUBIC FIELDS:
{p=2;Nn=3;vHK=2;ell=17;mKn=2;for(R=2,10^4,PK=x^3-R;
if(polisirreducible(PK)==0,next);K=bnfinit(PK,1);
r=matsize(idealfactor(K,ell))[1];
\\Testing the order of the p-class group of K compared to vHK:
HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;
\\Test for elimination of the stability from K:
if(n==1 & valuation(HKn,p)==valuation(HK,p),break);
if(n==1,print();print("PK=",PK," CK0=",CK0[2]," ell=",ell," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,p^n,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))}
p=2  PK=x^3-43  CK0=[12]  ell=17  r=2
CK1=[12,6]=[4,2]
h_1^[(S-1)^1]=[0,1]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,1]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[12,12]=[4,4]
h_1^[(S-1)^1]=[0,1]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,2]
norm in K2/K of the component 2 of CK2:[0,0]
Incomplete capitulation, m(K2)=2, e(K2)=2
CK3=[12,12]=[4,4]
h_1^[(S-1)^1]=[0,3]   h_2^[(S-1)^1]=[0,0]
```

```
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Complete capitulation (stability from K2), m(K3)=2, e(K3)=2

p=2  PK=x^3-113  CK0=[2,2]  ell=17  r=2
CK1=[6,2,2]=[2,2,2]
h_1^[(S-1)^1]=[0,1,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,1,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=1
CK2=[6,2,2,2,2]=[2,2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,1,0] h_2^[(S-1)^1]=[0,1,1,1,0]
h_3^[(S-1)^1]=[0,1,1,1,0] h_4^[(S-1)^1]=[1,1,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0]
h_1^[(S-1)^2]=[1,1,0,0,0] h_2^[(S-1)^2]=[1,1,0,0,0]
h_3^[(S-1)^2]=[1,1,0,0,0] h_4^[(S-1)^2]=[0,1,1,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,1,1,0,0]
norm in K2/K of the component 2 of CK2:[0,1,1,0,0]
norm in K2/K of the component 3 of CK2:[0,1,1,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0]
Incomplete capitulation, m(K2)=3, e(K2)=1
CK3=[12,2,2,2,2]=[4,2,2,2,2]
h_1^[(S-1)^1]=[2,1,0,0,1] h_2^[(S-1)^1]=[0,0,1,1,0]
h_3^[(S-1)^1]=[0,1,0,0,0] h_4^[(S-1)^1]=[2,0,1,1,1]
h_5^[(S-1)^1]=[0,1,0,0,1]
h_1^[(S-1)^2]=[0,1,1,1,1] h_2^[(S-1)^2]=[2,1,1,1,1]
h_3^[(S-1)^2]=[0,0,1,1,0] h_4^[(S-1)^2]=[2,0,1,1,0]
h_5^[(S-1)^2]=[0,1,1,1,1]
norm in K3/K of the component 1 of CK3:[0,0,0,0,0]
norm in K3/K of the component 2 of CK3:[0,0,0,0,0]
norm in K3/K of the component 3 of CK3:[0,0,0,0,0]
norm in K3/K of the component 4 of CK3:[0,0,0,0,0]
norm in K3/K of the component 5 of CK3:[0,0,0,0,0]
Complete capitulation, m(K3)=4, e(K3)=2

p=2  PK=x^3-122  CK0=[12]  ell=17  r=2
CK1=[12,4]=[4,4]
h_1^[(S-1)^1]=[0,1]      h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[0,2]      h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,1]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=3, e(K1)=2
CK2=[12,4]=[4,4]
h_1^[(S-1)^1]=[0,3]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[0,2]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,2]
norm in K2/K of the component 2 of CK2:[0,0]
Incomplete capitulation, m(K2)=3, e(K2)=2
CK3=[12,4]=[4,4]
h_1^[(S-1)^1]=[0,0]      h_2^[(S-1)^1]=[3,2]
h_1^[(S-1)^2]=[0,0]      h_2^[(S-1)^2]=[2,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Complete capitulation (stability from K1), m(K3)=3, e(K3)=2

p=2  PK=x^3-141  CK0=[4,2]  ell=17  r=2
CK1=[8,2]
h_1^[(S-1)^1]=[0,0]      h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=1, e(K1)=3
```

```
CK2=[16,2]
h_1^[(S-1)^1]=[0,0]      h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,0]
Incomplete capitulation, m(K2)=1, e(K2)=4
CK3=[288,18]=[32,2]
h_1^[(S-1)^1]=[0,0]      h_2^[(S-1)^1]=[0,0]
norm in K3/K of the component 1 of CK3:[8,0]
norm in K3/K of the component 2 of CK3:[0,0]
Incomplete capitulation, m(K3)=1, e(K3)=5
p=2  PK=x^3-174  CK0=[6,2]  ell=17  r=2
CK1=[12,6,2]=[4,2,2]
h_1^[(S-1)^1]=[2,1,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,1,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[2040,12,2,2]=[8,4,2,2]
h_1^[(S-1)^1]=[4,0,1,1]  h_2^[(S-1)^1]=[0,0,0,1]
h_3^[(S-1)^1]=[0,2,0,1]  h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,1]  h_2^[(S-1)^2]=[0,2,0,0]
h_3^[(S-1)^2]=[0,2,0,0]  h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,2,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=4, e(K2)=3
CK3=[4080,12,2,2]=[16,4,2,2]
```

Unfortunately, the computations for $n = 3$ in this last example take too much time; probably, the capitulation is still incomplete. We are going to examine separately this field:

6.5.2. *Case of* $K = \mathbb{Q}(\sqrt[3]{174})$, $\ell \equiv 1 \pmod{16}$. Varying ℓ, we find many capitulations in the layer K_2:

```
p=2  PK=x^3-174  CK0=[6,2]  ell=193  r=1
CK1=[12,6]=[4,2]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[2,0]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[12,6]=[4,2]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
p=2  PK=x^3-174  CK0=[6,2]  ell=353  r=2
CK1=[48,6]=[16,2]
h_1^[(S-1)^1]=[6,0]   h_2^[(S-1)^1]=[8,0]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[8,0]
norm in K1/K of the component 2 of CK1:[8,0]
Incomplete capitulation, m(K1)=4, e(K1)=4
CK2=[48,6,3,3]=[16,2]
h_1^[(S-1)^1]=[6,0,0,0]  h_2^[(S-1)^1]=[8,0,0,0]
h_1^[(S-1)^2]=[4,0,0,0]  h_2^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=4, e(K2)=4
p=2  PK=x^3-174  CK0=[6,2]  ell=401  r=2
```

