Algebraic norm and capitulation of p-class groups in ramified cyclic p-extensions

Georges Gras

ALGEBRAIC NORM AND CAPITULATION OF p-CLASS GROUPS IN RAMIFIED CYCLIC p-EXTENSIONS

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Abstract. We examine the phenomenon of capitulation of the p-class group $\mathscr{H}_K$ of a number field K in totally ramified cyclic p-extensions L/K of degree p^N and Galois group G. Using an elementary property of the algebraic norm $\mathcal{N}_{L/K}$, we show that the kernel of capitulation is in relation with the "complexity" of the structure of $\mathscr{H}_L$ measured via its exponent $p^{e(L)}$ and the length $m(L)$ of the usual filtration $\{\mathscr{H}_L^i\}_{i\geq 0}$ associated to $\mathscr{H}_L$ as $\mathbb{Z}_p[G]$-module. We prove that a sufficient condition of capitulation is given by $e(L) \in [1, N-s]$ if $m(L) \in [p^s, p^{s+1}-1]$ for $s \in [0, N-1]$ (Theorem 1.1 (i)); this improves the case of "stability" $\#\mathscr{H}_L = \#\mathscr{H}_K$ (i.e., $m(L)=1$, $s=0$, $e(L)=e(K)\leq N$) (Theorem 1.1 (ii)). A sufficient condition of partial capitulation is also made explicit. Numerical examples with directly usable PARI programs, showing most often capitulations, are given over cubic fields with $p=2$ and real quadratic fields with $p=3$, taking the simplest possible p-extensions $L \subset \mathbb{Q}(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$. Conjectures on the existence of non-zero densities are proposed (Conjectures 1.2, 2.4). Capitulations are also evoked for other invariants.

Contents

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1. Introduction

1.1. **Statement of the main result.** Let L/K be any cyclic p-extension of number fields ($p \geq 2$ prime), of degree p^N, $N \geq 1$, of Galois group G, and let $\mathscr{H}_K$, $\mathscr{H}_L$ be the p-class groups of K, L, respectively.

Let $\mathbf{J}_{L/K} : \mathscr{H}_K \to \mathscr{H}_L^G$ be the transfer map (or extension of classes) and let $\mathbf{N}_{L/K} : \mathscr{H}_L \to \mathscr{H}_K$ be the arithmetic norm. Let $H_K^{\rm nr}$ be the p-Hilbert class field of K and $L^{\rm nr} := L \cap H_K^{\rm nr}$. We know that $\mathscr{H}_K^0 := \mathbf{N}_{L/K}(\mathscr{H}_L)$ is the subgroup of $\mathscr{H}_K$ which corresponds to $\mathrm{Gal}(H_K^{\rm nr}/L^{\rm nr})$ by the Artin map of class field theory; thus, if L/K is totally ramified, then $L^{\rm nr} = K$ and $\mathscr{H}_K^0 = \mathscr{H}_K$. Denote by $\mathrm{rank}_p(A) := \dim_{\mathbb{F}_p}(A/A^p)$ the p-rank of any finite abelian group A.

We can state:

Theorem 1.1. *Let L/K be a cyclic p-extension of number fields, of degree p^N, $N \geq 1$, of Galois group $G =: \langle\sigma\rangle$. One assumes $\mathscr{H}_K^0 \neq 1$:*

(i) A sufficient condition for a partial capitulation of $\mathscr{H}_K^0$ in L, is that there exists $h \in \mathscr{H}_L$ such that $\mathbf{N}_{L/K}(h) \neq 1$, h of order p^e and annihilated by $(\sigma-1)^m$, for integers e, m with $e \in [1, N-s]$ if $m \in [p^s, p^{s+1}-1]$ for $s \in [0, N-1]$.

Let $p^{e(L)}$ be the exponent of $\mathscr{H}_L$ and let $m(L)$ be the minimal integer such that $(\sigma-1)^{m(L)}$ annihilates $\mathscr{H}_L$. Then, a sufficient condition for the complete capitulation of $\mathscr{H}_K^0 := \mathbf{N}_{L/K}(\mathscr{H}_L)$ in L, is that $e(L) \in [1, N-s]$ if $m(L) \in [p^s, p^{s+1}-1]$ for $s \in [0, N-1]$.

(ii) Let K_n, $n \in [0, N]$, be the subfield of L of degree p^n over K and put $G_n := \mathrm{Gal}(K_n/K)$. If L/K is totally ramified, then the condition $\#\mathscr{H}_{K_1} = \#\mathscr{H}_K$ implies $\mathscr{H}_{K_n} = \mathscr{H}_{K_n}^{G_n} \simeq \mathscr{H}_K$, for all $n \in [0, N]$, and the complete capitulation of $\mathscr{H}_K$ in $K_{e(K)}$ if $e(K) \leq N$. If $\mathrm{rank}_p(\mathscr{H}_{K_1}) = \mathrm{rank}_p(\mathscr{H}_K)$, then the p-rank is constant along the tower.

Point (i) will be given by Corollary 2.10 to Theorem 2.7. Then (ii) comes from [Gras2022[a], Theorem 3.1 & Section 6, § (b)] generalizing [Fuku1994, KrSch1995, Band2007, LOXZ2022, MiYa2021]; the stability of the class groups corresponds, in point (i), to the case $m(K_n) = 1$ (from $\mathscr{H}_{K_n} = \mathscr{H}_{K_n}^{G_n}$), whence $s = 0$ and $e(K) \leq N$ (from $e(L) = e(K)$ due to the isomorphisms $\mathscr{H}_{K_n} \simeq \mathscr{H}_K$ given by the arithmetic norms); this particular case is called the p-class groups stability in the tower L/K. The case of the p-ranks for the case of $\mathbb{Z}_p$-extensions was given by Fukuda [Fuku1994], then found again by Bandini [Band2007]. It holds for any cyclic totally ramified p-extension.

If $\mathscr{H}_K$ capitulates in K_n, necessarily $n \geq e(K)$ since $\mathbf{N}_{K_n/K} \circ \mathbf{J}_{K_n/K} = p^n$. Of course, the properties of stability may occur from a layer K_{n_0}.

Theorem 1.1 expresses that, if the "complexity" of $\mathscr{H}_L$ is not too important, then $\mathscr{H}_K$ capitulates in L. Conversely, if one knows that capitulation is impossible (e.g., framework of abelian imaginary fields and minus p-class groups $\mathscr{H}^-$), then the complexity of the $\mathscr{H}_{K_n}^-$'s is much more important compared to that of $\mathscr{H}_K^-$ and strictly increases with n (see Theorem 3.2).

1.2. **History and aims of the paper.** The problem of capitulation [1] of $\mathscr{H}_K$ in L (measured by the capitulation kernel $\mathrm{Ker}(\mathbf{J}_{L/K} : \mathscr{H}_K \to \mathscr{H}_L)$) has been studied in a very large number of publications, as precisely in the purpose of the factorization problem, for Dedekind rings, exposed in [Mart2011];

[1] I recently learned (from a Lemmermeyer text) that the word *capitulation* was coined by Arnold Scholz. It is possible that this term may be considered as incongruous; a solution is to consider that a non principal ideal $\mathfrak{a}$ is a troublemaker with respect to elementary arithmetic, in which case, the terminology is perfectly understandable. Conjectures about ideal capitulations have perhaps a moral significance.

it is impossible to give a complete bibliography, but one may cite, among many other contributions subsequent to the historical works of Hilbet–Scholz–Taussky: [Kisil1970, Tera1971, Bond1981, HeSc1982, Schm1985, GrJa1985, Jaul1986, Jaul1988, Iwas1989, Suzu1991, Maire1996, Gras1997, Maire1998, Kuri1999, GrWe2000, KoMo2000, GrWe2003, Gonz2006, Vali2008, Bosc2009, Mayer2014, AZT2016, Bisw2016, Jaul2019^b, Jaul2022], in which the reader will find more history and references.

Most of these papers are related to Hilbert's Theorem 94 on capitulation in the Hilbert class field $H_K^{\rm nr}$, which is not our purpose since, on the contrary, we shall study totally ramified cyclic p-extensions L/K and more precisely the simplest tamely ramified p-extensions $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$ prime, $[L:K] = p^N$, which, surprisingly, are often capitulation fields of $\mathscr{H}_K$ when K is totally real (even if K is not totally imaginary).

Many classical articles give cohomological expressions of the capitulation in terms of global units as the fact that, in the non-ramified case, the capitulation kernel $\mathrm{Ker}(\mathbf{J}_{L/K})$ is isomorphic to a subgroup of $\mathrm{H}^1(G, \mathbf{E}_L)$, where $\mathbf{E}_L$ is the group of units of L (see, e.g., [Jaul1986, Chap. III, § 1] and [Jaul1988] for more comments and references).

In the same spirit, using sets of places S, T and tamely ramified Galois extensions, the result of Maire [Maire1996, Théorème 4.1] gives, in our context $L \subset K(\mu_\ell)$, injective maps $\mathscr{H}_{L,(\ell)}/\mathbf{J}_{L/K}(\mathscr{H}_{K,(\ell)}) \hookrightarrow \mathrm{H}^2(G, \mathscr{E}_{L,(\ell)})$, where $\mathscr{E} := \mathbf{E} \otimes \mathbb{Z}_p$ for groups of units and where $\mathscr{H}_{K,(\ell)}$, $\mathscr{H}_{L,(\ell)}$ are ray-class groups and $\mathscr{E}_{L,(\ell)}$ the group of units congruent to 1 modulo (ℓ).

But the aspect "global units" is more difficult since the behavior of the unit groups in L/K is less known compared to that of p-class groups, even if there are some links; indeed, we have the following classical exact sequence:

$$1 \to \mathbf{J}_{L/K}(\mathscr{H}_K) \cdot \mathscr{H}_L^{\rm ram} \to \mathscr{H}_L^G \to \mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times)/\mathbf{N}_{L/K}(\mathscr{E}_L) \to 1,$$

where $\mathscr{H}_L^{\rm ram} \subseteq \mathscr{H}_L^G$ is generated by means of classes of the ramified prime ideals of L; in the right term, if $\mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times)$ depends on easier local considerations, $\mathbf{N}_{L/K}(\mathscr{E}_L)$ is in general unknown. On the contrary, $\mathbf{J}_{L/K}(\mathscr{H}_K)$, $\mathscr{H}_L^{\rm ram}$, are subgroups of $\mathscr{H}_L^G$ and the order of this group is known from the Chevalley–Herbrand formula [Chev1933, pp. 4002–405] (see [Gras1978] for isotopic components, [Jaul1986, Chapitre III, p. 167] with ramification and decomposition, [Lemm2013] in the spirit of Jaulent's work); then general higher fixed points formulas [Gras2017^a, LiYu2020] allow the algorithmic computation of $\mathscr{H}_L$ from a natural filtration.

In our previous paper [Gras2022^a], giving extensive numerical computations with [PARI], we have proposed the following conjecture, whose main consequence should be an obvious immediate proof, in the semi-simple case, of the real abelian Main Conjecture "$\mathscr{H}_\varphi = (\mathscr{E}_\varphi : \mathscr{F}_\varphi)$" in terms of index of Leopoldt's cyclotomic units [2]:

Conjecture 1.2. *(i) Let K be any totally real number field and let $\mathscr{H}_K$ be its p-class group, of exponent $p^{e(K)}$.*

- *There exist infinitely many primes $\ell \equiv 1 \pmod{2p^N}$, $N \geq e(K)$, such that $\mathscr{H}_K$ capitulates in $K(\mu_\ell)$.*
- *There exist infinitely many primes $\ell \equiv 1 \pmod{2p^N}$, N large enough, such that capitulation of $\mathscr{H}_K$ in $K(\mu_\ell)$ is due to a stability from some layer.*

[2] The complete statement being the following [Gras2022^b, Section 1.4, then Theorem 4.6]: Assume that K is a real cyclic extension of $\mathbb{Q}$, of prime-to-p degree. Let $\ell \equiv 1 \pmod{2p^N}$ be a prime number totally inert in $K/\mathbb{Q}$ and let $L \subset K(\mu_\ell)$ be the subfield of degree p^N over K. Then, if $\mathscr{H}_K$ capitulates in L, the "Main Conjecture" on the p-adic components $\mathscr{H}_{K,\varphi}$, of $\mathscr{H}_K$, holds (i.e., we have $\#\mathscr{H}_{K,\varphi} = (\mathscr{E}_{K,\varphi} : \mathscr{F}_{K,\varphi})$ for all p-adic character φ of K).

(ii) Let $\ell \equiv 1 \pmod{2p^N}$, $N \geq 1$, be a fixed prime number and let $\mathscr{K}_{d,N}$ be the family of totally real number fields K, of degree $d \geq 2$, whose p-class group $\mathscr{H}_K$ is of exponent $p^{e(K)}$ with $1 \leq e(K) \leq N$. There exist infinitely many $K \in \mathscr{K}_{d,N}$ such that $\mathscr{H}_K$ capitulates in $K(\mu_\ell)$.

This restriction to the family of p-extensions L/K, $L \subset K(\mu_\ell)$, is another point of view compared to the case of abelian capitulations ([Gras1997], [Kuri1999], [Bosc2009], [Jaul2022]). Indeed, all techniques in these papers need to built a finite set of abelian p-extensions L_k of $\mathbb{Q}$, ramified at various primes, requiring many local arithmetic conditions existing from Chebotarev theorem, whose compositum with K gives a capitulation field of $\mathscr{H}_K$; the method must apply to any abelian field K (of suitable signature), of arbitrary increasing degree, obtained in an iterative process giving, for instance, that the maximal real subfield of $\mathbb{Q}\big(\bigcup_{f>0}\mu_f\big)$ is principal (see in [Bosc2009] the most general statements).

2. Complexity of $\mathscr{H}_L$ and capitulation of $\mathscr{H}_K$

Let L/K be a cyclic p-extension of degree p^N, $N \geq 1$, of Galois group $G =: \langle\sigma\rangle$.

Recall that $\mathbf{N}_{L/K}(\mathscr{H}_L) = \mathscr{H}_K^0$, subgroup of $\mathscr{H}_K$ image of $\mathrm{Gal}(H_K^{\mathrm{nr}}/L^{\mathrm{nr}})$, where H_K^{nr} is the p-Hilbert class field of K and $L^{\mathrm{nr}} = L \cap H_K^{\mathrm{nr}}$. Let:

$$\nu_{L/K} := \sum_{i=0}^{p^N-1} \sigma^i,$$

be the algebraic norm in L/K. From the law of decomposition in L/K of an unramified prime ideal $\mathfrak{q}$ of K, of residue degree f, we get, for $\mathfrak{Q} \mid \mathfrak{q}$ in L, $\nu_{L/K}(\mathfrak{Q}) = \prod_{i=0}^{p^N-1} \mathfrak{Q}^{\sigma^i} = ((\mathfrak{q})_L)^f = (\mathfrak{q}^f)_L = (\mathbf{N}_{L/K}(\mathfrak{Q}))_L$; whence:

$$\nu_{L/K}(\mathscr{H}_L) = \mathbf{J}_{L/K} \circ \mathbf{N}_{L/K}(\mathscr{H}_L) = \mathbf{J}_{L/K}(\mathscr{H}_K^0). \tag{2.1}$$

Thus, $\mathscr{H}_K^0$ (or $\mathscr{H}_K$ if L/K is totally ramified) capitulates in L if and only if $\nu_{L/K}(\mathscr{H}_L) = 1$. A partial capitulation occurs as soon as $\#\nu_{L/K}(\mathscr{H}_L) < \#\mathscr{H}_K^0$.

So, the action of the algebraic norm characterizes the capitulation (complete or incomplete) and it is clear that the result mainly depends on the $\mathbb{Z}_p[G]$ structure of $\mathscr{H}_L$ which is expressed by means of the canonical associated filtration $\{\mathscr{H}_L^i\}_{i\geq 0}$ defining a classical algorithm of computation of $\mathscr{H}_L$ that we shall recall.

2.1. Filtration of $\mathscr{H}_L$ in the totally ramified case.

Let L/K be a cyclic p-extension of degree p^N, $N \geq 1$, and Galois group $G = \langle\sigma\rangle$. To avoid technical writings, we assume that any prime ideal $\mathfrak{l}$ of K, ramified in L/K, is totally ramified, and that there are $r \geq 1$ such ramified prime ideals. Let $\mathbf{E}_K$ be the group of units of K and put $\mathscr{E}_K := \mathbf{E}_K \otimes \mathbb{Z}_p$.

2.1.1. *The higher rank Chevalley–Herbrand formulas.* The generalizations of the Chevalley–Herbrand formula to the corresponding filtration, $\{\mathscr{H}_L^i\}_{i\geq 0}$, gives rise to the following expressions, where:

$$\mathscr{H}_L^0 = 1, \quad \mathscr{H}_L^1 := \mathscr{H}_L^G, \quad \mathscr{H}_L^{i+1}/\mathscr{H}_L^i := (\mathscr{H}_L/\mathscr{H}_L^i)^G,\ i \geq 0, \tag{2.2}$$

up to $i = m(L) = \min\{m \geq 0,\ \mathscr{H}_L^{(\sigma-1)^m} = 1\}$, for which $\mathscr{H}_L^{m(L)} = \mathscr{H}_L$.

Denote by $\mathscr{I}_L^i$ a $\mathbb{Z}_p[G]$-module of ideals of L, of finite type, generating $\mathscr{H}_L^i$, with $\mathscr{I}_L^0 = 1$, $\mathscr{I}_L^{i+1} \supseteq \mathscr{I}_L^i$, for all $i \geq 0$; the $\mathscr{I}_L^i$ are defined up to the group

of principal ideals of L, thus $\mathbf{N}_{L/K}(\mathscr{I}_L^i)$ defined up to $\big(\mathbf{N}_{L/K}(L^\times)\big)$. This filtration has the following properties [Gras2017[a], Theorem 3.6]:

$$\text{(2.3)}\qquad \begin{cases} (i)\ \ \#\mathscr{H}_L^1 = \#\mathscr{H}_K \times \dfrac{p^{N(r-1)}}{(\mathscr{E}_K : \mathscr{E}_K \cap \mathbf{N}_{L/K}(L^\times))}, \\ (ii)\ \ \#(\mathscr{H}_L^{i+1}/\mathscr{H}_L^i) = \dfrac{\#\mathscr{H}_K}{\#\mathbf{N}_{L/K}(\mathscr{H}_L^i)} \times \dfrac{p^{N(r-1)}}{(\mathbf{\Lambda}_K^i : \mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times))}, \\ \qquad\qquad \mathbf{\Lambda}_K^i := \{x \in K^\times,\ (x) \in \mathbf{N}_{L/K}(\mathscr{I}_L^i)\}, \\ (iii)\ \ \mathscr{H}_L^i = \{h \in \mathscr{H}_L,\ h^{(\sigma-1)^i} = 1\},\ \text{ for all } i \geq 0, \\ (iv)\ \ \#(\mathscr{H}_L^{i+1}/\mathscr{H}_L^i) \leq \#\mathscr{H}_L^1,\ \text{ for all } i \geq 0, \\ (v)\ \ \#\mathscr{H}_L = \prod_{i=0}^{m(L)-1} \#(\mathscr{H}_L^{i+1}/\mathscr{H}_L^i) \leq (\#\mathscr{H}_L^1)^{m(L)}. \end{cases}$$

The $\mathbf{\Lambda}_K^i$'s are subgroups of $K^\times$ containing $\mathbf{E}_K$, with $\mathbf{\Lambda}_K^0 = \mathbf{E}_K$. In particular, any $x \in \mathbf{\Lambda}_K^i$ is local norm in L/K at all the non-ramified places. So, for any $(x) = \mathbf{N}_{L/K}(\mathfrak{A})$, $\mathfrak{A} \in \mathscr{I}_L^i$, which is also local norm at the ramified places, $x = \mathbf{N}_{L/K}(y)$, $y \in L^\times$ (Hasse's norm theorem) and there exists an ideal $\mathfrak{B}$ of L such that $\mathfrak{A} = (y)\mathfrak{B}^{\sigma-1}$; this constitutes an algorithm by addition of the $\mathfrak{B}$'s to $\mathscr{I}_L^i$ to get $\mathscr{I}_L^{i+1}$ then $\mathscr{H}_L^{i+1}$.[3] Since $\mathbf{\Lambda}_K^0 = \mathbf{E}_K$ is a $\mathbb{Z}$-module of finite type, this algorithm allows to construct $\mathbf{\Lambda}_K^i$ of finite type for all i, with $\mathbf{\Lambda}_K^i \subseteq \mathbf{\Lambda}_K^{i+1}$ (indeed, $\mathbf{N}_{L/K}(\mathscr{I}_L^i)$ is of finite type, there is a finite number of relations of principality between the generators and $\mathbf{\Lambda}_K^i/\mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times)$ is of exponent $\leq p^N$).

The i-sequence $\#(\mathscr{H}_L^{i+1}/\mathscr{H}_L^i)$, $0 \leq i \leq m(L)$, is decreasing, from $\#\mathscr{H}_L^1$ up to 1, because of the injective maps $\mathscr{H}_L^{i+1}/\mathscr{H}_L^i \hookrightarrow \mathscr{H}_L^i/\mathscr{H}_L^{i-1} \hookrightarrow \cdots \hookrightarrow \mathscr{H}_L^1$ due to the action of $\sigma - 1$, giving the inequality in (v).

The first (resp. second) factor in (ii) is called the class (resp. norm) factor.

2.1.2. *Properties of the class and norm factors.* Since the ramified places v (= prime ideals) of K are assumed to be totally ramified in L/K, their inertia groups $I_v(L/K)$ in L/K are isomorphic to G. Let $\omega_{L/K}$ be the map which associates with $x \in \mathbf{\Lambda}_K^i$ the family of Hasse's norm symbols $\big(\frac{x\,,\,L/K}{v}\big) \in I_v(L/K)$. Since x is local norm at the unramified places, $\omega_{L/K}(\mathbf{\Lambda}_K^i)$ is contained in:

$$\Omega_{L/K} := \big\{(\tau_v)_v \in \textstyle\bigoplus_v I_v(L/K),\ \ \prod_v \tau_v = 1\big\} \simeq G^{r-1}$$

(product formula); then $\mathrm{Ker}(\omega_{L/K}) = \mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times)$.

It follows that $\#\omega_{L/K}(\mathbf{\Lambda}_K^i) = (\mathbf{\Lambda}_K^i : \mathbf{\Lambda}_K^i \cap \mathbf{N}_{L/K}(L^\times))$ divides $p^{N(r-1)}$.

Denote by K_n, $0 \leq n \leq N$, the subfields of L of degree p^n over K and let $G_n := \mathrm{Gal}(K_n/K) =: \langle\sigma_n\rangle$. All the previous definitions and formulas apply to the K_n's; we shall denote by $\mathbf{\Lambda}_K^i(n)$ the invariant corresponding to K_n/K instead of L/K; we have $\mathbf{\Lambda}_K^i(n) = \{x \in K^\times,\ \ (x) \in \mathbf{N}_{K_n/K}(\mathscr{I}_{K_n}^i)\}$. So $\mathbf{\Lambda}_K^i = \mathbf{\Lambda}_K^i(N)$ in the previous notations.

