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Alice Guionnet, Édouard Maurel-Segala

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MATRIX MODELS AT LOW TEMPERATURE

ALICE GUIONNET AND ÉDOUARD MAUREL-SEGALA

ABSTRACT. In this article we investigate the behavior of multi-matrix unitary invariant models under a potential $V_\beta = \beta U + W$ when the inverse temperature β becomes very large. We first prove, under mild hypothesis on the functionals U, W that as soon as these potentials are "confining" at infinity, the sequence of spectral distribution of the matrices are tight when the dimension goes to infinity. Their limit points are solutions of Dyson-Schwinger's equations. Next we investigate a few specific models, most importantly the "strong single variable model" where U is a sum of potentials in a single matrix and the "strong commutator model" where $U = -[X, Y]^2$.

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1. INTRODUCTION

Matrix integrals and their limit for large N play an important role in both mathematics and physics. Random matrices were first introduced in statistics by Wishart [49] to analyze a large array of noisy data, a viewpoint that is now particularly relevant and useful for principal component analysis and statistical learning [7, 18]. Wigner [48] and Dyson [16] later suggested that the statistics of their eigenvalues model very well those of the high energy levels in heavy nuclei. Even more surprisingly, Montgomery

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[41] showed that random matrices are closely related to the zeros of the Riemann Zeta function, a conjecture that nowadays provides great intuition for many mathematical results, see e.g [37, 3]. Random matrices are moreover intimately related to integrable systems to which they furnish key examples. Random matrices have also played a central role in the theory of operator algebras since Voiculescu [46, 45] proved that they are asymptotically free. In physics, they were introduced into quantum field theory by 't Hooft [43], then used in two-dimensional quantum gravity and string theory [17, 15, 35], and related with algebraic geometry [19, 8, 21]. In both free probability, statistical learning and physics, a central question is to estimate, in the large N -limit, a matrix integral of the form

$$I_N^\ell(V) = \int \exp\{-N\text{Tr}V(X_1^N, X_2^N, \dots, X_\ell^N)\} dX_1^N \dots dX_\ell^N$$

where the integration holds over Hermitian $N \times N$ matrices, V is apolynomial and dX_i^N denotes the Lebesgue measure on this space. It turns out that this was done only in few cases. The case where the integral holds only on one matrix has been thoroughly studied. Indeed, in this case one can diagonalize the matrix and find that

$$I_N^1(V) = C_N \int \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 e^{-N \sum V(\lambda_i)} \prod_{1 \leq i \leq N} d\lambda_i,$$

where C_N is a universal constant. $F_N^1(V) = N^{-2} \ln I_N^1(V)$ can be shown to converge as N goes to infinity. Its limit was computed by using a saddle point method or large deviations techniques [5] as soon as V is continuous and goes to infinity fast enough, yielding

$$\begin{aligned} F^1(V) &:= \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \int \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 e^{-N \sum V(\lambda_i)} \prod_{1 \leq i \leq N} d\lambda_i \\ (1) \quad &= - \inf \left\{ \int \int \left(\frac{1}{2} V(x) + \frac{1}{2} V(y) - \ln |x - y| \right) d\mu(x) d\mu(y) \right\} \end{aligned}$$

where the supremum is taken over probability measures μ on the real line. Large N corrections to this limit can also be computed, at least in some nice regimes [33, 11, 9, 13]. When more than one matrix is involved, even the very existence of the limit of the renormalized free energy $F_N^\ell(V) = N^{-2} \ln I_N^\ell(V)$ is unknown. The existence of such a limit in great generality would already be of importance for free entropy theory [47] since it would show that one can replace limsup by liminf in the definition of microstates free entropy. One class of multi-matrix integrals is still amenable to asymptotic analysis: those where matrices interact via a quadratic interaction such that the interaction can be expressed in terms of the Harich-Chandra-Itzykson-Zuber integral or more generally spherical integrals [31, 25, 20, 40, 27, 12]. Apart from these special cases, the case where the potential V is a small modification of a quadratic potential, which is also called a "high temperature" regime, is also well understood. Indeed, it has been foreseen that matrix integrals expand

as generating functions for enumerating maps [43, 17] at a formal level. In [28, 29], it was shown that this expansion holds asymptotically under general hypotheses, so that the free energy limit can be expressed as a generating function for the enumeration of planar maps. The central tool to derive these asymptotic estimates is based on the fact that the matrices remain bounded and that the limit points of their moments satisfy the so-called Dyson-Schwinger's equations. It can then be shown that these equations asymptotically have a unique solution when the potential is a small perturbation of the quadratic potential, a solution that expands analytically in the small parameters of the potential. Such a uniqueness fails in the general case, even in the case of a single matrix [10, 11]. In the latter case, it can be seen that for general potentials the limiting distribution of the spectrum has a disconnected support and that the Dyson-Schwinger's equations have a unique solution only if the mass in each connected component is fixed. The limiting distribution is the unique minimizer in (1), but the Dyson-Schwinger equation describes critical points, which in fact are much more numerous. Therefore, except for these few cases, the asymptotic analysis of most models is still a wide open problem and even efficient numerical methods have to be found [34]. In this article, we start the study of multi-matrix models in the "low" temperature regime where the potential dominates the measure. We will indeed consider the case where we can decompose V into

$$V_\beta = \beta U + W$$

with two polynomials U, W and β very large. The driving idea is that the situation where β is infinite should somehow be easier to analyze, as should the situation where β is large, as a perturbation of the latter.

We will first give sufficient conditions for U and W such that the integral $I_N^\ell(V_\beta)$ is finite and mostly supported on bounded matrices. We will also show that this integral is concentrated on matrices satisfying the so-called Dyson-Schwinger's equations. However, these equations generally have multiple solutions in the low temperature situation, even in the one-matrix case. We then characterize special cases. When U is strictly convex, we show that the analysis can be reduced to the high temperature situation studied in [28]. When U is the sum of polynomials in one matrix, we can also estimate $I_N^\ell(V_\beta)$ for sufficiently large β . The crux of our study will be to consider the case where $-U$ is the square of the commutator in two matrices X and Y and W decomposes as the sum of two polynomials in X and Y respectively. This type of model is also considered in [34, 36]: It can be solved when W is quadratic but not in general. In the limit N and then β goes to infinity, our matrix model will clearly concentrate on operators x and y which commute. However, there are many ways to achieve this: for example these operators could have the same eigenbasis or one of them could be the multiple of the identity. Our results show that the optimal scenario also depends on W and give the asymptotics of $F_N^2(V_\beta)$ for N and then β going to infinity. We also

give a general criterion for the tightness of these multi-matrix models. Our paper considers the Hermitian case but could easily be extended to symmetric matrices (see e.g. [39] and [14] for other generalizations of the loop equations techniques to orthogonal and symmetric matrices).

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2. NOTATIONS AND MAIN RESULTS

We let $\mathbb{C}\langle X_1, \dots, X_\ell \rangle$ be the space of non-commutative polynomials in $\mathbf{X} = (X_1, \dots, X_\ell)$ with complex coefficients. We will in general consider V to be a polynomial but for some applications we will need that the potentials belong to a more general class of smooth functions given by:

$$\mathbb{C}\langle X_1, \dots, X_\ell \rangle^c = \left\{ P(X_1, \dots, X_\ell) = \sum_{i=1}^k z_i X_{j_1^i} \cdots X_{j_{d_i}^i} + \sum_{j=1}^{\ell} \vartheta_j(X_j) \right\}$$

where z_j are complex numbers, k, j_i^d , the j_m^i are integer numbers in $\{1, \dots, \ell\}$ and ϑ_j are smooth real-valued functions. $\mathbb{C}\langle X_1, \dots, X_\ell \rangle_{sa}^c$ will denote the subset of self-adjoint elements of $\mathbb{C}\langle X_1, \dots, X_\ell \rangle^c$, namely functions $V \in \mathbb{C}\langle X_1, \dots, X_\ell \rangle^c$ such that $V = V^*$, where $*$ is the involution such that for any integer number d , any $j_1, \dots, j_d \in [1, \ell]^d$, any complex number z and any smooth function $\vartheta : \mathbb{R} \mapsto \mathbb{R}$, we have

$$(z X_{j_1} \cdots X_{j_d})^* = \bar{z} X_{j_d} X_{j_{d-1}} \cdots X_{j_1}, \quad \vartheta(X_j)^* = \vartheta(X_j).$$

$\mathbb{C}\langle X_1, \dots, X_\ell \rangle^c$ is equipped with a unit denoted by 1. A polynomial P is cyclically invariant iff it is the sum of terms of the form

$$z \sum_{\ell} X_{i_\ell} \cdots X_{i_k} X_{i_1} \cdots X_{i_{\ell-1}}$$

for some complex number z and $(i_1, \dots, i_k) \in [1, \ell]^k$, $k \in \mathbb{N}$. $\mathbb{C}\langle X_1, \dots, X_\ell \rangle^c$ is equipped with the unit I .

Let $V_\beta = \beta V + W$ be a one parameter family of non-commutative polynomials so that V and W belong to $\mathbb{C}\langle X_1, \dots, X_\ell \rangle_{sa}^c$ and β is a non-negative real number. We study the matrix model given by the ℓ -tuple $\mathbf{X}^N = (X_1^N, \dots, X_\ell^N)$ of $N \times N$ Hermitian matrices with law

$$(2) \quad d\mathbb{Q}_{V_\beta}^N(\mathbf{X}^N) = \frac{1}{Z_{V_\beta}^N} \exp\left(-N \operatorname{Tr} V_\beta(\mathbf{X}^N)\right) d\mathbf{X}^N$$

when the dimension goes to infinity and then β , the analogue of the inverse temperature in statistical physics, goes to ∞ . $d\mathbf{X}^N$ is the Lebesgue measure on the set $\mathcal{H}^N(\mathbb{C})$ of $N \times N$ Hermitian matrices and we have denoted in

short $X^N = (X_1^N, \dots, X_\ell^N)$. Moreover, $Z_{V_\beta}^N$ is the partition function given by

$$(3) \quad Z_{V_\beta}^N = \int \exp\left(-N \operatorname{Tr} V_\beta(\mathbf{X}^N)\right) d\mathbf{X}^N$$

We may and shall assume that V is cyclically invariant, since the trace is left invariant by cyclic permutation, namely we may replace any monomial $q(X) = X_{i_1} \cdots X_{i_k}$ in V by

$$c(q)(\mathbf{X}) := \frac{1}{k} \sum_{p=1}^k X_{i_p} \cdots X_{i_k} X_{i_1} \cdots X_{i_{p-1}}.$$

Our first result gives sufficient conditions on the potential V to insure that matrices distributed according to \mathbb{Q}_V^N stay bounded with large probability. This will prove that the sequence of moment distribution of the $\mathbf{X}^N = (X_i^N)_{1 \leq i \leq \ell}$ is tight under \mathbb{Q}_V^N . Let us denote \mathcal{D}_i the cyclic derivative with respect to the i th variable so that \mathcal{D}_i is linear on the space of non commutative polynomials and

$$\mathcal{D}_i(X_{i_1} \cdots X_{i_k}) = \sum_{p:i_p=i} X_{i_{p+1}} \cdots X_{i_k} X_{i_1} \cdots X_{i_{p-1}}.$$

We extend \mathcal{D}_i to smooth functions in one variable by setting $\mathcal{D}_i \vartheta(X_j) = 1_{i=j} \vartheta'(X_j)$. We will denote

$$X_i \cdot \mathcal{D}_i V = \frac{1}{2} (X_i \mathcal{D}_i V + \mathcal{D}_i V X_i).$$

We denote by $\|A\|_\infty$ the operator norm of a matrix A :

$$\|A\|_\infty = \sup_{\|u\|_2=1} \langle Au, Au \rangle^{1/2}.$$

We can finally define the notion of trapping potentials that will insure that matrices stay bounded under \mathbb{Q}_V^N .

Definition 2.1. *Let η be a non-negative real number and A be a finite real number and $I = (I_1, \dots, I_d)$ be a partition of $\{1, \dots, \ell\}$. Let us denote for $1 \leq j \leq d$, $Z_j^N = \sum_{i \in I_j} (X_i^N)^2$.*

We say that a potential $V \in \mathbb{C}\langle X_1, \dots, X_\ell \rangle_{sa}^c$ is (η, A, I) -trapping if for all $k, N \in \mathbb{N}^$, for all $\mathbf{X}^N = (X_1^N, \dots, X_\ell^N) \in \mathcal{H}^N(\mathbb{C})^\ell$,*

$$\operatorname{Tr} \left(\sum_{j=1}^d (Z_j^N)^k \sum_{i \in I_j} X_i^N \cdot \mathcal{D}_i V(X^N) \right) \geq \operatorname{Tr} \left(\eta \sum_{j=1}^d (Z_j^N)^{k+1} - A \left(1 + \sum_{j=1}^d (Z_j^N)^k \right) \right).$$

Our main interest in this definition is the following theorem, which shows that when the potential V is (η, A, I) -trapping, the spectrum of the ℓ tuple of matrices \mathbf{X}^N stays bounded with large probability.

Theorem 2.2. *Assume that there exists a positive real number η and a finite real number A such that V in $\mathbb{C}\langle X_1, \dots, X_\ell \rangle_{sa}^c$ is (η, A, I) -trapping. Then with $L = 32 \max((1 + |A|)d\eta^{-1}, 1)$, under \mathbb{Q}_V^N , almost surely*

$$\limsup_{N \rightarrow \infty} \max_{1 \leq i \leq \ell} \|X_i^N\|_\infty \leq L.$$

Moreover, for every $\kappa \in (0, 1)$, we have

$$(4) \quad \limsup_{N \rightarrow \infty} N^{\kappa-1} \ln \mathbb{Q}_V^N \left(\max_{1 \leq i \leq \ell} \|X_i^N\|_\infty \geq L^{\frac{1}{2} + \frac{1}{\kappa}} \right) \leq -\frac{1}{4}.$$

Remark 2.3. *If $X = CY + aI$ with C invertible and bounded it is enough to prove that the potential V is trapping for the variables Y since we then get the conclusion of Theorem 2.2 for the variables X as well since*

$$\max_{1 \leq i \leq \ell} \|X_i^N\|_\infty \leq \|C\|_\infty \max_{1 \leq i \leq \ell} \|Y_i^N\|_\infty + |a|$$

In this article, we will study the empirical distribution $\hat{\mu}^N$ of the matrices \mathbf{X}^N under \mathbb{Q}_V^N . For a ℓ -tuple $\mathbf{Y} = (Y_1, \dots, Y_\ell)$ of self-adjoint $N \times N$ matrices, we denote by $\hat{\mu}_{\mathbf{Y}}$ the empirical distribution defined as the linear form on $\mathbb{C}\langle X_1, \dots, X_\ell \rangle^c$ such that for all $P \in \mathbb{C}\langle X_1, \dots, X_\ell \rangle^c$,

$$\hat{\mu}_{\mathbf{Y}}(P) := \frac{1}{N} \text{Tr}(P(Y_1, \dots, Y_\ell)).$$

Throughout this article, we denote in short $\hat{\mu}^N = \hat{\mu}_{\mathbf{X}^N}$ with \mathbf{X}^N the random matrices with law \mathbb{Q}_V^N . We see $\hat{\mu}^N$ as a non-commutative law [2, Section 5.2], namely as a linear form τ on $\mathbb{C}\langle X_1, \dots, X_\ell \rangle^c$ such that for any $P, Q \in \mathbb{C}\langle X_1, \dots, X_\ell \rangle^c$,

$$\tau(PP^*) \geq 0, \tau(1) = 1, \tau(PQ) = \tau(QP).$$

We equip the space of non-commutative laws with its weak topology. An important consequence of Theorem 2.2 is that the empirical distribution $\hat{\mu}^N$ is tight [2, Lemma 5.4.6].

Corollary 2.4. *If V is (η, A, I) -trapping, the empirical distribution $\hat{\mu}^N$ is sequentially tight.*

We hereafter investigate examples of potentials which are (η, A, I) -trapping. Note that they may be replaced by cyclically equivalent potentials without loss of generality.

Theorem 2.5. (1) *If U is (η, A, I) -trapping and W is (η', A', I) -trapping and $\beta \geq 0$, $\beta U + W$ is $(\beta\eta + \eta', \beta A + A', I)$ -trapping.*

(2) *If V is $(\eta, A, \{\{1\}, \dots, \{\ell\}\})$ -trapping then for any non negative real numbers α_{ij} , the potential*

$$V(\mathbf{X}) - \sum_{1 \leq i < j \leq \ell} \alpha_{ij} [X_i, X_j]^2$$

is also $(\eta, A, \{\{1\}, \dots, \{\ell\}\})$ -trapping. Here, $[X_i, X_j] = X_i X_j - X_j X_i$ denotes the commutator.