```
CK1=[60,6,3]=[4,2]
h_1^[(S-1)^1]=[2,0,0]    h_2^[(S-1)^1]=[2,0,0]
h_1^[(S-1)^2]=[0,0,0]    h_2^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0]
norm in K1/K of the component 2 of CK1:[2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=1
CK2=[60,6,3]=[4,2]
h_1^[(S-1)^1]=[0,0,0]   h_2^[(S-1)^1]=[2,0,0]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Complete capitulation, m(K2)=2, e(K2)=1
```

```
p=2  PK=x^3-174  CK0=[6,2]  ell=577  r=1
CK1=[84,6,2]=[4,2,2]
h_1^[(S-1)^1]=[0,1,1]  h_2^[(S-1)^1]=[2,1,1]  h_3^[(S-1)^1]=[2,1,1]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[2,1,1]
norm in K1/K of the component 2 of CK1:[2,1,1]
norm in K1/K of the component 3 of CK1:[2,1,1]
Incomplete capitulation, m(K1)=2, e(K1)=2
CK2=[168,6,6,3]=[8,2,2]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[4,0,0,0] h_3^[(S-1)^1]=[4,1,0,0]
h_1^[(S-1)^2]=[4,0,0,0] h_2^[(S-1)^2]=[0,0,0,0] h_3^[(S-1)^2]=[4,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0] h_2^[(S-1)^3]=[0,0,0,0] h_3^[(S-1)^3]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=3, e(K2)=3
```

The last case $\ell = 577$ shows that the complexity of the $\mathscr{H}_{K_n}$'s is increasing, but nevertheless leads to complete capitulation in K_2; so conditions of Theorem 1.1 (i) are not necessary for capitulation. Indeed, we obtain the following information on the structure of the $\mathscr{H}_{K_n}$'s for $K = \mathbb{Q}(\sqrt[3]{174})$ and $\ell = 577$:

In K_1, $m(K_1) = 2$, $s(K_1) = 1$ but $e(K_1) = 2$. In K_2, $m(K_2) = 3$, $s(K_1) = 1$, $e(K_2) = 3$; thus $\nu_{K_2/K}$ acts like $4(\sigma-1)^2 + 6(\sigma-1) + 4$. The above data show that this reduces to the annihilation by $A = 6(\sigma-1) + 4$; indeed, $h_1^A = h_1^{12} h_1^4 = 1$, $h_i^A = 1$ for the other generators.

6.5.3. *Pure quintic fields, $L \subset K(\mu_{17})$.* Replacing, in the program, the polynomial $\mathsf{PK} = \mathsf{x}^3 - \mathsf{R}$ by $\mathsf{PK} = \mathsf{x}^5 - \mathsf{R}$ (or with any odd degree), we get analogous results, as:

```
p=2  PK=x^5-13  CK0=[4]  ell=17  r=2
CK1=[40]=[8]
h_1^[(S-1)^1]=[0]  h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[2]
No capitulation, m(K1)=1, e(K1)=3
CK2=[40]=[8]
h_1^[(S-1)^1]=[0]  h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[4]
Incomplete capitulation, m(K2)=1, e(K2)=3
Complete capitulation in K3 (stability from K1)
```

```
p=2  PK=x^5-122  CK0=[10,2]  ell=17  r=2
CK1=[10,2,2]=[2,2,2]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[1,1,0]
h_1^[(S-1)^2]=[0,0,0] h_2^[(S-1)^2]=[0,0,0] h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[1,1,0]
Incomplete capitulation, m(K1)=2, e(K1)=1
CK2=[20,2,2]=[4,2,2]
```

```
h_1^[(S-1)^1]=[2,0,1] h_2^[(S-1)^1]=[2,0,1] h_3^[(S-1)^1]=[2,0,0]
h_1^[(S-1)^2]=[2,0,0] h_2^[(S-1)^2]=[2,0,0] h_3^[(S-1)^2]=[0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2
```

7. Tables for real quadratic fields and $p = 3$

We consider various (totally ramified) cyclic p-extensions L/K, where $L = KL_0$, $L_0/\mathbb{Q}$ cyclic of p-power degree. As for the case of cubic base fields, we favor the case $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$.

7.1. **Quadratic fields, $L \subset K(\mu_\ell)$, $\ell \in \{19, 109, 163\}$.** Thus, L_0 is the real subfield of maximal 3-power degree of $\mathbb{Q}(\mu_\ell)$. We eliminate the cases of stability in K_1/K. The images $\mathbf{J}_{K_n/K}(\mathscr{H}_K)$ are computed for $n = 1$ and $n = 2$. The number $r \in \{1, 2\}$ is the number of prime ideals above ℓ in K:

```
MAIN PROGRAM FOR REAL QUADRATIC FIELDS
{p=3;Nn=2;bm=2;Bm=10^8;vHK=1;ell=109;mKn=2;
for(m=bm,Bm,if(core(m)!=m,next);PK=x^2-m;K=bnfinit(PK,1);
\\Testing the order of the p-class group of K compared to vHK:
HK=K.no;if(valuation(HK,p)<vHK,next);
CK0=K.clgp;r=(kronecker(m,ell)+3)/2;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
\\Test for elimination of the stability from K:
if(n==1 & valuation(HKn,p)==valuation(HK,p),break);
if(n==1,print();print("PK=",PK," CK0=",CK0[2]," ell=",ell," r=",r));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))}
```

7.1.1. *Case $\ell = 19$ ($N = 2$).*

```
PK=x^2-142  CK0=[3]  ell=19  r=2
CK1=[9]
h_1^[(S-1)^1]=[0]        h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=1, e(K1)=2
CK2=[9]
h_1^[(S-1)^1]=[0]        h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=1, e(K2)=2

PK=x^2-229  CK0=[3]  ell=19  r=2
CK1=[9]
h_1^[(S-1)^1]=[0]        h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=1, e(K1)=2
CK2=[9]
h_1^[(S-1)^1]=[0]        h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=1, e(K2)=2