Lemma 2.1. *For any i fixed, we may assume that $\mathbf{\Lambda}_K^i(n+1) \subseteq \mathbf{\Lambda}_K^i(n)$, for all $n \in [0, N-1]$.*

Proof. For all $n \in [0, N-1]$, we have the following diagram, where the norm $\mathbf{N}_{K_{n+1}/K_n}$, on $\mathscr{H}_{K_{n+1}}$ and $(\mathscr{H}_{K_{n+1}})^{(\sigma_{n+1}-1)^i}$, is surjective (total ramification),

[3] For explicit class field theory, Hasse's norm theorem, norm residue symbols, product formula, see, e.g., [Gras2005, Theorem II.6.2, Definition II.3.1.2, Theorems II.3.1.3, 3.4.1].

but its restriction to $\mathscr{H}^i_{K_{n+1}}$ may be non injective nor surjective:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{H}^i_{K_{n+1}} & \longrightarrow & \mathscr{H}_{K_{n+1}} & \xrightarrow{(\sigma_{n+1}-1)^i} & (\mathscr{H}_{K_{n+1}})^{(\sigma_{n+1}-1)^i} & \longrightarrow & 1 \\
 & & \downarrow & & \mathbf{N}_{K_{n+1}/K_n}\downarrow & & \mathbf{N}_{K_{n+1}/K_n}\downarrow & & \\
1 & \longrightarrow & \mathscr{H}^i_{K_n} & \longrightarrow & \mathscr{H}_{K_n} & \xrightarrow{(\sigma_n-1)^i} & (\mathscr{H}_{K_n})^{(\sigma_n-1)^i} & \longrightarrow & 1\,.
\end{array}$$

We have $\mathbf{N}_{K_{n+1}/K_n}(\mathscr{H}^i_{K_{n+1}}) \subseteq \mathscr{H}^i_{K_n}$; so, for any ideal $\mathfrak{A}_{n+1} \in \mathscr{I}^i_{K_{n+1}}$, one may write $\mathbf{N}_{K_{n+1}/K_n}(\mathfrak{A}_{n+1}) = (\alpha_n)\,\mathfrak{A}_n$, where $\alpha_n \in K_n^\times$ and $\mathfrak{A}_n \in \mathscr{I}^i_{K_n}$, in which case *modifying* the definition of $\mathscr{I}^i_{K_n}$ modulo principal ideals of K_n, one may assume $\mathbf{N}_{K_{n+1}/K_n}(\mathscr{I}^i_{K_{n+1}}) \subseteq \mathscr{I}^i_{K_n}$ whence $\mathbf{N}_{K_{n+1}/K}(\mathscr{I}^i_{K_{n+1}}) \subseteq \mathbf{N}_{K_n/K}(\mathscr{I}^i_{K_n})$; this modifies $\mathbf{\Lambda}^i_K(n)$ modulo $\mathbf{N}_{K_n/K}(K_n^\times)$ which does not modify $\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))$ in the formula giving $\#\big(\mathscr{H}^{i+1}_{K_n}/\mathscr{H}^i_{K_n}\big)$. Using the process from the top, we obtain $\mathbf{\Lambda}^i_K(N) \subseteq \mathbf{\Lambda}^i_K(N-1) \subseteq \cdots \subseteq \mathbf{\Lambda}^i_K(1) \subseteq \mathbf{\Lambda}^i_K(0)$. □

Lemma 2.2. *For i fixed, the integers $\#\big(\mathscr{H}^{i+1}_{K_n}/\mathscr{H}^i_{K_n}\big)$ define an increasing n-sequence from $\#\big(\mathscr{H}^{i+1}_{K}/\mathscr{H}^i_{K}\big) = 1$ up to $\#\big(\mathscr{H}^{i+1}_{L}/\mathscr{H}^i_{L}\big)$; the $\#\mathscr{H}^i_{K_n}$'s define an increasing n-sequence from $\#\mathscr{H}^i_K = \mathscr{H}_K$ up to $\#\mathscr{H}^i_L$ (note that $\#\mathscr{H}^0_{K_n} = 1$ for all n).*

Proof. Consider, for $i \geq 0$ fixed, the two factors of the finite n-sequence (for $n = 0$, the two factors are trivial):

$$\#\big(\mathscr{H}^{i+1}_{K_n}/\mathscr{H}^i_{K_n}\big) = \frac{\#\mathscr{H}_K}{\#\mathbf{N}_{K_n/K}(\mathscr{H}^i_{K_n})} \times \frac{p^{n(r-1)}}{\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))}.$$

As $\mathbf{N}_{K_{n+1}/K}(\mathscr{H}^i_{K_{n+1}}) \subseteq \mathbf{N}_{K_n/K}(\mathscr{H}^i_{K_n})$, $p^{c^i_{K_n}} := \dfrac{\#\mathscr{H}_K}{\#\mathbf{N}_{K_n/K}(\mathscr{H}^i_{K_n})}$ defines an increasing n-sequence from 1 up to a value $p^{c^i_L} \mid \#\mathscr{H}_K$. The norm factor:

$$p^{\rho^i_{K_n}} := \frac{p^{n(r-1)}}{\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))}$$

defines an increasing n-sequence from 1 up to $p^{\rho^i_L}$ since, from Lemma 2.1:

$$p^{\rho^i_{K_{n+1}}-\rho^i_{K_n}} = p^{r-1}\frac{\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))}{\#\omega_{K_{n+1}/K}(\mathbf{\Lambda}^i_K(n+1))} \geq p^{r-1}\frac{\#\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))}{\#\omega_{K_{n+1}/K}(\mathbf{\Lambda}^i_K(n))};$$

then, in the restriction $\Omega_{K_{n+1}/K} \longrightarrow\!\!\!\!\!\rightarrow \Omega_{K_n/K}$ (whose kernel is of order p^{r-1} because of the total ramification of each place), the image of $\omega_{K_{n+1}/K}(\mathbf{\Lambda}^i_K(n))$ is $\omega_{K_n/K}(\mathbf{\Lambda}^i_K(n))$ because of the properties of Hasse's symbols, whence:

$$p^{\rho^i_{K_{n+1}}-\rho^i_{K_n}} \geq 1$$

and the result for the n-sequence $p^{\rho^i_{K_n}}$, with maximal value $p^{\rho^i_L}$. The first claim of the lemma holds for the n-sequence $\#\big(\mathscr{H}^{i+1}_{K_n}/\mathscr{H}^i_{K_n}\big)$; for $n = N$, one gets the formula $\#\big(\mathscr{H}^{i+1}_{L}/\mathscr{H}^i_{L}\big) = p^{c^i_L}\cdot p^{\rho^i_L}$.

Assuming, by induction, that the n-sequence $\#\mathscr{H}^i_{K_n}$ is increasing, the property follows for the n-sequence $\#\mathscr{H}^{i+1}_{K_n}$. □

Remarks 2.3. (i) The i-sequences $p^{c^i_L}$, $p^{\rho^i_L}$, and $\#(\mathscr{H}^{i+1}_L/\mathscr{H}^i_L) = p^{c^i_L}\cdot p^{\rho^i_L}$ are decreasing up to a divisor of $\#\mathscr{H}_K$, $p^{N(r-1)}$, $\#\mathscr{H}_K\cdot p^{N(r-1)}$, respectively.

(ii) The n-sequence $m(K_n)$ is an increasing sequence from 0 up to $m(L)$. Then the $\#\mathscr{H}_{K_n}$'s define an increasing n-sequence from $\#\mathscr{H}_K$ up to $\#\mathscr{H}_L$ since we have:

$$\#\mathscr{H}_{K_{n+1}} \geq \#\big(\mathscr{H}^{\mathrm{Gal}(K_{n+1}/K_n)}_{K_{n+1}}\big) = \#\mathscr{H}_{K_n}\,\frac{p^{r-1}}{\omega_{K_{n+1}/K_n}(\mathscr{E}_{K_n})} \geq \#\mathscr{H}_{K_n}.$$

The integers $e(K_n)$ and $\dim_{\mathbb{F}_p}(\mathscr{H}_{K_n}/\mathscr{H}_{K_n}^p)$ define increasing n-sequences.

The interest of this filtration is that standard probabilities may be applied, at each layer n, to the algorithm computing $\mathscr{H}_{K_n}^{i+1}$ from $\mathscr{H}_{K_n}^i$ by means of the two factors $p^{c^i_{K_n}}$ and $p^{\rho^i_{K_n}}$, giving plausible heuristics in the spirit of works of [KoPa2022, Smith2022]. So we propose the following conjecture:

Conjecture 2.4. *Let L/K be a cyclic p-extension of degree p^N, $N \geq 1$, of Galois group G; we assume that L/K is ramified at $r \geq 1$ places of K, totally ramified in L/K. Then, the orders of each of the two factors (class and norm) in the i-sequence $\#(\mathscr{H}_L^{i+1}/\mathscr{H}_L^i)$, follow binomial laws as i increases (based, for $x \in \mathbf{\Lambda}_K^i$, on the relations $(x) = \mathbf{N}_{L/K}(\mathfrak{A})$, $x = \mathbf{N}_{L/K}(y)$ and $\mathfrak{A} = (y)\mathfrak{B}^{\sigma-1}$ when they apply for x, depending on Hasse's symbols), with the following probabilities:*

- *Let $\gamma \in G^{r-1}$; the probability, for $x \in \Lambda_K^i$, that $\omega_{L/K}(x) = \gamma$, is $\frac{1}{p^{N(r-1)}}$.*
- *Let $c \in \mathscr{H}_K$; the probability, for an ideal $\mathfrak{A}$ of L, that the p-class of $\mathbf{N}_{L/K}(\mathfrak{A})$ equals c, is $\frac{1}{\#\mathscr{H}_K}$.*

Remarks 2.5. (i) If the ramified places are not totally ramified in L/K, the previous formulas (2.3) must be modified according to [Gras2017[a], Formula 29], but the definitions of the invariants $m(L)$ and $e(L)$ are unchanged; only the group $\mathscr{H}_K$ must be replaced by $\mathscr{H}_K^0$, using Formula (2.1), $\nu_{L/K}(\mathscr{H}_L) = \mathbf{J}_{L/K}(\mathscr{H}_K^0)$, and the fact that $\mathbf{J}_{L/K}(\mathscr{H}_K^0) = 1$ if and only if $\nu_{L/K}(\mathscr{H}_L) = 1$.

(ii) Another way to work with totally ramified extensions is to replace the base field K by $L^{\rm nr} = L \cap H_K^{\rm nr}$; but in that case one try to principalize $\mathscr{H}_{L^{\rm nr}}$ and the algebraic norm becomes $\nu_{L/L^{\rm nr}}$ instead of $\nu_{L/K}$.

2.2. **Program computing the filtrations $\{\mathscr{H}_{K_n}^i\}_{i\geq 1}$.** The following program may be used for the calculation of the Galois structure of the $\mathscr{H}_{K_n}$'s, $K_n \subseteq L \subset K(\mu_\ell)$, whatever the base field K given, as usual, by means of a monic polynomial of $\mathbb{Z}[x]$ (here, of prime-to-p degree to simplify the computation of a generator S of $\mathrm{Gal}(K_n/K)$).

For this, one must indicate the prime p in p, the number Nn of layers K_n considered, the polynomial PK defining K, a prime ell congruent to 1 modulo $2p^N$, $\mathsf{N} \geq \mathsf{Nn}$, and a value mKn for computing the $\mathsf{h_j^i} := \mathsf{h_j^{(S-1)^i}}$ for $1 \leq \mathsf{i} \leq \mathsf{mKn}$, where the $\mathsf{h_j}$'s are the generators of the whole class group $\mathbf{H}_{K_n}$ given by PARI (in $\mathsf{CKn} = \mathsf{Kn.clgp}$), and where the generator S of G_n is chosen in $\mathsf{G} = \mathsf{nfgaloisconj(Kn)}$ testing the orders.

So $\mathscr{H}_{K_n}^i = \mathrm{Ker}(\mathscr{H}_{K_n} \to \mathscr{H}_{K_n}^{(\sigma_n-1)^i})$, $1 \leq i \leq m(K_n)$ (see (2.3) (iii)). The invariant $m(K_n)$ is obtained (assuming mKn large enough) for the first i giving zero matrices in the test of principality of the $\mathsf{h_j^i}$'s.

Since PARI works with generators h_i of $\mathbf{H}_{K_n}$, of orders $p^{n_i}t_i$, $p \nmid t_i$ (given in $\mathsf{CKn[2]}$), we consider $t = \prod t_i$, the non p-part of $\mathsf{HKn} = \mathsf{Kn.no}$ (the whole class number) and put $\overline{h}_i = h_i^t$ giving generators of $\mathscr{H}_{K_n}$; thus, for any data $[\mathsf{e_1}, \ldots, \mathsf{e_r}]$ given by $\mathsf{bnfisprincipal(Kn, Y)[1]}$ for an ideal Y whose class is $\mathsf{h_1^{e_1} \cdots h_r^{e_r}}$, the program gives, instead, the list $\mathsf{E} := [\overline{\mathsf{e}}_1, \ldots, \overline{\mathsf{e}}_\mathsf{r}]$ defining the class of $\mathsf{Y^t}$ (in $\mathscr{H}_{K_n}$) as $\overline{\mathsf{h}}_1^{\overline{\mathsf{e}}_1} \cdots \overline{\mathsf{h}}_\mathsf{r}^{\overline{\mathsf{e}}_\mathsf{r}}$, where $\overline{\mathsf{e}}_\mathsf{i} = \mathsf{lift(Mod(e_i, p^{n_i}))}$; this does not modify the Galois structure of $\mathscr{H}_{K_n}$ and the outputs are more readable. So, the ideal Y is p-principal if and only if $\mathsf{E} = [0, \ldots, 0]$.

The corresponsing outputs are writen under the form $\overline{h}_i^{(\sigma-1)^j} = [\overline{e}_1, \ldots, \overline{e}_r]$ instead of $\overline{h}_i^{(\sigma-1)^j} = \overline{h}_1^{\overline{e}_1} \cdots \overline{h}_r^{\overline{e}_r}$.

Below, we take as example a cyclic cubic field of conductor $\mathsf{f} = 703$, for which $\mathscr{H}_K \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathsf{ell} = 97$, $\mathsf{mKn} = 3$; the results are given for $n = 1$ and $n = 2$ and rho is the number of prime ideals of K dividing ℓ:

```
PROGRAM COMPUTING THE h_i^[(S-1)^j]:
{p=2;Nn=2;PK=x^3+x^2-234*x-729;ell=97;mKn=3;
K=bnfinit(PK,1);CK0=K.clgp;rho=matsize(idealfactor(K,ell))[1];
print("p=",p," Nn=",Nn," PK=",PK," ell=",ell,
" mKn=",mKn," CK0=",CK0[2]," rho=",rho);
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);CKn=Kn.clgp;dn=poldegree(Pn);
print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
\\Search of a generator S of Gal(Kn/K):
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
\\Computation of the image of CKn by (S-1)^j:
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
\\computation in Ehij of the modified exponents of B:
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij));print()))}
```

```
p=2  Nn=2  PK=x^3+x^2-234*x-729  ell=97  mKn=3  CK0=[6,2]  rho=1
CK1=[6,2,2,2]
h_1^[(S-1)^1]=[1,1,0,0]       h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,0,1,1]       h_4^[(S-1)^1]=[0,0,1,1]
h_1^[(S-1)^2]=[0,0,0,0]       h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]       h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0]       h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0]       h_4^[(S-1)^3]=[0,0,0,0]
CK2=[12,4,2,2]
h_1^[(S-1)^1]=[0,2,1,1]       h_2^[(S-1)^1]=[0,2,1,0]
h_3^[(S-1)^1]=[2,2,0,0]       h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,2,0,0]       h_2^[(S-1)^2]=[2,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0]       h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0]       h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0]       h_4^[(S-1)^3]=[0,0,0,0]
```

This gives $m(K_1) = 2$, $\mathscr{H}_{K_1}^{\sigma_1-1} = \langle h_1h_2, h_3h_4\rangle$, $\mathscr{H}_{K_1}^{G_1} = \langle h_1h_2^{-1}, h_3h_4^{-1}\rangle = \langle h_1h_2, h_3h_4\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Then $m(K_2) = 3$, $\mathscr{H}_{K_2}^{\sigma_2-1} = \langle h_2^2h_3h_4, h_2^2h_3, h_1^2h_2^2, h_1^2\rangle = \langle h_3, h_4, h_1^2, h_2^2\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^4$, whence $\mathscr{H}_{K_2}^{G_2} = (\mathscr{H}_{K_2})^2 \simeq (\mathbb{Z}/2\mathbb{Z})^2$. We have $\mathscr{H}_{K_2}^{(\sigma_2-1)^2} = \langle h_1^2, h_2^2\rangle \simeq (\mathscr{H}_{K_2})^2 \simeq (\mathbb{Z}/2\mathbb{Z})^2$, $\mathscr{H}_{K_2}^2 = \langle h_3, h_4, h_1^2, h_2^2\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^4$ and $\mathscr{H}_{K_2}^3 = 1$.

These computations will prove later a partial capitulation of $\mathscr{H}_K$ in K_1 and a complete capitulation of $\mathscr{H}_K$ in K_2.

Remark 2.6. An astonishing fact, in a diophantine viewpoint, is that, when the class of $\mathfrak{a}$ capitulates in some K_n, the writing of the generator $\alpha \in K_n^\times$, of the extended ideal $(\mathfrak{a})_{K_n}$ of K, on the $\mathbb{Q}$-basis Kn.zk of the field K_n, needs most often oversized coefficients (several thousand digits and, often, PARI proves the principality without giving these coefficients). If the reader wishes to verify this fact, it suffices to add the instruction print(bnfisprincipal(Kn, Y)) giving the whole data for the ideal Y considered.

2.3. **Decomposition of the algebraic norm $\nu_{L/K}$.** Put $x = \sigma - 1$; then

$$\nu_{L/K} = \sum_{i=0}^{p^N-1} \sigma^i = \sum_{i=0}^{p^N-1} (x+1)^i = \frac{(x+1)^{p^N} - 1}{x} = \sum_{i=1}^{p^N} \tbinom{p^N}{i} x^{i-1}.$$

We have the following elementary property which is perhaps known in Iwasawa's theory, but we have not found suitable references; see however [Jaul1986, IV.2 (b)], [Wash1997, § 13.3] or [BaCa2016, Cald2020]:

Theorem 2.7. *The prime number $p \geq 2$ and the integer $N \geq 1$ being given, the algebraic norm $\nu_{L/K} = \sum_{i=0}^{p^N-1} \sigma^i$ is of the form:*

$$\nu_{L/K} = (\sigma-1)^k \cdot A_k(\sigma-1,p) + p^{f(k)} \cdot B_k(\sigma-1,p), \text{ for all } k \in [1, p^N-1],$$

A_k, $B_k \in \mathbb{Z}[\sigma-1,p]$, where $f(k) = N-s$ if $k \in [p^s, p^{s+1}-1]$, $s \in [0, N-1]$.

Proof. From:

$$\begin{aligned}\nu_{L/K} = \tbinom{p^N}{1} + x\tbinom{p^N}{2} + \cdots + x^{k-1}\tbinom{p^N}{k} \\ + x^k\Big[\tbinom{p^N}{k+1} + x\tbinom{p^N}{k+2} + \cdots + x^{p^N-1-k}\tbinom{p^N}{p^N}\Big],\end{aligned}$$

we deduce that:

$$A_k(\sigma-1,p) = \tbinom{p^N}{k+1} + (\sigma-1)\tbinom{p^N}{k+2} + \cdots + (\sigma-1)^{p^N-1-k}\tbinom{p^N}{p^N}.$$

The computation of $B_k(\sigma-1,p)$ depends on the p-adic valuations of the $\binom{p^N}{j}$, $j \in [1,k]$. To find the maximal factor $p^{f(k)}$ dividing all the coefficients of the polynomial:

$$\tbinom{p^N}{1} + (\sigma-1)\tbinom{p^N}{2} + \cdots + (\sigma-1)^{k-1}\tbinom{p^N}{k},$$

in other words, to find the p-part of $\gcd\Big(\binom{p^N}{1}, \binom{p^N}{2}, \ldots, \binom{p^N}{k}\Big)$, we consider $s \in [0, N-1]$. Let v the p-adic valuation map.

Lemma 2.8. *One has $v\big(\binom{p^N}{p^s}\big) = N-s$, for all $s \in [0, N-1]$.*

Proof. We have $\binom{p^N}{p^s} = \frac{p^N!}{p^s! \cdot (p^N-p^s)!}$; then, using the well-known formula:

$$v(m!) = \frac{m - S(m)}{p-1}, \ \ m \geq 1,$$

where $S(m)$ is the sum of the digits in the writing of m in base p, we get $v(p^N!) = \frac{p^N-1}{p-1}$, $v(p^s!) = \frac{p^s-1}{p-1}$ and $v((p^N-p^s)!) = \frac{p^N-p^s-(p-1)(N-s)}{p-1}$ since p^N-p^s may be written $p^s(p^{N-s}-1)$ with:

$$p^{N-s}-1 = 1(p-1)+p(p-1)+p^2(p-1)+\cdots+p^{N-s-2}(p-1)+p^{N-s-1}(p-1),$$

giving $N-s$ times the digit $p-1$. Whence:

$$v\big(\tbinom{p^N}{p^s}\big) = \frac{1}{p-1}\big(p^N - 1 - (p^s-1) - (p^N-p^s-(p-1)(N-s))\big) = N-s,$$

for all $s \in [0, N-1]$. □

Lemma 2.9. *For all $k \in [p^s+1, p^{s+1}-1]$, $s \in [0, N_1]$, we have $v\big(\binom{p^N}{k}\big) \geq N-s$.*

Proof. Consider $\binom{p^N}{k}\binom{p^N}{p^s}^{-1}$, $k \in [p^s+1, p^{s+1}-1]$ (the interval is empty for $p=2$, $s=0$; so we drop this obvious case), to show that its valuation is non-negative:

$$\begin{aligned}\frac{\binom{p^N}{k}}{\binom{p^N}{p^s}} &= \frac{p^N!}{k!\,(p^N-k)!} \times \frac{p^s!\,(p^N-p^s)!}{p^N!} \\ &= \frac{p^s!}{k!} \times \frac{(p^N-p^s)!}{(p^N-k)!} \\ &= \frac{1}{(p^s+1)(p^s+2)\cdots(p^s+(k-p^s))} \times \frac{(p^N-p^s)!}{(p^N-k)!} \\ &= \frac{(p^N-k+1)(p^N-k+2)\cdots(p^N-k+(k-p^s))}{(p^s+1)(p^s+2)\cdots(p^s+(k-p^s))}.\end{aligned}$$

Put $k=p^s+h$, $h\in[1,p^s(p-1)-1]$; then we can write:

$$\begin{aligned}\frac{\binom{p^N}{k}}{\binom{p^N}{p^s}}&=\frac{[p^N-(p^s+h)+1][p^N-(p^s+h)+2]\cdots[p^N-(p^s+h)+(h)]}{[p^s+1][p^s+2]\cdots[p^s+h]}\\&=\frac{[p^N-(p^s+h)+1][p^N-(p^s+h)+2]\cdots[p^N-(p^s+h)+(h)]}{[p^s+h][p^s+h-1]\cdots[p^s+h-(h-1)]}\\&=\frac{[p^N-(p^s+h)+1]}{[(p^s+h)-1]}\,\frac{[p^N-(p^s+h)+2]}{[(p^s+h)-2]}\cdots\frac{[p^N-(p^s+h)+(h-1)]}{[(p^s+h)-(h-1)]}\\&\quad\times\frac{p^N-p^s}{p^s+h}.\end{aligned}$$

We remark that each factor of the form $\frac{[p^N-(p^s+h)+j]}{[(p^s+h)-j]}$ is a p-adic unit for $j\in[1,h-1]$. Now, consider the last factor $\frac{p^N-p^s}{p^s+h}=\frac{p^s}{p^s+h}$, up to a p-adic unit; since $h\le p^s(p-1)-1$ one can put $h=\lambda p^u$, $p\nmid\lambda$, $u\le s$; the case $u<s$ is obvious and if $u=s$, the relation $h\le p^s(p-1)-1$ implies $\lambda\le p-2$, thus $p^s+h=p^s(1+\lambda)$ with $1+\lambda\le p-1$ and $\frac{p^s}{p^s+h}$ is a p-adic unit, whence the lemma. □

This leads to the expression of $f(k)$ on $\bigcup_{s=0}^{N-1}[p^s,p^{s+1}-1]=[1,p^N-1]$, to the proof of the theorem and then that of Theorem 1.1 (i). □

The following corollary, proving Theorem 1.1 (ii), is of easy use in practice:

Corollary 2.10. *(i) A sufficient condition for a partial capitulation of $\mathscr{H}_K^0$ in L, is that there exists $h\in\mathscr{H}_L$ such that $\mathbf{N}_{L/K}(h)\ne1$, h of order p^e, annihilated by $(\sigma-1)^m$, for integers e,m such that $e\in[1,N-s]$ if $m\in[p^s,p^{s+1}-1]$ for $s\in[0,N-1]$.*

(ii) Let $m(L)$ be the minimal integer such that $\mathscr{H}_L^{m(L)}=\mathscr{H}_L$ and let $p^{e(L)}$ be the exponent of $\mathscr{H}_L$. A sufficient condition for the complete capitulation of $\mathscr{H}_K^0$ in L, is that $e(L)\in[1,N-s]$ if $m(L)\in[p^s,p^{s+1}-1]$ for $s\in[0,N-1]$.

Proof. If $h\in\mathscr{H}_L$, $\nu_{L/K}(h)=\mathbf{J}_{L/K}(\mathbf{N}_{L/K}(h))=\left(h^{(\sigma-1)^k}\right)^A\times\left(h^{p^{f(k)}}\right)^B$, for all $k\in[1,p^N-1]$. Thus, $\mathbf{J}_{L/K}$ is non-injective as soon as h fulfills the conditions stated in (i) with $k=m\in[p^s,p^{s+1}-1]$ and $f(k)=N-s\ge e$. For the triviality of $\nu_{L/K}$, it suffices that $m=m(L)$ and $e=e(L)$ be solution. □

In other words, if the length $m(L)$ of the filtration is not too large as well as the exponent $p^{e(L)}$ of $\mathscr{H}_L$, then using suitable values of k, we obtain $\nu_{L/K}(h)=1$ for all $h\in\mathscr{H}_L$ (or only for some ones), whence complete (or partial) capitulation of $\mathscr{H}_K^0$ in L (or capitulation complete or incomplete of $\mathscr{H}_K$ if L/K is totally ramified).

Another way to interpret this result is to say that if N is large enough and if the Galois complexity of the p-class groups $\mathscr{H}_{K_n}$, for the layers K_n, does not increase too much, then $\mathscr{H}_K$ capitulates in L (in the totally ramified case).

For this, we introduce the following definition:

Definition 2.11. Let L/K be any cyclic p-extension totally ramified of degree p^N, $N\ge1$, and let σ be a generator of $\mathrm{Gal}(L/K)$. Let $\mathscr{H}_K$ and $\mathscr{H}_L$ be the p-class groups of K and L, respectively. We denote by $p^{e(L)}$ the exponent of $\mathscr{H}_L$ and by $m(L)$ the length of the filtration $\left\{\mathscr{H}_L^i\right\}_{i\ge0}$ (i.e., the least integer m such that $(\sigma-1)^m$ annihilates $\mathscr{H}_L$, cf. § 2.1.1). Then we say that L/K is of *smooth complexity* if the conditions $e(L)\in[1,N-s$ if $m(L)\in[p^s,p^{s+1}-1]$ for $s\in[0,N-1]$, are fulfilled.

2.3.1. *Program giving the decompositions of* $\nu_{L/K}$. The following program put $\nu_{L/K}$ in the form $P(x,p) = x^k A(x,p) + p^{f(k)} B(x,p)$, $1 \leq k \leq p^N - 1$, where $x = \sigma - 1$ and p are considered as indeterminate variables. This is necessary to have universal expressions for any $\mathscr{H}_L$ as $\mathbb{Z}_p[G]$-module; in other words, we do not reduce modulo p the coefficients of A and B. One must note that, except the cases $k = 1$, $B = 1$ and $k = p^N - 1$, $A = 1$, the polynomials A and B are not invertible in the group algebra, which allows improvements of the standard reasoning with $(k,\ p^{f(k)})$.