- (3) We say that a potential V in $\mathbb{C}\langle X_1, \dots, X_\ell \rangle$ is **confining** if there is a partition (I_1, \dots, I_d) of $\{1, \dots, \ell\}$ so that we can write V as

$$V(X) = \sum_{i=1}^{\ell} P_i(X_i) + \sum_{u=1}^p d_u q_u(X) + \sum_{v=1}^r R_v \left(\sum_{i \in I_{m_v}} \alpha_{vi} X_i \right)$$

where P_i are polynomials whose term of highest degree has degree $2D_j \geq 2$ with a positive coefficient, R_v are polynomials with even highest degree with positive coefficient and the α_{ij} are real numbers, $1 \leq m_v \leq d$, the q_u are monomials of degree at most $2 \min_{i \in I_m} D_i - 1$ for all m such that there exists i in I_m such that X_i appears in q_u .

If V is confining, it is (η, A, I) -trapping for some η, A with $\eta > 0$.

Remark 2.6. Theorem 2.5 gives a quite general and technical condition in order to be confining and therefore trapping. Let us show it encompasses many natural examples.

- (1) If $V(\mathbf{X}) = \sum_{i=1}^{\ell} P_i(X_i) + \sum_{u=1}^p d_u q_u(\mathbf{X})$ with the same hypothesis on the P_i and the q_u than in Theorem 2.5 then V is confining with the trivial partition $I = \{\{1\}, \dots, \{\ell\}\}$.

For example $V(X, Y) = c_X X^{2D_X} + c_Y Y^{2D_Y} + V_X(X) + V_Y(Y) + W(X, Y)$ is confining if $c_X, c_Y > 0$ and V_X is of degree at most $2D_X - 1$, V_Y is of degree at most $2D_Y - 1$ and W is of degree at most $2 \min(D_X, D_Y) - 1$.

Another example of confining polynomial is for instance

$$P(X_1, X_2, X_3) = X_1^4 + X_2^4 + X_1 X_2^2 + X_3^2.$$

- (2) Any polynomial which is a sum of terms of type $\sum P_i(\sum_j v_{ij} X_j)$ with P a polynomial going to $+\infty$ at $\pm\infty$ and v of rank ℓ can be rewritten as a confining polynomial with partition $\{1, \dots, \ell\}$ after a linear change of variables. This implies that polynomials of the form

$$V(\mathbf{X}) = \sum_i P_i \left(\sum_j v_{ij} X_j \right) + \sum_{ij} u_{ij} X_i X_j + \sum_{u=1}^p d_u q_u(\mathbf{X})$$

can be rewritten as a confining polynomial with partition $\{1, \dots, \ell\}$ after a linear change of variable if u is a symmetric positive matrix, the P_i 's are going to $+\infty$ at $\pm\infty$ and the q_u are of degree at most $\min \deg P_i - 1$.

- (3) We would have liked trace convex polynomials to be confining, where V is trace convex iff $\mathbf{X} \rightarrow \text{Tr } V(\mathbf{X})$ is convex on the space of ℓ -tuples of $N \times N$ Hermitian matrices for any dimension N . Unfortunately, this is not the case as $X^2 + Y^2 + X^2 Y^2$ is trace convex but not confining (and not trapping). Note however that such polynomials are known to satisfy the conclusions of Theorem 2.2 thanks to Brascamp-Lieb inequalities, see [24, Section 6.5]

Our next result shows that, if V is trapping, any limit point of $\hat{\mu}^N$ satisfy the so-called Dyson-Schwinger's equations. To describe the later we first define the non-commutative derivative ∂_i with respect to the i th variable so that for all choices of monomials $q(\mathbf{X}) = X_{i_1} \dots X_{i_k}$ with $i_j \in \{1, \dots, \ell\}$ we have

$$\partial_i X_{i_1} \dots X_{i_k} = \sum_{j:i_j=i} X_{i_1} \dots X_{i_{j-1}} \otimes X_{i_{j+1}} \dots X_{i_k}.$$

This definition extends to smooth functions ϑ of one variable by setting

$$\vartheta(x, y) = \frac{\vartheta(x) - \vartheta(y)}{x - y}.$$

Theorem 2.7. (*Dyson-Schwinger's equations*) *Under the assumptions of Theorem 2.2, any limit point τ_V of $\hat{\mu}^N$ under \mathbb{Q}_V^N satisfies the Dyson-Schwinger's equations: for any $P \in \mathbb{C}\langle X_1, \dots, X_\ell \rangle^c$, any $i \in \{1, \dots, \ell\}$,*

$$(5) \quad \tau_V(\mathcal{D}_i V P) = \tau_V \otimes \tau_V(\partial_i P).$$

Moreover, if $V = \beta U + W$ with U and W (η, A, I) trapping for some $\eta > 0$ and A finite, then there exists a finite B (depending only on A and η) such that for $\beta > 0$ large enough, any $i \in \{1, \dots, \ell\}$, and any integer number k ,

$$(6) \quad \tau_{V_\beta}(|\mathcal{D}_i U|^{2k}) \leq \left(\frac{B}{\beta}\right)^k.$$

The last result (6) shows, as could be expected, that under reasonable assumptions the limit points τ_{V_β} concentrate on the critical points of the potential U when β goes to infinity. Its proof is a direct consequence of Proposition 4.2.

The Dyson-Schwinger's equations are not sufficient to characterize the limit in general since they may not have a unique solution. Efficient numerical methods have recently been proposed to investigate the solutions [34].

We therefore study special cases where the potential is given by $V_\beta = \beta U + W$ for some fixed polynomials U and V and β large. In Section 5, we study the case where U is strictly convex. We then show that the matrices following $\mathbb{Q}_{V_\beta}^N$ converge to multiples of the identity when N and then β go to infinity and moreover for β large but finite the empirical distribution $\hat{\mu}^N$ converges almost surely and its moments can be expanded in powers of $\beta^{-1/2}$. In Section 6, we consider the case where $U(\mathbf{X}) = \sum U^i(X_i)$ is the sum of polynomials in one variable that can achieve their minimal value at several points and W is small when the spectrum of the X_i are close to these minimizers. We then show, see Theorem 6.6, that for β large enough, the spectrum of the matrices X_i^N under \mathbb{Q}_V^N converges as N goes to infinity. To simplify the statement of our result in this introduction, let us assume that U^i takes its minimum value at $\{x_j^i\}_{1 \leq j \leq m_i}$ and that these minimizers are non degenerate in the sense that $\min_j (U^i)''(x_j^i) > 0$.

Theorem 2.8. *Under some technical hypotheses over U and W (see Assumption 6.4), for β large enough, the empirical distribution $\hat{\mu}_N$ of \mathbf{X}^N converges $\mathbb{Q}_{V_\beta}^N$ -almost surely towards a non-commutative law τ_β . Moreover, for any polynomial P , $\tau_\beta(P)$ expands as a converging series in $\beta^{-\frac{1}{2}}$. τ_∞ is the law of ℓ free variables with law $\frac{1}{m_i} \sum_{j=1}^{m_i} \delta_{x_j^i}$ for $i \in \{1, \dots, \ell\}$.*

Furthermore, we show in Theorem 6.2 that if the minimizers of U^i are degenerate, the mass of the Dirac at a minimizer x depends on the flatness of U^i close to x , namely on the exponent k such that $U^i(y) - U^i(x) \simeq c(x-y)^{2k}$ for y close to x , but not on the constant c provided it is positive.

Finally we consider in Section 7 the case where U is the square of the commutator and more precisely

$$(7) \quad V_\beta(\mathbf{X}) = -\beta(X_1X_2 - X_2X_1)^2 + V_1(X_1) + V_2(X_2).$$

Such a matrix-model can be considered as a toy model for Yang Mills theory [34] where this model is said to be "unsolvable". Our main result is that the non-commutative distribution of X_1^N, X_2^N under $\mathbb{Q}_{V_\beta}^N$ converges when N goes to infinity and then β goes to infinity towards two commutative variables whose spectrum lies at the minimizers of V_1 and V_2 . The central point of our result is that the mass in these minimizers is nontrivial and again depends on the flatness of the potentials in a neighborhood of the minimizers. In fact, we might have expected one of the matrices to converge to a multiple of the identity to make the commutator small, as would be the case for instance if we had a three matrix model with $U(X_1, X_2, X_3) = -(X_1X_2 - X_2X_1)^2 - (X_1X_3 - X_3X_1)^2$ where making X_1 close to the identity would have a higher probability than making it commute with both X_1 and X_2 (see a more detailed discussion in section 7). More precisely, we have the following statement.

Theorem 2.9. *We let V_1 and V_2 be two non-negative polynomials of one variable, going to $+\infty$ at infinity. Let $z_1^i < z_2^i < \dots < z_k^i$ be the zeroes of V_i and assume there exists $c_1^i, \dots, c_k^i \in \mathbb{R}^{+*}$ such that, for every $j \in \{1, \dots, k\}$,*

$$V_i(x) = c_j^i(x - z_j^i)^2 + o(x - z_j^i)^2.$$

Let V_β be given by (7) and X_1, X_2 following $\mathbb{Q}_{V_\beta}^N$. Then, for every $\epsilon > 0$ small enough, almost surely, for every $j \in \{1, \dots, k\}$, we have

$$\lim_{\beta \rightarrow \infty} \limsup_{N \rightarrow \infty} \hat{\mu}_{X_i}^N([z_j^i - \epsilon, z_j^i + \epsilon]) = \frac{(c_j^i)^{-\frac{1}{2}}}{\sum_{\ell} (c_\ell^i)^{-\frac{1}{2}}}.$$

The same result holds if the above limsup is replaced by a liminf.

3. TIGHTNESS OF MATRIX MODELS AT FINITE TEMPERATURE

In this section, we prove Theorems 2.5 and 2.2. This is based on new techniques to bound probabilities of rare events developed in the appendix, see Section 8.1.

3.1. Proof of Theorem 2.5. The first point is clear. For the second point, we just observe that $-[X_1, X_2]^2 = -(X_1X_2 - X_2X_1)^2$ satisfies the $(0, 0, \{\{1\}, \dots, \{\ell\}\})$ -trapping condition. To see this, remark that it amounts to check that for any integer number k :

$$\mathrm{Tr}(X_1^{2k+1}X_2X_1X_2 + X_2^{2k+1}X_1X_2X_1) \leq \mathrm{Tr}(X_1^{2k+2}X_2^2 + X_2^{2k+2}X_1^2).$$

This is a direct consequence of Lemma 8.5. Now if V is $(\eta, A, \{\{1\}, \dots, \{\ell\}\})$ -trapping then by the first point for any $\alpha > 0$,

$$V(\mathbf{X}) - \alpha[X_1, X_2]^2$$

is also $(\eta, A, \{\{1\}, \dots, \{\ell\}\})$ -trapping. The generalization to any number of square commutators is immediate.

We finally prove the third point. Let us consider a confining potential V of the form :

$$V(X) = \sum_{i=1}^{\ell} P_i(X_i) + \sum_{u=1}^p d_u q_u(X) + \sum_{v=1}^r R_v \left(\sum_{i \in I_{m_v}} \alpha_{vi} X_i \right)$$

Observe that

$$\sum_{i \in I_m} X_i \cdot \mathcal{D}_i R_v \left(\sum_{i \in I_{m_v}} \alpha_{vi} X_i \right) = 1_{m=m_v} \tilde{R}_v \left(\sum_{i \in I_{m_v}} \alpha_{vi} X_i \right)$$

where $\tilde{R}_v(X) = X R'_v(X)$ is also a polynomial with a positive even term of highest degree, therefore it is bounded from below by some constant $-c_v$. Hence, for $j \in \{1, \dots, m\}$, we have (since $(Z_m^N)^k$ is non-negative)

$$\sum_{m=1}^d \mathrm{Tr} \left((Z_m^N)^k \sum_{i \in I_m} X_i^N \cdot \mathcal{D}_i R_v \left(\sum_{k \in I_{m_v}} \alpha_{jk} X_k^N \right) \right) \geq -c_v \mathrm{Tr}(Z_{m_v}^N)^k,$$

i.e. $R_v \left(\sum_{i \in I_{m_v}} \alpha_{vi} X_i \right)$ satisfies the $(0, c_v, I)$ -trapping condition. With this in mind, we need only worry about the other part of the polynomial V and then apply the first point of the theorem. We can assume up to rescaling that the polynomials P_i are monic. We consider a monomial q_k in the decomposition of V and denote it by q for simplicity. We denote by $S(q)$ the smallest subset of $i \in [1, \ell]$ so that q only depends on $X_i, i \in S(q)$. Let $s(q)$ be the set of j such that $I_j \cap S(q) \neq \emptyset$ and define $D_q = \min_{i \in s(q)} \min_{i \in I_j} D_i$.

(8)

$$\mathrm{Tr} \left(\sum_{j=1}^d (Z_j^N)^k \sum_{i \in I_j} X_i^N \cdot \mathcal{D}_i q(X^N) \right) = \sum_{j \in s(q)} \mathrm{Tr} \left((Z_j^N)^k \sum_{i \in I_j} X_i^N \cdot \mathcal{D}_i q(X^N) \right).$$

Any monomial $X_{i_1} \cdots X_{i_t}$ coming in the linear decomposition of $\mathcal{D}_i q$ has degree t bounded above by $2D_q - 1$ by assumption. Observe that, by Hölder's non commutative inequality with parameters $(p_0, p_1, \dots, p_t) = (1 + \frac{t}{2k}, 2k + t, \dots, 2k + t)$, we find

$$\begin{aligned} |\mathrm{Tr}(((Z_j^N)^{1/2})^{2k} X_{i_1}^N \cdots X_{i_t}^N)| &\leq \left(\mathrm{Tr}(Z_j^N)^{k+\frac{t}{2}}\right)^{\frac{2k}{2k+t}} \prod_{m=1}^t \left(\mathrm{Tr}|X_{i_m}|^{2k+t}\right)^{\frac{1}{2k+t}} \\ &\leq \sum_{j \in s(q)} \mathrm{Tr}(Z_j^N)^{k+\frac{t}{2}} \\ &\leq \varepsilon \mathrm{Tr} \sum_{j \in s(q)} (Z_j^N)^{k+D_q} + \varepsilon^{-1} \mathrm{Tr} \sum_{j \in s(q)} (Z_j^N)^k \end{aligned}$$

where in the second line we used that for $s \in [1, t]$,

$$\mathrm{Tr}(|X_{i_s}|^k) \leq \max_{j \in s(q)} \mathrm{Tr}((Z_j^N)^{k/2})$$

for all integer number k . In the last line we picked $\varepsilon \in (0, 1)$ to be chosen later and used the inequality valid for $t \in [1, 2D_q - 1]$

$$x^{k+t/2} \leq (\varepsilon^2)^{D_q-t/2} x^{k+D_q} \mathbf{1}_{\varepsilon^2 x > 1} + \varepsilon^{-t} x^k \mathbf{1}_{\varepsilon^2 x \leq 1}$$

and took the worst case scenario for each term. Using this inequality for each term coming from the q , we deduce that there exists a finite constant $C(q)$ such that

$$\begin{aligned} (9) \quad &\mathrm{Tr}\left(\sum_{j=1}^d (Z_j^N)^k \sum_{i \in I_j} X_i^N \cdot \mathcal{D}_i q(X^N)\right) \\ &\geq -C(q) \left(\varepsilon \sum_{j \in s(q)} \mathrm{Tr}(Z_j^N)^{k+D_q} + \varepsilon^{-1} \mathrm{Tr} \sum_{j \in s(q)} (Z_j^N)^k \right). \end{aligned}$$

Moreover, putting $\bar{D}_j = \min_{i \in I_j} D_i$, and since $x^{2D_i} \geq x^{2\bar{D}_j} - 1$ for all real number x and $i \in I_j$ as $D_i \geq \bar{D}_j$ for $i \in s(q)$, we have

$$\begin{aligned} (10) \quad &\sum_{j=1}^d \mathrm{Tr}((Z_j^N)^k \sum_{k \in I_j} X_k \cdot \mathcal{D}_k \sum_{i=1}^{\ell} X_i^{2D_i}) \geq 2 \sum_{j=1}^d \bar{D}_j \mathrm{Tr}((Z_j^N)^k \sum_{i \in I_j} X_i^{2\bar{D}_j}) \\ &\quad - 2 \sum_j \bar{D}_j |I_j| \mathrm{Tr}(Z_j^N)^k \end{aligned}$$

But, using Lemma 8.6, we find that for all $j \in \{1, \dots, d\}$,

$$(11) \quad \ell^{\bar{D}_j} \mathrm{Tr}((Z_j^N)^k \sum_{i \in I_j} X_i^{2\bar{D}_j}) \geq \mathrm{Tr}((Z_j^N)^{k+\bar{D}_j})$$

Putting (9) and (11) together we find finite constants C', C'' such that

$$\begin{aligned} \mathrm{Tr}((Z_j^N)^k \sum_{i \in I_j} X_i \cdot \mathcal{D}_i V) &\geq \sum_{j=1}^d \frac{2\bar{D}_j}{\ell^{\bar{D}_j}} (\mathrm{Tr}((Z_j^N)^{k+\bar{D}_j}) - C' \varepsilon^{-1} \mathrm{Tr}(Z_j^N)^k) \\ &\quad - C\varepsilon \sum_{r=1}^n \sum_{j \in s(q_r)} \mathrm{Tr}(Z_j^N)^{k+D_{q_r}} \\ &\geq \frac{1}{\ell^{\max \bar{D}_j}} \sum_{j=1}^d \mathrm{Tr}((Z_j^N)^{k+1}) - C'' \sum_{j=1}^d \mathrm{Tr}((Z_j^N)^k) \end{aligned}$$

where we finally choose ε small enough, used that $D_{q_r} < \bar{D}_i$ for all $i \in s(q_r)$ and that $\min D_i \geq 1$ to find a finite constant C'' so that the above holds. This completes the proof.