PK=x^2-346  CK0=[6]  ell=19  r=2
CK1=[18]=[9]
```

```
h_1^[(S-1)^1]=[0]        h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=1, e(K1)=2
CK2=[18]=[9]
h_1^[(S-1)^1]=[0]        h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=1, e(K2)=2
PK=x^2-359  CK0=[3]  ell=19  r=2
CK1=[9]
h_1^[(S-1)^1]=[0]        h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=1, e(K1)=2
CK2=[27]
h_1^[(S-1)^1]=[0]        h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[9]
No capitulation, m(K2)=1, e(K2)=3
PK=x^2-574  CK0=[6]  ell=19  r=2
CK1=[18]=[9]
h_1^[(S-1)^1]=[6]        h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=2, e(K1)=2
CK2=[18]=[9]
h_1^[(S-1)^1]=[6]        h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=2, e(K2)=2
PK=x^2-761  CK0=[3]  ell=19  r=2
CK1=[9]
h_1^[(S-1)^1]=[0]        h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=1, e(K1)=2
CK2=[27]
h_1^[(S-1)^1]=[18]       h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[9]
No capitulation, m(K2)=2, e(K2)=3
PK=x^2-786  CK0=[6]  ell=19  r=2
CK1=[18]=[9]
h_1^[(S-1)^1]=[6]         h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=2, e(K1)=2
CK2=[54]=[27]
h_1^[(S-1)^1]=[15]        h_1^[(S-1)^2]=[9]
norm in K2/K of the component 1 of CK2:[9]
No capitulation, m(K2)=3, e(K2)=3
PK=x^2-895  CK0=[6]  ell=19  r=1
CK1=[18]=[9]
h_1^[(S-1)^1]=[6]          h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=2, e(K1)=2
CK2=[54]=[27]
h_1^[(S-1)^1]=[24]          h_1^[(S-1)^2]=[9]
norm in K2/K of the component 1 of CK2:[9]
No capitulation, m(K2)=3, e(K2)=3
PK=x^2-934  CK0=[3]  ell=19  r=1
CK1=[9]
h_1^[(S-1)^1]=[3]          h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=2, e(K1)=2
CK2=[27]
h_1^[(S-1)^1]=[12]          h_1^[(S-1)^2]=[9]
norm in K2/K of the component 1 of CK2:[9]
No capitulation, m(K2)=3, e(K2)=3
PK=x^2-1090  CK0=[6,2]  ell=19  r=2
CK1=[18,2]=[9]
```

```
h_1^[(S-1)^1]=[6,0]      h_1^[(S-1)^2]=[0,0]
h_2^[(S-1)^1]=[0,0]      h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation, m(K1)=2, e(K1)=2
CK2=[18,2]=[9]
h_1^[(S-1)^1]=[6,0]      h_1^[(S-1)^2]=[0,0]
h_2^[(S-1)^1]=[0,0]      h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

7.1.2. *Case* $\ell = 109$ *(*$N = 3$*).*

```
p=3  PK=x^2-142  CK0=[3]  ell=109  r=1
CK1=[18,2]=[9]
h_1^[(S-1)^1]=[3,0]       h_1^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
No capitulation, m(K1)=2, e(K1)=2
CK2=[54,2]=[27]
h_1^[(S-1)^1]=[24,0]      h_1^[(S-1)^2]=[9,0]
norm in K2/K of the component 1 of CK2:[9,0]
No capitulation, m(K2)=3, e(K2)=3
```

```
p=3  PK=x^2-223  CK0=[3]  ell=109  r=2
CK1=[9]
h_1^[(S-1)^1]=[6]         h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=2, e(K1)=2
CK2=[9]
h_1^[(S-1)^1]=[3]         h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=3  PK=x^2-229  CK0=[3]  ell=109  r=1
CK1=[9]
h_1^[(S-1)^1]=[3]         h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=2, e(K1)=2
CK2=[9]
h_1^[(S-1)^1]=[6]         h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=3  PK=x^2-254  CK0=[3]  ell=109  r=2
CK1=[3,3]
h_1^[(S-1)^1]=[2,2]         h_2^[(S-1)^1]=[1,1]
h_1^[(S-1)^2]=[0,0]         h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation, m(K1)=2, e(K1)=1
CK2=[3,3]
h_1^[(S-1)^1]=[1,1]         h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]         h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=1
```

```
p=3  PK=x^2-427  CK0=[6]  ell=109  r=2
CK1=[18]=[9]
h_1^[(S-1)^1]=[0]         h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation, m(K1)=1, e(K1)=2
CK2=[18]=[9]
h_1^[(S-1)^1]=[0]         h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=1, e(K2)=2
```

```
p=3  PK=x^2-574  CK0=[6]  ell=109  r=2
CK1=[18,2,2]=[9]
h_1^[(S-1)^1]=[0,0,0]    h_1^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
No capitulation, m(K1)=1, e(K1)=2
CK2=[54,2,2]=[27]
h_1^[(S-1)^1]=[0,0,0]    h_1^[(S-1)^2]=[0,0,0]
norm in K2/K of the component 1 of CK2:[9,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
No capitulation, m(K2)=1, e(K2)=3
```

7.1.3. *Case* $\ell = 163$ *(*$N = 4$*).*

```
p=3  PK=x^2-79  CK0=[3]  ell=163  r=1
CK1=[18,2]=[9]
h_1^[(S-1)^1]=[3,0]         h_1^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation, m(K1)=2, e(K1)=2
CK2=[18,2]=[9]
h_1^[(S-1)^1]=[3,0]         h_1^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
p=3  PK=x^2-223  CK0=[3]  ell=163  r=2
CK1=[18,2]=[9]
h_1^[(S-1)^1]=[3,0]        h_1^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation, m(K1)=2, e(K1)=2
CK2=[18,2]=[9]
h_1^[(S-1)^1]=[3,0]        h_1^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
p=3  PK=x^2-254  CK0=[3]  ell=163  r=2
CK1=[18,2]=[9]
h_1^[(S-1)^1]=[0,0]        h_1^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation, m(K1)=1, e(K1)=2
CK2=[18,2]=[9]
h_1^[(S-1)^1]=[0,0]        h_1^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=1, e(K2)=2
```