One must precise the numerical prime number p in Prime and N in N:

```
PROGRAM OF DECOMPOSITION OF THE ALGEBRAIC NORM
{Prime=2;N=3;P=0;for(i=1,Prime^N,C=binomial(Prime^N,i);
v=valuation(C,Prime);c=C/Prime^v;P=P+c*p^v*x^(i-1));print("P=",P);
for(k=1,Prime^N-1,B=lift(Mod(P,x^k));w=valuation(B,p);print();
print("P=x^",k,".A+p^",w,".B");print("A=",(P-B)/x^k);print("B=",B/p^w))}
```

2.3.2. *Case* $p = 2, N = 1$*:* $P = x + p$.

```
P=x+p
A=1
B=1
```

2.3.3. *Case* $p = 3, N = 1$*:* $P = x^2 + p * x + p$.

```
P=x^1.A+p^1.B          P=x^2.A+p^1.B
A=x+p                  A=1
B=1                    B=x+1
```

2.3.4. *Case* $p = 2, N = 2$*:* $P = x^3 + p^2 * x^2 + 3 * p * x + p^2$.

```
P=x^1.A+p^2.B          P=x^2.A+p^1.B          P=x^3.A+p^1.B
A=x^2+p^2*x+3*p        A=x+p^2                A=1
B=1                    B=3*x+p                B=p*x^2+3*x+p
```

For the next examples, we only write P.

2.3.5. *Case* $p = 3, N = 2$*:*

```
P=x^8+p^2*x^7+4*p^2*x^6+28*p*x^5+14*p^2*x^4+14*p^2*x^3+28*p*x^2+4*p^2*x+p^2
```

2.3.6. *Case* $p = 2, N = 3$*:*

```
P=x^7+p^3*x^6+7*p^2*x^5+7*p^3*x^4+35*p*x^3+7*p^3*x^2+7*p^2*x+p^3
```

2.3.7. *Case* $p = 2, N = 4$*:*

```
P=x^15+p^4*x^14+15*p^3*x^13+35*p^4*x^12+455*p^2*x^11+273*p^4*x^10
  +1001*p^3*x^9+715*p^4*x^8+6435*p*x^7+715*p^4*x^6+1001*p^3*x^5+273*p^4*x^4
  +455*p^2*x^3+35*p^4*x^2+15*p^3*x+p^4
```

2.3.8. *Case* $p = 3, N = 3$*:*

```
P= x^26+p^3*x^25+13*p^3*x^24+325*p^2*x^23+650*p^3*x^22+2990*p^3*x^21
  +32890*p^2*x^20+32890*p^3*x^19+82225*p^3*x^18+1562275*p*x^17
  +312455*p^3*x^16+482885*p^3*x^15+1931540*p^2*x^14+742900*p^3*x^13
  +742900*p^3*x^12+1931540*p^2*x^11+482885*p^3*x^10+312455*p^3*x^9
  +1562275*p*x^8+82225*p^3*x^7+32890*p^3*x^6+32890*p^2*x^5+2990*p^3*x^4
  +650*p^3*x^3+325*p^2*x^2+13*p^3*x+p^3
```

Thus, as soon as $\nu_{L/K} = (\sigma - 1)^k \times A_k(\sigma - 1, p) + p^{f(k)} \times B_k(\sigma - 1, p)$, with $m(L) \leq k$ and $e(L) \leq f(k)$, then $\mathscr{H}_K$ capitulates in L. But the reciprocal does not hold and capitulation (complete or incomplete) may occur whatever the invariants $m(L)$ and $e(L)$. We shall give examples where the above property applies, apart from the obvious case of stability. More generally, on may reduce P modulo $x^{m(L)}$ and modulo $p^{e(L)}$, which may give interesting relations.

2.4. **Non-trivial uses of the decomposition of $\nu_{L/K}$.** In what follows, the extensions L/K are always totally ramified.

Example 2.12. Let's begin with an example where the structure of $\mathscr{H}_{K_n}$ grows sufficiently with n, giving no capitulation up to $n = 3$; next, another choice of ℓ leads to capitulation, but where Theorem 1.1 (i) does not apply.

(i) We consider the cyclic cubic field K of conductor 703, $\mathsf{p} = 2$, $\mathsf{mKn} = 3$ and $\mathsf{ell} = 17$. Then $\mathscr{H}_K \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and we get, from the Program 4.1 (with several hours for the layer $n = 3$):

```
p=2  Nn=3  f=703  PK=x^3+x^2-234*x-729  mKn=3  CK0=[6,2]  ell=17  rho=1
CK1=[12,4]
h_1^[(S-1)^1]=[0,2]      h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]      h_2^[(S-1)^2]=[0,0]
h_1^[(S-1)^3]=[0,0]      h_2^[(S-1)^3]=[0,0]
norm in K1/K of the component 1 of CK1:[2,2]
norm in K1/K of the component 2 of CK1:[2,0]
No capitulation in K1, m(K1)=2, e(K1)=2
CK2=[24,8]
h_1^[(S-1)^1]=[0,2]      h_2^[(S-1)^1]=[6,6]
h_1^[(S-1)^2]=[4,4]      h_2^[(S-1)^2]=[4,0]
h_1^[(S-1)^3]=[0,0]      h_2^[(S-1)^3]=[0,0]
norm in K2/K of the component 1 of CK2:[4,4]
norm in K2/K of the component 2 of CK2:[4,0]
No capitulation in K2, m(K2)=3, e(K2)=3
CK3=[48,16]
h_1^[(S-1)^1]=[8,10]     h_2^[(S-1)^1]=[10,14]
h_1^[(S-1)^2]=[4,12]     h_2^[(S-1)^2]=[12,8]
norm in K3/K of the component 1 of CK3:[8,8]
norm in K3/K of the component 2 of CK3:[8,0]
No capitulation in K3, m(K2)=4, e(K2)=4
```

For $n = 1$, the algebraic norm $\nu_{K_1/K}$ is given by $\mathsf{P} = \mathsf{x} + 2$. There is no capitulation and no pair $(k, f(k))$ satisfying the conditions since $m(K_1) = 2$ $(s = 1)$ and $e(K_1) = 2 > n - s = 1 - 1 = 0$.

For $n = 2$, $\nu_{K_2/K}$ is given by $\mathsf{P} = \mathsf{x}^3 + 2^2 * \mathsf{x}^2 + 3 * 2 * \mathsf{x} + 2^2$. There is no capitulation and conditions are not satisfied since $m(K_2) = 3$ $(s = 1)$ and $e(K_2) = 3 > n - s = 2 - 1 = 1$.

For $n = 3$, $\nu_{K_3/K}$ is given by:

$$\mathsf{P} = \mathsf{x}^7 + 2^3 * \mathsf{x}^6 + 7 * 2^2 * \mathsf{x}^5 + 7 * 2^3 * \mathsf{x}^4 + 35 * 2 * \mathsf{x}^3 + 7 * 2^3 * \mathsf{x}^2 + 7 * 2^2 * \mathsf{x} + 2^3,$$

+ but there is no term of the form $2^4 B$ for $e(K_3) = 4$.

In this case the complexity increases due to successive exponents $1, 2, 3, 4$. We ignore what happens for $n = 4$.

(ii) Changing $\ell = 17$ into $\ell = 97$ gives complete capitulation in K_2 because of a smooth complexity, but higher than a stability:

```
p=2  Nn=2  f=703  PK=x^3+x^2-234*x-729  CK0=[6,2]  ell=97  rho=1
CK1=[6,2,2,2]
h_1^[(S-1)^1]=[1,1,0,0]      h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,0,1,1]      h_4^[(S-1)^1]=[0,0,1,1]
h_1^[(S-1)^2]=[0,0,0,0]      h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]      h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,0,0]
norm in K1/K of the component 2 of CK1:[1,1,0,0]
norm in K1/K of the component 3 of CK1:[0,0,1,1]
norm in K1/K of the component 4 of CK1:[0,0,1,1]
No capitulation in K1, m(K1)=2, e(K1)=1
CK2=[12,4,2,2]
h_1^[(S-1)^1]=[0,2,1,1]      h_2^[(S-1)^1]=[0,2,1,0]
h_3^[(S-1)^1]=[2,2,0,0]      h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,2,0,0]      h_2^[(S-1)^2]=[2,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0]      h_4^[(S-1)^2]=[0,0,0,0]
```

```
h_1^[(S-1)^3]=[0,0,0,0]      h_2^[(S-1)^3]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation in K2, m(K2)=3, e(K2)=2
```

For $n = 2$, $\mathsf{P} = \mathsf{x}^3 + 2^2 * \mathsf{x}^2 + 3 * 2 * \mathsf{x} + 2^2$, $m(K_2) = 3$ and $e(K_2) = 2$, then, modulo $(x^3, 2^2)$, it follows that $3 * 2 * \mathsf{x}$ annihilates $\mathscr{H}_{K_2}$, which is confirmed by the data. The conditions of application of Theorem 1.1 (i) are for $m(K_2) = 3$, $s = 1$ and $e(K_2) \leq 2 - s$, which are not fulfilled. So, Theorem 1.1 (i) gives a sufficient condition, not necessary.

Example 2.13. We consider the cyclic cubic field K of conductor 1777, $p = 2$ and $\ell = 17$. Then $\mathscr{H}_K \simeq (\mathbb{Z}/2^2\mathbb{Z})^2$ of exponent 4.

```
p=2  Nn=3  f=1777  PK=x^3+x^2-592*x+724  CK0=[4,4]  ell=17  rho=3
CK1=[8,8]
h_1^[(S-1)^1]=[0,0]     h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation in K1, m(K1)=1, e(K1)=3
CK2=[8,8]
h_1^[(S-1)^1]=[0,0]     h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Incomplete capitulation, m(K2)=1, e(K2)=3
CK3=[8,8]
h_1^[(S-1)^1]=[0,0]     h_2^[(S-1)^1]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Complete capitulation in K3, m(K3)=1, e(K3)=3
```

In this case, the stability from K_1 implies necessarily the capitulation in K_3 (so the third computation is for checking). Moreover, for all $n \geq 1$, $\mathscr{H}_{K_n}$ is annihilated by $\sigma - 1$, whence $m(K_n) = 1$, $s = 0$, $e(K_3) = n - s$, $\mathscr{H}_{K_n} = \mathscr{H}_{K_n}^{G_n}$ as expected from Theorem 1.1 (ii) and given by Program 2.2.

Example 2.14. (i) We consider the quadratic field $\mathbb{Q}(\sqrt{142})$ with $p = 3$ and various primes $\ell \equiv 1 \pmod 3$, $\ell \not\equiv 1 \pmod 9$, so that $N = 1$, $L = K_1$, and the conditions of Theorem 1.1 (i) are $e(K_1) = 1 \leq 1 - s$, whence $s = 0$ and $m(K_1) \in [1, 2]$:

```
p=3  PK=x^2-142  N=1  CK0=[3]  ell=13  rho=2
CK1=[3,3]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation in K1, m(K1)=1, e(K1)=1
p=3  PK=x^2-142  N=1  CK0=[3]  ell=1123  rho=2
CK1=[21,3]
h_1^[(S-1)^1]=[1,2]   h_2^[(S-1)^1]=[1,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation in K1, m(K1)=2, e(K1)=1
p=3  PK=x^2-142  N=1  CK0=[3]  ell=208057  rho=2
CK1=[3,3,3,3]
h_1^[(S-1)^1]=[2,2,0,0]   h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,1,1,1]   h_4^[(S-1)^1]=[2,1,2,2]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
```

```
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation in K1, m(K1)=2, e(K1)=1
```

(ii) For $p = 5$ and $N = 1$ the conditions become $e(K_1) = 1 \leq 1 - s$, whence $s = 0$, with $m(K_1) \in [1, 4]$, which offers more possibilities:

```
p=5  PK=x^2-401  N=1  CK0=[5]  ell=1231  rho=2
CK1=[5,5]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation in K1, m(K1)=1, e(K1)=1
```

```
p=5  PK=x^2-401  N=1  CK0=[5]  ell=1741  rho=1
CK1=[5,5]
h_1^[(S-1)^1]=[3,3]    h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation in K1, m(K1)=2, e(K1)=1
```

```
p=5  PK=x^2-401  CK0=[5]  ell=4871  rho=1
CK1=[10,10,10,2]
h_1^[(S-1)^1]=[4,0,4,0]  h_2^[(S-1)^1]=[1,4,0,0]
h_3^[(S-1)^1]=[3,4,2,0]  h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[3,1,4,0]  h_2^[(S-1)^2]=[3,1,4,0]
h_3^[(S-1)^2]=[2,4,1,0]  h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0]  h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0]  h_4^[(S-1)^3]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Complete capitulation in K1, m(K1)=3, e(K1)=1
```

Example 2.15. We consider the cubic field of conductor $f = 20887$ with $p = 2$ and $\ell = 17$:

```
p=2  Nn=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]  ell=17  rho=3
CK1=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=1, e(K1)=3
```

We note that $\mathbf{N}_{K_1/K}(h_i) \neq 1$ for $i = 3, 4$, otherwise $\mathbf{N}_{K_1/K}(\mathscr{H}_{K_1})$ would be of 2-rank 2 instead of 4 (absurd). Since $m(K_1) = 1$ (all classes are invariant), Theorem 1.1 (i) applies non-trivially for the classes h_3, h_4 of order 2 ($m = 1$, $s = 0$, $e \in [1, 1]$, which is indeed the case).

Let's give the complete data proving the capitulation of the two classes of K of order 2; the instruction $\mathsf{CK0} = \mathsf{K.clgp}$ gives:

```
[64,[4,4,2,2],[[2897,2889,2081;0,1,0; 0,0,1],[2897,825,2889;0,1,0;0,0,1],
[17,16,13;0,1,0;0,0,1],[53,36,44;0,1,0; 0,0,1]]]
```

it describes $\mathscr{H}_K$ with 4 representative ideals of generating classes; that of order 2 are:

$$\mathfrak{a}_3 = [17, 16, 13; 0, 1, 0; 0, 0, 1], \quad \mathfrak{a}_4 = [53, 36, 44; 0, 1, 0; 0, 0, 1];$$

the following 6 large coefficients (on the integral basis computed by PARI) give integers α_i of L with the relations $(\mathfrak{a}_i)_L = (\alpha_i)$:

```
[[0,0,0,0]~,[4482450896,-1173749328,81969609,69123722,7646555,39729395]~]
norm in K1/K of the component 3 of CK1:List([0,0,0,0])
[[0,0,0,0]~,[-4877380814,1968946273,-1411818,102996743,38571732,40207952]~]
norm in K1/K of the component 4 of CK1:List([0,0,0,0])
```

At the layer $n = 2$, the result is similar, but shows that the classes of order 4 of $\mathscr{H}_K$ never capitulate:

```
CK2=[16,16,2,2]
h_1^[(S-1)^1]=[8,0,0,0] h_2^[(S-1)^1]=[0,8,0,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=2, e(K2)=4
```

2.5. **Case where no capitulation can happen.** For instance, if K is an imaginary quadratic field and L/K a cyclic extension, totally ramified, of degree p^N, with $L/\mathbb{Q}$ abelian, we know that there is never capitulation (except few cases for $p = 2$). This implies necessarily that, as soon as $\mathscr{H}_K \neq 1$, the structure of $\mathscr{H}_L$ does not allow the previous use of the algebraic norm and the complexity is not smooth according to the Definition 2.11; in the following example, for $p = 3$, $n = 1$, 2, we examine the structure of $\mathscr{H}_{K_n}$:

Example 2.16. We consider $K = \mathbb{Q}(\sqrt{-199})$, $p = 3$ and $\ell = 19$ and use a modified version on Program 6 given further:

```
{p=3;Nn=2;m=199;ell=19;mKn=2;PK=x^2+m;K=bnfinit(PK,1);
CK0=K.clgp;rho=(kronecker(-m,ell)+3)/2;print("rho=",rho);
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
dn=poldegree(Pn);Kn=bnfinit(Pn,1);if(n==1,print();
print("PK=",PK," CK0=",CK0[2]," ell=",ell," rho=",rho));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu)))}
```

```
p=3  Nn=2  PK=x^2+199  CK0=[9]  ell=19  rho=1
CK1=[513]=[27.19]
h_1^[(S-1)^1]=[9]
h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
m(K1)=2, e(K1)=3
CK2=[749493,19,19]=[81.19.487,19,19]
h_1^[(S-1)^1]=[45,0,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
m(K2)=2, e(K1)=4
```

(i) Case $n = 1$. We have $\mathscr{H}_K \simeq \mathbb{Z}/9\mathbb{Z}$ and $\mathscr{H}_{K_1} \simeq \mathbb{Z}/3^3\mathbb{Z}$; so, $\#\mathscr{H}_{K_1}^{G_1} = \#\mathscr{H}_K = 9$ since ℓ is inert in K (trivial norm factor); thus, necessarily $m(K_1) = 2$ and $e(K_1) = 3$ (i.e., x^2 and 3^3 annihilate $\mathscr{H}_{K_1}$).

But for $n = 1$, $\mathsf{P} = \mathsf{x}^2 + 3 * \mathsf{x} + 3 \equiv 3 * \mathsf{x} + 3 \pmod{\mathsf{x}^2}$ and $\mathscr{H}_{K_1}^3 \simeq \mathbb{Z}/9\mathbb{Z}$ is not annihilated by $x + 1$ (invertible). So, $\nu_{K_1/K}(\mathscr{H}_{K_1}) \simeq \mathscr{H}_K$ as expected.

(ii) Case $n=2$. Then $\mathscr{H}_{K_2}\simeq\mathbb{Z}/3^4\mathbb{Z}]$; we have $\#\mathscr{H}_{K_2}^{G_2}=\#\mathscr{H}_K=9$, giving $\mathscr{H}_{K_2}^{\sigma_2-1}=\mathscr{H}_{K_2}^{G_2}$, whence $m(K_2)=2$ and $e(K_2)=4$. Similarly, the data confirm this. The decompositions of P being of the form:

```
x^8+3^2*x^7+4*3^2*x^6+28*3*x^5+14*3^2*x^4+14*3^2*x^3+28*3*x^2+4*3^2*x+3^2
```

the conditions are not fulfilled to apply Theorem 1.1 (i) ($m(K_2)=2$, $s=0$, $e(K_2)=4$). More precisely, $\mathsf{P}\equiv 4*3^2*\mathsf{x}+3^2 \pmod{\mathsf{x}^2}$ and $\mathscr{H}_{K_2}^9\simeq\mathbb{Z}/3^2\mathbb{Z}$ is not annihilated by $4*x+1$ (invertible). So $\nu_{K_2/K}(\mathscr{H}_{K_2})\simeq\mathscr{H}_K$ as for the case $n=1$.

Example 2.17. We consider $K=\mathbb{Q}(\sqrt{-199})$, $\ell=37$.

```
p=3  Nn=2  PK=x^2+199  CK0=[9]  ell=37   rho=1
CK1=[54,6,3]
h_1^[(S-1)^1]=[21,1,0]   h_2^[(S-1)^1]=[18,0,1]   h_3^[(S-1)^1]=[18,0,0]
h_1^[(S-1)^2]=[0,0,1]    h_2^[(S-1)^2]=[18,0,0]   h_3^[(S-1)^2]=[0,0,0]
h_1^[(S-1)^3]=[18,0,0]   h_2^[(S-1)^3]=[0,0,0]
h_1^[(S-1)^4]=[0,0,0]
norm in K1/K of the component 1 of CK1:[12,0,1]
norm in K1/K of the component 2 of CK1:[18,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
m(K1)=4, e(K1)=3
CK2=[42442542,18,9]
m(K2)=?, e(K1)=4
```

(i) Case $n=1$. In this case, $\mathscr{H}_{K_1}\simeq\mathbb{Z}/3^3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$; the above data shows that $m(K_1)=4$ for $e(K_1)=3$, and a more complex structure, since $s=1$, but $e(K_1)=3$. We have $\nu_{K_1/K}(\mathscr{H}_{K_1})\simeq\mathscr{H}_K$.

(ii) Case $n=2$. For K_2, $\mathscr{H}_{K_2}\simeq\mathbb{Z}/3^4\mathbb{Z}\times\mathbb{Z}/3^2\mathbb{Z}\times\mathbb{Z}/3^2\mathbb{Z}$. So $m(K_2)=4$, $s=1$ with $e(K_2)=4$.

3. Other contexts for capitulation kernels

Questions of capitulation of other arithmetic invariants are at the origin of many papers (e.g., [Maire1996, KoMo2000, Brig2007, Vali2008, Jaul2016, GJN2016, Jaul2019^a, Jaul2019^b, KhPr2000, Jaul2022, Gras2022^a] and their references); they are related to generalized p-class groups with conditions of ramification and decomposition, wild kernels of K-theory, torsion groups in p-ramification theory, Tate–Chafarevich groups, Bertrandias–Payan modules about the embedding problem, logarithmic class groups[4]. The same techniques, using the algebraic norm, may be applied; the results essentially depend on the properties of $\mathbf{N}_{L/K}$ and $\mathbf{J}_{L/K}$.

3.1. Arithmetic invariants that do not capitulate.

We note that some invariants never capitulate. For instance we will evoke the case of the torsion group $\mathscr{T}_K$ of the Galois group of the maximal abelian p-ramified pro-p-extension of a number field K; then under Leopoldt's conjecture, the transfer map is always injective, whatever the extensions of number fields considered. This has some consequences because of the formula:

$$\#\mathscr{T}_K=\#\mathscr{W}_K\cdot\#\mathscr{R}_K\cdot\#\widetilde{\mathscr{H}_K},$$

where $\mathscr{W}_K$ is a canonical invariant built on the groups of (local and global) roots of unity of K, $\mathscr{R}_K$ is the normalized p-adic regulator, and $\widetilde{\mathscr{H}_K}$ a subgroup of $\mathscr{H}_K$ (see, for instance [Gras2005, Theorem IV.2.1], [Gras2018, Diagram § 3 and § 5], [Gras2021^a]). Note that for Bertrandias–Payan modules the transfers $\mathbf{J}_{L/K}$ are injective, except few special cases discussed in [GJN2016].

[4] Invariant defined in [Jaul1994] and being, in the class field theory viewpoint, isomorphic to $\mathrm{Gal}(H_K^{\mathrm{lc}}/K^{\mathrm{cyc}})$, where H_K^{lc} is the maximal abelian locally cyclotomic pro-p-extension of K and K^{cyc} its cyclotomic $\mathbb{Z}_p$-extension

Thus, as we have seen, in any cyclic p-extension L/K of Galois group G, the complexity of the invariants $\mathscr{T}_{K_n}$ must be increasing with n, regarding that of K; this may be checked by means the following general framework:

Definition 3.1. Assume given a family $\mathscr{X}_k$ of finite invariants of p-power order, indexed by the set of number fields k (or a suitable subset), fulfilling the following conditions:

(i) For any Galois extension k'/k, of Galois group G, $\mathscr{X}_{k'}$ is a $\mathbb{Z}_p[G]$-module;

(ii) There exist arithmetic norms $\mathbf{N}_{k'/k} : \mathscr{X}_{k'} \to \mathscr{X}_k^0 \subseteq \mathscr{X}_k$ and transfer maps $\mathbf{J}_{k'/k} : \mathscr{X}_k \to \mathscr{X}_{k'}$, such that $\mathbf{J}_{k'/k} \circ \mathbf{N}_{k'/k} = \nu_{k'/k}$.

When G is a cyclic p-group, we define the associated filtration $\{\mathscr{X}_{k'}^i\}_{i\geq 0}$ defined by $\mathscr{X}_{k'}^{i+1}/\mathscr{X}_{k'}^i := (\mathscr{X}_{k'}/\mathscr{X}_{k'}^i)^G$, for all $i \geq 0$.

Thus, in a cyclic p-extension L/K, the conditions $e(L) \in [1, N-s]$ for $m(L) \in [p^s, p^{s+1}-1]$, $s \in [0, N-1]$, of Theorem 1.1 (i), apply in the same way, independently of the fact of being able to calculate the orders of the $\mathscr{X}_L^i$'s by means of a suitable algorithm moving from $\mathscr{X}_L^i$ to $\mathscr{X}_L^{i+1}$.

Theorem 3.2. *Let L/K be a cyclic p-extension of degree p^N, $N \geq 1$, and let K_n be the subfield of L of degree p^n over K, $n \in [0, N]$. We assume that, for all $n \in [0, N-1]$, the arithmetic norms $\mathbf{N}_{K_{n+1}/K_n}$ are surjective and that the transfer maps $\mathbf{J}_{K_{n+1}/K_n}$ are injective.*

Then $\#\mathscr{X}_{n+h} \geq \#\mathscr{X}_n \cdot \#\mathscr{X}_n[p^h]$, for all $n \in [0, N]$ and $h \in [0, N-n]$.