3.2. Proof of Theorem 2.2. Our proof uses bounds based on change of variables formula that we prove in the appendix, see Lemma 8.4. We fix $L > 0$ and let for a ℓ -tuple of $N \times N$ Hermitian matrices $\mathbf{X}^N = (X_1^N, \dots, X_\ell^N)$,

$$T(\mathbf{X}^N) = \inf \left\{ k : \frac{1}{dN} \mathrm{Tr} \sum_{j=1}^d (Z_j(\mathbf{X}^N))^k \geq L^k \right\},$$

where we recall that $Z_j(\mathbf{X}^N) = \sum_{i \in I_j} (X_i^N)^2$. Note that by Jensen's inequality, $k \rightarrow \frac{1}{dN} \mathrm{Tr} \sum_{j=1}^d L^{-k} (Z_j^N)^k$ is increasing. To prove the second inequality in (4), it is enough to show that for $\kappa \in (0, 1)$ and N large enough

$$(12) \quad \mathbb{Q}_V^N \left(\mathbf{X}^N : T(\mathbf{X}^N) \leq \frac{\kappa \ln N}{2 \ln(2L)} \right) \leq \frac{\kappa \ln N}{2 \ln L} e^{-\frac{1}{2} N^{1-\kappa}}.$$

Indeed, observe that for any $M > 0$, any integer number k , we have

$$\left\{ \mathbf{X}^N : \max_i \|X_i^N\|_\infty \geq M \right\} \subset \left\{ \mathbf{X}^N : \frac{1}{N} \mathrm{Tr} \left(\sum_{j=1}^d (Z_j(\mathbf{X}^N))^k \right) \geq \frac{1}{N} M^{2k} \right\}.$$

But on $\{T(\mathbf{X}^N) > \frac{\kappa \ln N}{2 \ln(2L)}\}$ we know that for all $k \leq \frac{\kappa \ln N}{2 \ln(2L)}$

$$\frac{1}{dN} \mathrm{Tr} \left(\sum_{j=1}^d (Z_j(\mathbf{X}^N))^k \right) \leq L^k.$$

Therefore, on $\{\max_i \|X_i^N\|_\infty \geq M\} \cap \{T(\mathbf{X}^N) > \frac{\kappa \ln N}{2 \ln(2L)}\}$, we must have $dL^k \geq \frac{1}{N} M^{2k}$ for all $k \leq \frac{\kappa \ln N}{2 \ln(2L)}$. This is impossible unless

$$\ln \frac{M^2}{L} \leq \inf_{k \leq \frac{\kappa \ln N}{2 \ln(2L)}} \frac{\ln(dN)}{k} = \frac{2}{\kappa} \left(1 + \frac{\ln d}{\ln N} \right) \ln 2L,$$

for instance when $M = (2L)^{\frac{1}{2} + \frac{1}{\kappa}}$ and $N \geq d^2$. Then, we deduce that $\{T(\mathbf{X}^N) > \frac{\kappa \ln N}{2 \ln(2L)}\} \cap \{\max_i \|X_i^N\|_\infty \geq (2L)^{\frac{1}{2} + \frac{1}{\kappa}}\}$ is empty. Therefore we

conclude

$$\mathbb{Q}_V^N \left(\mathbf{X}^N : \max_i \|X_i^N\|_\infty \geq (2L)^{\frac{1}{2} + \frac{1}{\kappa}} \right) \leq \mathbb{Q}_V^N \left(\mathbf{X}^N : T(\mathbf{X}^N) \leq \frac{\kappa \ln N}{2 \ln 2L} \right),$$

from which the theorem follows from (12). To prove the latter, we first notice that

$$(13) \quad \mathbb{Q}_V^N \left(\mathbf{X}^N : T(\mathbf{X}^N) \leq \frac{\kappa \ln N}{2 \ln 2L} \right) \leq \sum_{k=0}^{\frac{\kappa \ln N}{2 \ln 2L}} q_k^N$$

where $q_k^N = \mathbb{Q}_V^N \left(\mathbf{X}^N \in S_{k,a} \right)$ if $S_{k,a}$ is the measurable set of $N \times N$ Hermitian matrices $\mathbf{X}^N = (X_1^N, \dots, X_\ell^N)$ given by

$$S_{k,a} = \left\{ \frac{1}{dN} \operatorname{Tr} \sum_{j=1}^d (Z_j(\mathbf{X}^N))^k > a_k \right\} \cap \bigcap_{1 \leq u \leq k-1} \left\{ \frac{1}{dN} \operatorname{Tr} \sum_{j=1}^d (Z_j(\mathbf{X}^N))^u \leq a_u \right\}$$

with a sequence $(a_u)_{u \in \mathbb{N}}$ to be chosen later. We prove by induction over $k \leq \frac{\kappa \ln N}{2 \ln 2L}$ that we can find non-negative real numbers $a_k, k \leq \frac{\kappa \ln N}{2 \ln 2L}$ such that for N large enough

$$(14) \quad q_k^N \leq e^{-\frac{1}{2}N^{1-\kappa}}$$

and moreover that for all $k, a_k \leq L^k$, with L as in the theorem. The same reasoning would give an upper bound of the expected order e^{-cN} but only up to k of order $\sqrt{\ln N}$ which is not sufficient to bound the spectral radius. Note that when $k = 0$ the bound is satisfied for any $a_0 > 1$. Assume that up to rank k , we have found a_0, \dots, a_k such that $a_u \leq L^u$ for all $u \leq k$ and a_u verifies (14). We next construct a_{k+1} so that the same property holds. We look at the dynamic of Lemma 8.4 up to time $t_N = N^{-1-\kappa}$ on each set of variables indexed by I_j , namely for $i \in I_j$ we set, for $\mathbf{X}^N = (X_1^N, \dots, X_\ell^N)$ a ℓ -tuple of Hermitain $N \times N$ matrices :

$$\begin{cases} X_i^N(0) &= X_i^N \\ \partial_t X_i^N(t) &= -((Z_j^N(t))^k X_i^N(t) + X_i^N(t)(Z_j^N(t))^k) \\ Z_j^N(t) &= \sum_{i \in I_j} (X_i^N(t))^2 \end{cases}$$

As proved in Lemma 8.4, we have

$$\begin{aligned} q_{k+1}^N &= \mathbb{Q}_V^N(\mathbf{X}^N \in S_{k,a}) \\ &\leq \sup_{\mathbf{X}^N \in S_{k,a}} \left\{ \exp\left(-2 \int_0^{t_N} \left(N \operatorname{Tr} \sum_j (Z_j^N(s))^k \sum_{i \in I_j} X_i^N(s) \cdot \mathcal{D}_i V(X^N(s)) \right. \right. \right. \\ &\quad \left. \left. \left. - \operatorname{Tr} \otimes \operatorname{Tr} \sum_{j,i \in I_j} \partial_i (X_i^N(s)(Z_j^N(s))^k) \right) ds \right) \right\} \end{aligned}$$

We next bound the term in the exponential by using the hypothesis that V is (η, A, I) -trapping:

$$\begin{aligned}
(15) \quad & N \operatorname{Tr} \sum_{j=1}^d (Z_j^N(s))^k \sum_{i \in I_j} X_i^N(s) \mathcal{D}_i V(X^N(s)) - \operatorname{Tr} \otimes \operatorname{Tr} \sum_{j, i \in I_j} \partial_i (X_i^N(s) (Z_j^N(s))^k) \\
& \geq N \sum_{j=1}^d \operatorname{Tr} (\eta (Z_j^N(s))^{k+1} - A(1 + Z_j^N(s)^k)) \\
& \quad - \sum_{j=1}^d \sum_{p=0}^k \operatorname{Tr} (Z_j^N(s))^p \operatorname{Tr} (Z_j^N(s))^{k-p} \\
& \quad - \sum_{j=1}^d \sum_{i \in I_j} \sum_{p=0}^{k-1} \operatorname{Tr} X_i^N(s) (Z_j^N(s))^p \operatorname{Tr} X_i^N(s) (Z_j^N(s))^{k-1-p}
\end{aligned}$$

Now observe that $\operatorname{Tr} \otimes \operatorname{Tr} ((X \otimes 1 - 1 \otimes X)^2 Z^a \otimes Z^b)$ is non-negative since $(X \otimes 1 - 1 \otimes X)^2$ and $Z^a \otimes Z^b$ are non-negative operators in $\mathcal{H}^N(\mathbb{C})^{\otimes 2}$ for all integer numbers a, b . As a consequence, we find that

$$\begin{aligned}
& \operatorname{Tr} (X_i^N(s))^2 (Z_j^N(s))^p \operatorname{Tr} (Z_j^N(s))^{k-1-p} + \operatorname{Tr} (Z_j^N(s))^p \operatorname{Tr} (X_i^N(s))^2 (Z_j^N(s))^{k-1-p} \\
& \geq 2 \operatorname{Tr} X_i^N(s) (Z_j^N(s))^p \operatorname{Tr} X_i^N(s) (Z_j^N(s))^{k-1-p}.
\end{aligned}$$

Thus, we deduce the bound

$$\begin{aligned}
& \sum_{i \in I_j} \sum_{p=0}^{k-1} \operatorname{Tr} X_i^N(s) (Z_j^N(s))^p \operatorname{Tr} X_i^N(s) (Z_j^N(s))^{k-1-p} \\
& \leq \sum_{p=0}^k \operatorname{Tr} (Z_j^N(s))^p \operatorname{Tr} (Z_j^N(s))^{k-p}.
\end{aligned}$$

Now Lemma 8.4 tells us that $s \rightarrow \operatorname{Tr} (Z_j^N(s))^n$ is decreasing on $[0, \frac{1}{N^{\frac{k}{k+1}}}]$, so that we deduce if $t \leq \frac{1}{N^{\frac{k}{k+1}}}$, that for $s \leq t$

$$\begin{aligned}
(16) \quad & N \sum_{\substack{j \in [1, d] \\ i \in I_j}} \left(\operatorname{Tr} (Z_j^N(s))^k X_i^N(s) \mathcal{D}_i V(X^N(s)) - \operatorname{Tr} \otimes \operatorname{Tr} \partial_i (X_i^N(s) (Z_j^N(s))^k) \right) \\
& \geq N \sum_{j=1}^d \operatorname{Tr} (\eta (Z_j^N(s))^{k+1} - A(1 + (Z_j^N(0))^k)) \\
& \quad - 2 \sum_{p=0}^k \sum_{j=1}^d \operatorname{Tr} (Z_j^N(0))^p \operatorname{Tr} (Z_j^N(0))^{k-p}
\end{aligned}$$

Finally, on the set $\{\frac{1}{N} \operatorname{Tr} \sum_{j=1}^d (Z_j^N(0))^{k+1} > a_{k+1}\}$ there is at least one j such that $\frac{1}{N} \operatorname{Tr} (Z_j^N(0))^{k+1} > a_{k+1}/d$, so that by Lemma 8.4 for any $\kappa > 0$,

all $s \in [0, \frac{1}{N^{1+\kappa}}]$, we have

$$\frac{1}{N} \operatorname{Tr} \sum_{j=1}^d (Z_j^N(s))^{k+1} \geq \frac{a_{k+1}}{d} - \frac{4(k+1)}{N^{\kappa+\frac{1}{k+1}}} \left(\frac{a_{k+1}}{d}\right)^{\frac{2k+1}{k+1}}$$

Therefore, on the set $S_{k,a}$, (15) gives, with $t = t_N = N^{-1-\kappa}$, that q_{k+1}^N is bounded as follows:

$$(17) \quad q_{k+1}^N \leq e^{-N^{1-\kappa} F_{k+1}(a)}$$

with

$$F_{k+1}(a) := 2\left(\eta \left(\frac{a_{k+1}}{d} - 4(k+1)N^{-\kappa-\frac{1}{k+1}} \left(\frac{a_{k+1}}{d}\right)^{\frac{2k+1}{k+1}}\right) - A(1+a_k) - 2 \sum_{p=0}^k a_p a_{k-p}\right).$$

We next define $a_0 = 1$ and by induction set

$$a_{k+1} = \frac{d}{\eta} (1 + |A|(1 + a_k) + 2 \sum_{p=0}^k a_p a_{k-p}).$$

We next show by induction that for all integer numbers k ,

$$(18) \quad a_k \leq C_k \left(\frac{4(1+|A|)d}{\eta}\right)^k$$

with C_k the Catalan numbers. We assume without loss of generality that $(1+|A|)d \geq \eta$ (since an (η, A) -trapping potential is also (η', A) trapping for any $\eta' = (1+|A|)d < \eta$). (18) is true for $k = 0$. We assume by induction it is true up to k , then

$$\begin{aligned} a_{k+1} &\leq \frac{d}{\eta} \left(1 + |A|(1 + 4^k C_k \left(\frac{(1+|A|)d}{\eta}\right)^k) + \frac{1}{2} 4^{k+1} C_{k+1} \left(\frac{(1+|A|)d}{\eta}\right)^k\right) \\ &\leq C_{k+1} \left(\frac{4(1+|A|)d}{\eta}\right)^{k+1} \frac{1}{(1+|A|)} \left(\frac{1+|A|}{C_{k+1} \left(\frac{4(1+|A|)d}{\eta}\right)^k} + \frac{|A|C_k}{4C_{k+1}} + \frac{1}{2}\right) \\ &\leq C_{k+1} \left(\frac{4(1+|A|)d}{\eta}\right)^{k+1} \end{aligned}$$

provided

$$(19) \quad \frac{1}{(1+|A|)} \left(\frac{1+|A|}{4^{k+1} C_{k+1} \left(\frac{(1+|A|)d}{\eta}\right)^k} + \frac{|A|C_k}{4C_{k+1}} + \frac{1}{2}\right) \leq 1.$$

Since we assumed $(1+|A|)d \geq \eta$ the above inequality is clearly true for all $k \geq 0$. Hence, the induction is satisfied. As a consequence, since $C_k \leq 4^k$, we deduce that $a_k \leq 4^{2k} \left(\frac{(1+|A|)d}{\eta}\right)^k = L^k$. Moreover, for such a choice of $a_k, k \geq 0$, we can lower bound the exponent in (17):

$$F_{k+1}(a) \geq 1 - 2\eta \frac{4(k+1)L^{2k+1}}{N^{\kappa+\frac{1}{k+1}}} d^{\frac{2k+1}{k}}.$$

The right hand side is bounded from below by $1/2$ for N large enough provided k is small enough :

$$2k \leq \kappa \frac{\ln N}{\ln(2L)} .$$

From here, (12) and (4) follow readily.

4. ASYMPTOTIC LOCALIZATION AT LOW TEMPERATURE IN THE GENERAL CASE VIA SCHWINGER DYSON'S EQUATION

Let us consider a potential V which is (η, A, I) -trapping for some $\eta > 0$, A finite and a partition I of $\{1, \dots, \ell\}$. By Corollary 2.4, we know that the empirical distribution $\hat{\mu}^N$ is tight almost surely under \mathbb{Q}_V^N . Let τ_V be an almost sure limit point.