7.1.4. *Examples with* $\mathrm{rk}_3(\mathscr{H}_K) = 2$. Due to a very large calculation time for the degrees $[K_2 : \mathbb{Q}] = 18$, we have only some results showing that, as for the case of cubic fields and $p = 2$ (degrees $[K_2 : \mathbb{Q}] = 12$) capitulation does occur at the level $n = 2$:

```
p=3  PK=x^2-23659  CK0=[6,3]  ell=19  r=2
CK1=[18,3]=[9,3]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=2 ,e(K1)=2
CK2=[18,3]=[9,3]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
```

```
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
p=3  PK=x^2-23659  CK0=[6,3]  ell=37  r=2
CK1=[18,3,3]=[9,3,3]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[6,0,1]  h_3^[(S-1)^1]=[6,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[6,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0]
norm in K1/K of the component 2 of CK1:[3,0,0]
norm in K1/K of the component 3 of CK1:[3,0,0]
Incomplete capitulation, m(K1)=3 ,e(K1)=2
CK2=[18,3,3]=[9,3,3]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[3,1,1]  h_3^[(S-1)^1]=[0,2,2]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[3,0,0]  h_3^[(S-1)^2]=[6,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Complete capitulation, m(K2)=3, e(K2)=2
p=3  PK=x^2-32009  CK0=[3,3]  ell=19  r=1
CK1=[9,3]
h_1^[(S-1)^1]=[3,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=2 ,e(K1)=2
CK2=[9,3]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
p=3  PK=x^2-32009  CK0=[3,3]  ell=37  r=2
CK1=[9,3]
h_1^[(S-1)^1]=[3,0]        h_2^[(S-1)^1]=[6,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=2 ,e(K1)=2
CK2=[9,3]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

7.2. **Quadratic fields, $L \subset K(\mu_{109\cdot 163})$.** In this subsection, we consider cyclic 3-extensions $L \subset K(\mu_{109\cdot 163})$ with the two totally ramified primes 109 and 163, giving larger 3-class groups, especially when these primes split in the quadratic field $K = \mathbb{Q}(\sqrt{m})$. We limit ourselves to the layer K_1 of degree 3 for which a defining polynomial is for instance $\mathsf{QK1} = \mathsf{x}^3 - \mathsf{x}^2 - 5922 * \mathsf{x} - 17109$. The number r of ramified primes is equal to 2, 3 or 4 depending on the splitting of 109 and 163.

We have, quite often, $\#\mathscr{H}_{K_1}^{G_1} = \#\mathscr{H}_K$ $(m(K_1) = 1)$; we give an excerpt of cases where $m(K_1) \geq 2$, but we have only two examples of $m(K_1) \geq 3$ ($m = 116279$ and $m = 370878$) in the interval selected ($m < 10^6$).

```
{p=3;bm=2;Bm=10^6;vHK=3;mKn=2;for(m=bm,Bm,if(core(m)!=m,next);
PK=x^2-m;K=bnfinit(PK,1);HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
QK1=x^3-x^2-5922*x-17109;Pn=polcompositum(PK,QK1)[1];Kn=bnfinit(Pn,1);
HKn=Kn.no;dn=poldegree(Pn);if(valuation(HKn,p)==valuation(HK,p),break);
r=(kronecker(m,109)+3)/2+(kronecker(m,163)+3)/2;CKn=Kn.clgp;
print("p=3"," PK=",PK," CK0=",CK0[2]," r=",r);print("CK",1,"=",CKn[2]);
```

```
rKn=matsize(CKn[2])[2];G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;
while(Z!=Id,Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",1,"/K of the component ",i,
" of CK",1,":",Enu)))}
p=3  PK=x^2-8761  CK0=[27]  r=2
CK1=[27,3]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)=1, e(K1)=3
p=3  PK=x^2-36073  CK0=[27]  r=2
CK1=[27,3,3]
h_1^[(S-1)^1]=[0,2,1]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=3
p=3  PK=x^2-65029  CK0=[27]  r=3
CK1=[81,3]
h_1^[(S-1)^1]=[0,0]  h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]  h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation, m(K1)=1, e(K1)=4
p=3  PK=x^2-116054  CK0=[27]  r=4
CK1=[81,3,3]
h_1^[(S-1)^1]=[0,0,2]   h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
No capitulation, m(K1)=2, e(K1)=4
p=3  PK=x^2-116279  CK0=[27]  r=3
CK1=[81,3,3]
h_1^[(S-1)^1]=[54,2,2]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[54,0,0]
h_1^[(S-1)^2]=[27,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[30,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
No capitulation, m(K1)=3, e(K1)=4
p=3  PK=x^2-156566  CK0=[9,3]  r=2
CK1=[27,3,3]
h_1^[(S-1)^1]=[18,0,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[18,0,0]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=3
p=3  PK=x^2-255973  CK0=[9,3]  r=2
CK1=[9,9,3]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[0,6,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,3,0]
```

```
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2

p=3  PK=x^2-339887  CK0=[27]  r=3
CK1=[81,3,3,3]
h_1^[(S-1)^1]=[27,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[27,1,0,2]   h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=4