Proof. Put $G_n^{n+h} := \mathrm{Gal}(k_{n+h}/k_n)$ and so on for the maps $\mathbf{N}, \mathbf{J}$. From the exact sequence:

$$1 \to \mathbf{J}_n^{n+h}\mathscr{X}_n \to \mathscr{X}_{n+h} \to \mathscr{X}_{n+h}/\mathbf{J}_n^{n+h}\mathscr{X}_n \to 1,$$

we get:

$$\begin{aligned} 1 \to \mathscr{X}_{n+h}^{G_n^{n+h}}/\mathbf{J}_n^{n+h}\mathscr{X}_n &\to (\mathscr{X}_{n+h}/\mathbf{J}_n^{n+h}\mathscr{X}_n)^{G_n^{n+h}} \\ &\to \mathrm{H}^1(G_n^{n+h}, \mathbf{J}_n^{n+h}\mathscr{X}_n) \to \mathrm{H}^1(G_n^{n+h}, \mathscr{X}_{n+h}), \end{aligned}$$

where $\mathrm{H}^1(G_n^{n+h}, \mathbf{J}_n^{n+h}\mathscr{X}_n) = (\mathbf{J}_n^{n+h}\mathscr{X}_n)[p^h] \simeq \mathscr{X}_n[p^h]$ (injectivity of $\mathbf{J}_n^{n+h}$), and:

$$\#\mathrm{H}^1(G_n^{n+h}, \mathscr{X}_{n+h}) = \#\mathrm{H}^2(G_n^{n+h}, \mathscr{X}_{n+h}) = \#(\mathscr{X}_{n+h}^{G_n^{n+h}}/\mathbf{J}_n^{n+h}\mathscr{X}_n),$$

since $\nu_n^{n+h}\mathscr{X}_{n+h} = \mathbf{J}_n^{n+h} \circ \mathbf{N}_n^{n+h}\mathscr{X}_{n+h} = \mathbf{J}_n^{n+h}\mathscr{X}_n$ (surjectivity of $\mathbf{N}_n^{n+h}$), giving an exact sequence of the form:

$$1 \to A \to (\mathscr{X}_{n+h}/\mathbf{J}_n^{n+h}\mathscr{X}_n)^{G_n^{n+h}} \to \mathscr{X}_n[p^h] \to A', \text{ with } \#A' = \#A.$$

We then obtain the inequality $\#\mathscr{X}_{n+h} \geq \#\mathscr{X}_n \cdot \#\mathscr{X}_n[p^h]$. □

Corollary 3.3. *Let p^{e_n} be the exponent of $\mathscr{X}_n$, $n \in [0, N]$, and assume N arbitrary large. Then, as soon as $e_n \in [0, N-n]$, we have $\#\mathscr{X}_{n+e_n} \geq (\#\mathscr{X}_n)^2$. In any Iwasawa's theory context, if $\mu = 0$, then $e_n \leq \lambda$ for all $n \gg 0$. In particular, if $\lambda = \mu = 0$, then $\mathscr{X}_n = 1$ for all $n \geq 0$.*

Proof. The first claim is obvious; if $\mu = 0$ in the formula $\#\mathscr{X}_n = p^{\lambda n + \mu p^n + \nu}$ for all n large enough, the relation $\#\mathscr{X}_{n+1} \geq \#\mathscr{X}_n \cdot p^{e_n}$ implies the second one. □

So we must notice, in the practice, that Theorem 1.1 (i) does not apply since the complexity of the $\mathscr{X}_{K_n}$ crucially increases with n.

Example 3.4. We use the program [Gras2019[a], Corollary 2.2, Program I, § 3.2] computing the group structure of the $\mathscr{T}_{K_n}$'s for quadratic fields K, $p = 2$, $K_n \subset K(\mu_\ell)$ (so the number $r_2 + 1$ of independent $\mathbb{Z}_p$-extensions is 1

for $m > 0$ and $2^n + 1$ for $m < 0$). One must chose an arbitrary constant E, assuming $E > e_n + 1$, to be controlled a posteriori:

```
MAIN PROGRAM COMPUTING THE STRUCTURE OF TKn (real quadratic fields):
{p=2;ell=257;Nn=4;E=16;for(m=2,150,if(core(m)!=m,next);PK=x^2-m;
print("p=",p," PK=",PK," ell=",ell);for(n=0,Nn,Qn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);Knmod=bnrinit(Kn,p^E);
CKnmod=Knmod.cyc;TKn=List;d=matsize(CKnmod)[2];for(j=1,d-1,c=CKnmod[d-j+1];
w=valuation(c,p);if(w>0,listinsert(TKn,p^w,1)));print("TK",n,"=",TKn)))}
```

In a very simple context ($K = \mathbb{Q}(\sqrt{m})$, $L = K(\mu_{257})$, $p = 2$), the complexity of the torsion groups $\mathscr{T}_{K_n}$ is growing dramatically (for the p^k-ranks as well as the exponents) as shown by the following excerpts:

```
p=2  PK=x^2-2  ell=257
TK0=[]
TK1=[8,8]
TK2=[16,16,4,4,2,2]
TK3=[32,32,8,8,4,4,4,4,2,2,2,2]
TK4=[64,64,16,16,4,4,4,4,4,4,4,4,4,4,4,4,2,2,2,2]
```

```
p=2 ell=257 PK=x^2-73
TK0=[2]
TK1=[64,8,2,2]
TK2=[128,16,8,4,4,2,2,2]
TK3=[256,32,16,16,8,8,8,4,2,2,2,2,2,2,2,2]
TK4=[512,64,32,32,16,16,16,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
```

```
p=2  ell=257  PK=x^2-105
TK0=[2,2]
TK1=[16,8,2,2]
TK2=[32,16,8,4,2,2,2,2]
TK3=[64,32,8,8,8,8,8,2,2,2,2,2,2,2,2,2]
TK4=[128,64,16,16,16,16,16,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
```

```
p=2 ell=257 PK=x^2-113
TK0=[4]
TK1=[128,16,4]
TK2=[256,32,8,4,4,4,2]
TK3=[512,64,16,8,8,8,2,2,2,2,2,2,2,2,2]
TK4=[1024,128,32,16,16,16,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
```

In that example, the analogue of the Chevalley–Herbrand formula gives, in K_n/K, $\#\mathscr{T}_{K_n}^{G_n} = \mathscr{T}_K \cdot p^{\rho\, n}$, where $\rho = 1$ or 2 is the number of primes $\mathfrak{l} \mid \ell$ in $K/\mathbb{Q}$ [Gras2005, Theorem IV.3.3, Exercise 3.3.1]; unfortunately, we do not know formulas, similar to that of (2.3), for the orders of the $\mathscr{T}_{K_n}^i$ for $i > 1$.

Consider, for instance, the above case of $m = 113$, $L = K_4$, $\#\mathscr{T}_L = 2^{59}$, $\rho = 2$, $\nu_{L/K}(\mathscr{T}_L) \simeq \mathbb{Z}/4\mathbb{Z}$, with the associated polynomial:

```
P=x^15+p^4*x^14+15*p^3*x^13+35*p^4*x^12+455*p^2*x^11+273*p^4*x^10
  +1001*p^3*x^9+715*p^4*x^8+6435*p*x^7+715*p^4*x^6+1001*p^3*x^5
  +273*p^4*x^4+455*p^2*x^3+35*p^4*x^2+15*p^3*x+p^4
```

Since $\#(\mathscr{T}_L^{i+1}/\mathscr{T}_L^i) \leq \#\mathscr{T}_L^G = 4 \cdot 2^8 = 2^{10}$, this gives $m(L) \geq 10$ ($s \geq 3$). These computations show that $m(L)$ and $e(L) = 10$ are large. Moreover, the conditions of Theorem 1.1 (i), $e(L) \leq 4 - s$ can not be satisfied.

Taking imaginary quadratic fields does not modify the behavior of the $\mathscr{T}_{K_n}$ since, for all n, $\mathbf{J}_{K_n/K}$ is still injective and $\mathbf{N}_{K_n/K}$ surjective:

```
MAIN PROGRAM COMPUTING THE STRUCTURE OF TKn (imaginary quadratic fields):
{p=2;ell=257;Nn=3;E=16;for(m=2,150,if(core(m)!=m,next);PK=x^2+m;print("p=",p,
" ell=",ell," PK=",PK);for(n=0,Nn,r2=2^n+1;PKn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,PKn)[1];Kn=bnfinit(Pn,1);Knmod=bnrinit(Kn,p^E);
CKnmod=Knmod.cyc;TKn=List;d=matsize(CKnmod)[2];for(j=1,d-r2,c=CKnmod[d-j+1];
w=valuation(c,p);if(w>0,listinsert(TKn,p^w,1)));print("TK",n,"=",TKn)))}
```

```
p=2  PK=x^2+2  ell=257
TK0=[]
```

```
TK1=[16,2]
TK2=[16,4,4,4,4]
TK3=[32,8,4,4,4,4,4,4,4]
TK4=[64,16,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4]

p=2 ell=257 PK=x^2+3
TK0=[]
TK1=[16]
TK2=[32,8,4]
TK3=[64,16,4,4,4,4,4]
TK4=[128,32,4,4,4,4,4,4,4,4,4,4,4,4,4]

p=2 ell=257 PK=x^2+7
TK0=[2]
TK1=[8,2,2]
TK2=[16,4,2,2,2,2,2]
TK3=[32,8,2,2,2,2,2,2,2,2,2,2,2,2,2]
TK4=[64,16,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
```

For cyclic cubic fields and $p = 2$, $\ell = 257$, we obtain for instance:

```
MAIN PROGRAM COMPUTING THE STRUCTURE OF TKn (cyclic cubic fields):
{p=2;ell=257;Nn=3;E=16;bf=7;Bf=10^3;
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);print("p=",p," f=",f,
" PK=",PK," ell=",ell);for(n=0,Nn,Qn=polsubcyclo(ell,p^n);
Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);Knmod=bnrinit(Kn,p^E);
CKnmod=Knmod.cyc;TKn=List;d=matsize(CKnmod)[2];for(j=1,d-1,c=CKnmod[d-j+1];
w=valuation(c,p);if(w>0,listinsert(TKn,p^w,1)));print("TK",n,"=",TKn)))))}

p=2  f=31  PK=x^3+x^2-10*x-8  ell=257
TK0=[2,2]
TK1=[8,2,2,2,2]
TK2=[16,4,2,2,2,2,2,2,2,2,2]
TK3=[32,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]

p=2  f=43  PK=x^3+x^2-14*x+8  ell=257
TK0=[2,2]
TK1=[16,16,8,2,2]
TK2=[32,32,16,4,4,4,2,2,2,2,2]
TK3=[64,64,32,8,8,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]

p=2  f=171  PK=x^3-57*x-152  ell=257
TK0=[8,8]
TK1=[16,16,8,2,2]
TK2=[32,32,16,4,2,2,2,2,2,2,2]
TK3=[64,64,32,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]

p=2  f=277  PK=x^3+x^2-92*x+236  ell=257
TK0=[4,4]
TK1=[8,4,4,4,4]
TK2=[16,8,8,4,4,4,4,4,2,2,2]
TK3=[32,16,16,8,8,8,8,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2]
```

3.2. **Capitulation in $\mathbb{Z}_p$-extensions.** The problem of capitulations in a $\mathbb{Z}_p$-extension $\widetilde{K} = \bigcup_{n\geq 0} K_n$ of K has a long history from Iwasawa pioneering works showing, for instance, that the capitulation kernels $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n})$ have a bounded order as $n \to \infty$ [Iwas1973, Theorem 10, § 5]. The reader may refer for instance to [GrJa1985, BaCa2016, Cald2020] for classical context of p-class groups and to [KoMo2000, Vali2008] for wild kernels, [Jaul2016, Jaul2019[a]] for logarithmic class groups.

3.2.1. *Survey of known results under the assumption $\mu = 0$.* If $\mu = 0$, in the writing $\#\mathscr{H}_{K_n} = p^{\lambda n+\mu p^n+\nu}$ for $n \gg 0$, the following properties are proved in

[GrJa1985, Théorème, p. 214]. Let $\mathfrak{H}_{\widetilde{K}} := \varprojlim_n \mathscr{H}_{K_n}$ (for the arithmetic norms) be the Galois group of the maximal unramified abelian pro-p-extension of $\widetilde{K}$ and let $\mathscr{H}_{\widetilde{K}} := \varinjlim_n \mathscr{H}_{K_n}$ (for the transfer maps) be its p-class group; then $\mathfrak{H}_{\widetilde{K}} \simeq T \bigoplus \mathbb{Z}_p^{\lambda}$, where T is a finite group. Moreover, for all $n \gg 0$:

- $\mathbf{N}_{\widetilde{K}/K_n} : \mathfrak{H}_{\widetilde{K}} \to \mathscr{H}_{K_n}$ induces an isomorphism of T onto $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n})$.
- $\mathscr{H}_{K_n} \simeq \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \oplus \mathbf{J}_{\widetilde{K}/K_n}(\mathscr{H}_{K_n}) \simeq \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{n+\alpha_i}\mathbb{Z}$,

for some relative integers α_i.

From now on, we take the base field "$K = K_{n_0}$", n_0 large enough, in such a way that $\widetilde{K}/K$ is totally ramified and such that all the above properties are fulfilled from $n = 0$. Thus, $\#\mathscr{H}_{K_n} = p^{\lambda n+\nu}$, $\lambda, \nu \geq 0$ with $p^{\nu} = \#\mathscr{H}_K$, and:

$$\mathscr{H}_{K_n} \simeq \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{n+\alpha_i}\mathbb{Z}, \text{ with } \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \simeq T \ , \ \ \forall n \geq 0. \tag{3.1}$$

In particular, in this new context, $\alpha_i \geq 0$ and:

$$\mathscr{H}_K \simeq \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K}) \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{\alpha_i}\mathbb{Z}, \ \ \alpha_i \geq 0. \tag{3.2}$$

Proposition 3.5. *Under the above choice of the base field K in the $\mathbb{Z}_p$-extension $\widetilde{K}$ and assuming $\mu = 0$, the capitulation of $\mathscr{H}_K$ in $\widetilde{K}$ is equivalent for all $n \geq 0$ to the isomorphism $\mathscr{H}_{K_n} \simeq \mathscr{H}_K \oplus \mathbf{J}_{\widetilde{K}/K_n}(\mathscr{H}_{K_n}) \simeq \mathscr{H}_K \oplus \big(\mathbb{Z}/p^n\mathbb{Z}\big)^{\lambda}$.*

Proof. If $\mathscr{H}_K$ capitulates in $\widetilde{K}$, then $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K}) = \mathscr{H}_K$ and, from (3.2), $\alpha_i = 0$ for all $i \in [1, \lambda]$. So $\mathscr{H}_{K_n} \simeq \mathscr{H}_K \oplus \big(\mathbb{Z}/p^n\mathbb{Z}\big)^{\lambda}$, for all $n \geq 0$, since each capitulation kernel $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n})$ is isomorphic to T, whence isomorphic to $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K}) = \mathscr{H}_K$ (isomorphisms given by the arithmetic norms).

Reciprocally, assume that $\mathscr{H}_{K_n} \simeq \mathscr{H}_K \oplus \big(\mathbb{Z}/p^n\mathbb{Z}\big)^{\lambda}$; then, from (3.1):

$$\mathscr{H}_{K_n} = \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \oplus \bigoplus_{i=1}^{\lambda} \mathbb{Z}/p^{n+\alpha_i}\mathbb{Z} \simeq \mathscr{H}_K \oplus \big(\mathbb{Z}/p^n\mathbb{Z}\big)^{\lambda}.$$

Comparing the structures gives $\alpha_i = 0$ for all $i \in [1, \lambda]$ and $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_n}) \simeq \mathscr{H}_K$ for all n, whence the capitulation of $\mathscr{H}_K$ in $\widetilde{K}$. □

If the proposition applies, for all $n \geq 1$, the capitulation in incomplete; for instance $\mathscr{H}_{K_1} \simeq \mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_1}) \oplus (\mathbb{Z}/p\mathbb{Z})^{\lambda}$ with $\mathrm{Ker}(\mathbf{J}_{\widetilde{K}/K_1}) \simeq \mathscr{H}_K$.

3.2.2. *Case of the cyclotomic $\mathbb{Z}_p$-extension of K.* Assume that K is totally real and let $K_\infty = \bigcup_{n \geq 0} K_n$ be the cyclotomic $\mathbb{Z}_p$-extension of K, assuming the previous choice of K in K_∞. Greenberg's conjecture [Gree1976] for K_∞ ($\lambda = \mu = 0$) is equivalent to the stability of the $\#\mathscr{H}_{K_n}$'s from K, giving capitulations of all the class groups in K_∞ from $n = 0$, then equalities $\mathscr{H}_{K_n} = \mathscr{H}_{K_n}^{G_n}$ for all $n \geq 0$, the isomorphisms $\mathscr{H}_{K_m} \simeq \mathscr{H}_{K_n} \simeq \mathscr{H}_K$, for all $m \geq n \geq 0$; thus, for $n \geq 1$, $m(K_n) = 1$ ($s = 0$) with $e(K_n) = e(K)$, which is exactly the limit case of application of Theorem 1.1 (i) for $n \geq e(K)$.

In [KrSch1995, Paga2022] such properties of stability are used to check the conjecture by means of analytic formulas.

In [Jaul2016, Jaul2019^a, Jaul2019^b], it is proved that Greenberg's conjecture is equivalent to the capitulation of the logarithmic class group $\widetilde{\mathscr{T}}_K$ in K_∞; this may be effective if, by chance, a capitulation occurs in the firsts layers over the base field K; indeed, this criterion is probably the only one giving an algorithmic test (using [BeJa2016]) from the base field.

In what follows, we will analyze the generalized Chevalley–Herbrand formula in K_∞/K.

Conventions 3.6. *Taking in the sequel, as totally real base field K, a suitable layer "K_{n_0}" in K_∞, we may assume, without restricting the generality, the following properties of K_∞/K:*

(i) p is totally ramified in K_∞/K;

(ii) Iwasawa's formula $\#\mathscr{H}_{K_n} = p^{\lambda n+\mu p^n+\nu}$ is valid for all $n \geq 0$ (this gives new invariants of the form $(\lambda,\ \mu p^{n_0},\ \nu+\lambda n_0)$ that we still denote (λ,μ,ν)).

Then, as for the "tame casee", formulas (2.3) hold with $r = \#\{\mathfrak{p},\ \mathfrak{p}\,|\,p \text{ in } K\}$ and the filtration still depends on the class and norm factors. However, the norm factors can be interpreted, from generalizations of Taya results, as divisor of the normalized p-adic regulator of K, as follows from class field theory:

Definitions 3.7. *(i) Let $H_{K_n}^{\rm gen}$ be the genus field of K_n (i.e., the subfield of the p-Hilbert class field $H_{K_n}^{\rm nr}$, abelian over K and maximal, whence the subfield of $\mathrm{Gal}(H_{K_n}^{\rm nr}/K)$ fixed by the image of $\mathscr{H}_{K_n}^{\sigma_n-1}$), and let $K_\infty H_{K_n}^{\rm gen}$, which is abelian p-ramified over K, whence $K_\infty H_{K_n}^{\rm gen} \subseteq H_K^{\rm pr}$, the maximal p-ramified abelian pro-p-extension of K. So $\mathscr{T}_K := \mathrm{Gal}(H_K^{\rm pr}/K_\infty)$ is finite under Leopoldt's conjecture.*

We define $H_{K_\infty}^{\rm gen} := \bigcup_n K_\infty H_{K_n}^{\rm gen}$ and put $\mathscr{G}_K := \mathrm{Gal}(H_{K_\infty}^{\rm gen}/K_\infty)$.

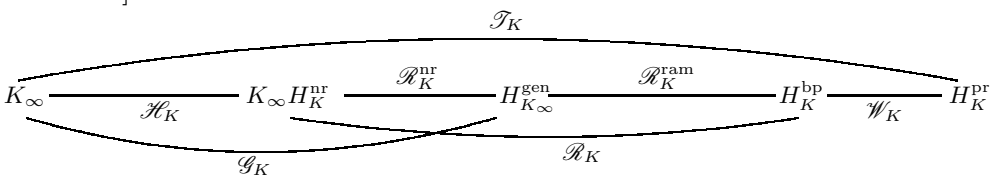
We denote by K_{n_1}, $n_1 \geq 0$, the minimal layer such that $K_\infty H_{K_{n_1}}^{\rm gen} = H_{K_\infty}^{\rm gen}$ (even with the above conventions, K_{n_1} may be distinct from K).

(ii) Let $H_K^{\rm bp}$ be the Bertrandias–Payan field fixed by $\mathscr{W}_K \simeq \big(\oplus_{v|p}\mu_{K_v}\big)/\mu_K$, where K_v is the v-completion of K and μ_k the group of pth-roots of unity of the field k (local or global); if U_v is the group of principal units of K_v, then $\mu_{K_v} = \mathrm{tor}_{\mathbb{Z}_p}(U_v)$.

(iii) Let $\iota\mathscr{E}_K$ be the image of $\mathscr{E}_K$ in $U_K := \prod_{v|p} U_v$ and let $I_v(H_K^{\rm pr}/K_\infty) := \mathrm{tor}_{\mathbb{Z}_p}(U_v/\iota\mathscr{E}_K \cap U_v)$ be the inertia groups of v in $H_K^{\rm pr}/K_\infty$; the subgroup of $\mathscr{T}_K$ generated by these inertia groups fixes $H_{K_\infty}^{\rm gen}$.

(iv) Let $\mathscr{R}_K^{\rm nr} := \mathrm{Gal}(H_{K_\infty}^{\rm gen}/K_\infty H_K^{\rm nr})$ and let $\mathscr{R}_K^{\rm ram} := \mathrm{Gal}(H_K^{\rm bp}/H_{K_\infty}^{\rm gen})$, where $\mathscr{R}_K := \mathrm{Gal}(H_K^{\rm bp}/K_\infty H_K^{\rm nr})$ is the normalized p-adic regulator defined in [Gras2018, Section 5].

These definitions may be summarized by the following diagram [Gras2021[a], Section 2]:



Recall, under the above conventions about the choice of the base field K, some results, given in [Gras2017[b], Gras2019[b]] and generalizing some particular results of Taya [Taya1996, Taya1999, Taya2000]:

Proposition 3.8. *For all $n \geq 0$, the norm factor $\dfrac{p^{n\cdot(r-1)}}{\omega_{K_n/K}(\mathscr{E}_K)}$ divides $\#\mathscr{R}_K^{\rm nr}$ and $\#\mathscr{H}_{K_n}^{G_n} = \#\mathscr{H}_K \times \dfrac{p^{n\cdot(r-1)}}{\omega_{K_n/K}(\mathscr{E}_K)}$ divides $\#\mathscr{G}_K = \#\mathscr{H}_K \cdot \#\mathscr{R}_K^{\rm nr}$; then, equality holds for all $n \geq n_1$. If $\mathscr{H}_K = \mathscr{R}_K^{\rm nr} = 1$, then $\lambda = \mu = 0$.*

Thus, the norm factors $\dfrac{p^{n\cdot(r-1)}}{\omega_{K_n/K}(\mathbf{\Lambda}_K^i(n))}$, associated to the filtration, divide $\#\mathscr{R}_K^{\rm nr}$, which allows computations in the base field K without Hasse's symbols.

Recall that $m(K_n)$ is the length of the filtration for K_n and that $m(K)=0$; so, formulas (2.3) apply in general for $L=K_n$, $n\geq 1$:

Proposition 3.9. *Let v_p denotes the p-adic valuation; then, under the previous conventions about the base field K, we have for all $n\geq 1$:*

$$m(K_n)\leq \lambda\cdot n+\mu\cdot p^n+\nu\leq v_p(\#\mathscr{H}_K\cdot\#\mathscr{R}_K^{\rm nr})\cdot m(K_n).$$

Corollary 3.10. *Greenberg's conjecture is equivalent (under Leopoldt's conjecture) to each of the following properties:*

(i) $m(K_n)$ is bounded over n.

(ii) $\mathscr{H}_{K_n}=\mathscr{H}_{K_n}^{G_n}$ $(\forall\, n\geq 0)$ and $\mathscr{R}_K^{\rm nr}=1$.

Proof. (i) is obvious.

(ii) If $\lambda=\mu=0$, then there is stability from K and $\mathscr{H}_{K_n}=\mathscr{H}_{K_n}^{G_n}\simeq\mathscr{H}_K$ for all n. There exists $n_1\gg 0$ such that $\mathrm{Gal}(K_\infty H_{K_{n_1}}^{\rm gen})=\mathscr{G}_K$ by definition; thus, since $\#\mathscr{H}_{K_n}^{G_n}=\#\mathscr{G}_K$, for all $n\geq n_1$, it follows that $\mathscr{R}_K^{\rm nr}=1$.

Reciprocally, $\mathscr{R}_K^{\rm nr}=1$ implies $\#\mathscr{H}_{K_n}^{G_n}=\#\mathscr{H}_K$ for all $n\geq 0$; so the second element of the filtration is of order $\dfrac{\#\mathscr{H}_K}{\mathbf{N}_{K_n/K}(\mathscr{H}_{K_n}^{G_n})}=\dfrac{\#\mathscr{H}_K}{\mathbf{N}_{K_n/K}(\mathscr{H}_{K_n})}=1$, using $\mathscr{H}_{K_n}^{G_n}=\mathscr{H}_{K_n}$ and $\mathbf{N}_{K_n/K}(\mathscr{H}_{K_n})=\mathscr{H}_K$. Whence $m(K_n)=1$ is bounded. □

We can wonder if Greenberg's conjecture is equivalent, under Conventions 3.6 and $\mu=0$, to $\nu_{K_n/K}(\mathscr{H}_{K_n})=1$, for all $n\geq e(K)$, obtained with the two particular conditions $m(K_n)=1$ (i.e., $s=0$) and $e(K_n)=e(K)$.

Indeed, under Greenberg's conjecture, $\mathscr{H}_{K_n}\simeq\mathscr{H}_K$ (via the $\mathbf{N}_{K_n/K}$'s), for all $n\geq 0$, which characterizes the stability with $m(K_n)=1$ and $e(K_n)=e(K)$, precisely the kind of annihilation of the $\mathscr{H}_{K_n}$'s by $\nu_{K_n/K}$ with $m(K_n)=1$ (i.e., $s=0$) and $e(K_n)=e(K)$.