Theorem 4.1. (*Dyson-Schwinger's equations*) *Under the assumptions of Theorem 2.2, any limit point τ_V of $\hat{\mu}^N$ under \mathbb{Q}_V^N satisfies the Dyson-Schwinger's equations: for any polynomial P in $\mathbb{C}\langle X_1, \dots, X_\ell \rangle$, any $i \in \{1, \dots, \ell\}$,*

$$(20) \quad \tau_V(\mathcal{D}_i V P) = \tau_V \otimes \tau_V(\partial_i P) .$$

When V is a small perturbation of a quadratic polynomial, it was shown in [28] that the Dyson-Schwinger's equations have a unique solution. This is not true in general, even in the one-matrix case. Indeed, if V has two wells, the limit of the empirical measure of the eigenvalues of the matrix may have disconnected support and there is a solution to the Dyson-Schwinger equation for any choice of masses in the different connected pieces of the support [10].

Proof. We use the change of variables of Lemma 8.3 with $g_j(X) = 1_{j=i}P(X^\varepsilon)$ for a non-commutative polynomial P , and $X^\varepsilon = (f_\varepsilon(X_1), \dots, f_\varepsilon(X_\ell))$ with $f_\varepsilon(x) = (\varepsilon + x^2)^{-1/2}x$ for some $\varepsilon > 0$. g is uniformly bounded (with a uniform bound depending on ε) so that for $t \leq \frac{1}{N}$, for all $j \in \{1, \dots, \ell\}$

$$(21) \quad X_j^N(t) = X_j^N + O_\varepsilon\left(\frac{1}{N}\right).$$

Let us set for some $\delta > 0$,

$$A_\delta = \{\mathbf{X}^N \in (\mathcal{H}^N(\mathbb{C}))^\ell : \frac{1}{N} \text{Tr}(\mathcal{D}_i V) - \frac{1}{N} \text{Tr} \otimes \frac{1}{N} \text{Tr}(\partial_i) \geq \delta\} .$$

It is easy to see that since g_i is uniformly Lipschitz, by (21), we have uniformly on A_δ ,

$$\int_0^{\frac{1}{N}} (N \text{Tr} \mathcal{D}_i V(X_s^N) g(X_s^N) + \text{Tr} \otimes \text{Tr} \partial_i g(X_s^N)) ds \geq N\delta + O_\varepsilon(1) .$$

Hence, Lemma 8.3 implies that

$$\mathbb{Q}_V^N(A_\delta) \leq e^{-N\delta + O_\varepsilon(1)} .$$

The same holds when we replace g by $-g$ and therefore almost surely by Borel Cantelli's lemma

$$\limsup_{N \rightarrow \infty} \left| \frac{1}{N} \operatorname{Tr}(\mathcal{D}_i V.g) - \frac{1}{N} \operatorname{Tr} \otimes \frac{1}{N} \operatorname{Tr}(\partial_i g) \right| = 0.$$

On the other hand, put $B_D = \{\mathbf{X}^N \in (\mathcal{H}^N(\mathbb{C}))^\ell : \frac{1}{N} \operatorname{Tr}(X_i^{2D}) \leq L^{2D}\}$ for L chosen as in Theorem 2.2, with $2D$ larger or equal than the sum of the degree of V and the maximum degree of the P_i 's. Theorem 2.2 implies that $\mathbb{Q}_V^N(B_D^c) \leq e^{-N/2}$ for N large enough. On the other hand, on B_D , it is not hard to see that

$$\left| \frac{1}{N} \operatorname{Tr}(\mathcal{D}V.(g - P)) \right| \leq O_L(\varepsilon), \text{ and } \left| \frac{1}{N} \operatorname{Tr} \otimes \frac{1}{N} \operatorname{Tr}(\partial(g - P)) \right| \leq O_L(\varepsilon),$$

with $O_L(\varepsilon)$ going to zero uniformly on N as ε goes to zero. As a consequence, almost surely, we have

$$\limsup_{N \rightarrow \infty} \left| \frac{1}{N} \operatorname{Tr}(\mathcal{D}_i V.P) - \frac{1}{N} \operatorname{Tr} \otimes \frac{1}{N} \operatorname{Tr}(\partial_i P) \right| = 0.$$

This implies that any almost sure limit point of $\hat{\mu}^N$ satisfies the limiting Dyson-Schwinger's equations (20). \square

We next consider general potentials in the low temperature situation where $\beta = 1/T$ is large and

$$V_\beta = \beta U + W$$

with U, W self-adjoint polynomials independent of β .

We will first study the limit points of $\hat{\mu}^N$ under $\mathbb{Q}_{V_\beta}^N$ and show that they concentrate near the critical points of U , namely the points where $\mathcal{D}_i U$ is small, when β is large enough. Theorem 2.2 allows us to localize the state we are looking at on a fixed support. Our result takes the following form:

Proposition 4.2 (Control of the support, Low temperature). *Assume that we are given $L < \infty, \beta_0 > 0$ and for all $\beta > \beta_0$ a non-commutative distribution ν_β solution of the Dyson-Schwinger equation (20) with potential $V_\beta = \beta U + W$ and with support in $[-L; L]$ in the sense that for all $i \in [1, \ell]$, all $k \in \mathbb{N}$*

$$\nu_\beta(X_i^{2k}) \leq L^{2k}.$$

Then there exists $B > 0$ such that for $\beta > \beta_0$, for all integer number k

$$\max_{1 \leq i \leq \ell} \nu_\beta \left(|\mathcal{D}_i U|^{2k} \right) \leq \left(\frac{B}{\beta} \right)^k.$$

Moreover one can take $B = 4 \max_i \{ \|\mathcal{D}_i U \mathcal{D}_i W\|_L + 2 \|\partial_i \mathcal{D}_i U\|_L \}$ where $\|P\|_L = \sup_{\max_i \|x_i\| \leq L} \|P(x)\|$ if the supremum is taken over all C^ -algebras and operators $\{x_1, \dots, x_\ell\}$ with norm bounded by L , and we set $\|\sum_i P_i \otimes Q_i\|_L = \sum_i \|P_i\|_L \|Q_i\|_L$.*

Proof. According to the Dyson-Schwinger equation, (20) in Theorem 4.1, applied to $P_i = (\mathcal{D}_i U)^{2k-1}$,

$$\begin{aligned} \nu_\beta((\mathcal{D}_i U)^{2k-1}(\mathcal{D}_i U + \frac{1}{\beta}\mathcal{D}_i W)) &= \beta^{-1}\nu_\beta \otimes \nu_\beta \left(\partial_i(\mathcal{D}_i U)^{2k-1} \right) \\ &= \beta^{-1} \sum_{j=1}^{2k-1} \nu_\beta \otimes \nu_\beta \left(((\mathcal{D}_i U)^{j-1} \otimes 1)(\partial_i \mathcal{D}_i U)(1 \otimes (\mathcal{D}_i U)^{2k-1-j}) \right) \end{aligned}$$

We use this equality to prove by induction that with $B \geq 4 \max_i \{\|\mathcal{D}_i U \mathcal{D}_i W\|_L + 2\|\partial_i \mathcal{D}_i U\|_L\}$ for all $p \geq 0$

$$(22) \quad \nu_\beta((\mathcal{D}_i U)^{2p}) \leq C_p \left(\frac{B}{4\beta}\right)^p$$

where C_p denotes the Catalan numbers. The result follows since $C_p \leq 4^p$. The property is obviously satisfied for $p = 0$. Assume that it is true up to $p = k - 1$. This implies that for any $j \leq 2k - 2$, for any polynomial P

$$|\nu_\beta((\mathcal{D}_i U)^j P)| \leq \|P\|_L \nu_\beta(|\mathcal{D}_i U|^j) \leq \|P\|_L \nu_\beta(|\mathcal{D}_i U|^{2\lceil \frac{j}{2} \rceil})^{2\lceil \frac{j}{2} \rceil} \leq \|P\|_L C_{\lceil \frac{j}{2} \rceil} \left(\frac{B}{4\beta}\right)^{\frac{j}{2}}.$$

Moreover

$$|\nu_\beta((\mathcal{D}_i U)^{2k-1} \mathcal{D}_i W)| \leq \|\mathcal{D}_i U \mathcal{D}_i W\|_L \nu_\beta(|\mathcal{D}_i U|^{2k-2})$$

Plugging this estimate in the Dyson-Schwinger equation and using the induction hypothesis we find that :

$$\begin{aligned} &\left(\left(\frac{B}{4\beta}\right)^k C_k\right)^{-1} \nu_\beta((\mathcal{D}_i U)^{2k}) \\ &\leq \frac{4C_{k-1}}{BC_k} \|\mathcal{D}_i U \mathcal{D}_i W\|_L + \frac{4}{BC_k} \|\partial_i \mathcal{D}_i U\|_L \sum_{j=1}^{2k-1} C_{\lceil \frac{j-1}{2} \rceil} C_{\lceil \frac{2k-1-j}{2} \rceil} \\ &\leq \frac{4}{B} \left(\max_i \{\|\mathcal{D}_i U \mathcal{D}_i W\|_L + 2\|\partial_i \mathcal{D}_i U\|_L\} \right). \end{aligned}$$

Hence, the induction hypothesis is satisfied as soon as B is greater than $4 \max_i \{\|\mathcal{D}_i U \mathcal{D}_i W\|_L + 2\|\partial_i \mathcal{D}_i U\|_L\}$. \square

Remark 4.3. Note that if U is homogenous of degree d and cyclically invariant,

$$U(\mathbf{X}) = \frac{1}{d} \sum_i X_i \mathcal{D}_i U.$$

Therefore if we have a bound on the $\nu_\beta(X_i^{2k})$'s, the bound on $\nu_\beta((\mathcal{D}_i U)^{2k})$ from the previous Lemma implies a control on $\nu_\beta(U^{2k})$. This is the content of the next Corollary:

Corollary 4.4. *With the same hypothesis and notation than in Proposition 4.2, and if furthermore U is homogenous of degree d (in the sense that all its monomials have the same total degree) and cyclically invariant, then for $\beta > \beta_0$, for every integer number k ,*

$$\nu_\beta(U^{2k}) \leq \left(\frac{L^2 B}{\beta}\right)^k$$

with $B = 4 \max_i \{\|\mathcal{D}_i U \mathcal{D}_i W\|_L + 2\|\partial_i \mathcal{D}_i U\|_L\}$.

In the case above, we see that under ν_β , U converges to zero. In a more general situation, we have only the following concentration of the mean of U .

Proposition 4.5. *Let $V_\beta = \beta U + W$ be a self-adjoint potential such that, for some $\beta_0 > 0$, the partition function (3) is finite :*

$$(23) \quad \limsup_{N \rightarrow \infty} \frac{1}{N^2} \ln Z_{V_{\beta_0}}^N < -\infty$$

and there exist real numbers $\alpha_1, \dots, \alpha_\ell$ such that

$$m := \inf_{\mathbf{X} \in \mathcal{H}^N(\mathbb{C})^\ell} \frac{1}{N} \text{Tr} U(\mathbf{X}) = \frac{1}{N} \text{Tr}(U(\alpha_1 \text{Id}, \dots, \alpha_\ell \text{Id})) = U(\alpha_1, \dots, \alpha_\ell).$$

Then, there exists a finite constant C such that for $\beta > \beta_0$ and any non-negative real number M ,

$$\limsup_{N \rightarrow \infty} \frac{1}{N^2} \ln \mathbb{Q}_V^N \left(\frac{1}{N} \text{Tr} U(\mathbf{X}^N) \geq m + \frac{M + \ell \ln \beta}{\beta} \right) < C - M.$$

As a consequence, if we assume furthermore that we have an accumulation point τ_β of the empirical distribution for all β sufficiently large. Then any accumulation point τ of τ_β when $\beta \rightarrow \infty$ is such that $\tau(U) = m$.

Proof. Let $2D$ be the degree of U . We can assume without loss of generality, that up to some translation and the addition of a constant term, that $\alpha_1 = \dots = \alpha_\ell = 0$ and that $m = \text{Tr}(U(\alpha_1 \text{Id}, \dots, \alpha_\ell \text{Id})) = 0$.

First step : upper bound on U . Since $\text{Tr} U(\mathbf{X})$, as a function of the coefficients of the matrices \mathbf{X} , is a polynomial function that reaches its local minimum in 0 and whose value is 0, then necessarily $\text{Tr} U$ has no constant term and no terms of degree 1. Therefore U is a sum of monomials of degree at least 2. Let $q = X_{i_1} \dots X_{i_k}$ be a monomial of degree $k \leq 2D$ in U , and observe that by non-commutative Hölder's inequality:

$$|\text{Tr} q(X)| \leq ((\text{Tr} |X_{i_1}|^k) \dots (\text{Tr} |X_{i_k}|^k))^{\frac{1}{k}} \leq \max_i \text{Tr} |X_i|^k \leq \sum_i \text{Tr}(X_i^2 + X_i^{2D})$$

since for every $2 \leq k \leq 2D$, $|x|^k \leq x^2 + x^{2D}$. By combining such inequalities we deduce that there is some finite constant $0 < A < \infty$ (which only depend on U) such that for all $\mathbf{X} \in (\mathcal{H}^N(\mathbb{C}))^\ell$:

$$0 \leq \text{Tr} U(\mathbf{X}) \leq A \text{Tr} \sum_i \text{Tr}(X_i^2 + X_i^{2D}).$$

Second step : lower bound on the partition function $Z_{V_\beta}^N$. First make the change of variables $\sqrt{\beta}X_i = Y_i$ and then use Jensen's inequality:

$$\begin{aligned}
\frac{1}{N^2} \ln Z_{V_\beta}^N &\geq \frac{1}{N^2} \ln \int e^{-N \operatorname{Tr} W} e^{-N\beta A \operatorname{Tr} \sum_i \operatorname{Tr}(X_i^2 + X_i^{2D})} dX_i \\
&\geq -\ell \ln \beta + \frac{1}{N^2} \ln \int e^{-N \operatorname{Tr} W(\mathbf{Y}/\sqrt{\beta})} \prod_i e^{-NA \operatorname{Tr} \operatorname{Tr}(Y_i^2 + \beta^{1-D} Y_i^{2D})} dY_i \\
&\geq -\ell \ln \beta + \frac{1}{N^2} \ln Z_{A \sum_i Y_i^2}^N \\
&\quad - \int \left(\frac{1}{N} \operatorname{Tr} W\left(\frac{\mathbf{Y}}{\sqrt{\beta}}\right) + A\beta^{1-D} \sum_i \frac{1}{N} \operatorname{Tr}(Y_i^{2D}) \right) \prod_i \frac{e^{-NA \operatorname{Tr}(Y_i^2)} dY_i}{Z_{AY^2}^N} \\
&\geq -\ell \ln \beta + O(1)
\end{aligned}$$

where we used that moments of Gaussian matrices are bounded, as well as the partition function which was computed by Selberg [1, 42]. Here, the term $O(1)$ is uniform in $\beta > 0$.

Third step : conclusion. Finally, by Tchebychev's inequality we find when $\beta \geq \beta_0$

$$\begin{aligned}
&\frac{1}{N^2} \ln \mathbb{Q}_{V_\beta}^N \left(\frac{1}{N} \operatorname{Tr} U(\mathbf{X}^N) \geq \frac{\ell \ln \beta + M}{\beta} \right) \\
&\leq \frac{1}{N^2} \ln \frac{Z_{V_{\beta_0}}^N}{Z_{V_\beta}^N} \int e^{-N^2(\beta - \beta_0) \frac{\ell \ln \beta + M}{\beta}} d\mathbb{Q}_{V_{\beta_0}}^N \\
&\leq O(1) + \ell \ln \beta - (\beta - \beta_0) \frac{\ell \ln \beta + M}{\beta} = O(1) - M
\end{aligned}$$

where we used that $\frac{1}{N^2} \ln \frac{Z_{V_{\beta_0}}^N}{Z_{V_\beta}^N} \leq \ell \ln \beta + O(1)$ by the second step and (23). \square

5. STRONG UNIFORMLY CONVEX POTENTIALS

The three remaining sections are devoted to the study of particular cases of low temperature asymptotics. We begin by the simplest case as in a way it can be converted to the well known high temperature regime. Indeed, in this section we look at a ℓ -uple of random $N \times N$ Hermitian matrices $\mathbf{X}^N = (X_1^N, \dots, X_\ell^N)$ under the law :

$$(24) \quad \mathbb{Q}_{V_\beta}^N(d\mathbf{X}^N) = \frac{1}{Z_N^\beta} \exp\left(-N \operatorname{Tr} V_\beta(\mathbf{X}^N)\right) d\mathbf{X}^N$$

where V_β is a self-adjoint polynomial that satisfy the following convexity assumption.