p=3  PK=x^2-370878  CK0=[54]  r=4
CK1=[1134,6,6,3]=[81,3,3,3]
h_1^[(S-1)^1]=[27,2,2,0]    h_2^[(S-1)^1]=[0,0,1,0]
h_3^[(S-1)^1]=[0,0,0,0]     h_4^[(S-1)^1]=[0,0,2,0]
h_1^[(S-1)^2]=[0,0,2,0]     h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]     h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,2,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
No capitulation, m(K1)=3, e(K1)=4
```

As for the previous case of cubic fields with $p = 2$, since the complexity of K_1/K (in the meaning of Definition 2.13) is non-smooth in most cases because of larger $\mathscr{H}_{K_1}$'s, there is often no capitulation in K_1, which confirms, once again, the heuristic about capitulation versus complexity.

8. Isotopic components and capitulation

Consider a real cyclic field K of prime-to-p degree d and $L = L_0K$ with $L_0/\mathbb{Q}$ real cyclic of degree p^N, $N \geq 1$. Then $L/\mathbb{Q}$ is cyclic of degree $D = d \cdot p^N$ with Galois group $\Gamma = g \oplus G$ where $g = \mathrm{Gal}(L/L_0)$ and $G = \mathrm{Gal}(L/K)$. The field L is associated to an irreducible rational character χ, sum of irreducible p-adic characters φ of "order" the order D of any $\psi \mid \varphi$ of degree 1.

This non semi-simple context is problematic for the definition of isotopic p-adic components of the form $\mathscr{H}_{L,\varphi}$ and $\mathscr{H}_{K_n,\varphi_n}$ for the subfields K_n of L with corresponding rational and p-adic characters χ_n and $\varphi_n \mid \chi_n$; this is extensively developed in [Gras2021^b]. So we just recall the definitions and explain how the phenomenon of capitulation gives rise to difficulties about the classical algebraic definition of the literature, compared to the arithmetic one that we have introduced to state the Main Conjecture in the general case.

Indeed, classical works deal with an algebraic definition of the φ-components of p-class groups, which presents an inconsistency regarding analytic formulas; this definition is, for Γ cyclic of order $D = d\,p^N$ and for all $\varphi \mid \chi$:

$$\mathscr{H}_{L,\varphi}^{\mathrm{alg}} := \mathscr{H}_L \bigotimes_{\mathbb{Z}_p[\Gamma]} \mathbb{Z}_p[\mu_D],$$

with the $\mathbb{Z}_p[\mu_D]$-action $\tau \in \Gamma \mapsto \psi(\tau)$, with $\psi \mid \varphi$ of order D (see [Solo1990, Lemma II.2] or [Grei1992, Definition, p. 451]). We then have proved, with this definition [Gras2021^b, § 3.2.4, Theorem 3.7], the following interpretation:

$$\begin{cases} \mathscr{H}_{L,\chi}^{\mathrm{alg}} = \{x \in \mathscr{H}_L,\ P(\sigma_\chi) \cdot x = 1\} = \{x \in \mathscr{H}_L,\ \nu_{L/k}(x) = 1,\ \forall\, k \subsetneq L\}, \\ \mathscr{H}_{L,\varphi}^{\mathrm{alg}} = \{x \in \mathscr{H}_L,\ \nu_{L/k}(x) = 1,\ \forall\, k \subsetneq L\} \otimes_{\mathbb{Z}_p[\Gamma]} \mathbb{Z}_p[\mu_D], \end{cases}$$

where P is the Dth cyclotomic polynomial and σ_χ a generator of $\mathrm{Gal}(L/\mathbb{Q})$, which gives rise to our corresponding arithmetic definitions:

$$\begin{cases} \mathscr{H}_{L,\chi}^{\mathrm{ar}} := \{x \in \mathscr{H}_L,\ \ \mathbf{N}_{L/k}(x) = 1,\ \forall\, k \subsetneqq L\}, \\ \mathscr{H}_{L,\varphi}^{\mathrm{ar}} := \{x \in \mathscr{H}_L,\ \ P_\varphi(\sigma)\cdot x = 1 \quad \& \quad \mathbf{N}_{L/k}(x) = 1,\ \forall\, k \subsetneqq L\}, \end{cases}$$

where $P_\varphi \mid P$ is the corresponding local cyclotomic polynomial associated to the above action $\tau \in \Gamma \mapsto \psi(\tau)$, with $\psi \mid \varphi$.

We then have (since L/K is totally ramified):

$$\#\mathscr{H}_L^{\mathrm{ar}} = \prod_{\chi \in \mathbf{R}_L} \#\mathscr{H}_{L,\chi}^{\mathrm{ar}}, \tag{8.1}$$

where $\mathbf{R}_L$ is the set of irreducible rational characters of L. More precisely, χ is of the form $\chi_0\,\chi_n$ for the rational characters χ_0 of K and the rational characters χ_n of L_0 of order p^n, $n \in [1, N]$; then $\mathscr{H}_{L,\chi}^{\mathrm{ar}} = \mathscr{H}_{K_{\chi_0} L_{0,\chi_n},\chi}^{\mathrm{ar}}$, where $K_{\chi_0} \subseteq K$ (resp. $L_{0,\chi_n} \subseteq L_0$) correspond to χ_0 (resp. χ_n).

This notion leads to an unexpected semi-simplicity, especially in accordance with analytic formulas, which enforces the Main Conjecture in that case:

$$\mathscr{H}_{L,\chi}^{\mathrm{ar}} = \bigoplus_{\varphi \mid \chi} \mathscr{H}_{L,\varphi}^{\mathrm{ar}},\ \text{for all } \chi = \chi_0\chi_n \in \mathbf{R}_L,\ \varphi = \varphi_0\varphi_n,\ \varphi_0 \mid \chi_0,\ \varphi_n \mid \chi_n.$$

The $\mathbb{Z}_p[\Gamma]$-modules of the form $\mathscr{H}_{L,\chi}^{\mathrm{ar}}$ (resp. $\mathscr{H}_{L,\varphi}^{\mathrm{ar}}$), annihilated by all the arithmetic norms $\mathbf{N}_{L/k}$, are called arithmetic χ-objects (resp. φ-objects).

We have $\mathscr{H}_{L,\varphi}^{\mathrm{ar}} = \mathscr{H}_{L,\varphi}^{\mathrm{alg}}$ as soon as the $\mathbf{J}_{L/k}$'s are injective for all $k \subsetneqq L$, but as we have seen, this does not hold in general when $K \subseteq k \subsetneqq L$ since there is often partial capitulation. One can even say that the classic admitted definition is ineffective and fallacious in the real case for p-class groups.

Remark 8.1. Let χ be the rational character associated to L. Our Main Conjecture [Gras1977] (not yet proven in the non semi-simple case contrary to some claims) requires that the equality of orders of χ-objects:

$$\#\mathscr{H}_{L,\chi}^{\mathrm{ar}} = \#(\mathscr{E}_L/\mathscr{E}_L^0 \cdot \mathscr{F}_L)\ \text{(see [Gras2021}^b\text{, Theorem 7.5 (i)])},$$

be valid for the φ-components, for all $\varphi \mid \chi$; in these formula, $\mathscr{E}_L^0$ is the subgroup of $\mathscr{E}_L$ generated by the units of the strict subfields of L and $\mathscr{F}_L$ is the group of classical Leopoldt's cyclotomic units (see [Gras2021^b, Examples 3.12, 3.13]).

In the case of cubic fields and $p = 2$ or in the case of real quadratic fields, $\chi = \varphi$, so that the Main Conjecture is trivial, but not the definition of arithmetic φ-objects regarding the algebraic ones.

Let's give numerical examples showing the consequences of capitulation for the non-arithmetic definitions:

Example 8.2. Consider $K = \mathbb{Q}(\sqrt{4409})$, $p = 3$, $\ell = 19$ and $L = K_2 \subset K(\mu_\ell)$ of degree 9 over K. We are going to see that $\mathscr{H}_{K_n} \simeq \mathbb{Z}/9\mathbb{Z}$ for all $n \leq 2$ (stability); Program 7.1 gives:

```
p=3  PK=x^2-4409  CK0=[9]  ell=19  r=2
CK1=[9]
h_1^[(S-1)^1]=[0]
norm in K1/K of the component 1 of CK1:[3]
Incomplete capitulation, m(K1)=1, e(K1)=2
CK2=[9]
h_1^[(S-1)^1]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation, m(K2)=1, e(K2)=2
```

The capitulation, incomplete in K_1, is complete in K_2 as expected (stability from K giving $\mathscr{H}_{K_n}=\mathscr{H}_{K_n}^{G_n}$ for $n\in\{1,2\}$).

We use obvious notations for the characters defining the fields K_n. Since arithmetic norms are surjective, the above computations prove that:

$$\begin{cases}\nu_{K_2/K_1}(\mathscr{H}_{K_2})=(\mathscr{H}_{K_2})^{1+\sigma_2^3+\sigma_2^6}=(\mathscr{H}_{K_2})^3\simeq\mathbb{Z}/3\mathbb{Z},\\ \nu_{K_1/K}(\mathscr{H}_{K_1})=(\mathscr{H}_{K_1})^{1+\sigma_1+\sigma_1^2}=(\mathscr{H}_{K_1})^3\simeq\mathbb{Z}/3\mathbb{Z},\end{cases}$$

whence:

$$\begin{cases}\mathscr{H}_{\chi_2}^{\rm ar}=\{x\in\mathscr{H}_{K_2},\ \mathbf{N}_{K_2/K_1}(x)=1\}=1,\\ \mathscr{H}_{\chi_2}^{\rm alg}=\{x\in\mathscr{H}_{K_2},\ \nu_{K_2/K_1}(x)=1\}=\mathscr{H}_{K_2}[3]\simeq\mathbb{Z}/3\mathbb{Z}.\\ \mathscr{H}_{\chi_1}^{\rm ar}=\{x\in\mathscr{H}_{K_1},\ \mathbf{N}_{K_1/K}(x)=1\}=1,\\ \mathscr{H}_{\chi_1}^{\rm alg}=\{x\in\mathscr{H}_{K_1},\ \nu_{K_1/K}(x)=1\}=\mathscr{H}_{K_1}[3]\simeq\mathbb{Z}/3\mathbb{Z}.\end{cases}$$

Formula (8.1) gives the product of orders of the χ-components $\mathscr{H}_{\chi_n}^{\rm ar}$:

$$\#\mathscr{H}_{K_2}=\#\mathscr{H}_{\chi_0}^{\rm ar}\cdot\#\mathscr{H}_{\chi_1}^{\rm ar}\cdot\#\mathscr{H}_{\chi_2}^{\rm ar},\ \ \#\mathscr{H}_{K_1}=\#\mathscr{H}_{\chi_0}^{\rm ar}\cdot\#\mathscr{H}_{\chi_1}^{\rm ar},$$

of the form (since $\#\mathscr{H}_{\chi_0}^{\rm ar}=\#\mathscr{H}_K=9$) $\#\mathscr{H}_{K_2}=9\times1\times1$, $\#\mathscr{H}_{K_1}=9\times1$.

These formulas are not fulfilled in the algebraic sense, because:

$$\#\mathscr{H}_{\chi_0}^{\rm alg}\cdot\#\mathscr{H}_{\chi_1}^{\rm alg}\cdot\#\mathscr{H}_{\chi_2}^{\rm alg}=9\times3\times3=3^4,\ \#\mathscr{H}_{\chi_0}^{\rm alg}\cdot\#\mathscr{H}_{\chi_1}^{\rm alg}=9\times3=3^3.$$

Example 8.3. This example is analogous for a cyclic cubic field, $p=2$, $\ell=17$, except that capitulation takes place from K_1; Program 6.1.1 gives:

```
p=2  f=1951  PK=x^3+x^2-650*x-289  CK0=[2,2]  ell=17  r=3

CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation, m(K1)=2, e(K1)=2

CK2=[4,4,4,4]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[2,0,2,0] h_4^[(S-1)^1]=[0,2,0,2]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2