Reciprocally, from Proposition 3.5, if $\mathscr{H}_K$ capitulates, then $\nu_{K_n/K}(\mathscr{H}_{K_n})=\mathbf{J}_{K_n/K}\circ\mathbf{N}_{K_n/K}(\mathscr{H}_{K_n})=1$ and $e(K_n)=e(K)$ implies $\lambda=0$.

But, unfortunately, in the practice, these phenomenon (if any) holds from an unknown layer.

This may be suggested by the following example of the cyclic cubic field of conductor $f=2689$, of 2-class group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and its cyclotomic $\mathbb{Z}_2$-extension, giving $\mathscr{H}_{K_1}\simeq\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$, $\mathscr{H}_{K_2}\simeq\mathbb{Z}/8\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$, and $\mathscr{H}_{K_3}\simeq\mathbb{Z}/8\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$ (see details of the PARI programs and data in § 4.3):

```
p=2  f=2689 PK=x^3+x^2-896*x+5876 CK0=[2,2]
CK1=[28,4]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
No capitulation of CK0 in K1
CK2=[56,8]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
No capitulation of CK0 in K2
CK3=[56,8]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Complete capitulation of CK0 in K3, m(K3)=1, e(K2)=3
```

At any layer, σ_n-1 annihilates $\mathscr{H}_{K_n}$ and capitulation in K_∞ holds for all n. Stability occurs from K_2 giving a checking of Greenberg's conjecture.

Remark 3.11. It is interesting to check the numerical examples given by Fukuda [Fuku1994] for real quadratic fields and $p = 3$ (stability from K) with the simplified usual program and suitable polynomials defining the layers of the cyclotomic $\mathbb{Z}_3$-extension; we limit the computations to the first layer and use the fact that the $\mathscr{H}_{K_n}$'s are cyclic:

```
{p=3;Lm=List([3137,3719,4409,6809,7226,9998]);for(k=1,6,PK=x^2-Lm[k];
K=bnfinit(PK,1);CK0=K.clgp;Pn=polcompositum(PK,polsubcyclo(p^2,p))[1];
Kn=bnfinit(Pn,1);CKn=Kn.clgp;print();G=nfgaloisconj(Kn);Id=x;for(k=1,6,
Z=G[k];ks=1;while(Z!=Id,Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);
if(ks==p,S=G[k];break));A0=CKn[3][1];A=1;for(t=1,p,
As=nfgaloisapply(Kn,S,A);A=idealmul(Kn,A0,As));
C=bnfisprincipal(Kn,A)[1];print("PK=",PK," CK0=",CK0[2],
" CK",1,"=",CKn[2]," norm in K",1,"/K of CK",1,":",C)}
```

```
PK=x^2-3137   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
PK=x^2-3719   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
PK=x^2-4409   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
PK=x^2-6809   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
PK=x^2-7226   CK0=[18]   CK1=[18]   norm in K1/K of CK1:[3]~
PK=x^2-9998   CK0=[9]    CK1=[9]    norm in K1/K of CK1:[3]~
```

The stability from K implies Greenberg's conjecture and capitulation of $\mathscr{H}_K$ in K_2, with an incomplete capitulation in K_1.

4. Examples for cyclic cubic K and $p = 2$

We consider various totally ramified cyclic p-extensions L/K, where $L = KL_0$, $L_0/\mathbb{Q}$ cyclic, especially $L_0 \subset \mathbb{Q}(\mu_\ell)$, $\ell \equiv 1 \pmod{2p^N}$.

4.1. Cyclic cubic fields K, $L \subset K(\mu_{17})$, $L \subset K(\mu_{97})$.

4.1.1. *Case* $\ell = 17$. In that examples, L_0 is the real subfield of degree 8 of $\mathbb{Q}(\mu_\ell)$. We eliminate the cases of stability $\#\mathscr{H}_{K_1} = \#\mathscr{H}_K$ since complete capitulation holds in a suitable layer if $e(K) \leq 3$. The number vHK eliminates fields such that $\#\mathsf{CKn} < \mathsf{p}^{\mathsf{vHK}}$; it may be chosen at will. The images $\mathbf{J}_{K_n/K}(\mathscr{H}_K)$ are computed for $n = 1$ and $n = 2$. The number $\rho \in \{1, 3\}$ is the number of prime ideals above ℓ in K.

We give an excerpt of the various cases obtained (all these examples show the random framework of the structures and of the capitulation, complete or incomplete); the deductions of $\mathsf{m(Kn)}, \mathsf{e(Kn)}$ from the outputs, are left to the reader. We indicate if $\mathscr{H}_K$ capitulates in K_3 (no computed) which holds as soon as $\#\mathscr{H}_{K_2} = \#\mathscr{H}_{K_1}$ (stability from K_1) and $e(K) \leq 2$:

```
MAIN PROGRAM FOR CYCLIC CUBIC FIELDS
{p=2;Nn=2;bf=7;Bf=10^4;vHK=2;ell=17;mKn=2;
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
K=bnfinit(PK,1);rho=matsize(idealfactor(K,ell))[1];
\\Test over the order of the p-class group of K:
HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
\\Test for elimination of the stability from K:
if(n==1 & valuation(HKn,p)==valuation(HK,p),break);
if(n==1,print("f=",f," PK=",PK," CK0=",CK0[2]," ell=",ell," rho=",rho));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
\\Search of a generator S of Gal(Kn/K):
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
\\Computation of the filtration:
```

```
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
\\Computation of the algebraic norms:
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))))}
```

```
p=2  f=607  PK=x^3+x^2-202*x-1169  CK0=[2,2]  ell=17 rho=1
CK1=[2,2,2,2]
h_1^[(S-1)^1]=[1,0,0,1]   h_2^[(S-1)^1]=[0,1,1,1]
h_3^[(S-1)^1]=[1,1,1,0]   h_4^[(S-1)^1]=[1,0,0,1]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,0,0,1]
norm in K1/K of the component 2 of CK1:[0,1,1,1]
norm in K1/K of the component 3 of CK1:[1,1,1,0]
norm in K1/K of the component 4 of CK1:[1,0,0,1]
CK2=[2,2,2,2]
h_1^[(S-1)^1]=[1,0,1,0]   h_2^[(S-1)^1]=[0,1,0,1]
h_3^[(S-1)^1]=[1,0,1,0]   h_4^[(S-1)^1]=[0,1,0,1]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=1
```

```
p=2  f=703  PK=x^3+x^2-234*x-729  CK0=[6,2]  ell=17  rho=1
CK1=[12,4]
h_1^[(S-1)^1]=[0,2]   h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,2]
norm in K1/K of the component 2 of CK1:[2,0]
CK2=[24,8]
h_1^[(S-1)^1]=[0,2]   h_2^[(S-1)^1]=[6,6]
h_1^[(S-1)^2]=[4,4]   h_2^[(S-1)^2]=[4,0]
norm in K2/K of the component 1 of CK2:[4,4]
norm in K2/K of the component 2 of CK2:[4,0]
No capitulation, m(K2)>2, e(K2)=3
```

```
p=2  f=1009  PK=x^3+x^2-336*x-1719  CK0=[2,2]  ell=17  rho=1
CK1=[28,4]
h_1^[(S-1)^1]=[0,2]   h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,2]
norm in K1/K of the component 2 of CK1:[2,0]
CK2=[28,4]
h_1^[(S-1)^1]=[0,2]   h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=1197  PK=x^3-399*x+2926  CK0=[6,6]  ell=17  rho=3
CK1=[12,12]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[12,12]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
```

```
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=1, e(K2)=2
p=2  f=1777  PK=x^3+x^2-592*x+724  CK0=[4,4]  ell=17  rho=3
CK1=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Complete capitulation in K3, m(K2)=1, e(K2)=3
p=2  f=1789  PK=x^3+x^2-596*x-5632  CK0=[2,2]  ell=17  rho=1
CK1=[24,8]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[4,0]
norm in K1/K of the component 2 of CK1:[0,4]
CK2=[312,8]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)>2, e(K2)=3
p=2  f=1951  PK=x^3+x^2-650*x-289  CK0=[2,2]  ell=17  rho=3
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[4,4,4,4]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[2,0,2,0] h_4^[(S-1)^1]=[0,2,0,2]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation in K1, m(K2)=2, e(K2)=2
p=2  f=2077  PK=x^3+x^2-692*x-7231  CK0=[6,2]  ell=17  rho=1
CK1=[6,2,2,2]
h_1^[(S-1)^1]=[1,1,1,0] h_2^[(S-1)^1]=[0,0,1,1]
h_3^[(S-1)^1]=[1,1,0,1] h_4^[(S-1)^1]=[1,1,0,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,1,0]
norm in K1/K of the component 2 of CK1:[0,0,1,1]
norm in K1/K of the component 3 of CK1:[1,1,0,1]
norm in K1/K of the component 4 of CK1:[1,1,0,1]
CK2=[6,2,2,2,2,2]
h_1^[(S-1)^1]=[1,0,0,0,1,0] h_2^[(S-1)^1]=[0,1,0,0,0,1]
h_3^[(S-1)^1]=[1,1,0,1,0,1] h_4^[(S-1)^1]=[1,0,0,1,0,0]
h_5^[(S-1)^1]=[1,0,0,1,0,0] h_6^[(S-1)^1]=[0,0,1,1,0,1]
h_1^[(S-1)^2]=[0,0,0,1,1,0] h_2^[(S-1)^2]=[0,1,1,1,0,0]
h_3^[(S-1)^2]=[0,1,1,0,1,0] h_4^[(S-1)^2]=[0,0,0,1,1,0]
h_5^[(S-1)^2]=[0,0,0,1,1,0] h_6^[(S-1)^2]=[0,1,1,1,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
```

```
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)>2, e(K2)=1
```

```
p=2  f=2817  PK=x^3-939*x+6886  CK0=[12,4]  ell=17  rho=1
CK1=[84,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[84,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=1, e(K2)=2
```

```
p=2  f=3357  PK=x^3-1119*x+9325  CK0=[6,2]  ell=17  rho=3
CK1=[6,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[12,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation in K1, m(K2)=2, e(K2)=2
```

```
p=2  f=3409  PK=x^3+x^2-1136*x-10732  CK0=[6,2]  ell=17  rho=3
CK1=[6,2,2,2]
h_1^[(S-1)^1]=[1,1,1,0] h_2^[(S-1)^1]=[1,1,0,1]
h_3^[(S-1)^1]=[0,0,1,1] h_4^[(S-1)^1]=[0,0,1,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,1,0]
norm in K1/K of the component 2 of CK1:[1,1,0,1]
norm in K1/K of the component 3 of CK1:[0,0,1,1]
norm in K1/K of the component 4 of CK1:[0,0,1,1]
CK2=[6,2,2,2]
h_1^[(S-1)^1]=[1,1,0,1] h_2^[(S-1)^1]=[1,1,0,1]
h_3^[(S-1)^1]=[0,0,0,1] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=1
```

```
p=2  f=5479  PK=x^3+x^2-1826*x+13799  CK0=[2,2]  ell=17  rho=1
CK1=[2,2,2,2]
h_1^[(S-1)^1]=[1,0,0,1] h_2^[(S-1)^1]=[1,1,1,0]
h_3^[(S-1)^1]=[0,1,1,1] h_4^[(S-1)^1]=[1,0,0,1]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,0,0,1]
norm in K1/K of the component 2 of CK1:[1,1,1,0]
```

```
norm in K1/K of the component 3 of CK1:[0,1,1,1]
norm in K1/K of the component 4 of CK1:[1,0,0,1]
CK2=[4,4,4,4]
h_1^[(S-1)^1]=[3,1,3,1] h_2^[(S-1)^1]=[3,3,0,3]
h_3^[(S-1)^1]=[0,2,2,2] h_4^[(S-1)^1]=[0,2,3,0]
h_1^[(S-1)^2]=[0,2,2,0] h_2^[(S-1)^2]=[2,2,2,0]
h_3^[(S-1)^2]=[2,2,2,2] h_4^[(S-1)^2]=[2,0,2,0]
norm in K2/K of the component 1 of CK2:[0,0,2,0]
norm in K2/K of the component 2 of CK2:[2,2,2,2]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[2,2,0,2]
No capitulation, m(K2)>2, e(K2)=2
```

```
p=2  f=6247  PK=x^3+x^2-2082*x-35631  CK0=[4,4]  ell=17  rho=1
CK1=[24,8,2,2]
h_1^[(S-1)^1]=[0,2,0,1] h_2^[(S-1)^1]=[6,6,1,0]
h_3^[(S-1)^1]=[0,4,0,0] h_4^[(S-1)^1]=[4,0,0,0]
h_1^[(S-1)^2]=[0,4,0,0] h_2^[(S-1)^2]=[4,4,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,2,0,1]
norm in K1/K of the component 2 of CK1:[6,0,1,0]
norm in K1/K of the component 3 of CK1:[0,4,0,0]
norm in K1/K of the component 4 of CK1:[4,0,0,0]
CK2=[24,8,2,2]
h_1^[(S-1)^1]=[0,6,1,1] h_2^[(S-1)^1]=[2,6,0,1]
h_3^[(S-1)^1]=[4,4,0,0] h_4^[(S-1)^1]=[0,4,0,0]
h_1^[(S-1)^2]=[0,4,0,0] h_2^[(S-1)^2]=[4,4,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,4,0,0]
norm in K2/K of the component 2 of CK2:[4,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation in K3, m(K2)>2, e(K2)=3
```

```
p=2  f=7687  PK=x^3+x^2-2562*x-48969  CK0=[2,2,2,2]  ell=17  rho=1
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0]
CK2=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0] h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=8629  PK=x^3+x^2-2876*x-50176  CK0=[14,2]  ell=17  rho=1
CK1=[56,8]
h_1^[(S-1)^1]=[2,4]   h_2^[(S-1)^1]=[4,6]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[4,4]
norm in K1/K of the component 2 of CK1:[4,0]
CK2=[112,16]
h_1^[(S-1)^1]=[6,4]   h_2^[(S-1)^1]=[12,10]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[0,8]
norm in K2/K of the component 2 of CK2:[8,8]
No capitulation, m(K2)>2, e(K2)=4
```

```
p=2  f=9247  PK=x^3+x^2-3082*x-27056  CK0=[12,4]  ell=17  rho=3
CK1=[24,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[48,16]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
No capitulation, m(K2)=1, e(K2)=4

p=2  f=9283  PK=x^3+x^2-3094*x-5501  CK0=[2,2]  ell=17  rho=1
CK1=[48,16]
h_1^[(S-1)^1]=[6,0]   h_2^[(S-1)^1]=[0,6]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[8,0]
norm in K1/K of the component 2 of CK1:[0,8]
CK2=[48,16]
h_1^[(S-1)^1]=[6,0]   h_2^[(S-1)^1]=[0,6]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)>2, e(K2)=4
(...)
p=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]  ell=17  rho=3
CK1=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=1, e(K1)=3
CK2=[16,16,2,2]
h_1^[(S-1)^1]=[8,0,0,0] h_2^[(S-1)^1]=[0,8,0,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=2, e(K2)=4

p=2  f=25119  PK=x^3-8373*x+2791  CK0=[12,4]  ell=17  rho=1
CK1=[12,4,2,2]
h_1^[(S-1)^1]=[2,2,0,1] h_2^[(S-1)^1]=[0,0,1,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,1]
norm in K1/K of the component 2 of CK1:[0,2,1,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[24,8,2,2,2,2]
h_1^[(S-1)^1]=[6,4,0,0,1,1] h_2^[(S-1)^1]=[4,6,1,0,0,1]
h_3^[(S-1)^1]=[0,4,0,0,0,0] h_4^[(S-1)^1]=[4,4,0,1,1,0]
h_5^[(S-1)^1]=[0,0,0,1,1,0] h_6^[(S-1)^1]=[0,4,1,0,0,0]
h_1^[(S-1)^2]=[4,4,1,1,1,0] h_2^[(S-1)^2]=[0,4,1,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0] h_4^[(S-1)^2]=[4,4,0,0,0,0]
h_5^[(S-1)^2]=[4,4,0,0,0,0] h_6^[(S-1)^2]=[0,4,0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
```

```
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Incomplete capitulation, m(K2)>2, e(K2)=3

p=2  f=29467  PK=x^3+x^2-9822*x-20736  CK0=[84,4]  ell=17  rho=3
CK1=[168,8,2,2]
h_1^[(S-1)^1]=[4,0,1,0] h_2^[(S-1)^1]=[4,4,1,1]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[6,0,1,0]
norm in K1/K of the component 2 of CK1:[4,6,1,1]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[336,16,2,2]
h_1^[(S-1)^1]=[0,8,1,0] h_2^[(S-1)^1]=[8,8,0,1]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
No capitulation, m(K2)=2, e(K2)=4

p=2  f=33061  PK=x^3+x^2-11020*x-262039  CK0=[6,2,2,2]  ell=17  rho=1
CK1=[12,4,2,2,2,2]
h_1^[(S-1)^1]=[0,0,1,0,1,1] h_2^[(S-1)^1]=[0,2,1,1,0,1]
h_3^[(S-1)^1]=[2,0,0,0,0,0] h_4^[(S-1)^1]=[0,2,0,0,0,0]
h_5^[(S-1)^1]=[0,2,0,0,0,0] h_6^[(S-1)^1]=[2,2,0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0] h_6^[(S-1)^2]=[0,0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,1,0,1,1]
norm in K1/K of the component 2 of CK1:[0,0,1,1,0,1]
norm in K1/K of the component 3 of CK1:[2,0,0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0,0,0]
norm in K1/K of the component 5 of CK1:[0,2,0,0,0,0]
norm in K1/K of the component 6 of CK1:[2,2,0,0,0,0]
CK2=[24,8,4,4,2,2]
h_1^[(S-1)^1]=[6,4,2,2,0,1] h_2^[(S-1)^1]=[6,0,2,0,1,0]
h_3^[(S-1)^1]=[2,2,0,2,0,0] h_4^[(S-1)^1]=[2,0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,2,2,0,0] h_6^[(S-1)^1]=[0,0,0,2,0,0]
h_1^[(S-1)^2]=[4,4,0,2,0,0] h_2^[(S-1)^2]=[0,4,2,2,0,0]
h_3^[(S-1)^2]=[4,0,0,0,0,0] h_4^[(S-1)^2]=[4,0,0,0,0,0]
h_5^[(S-1)^2]=[0,4,0,0,0,0] h_6^[(S-1)^2]=[4,0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[4,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[4,4,0,0,0,0]
norm in K2/K of the component 4 of CK2:[4,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Incomplete capitulation, m(K2)>2, e(K2)=3

p=2  f=37087  PK=x^3+x^2-12362*x-401089  CK0=[2,2,2,2]  ell=17  rho=3
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
```

```
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)=1, e(K2)=3
p=2  f=44857  PK=x^3+x^2-14952*x-704421  CK0=[6,2,2,2]  ell=17  rho=3
CK1=[12,12,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[2,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[2,2,0,0]
CK2=[12,12,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[2,2,0,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
p=2  f=48769  PK=x^3+x^2-16256*x-7225  CK0=[24,8]  ell=17  rho=3
CK1=[48,16]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[48,16]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Incomplete capitulation, m(K2)=1, e(K2)=4
p=2  f=55609  PK=x^3+x^2-18536*x-823837  CK0=[4,4,2,2]  ell=17  rho=3
CK1=[56,8,2,2]
h_1^[(S-1)^1]=[4,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[4,0,0,0] h_4^[(S-1)^1]=[0,4,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[6,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[4,0,0,0]
norm in K1/K of the component 4 of CK1:[0,4,0,0]
CK2=[56,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,4,0,0] h_4^[(S-1)^1]=[4,4,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation in K3, m(K2)=2, e(K2)=3
```

Let's give some comments on interesting examples found above:

```
p=2  f=9283  PK=x^3+x^2-3094*x-5501  CK0=[2, 2]  ell=17  rho=1
CK1=[48,16]
h_1^[(S-1)^1]=[6,0]  h_2^[(S-1)^1]=[0,6]
h_1^[(S-1)^2]=[4,0]  h_2^[(S-1)^2]=[0,4]
h_1^[(S-1)^3]=[8,0]  h_2^[(S-1)^3]=[0,8]
h_1^[(S-1)^4]=[0,0]  h_2^[(S-1)^4]=[0,0]
norm in K1/K of the component 1 of CK1:[8,0]
norm in K1/K of the component 2 of CK1:[0,8]
No capitulation,m(K1)=4,e(K1)=4
CK2=[48,16]
```

```
h_1^[(S-1)^1]=[6,0]  h_2^[(S-1)^1]=[0,6]
h_1^[(S-1)^2]=[4,0]  h_2^[(S-1)^2]=[0,4]
h_1^[(S-1)^3]=[8,0]  h_2^[(S-1)^3]=[0,8]
h_1^[(S-1)^4]=[0,0]  h_2^[(S-1)^4]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation,m(K2)=4,e(K2)=4
```

There is complete capitulation, even if conditions of Theorem 1.1 (i) are not fulfilled for the K_n/K's. Moreover, the exponent of $\mathscr{H}_{K_1}$ is 2^4 giving a larger complexity in K_1/K, but in K_n, $n \geq 2$, the exponent is still 2^4 (no increasing of the complexity). If K_5 had existed, we would have get the following data in $K'_4 = K_5$ with the base field $K' = K_1$ (i.e., $N' = 4$) and the relation $\nu_{K'_4/K'}(\mathscr{H}_{K'_n}) = 1$ (one has the canonical isomorphisms between the $\mathscr{H}_{K'_n}$ and the identities $\mathscr{H}_{K'_n} = \mathscr{H}_{K'_n}^{G'_n}$ for all $n > 1$):

```
CK6=[16,16] (up to odd factors)
h'_1^[(S'-1)^1]=[0,0]  h'_2^[(S'-1)^1]=[0,0]
h'_1^[(S'-1)^2]=[0,0]  h'_2^[(S'-1)^2]=[0,0]
h'_1^[(S'-1)^3]=[0,0]  h'_2^[(S'-1)^3]=[0,0]
h'_1^[(S'-1)^4]=[0,0]  h'_2^[(S'-1)^4]=[0,0]
norm in K'5/K' of the component 1 of CK'5:[0,0]
norm in K'5/K' of the component 2 of CK'5:[0,0]
Complete capitulation, m(K'5)=1, e(K'5)=4
```

So, for this layer $L' = K'_4$, Theorem 1.1 (i) applies with $e(K'_5) = 4 \in [4 - s']$ since $m(L') = 1$ ($s' = 0$), and $\mathscr{H}_{K'_1}$ capitulates in K'_4.