Assumption 5.1. *Let C be a real number. We say that a self-adjoint polynomial $V \in \mathbb{C}\langle X_1, \dots, X_\ell \rangle$ is C -convex if for any dimension N , the Hessian*

of the function $\varphi_V(\mathbf{X}^N) = \text{Tr}V(\mathbf{X}^N)$ on the set of ℓ -tuple \mathbf{X}^N of ℓ $N \times N$ Hermitian matrices is bounded from below uniformly :

$$\text{Hess}(\varphi_V(\mathbf{X}^N)) \geq C.Id$$

for any ℓ -tuple of $N \times N$ Hermitian matrices \mathbf{X}^N and any integer number N . Then we assume that

$$V_\beta(\mathbf{X}) = \beta U(\mathbf{X}) + W(\mathbf{X})$$

where U and W are self-adjoint polynomials such that:

- U is c -convex for some $c > 0$ (therefore $\mathbf{X} : (\mathcal{H}_N(\mathbb{C}))^\ell \mapsto \text{Tr} U(\mathbf{X})$ is strictly convex),
- W is A -convex for some A finite (but possibly negative).

It is not hard to check the following example.

Example 5.2. If $U(\mathbf{X}) = \sum_i P_i(X_i)$ with polynomials P_i with real coefficient such that $P_i''(x) \geq c > 0$ for all $x \in \mathbb{R}$ and $W(X) = \tilde{W}(X) + \sum_i a_i X_i^{2D}$ for an integer number $D \geq 1$, some polynomial \tilde{W} with degree strictly smaller than $2D$ and positive real numbers $(a_i)_{1 \leq i \leq \ell}$, then (U, W) satisfies our hypothesis with constant c and some finite A .

The main result of this section is

Theorem 5.3. Under Assumption 5.1, there exists a finite positive constant β_0 such that for $\beta > \beta_0$, the non-commutative distribution of \mathbf{X}^N converges almost surely under $\mathbb{Q}_{V_\beta}^N$ towards a non-commutative law τ_{V_β} : for any polynomial $P \in \mathbb{C}\langle X_1, \dots, X_\ell \rangle$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr} \left(P(\mathbf{X}^N) \right) = \tau_{V_\beta}(P) \quad a.s.$$

We next characterize the limit τ_{V_β} . Observe that φ_U achieves its minimum value at a unique point in any dimension by uniform strict convexity. In fact we can prove that these minimizers are characterized as follows.

Lemma 5.4. Assume that U is c -convex with $c > 0$. Let $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}^\ell$ be the unique minimizer of U on \mathbb{R}^ℓ . Then, for any integer number N , $\varphi_U(\mathbf{X}^N)$ is minimized on the set of ℓ -tuples of $N \times N$ Hermitian matrices at the point $(\alpha_1 I_N, \dots, \alpha_\ell I_N)$ where I_N is the identity in the space of $N \times N$ matrices.

Proof. We first fix the dimension N . Then, by strict convexity we know that φ_U is minimized at a unique ℓ -tuple (A_1^N, \dots, A_ℓ^N) of Hermitian matrices. But by invariance of φ_U by the change of variables $(X_1, \dots, X_\ell) \rightarrow (OX_1O^*, \dots, OX_\ell O^*)$ for any unitary matrix O , and uniqueness of the minimizer, we see that for all $i \in [1, \ell]$, $OA_i^N O^* = A_i^N$. This implies that A_i^N is a multiple of identity, $A_i^N = \alpha_i^N I$. But then $\varphi_U(A_1^N, \dots, A_\ell^N) = NV(\alpha_1^N, \dots, \alpha_\ell^N)$ so that $(\alpha_1^N, \dots, \alpha_\ell^N)$ minimizes U and therefore equals $(\alpha_1, \dots, \alpha_\ell)$. \square

To prove Theorem 5.3, we first do a change of variables related with the matrix

$$(25) \quad C_U(ij) = \partial_{x_i} \partial_{x_j} U(x_1, \dots, x_\ell)|_{x=\alpha} \geq c.Id$$

In the following, for any ℓ -tuple of self-adjoint operators $\mathbf{X} = (X_1, \dots, X_\ell)$, and any matrix $M \in \mathcal{H}^\ell(\mathbb{C})$, we denote by $M\mathbf{X}$ the ℓ -tuple of self-adjoint operators $(\sum_{j=1}^\ell M_{ij}X_j)_{1 \leq i \leq \ell}$. We let

$$\mathbf{Y} = \sqrt{\beta}C_U^{1/2}(\mathbf{X} - \alpha I).$$

In these variables, let us write the decomposition

$$(26) \quad \beta U(\mathbf{X}) + W(\mathbf{X}) = \sum_{i=1}^\ell Y_i^2 + \sum_{i=1}^p \frac{c_i}{\sqrt{\beta}^{n_i}} q_i(\mathbf{Y})$$

where the n_i are integers numbers greater or equal to one and the q_i 's are monomials coming either from W (and then with degree n_i) or from U (and then with degree $n_i + 2$).

We next characterize the distribution of $\mathbf{Y} = \sqrt{\beta}C_U^{1/2}(\mathbf{X} - \alpha I)$ under τ_{V_β} as a generating function for the enumeration of planar maps. Recall [28] that a star of type $q(X) = X_{i_1} \cdots X_{i_r}$ is a vertex embedded into the plane with r colored half-edges rooted at the first half-edge of color i_1 , second of color i_2 , until the last one of color i_r (following the orientation of the plane). A planar map build over k_i stars of type q_i , $1 \leq i \leq p$, is a matching of the half-edges of the same color resulting into a connected graph with genus zero. $M((k_i, q_i)_{1 \leq i \leq p})$ denotes the number of planar maps build over k_i stars of type q_i , $1 \leq i \leq p$, counted with labelled half-edges. If $M((k_i, q_i)_{1 \leq i \leq p})$ denotes the number of planar maps build over k_i stars of type q_i , we have

Proposition 5.5. *Under Hypothesis 5.1, with $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}^\ell$ be the unique minimizer of U on \mathbb{R}^ℓ and C_U given by (25), there exists a finite positive constant β_0 such that for $\beta > \beta_0$, such that for any monomial q*

$$\tau_{V_\beta}(q(\sqrt{\beta}C_U^{1/2}(\mathbf{X} - \alpha I))) = \sum \prod_{1 \leq i \leq p} \frac{(c_i \sqrt{\beta}^{-n_i})^{k_i}}{k_i!} M((1, q), (k_i, q_i)_{1 \leq i \leq p}).$$

Proposition 5.5 implies Theorem 5.3. We can also use [32] to see that $\tau_{\beta, U, W}$ is a push-forward of free semi-circle laws by smooth functions (namely series which are uniformly converging when evaluated at self-adjoint operators with norm bounded by 3). Hence we deduce that

Corollary 5.6. *Under the assumptions of Theorem 5.3, $\tau_{\beta, U, W}$ is described as follows. If S_1, \dots, S_d are ℓ free semi-circular variables, τ_{V_β} is the non-commutative distribution of*

$$X_i = \alpha_i I + \frac{1}{\sqrt{\beta}} S_i + \frac{1}{\beta} F_i(S_1, \dots, S_\ell), 1 \leq i \leq \ell,$$

where F_1, \dots, F_ℓ are functions which expand as converging power series in $1/\sqrt{\beta}$:

$$F_i(X_1, \dots, X_\ell) = \sum_{n \geq 0} \frac{1}{\sqrt{\beta}^n} P_n(X_1, \dots, X_\ell)$$

with non commutative polynomials P_n such that $\sum_{\ell, n} \sqrt{\beta}^{-n} \|P_n\|$ is finite. Here, the operator norm is taken uniformly over all ℓ -tuple of self-adjoint operators bounded by 3. In particular the C^* and the von Neumann algebra of τ_{V_β} is the same than those of free semi-circular provided β is finite and large.

6. STRONG SINGLE VARIABLES POTENTIALS

In this section we study more closely the model where the strong potential constrains the eigenvalues of $p \leq \ell$ matrices, but this constraint is not convex. More precisely we assume:

Assumption 6.1.

$$V_\beta(\mathbf{X}) = \beta(U^1(X_1) + \dots + U^p(X_p)) + W(\mathbf{X})$$

where $p \leq \ell$, U^i and W are self-adjoint polynomials such that:

- There exists $\beta_0 > 0$, $\eta > 0$, $A < \infty$ and a partition I so that for all $\beta \geq \beta_0$, V_β is (η, A, I) -trapping.
- For $i \in \{1, \dots, p\}$, U^i achieves its minimal value at real numbers $x_1^i, \dots, x_{m_i}^i$ with $U^i(x) \sim_{x_j} c_j^i (x - x_j^i)^{2k_j^i}$ when x belongs to a small neighborhood of x_j^i , for some positive constant c_j^i and integer numbers k_j^i .

We then prove that the spectrum of the matrix X_i^N following $\mathbb{Q}_{V_\beta}^N$ concentrates in a neighborhood of the minimizers of U^i for $i \in \{1, \dots, p\}$. To state our result, we will denote $\hat{\mu}_X^N = N^{-1} \sum_{i=1}^N \delta_{\lambda_i(X)}$ the empirical measure of the eigenvalues of a $N \times N$ normal matrix X . Moreover, we recall that $\|X_i\|_\infty$ denotes the operator norm of the matrix X_i and we set, for $\mathbf{X} = (X_1, \dots, X_\ell)$, $\|\mathbf{X}\|_\infty = \max_{1 \leq i \leq \ell} \|X_i\|_\infty$.

Theorem 6.2. *Assume V_β satisfies Assumption 6.1. For a positive real number R , we denote by*

$$K_{\beta, R}^i = \cup_{1 \leq j \leq m_i} [x_j^i - R(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j^i}}, x_j^i + R(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j^i}}].$$

Then, for all $\kappa \in (0, 1)$, there exists $R > 0$ and $\beta_0 > 0$, so that for all $\beta \geq \beta_0$, for all $i \in \{1, \dots, p\}$,

$$(27) \quad \limsup_N \frac{1}{N^{1-\kappa}} \ln \mathbb{Q}_{V_\beta}^N \left(\hat{\mu}_{X_i^N}^N((K_{\beta, R}^i)^c) \neq 0 \right) < 0.$$

Besides, if $K_i = \sum_j k_j^i$, there exists β_0 finite such that for $\beta \geq \beta_0$, for every $j \in \{1, \dots, m_i\}$, we have almost surely:

$$\limsup_N \sqrt{\ln \beta} |\hat{\mu}_{X_i^N}^N([x_j^i - \max_j R(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j^i}}, x_j^i + \max_j R(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j^i}}]) - \frac{k_j^i}{K_i}| < \infty.$$

To describe more precisely the asymptotic distribution of our matrix model, recall [6] that in the case of a single matrix model and a potential V (corresponding to $\ell = 1$), the empirical measure of the eigenvalues $\hat{\mu}_X^N$ converges almost surely to the unique minimizer μ_V of

$$(28) \quad E_V(\mu) = \frac{1}{2} \int \int \left(V(x) + V(y) - \frac{1}{2} \ln |x - y| \right) d\mu(x) d\mu(y).$$

Such a convergence is also true in the multi-matrix model satisfying Hypothesis 6.1 in the limit where β goes to infinity in the following sense. For $r > 0$ and an integer number k , let $\sigma_r^k = \mu_{rx^{2k}}$ be the unique probability measure minimizing $E_{rx^{2k}}$. Observe that when $k = 1$, σ_r^k is the semi-circle law with variance r^{-1} .

Corollary 6.3. *Assume V_β satisfies Assumption 6.1. Then there exists β_0 finite such that for $\beta \geq \beta_0$ the empirical distribution $\hat{\mu}^N$ of $\mathbf{X}^N = (X_1^N, \dots, X_\ell^N)$ is tight almost surely. Denote ν^β a limit point. Then, there exists $R \in (0, +\infty)$ such that for β large enough, for any $i \in \{1, \dots, p\}$*

$$\nu_{X_i}^\beta(K_{\beta, R}^i) = 1.$$

Moreover, the filling fraction $\alpha_j^i(\beta) := \nu_{X_i}^\beta([x_j^i - R(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j^i}}, x_j^i + R(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j^i}}])$ satisfies

$$|\alpha_j^i(\beta) - \frac{k_j^i}{K_i}| \leq \frac{R}{\sqrt{\ln \beta}}.$$

Finally, the distribution of X_i under ν_β is close to $\sigma_{r_{i,j}^\beta}^{k_j^i}$ near x_j^i : for any smooth compactly supported function f

$$\left| \alpha_j^i(\beta)^{-1} \int \left(f(\beta^{\frac{1}{2k_j^i}}(x - x_j^i)) \right) d\nu_{X_i}^\beta(x) - \int f(x) d\sigma_{r_{i,j}^\beta}^{k_j^i}(x) \right| \leq \frac{\|f\|_{1/2}}{\min_j \beta^{\frac{1}{2k_j^i}}}$$

where $r_{i,j}^\beta = \frac{\alpha_j^i(\beta)}{2c_j^i}$ and $\|f\|_{1/2} = (\int_0^\infty t |\hat{f}_t|^2 dt)^{1/2}$.

The proof of Theorem 6.2 follows again from a wise change of variables and Lemma 8.3. We first construct this change of variables in the next subsection: it is based on the explicit joint law of the eigenvalues of a one dimensional matrix model and change of variables involving the eigenvalues of a single matrix. Indeed, the interaction between the matrices being of lower strength, such change of variables will be sufficient for our purpose.

6.1. Proofs of Theorem 6.2 and Corollary 6.3. To simplify the notations we will take $p = 1$, $k_j^1 = k_j$, $x_j^1 = x_j$. The proof extends readily to the general case.

Proof of Theorem 6.2 We denote by $\{x_1, \dots, x_k\}$ the minimizers of U^1 . Without loss of generality we may assume that the minimum value of U^1 is zero and it is achieved at the origin so that $U^1(0) = 0$. We set for $\eta > 0$

$$I = \{x \in \mathbb{R} : U^1(x) \in [0, \eta]\}$$

to be a neighborhood of the minimizers of U^1 . η is a constant depending on β that we will choose later. We let $\varepsilon > 0$ be such that $U^1(x) < \eta/2$ for $|x| < \varepsilon$. We shall prove the theorem by sending all eigenvalues outside of I in $[-\varepsilon, \varepsilon]$. To this end we first remark that for each $M > 0$

$$(29) \quad \begin{aligned} & \mathbb{Q}_{V_\beta}^N \left(\left\{ \hat{\mu}_{X_1}^N(I) \neq 0 \right\} \cap \{\|\mathbf{X}\|_\infty \leq M\} \right) \\ &= \sum_{n=1}^N \mathbb{Q}_{V_\beta}^N \left(\left\{ \hat{\mu}_{X_1}^N(I) = \frac{n}{N} \right\} \cap \{\|\mathbf{X}\|_\infty \leq M\} \right). \end{aligned}$$

It is sufficient to show that each term in the above right hand side decays like some stretched exponential provided $M \geq L$ is chosen large enough as in Theorem 2.2 so that

$$(30) \quad \mathbb{Q}_{V_\beta}^N (\{\|\mathbf{X}\|_\infty \geq M\}) \leq e^{-\frac{1}{2}N^{1-\kappa}}$$

for some $\kappa \in (0, 1)$. We assume without loss of generality that $M \geq 1$. To bound each term in the right hand side of (29) we perform the previous change of variables, keeping fixed the n eigenvalues in I , as follows. Let (x_1, \dots, x_n) be the eigenvalues of X_1 inside I and let us apply the change of variables ϕ constructed in Lemma 8.10 on the remaining eigenvalues, with $\alpha = 0$ and ε small enough so that $U(x) \in [0, \eta/2]$ for x in $[-\varepsilon, \varepsilon]$. Then, we have constructed ϕ so that

$$J_\phi \geq \left(\frac{\varepsilon}{4e} \right)^{3N(N-n)} e^{-2N \operatorname{Tr} \ln(1+|X_1|1_{X_1 \notin I}) - 2(N-n) \operatorname{Tr} \ln(1+|X_1|1_{X \in I})}$$

and the spectrum of $U^1(\phi(X_1))$ is included in $[0, \eta]$. Note also that each eigenvalue y outside of I is sent in $[-\varepsilon, \varepsilon]$ and thus $U^1(y) - U^1(\phi(y)) \geq \eta/2 + \ln(1 + |y|1_{|y| > A_U})$ for all $y \in I^c$ and a fixed large A_U which depends only on U^1 (this comes from the fact that U^1 grows at least like a quadratic at infinity). As a consequence, we get

$$\frac{e^{-N\beta \operatorname{Tr}(U^1(X_1))}}{e^{-N\beta \operatorname{Tr}(U^1(\phi(X_1)))}} \leq e^{-\beta N(N-n)\eta/2 - \beta N \operatorname{Tr} \ln(1+|X_1|1_{X_1 \notin I, |X_1| > A_U})}$$