CK3=[8,8,4,4]
h_1^[(S-1)^1]=[0,0,2,0] h_2^[(S-1)^1]=[2,2,2,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[6,2,0,2]
h_1^[(S-1)^2]=[4,0,0,0] h_2^[(S-1)^2]=[0,4,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K3/K of the component 1 of CK3:[0,0,0,0]
norm in K3/K of the component 2 of CK3:[0,0,0,0]
norm in K3/K of the component 3 of CK3:[0,0,0,0]
norm in K3/K of the component 4 of CK3:[0,0,0,0]
Complete capitulation, m(K3)=3, e(K3)=3
```

Numerical data give, with $\mathbf{Z}=\mathbb{Z}_2[\exp(\frac{2i\pi}{3})]$:

$$\begin{cases}\nu_{K_3/K_2}(\mathscr{H}_{K_3})=\mathscr{H}_{K_3}^{1+\sigma_3^4}=\mathscr{H}_{K_3}^2\simeq\mathbf{Z}/4\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z},\\ \nu_{K_2/K_1}(\mathscr{H}_{K_2})=\mathscr{H}_{K_2}^{1+\sigma_2^2}=\mathscr{H}_{K_2}^2\simeq\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z},\\ \nu_{K_1/K}(\mathscr{H}_{K_1})=\mathscr{H}_{K_1}^{1+\sigma_1}=1.\end{cases}$$

Whence:

$$\begin{cases}\mathscr{H}_{\chi_3}^{\rm ar}=\{x\in\mathscr{H}_{K_3},\ \mathbf{N}_{K_3/K_2}(x)=1\}\simeq\mathbf{Z}/2\mathbf{Z},\\ \mathscr{H}_{\chi_3}^{\rm alg}=\{x\in\mathscr{H}_{K_3},\ \nu_{K_3/K_2}(x)=1\}=\mathscr{H}_{K_3}[2]\simeq\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z},\\ \mathscr{H}_{\chi_2}^{\rm ar}=\{x\in\mathscr{H}_{K_2},\ \mathbf{N}_{K_2/K_1}(x)=1\}\simeq\mathbf{Z}/2\mathbf{Z},\\ \mathscr{H}_{\chi_2}^{\rm alg}=\{x\in\mathscr{H}_{K_2},\ \nu_{K_2/K_1}(x)=1\}=\mathscr{H}_{K_2}[2]\simeq\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z},\\ \mathscr{H}_{\chi_1}^{\rm ar}=\{x\in\mathscr{H}_{K_1},\ \mathbf{N}_{K_1/K}(x)=1\}\simeq\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z},\\ \mathscr{H}_{\chi_1}^{\rm alg}=\{x\in\mathscr{H}_{K_1},\ \nu_{K_1/K}(x)=1\}=\mathscr{H}_{K_1}\simeq\mathbf{Z}/4\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}.\end{cases}$$

Which gives, noting that $\#(\mathbf{Z}/2^k\mathbf{Z})=4^k$:
$\#\mathscr{H}_{K_3}=\#\mathscr{H}_{\chi_0}^{\rm ar}\cdot\#\mathscr{H}_{\chi_1}^{\rm ar}\cdot\#\mathscr{H}_{\chi_2}^{\rm ar}\cdot\#\mathscr{H}_{\chi_3}^{\rm ar}=2^{10}$, $\#\mathscr{H}_{K_2}=\#\mathscr{H}_{\chi_0}^{\rm ar}\cdot\#\mathscr{H}_{\chi_1}^{\rm ar}\cdot\#\mathscr{H}_{\chi_2}^{\rm ar}=2^8$, $\#\mathscr{H}_{K_1}=\#\mathscr{H}_{\chi_0}^{\rm ar}\cdot\#\mathscr{H}_{\chi_1}^{\rm ar}=2^6$,
contrary to:
$\#\mathscr{H}_{\chi_0}^{\rm alg}\cdot\#\mathscr{H}_{\chi_1}^{\rm alg}\cdot\#\mathscr{H}_{\chi_2}^{\rm alg}\cdot\#\mathscr{H}_{\chi_3}^{\rm alg}=2^2\cdot2^6\cdot2^4\cdot2^4=2^{16}$, $\#\mathscr{H}_{\chi_0}^{\rm alg}\cdot\#\mathscr{H}_{\chi_1}^{\rm alg}\cdot\#\mathscr{H}_{\chi_2}^{\rm alg}=2^2\cdot2^6\cdot2^4=2^{12}$, $\#\mathscr{H}_{\chi_0}^{\rm alg}\cdot\#\mathscr{H}_{\chi_1}^{\rm alg}=2^2\cdot2^6=2^8$.

We will illustrate, in K_1, the analytic equality, discussed in Remark 8.1, to compare the two notions of χ-objects O (i.e., $O^{\rm alg}$ versus $O^{\rm ar}$):

$$\#\mathscr{H}_{\chi_1}^{\rm ar}=\#(\mathscr{E}_{K_1}/\mathscr{E}_{K_1}^0\cdot\mathscr{F}_{K_1}).$$

Let $k:=\mathbb{Q}(\sqrt{17})$, $\mathrm{Gal}(K_1/k)=:\{1,\tau,\tau^2\}$ and $G_1=\{1,\sigma\}$. Since 17 splits in K and 1951 splits in k, the generating cyclotomic unit η of K_1 is of norm 1, both in K_1/k and K_1/K, so it generates $\mathscr{F}_{\chi_1}$ that we write $\langle\eta^\tau,\eta^{\tau^2}\rangle_{\mathbb{Z}}$.

Computing in $\mathbb{Q}(\mu_{17\cdot1951})/K_1$ gives (taking logarithms for convenience):

$\log(\eta)=-5.047187756810161779147188420685320 7885$,
$\log(\eta^\tau)=+32.072728696925313868267792411432213485$,
$\log(\eta^{\tau^2})=-27.025540940115152089120603990746892715$,

then:

$\log(\eta)=-5.047187756810161779147188420685320 7885$,
$\log(\eta^\sigma)=5.047187756810161779147188420685320 7885$.

The group $\mathscr{E}_{K_1}$ of units of K_1 given by PARI is of the form:

$$\mathscr{E}_{K_1}=\mathscr{E}_k\oplus\mathscr{E}_K\oplus\langle e_4,e_5\rangle_{\mathbb{Z}},$$

where $\mathbf{N}_{K_1/K}(e_4)=\mathbf{N}_{K_1/K}(e_5)=1$, $\mathbf{N}_{K_1/k}(e_4)=\mathbf{N}_{K_1/k}(e_5)=1$, so that $\mathscr{E}_{\chi_1}=\langle e_4,e_5\rangle_{\mathbb{Z}}$.

Similarly we get:

$\log(e_4)=-8.018182174231328467066948102858573234 5$,

$\log(e_5)=+6.756385235028788022280150997686723176 6$,

yielding immediately $\eta^\tau=e_4^{-4}$, $\eta^{\tau^2}=e_5^{-4}$. Thus, in this example:

$$\mathscr{E}_{K_1}/\mathscr{E}_{K_1}^0\cdot\mathscr{F}_{K_1}=\mathscr{E}_{\chi_1}/\mathscr{F}_{\chi_1}=\langle e_4,e_5\rangle_{\mathbb{Z}}/\langle\eta^\tau,\eta^{\tau^2}\rangle_{\mathbb{Z}}\simeq\mathbf{Z}/4\mathbf{Z},$$

of order 16. So, we have $\#\mathscr{H}_{\chi_1}^{\rm ar}=\#(\mathscr{E}_{\chi_1}/\mathscr{F}_{\chi_1})$, but with the different structures $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/4\mathbf{Z}$, respectively, which relativizes the interest of algebraic

definitions, regarding analytic formulas, since $\mathscr{H}_{\chi_1}^{\rm alg} = \mathscr{H}_{K_1} \simeq \mathbf{Z}/4\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ in that example.

Remark 8.4. More generally, Galois cohomology groups are based on algebraic definitions of the norms, so that results strongly depend on capitulation phenomena. For instance, let L/K be a cyclic p-extension of Galois group G; assuming, to simplify, that all the ramified prime ideals of K ramified in L/K are totally ramified, then:

$$\mathrm{H}^1(G,\mathscr{H}_L) = \mathrm{Ker}_{\mathscr{H}_L}(\nu_{L/K})/\mathscr{H}_L^{\sigma-1} \quad \& \quad \mathrm{H}^2(G,\mathscr{H}_L) = \mathscr{H}_L^G/\nu_{L/K}(\mathscr{H}_L),$$

are of same order $\frac{\#\mathscr{H}_L^G}{\#\mathbf{J}_{L/K}(\mathscr{H}_K)} = \frac{\#\mathscr{H}_K}{\#\mathbf{J}_{L/K}(\mathscr{H}_K)} \times \frac{p^{N\,(r-1)}}{\#\omega_{L/K}(\mathscr{E}_K)}$.