Some other examples as:

```
p=2  f=7687  PK=x^3+x^2-2562*x-48969  CK0=[2,2,2,2]  ell=17  rho=1
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0]
CK2=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0] h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[2,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
p=2  f=44857  PK=x^3+x^2-14952*x-704421  CK0=[6,2,2,2]  ell=17  rho=3
CK1=[12,12,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0] h_4^[(S-1)^1]=[2,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[2,2,0,0]
CK2=[12,12,2,2]
h_1^[(S-1)^1]=[0,0,0,0] h_2^[(S-1)^1]=[2,2,0,0]
h_3^[(S-1)^1]=[2,2,0,0] h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
```

```
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

suggest that the size of the p-rank is not an obstruction to a capitulation in such cyclic sub-p-extensions of $K(\mu_\ell)$; here, the 2-ranks are even because of the structure of $\mathbb{Z}_2[\mu_3]$-module of the $\mathscr{H}_{K_n}$'s. In the above cases, the capitulation is obtained by means of a stability in larger layers.

4.1.2. *Case* $\ell = 97$. Similarly, we give a table for $\ell = 97$ allowing capitulations up to K_4. One finds much more cases of capitulation (not in the table below since they are very numerous); it seems clearly that a larger value of N intervenes in the phenomenon of capitulation, even for the same values of $n = 1, 2$:

```
p=2  f=349 PK=x^3+x^2-116*x-517 CK0=[2,2] ell=97 rho=1
CK1=[4,4]
h_1^[(S-1)^1]=[2,2]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,2]
norm in K1/K of the component 2 of CK1:[2,2]
CK2=[4,4]
h_1^[(S-1)^1]=[2,2]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=547 PK=x^3+x^2-182*x-81 CK0=[2,2] ell=97 rho=1
CK1=[2,2,2,2]
h_1^[(S-1)^1]=[1,1,1,0]   h_2^[(S-1)^1]=[0,1,0,1]
h_3^[(S-1)^1]=[1,0,1,1]   h_4^[(S-1)^1]=[0,1,0,1]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,1,0]
norm in K1/K of the component 2 of CK1:[0,1,0,1]
norm in K1/K of the component 3 of CK1:[1,0,1,1]
norm in K1/K of the component 4 of CK1:[0,1,0,1]
CK2=[2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,1,0,0]   h_4^[(S-1)^1]=[1,1,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=1
```

```
p=2  f=607 PK=x^3+x^2-202*x-1169 CK0=[2,2] ell=97 rho=1
CK1=[8,8]
h_1^[(S-1)^1]=[6,4]   h_2^[(S-1)^1]=[4,2]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K1/K of the component 1 of CK1:[0,4]
norm in K1/K of the component 2 of CK1:[4,4]
CK2=[104,8]
h_1^[(S-1)^1]=[6,4]   h_2^[(S-1)^1]=[4,2]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,4]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)>2, e(K2)=3
```

```
p=2  f=703 PK=x^3+x^2-234*x-729 CK0=[6,2] ell=97 rho=1
CK1=[6,2,2,2]
h_1^[(S-1)^1]=[1,1,0,0]   h_2^[(S-1)^1]=[1,1,0,0]
h_3^[(S-1)^1]=[0,0,1,1]   h_4^[(S-1)^1]=[0,0,1,1]
```

```
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,1,0,0]
norm in K1/K of the component 2 of CK1:[1,1,0,0]
norm in K1/K of the component 3 of CK1:[0,0,1,1]
norm in K1/K of the component 4 of CK1:[0,0,1,1]
CK2=[12,4,2,2]
h_1^[(S-1)^1]=[2,0,1,0]   h_2^[(S-1)^1]=[0,2,0,1]
h_3^[(S-1)^1]=[0,2,0,0]   h_4^[(S-1)^1]=[2,2,0,0]
h_1^[(S-1)^2]=[0,2,0,0]   h_2^[(S-1)^2]=[2,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)>2, e(K2)=2
```

```
p=2  f=1957 PK=x^3+x^2-652*x+6016 CK0=[6,2] ell=97 rho=3
CK1=[12,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[24,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
No capitulation, m(K2)=1, e(K2)=3
```

```
p=2  f=4207 PK=x^3+x^2-1402*x+14335 CK0=[6,2] ell=97 rho=3
CK1=[12,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0]   h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[0,2,0,0]   h_4^[(S-1)^1]=[2,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,0]
norm in K1/K of the component 2 of CK1:[2,2,0,0]
norm in K1/K of the component 3 of CK1:[0,2,0,0]
norm in K1/K of the component 4 of CK1:[2,2,0,0]
CK2=[12,4,4,4]
h_1^[(S-1)^1]=[0,2,0,0]   h_2^[(S-1)^1]=[2,2,0,0]
h_3^[(S-1)^1]=[0,0,0,2]   h_4^[(S-1)^1]=[0,0,2,2]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=4639 PK=x^3+x^2-1546*x+6529 CK0=[2,2] ell=97 rho=1
CK1=[4,4]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
CK2=[4,4]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=9391 PK=x^3+x^2-3130*x-24347 CK0=[2,2] ell=97 rho=3
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0]   h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[2,0,0,0]   h_4^[(S-1)^1]=[2,2,0,0]
```

```
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[2,2,0,0]
CK2=[4,4,4,4,2,2]
h_1^[(S-1)^1]=[0,0,0,2,0,1]   h_2^[(S-1)^1]=[0,0,2,0,1,0]
h_3^[(S-1)^1]=[2,0,2,0,1,0]   h_4^[(S-1)^1]=[0,0,2,0,0,1]
h_5^[(S-1)^1]=[0,2,2,0,0,0]   h_6^[(S-1)^1]=[2,0,0,2,0,0]
h_1^[(S-1)^2]=[2,0,0,2,0,0]   h_2^[(S-1)^2]=[0,2,2,0,0,0]
h_3^[(S-1)^2]=[0,2,2,0,0,0]   h_4^[(S-1)^2]=[2,0,0,2,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0]   h_6^[(S-1)^2]=[0,0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)>2, e(K2)=2

p=2  f=10513 PK=x^3+x^2-3504*x-80989 CK0=[8,8] ell=97 rho=3
CK1=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Complete capitulation in K3, m(K2)=1, e(K2)=3

p=2  f=11149 PK=x^3+x^2-3716*x+39228 CK0=[2,2] ell=97 rho=3
CK1=[12,4]
h_1^[(S-1)^1]=[2,2]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,2]
norm in K1/K of the component 2 of CK1:[2,2]
CK2=[12,4]
h_1^[(S-1)^1]=[0,2]   h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=2, e(K2)=2

p=2  f=15823 PK=x^3+x^2-5274*x+141821 CK0=[4,4] ell=97 rho=3
CK1=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation, m(K2)=1, e(K2)=2

p=2  f=19177 PK=x^3+x^2-6392*x-79549 CK0=[6,2] ell=97 rho=3
CK1=[24,8,2,2]
h_1^[(S-1)^1]=[6,4,0,1]   h_2^[(S-1)^1]=[0,6,1,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[4,0,0,0]   h_2^[(S-1)^2]=[0,4,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,4,0,1]
norm in K1/K of the component 2 of CK1:[0,0,1,0]
```

```
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[336,112,4,4]
h_1^[(S-1)^1]=[2,4,1,2]   h_2^[(S-1)^1]=[12,6,1,1]
h_3^[(S-1)^1]=[0,8,0,2]   h_4^[(S-1)^1]=[8,0,2,2]
h_1^[(S-1)^2]=[4,8,2,2]   h_2^[(S-1)^2]=[8,12,0,2]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[8,8,2,0]
norm in K2/K of the component 2 of CK2:[8,0,2,2]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)>2, e(K2)=2
```

```
p=2  f=20419 PK=x^3+x^2-6806*x-3025 CK0=[42,2] ell=97 rho=3
CK1=[84,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0]   h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0]
CK2=[84,4,4,4,2,2]
h_1^[(S-1)^1]=[0,0,2,0,1,0]   h_2^[(S-1)^1]=[0,2,0,2,1,0]
h_3^[(S-1)^1]=[2,2,2,2,0,0]   h_4^[(S-1)^1]=[0,2,2,2,1,1]
h_5^[(S-1)^1]=[2,0,0,2,0,0]   h_6^[(S-1)^1]=[0,0,2,2,0,0]
h_1^[(S-1)^2]=[2,0,0,2,0,0]   h_2^[(S-1)^2]=[2,0,0,2,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0]   h_4^[(S-1)^2]=[2,0,2,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0]   h_6^[(S-1)^2]=[0,0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)>2, e(K2)=2
```

```
p=2  f=20887 PK=x^3+x^2-6962*x-225889 CK0=[4,4,2,2] ell=97 rho=3
CK1=[8,8,2,2]
h_1^[(S-1)^1]=[4,4,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[4,4,0,0]   h_4^[(S-1)^1]=[4,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[6,4,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[4,4,0,0]
norm in K1/K of the component 4 of CK1:[4,0,0,0]
CK2=[8,8,2,2]
h_1^[(S-1)^1]=[0,4,0,0]   h_2^[(S-1)^1]=[4,4,0,0]
h_3^[(S-1)^1]=[4,0,0,0]   h_4^[(S-1)^1]=[4,4,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation in K3, m(K2)=2, e(K2)=3
```

```
p=2  f=21931 PK=x^3+x^2-7310*x-3249 CK0=[12,12] ell=97 rho=3
CK1=[24,24]
h_1^[(S-1)^1]=[4,4]   h_2^[(S-1)^1]=[4,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[6,4]
norm in K1/K of the component 2 of CK1:[4,2]
CK2=[24,24]
h_1^[(S-1)^1]=[0,4]   h_2^[(S-1)^1]=[4,4]
```

```
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Complete capitulation in K3, m(K2)=2, e(K2)=3
p=2  f=24589 PK=x^3+x^2-8196*x-33696 CK0=[6,2] ell=97 rho=3
CK1=[6,2,2,2]
h_1^[(S-1)^1]=[1,0,0,1]   h_2^[(S-1)^1]=[0,1,1,0]
h_3^[(S-1)^1]=[0,1,1,0]   h_4^[(S-1)^1]=[1,0,0,1]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[1,0,0,1]
norm in K1/K of the component 2 of CK1:[0,1,1,0]
norm in K1/K of the component 3 of CK1:[0,1,1,0]
norm in K1/K of the component 4 of CK1:[1,0,0,1]
CK2=[6,2,2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0,0,0]
h_3^[(S-1)^1]=[1,1,0,1,0,0]   h_4^[(S-1)^1]=[1,1,0,0,0,0]
h_5^[(S-1)^1]=[0,1,0,1,1,1]   h_6^[(S-1)^1]=[1,1,0,1,1,1]
h_1^[(S-1)^2]=[0,0,0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0,0,0]
h_3^[(S-1)^2]=[1,1,0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0,0,0]
h_5^[(S-1)^2]=[0,1,0,0,0,0]   h_6^[(S-1)^2]=[0,1,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Complete capitulation, m(K2)>2, e(K2)=1
p=2  f=25171 PK=x^3+x^2-8390*x+273152 CK0=[14,2] ell=97 rho=3
CK1=[84,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0]   h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[84,4,4,4]
h_1^[(S-1)^1]=[0,0,2,0]   h_2^[(S-1)^1]=[0,0,2,0]
h_3^[(S-1)^1]=[0,0,2,0]   h_4^[(S-1)^1]=[2,0,2,2]
h_1^[(S-1)^2]=[0,0,0,0]   h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

4.2. **Cyclic cubic fields K, $L \subset K(\mu_{17\cdot 97})$.** We use the same program with suitable defining polynomials, of degrees 2 and 4, respectively, so that $L_0/\mathbb{Q}$ be cyclic with total ramification of 17 and 97. Since two primes $\ell_1 = 17$ and $\ell_2 = 97$ ramify in K_n/K, the factor norm is in general non trivial, which gives larger 2-class groups but, a priori, this does not modify the notion of complexity in the tower:

```
{p=2;Nn=2;bf=7;Bf=10^6;vHK=4;mKn=2;L4=List;
L4=[x^2-x-412,x^4-x^3-618*x^2+1752*x+8960];
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
```

```
K=bnfinit(PK,1);HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
rho=matsize(idealfactor(K,17))[1]+matsize(idealfactor(K,97))[1];
for(n=1,Nn,Qn=L4[n];Pn=polcompositum(PK,Qn)[1];Kn=bnfinit(Pn,1);
HKn=Kn.no;dn=poldegree(Pn);if(n==1 & valuation(HKn,p)==vHK,break);
if(n==1,print();print("f=",f," PK=",PK," CK0=",CK0[2],
" ell=",ell," rho=",rho));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))))}
```

```
p=2  f=1777  PK=x^3+x^2-592*x+724  CK0=[4,4]
CK1=[8,8,2]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
CK2=[16,16,4]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
No capitulation, m(K2)=1, e(K2)=3
```

```
p=2  f=2817  PK=x^3-939*x+6886  CK0=[12,4]
CK1=[444,4,2,2,2]
h_1^[(S-1)^1]=[2,0,1,0,1] h_2^[(S-1)^1]=[0,2,1,1,0]
h_3^[(S-1)^1]=[0,0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,1,0,1]
norm in K1/K of the component 2 of CK1:[0,0,1,1,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0,0]
norm in K1/K of the component 5 of CK1:[0,0,0,0,0]
CK2=[888,8,4,2,2]
h_1^[(S-1)^1]=[2,4,2,1,0] h_2^[(S-1)^1]=[0,2,2,0,1]
h_3^[(S-1)^1]=[0,4,0,0,0] h_4^[(S-1)^1]=[4,4,0,0,0]
h_5^[(S-1)^1]=[4,0,0,0,0]
h_1^[(S-1)^2]=[0,4,0,0,0] h_2^[(S-1)^2]=[4,4,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0]
Complete capitulation, m(K2)>2, e(K2)=3
```

```
p=2  f=4297  PK=x^3+x^2-1432*x+20371  CK0=[4,4]
CK1=[4,4,2]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
CK2=[4,4,4]
h_1^[(S-1)^1]=[0,0,0] h_2^[(S-1)^1]=[0,0,0] h_3^[(S-1)^1]=[0,0,0]
```

```
norm in K2/K of the component 1 of CK2:[0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Complete capitulation, m(K2)=1, e(K2)=2
p=2  f=7687  PK=x^3+x^2-2562*x-48969  CK0=[2,2,2,2]
CK1=[4,4,2,2,2]
h_1^[(S-1)^1]=[2,2,0,0,0] h_2^[(S-1)^1]=[0,0,0,0,0]
h_3^[(S-1)^1]=[2,2,0,0,0] h_4^[(S-1)^1]=[2,0,0,0,0]
h_5^[(S-1)^1]=[2,2,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0,0]
norm in K1/K of the component 3 of CK1:[2,2,0,0,0]
norm in K1/K of the component 4 of CK1:[2,0,0,0,0]
norm in K1/K of the component 5 of CK1:[2,2,0,0,0]
CK2=[4,4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0,0] h_2^[(S-1)^1]=[2,0,0,0,0]
h_3^[(S-1)^1]=[2,0,2,0,0] h_4^[(S-1)^1]=[0,0,2,0,0]
h_5^[(S-1)^1]=[2,0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
p=2  f=10513  PK=x^3+x^2-3504*x-80989  CK0=[8,8]
CK1=[8,8,2]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
CK2=[8,8,4]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Incomplete capitulation, m(K2)=1, e(K2)=3
p=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]
CK1=[8,8,2,2,2]
h_1^[(S-1)^1]=[0,0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0,0]
norm in K1/K of the component 5 of CK1:[0,0,0,0,0]
CK2=[16,16,4,2,2]
h_1^[(S-1)^1]=[0,0,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0,0]   h_4^[(S-1)^1]=[0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0]
Incomplete capitulation, m(K2)=1, e(K2)=4
p=2  f=21931  PK=x^3+x^2-7310*x-3249  CK0=[12,12]
CK1=[12,12,2,2,2,2,2]
```

```
h_1^[(S-1)^1]=[2,0,0,1,0,0,0] h_2^[(S-1)^1]=[0,2,0,0,1,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0,0,0] h_6^[(S-1)^1]=[0,0,0,0,0,0,0]
h_7^[(S-1)^1]=[0,0,0,0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,1,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0,1,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0,0,0,0]
norm in K1/K of the component 5 of CK1:[0,0,0,0,0,0,0]
norm in K1/K of the component 6 of CK1:[0,0,0,0,0,0,0]
norm in K1/K of the component 7 of CK1:[0,0,0,0,0,0,0]
CK2=[24,24,4,4,4,2,2]
h_1^[(S-1)^1]=[6,0,3,0,3,1,1] h_2^[(S-1)^1]=[2,0,2,3,1,1,0]
h_3^[(S-1)^1]=[4,0,0,2,2,0,0] h_4^[(S-1)^1]=[0,0,0,0,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0,0,0] h_6^[(S-1)^1]=[4,0,2,0,2,0,0]
h_7^[(S-1)^1]=[4,0,0,2,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0,0,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0,0,0] h_6^[(S-1)^2]=[0,0,0,0,0,0,0]
h_7^[(S-1)^2]=[0,0,0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,2,0,2,0,0]
norm in K2/K of the component 2 of CK2:[4,4,0,2,2,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0,0]
norm in K2/K of the component 7 of CK2:[0,0,0,0,0,0,0]
Incomplete capitulation, m(K2)=2, e(K2)=3
```

4.3. **Cyclic cubic fields K, $L = K(\sqrt{2+\sqrt{2}})$.** In that case, L is a subfield of the cyclotomic $\mathbb{Z}_2$-extension of K. The program is unchanged, except that the cyclotomic polynomials $\mathsf{Qn} = \mathsf{polsubcyclo}(\mathsf{ell}, \mathsf{p}^{\mathsf{n}})$ are replaced by $\mathsf{Q1} = \mathsf{x}^2 - 2$ and $\mathsf{Q2} = \mathsf{x}^4 - 4 * \mathsf{x}^2 + 2$; we find often cases of capitulation from stability, noting that "N" is unlimited:

```
{p=2;Nn=2;bf=7;Bf=10^6;vHK=4;mKn=2;
for(f=bf,Bf,h=valuation(f,3);if(h!=0 & h!=2,next);F=f/3^h;
if(core(F)!=F,next);F=factor(F);Div=component(F,1);d=matsize(F)[1];
for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
K=bnfinit(PK,1);rho=matsize(idealfactor(K,p))[1];HK=K.no;
if(valuation(HK,p)<vHK,next);CK0=K.clgp;for(n=1,Nn,Qn=x;
for(i=1,n,Qn=Qn^2-2);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
if(n==1,print();print("f=",f," PK=",PK," CK0=",CK0[2]," rho=",rho));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))))}
```

```
p=2  f=1777  PK=x^3+x^2-592*x+724  CK0=[4,4]  rho=3
CK1=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
```

```
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[4,4]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation in K3, m(K2)=1, e(K2)=2
```

```
p=2  f=7687  PK=x^3+x^2-2562*x-48969  CK0=[2,2,2,2]  rho=1
CK1=[12,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0]     h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]     h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]     h_2^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,0]
norm in K1/K of the component 2 of CK1:[2,2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[24,8,2,2]
h_1^[(S-1)^1]=[0,6,0,0]   h_2^[(S-1)^1]=[6,2,0,0]
h_3^[(S-1)^1]=[0,4,0,0]   h_4^[(S-1)^1]=[4,0,0,0]
h_1^[(S-1)^2]=[4,4,0,0]   h_2^[(S-1)^2]=[4,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]   h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,4,0,0]
norm in K2/K of the component 2 of CK2:[4,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)>2, e(K2)=3
```

```
p=2  f=10513  PK=x^3+x^2-3504*x-80989  CK0=[8,8]  rho=1
CK1=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0])
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[8,8]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
Complete capitulation in K4, m(K2)=1, e(K2)=3
```

```
p=2  f=16363  PK=x^3+x^2-5454*x-16969  CK0=[2,2,2,2]  rho=1
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,2,0,0]    h_2^[(S-1)^1]=[2,0,0,0]
h_3^[(S-1)^1]=[2,0,0,0]    h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,2,0,0]
norm in K1/K of the component 2 of CK1:[2,2,0,0]
norm in K1/K of the component 3 of CK1:[2,0,0,0]
norm in K1/K of the component 4 of CK1:[0,2,0,0]
CK2=[4,4,2,2]
h_1^[(S-1)^1]=[0,2,0,0]    h_2^[(S-1)^1]=[2,2,0,0]
h_3^[(S-1)^1]=[2,2,0,0]    h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation, m(K2)=2, e(K2)=2
```

```
p=2  f=16627  PK=x^3+x^2-5542*x-159496  CK0=[12,4]  rho=3
CK1=[24,24]
h_1^[(S-1)^1]=[0,0]  h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[24,24]
h_1^[(S-1)^1]=[0,0]  h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
```

```
Complete capitulation in K3, m(K2)=1, e(K2)=3


p=2  f=20599  PK=x^3+x^2-6866*x+216671  CK0=[28,4]  rho=1
CK1=[28,4,2,2]
h_1^[(S-1)^1]=[0,0,0,1]    h_2^[(S-1)^1]=[0,2,1,1]
h_3^[(S-1)^1]=[0,0,0,0]    h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,1]
norm in K1/K of the component 2 of CK1:[0,0,1,1]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[56,8,2,2]
h_1^[(S-1)^1]=[0,6,0,1]    h_2^[(S-1)^1]=[2,6,1,1]
h_3^[(S-1)^1]=[0,0,0,0]    h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[4,4,0,0]    h_2^[(S-1)^2]=[4,0,0,0]
norm in K2/K of the component 1 of CK2:[4,4,0,0]
norm in K2/K of the component 2 of CK2:[4,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation, m(K2)>2, e(K2)=3

p=2  f=20887  PK=x^3+x^2-6962*x-225889  CK0=[4,4,2,2]  rho=1
CK1=[8,8,2,2]
h_1^[(S-1)^1]=[0,0,0,0]    h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,4,0,0]    h_4^[(S-1)^1]=[4,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,0]
norm in K1/K of the component 2 of CK1:[0,2,0,0]
norm in K1/K of the component 3 of CK1:[0,4,0,0]
norm in K1/K of the component 4 of CK1:[4,0,0,0]
CK2=[8,8,2,2]
h_1^[(S-1)^1]=[0,4,0,0]    h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[0,4,0,0]    h_4^[(S-1)^1]=[4,4,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,0,0,0]
norm in K2/K of the component 2 of CK2:[0,4,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation in K3, m(K2)=2, e(K2)=3


p=2  f=31513  PK=x^3+x^2-10504*x-417839  CK0=[28,4]  rho=1
CK1=[84,4,2,2]
h_1^[(S-1)^1]=[0,0,0,1]    h_2^[(S-1)^1]=[0,2,1,1]
h_3^[(S-1)^1]=[0,0,0,0]    h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[2,0,0,1]
norm in K1/K of the component 2 of CK1:[0,0,1,1]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[168,8,2,2,2,2]
h_1^[(S-1)^1]=[2,0,1,0,1,1] h_2^[(S-1)^1]=[0,6,0,0,0,1]
h_3^[(S-1)^1]=[0,0,1,0,0,1] h_4^[(S-1)^1]=[4,0,0,1,1,0]
h_5^[(S-1)^1]=[4,4,0,1,1,0] h_6^[(S-1)^1]=[4,4,1,0,0,1]
h_1^[(S-1)^2]=[4,0,0,1,1,0] h_2^[(S-1)^2]=[4,0,1,0,0,1]
h_3^[(S-1)^2]=[4,4,0,0,0,0] h_4^[(S-1)^2]=[0,4,0,0,0,0]
h_5^[(S-1)^2]=[0,4,0,0,0,0] h_6^[(S-1)^2]=[4,4,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,4,0,0,0,0]
norm in K2/K of the component 2 of CK2:[4,4,0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0,0]
norm in K2/K of the component 6 of CK2:[0,0,0,0,0,0]
Incomplete capitulation, m(K2)>2, e(K2)=3
```

For some ambiguous cases, one must determine CK3, which takes several hours (or more); for instance, we got the following result, suggesting that everything regularizes if n is large enough, as Greenberg's conjecture predicts:

```
p=2  f=2689 PK=x^3+x^2-896*x+5876 CK0=[2,2] rho=3
CK1=[28,4]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,2]
CK2=[56,8]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,4]
CK3=[56,8]
h_1^[(S-1)^1]=[0,0]    h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]    h_2^[(S-1)^2]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Complete capitulation, m(K3)=1, e(K2)=3
```

5. Examples for pure cubic K and $p = 2$

5.1. **Pure cubic fields K, $L \subset K(\mu_{17})$.** We consider the set of pure cubic fields $K = \mathbb{Q}(\sqrt[3]{R})$; although these fields are not totally real, it is known that capitulation may exist in compositum $L = KL_0$, with suitable abelian p-extensions $L_0/\mathbb{Q}$ (conjectured in [Gras1997], proved in [Bosc2009]); so we limit ourselves to the usual $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{p^N}$. The extension $L/\mathbb{Q}$ is not Galois, but, by chance, the instruction $\mathsf{G} = \mathsf{nfgaloisconj}(\mathsf{Kn})$ of PARI computes the group of automorphisms, whence $\mathrm{Gal}(L/K)$ in our case; this simplifies the search of S of order p^n. Taking $\mathsf{p} = 2$, $\mathsf{Nn} = 3$ and restricting to fields K such that $\#\mathscr{H}_K \geq p^2$, $\ell = 17$, we obtain many capitulations:

```
MAIN PROGRAM FOR PURE CUBIC FIELDS:
{p=2;Nn=3;vHK=2;ell=17;mKn=2;for(R=2,10^4,PK=x^3-R;
if(polisirreducible(PK)==0,next);K=bnfinit(PK,1);
rho=matsize(idealfactor(K,ell))[1];
\\Test over the order of the p-class group of K:
HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;
\\Test for elimination of the stability from K:
if(n==1 & valuation(HKn,p)==valuation(HK,p),break);
if(n==1,print();print("PK=",PK," CK0=",CK0[2]," ell=",ell," rho=",rho));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,p^n,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))}
```

```
p=2  PK=x^3-43  CK0=[12]  ell=17  rho=2
CK1=[12,6]
h_1^[(S-1)^1]=[0,1]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,1]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation in K1, m(K1)=2, e(K1)=2
```

```
CK2=[12,12]
h_1^[(S-1)^1]=[0,1]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,2]
norm in K2/K of the component 2 of CK2:[0,0]
Incomplete capitulation in K2, m(K2)=2, e(K2)=2
CK3=[12,12]
h_1^[(S-1)^1]=[0,3]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
Stability from K2, complete capitulation in K3, m(K3)=2, e(K3)=2
```

```
p=2  PK=x^3-113  CK0=[2,2]  ell=17  rho=2
CK1=[6,2,2]
h_1^[(S-1)^1]=[0,1,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,1,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation in K1, m(K1)=2, e(K1)=1
CK2=[6,2,2,2,2]
h_1^[(S-1)^1]=[0,0,0,1,0] h_2^[(S-1)^1]=[0,1,1,1,0]
h_3^[(S-1)^1]=[0,1,1,1,0] h_4^[(S-1)^1]=[1,1,0,0,0]
h_5^[(S-1)^1]=[0,0,0,0,0]
h_1^[(S-1)^2]=[1,1,0,0,0] h_2^[(S-1)^2]=[1,1,0,0,0]
h_3^[(S-1)^2]=[1,1,0,0,0] h_4^[(S-1)^2]=[0,1,1,0,0]
h_5^[(S-1)^2]=[0,0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,1,1,0,0]
norm in K2/K of the component 2 of CK2:[0,1,1,0,0]
norm in K2/K of the component 3 of CK2:[0,1,1,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0,0]
norm in K2/K of the component 5 of CK2:[0,0,0,0,0]
Incomplete capitulation in K2, m(K2)>2, e(K2)=1
CK3=[12,2,2,2,2]
h_1^[(S-1)^1]=[2,1,0,0,1] h_2^[(S-1)^1]=[0,0,1,1,0]
h_3^[(S-1)^1]=[0,1,0,0,0] h_4^[(S-1)^1]=[2,0,1,1,1]
h_5^[(S-1)^1]=[0,1,0,0,1]
h_1^[(S-1)^2]=[0,1,1,1,1] h_2^[(S-1)^2]=[2,1,1,1,1]
h_3^[(S-1)^2]=[0,0,1,1,0] h_4^[(S-1)^2]=[2,0,1,1,0]
h_5^[(S-1)^2]=[0,1,1,1,1]
norm in K3/K of the component 1 of CK3:[0,0,0,0,0]
norm in K3/K of the component 2 of CK3:[0,0,0,0,0]
norm in K3/K of the component 3 of CK3:[0,0,0,0,0]
norm in K3/K of the component 4 of CK3:[0,0,0,0,0]
norm in K3/K of the component 5 of CK3:[0,0,0,0,0]
Complete capitulation in K3, no stability, m(K3)>2, e(K3)=2
```

```
p=2  PK=x^3-122  CK0=[12]  ell=17  rho=2
CK1=[12,4]
h_1^[(S-1)^1]=[0,1]      h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[0,2]      h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[2,1]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation in K1, m(K1)>2, e(K1)=2
CK2=[12,4]
h_1^[(S-1)^1]=[0,3]   h_2^[(S-1)^1]=[0,2]
h_1^[(S-1)^2]=[0,2]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,2]
norm in K2/K of the component 2 of CK2:[0,0]
Stability from K1, incomplete capitulation in K2, m(K2)>2, e(K2)=2
CK3=[12,4]
h_1^[(S-1)^1]=[0,0]      h_2^[(S-1)^1]=[3,2]
h_1^[(S-1)^2]=[0,0]      h_2^[(S-1)^2]=[2,0]
norm in K3/K of the component 1 of CK3:[0,0]
norm in K3/K of the component 2 of CK3:[0,0]
```

```
Stability from K1, complete capitulation in K3, m(K3)>2, e(K3)=2
p=2  PK=x^3-141  CK0=[4,2]  ell=17  rho=2
CK1=[8,2]
h_1^[(S-1)^1]=[0,0]      h_2^[(S-1)^1]=[0,0]
norm in K1/K of the component 1 of CK1:[2,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation in K1, m(K1)=1, e(K1)=3
CK2=[16,2]
h_1^[(S-1)^1]=[0,0]      h_2^[(S-1)^1]=[0,0]
norm in K2/K of the component 1 of CK2:[4,0]
norm in K2/K of the component 2 of CK2:[0,0]
Incomplete capitulation in K2, m(K2)=1, e(K2)=4
CK3=[288,18]
h_1^[(S-1)^1]=[0,0]      h_2^[(S-1)^1]=[0,0]
norm in K3/K of the component 1 of CK3:[8,0]
norm in K3/K of the component 2 of CK3:[0,0]
Incomplete capitulation in K3, m(K3)=1, e(K3)=5
p=2  PK=x^3-174  CK0=[6,2]  ell=17  rho=2
CK1=[12,6,2]
h_1^[(S-1)^1]=[2,1,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,1,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation in K1, m(K1)=2, e(K1)=2
CK2=[2040,12,2,2]
h_1^[(S-1)^1]=[4,0,1,1]  h_2^[(S-1)^1]=[0,0,0,1]
h_3^[(S-1)^1]=[0,2,0,1]  h_4^[(S-1)^1]=[0,2,0,0]
h_1^[(S-1)^2]=[0,0,0,1]  h_2^[(S-1)^2]=[0,2,0,0]
h_3^[(S-1)^2]=[0,2,0,0]  h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[4,2,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Incomplete capitulation in K2, m(K2)>2, e(K2)=3
CK3=[4080,12,2,2]
```