We can extend the change of variables ϕ to the other matrices by just setting it equal to the identity $\phi(X_i) = X_i, i \geq 2$. Noticing that $X_i - \phi(X_i)$ has

rank at most $N - n$, we find that if W is a polynomial of degree $2D$, we can find a finite constant C_1 which only depends on W such that

$$\frac{e^{-N \operatorname{Tr}(W(X))}}{e^{-N \operatorname{Tr}(W(\phi(X)))}} \leq \exp\{C_1(\|\mathbf{X}\|_\infty^{2D} + 1)(N - n)N\}$$

Therefore, Lemma 8.1 gives (up to eventually change a bit the constant C_V)

$$\begin{aligned} & Q_{V_\beta}^N \left(\left\{ \hat{\mu}_{X_1}^N(I) = \frac{n}{N} \right\} \cap \{\|\mathbf{X}\|_\infty \leq M\} \right) \\ & \leq e^{-\beta N(N-n)\eta/2 + C_1(M^{2D}+1)(N-n)N} \left(\frac{\varepsilon}{4e} \right)^{-3N(N-n)} \\ & \leq e^{N(N-n)(C(M) - \beta\eta/2 - 3\ln(\varepsilon))} \end{aligned}$$

where $C(M)$ is some finite constant which only depends on M, U^1 and W and in particular does not depend on β . Finally, we take $\eta = \beta^{-1}L \ln \beta$ with L large enough. Because U^1 is C^2 , ε is of order η and in particular $\ln \varepsilon = o(\eta\beta)$. Therefore $C(M) - \beta\eta/2 - 3\ln(\varepsilon)$ will be large and negative, say smaller than $-2^{-1}L \ln \beta$, for large β . Consequently by (29), we find

$$Q_{V_\beta}^N \left(\left\{ \hat{\mu}_{X_1}^N(I) \neq 0 \right\} \cap \{\|\mathbf{X}\|_\infty \leq M\} \right) \leq \sum_{n=1}^N e^{-N(N-n)\frac{L}{2} \ln \beta} \leq 2e^{-N\frac{L}{2} \ln \beta}$$

which implies by (30) that there exists $\kappa \in (0, 1)$

$$Q_{V_\beta}^N \left(\left\{ \hat{\mu}_{X_1}^N(I) = 1 \right\} \right) \geq 1 - e^{-N^{1-\kappa}}$$

and concludes the proof that almost surely $\hat{\mu}_{X_1}^N(I^c)$ vanishes for N large enough. It also implies (27) because U^1 is C^2 and with finitely many minimizers, for β large enough, the set $I = \{x : |U^1(x)| < \beta^{-1}L \ln \beta\}$ is included in the set

$$\cup_j \{x : |x - x_j| < L'(\beta(\ln \beta)^{-1})^{-\frac{1}{2k_j}}\}$$

for some L' large enough. Therefore, we deduce that almost surely

$$\hat{\mu}_{X_1}^N \left(\cup_{1 \leq j \leq k} [x_j - L'(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j}}, x_j + L'(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j}}] \right) = 1 \quad \text{for } N \text{ large enough}$$

We now estimate the filling fractions, namely the masses around each minimizers. From the previous considerations, if we set

$$A = \left\{ \hat{\mu}_{X_1}^N \left(\cup_{j=1}^k [x_j - L'(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j}}, x_j + L'(\frac{\beta}{\ln \beta})^{-\frac{1}{2k_j}}] \right) = 1 \right\} \cap \{\|\mathbf{X}\|_\infty \leq M\}$$

then $\mathbb{Q}_V^N(A^c)$ goes to zero exponentially fast. We will therefore restrict ourselves to the set A . We diagonalize X_1 and write for a unitary matrix V

$$X_1 = V^* x V + V^* \Lambda(\beta) V$$

with $x = \text{Diag}(x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_p)$ where x_i has multiplicity N_i and

$$\Lambda(\beta) = \begin{pmatrix} \beta^{-1/2k_1} \Lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \beta^{-1/2k_2} \Lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \beta^{-1/2k_1} \Lambda_p \end{pmatrix}$$

with Λ_i a $N_i \times N_i$ real diagonal matrices with entries in $[-L'(\ln \beta)^{\frac{1}{2k_j}}; L'(\ln \beta)^{\frac{1}{2k_j}}]$. Recalling our hypothesis on the minimizers x_i of U^1 ,

$$\beta U^1(x_i + \beta^{-1/2k_i} x) = c_i x^{2k_i} + \beta P_i(\beta^{-1/2k_i} x)$$

with c_i a positive constant and P_i a polynomial with a zero of order p_i strictly greater than $2k_i$ at 0. Note that on the set A :

$$\beta \text{Tr} U^1(X_1) = \sum_i c_i \text{Tr}(\Lambda_i^{2k_i}) + \text{Tr} \beta P_i(\beta^{-1/2k_i} \Lambda_i)$$

the second term is small when β is large since it is bounded by $CN\beta(\ln \beta/\beta)^{p_i/2k_i}$ for some integer number $p_i > 2k_i$. Setting $\mathbf{Y} = (X_2, \dots, X_\ell)$, we can give a new expression for $1_A d\mathbb{Q}_{V_\beta}^N$ which is equal to

$$\begin{aligned} &= \frac{1_A}{Z_{V_\beta}^N} e^{-\beta N \text{Tr} U^1(X_1) - N \text{Tr} W(X_1, \mathbf{Y})} dX_1 d\mathbf{Y} \\ &= \frac{1}{Z_{V_\beta}^N} 1_A e^{-N \left(\sum_i c_i \text{Tr}(\Lambda_i^{2k_i}) + \text{Tr} \beta P_i(\beta^{-1/2k_i} \Lambda_i) - \text{Tr} W(x + \Lambda(\beta), \mathbf{Y}) \right)} \Delta^2(x + \Lambda(\beta)) d\Lambda d\mathbf{Y} \\ (31) \quad &= \frac{1}{Z_{V_\beta}^N} 1_A e^{-N \left(\sum_i c_i \text{Tr}(\Lambda_i^{2k_i}) + o(\beta)N - \text{Tr} W(x, \mathbf{Y}) \right)} \Delta^2(x + \Lambda(\beta)) d\Lambda d\mathbf{Y} \end{aligned}$$

where $o(\beta)$ goes to zero like $\beta^{-\frac{1}{2 \max k_i}}$ as β goes to infinity while M is fixed.

In the square of the VanderMonde $\Delta^2(x + \Lambda(\beta))$ there are factors with the differences coming from two different global minima:

$$\begin{aligned} &\prod_{x_i + \beta^{-1/2k_i} y \in Sp(\Lambda_i)} \prod_{x_j + \beta^{-1/2k_j} z \in Sp(\Lambda_j)} |x_i - x_j + \beta^{-1/2k_i} (y - z)|^2 \\ &= |x_i - x_j|^{N_i N_j} e^{o(\beta) N_i N_j} \end{aligned}$$

where $|o(\beta)| \leq C\beta^{-\frac{1}{2 \max k_i}}$, and the factors coming from the same minimum:

$$\prod_{x_i + \beta^{-1/2k_i} y \in Sp(\Lambda_i)} \prod_{x_i + \beta^{-1/2k_i} z \in Sp(\Lambda_i)} \beta^{-\frac{1}{k_i}} |y - z|^2 = \beta^{-\frac{N_i(N_i-1)}{2k_i}} \Delta(\Lambda_i)^2$$

Finally we get,

$$(32) \quad 1_A \mathbb{Q}_{V_\beta}^N = \frac{1}{Z_{V_\beta}^N} 1_A \prod_i \beta^{-\frac{N_i(N_i-1)}{2k_i}} e^{N^2 o(\beta)} \prod_i e^{-N c_i \text{Tr}(\Lambda_i^{2k_i})} \Delta(\Lambda_i)^2 d\Lambda_i e^{O(N^2)} d\mathbf{Y}$$

where $X = V^* x V + V^* \Lambda(\beta) V$ in the above RHS and $O(N^2)$ bounds $N \text{Tr} W$ uniformly on A . We deduce a lower bound of the partition function by conditioning by $N_i/N = n_i$ to see that

$$(33) \quad Z_{V_\beta}^N \geq \beta^{-N^2 \sum_i \frac{n_i^2}{2k_i} + O(N^2)}$$

where we used [6, 2] to see that for all i , with the notations of (28), we have

$$\lim_{N_i \rightarrow \infty} \frac{1}{N_i^2} \ln \int e^{-N c_i \text{Tr}(\Lambda_i^{2k_i})} \Delta(\Lambda_i)^2 d\Lambda_i = -\inf\{E_{n_i^{-1} c_i x^{2k_i}}\}$$

where the supremum is taken over all probability measures μ on the real line. In particular, the limit does not depend on β large enough, even though we integrate a priori on $[-L'(\ln \beta)^{\frac{1}{2k_j}}; L'(\ln \beta)^{\frac{1}{2k_j}}]$, because the supremum of $E_{n_i^{-1} c_i x^{2k_i}}$ is taken at a compactly supported probability measure. We can optimize over the choice of n_i in (33), that is minimize $(n_i) \mapsto \sum_i \frac{n_i^2}{2k_i}$ while $\sum n_i = 1$. We find that we should take $n_i = n_i^* = k_i / \sum k_j$. Plugging back this lower bound on the partition function into (32) we find

$$\mathbb{Q}_{V_\beta}^N \left(\cap_i \left\{ \frac{N_i}{N} = n_i \right\} \cap A \right) \leq \exp\left\{-N^2 \ln \beta \left(\sum \frac{n_i^2}{k_i} - \sum \frac{(n_i^*)^2}{k_i} \right) + O(N^2)\right\}$$

Noticing that $\sum \frac{n_i^2}{k_i} - \sum \frac{(n_i^*)^2}{k_i} = \sum_i (n_i - n_i^*)^2 / k_i$, we conclude that there exists R finite so that for β big enough

$$\mathbb{Q}_{V_\beta}^N \left(\left\{ \max_i \left| \frac{N_i}{N} - n_i^* \right| > R \sqrt{\ln \beta}^{-1} \right\} \cap A \right) \leq e^{-N^2}$$

which completes the proof when $p = 1$. It is straightforward to generalize the proof to more general p as we can do the change of variables for any matrices separately to localize the spectrum of each of the p matrices and then expand around the minimizers the spectrum of each of these matrices.

Proof of Corollary 6.3. The first points are direct consequence of Theorem 6.2. The main point is hence to show convergence of the rescaled distribution near a minimizer. But, coming back to (31), if we condition by the event $\mathbf{N} = (N_1, \dots, N_p) = N(n_1, \dots, n_p) = N\mathbf{n}$, X_1 is fixed up to terms of order $\beta^{-\min \frac{1}{2k_i}}$, and therefore the interaction term $N \text{Tr} W(X_1, Y)$ is fixed up to order $\beta^{-\min \frac{1}{2k_i}} N^2$. Hence, the previous proof shows that for any measurable set B of the set of probability measures on the real line,

$$\mathbb{Q}_{V_\beta}^N (\{\hat{\mu}_{\Lambda_i} \in B\} \cap A | \mathbf{N}) = \frac{1}{\tilde{Z}_\beta^N} \int_{\hat{\mu}_{\Lambda_i} \in B} e^{-N(c_i \text{Tr}(\Lambda_i^{2k_i}) + o(\beta)N)} \Delta^2(\Lambda_i) d\Lambda dY$$

and therefore again by [6] the right hand side goes exponentially fast to zero unless

$$\inf_{\mu \in B} \{E_{c_i n_i^{-1} x^{2k_i}}(\mu)\} \leq \inf E_{c_i n_i^{-1} x^{2k_i}} + o(\beta).$$

Put $r_i = c_i/n_i$ and using that $\sigma_{r_i}^{k_i}$ satisfies

$$r_i x^{2k_i} - \int \ln |x - y| d\sigma_{r_i}^{k_i}(y) \geq C$$

everywhere, with equality $\sigma_{r_i}^{k_i}$ almost surely, we find that the above inequality holds iff for all $\mu \in B$,

$$D(\mu, \sigma_{r_i}^{k_i})^2 := - \int \ln |x - y| d(\mu - \sigma_{r_i}^{k_i})(x) d(\mu - \sigma_{r_i}^{k_i})(y) \leq o(\beta)$$

Moreover, for any probability measures μ, ν , we have [6]

$$D(\mu, \nu) = \left(\int_0^{+\infty} \frac{1}{t} \left| \int e^{itx} d(\mu - \nu) \right|^2 dt \right)^{1/2}.$$

so that D defines a distance on the set of probability measures and moreover, Cauchy-Schwartz inequality shows that

$$\left| \int f d(\mu - \nu) \right| \leq \|f\|_{1/2} D(\mu, \nu)$$

and the fact that $o(\beta)$ is of order at most $\beta^{-\frac{1}{2 \max k_i}}$ completes the proof.

6.2. Low temperature expansion. In this section we consider a special case of the previous one where the interaction term vanishes at the minimizers of U^1 . This additional hypothesis will allow us to obtain results when β is sufficiently large but finite.

Assumption 6.4.

$$V_\beta(X_1, \dots, X_\ell) = \beta \sum_{i=1}^p U^i(X_i) + \sum_{i=p+1}^{\ell} V^i(X_i) + Z(X_1, \dots, X_\ell)$$

where

- $(U^i)_{1 \leq i \leq p}$ and $W(\mathbf{X}) = \sum_{i=p+1}^{\ell} V^i(X_i) + Z(X_1, \dots, X_\ell)$ satisfy Assumption 6.1. Moreover $V^i, p+1 \leq i \leq \ell$ is uniformly convex and the k_j^i are equal to one.
- Z can be decomposed, up to cyclic invariance as $\sum_{i=1}^p Z_i(X_1, \dots, X_\ell)$ with some polynomials $Z_i(X_1, \dots, X_\ell)$ whose trace vanishes at any ℓ -tuple (X_1, \dots, X_ℓ) of operators such that the eigenvalues of X_i belong to the set $\{x_j^i, 1 \leq j \leq m_i\}$ for any choice of operators $X_j, j \neq i$.

Remark 6.5. • Potentials of the form

$$Z_i(X_1, \dots, X_\ell) = \prod_{j=1}^{m_i} (X_i - x_j^i) Q^i(X_1, \dots, X_\ell)$$

with polynomials Q^i fulfill the second point of Assumption 6.4.

- By closer inspection of the proof, we see that it is enough to assume that for any ℓ tuple of unitary matrices O_1, \dots, O_ℓ and diagonal matrices D_1, \dots, D_ℓ such that the eigenvalues of D_i belong to the set $\{x_j^i, 1 \leq j \leq m_i\}$ and for ϵ small enough $\hat{\mu}_{D_i}(\{x_j^i\})$ belongs to $[1/m_i - \epsilon, 1/m_i + \epsilon]$,

$$\frac{1}{N} \text{Tr}(Z(O_1 D_1 O_1^*, \dots, O_\ell D_\ell O_\ell^*)) = 0.$$

Also, our result could extend to more general integer numbers k_i^j but we chose to take $k_i^j = 1$ for simplicity.

- Moreover, we might have generalized our result to functions U^i, V^i, Z which are not polynomials in the sense that tightness could be proved directly in this setting without the trapping conditions. For consistency we however restrict ourselves to polynomials functions.

Then, we prove the following low temperature (namely large β) expansion, which is based on the fact that, up to rescaling, our hypothesis insures that the interaction is small with overwhelming probability under our hypothesis so that we can use [14, 30].