So, if $\mathrm{Ker}(\mathbf{J}_{L/K}) = 1$, one gets the order $\frac{p^{N\,(r-1)}}{\#\omega_{L/K}(\mathscr{E}_K)}$; if $\mathscr{H}_K$ capitulates, the order becomes $\#\mathscr{H}_K \times \frac{p^{N\,(r-1)}}{\#\omega_{L/K}(\mathscr{E}_K)} = \#\mathscr{H}_L^G$ and any intermediate situation does exist.

9. Conclusions and prospects

a) We have conjectured in Conjecture 1.2 (i) that, varying $\ell \equiv 1 \pmod{2p^N}$, N large enough, there are infinitely many cases with stability from a suitable layer in $K(\mu_\ell)$, yielding capitulation of $\mathscr{H}_K$ (Theorem 1.1 (ii)), which reinforces the general capitulation phenomenon; this would be coherent with Greenberg's conjecture, equivalent to the stability of the $\mathscr{H}_{K_n}$'s in the cyclotomic $\mathbb{Z}_p$-extension, from $n_0 \gg 0$. In other words, our conjecture may be seen as the "tame version", it being understood that our towers are finite, so that capitulation needs large N's allowing a kind of "finite Iwasawa's theory".

Furthermore, the more general criterion of Theorem 1.1 (i), using the algebraic norm by means of the invariants $m(L)$ and $e(L)$, yields capitulation without there necessarily being stability; it shows the link between capitulation and complexity of the filtration of the $\mathscr{H}_{K_n}$'s, likely to be governed by natural density results (Conjecture 2.4). It is reasonable to think that, restricting to primes $\ell \equiv 1 \pmod{2p^N}$ with $N \to \infty$, $N - s(L)$ becomes larger than $e(L)$ taking into account that $s(L) = \left\lceil \frac{\log(m(L))}{\log(p)} \right\rceil$ is logarithmic regarding $m(L)$ which essentially depends, from the algorithm defining $\mathscr{H}_L^{i+1}$ from $\mathscr{H}_L^i$, on the magnitude of $\mathscr{H}_K$ (exponent and p-rank) and of local norm symbols associated to the $r-1$ ramified prime ideals.

When capitulation is, on the contrary, structurally impossible (e.g., case of minus parts of p-class groups or case of torsion groups $\mathscr{T}_K$ of p-ramification theory), the complexity of the corresponding invariants necessarily increases in any totally ramified cyclic p-tower.

b) Due to the computations given in various frameworks in this paper, it is difficult to imagine that, for all $\ell \equiv 1 \pmod{2p^N}$, $\mathscr{H}_K$ does not capitulate in $K(\mu_\ell)$, all the more that we were limited to testing with few values of ℓ (among infinitely many !) and only for the levels $n \leq 3$. Similarly, we were limited to small primes p because of the degrees $[K_n : \mathbb{Q}] = [K : \mathbb{Q}]\, p^n$ for PARI calculations; but the nature of the theoretical results does not seem to depend on it; this is strengthened by the algorithmic aspect of formulas 2.2.

c) The remarkable circumstance of capitulations in these simplest tamely ramified cyclic p-extensions L/K, is certainly a basic principle for many arithmetic properties, as the following ones:

(i) The abelian Main Conjecture for real abelian fields, whose proof becomes trivial in the semi-simple case as soon as ℓ is taken inert in $K/\mathbb{Q}$ and if

$\mathscr{H}_K$ capitulates in some K_n's; indeed, in the case $r=1$ the filtration is only defined by the class factors $\#(\mathscr{H}_{K_n}^{i+1}/\mathscr{H}_{K_n}^{i}) = \frac{\#\mathscr{H}_K}{\#\mathbf{N}_{K_n/K}(\mathscr{H}_{K_n}^{i})}$, $i \in [0, m(K_n)]$.

(ii) In the non semi-simple Iwasawa context of the Main Conjecture, we refer for instance to [AsMaOu2017, Mazigh2016, Mazigh2017] and their references about generalization of Main Conjectures associated to Euler systems built over Stark units replacing Leopoldt's cyclotomic units, hoping that capitulation phenomena may give new insights in these theories using auxiliary cyclic p-extensions L/K in which exact sequence (1.1) is still valid.

(iii) Capitulations prevent to get efficient algebraic definitions of p-adic isotopic components (i.e., $\mathscr{H}_\varphi^{\rm alg}$) of arithmetic invariants in the non semi-simple real case; which suggests to replace algebraic norms by arithmetic ones in the definitions and thus to use instead arithmetic φ-objects (i.e., $\mathscr{H}_\varphi^{\rm ar}$).

(v) Some speculative questions can be asked, such as the following: An obstruction to a cyclotomic proof of Fermat's Last Theorem is Vandiver's conjecture. But if an ideal $\mathfrak{a}$ of $K := \mathbb{Q}(\zeta_p + \zeta_p^{-1})$ is non principal of the form, say $\mathfrak{a}^p = (a)$, $a \in K^\times$, the existence of a capitulation field of the form $L = K(\mu_\ell)$ in which the above relation becomes $\alpha_L^p = a \cdot \eta_L$, $\alpha_L, \eta_L \in L^\times$, would allow some classical reasonings in an *abelian context over* $\mathbb{Q}$ as that of [Wash1997, Chap. 1, 6, 9].

d) Because capitulation of p-class groups, in totally real ramified cyclic p-extensions L/K, is in connection with the class group complexity of the layers K_n, one may wonder if this has some repercussion on the very numerous heuristics on repartition of p-class groups when Galois groups are of order divisible by p (see, e.g., [BarLen2020, BarJoLe2022] dealing with some difficulties about the classical heuristics of Cohen–Lenstra–Martinet–Malle, and giving attempts to modify them). In other words, is the capitulation a governing principle for complexity, or, on the contrary, is the complexity a governing principle for capitulation ? This is a difficult question all the more that numerical examples show that if the tower L/K is hight enough, complexity seems to become as smooth as possible (for instance, stability from some layer), while the discriminants become oversized in the tower.

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