Unfortunately, the computations for $n = 3$ in this last example take too much time; probably, the capitulation is still incomplete. We shall examine separately this field:

5.2. **Pure cubic field** $K = \mathbb{Q}(\sqrt[3]{174})$, $\ell \equiv 1 \pmod{16}$. Varying ℓ, we find many capitulations in the layer K_2:

```
p=2  PK=x^3-174  CK0=[6,2]  ell=193  rho=1
CK1=[12,6]
h_1^[(S-1)^1]=[2,0]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[2,0]
Incomplete capitulation in K1, m(K1)=2, e(K1)=2
CK2=[12,6]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[2,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation in K2, m(K2)=2, e(K2)=2
p=2  PK=x^3-174  CK0=[6,2]  ell=353  rho=2
CK1=[48,6]
h_1^[(S-1)^1]=[6,0]   h_2^[(S-1)^1]=[8,0]
h_1^[(S-1)^2]=[4,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[8,0]
norm in K1/K of the component 2 of CK1:[8,0]
Incomplete capitulation in K1, m(K1)>2, e(K1)=4
CK2=[48,6,3,3]
```

```
h_1^[(S-1)^1]=[6,0,0,0]  h_2^[(S-1)^1]=[8,0,0,0]
h_3^[(S-1)^1]=[0,0,0,0]  h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[4,0,0,0]  h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]  h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation in K2, m(K2)>2, e(K2)=4
p=2  PK=x^3-174  CK0=[6,2]  ell=401  rho=2
CK1=[60,6,3]
h_1^[(S-1)^1]=[2,0,0]   h_2^[(S-1)^1]=[2,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0]
norm in K1/K of the component 2 of CK1:[2,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation in K1, m(K1)=2, e(K1)=1
CK2=[60,6,3]
h_1^[(S-1)^1]=[0,0,0]   h_2^[(S-1)^1]=[2,0,0]   h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]   h_3^[(S-1)^2]=[0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Complete capitulation in K2, m(K2)=2, e(K2)=1
p=2  PK=x^3-174  CK0=[6,2]  ell=577  rho=1
CK1=[84,6,2]
h_1^[(S-1)^1]=[0,1,1]   h_2^[(S-1)^1]=[2,1,1]   h_3^[(S-1)^1]=[2,1,1]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]   h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[2,1,1]
norm in K1/K of the component 2 of CK1:[2,1,1]
norm in K1/K of the component 3 of CK1:[2,1,1]
Incomplete capitulation in K1, m(K1)=2, e(K1)=2
CK2=[168,6,6,3]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[4,0,0,0]
h_3^[(S-1)^1]=[4,1,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[4,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[4,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
h_1^[(S-1)^3]=[0,0,0,0] h_2^[(S-1)^3]=[0,0,0,0]
h_3^[(S-1)^3]=[0,0,0,0] h_4^[(S-1)^3]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
Complete capitulation in K2, m(K2)=3, e(K2)=3
```

The last case $\ell = 577$ clearly shows that complexity of the Galois structure of the $\mathscr{H}_{K_n}$'s does not increase too much and that stability or conditions of Theorem 1.1 (i) are not necessary for capitulation. We obtain the following information on the structure of the $\mathscr{H}_{K_n}$'s for $K = \mathbb{Q}(\sqrt[3]{174})$ and $\ell = 577$:

In K_1, one find $m(K_1) = 2$, $e(K_1) = 2$ and $\mathscr{H}_{K_1}^{G_1} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For K_2, $m(K_2) = 3$, $e(K_2) = 3$; thus $\nu_{K_2/K}$ reduces to $4(\sigma-1)^2 + 6(\sigma-1) + 4$. The conditions of Theorem 1.1 (i) are not fulfilled. The above data shows that this reduces to the annihilation by $A = 6(\sigma-1) + 4$; but $h_1^A = h_1^{12} h_1^4 = 1$, $h_i^A = 1$ for the other generators.

6. Examples for real quadratic K and $p = 3$

We consider various (totally ramified) cyclic p-extensions L/K, where $L = KL_0$, $L_0/\mathbb{Q}$ cyclic of p-power degree. As for the case of cubic base fields, we favor the case $L \subset K(\mu_\ell)$, $\ell \equiv 1 \pmod{p^N}$.

6.1. **Quadratic fields K, $L \subset K(\mu_{109})$ and $L \subset K(\mu_{163})$.** Thus, L_0 is the real subfield of maximal 3-power degree of $\mathbb{Q}(\mu_\ell)$. We eliminate the cases of

stability in K_1/K (test $\mathsf{valuation(HKn,p)} == \mathsf{vHK}$, $\mathsf{vHK} = \mathsf{valuation(HK,p)}$). The images $\mathbf{J}_{K_n/K}(\mathscr{H}_K)$ are computed for $n = 1$ and $n = 2$. The number $\rho \in \{1, 2\}$ is the number of prime ideals above ℓ in K:

```
MAIN PROGRAM FOR REAL QUADRATIC FIELDS
{p=3;Nn=2;bm=2;Bm=10^8;vHK=1;ell=109;mKn=2;
for(m=bm,Bm,if(core(m)!=m,next);PK=x^2-m;K=bnfinit(PK,1);
\\Test over the order of the p-class group of K:
HK=K.no;if(valuation(HK,p)<vHK,next);
CK0=K.clgp;rho=(kronecker(m,ell)+3)/2;
for(n=1,Nn,Qn=polsubcyclo(ell,p^n);Pn=polcompositum(PK,Qn)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
\\Test for elimination of the stability from K:
if(n==1 & valuation(HKn,p)==valuation(HK,p),break);
if(n==1,print();print("PK=",PK," CK0=",CK0[2]," ell=",ell," rho=",rho));
CKn=Kn.clgp;print("CK",n,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p^n,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p^n,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",n,"/K of the component ",i,
" of CK",n,":",Enu))))}
```

6.1.1. *Case* $\ell = 109$.

```
p=3  PK=x^2-142  CK0=[3]  ell=109  rho=1
CK1=[18,2]
h_1^[(S-1)^1]=[3,0]         h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]         h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation in K1, m(K1)=2, e(K1)=2
CK2=[54,2]
h_1^[(S-1)^1]=[24,0]        h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[9,0]         h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[9,0]
norm in K2/K of the component 2 of CK2:[0,0]
No capitulation in K2, m(K2)>2, e(K1)=3

p=3  PK=x^2-223  CK0=[3]  ell=109  rho=2
CK1=[9]
h_1^[(S-1)^1]=[6]         h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation in K1, m(K2)=2, e(K1)=2
CK2=[9]
h_1^[(S-1)^1]=[3]         h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation in K2, m(K2)=2, e(K1)=2

p=3  PK=x^2-229  CK0=[3]  ell=109  rho=1
CK1=[9]
h_1^[(S-1)^1]=[3]         h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation in K1, m(K1)=2, e(K1)=2
CK2=[9]
h_1^[(S-1)^1]=[6]         h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation in K2, m(K2)=2, e(K2)=2

p=3  PK=x^2-254  CK0=[3]  ell=109  rho=2
CK1=[3,3]
h_1^[(S-1)^1]=[2,2]         h_2^[(S-1)^1]=[1,1]
h_1^[(S-1)^2]=[0,0]         h_2^[(S-1)^2]=[0,0]
```

```
norm in K1/K of the component 1 of CK1:[0,0]
norm in K1/K of the component 2 of CK1:[0,0]
Complete capitulation in K1, m(K1)=2, e(K1)=1
CK2=[3,3]
h_1^[(S-1)^1]=[1,1]          h_2^[(S-1)^1]=[2,2]
h_1^[(S-1)^2]=[0,0]          h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation in K2, m(K2)=2, e(K2)=1
p=3  PK=x^2-427  CK0=[6]  ell=109  rho=2
CK1=[18]
h_1^[(S-1)^1]=[0]          h_1^[(S-1)^2]=[0]
norm in K1/K of the component 1 of CK1:[3]
No capitulation in K1, m(K1)=1, e(K1)=2
CK2=[18]
h_1^[(S-1)^1]=[0]          h_1^[(S-1)^2]=[0]
norm in K2/K of the component 1 of CK2:[0]
Complete capitulation in K2, m(K2)=1, e(K2)=2
p=3  PK=x^2-574  CK0=[6]  ell=109  rho=2
CK1=[18,2,2]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation in K1, m(K1)=1, e(K1)=2
CK2=[54,2,2]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[0,0,0]  h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[0,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K2/K of the component 1 of CK2:[9,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
No capitulation in K2, m(K2)=1, e(K1)=3
```

6.1.2. *Case* $\ell = 163$.

```
p=3  PK=x^2-79  CK0=[3]  ell=163  rho=1
CK1=[18,2]
h_1^[(S-1)^1]=[3,0]          h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]          h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation in K1, m(K1)=2, e(K1)=2
CK2=[18,2]
h_1^[(S-1)^1]=[3,0]          h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]          h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation in K2, m(K2)=2, e(K2)=2
p=3  PK=x^2-223  CK0=[3]  ell=163  rho=2
CK1=[18,2]
h_1^[(S-1)^1]=[3,0]          h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]          h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation in K1, m(K1)=2, e(K1)=2
CK2=[18,2]
h_1^[(S-1)^1]=[3,0]          h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]          h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation in K2, m(K2)=2, e(K2)=2
PK=x^2-254  CK0=[3]  ell=163  rho=2
CK1=[18,2]
h_1^[(S-1)^1]=[0,0]          h_2^[(S-1)^1]=[0,0]
```

```
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation in K1, m(K1)=1, e(K1)=2
CK2=[18,2]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation in K2, m(K2)=1, e(K2)=2
```

6.1.3. *Examples with* $\mathrm{rank}_3(\mathscr{H}_K) = 2$. Due to a very large calculation time for the degrees $[K_2 : \mathbb{Q}] = 18$, we have only some results showing that, as for the case of cubic fields and $p = 2$ (degrees $[K_2 : \mathbb{Q}] = 12$) capitulation does occur at the layer $n = 2$:

```
PK=x^2-23659 CK0=[6,3]  ell=19  rho=2
CK1=[18,3]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation in K1, m(K1)=2 ,e(K1)=2
CK2=[18,3]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation in K2, m(K2)=2, e(K2)=2
```

```
PK=x^2-23659 CK0=[6,3] ell=37 rho=2
CK1=[18,3,3]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[6,0,1]  h_3^[(S-1)^1]=[6,0,0]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[6,0,0]  h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[6,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
CK2=[18,3,3]
h_1^[(S-1)^1]=[0,0,0]  h_2^[(S-1)^1]=[3,1,1]  h_3^[(S-1)^1]=[0,2,2]
h_1^[(S-1)^2]=[0,0,0]  h_2^[(S-1)^2]=[3,0,0]  h_3^[(S-1)^2]=[6,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0]
Complete capitulation in K2, m(K2)=3, e(K2)=2
```

```
PK=x^2-32009 CK0=[3,3] ell=19 rho=1
CK1=[9,3]
h_1^[(S-1)^1]=[3,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
CK2=[9,3]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation in K2, m(K2)=2, e(K2)=2
```

```
PK=x^2-32009 CK0=[3,3] ell=37 rho=2
CK1=[9,3]
h_1^[(S-1)^1]=[3,0]        h_2^[(S-1)^1]=[6,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
CK2=[9,3]
h_1^[(S-1)^1]=[0,0]        h_2^[(S-1)^1]=[3,0]
h_1^[(S-1)^2]=[0,0]        h_2^[(S-1)^2]=[0,0]
```

```
norm in K2/K of the component 1 of CK2:[0,0]
norm in K2/K of the component 2 of CK2:[0,0]
Complete capitulation in K2, m(K2)=2, e(K2)=2
```

6.2. **Quadratic fields K, $L \subset K(\mu_{109\cdot 163})$.** As for the previous case of cubic fields with $p = 2$, two primes $\ell_1 = 109$ and $\ell_2 = 163$ ramify in K_n/K, giving larger 3-class groups. We limit ourselves to the layer K_1 of degree 3. Most often, we have $\#\mathscr{H}_{K_1}^{G_1} = \#\mathscr{H}_K$ $(m(K_1) = 1)$; we give an excerpt of cases where $m(K_1) \geq 2$, but we have only two examples of $m(K_1) \geq 3$ in the interval selected ($m = 116279$ and $m = 370878$).

```
{p=3;bm=2;Bm=10^6;vHK=3;mKn=2;for(m=bm,Bm,if(core(m)!=m,next);
PK=x^2-m;K=bnfinit(PK,1);HK=K.no;if(valuation(HK,p)<vHK,next);CK0=K.clgp;
QK1=x^3-x^2-5922*x-17109;Pn=polcompositum(PK,QK1)[1];
Kn=bnfinit(Pn,1);HKn=Kn.no;dn=poldegree(Pn);
if(valuation(HKn,p)==valuation(HK,p),break);
print("p=3"," PK=",PK," CK0=",CK0[2]);
CKn=Kn.clgp;print("CK",1,"=",CKn[2]);rKn=matsize(CKn[2])[2];
G=nfgaloisconj(Kn);Id=x;for(k=1,dn,Z=G[k];ks=1;while(Z!=Id,
Z=nfgaloisapply(Kn,G[k],Z);ks=ks+1);if(ks==p,S=G[k];break));
for(i=1,rKn,X=CKn[3][i];Y=X;for(j=1,mKn,YS=nfgaloisapply(Kn,S,Y);
T=idealpow(Kn,Y,-1);Y=idealmul(Kn,YS,T);B=bnfisprincipal(Kn,Y)[1];
Ehij=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Ehij,c,ii));print("h_",i,"^[","(S-1)^",j,"]=",Ehij)));
for(i=1,rKn,A0=CKn[3][i];A=1;for(t=1,p,As=nfgaloisapply(Kn,S,A);
A=idealmul(Kn,A0,As));B=bnfisprincipal(Kn,A)[1];
Enu=List;for(ii=1,rKn,c=B[ii];w=valuation(CKn[2][ii],p);c=lift(Mod(c,p^w));
listput(Enu,c,ii));print("norm in K",1,"/K of the component ",i,
" of CK",1,":",Enu)))}
```

```
p=3  PK=x^2-8761  CK0=[27]
CK1=[27,3]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
Incomplete capitulation, m(K1)>1, e(K1)=3
```

```
p=3  PK=x^2-36073  CK0=[27]
CK1=[27,3,3]
h_1^[(S-1)^1]=[18,1,2]  h_2^[(S-1)^1]=[0,0,0]   h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]   h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=3
```

```
p=3  PK=x^2-65029  CK0=[27]
CK1=[81,3]
h_1^[(S-1)^1]=[0,0]   h_2^[(S-1)^1]=[0,0]
h_1^[(S-1)^2]=[0,0]   h_2^[(S-1)^2]=[0,0]
norm in K1/K of the component 1 of CK1:[3,0]
norm in K1/K of the component 2 of CK1:[0,0]
No capitulation, m(K1)=1, e(K1)=4
```

```
p=3  PK=x^2-116054  CK0=[27]
CK1=[81,3,3]
h_1^[(S-1)^1]=[54,0,2]  h_2^[(S-1)^1]=[0,0,0]   h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]   h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
No capitulation, m(K1)=2, e(K1)=4
```

```
p=3  PK=x^2-116279  CK0=[27]
CK1=[81,3,3]
h_1^[(S-1)^1]=[27,0,1]   h_2^[(S-1)^1]=[0,0,0]   h_3^[(S-1)^1]=[27,0,0]
h_1^[(S-1)^2]=[27,0,0]   h_2^[(S-1)^2]=[0,0,0]   h_3^[(S-1)^2]=[0,0,0]
```

```
norm in K1/K of the component 1 of CK1:[30,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
No capitulation, m(K1)>2, e(K1)=4

p=3  PK=x^2-156566  CK0=[9,3]
CK1=[27,3,3]
h_1^[(S-1)^1]=[18,0,0]   h_2^[(S-1)^1]=[0,0,0]   h_3^[(S-1)^1]=[18,0,0]
h_1^[(S-1)^2]=[0,0,0]    h_2^[(S-1)^2]=[0,0,0]   h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=*, e(K1)=3

p=3  PK=x^2-255973  CK0=[9,3]
CK1=[9,9,3]
h_1^[(S-1)^1]=[0,0,0]   h_2^[(S-1)^1]=[0,6,0]   h_3^[(S-1)^1]=[0,0,0]
h_1^[(S-1)^2]=[0,0,0]   h_2^[(S-1)^2]=[0,0,0]   h_3^[(S-1)^2]=[0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0]
norm in K1/K of the component 2 of CK1:[0,3,0]
norm in K1/K of the component 3 of CK1:[0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=2

p=3  PK=x^2-339887  CK0=[27]
CK1=[81,3,3,3]
h_1^[(S-1)^1]=[27,0,0,0]   h_2^[(S-1)^1]=[0,0,0,0]
h_3^[(S-1)^1]=[27,1,0,2]   h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0]    h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]    h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[3,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)=2, e(K1)=4

p=3  PK=x^2-370878  CK0=[54]
CK1=[1134,6,6,3]
h_1^[(S-1)^1]=[378,2,2,0]   h_2^[(S-1)^1]=[0,0,2,0]
h_3^[(S-1)^1]=[0,0,0,0]     h_4^[(S-1)^1]=[0,0,4,0]
h_1^[(S-1)^2]=[0,0,2,0]     h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0]     h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[168,0,2,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
Incomplete capitulation, m(K1)>2, e(K1)=4
```

7. Isotopic components and capitulation

Consider a real cyclic field K of prime-to-p degree d and $L = L_0K$ with $L_0/\mathbb{Q}$ real cyclic of degree p^N, $N \geq 1$. Then $L/\mathbb{Q}$ is cyclic of degree $D = d \cdot p^N$ with Galois group $\Gamma = g \times G$ where $g = \mathrm{Gal}(K/\mathbb{Q})$ and $G = \mathrm{Gal}(L_0/\mathbb{Q})$; L is associated to an irreducible rational character χ, sum of irreducible p-adic characters φ of Γ of "order" the order D of any $\psi \mid \varphi$ of degree 1.

This non semi-simple context is problematic for the definition of isotopic p-adic components of the form $\mathscr{H}_{L,\varphi}$; this is extensively developed in [Gras2021^b]. So we just recall the definitions and explain how the phenomenon of capitulation gives rise to difficulties about the classical algebraic definition of the literature, compared to the arithmetic one that we have introduced to state the Main Conjecture in the general case.

Indeed, classical works deal with an algebraic definition of the φ-components of p-class groups, which presents an inconsistency regarding analytic formulas; this definition is, for Γ cyclic of order D divisible by p and for all $\varphi \mid \chi$:

$$\mathscr{H}_{L,\varphi}^{\mathrm{alg}} := \mathscr{H}_L \otimes_{\mathbb{Z}_p[\Gamma]} \mathbb{Z}_p[\mu_D],$$

with the $\mathbb{Z}_p[\mu_D]$-action $\tau \in \Gamma \mapsto \psi(\tau)$, with $\psi \mid \varphi$ of order D (see [Solo1990, Lemma II.2] or [Grei1992, Definition, p. 451]).

We then have proved, with this definition [Gras2021^b, § 3.2.4, Theorem 3.7], the following first reduction of the problem:

$$\mathscr{H}_{L,\varphi}^{\rm alg} = \{x \in \mathscr{H}_L,\ \nu_{L/k}(x) = 1,\ \forall\, k \subsetneq L\} \otimes_{\mathbb{Z}_p[\Gamma]} \mathbb{Z}_p[\mu_D],$$

giving rise to our corresponding arithmetic definitions:

$$\begin{aligned}\mathscr{H}_{L,\chi}^{\rm ar} &:= \{x \in \mathscr{H}_L,\ \ \mathbf{N}_{L/k}(x) = 1,\ \forall\, k \subsetneq L\},\\ \mathscr{H}_{L,\varphi}^{\rm ar} &:= \{x \in \mathscr{H}_L,\ \mathbf{N}_{L/k}(x) = 1,\ \forall\, k \subsetneq L\} \otimes_{\mathbb{Z}_p[\Gamma]} \mathbb{Z}_p[\mu_D].\end{aligned}$$

We then have (since L/K is totally ramified):

$$\#\mathscr{H}_L^{\rm ar} = \prod_{\chi} \#\mathscr{H}_{L_\chi}^{\rm ar}, \tag{7.1}$$

where χ runs trough the set of irreducible rational characters of L and L_χ is the subfield of L fixed by $\mathrm{Ker}(\chi)$.

This notion leads to an unexpected semi-simplicity, especially in accordance with analytic formulas, which enforces the Main Conjecture in that case:

$$\mathscr{H}_{L,\chi}^{\rm ar} = \bigoplus_{\varphi \mid \chi} \mathscr{H}_{L,\varphi}^{\rm ar}.$$

We have $\mathscr{H}_{\varphi}^{\rm ar} = \mathscr{H}_{\varphi}^{\rm alg}$ as soon as the $\mathbf{J}_{L/k}$'s are injective for all $k \subsetneq L$, but as we have seen, this does not hold in general when $K \subseteq k \subsetneq L$ since there is often partial capitulation. One can even say that the classic admitted definition is ineffective and fallacious in the real case.

Let's give numerical examples showing the consequences of capitulation for these non-arithmetic definitions; for all them, p-adic characters are also rational:

Example 7.1. Consider $K = \mathbb{Q}(\sqrt{4409})$, $p = 3$, $\ell = 19$ and $L = K_2 \subset K(\mu_\ell)$ of degree 9 over K. The prime 2 splits in K, is inert in K_2/K and such that $\mathfrak{q} \mid 2$ in K generates $\mathscr{H}_K$ (cyclic of order 9); considering the extension $\mathfrak{Q}_i := (\mathfrak{q})_{K_i}$ of $\mathfrak{q}$ in K_i, we test its order in $\mathscr{H}_{K_i}$, $i = 1, 2$ (we are going to see that $\mathscr{H}_{K_i} \simeq \mathbb{Z}/9\mathbb{Z}$ for all i (stability), which is supported by the fact that $\mathbf{N}_{K_2/K}(\mathfrak{Q}_2) = \mathfrak{q}^9$ is principal, but $\mathbf{N}_{K_2/K}(\mathscr{H}_{K_2}) = \mathscr{H}_K$); the general program gives:

```
p=3  PK=x^2-4409  CK0=[9]  ell=19  rho=2
CK1=[9]
h_1^[(S-1)^1]=[0]
norm in K1/K of the component 1 of CK1:[3]
CK2=[9]
h_1^[(S-1)^1]=[0]
norm in K2/K of the component 1 of CK2:[0]
```

The capitulation, incomplete in K_1, is complete in K_2 (stability from K).