Theorem 6.6. *Assume Assumption 6.4 holds. Then, for β large enough, the empirical distribution $\hat{\mu}_N$ of \mathbf{X}^N converges $\mathbb{Q}_{V_\beta}^N$ -almost surely towards τ_β . Moreover, the moments under τ_β expand as a converging series in $\beta^{-\frac{1}{2}}$. τ_∞ is the law of ℓ free variables, the p th first having law $\frac{1}{m_i} \sum_{j=1}^{m_i} \delta_{x_j^i}$ and the $\ell - p$ last ones following the minimizer μ_{V^i} of the energy E_{V^i} .*

Proof. By Theorem 6.2, we can restrict the domain of integration to

$$\mathcal{B} = \cap_{i \leq p} \{\hat{\mu}_{X_i^N}(K_{\beta,R}^i) = 1\} \cap_{1 \leq j \leq m_i} \{|\hat{\mu}_{X_i^N}([x_j^i - \eta, x_j^i + \eta]) - \frac{1}{m_i}| \leq \frac{C}{\sqrt{\ln(\beta)}}\}$$

where C is finite and $\eta > \beta^{-d}$ with $d > \frac{1}{2}$ fixed. We next condition $\mathbb{Q}_{V_\beta}^N$ by the values of $N_j^i = N \hat{\mu}_{X_i^N}([x_j^i - \eta, x_j^i + \eta])$, $1 \leq j \leq m_i$, and diagonalize the matrices $X_i^N = O_i D_i O_i^*$ with unitary matrices O_i . Under this conditioning, D_i is the block diagonal matrix with N_j^i entries close to x_j^i . We let D_i^∞ be the matrix with N_j^i entries exactly equal to x_j^i : it is therefore fixed. We denote by $\Lambda_i = \beta^d (D_i - D_i^\infty)$ with $d \in (0, \min_i \frac{1}{2})$ so that the matrices Λ_i stay bounded on \mathcal{B} uniformly on β and let $\mathbf{Y}_i = (X_j, j \neq i)$. Moreover, by assumption, $\text{Tr} Z_i(O_i D_i^\infty O_i^*, \mathbf{Y}_i)$ vanishes so that

$$\begin{aligned} \text{Tr} Z_i(\mathbf{X}) &= \text{Tr}(Z_i(O_i D_i O_i^*, \mathbf{Y}_i) - Z_i(O_i D_i^\infty O_i^*, \mathbf{Y}_i)) \\ &= \beta^{-d} \text{Tr} Z_i \left(O_i \Lambda_i O_i^* \int_0^1 \mathcal{D}_{X_i} Z_i(O_i D_i^\infty O_i^* + \alpha \beta^{-d} O_i \Lambda_i O_i^*, \mathbf{Y}_i) d\alpha \right) \end{aligned}$$

Consequently, since this is true for all i , we find a polynomial P^β with coefficients which are polynomials in β^{-d} and such that

$$\text{Tr} Z((O_j D_j O_j^*)_{1 \leq i \leq j}) = \beta^{-d} \text{Tr}(P^\beta((O_j D_j^\infty O_j^*, O_j \Lambda_j O_j^*)_{j \leq p}, (O_j D_j O_j^*)_{j \geq p+1}))$$

Following [22, Theorem 2.3] and [30], when β is large enough, we know that

$$\begin{aligned} & \frac{1}{N^2} \int e^{N \text{Tr} Z(O_1 D_1 O_1^*, \dots, O_\ell D_\ell O_\ell^*)} dO_1 \cdots dO_\ell \\ &= \beta^{-d} F_\beta((\hat{\mu}_{D_i^\infty, \Lambda_i})_{1 \leq i \leq p}, (\hat{\mu}_{D_i})_{p+1 \leq i \leq \ell}) + o(1) \end{aligned}$$

where F_β is a smooth function of the joint distributions $\hat{\mu}_{D_i^\infty, \Lambda_i}$ of the diagonal matrices (D_i^∞, Λ_i) , $1 \leq i \leq p$, and of the empirical measures of the eigenvalues $\hat{\mu}_{D_i}$, $i \geq p+1$ of the remaining matrices. Moreover, F_β expands as a converging series in $\beta^{-1/2}$. Note that by construction for any polynomial $P(X, Y) = P_1(X)Q_1(Y) \cdots P_r(X)Q_r(Y)$,

$$\hat{\mu}_{D_i^\infty, \Lambda_i}(P) = \sum_{j=1}^{m_i} \frac{N_j^i}{N} \prod_{k=1}^r P_k(x_j^i) \hat{\mu}_{\Lambda_i^j}(\prod Q_k)$$

is in fact completely determined by the empirical measure of D_i , that is of $X_i = D_i^\infty + \beta^{-d} \Lambda_i$ for β large enough. $\hat{\mu}_{\Lambda_i^j}$ is the empirical measure of the eigenvalues of X_i close to x_j^i : for any bounded continuous function f

$$\hat{\mu}_{\Lambda_i^j}(f) = \hat{\mu}_{X_i}(f | |X_i - x_j^i| \leq \eta) = \frac{\hat{\mu}_{X_i}(f(X_i) 1_{|X_i - x_j^i| \leq \eta})}{\hat{\mu}_{X_i}(|X_i - x_j^i| \leq \eta)}.$$

Hence $\hat{\mu}_{D_i^\infty, \Lambda_i}$ is a smooth function of $\hat{\mu}_{X_i}$. it is a smooth function of the latter. We therefore can write

$$F_\beta((\hat{\mu}_{D_i^\infty, \Lambda_i})_{1 \leq i \leq p}, (\hat{\mu}_{D_i})_{p+1 \leq i \leq \ell}) = G_{\beta, (x_j^i)_{1 \leq j \leq m_i}}((\hat{\mu}_{\Lambda_i^j})_{\substack{1 \leq j \leq m_i \\ 1 \leq i \leq p}}, (\hat{\mu}_{D_j})_{p+1 \leq j \leq \ell}).$$

We deduce by large deviations techniques, see [6, 2], that if N_i/N goes to n_i ,

$$\mathbb{Q}_V^N(\mathcal{B} \cap \{\max_{1 \leq i \leq k} d(\hat{\mu}_{X_i}, \mu_i) \leq \delta\} | \frac{N_j^i}{N} = n_j^i) \simeq e^{-N^2 I_n((\mu_i)_{1 \leq i \leq \ell}) + o(N^2)}$$

where

$$\begin{aligned} I_n(\mu_i, 1 \leq i \leq \ell) &= \sum_{i=1}^p \sum_{j=1}^{m_i} n_j^i E_{c_j^i (n_j^i)^{-1} x_j^{2k_j^i}}(\mu_{\Lambda_i^j}) + \sum_{j=p+1}^{\ell} E_{V^i}(\mu_i) \\ &\quad - \beta^{-d} G_{\beta, (x_j^i)_{1 \leq j \leq m_i}}((\mu_{\Lambda_i^j})_{1 \leq j \leq m_i}, i \leq p, (\mu_j)_{p+1 \leq j \leq \ell}) \end{aligned}$$

where finally $\mu_i = \sum n_j^i \mu_{\Lambda_i^j}$, or in other words $\mu_{\Lambda_i^j}$ is the law μ_i conditioned to be in the neighborhood of the minimizing point x_j^i . The rate function for $\mathbb{Q}_{V_\beta}^N$ is deduced by optimizing over the n_i . Minimizers of such rate functions were studied in [22, Lemma 3.1]: it was shown that when the potentials V^i are uniformly convex and $k_j^i = 1$ and β is large enough then there exists a unique minimizer because the rate function stays strictly convex under transport maps. This implies convergence of $\hat{\mu}_{X_1}, \dots, \hat{\mu}_{X_\ell}$ towards some $\mu_1^\beta, \dots, \mu_\ell^\beta$ when β is large enough. To deduce the convergence of the joint distribution of $\mathbf{X} = (O_1 D_1 O_1^*, \dots, O_\ell D_\ell O_\ell^*)$ it is enough to show the joint convergence of the distribution of $\mathbf{O} = (O_1, \dots, O_\ell)$ knowing that the distributions of

D_1, \dots, D_ℓ converges towards $\mu_1^\beta, \dots, \mu_\ell^\beta$. But the law of \mathbf{O} is absolutely continuous with respect to Lebesgue measure and with density

$$d\mathbb{P}_{V_\beta}^N(\mathbf{O}) = \frac{1}{Z_{V_\beta}^N} e^{N\beta^{-d} \text{Tr}(P_\beta((O_j D_j^\infty O_j^*, O_j \Lambda_j O_j^*)_{j \leq p}, (O_j D_j O_j^*)_{j \geq p+1}))} d\mathbf{U},$$

with P_β a polynomial with coefficients bounded independently of β such that

$$\begin{aligned} & \beta^{-d} \text{Tr} P_\beta(O_j D_j^\infty O_j^*, O_j \Lambda_j O_j^*)_{j \leq p}, (O_j D_j O_j^*)_{j \geq p+1}) \\ &= \text{Tr}(Z((O_j(D_j^\infty + \Lambda_j)O_j^*)_{j \leq p}, (O_j D_j O_j^*)_{j \geq p+1})). \end{aligned}$$

We can now use [30, 14] to deduce that when β is large enough and the laws of $\hat{\mu}_{X_1}, \dots, \hat{\mu}_{X_\ell}$ converging towards some $\mu_1^\beta, \dots, \mu_\ell^\beta$, the joint distribution of $(O_i D_i^\infty O_i^*, O_i \Lambda_i O_i^*, i \leq p, U_{p+1} D_{p+1} U_{p+1}^*, \dots, U_\ell D_\ell U_\ell^*)$ converges towards a non-commutative law τ_β whose moments are iteratively determined by Dyson-Schwinger equations. Moreover, the moments under τ_β expand as a converging series in $\beta^{-1/2}$ and τ_∞ is the law of ℓ free variables, the p th first having law $\frac{1}{K_i} \sum k_j^i \delta_{x_j^i}$ and the $\ell - p$ last ones following the minimizer μ_{V_i} of the energy E_{V_i} . □

Notice that an analogue of Corollary 5.6 should also hold in the setting of Theorem 6.6 but its proof would necessitate to extend [32] to the case of perturbative potentials depending on random matrices and deterministic projections, a project that we do not pursue here.

7. STRONG COMMUTATOR INTERACTION

In this section we consider the case where the strong interaction is given by a commutator.

Assumption 7.1. *There exists polynomials V_1 and V_2 such that*

$$V_\beta(\mathbf{X}) = -\beta(X_1 X_2 - X_2 X_1)^2 + V_1(X_1) + V_2(X_2).$$

Moreover, V_1 and V_2 go to $+\infty$ at infinity (and therefore faster than cx^2 for some $c > 0$).

We already know that X_1 and X_2 commute asymptotically when N and then β go to infinity by Corollary 4.4, since it implies that the trace of the moments of the commutator $U(\mathbf{X}) = -(X_1 X_2 - X_2 X_1)^2$ goes to zero. However there are many ways to insure this is the case: for instance X_1 and X_2 can have asymptotically the same basis of eigenvectors but a non trivial spectrum or X_1 or X_2 could converge to a multiple of the identity. In fact it turns out this two strategies have the same small entropy: Because the eigenvalues of X_1 follow a Coulomb gaz, the event $\{\|X_1 - I\|_\infty \leq \varepsilon\}$ has probability of order $\varepsilon^{\beta N^2/2}$. But similarly if we diagonalize X_1 and write $X_2 = U D U^*$ with D and X_1 commuting, and U a unitary matrix, the eigenvalues of U follow a Coulomb gas and therefore the event $\{\|U - I\|_\infty \leq$

$\varepsilon\}$ has probability of order $\varepsilon^{\beta N^2/2}$. As noticed by D. Shlyakhtenko in a private communication, this implies that if we have a 3(or more)-matrix model such as

$$V(X_1, X_2, X_3) = -\beta[X_1, X_2]^2 - \beta[X_1, X_3]^2 + W(X_1, X_2, X_3)$$

then the most likely strategy is that X_1 is close to a multiple of identity rather than to force X_1 to commute with both X_2 and X_3 . We show in the sequel that with only one commutator the situation is more interesting and the matrices in general do not converge to a multiple of identity.

First we consider the case where one of the potential is quadratic. We define the following functional on the space of probability measures on \mathbb{R} :

$$J_\beta^V(\mu) = \int \int \left(\frac{1}{2}(V(x) + V(y)) - \ln \frac{\beta|x-y|^2}{1 + \beta|x-y|^2} \right) d\mu(x)d\mu(y)$$

and its off diagonal version $J_{\beta, \neq}^V(\mu)$ given by

$$J_{\beta, \neq}^V(\mu) = \int \int 1_{x \neq y} \left(\frac{1}{2}(V(x) + V(y)) - \ln \frac{\beta|x-y|^2}{1 + \beta|x-y|^2} \right) d\mu(x)d\mu(y)$$

Proposition 7.2 (Asymptotics if one potential is quadratic). *Let X^N, Y^N be two $N \times N$ random Hermitian or symmetric matrices with distribution:*

$$d\mathbb{Q}_{V_\beta}^N(X, Y) = \frac{1}{Z_{V_\beta}^N} \exp(-N \operatorname{Tr}(-\frac{\beta}{2}[X, Y]^2 + V(X) + \frac{Y^2}{2})) dX dY$$

where V is a polynomial going to $+\infty$ at infinity and β a non-negative real number. Then under $\mathbb{Q}_{V_\beta}^N$, the law of the eigenvalues x_1, \dots, x_N of X^N is given by:

$$Q_{V_\beta}^N(dx) = \frac{1}{Z_{V_\beta}^N} \exp(-N^2 J_{\beta, \neq}^V(\frac{1}{N} \sum_{i=1}^N \delta_{x_i})) dx_1 \dots dx_N$$

(1) Moreover,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln Z_{V_\beta}^N &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \int_{\mathbb{R}^N} \exp(-N^2 J_{\beta, \neq}^V(\frac{1}{N} \sum_{i=1}^N \delta_{x_i})) dx_1 \dots dx_N \\ &= -c_\beta^V := -\inf J_\beta^V. \end{aligned}$$

(2) Under $\mathbb{Q}_{V_\beta}^N$ (or $Q_{V_\beta}^N$), the law of $\hat{\mu}_X^N$ satisfies a large deviation principle with speed N^2 and good rate function $\mu \rightarrow J_\beta^V(\mu) - c_\beta^V$. In particular $\hat{\mu}_X^N$ converges almost surely towards the unique minimizer μ_β^V of J_β^V .