We use obvious notations for the characters defining the fields K_i, $i = 0, 1, 2$. Since arithmetic norms are surjective (because of the stability, they are isomorphisms and $\mathscr{H}_{K_i} = \mathscr{H}_{K_i}^{G_i}$), the above computations prove that:

$$\begin{cases} \nu_{K_2/K_1}(\mathscr{H}_{K_2}) = (\mathscr{H}_{K_2})^{1+\sigma_2^3+\sigma_2^6} = (\mathscr{H}_{K_2})^3 \simeq \mathbb{Z}/3\mathbb{Z},\\ \nu_{K_1/K}(\mathscr{H}_{K_1}) = (\mathscr{H}_{K_1})^{1+\sigma_1+\sigma_1^2} = (\mathscr{H}_{K_1})^3 \simeq \mathbb{Z}/3\mathbb{Z},\end{cases}$$

whence:

$$\begin{cases} \mathscr{H}_{\chi_2}^{\rm ar} = \{x \in \mathscr{H}_{K_2},\ \mathbf{N}_{K_2/K_1}(x) = 1\} = 1, \\ \mathscr{H}_{\chi_2}^{\rm alg} = \{x \in \mathscr{H}_{K_2},\ \nu_{K_2/K_1}(x) = 1\} = \mathscr{H}_{K_2}[3] \simeq \mathbb{Z}/3\mathbb{Z}. \\ \mathscr{H}_{\chi_1}^{\rm ar} = \{x \in \mathscr{H}_{K_1},\ \mathbf{N}_{K_1/K}(x) = 1\} = 1, \\ \mathscr{H}_{\chi_1}^{\rm alg} = \{x \in \mathscr{H}_{K_1},\ \nu_{K_1/K}(x) = 1\} = \mathscr{H}_{K_1}[3] \simeq \mathbb{Z}/3\mathbb{Z}. \end{cases}$$

Formula (7.1) gives the product of orders of the χ-components:

$$\#\mathscr{H}_{K_2} = \#\mathscr{H}_{\chi_0}^{\rm ar} \cdot \#\mathscr{H}_{\chi_1}^{\rm ar} \cdot \#\mathscr{H}_{\chi_2}^{\rm ar} \quad \& \quad \#\mathscr{H}_{K_1} = \#\mathscr{H}_{\chi_0}^{\rm ar} \cdot \#\mathscr{H}_{\chi_1}^{\rm ar},$$

of the form $\#\mathscr{H}_{K_2} = 9 \times 1 \times 1$ and $\#\mathscr{H}_{K_1} = 9 \times 1$ since $\#\mathscr{H}_{\chi_0}^{\rm ar} = \#\mathscr{H}_K = 9$.

These formulas are not fulfilled in the algebraic sense, because:

$$\#\mathscr{H}_{\chi_0}^{\rm alg} \cdot \#\mathscr{H}_{\chi_1}^{\rm alg} \cdot \#\mathscr{H}_{\chi_2}^{\rm alg} = 9 \times 3 \times 3 = 3^4 \quad \& \quad \#\mathscr{H}_{\chi_0}^{\rm alg} \cdot \#\mathscr{H}_{\chi_1}^{\rm alg} = 9 \times 3 = 3^3.$$

Our Main Conjecture requires that $\#\mathscr{H}_{\chi_i}^{\rm ar} = (\mathscr{E}_{K_i} : \mathscr{E}_{K_i}^0 \cdot \mathscr{F}_{K_i})$, where $\mathscr{E}_{K_i}^0$ is the subgroup of $\mathscr{E}_{K_i}$ generated by the units of the strict subfields of K_i and $\mathscr{F}_{K_i}$ the group of classical Leopoldt's cyclotomic units; these equalities are fulfilled in that numerical example (see [Gras2021^b, Examples 3.12, 3.13]).

Example 7.2. This example is analogous for a cyclic cubic field and $p = 2$, except that capitulation takes place from K:

```
p=2  f=1951  PK=x^3+x^2-650*x-289  CK0=[2,2]  ell=17  rho=3
CK1=[4,4,2,2]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[0,0,0,0] h_4^[(S-1)^1]=[0,0,0,0]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K1/K of the component 1 of CK1:[0,0,0,0]
norm in K1/K of the component 2 of CK1:[0,0,0,0]
norm in K1/K of the component 3 of CK1:[0,0,0,0]
norm in K1/K of the component 4 of CK1:[0,0,0,0]
CK2=[4,4,4,4]
h_1^[(S-1)^1]=[2,0,0,0] h_2^[(S-1)^1]=[0,2,0,0]
h_3^[(S-1)^1]=[2,0,2,0] h_4^[(S-1)^1]=[0,2,0,2]
h_1^[(S-1)^2]=[0,0,0,0] h_2^[(S-1)^2]=[0,0,0,0]
h_3^[(S-1)^2]=[0,0,0,0] h_4^[(S-1)^2]=[0,0,0,0]
norm in K2/K of the component 1 of CK2:[0,0,0,0]
norm in K2/K of the component 2 of CK2:[0,0,0,0]
norm in K2/K of the component 3 of CK2:[0,0,0,0]
norm in K2/K of the component 4 of CK2:[0,0,0,0]
```

Numerical data give:

$$\begin{cases} \nu_{K_2/K_1}(\mathscr{H}_{K_2}) = \mathscr{H}_{K_2}^2 \simeq (\mathbb{Z}/2\mathbb{Z})^4, \\ \nu_{K_2/K}(\mathscr{H}_{K_2}) = \nu_{K_1/K}(\mathscr{H}_{K_1}) = 1. \end{cases}$$

Whence:

$$\begin{cases} \mathscr{H}_{\chi_2}^{\rm ar} = \{x \in \mathscr{H}_{K_2},\ \mathbf{N}_{K_2/K_1}(x) = 1\} \simeq (\mathbb{Z}/2\mathbb{Z})^2, \\ \mathscr{H}_{\chi_2}^{\rm alg} = \{x \in \mathscr{H}_{K_2},\ \nu_{K_2/K_1}(x) = 1\} = \mathscr{H}_{K_2}[2] \simeq (\mathbb{Z}/2\mathbb{Z})^4. \\ \mathscr{H}_{\chi_1}^{\rm ar} = \{x \in \mathscr{H}_{K_1},\ \mathbf{N}_{K_1/K}(x) = 1\} \simeq (\mathbb{Z}/2\mathbb{Z})^4, \\ \mathscr{H}_{\chi_1}^{\rm alg} = \{x \in \mathscr{H}_{K_1},\ \nu_{K_1/K}(x) = 1\} = \mathscr{H}_{K_1} \simeq (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2. \end{cases}$$

Which gives:

$$\#\mathscr{H}_{K_2} = \#\mathscr{H}_{\chi_0}^{\rm ar} \cdot \#\mathscr{H}_{\chi_1}^{\rm ar} \cdot \#\mathscr{H}_{\chi_2}^{\rm ar} = 2^8 \quad \& \quad \#\mathscr{H}_{K_1} = \#\mathscr{H}_{\chi_0}^{\rm ar} \cdot \#\mathscr{H}_{\chi_1}^{\rm ar} = 2^6,$$

contrary to $\#\mathscr{H}_{\chi_0}^{\rm alg} \cdot \#\mathscr{H}_{\chi_1}^{\rm alg} \cdot \#\mathscr{H}_{\chi_2}^{\rm alg} = 2^6$ & $\#\mathscr{H}_{\chi_0}^{\rm alg} \cdot \#\mathscr{H}_{\chi_1}^{\rm alg} = 2^2$, of the form $2^2 \cdot 2^6 \cdot 2^4 = 2^{12}$ and $2^2 \cdot 2^6 = 2^8$, respectively; which relativizes the interest of algebraic definitions.

8. Conclusions and remarks

a) We have conjectured in Conjecture 1.2 (i) that, varying $\ell \equiv 1 \pmod{2p^N}$, N large enough, there are infinitely many cases with stability from a suitable layer in $K(\mu_\ell)$, yielding capitulation (Theorem 1.1 (ii)), which reinforces the simple capitulation phenomenon; this would be coherent with Greenberg's conjecture, equivalent to the stability of the $\mathscr{H}_{K_n}$ in the cyclotomic $\mathbb{Z}_p$-extension, for $n \gg 0$. In other words, the conjecture may be seen as the "tame version", it being understood that our towers are finite, so that capitulation needs large N's allowing a kind of "finite Iwasawa's theory". The more general criterion of Theorem 1.1 (i), using the algebraic norm, shows the link between capitulation and complexity of the filtration of the $\mathscr{H}_{K_n}$'s, likely to be governed by natural density results (Conjecture 2.4).

b) Due to the computations given in various frameworks in this paper, it is difficult to imagine that, for all $\ell \equiv 1 \pmod{2p^N}$, N large enough, $\mathscr{H}_K$ does not capitulate in $K(\mu_\ell)$, all the more that we were limited to testing with few values of ℓ (among infinitely many !) and only for the layers $n \leq 3$. It is reasonable to think that, restricting to primes ℓ with $N \gg 0$, $N - s$ is larger than $e(K_N)$ taking into account that s is logarithmic regarding $m(K_N)$ which essentially depends on the magnitude of $\mathscr{H}_K$ (exponent and p-rank).

Similarly, we were limited to small p's because of the degrees $[K_n : \mathbb{Q}] = [K : \mathbb{Q}]\, p^n$ for PARI calculations; but the nature of the theoretical results does not seem to depend on it; this is strengthened by the algorithmic aspect of formulas 2.3.

c) The remarkable circumstance of capitulations in these simplest ramified cyclic p-extensions and certainly in most arbitrary cyclic p-extensions L/K, is certainly a basic principle for many arithmetic properties, as the following ones:

(i) The abelian Main Conjecture for real abelian fields, whose proof becomes trivial, in the semi-simple case, as soon as ℓ is taken inert in $K/\mathbb{Q}$ and if $\mathscr{H}_K$ capitulates in some K_n.

(ii) Capitulations prevent to get standard algebraic definitions of p-adic isotopic components of arithmetic invariants in the non semi-simple case.

(iii) When capitulation is, on the contrary, structurally impossible (e.g., case of torsion groups of p-ramification theory), the complexity of the corresponding invariants increases in the towers.

References

[AZT2016] A. Azizi, A. Zekhnini, M. Taous, On the strongly ambiguous classes of some bi-quadratic number fields, Mathematica Bohemica **141**(3) (2016), 363–384. http://eudml.org/doc/286792 3

[BaCa2016] A. Bandini, F. Caldarola, Stabilization for Iwasawa modules in $\mathbb{Z}_p$-extensions, Rend. Sem. Mat. Univ. Padova **136** (2016), 137–155. https://doi.org/10.4171/RSMUP/136-10 8, 19

[Band2007] A. Bandini, A note on p-ranks of class groups in $\mathbb{Z}_p$-extensions, JP Journal of Algebra, Number Theory and Applications **9**(1) (2007), 95–103. https://arpi.unipi.it/handle/11568/925088 2

[BeJa2016] K. Belabas, J-F. Jaulent, The logarithmic class group package in PARI/GP, Publications Mathématiques de Besançon (Algèbre et théorie des nombres) (2016), 5–18. http://pmb.univ-fcomte.fr/2016/pmb_2016.pdf 20

[Bisw2016] S. Biswas, Capitulation, unit groups, and the cohomology of S-idèle classes, arXiv (2016) https://arxiv.org/abs/1610.00733 3

[Brig2007] C. Brighi, Capitulation des classes logarithmiques et étude de certaines tours de corps de nombres, Thèse, Université Paul Verlaine–Metz (2007). https://hal.univ-lorraine.fr/tel-01749002 16

[Bond1981] R. Bond, Some results on the Capitulation Problem, J. Number Theory **13**(2) (1981), 246–254. https://doi.org/10.1016/0022-314X(81)90007-X 3

[Bosc2009] S. Bosca, Principalization of ideals in Abelian extensions of number fields, Int. J. Number Theory **5** (2009), 527–539. https://doi.org/10.1142/S1793042109002213 3, 4, 42

[Cald2020] F. Caldarola, On the maximal finite Iwasawa submodule in $\mathbb{Z}_p$-extensions and capitulation of ideals, Rendiconti Sem. Mat. Univ. Pol. Torino **78**(1) (2020), 27–42. http://www.seminariomatematico.polito.it/rendiconti/78-1/Caldarola.pdf 8, 19

[Chev1933] C. Chevalley, Sur la théorie du corps de classes dans les corps finis et les corps locaux, Thèse no. 155, Jour. of the Faculty of Sciences Tokyo **2** (1933), 365–476. http://archive.numdam.org/item/THESE_1934__155__365_0/ 3

[Fuku1994] T. Fukuda, Remarks on $\mathbb{Z}_p$-extensions of number fields, Proc. Japan Acad. Ser. A **70**(8) (1994), 264–266. https://doi.org/10.3792/pjaa.70.264 2, 23

[GrJa1985] M. Grandet, J-F. Jaulent, Sur la capitulation dans une $\mathbb{Z}_\ell$-extension, J. reine angew. Math. 362 (1985), 213–217. http://eudml.org/doc/152777 3, 19, 20

[GJN2016] G. Gras, J-F. Jaulent, T. Nguyen Quang Do, Sur le module de Bertrandias–Payan dans une p-extension – Noyau de capitulation, pp. 25–44. Sur la capitulation pour le module de Bertrandias–Payan, pp. 45–58. Descente galoisienne et capitulation entre modules de Bertrandias–Payan, pp. 59–79. Publications Mathématiques de Besançon (Algèbre et théorie des nombres) (2016). https://doi.org/10.5802/pmb.o-3 https://doi.org/10.5802/pmb.o-4 https://doi.org/10.5802/pmb.o-5 16

[Gonz2006] C.D. González-Avilés, Capitulation, ambiguous classes and the cohomology of the units, J. reine angew. Math. 2007 **613** (2006), 75–97. https://www.researchgate.net/publication/2128194 3

[Gras1978] G. Gras, Nombre de φ-classes invariantes. Application aux classes des corps abéliens, Bulletin Soc. Math. France **106** (1978), 337–364. https://doi.org/10.24033/bsmf.1876 3

[Gras1997] G. Gras, Principalisation d'idéaux par extensions absolument abéliennes, J. Number Theory **62**(2) (1997), 403–421. https://doi.org/10.1006/jnth.1997.2068 3, 4, 42

[Gras2005] G. Gras, *Class Field Theory: from theory to practice*, corr. 2nd ed. Springer Monographs in Mathematics, Springer, xiii+507 pages (2005). 5, 16, 18

[Gras2017^a] G. Gras, Invariant generalized ideal classes – Structure theorems for p-class groups in p-extensions, Proc. Indian Acad. Sci. (Math. Sci.) **127**(1) (2017), 1–34. https://doi.org/10.1007/s12044-016-0324-1 3, 5, 7

[Gras2017^b] G. Gras, Approche p-adique de la conjecture de Greenberg pour les corps totalement réels, Ann. Math. Blaise Pascal **24**(2) (2017), 235–291. https://doi.org/10.5802/ambp.370 Numerical table: https://www.dropbox.com/s/tcqfp41plzl3u60/R 21

[Gras2018] G. Gras, The p-adic Kummer–Leopoldt Constant: Normalized p-adic Regulator, Int. J. of Number Theory, **14**(2) (2018), 329–337. https://doi.org/10.1142/S1793042118500203 16, 21

[Gras2019^a] G. Gras, On p-rationality of number fields. Applications–PARI/GP programs, Publications Mathématiques de Besançon (Algèbre et théorie des nombres) **2** (2019), 29–51. http://www.numdam.org/articles/10.5802/pmb.35/ 17

[Gras2019^b] G. Gras, Normes d'idéaux dans la tour cyclotomique et conjecture de Greenberg, Ann. math. du Québec **43** (2019), 249–280. https://doi.org/10.1007/s40316-018-0108-3 21

[Gras2021^a] G. Gras, Algorithmic complexity of Greenberg's conjecture, Arch. Math. **117** (2021), 277–289. https://doi.org/10.1007/s00013-021-01618-9 16, 21

[Gras2021^b] G. Gras, Application of the notion of φ-object to the study of p-class groups and p-ramified torsion groups of abelian extensions, arXiv (2021). https://arxiv.org/abs/2112.02865 50, 51, 52

[Gras2022^a] G. Gras, On the λ-stability of p-class groups along cyclic p-towers of a number field, Int. J. of Number Theory **18**(10) (2022), 2241–2263. https://doi.org/10.1142/S1793042122501147 2, 3, 16

[Gras2022^b] G. Gras, The Chevalley–Herbrand formula and the real abelian conjecture–New criterion using capitulation of the class group, arXiv (2022). https://arxiv.org/abs/2207.13911 3

[Gree1976] R. Greenberg, On the Iwasawa invariants of totally real number fields, Amer. J. Math. **98**(1) (1976), 263–284. https://doi.org/10.2307/2373625 20

[Grei1992] C. Greither, Class groups of abelian fields, and the main conjecture, Ann. Inst. Fourier **42**(3) (1992), 449–499. https://doi.org/10.5802/aif.1299 51

[GrWe2000] K.W. Gruenberg, A. Weiss, Capitulation and transfer kernels, J. Théorie Nombres Bordeaux **12**(1) (2000), 219–226.
http://www.numdam.org/item/JTNB_2000__12_1_219_0/ 3

[GrWe2003] K.W. Gruenberg, A. Weiss, Capitulation and Transfer Triples, Proc. London Math. Soc. **87**(2) (2003), 273–290.
https://doi.org/10.1112/S0024611503014199 3

[HeSc1982] F-P. Heider, B. Schmithals, Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen, J. Reine Angew. Math. **336** (1982), 1–25.
https://doi.org/10.1515/crll.1982.336.1 3

[Iwas1973] K. Iwasawa, On $\mathbb{Z}_\ell$-extensions of algebraic number fields, Ann. Math. **98** (1973), 243–326. https://doi.org/10.2307/1970784 19

[Iwas1989] K. Iwasawa, A note on capitulation problem for number fields, II, Proc. Japan Acad. Ser. A Math. Sci. **65**(6) (1989), 183–186.
https://doi.org/10.3792/pjaa.65.183 3

[Jaul1986] J-F. Jaulent, L'arithmétique des ℓ-extensions (Thèse d'état), Publications Mathématiques de Besançon (Algèbre et théorie des nombres) **1**(1) (1986), 1–357. https://doi.org/10.5802/pmb.a-42 3, 8

[Jaul1988] J-F. Jaulent, L'état actuel du problème de la capitulation, Sém. Théor. Nombres Bordeaux, exposé 17 (1987/1988), 1–33.
https://www.jstor.org/stable/44166467 3

[Jaul1994] J-F. Jaulent, Classes logarithmiques des corps de nombres, J. Théorie Nombres Bordeaux **6** (1994), 301–325.
https://jtnb.centre-mersenne.org/article/JTNB_1994__6_2_301_0.pdf 16

[Jaul2016] J-F. Jaulent, Classes logarithmiques et capitulation, Funct. Approx. Comment. Math. **54**(2) (2016), 227–239.
https://doi.org/10.7169/facm/2016.54.2.6 16, 19, 20

[Jaul2019^a] J-F. Jaulent, Note sur la conjecture de Greenberg, J. Ramanujan Math. Soc. **34**(1) (2019) 59–80.
http://www.mathjournals.org/jrms/2019-034-001/2019-034-001-005.pdf 16, 19, 20

[Jaul2019^b] J-F. Jaulent, Principalisation abélienne des groupes de classes logarithmiques, Funct. Approx. Comment. Math. **61**(2) (2019), 257–275.
https://doi.org/10.7169/facm/1765 3, 16, 20

[Jaul2022] J-F. Jaulent, Capitulation abélienne des groupes de classes de rayons, J. Number Theory **231** (2022), 316–332.
https://doi.org/10.1016/j.jnt.2021.05.003 3, 4, 16

[KhPr2000] C. Khare, D. Prasad, On the Steinitz module and capitulation of ideals, Nagoya Math. J. **160** (2000), 1–15. https://doi.org/10.1017/S0027763000007686 16

[Kisil1970] H. Kisilevsky, Some results related to Hilbert's theorem 94, J. Number Theory **2**(2) (1970), 199–206.
https://doi.org/10.1016/0022-314X(70)90020-X 3

[KoMo2000] M. Kolster, A. Movahhedi, Galois co-descent for étale wild kernels and capitulation, Annales Institut Fourier **50**(1) (2000), 35–65.
https://doi.org/10.5802/aif.1746 3, 16, 19

[KoPa2022] P. Koymans, C. Pagano, On the distribution of $Cl(K)[\ell^\infty]$ for degree ℓ cyclic fields, J. Eur. Math. Soc. **24**(4) (2022), 1189–1283.
https://doi.org/10.4171/JEMS/1112 7

[KrSch1995] J.S. Kraft, R. Schoof, Computing Iwasawa modules of real quadratic number fields, Compositio Math. **97**(1-2) (1995), 135–155.
http://eudml.org/doc/90370 2, 20
Erratum. Compositio Math., **103**(2) (1996), p. 241.
http://www.numdam.org/item?id=CM_1996__103_2_241_0

[Kuri1999] M. Kurihara, On the ideal class groups of the maximal real subfields of number fields with all roots of unity, J. Eur. Math. Soc. **1** (1999), 35–49.
https://doi.org/10.1007/PL00011159 3, 4

[Lemm2013] F. Lemmermeyer, The ambiguous class number formula revisited, Journal of Ramanujan Math. Soc. **28**(4) (2013), 415–421.
http://www.mathjournals.org/jrms/2013-028-004/2013-028-004-002.html 3

[LiYu2020] J. Li, C.F. Yu, The Chevalley–Gras formula over global fields, J. Théor. Nombres Bordeaux **32**(2) (2020), 525–543. https://doi.org/10.5802/jtnb.1133 3

[LOXZ2022] J. Li, Y. Ouyang, Y. Xu, S. Zhang, ℓ-Class groups of fields in Kummer towers, Publ. Mat. **66**(1) (2022), 235–267.
https://doi.org/10.5565/PUBLMAT6612210 2

[Mart2011] K. Martin, Nonunique factorization and principalization in number fields, Proceedings of the American Mathematical Society **139**(9) (2011), 3025–3038. https://doi.org/10.1090/S0002-9939-2011-11053-0 2

[Mayer2014] D.C. Mayer, Principalization algorithm via class group structure, J. Théorie Nombres Bordeaux **26**(2) (2014), 415–464. https://doi.org/10.5802/jtnb.874 3

[Maire1996] C. Maire, $T-S$ capitulation, Publ. Math. Fac. Sci. Besançon (Algèbre et théorie des nombres) (1996), article no. 2, 32 p. https://doi.org/10.5802/pmb.a-79 3, 16

[Maire1998] C. Maire, Une remarque sur la capitulation du groupe des classes au sens restreint, Publ. Math. Fac. Sci. Besançon (Algèbre et théorie des nombres) (1998), article no. 2, 11 p. https://doi.org/10.5802/pmb.a-87 3

[MiYa2021] Y. Mizusawa, K. Yamamoto, On p-class groups of relative cyclic p-extensions, Arch. Math. **117**(3) (2021), 253–260. https://doi.org/10.1007/s00013-021-01619-8 2

[Paga2022] L. Pagani, Greenberg's conjecture for real quadratic fields and the cyclotomic $\mathbb{Z}_2$-extension, Math. Comp. **91** (2022) 1437–1467. https://doi.org/10.1090/mcom/3712 20

[PARI] The PARI Group, *PARI/GP, version `2.5.3` (2013)*, Univ. Bordeaux. http://pari.math.u-bordeaux.fr/download.html 3

[Schm1985] B. Schmithals, Kapitulation der Idealklassen und Einheitenstruktur in Zahlkörpern, J. reine angew. math. **358** (1985), 43–60. https://doi.org/10.1515/crll.1985.358.43 3

[Smith2022] A. Smith, The distribution of ℓ^∞-Selmer groups in degree ℓ twist families, arXiv (2022). https://doi.org/10.48550/arXiv.2207.05674 7

[Solo1990] D. Solomon, On the class groups of imaginary abelian fields, Ann. Inst. Fourier **40**(3) (1990), 467–492. https://doi.org/10.5802/aif.1221 51

[Suzu1991] H. Suzuki, A generalization of Hilbert's Theorem 94, Nagoya Math. J. **121** (1991), 161–169. https://doi.org/10.1017/S0027763000003445 3

[Taya1996] H. Taya, On cyclotomic $\mathbb{Z}_p$-extensions of real quadratic fields, Acta Arithmetica **74**(2) (1996), 107–119. http://matwbn.icm.edu.pl/ksiazki/aa/aa74/aa7422.pdf 21

[Taya1999] H. Taya, On p-adic zêta functions and $\mathbb{Z}_p$-extensions of certain totally real number fields, Tohoku Math. J. **51**(1) (1999), 21–33. https://projecteuclid.org/download/pdf_1/euclid.tmj/1178224850 21

[Taya2000] H. Taya, Iwasawa invariants and class numbers of quadratic fields for the prime 3, Proc. Amer. Math. Soc. **128**(5), 1285–1292. https://doi.org/10.1090/S0002-9939-99-05177-1 21

[Tera1971] F. Terada, A principal ideal theorem in the genus field, Tohoku Math. J. (2) **23**(4) (1971), 697–718. https://doi.org/10.2748/tmj/1178242555 3

[Vali2008] R. Validire, Capitulation des noyaux sauvages étales, Thèse, Université de Limoges (24 Juin 2008). https://tel.archives-ouvertes.fr/tel-00343427/document 3, 16, 19

[Wash1997] L.C. Washington, *Introduction to Cyclotomic Fields*, GTM 83, Springer enlarged second edition 1997. 8

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