(3) Assume that V achieves its minimal value 0 at real numbers $\{x_1, \dots, x_r\}$ so that $V(x) = c_i(x-x_i)^{2p_i}(1+o(1))$ for some integer numbers $p_i \geq 1$

and positive real number c_i . Let $p = \max p_i$. Then, there exists a finite constant $A(V)$ such that

$$(34) \quad \limsup \frac{1}{N} \ln \mathbb{Q}_{V_\beta}^N (\|Y\|_\infty > A(V) \beta^{-\frac{p}{5(2p+1)}}) < 0.$$

Proof. The fact that the law of (x_1, \dots, x_N) has distribution $\mathbb{Q}_{V_\beta}^N$ follows from the fact that we can assume without loss of generality that X is diagonal as the law of Y is invariant under conjugation by the unitary group and then

$$\mathrm{Tr}(-[X, Y]^2) = \mathrm{Tr}(-(XY - YX)^2) = 2 \sum_{i < j} (x_i - x_j)^2 |Y_{ij}|^2.$$

Integrating over the entries of the matrix Y , we get the distribution

$$\frac{1}{Z_\beta^N} \prod_{i < j} (1 + \beta(x_i - x_j)^2)^{-1} e^{-N \sum_{i=1}^N V(x_i)} \prod_{i < j} |x_i - x_j|^2 \prod dx_i$$

The large deviation principle and the convergence of the free energy are a direct consequence of a general large deviation principle proved by Garcia Zelada [23] and Ben Arous-Guionnet [6, 2]. It is classical to prove that J_β^V is a good rate function (J_β differs from the usual Beta-ensemble rate function by a smooth smooth and its behavior at infinity is balanced by the logarithmic entropy). We show below that it is strictly convex and therefore has a unique minimizer. Because J_β^V is a sum of the linear term $\mu \rightarrow \int V d\mu$ and J_β^0 , it is enough to show that J_β^0 is strictly convex. The argument is similar to the one used for Beta-ensembles [2, Lemma 2.6.2]. Let ε be a positive real number. We define the regularization J_β^ε of J_β^0 by setting, for any probability measure μ ,

$$J_\beta^\varepsilon(\mu) = - \int \int \ln \frac{\varepsilon + \beta|x - y|^2}{1 + \beta|x - y|^2} d\mu(x) d\mu(y)$$

Then, by monotone convergence theorem, for any probability measure μ ,

$$J_\beta^0(\mu) = \lim_{\varepsilon \rightarrow 0} J_\beta^\varepsilon(\mu).$$

Moreover, for $0 < \alpha < 1$, we find :

$$\begin{aligned} & (\alpha(1 - \alpha))^{-1} (\alpha J_\beta^\varepsilon(\mu) + (1 - \alpha) J_\beta^\varepsilon(\nu) - J_\beta^\varepsilon(\alpha\mu + (1 - \alpha)\nu)) \\ &= \int_{\mathbb{R}^2} \left(\ln \frac{1 + \beta|x - y|^2}{\varepsilon + \beta|x - y|^2} \right) d(\mu - \mu')(x) d(\mu - \mu')(y) \\ &= \int_{\mathbb{R}^2} \int_\varepsilon^1 \frac{da}{a + \beta|x - y|^2} d(\mu - \mu')(x) d(\mu - \mu')(y) \\ &= \int_{\mathbb{R}^2} \int_\varepsilon^1 \left(\int_0^\infty \exp(-(a + \beta|x - y|^2)t) dt \right) d(\mu - \mu')(x) d(\mu - \mu')(y) da \\ &= \int_{\mathbb{R}^2} \int_\varepsilon^1 \int_0^\infty \left(\int_{g \in \mathbb{R}} e^{-at} e^{ig\sqrt{t\beta}(x-y)} d\gamma(g) \right) dt da d(\mu - \mu')(x) d(\mu - \mu')(y) \end{aligned}$$

where γ is the standard Gaussian measure. Thanks to the regularization, we can use Fubini's theorem. As a consequence, we find that

$$\begin{aligned} & (\alpha(1-\alpha))^{-1}(\alpha J_\beta^\varepsilon(\mu) + (1-\alpha)J_\beta^\varepsilon(\nu) - J_\beta^\varepsilon(\alpha\mu + (1-\alpha)\nu)) \\ &= \int_\varepsilon^1 \int_0^\infty \int_{g \in \mathbb{R}} e^{-at} \left| \int_{\mathbb{R}} e^{ig\sqrt{t\beta}x} d(\mu - \mu')(x) \right|^2 d\gamma(g) dt da \end{aligned}$$

The right hand side is always non-negative so that J_β^ε is convex for every $\varepsilon > 0$ and therefore J_β^0 is convex by passing to the limit. J_β^0 is in fact strictly convex since, if J_β^0 is finite at μ and ν , $J_\beta^0(\alpha\mu + (1-\alpha)\nu)$ is also finite and

$$\begin{aligned} & (\alpha(1-\alpha))^{-1}(\alpha J_\beta^0(\mu) + (1-\alpha)J_\beta^0(\nu) - J_\beta^0(\alpha\mu + (1-\alpha)\nu)) \\ &= \int_0^1 \int_0^\infty \int_{\mathbb{R}} e^{-at} \left| \int_{\mathbb{R}} e^{ig\sqrt{t\beta}x} d(\mu - \mu')(x) \right|^2 d\gamma(g) dt da \end{aligned}$$

The RHS can only vanishes if μ and ν have the same Fourier transforms almost surely which implies $\mu = \nu$. This proves the strict convexity of J_β^V and the uniqueness of its minimizer μ_β^V . As a consequence, if d is a distance compatible with the weak topology, for any $\delta > 0$, there exists $c_\delta > 0$ such that for N large enough

$$\mathbb{Q}_{V_\beta}^N \left(d(\hat{\mu}_X^N, \mu_\beta^V) > \delta \right) \leq e^{-c_\delta N^2}.$$

We take $d(\mu, \nu) = \sup\{|\int f d(\mu - \nu)|\}$ where the supremum is taken over functions f with Lipschitz norm bounded by one.

We finally deduce the bound on $\|Y\|_\infty$. By Lemma 8.13, we know that there exists a finite constant C such that for any interval I_β of length $2\varepsilon_\beta$ with $\varepsilon_\beta^2 \beta \rightarrow \infty$,

$$\mu_\beta^V(I_\beta) \leq C \beta^{\frac{1}{4(2p+1)}} \sqrt{\varepsilon_\beta}.$$

We take $I = [x - \varepsilon_\beta, x + \varepsilon_\beta]$ for a real number x and approximate the indicator function of I_β by a Lipschitz function with support in $[x - 2\varepsilon_\beta, x + 2\varepsilon_\beta]$ and Lipschitz norm bounded by $1/\varepsilon_\beta$. Therefore, for any $\delta > 0$, on $d(\hat{\mu}_X^N, \mu_\beta^V) \leq \delta$, we have

$$(35) \quad \hat{\mu}_X^N([x - \varepsilon_\beta, x + \varepsilon_\beta]) \leq C \beta^{\frac{1}{4(2p+1)}} \sqrt{\varepsilon_\beta} + \frac{1}{\varepsilon_\beta} \delta$$

with probability greater than $e^{-c_\delta N^2}$ for some $c_\delta > 0$. This upper bound can be made uniform on x by using a finite covering of $[-M, M]$ and the fact that $\|X\|_\infty$ stays bounded by some finite M with overwhelming probability since V_β is (η, A) -trapping for some $\eta > 0$, A finite and the trivial partition $\{\{1\}, \{2\}\}$ by Theorem 2.5. We now fix $\delta = C \beta^{\frac{1}{2(2p+1)}} \varepsilon_\beta^{3/2}$. As a consequence, we find a positive finite constant c' such that, with probability greater than

$1 - e^{-cN}$ for some $c > 0$ and N large, if we order the $(x_i)_{1 \leq i \leq N}$ we have

$$(36) \quad \inf \left\{ |x_i - x_j|, |i - j| \geq C\beta^{\frac{1}{4(2p+1)}} \sqrt{\varepsilon_\beta} N \right\} \geq 2\varepsilon_\beta.$$

Indeed, otherwise the interval $I_\beta = [x_i; x_j]$ with x_i, x_j achieving the above minimum would contradict (35). Next, we consider the matrix Y conditionally to the $(x_i)_{1 \leq i \leq N}$ satisfying (36). We first observe that, under this conditional law, the Y_{ij} are independent centered Gaussian variables with variance $N^{-1}(1 + 2\beta(x_i - x_j)^2)^{-1}$. Define $\eta_\beta = C\beta^{\frac{1}{4(2p+1)}} \sqrt{\varepsilon_\beta}$, we make the following decomposition

$$Y = \Delta + G$$

where Δ is the band matrix with entries $(Y_{ij}1_{|i-j| \leq \eta_\beta N})_{i,j}$ and G collects the remaining entries. The variance of these entries is bounded by $1/N$ and therefore, by Lemma 8.7,

$$\mathbb{P}(\|\Delta\|_\infty \geq C\sqrt{\eta_\beta}) \leq e^{-\eta_\beta N}.$$

On the other hand, when (36) holds, the entries of G have variance bounded from above by $N^{-1}(1 + 2\beta(\varepsilon_\beta)^2)^{-1}$. Therefore, if W is a $N \times N$ matrix from the GOE, [26][Lemma 5.4] shows that

$$\|G\|_\infty \leq (1 + 2\beta(\varepsilon_\beta)^2)^{-1/2} \|W\|_\infty$$

so that the large deviation principle for $\|W\|_\infty$ (see [4] or [2, Theorem 2.6.6]) implies that there exists finite non negative constants c, C such that

$$\mathbb{P}\left(\|G\|_\infty \geq 4\left(1 + 2\beta(\varepsilon_\beta)^2\right)^{-1/2}\right) \leq C \exp\{-cN\}.$$

Taking $\varepsilon_\beta = \beta^{-\frac{1}{5}\frac{8p+5}{2(2p+1)}}$ (it is easy to check that we always have $\varepsilon_\beta^2 \beta \rightarrow \infty$) and putting these two bounds together gives a positive finite constant C such that for β and N large enough

$$(37) \quad \mathbb{P}\left(\|Y\|_\infty \geq C\beta^{-\frac{p}{5(2p+1)}}\right) \leq Ce^{-C\beta^{-\frac{p}{10(2p+1)}} N}.$$

□

Amazingly, in the simpler case where both potentials are quadratic, the asymptotic distribution of the matrices can be computed explicitly.

Proposition 7.3 (Asymptotics if both potentials are quadratic). *Let X^N, Y^N be two random matrices following*

$$d\mathbb{Q}_\beta^N(X, Y) = \frac{1}{Z^N(\beta)} \exp(-N \operatorname{Tr}(-\beta[X, Y]^2 + \frac{X^2}{2} + \frac{Y^2}{2})) dX dY.$$

Then under the law \mathbb{Q}_β^N , the sequence of empirical distribution $\hat{\mu}_{\beta^{\frac{1}{6}}X, \beta^{\frac{1}{6}}Y}^N$ of $(\beta^{\frac{1}{6}}X, \beta^{\frac{1}{6}}Y)$ is tight almost surely. Besides, if τ_β is an accumulation point of $\hat{\mu}^N$ then τ_β converges when $\beta \rightarrow \infty$ towards the commutative distribution

of the first two coordinates of a point taken uniformly in $B_{\mathbb{R}^3}(0, 2^{1/4}(3\pi)^{1/6})$. In other words, for any $P \in \mathbb{C}\langle X, Y \rangle$, almost surely,

$$\lim_{\beta \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \text{Tr} P(\beta^{1/6} X^N, \beta^{1/6} Y^N) = \frac{\int_{B_{\mathbb{R}^3}(0, (3\pi)^{1/3})} P(x, y) dx dy dz}{|B_{\mathbb{R}^3}(0, (3\pi)^{1/3})|}$$

and the same result holds if we replace the lim sup by a lim inf.

Remark that the uniform law on $B_{\mathbb{R}^3}(0, (3\pi)^{1/3})$ is in a way a 2-dimensional analog of the semi-circular law seen as the first coordinate of a uniform point on $B_{\mathbb{R}^2}(0, 2)$.

Proof. This is a consequence of the previous results. We know that $\hat{\mu}_{\beta^{1/6} X, \beta^{1/6} Y}^N$ is tight (since both X^N and Y^N have a bounded spectral norm with large probability by (34)) and let τ_β be a limit point. By the previous proposition, the distribution of x and y under τ_β is the minimizer $\mu_\beta^{x^2/2}$ of the rate function $J_\beta^{x^2/2}$. By the second point of Proposition 8.13, we know that the rescaling $\beta^{1/6} \# \mu_\beta^{x^2/2}$ of the equilibrium measure by $\beta^{1/6}$ is supported in $[-L_V, L_V]$. Therefore the distribution of $(\beta^{1/6} x, \beta^{1/6} y)$ under τ_β is tight. Let τ_∞ be a limit point. Moreover, $-[X^N, Y^N]^2$ is a non-negative matrix with a first moment converging to 0 at speed $\frac{\ln \beta}{\beta}$ by Proposition 4.5. Therefore $\tau_\beta(-[\beta^{1/6} x, \beta^{1/6} y]^2) = O(\ln \beta \beta^{-1/3})$ still goes to 0 when β goes to infinity. Hence, τ_∞ is the distribution of commuting variables. Lemma 8.13.4) shows that the distribution of x under τ_∞ is the probability distribution with density $(2\pi)^{-1}(A - x^2/2)_+$ with $A = \frac{1}{2}(3\pi)^{2/3}$. Finally there is only one law of two commutative variables on \mathbb{R}^2 with a distribution invariant by rotation with a fixed marginal distribution. Indeed, the law of two commutative variables is encoded by a probability measure P on \mathbb{R}^2 . Because of the invariance by rotation of P , we see that for any vector $\lambda \in \mathbb{R}^2$, $\langle \lambda, (x, y) \rangle$ has the same law that $\|\lambda\|_2 x$ if (x, y) follows P , which is thus uniquely defined if the law of x is prescribed. Since the distribution of the first two coordinates of an uniform points $B_{\mathbb{R}^3}(0, (3\pi)^{1/3})$ is invariant by rotation and they have this marginal $(2\pi)^{-1}(A - x^2/2)_+$, this must be the distribution τ_∞ . □

Theorem 7.4. *We let V and W be two non-negative polynomials of one variable, going to $+\infty$ at infinity. Let $z_1 < z_2 < \dots < z_k$ be the zeroes of V and assume there exists $c_1, \dots, c_k \in \mathbb{R}^{+*}$ such that, for every $j \in \{1, \dots, k\}$,*

$$V(x) = c_j(x - z_j)^2 + o(x - z_j)^2$$

Assume moreover that W achieves its minimum value (equal to 0) at y_1, \dots, y_ℓ and there exists finite integer numbers p_1, \dots, p_ℓ and positive real numbers d_1, \dots, d_ℓ such that $W(x) \simeq d_j(x - y_j)^{2p_j}$ in a neighborhood of y_j . Let

$$d\mathbb{Q}_{V_\beta}^N(X, Y) = (\mathbb{Z}_{V_\beta}^N)^{-1} e^{-N \text{Tr}(-\beta[X, Y]^2 + W(X) + V(Y))} dX dY.$$

Then, for all $\varepsilon > 0$, $\mathbb{Q}_{V_\beta}^N$ almost surely, we have

$$(38) \quad \lim_{\beta \rightarrow \infty} \liminf_{N \rightarrow \infty} \hat{\mu}_Y^N(\cup_j [z_j - \varepsilon, z_j + \varepsilon]) = 1$$

Moreover, for every $j \in \{1, \dots, k\}$ and $\varepsilon > 0$, we have almost surely

$$\lim_{\beta \rightarrow \infty} \liminf_{N \rightarrow \infty} \hat{\mu}_Y^N([z_j - \varepsilon, z_j + \varepsilon]) = \alpha_j^* := \frac{(c_j)^{-\frac{1}{2}}}{\sum_\ell (c_\ell)^{-\frac{1}{2}}}$$

and the above \liminf can be replaced by a \limsup . Besides,

$$\left(\frac{2\beta N}{\pi}\right)^{N(N-1)/2} Z_{V_\beta}^N = \exp\left(-N^2(c\beta^{-\frac{1}{3}} + o(\beta^{-\frac{1}{3}})) + o(N^2)\right)$$

where $c = c_W(\sum_j c_j^{-\frac{1}{2}})^{-\frac{2}{3}}$ and c_W is a constant depending only on W defined in Lemma 8.13.

Theorem 2.9 follows from Theorem 7.4 by assuming that the integer numbers p_i are all equal to 2 and exchanging the roles of X and Y .

Proof. Compactification : We first show that we can replace W, V by uniformly Lipschitz finite variation functions, up to a quadratic term. First notice that our assumption on V and W and Theorem 2.5 implies that the potential $V_\beta(X, Y) = -\beta[X, Y]^2 + W(X) + V(Y)$ is (η, A) -trapping for some $\eta > 0$, A finite and the trivial partition $\{\{1\}, \{2\}\}$. We may assume $A \geq 0$ without loss of generality. Therefore, Theorem 2.2 implies that the operator norms of both X and Y stay bounded by $L^{\frac{1}{\kappa} + \frac{1}{2}}$ with overwhelming probability, with some L depending only on η and A . As a consequence, taking $M \geq L^{\frac{1}{\kappa} + \frac{1}{2}}$, we can replace $\mathbb{Q}_{V_\beta}^N$ by $\mathbb{Q}_{V_\beta}^{N, M} = 1_{B_M} \mathbb{Q}_{V_\beta}^N / \mathbb{Q}_{V_\beta}^N(B_M)$ where $B_M = \{\|X\|_\infty \leq M\} \cap \{\|Y\|_\infty \leq M\}$. On B_M , we can replace V and W by \tilde{V} and \tilde{W} given for any real numbers x, y by:

$$\tilde{V}(y) = \frac{\eta}{4}y^2 + V(\varphi_M(y)) - \frac{\eta}{4}\varphi_M(y)^2, \tilde{W}(x) = \frac{\eta}{4}x^2 + W(\varphi_M(x)) - \frac{\eta}{4}\varphi_M(x)^2$$

where $\varphi_M(x) = x$ on $|x| \leq M$. We choose next φ_M outside of the ball so that that $\varphi_M(x)x$ is non negative for all real number x , φ_M is increasing and converging to $\pm M'$ when x goes to $\pm\infty$ with some $M' > M$ (e.g $M+1$). We now wish to remove the restriction to the ball B_M to consider instead $\mathbb{Q}_{V_\beta}^N$ with $\tilde{V}_\beta(X, Y) = -\beta[X, Y]^2 + \tilde{W}(X) + \tilde{V}(Y)$. To this end we need to check that we can choose M, M' large enough so that $\mathbb{Q}_{V_\beta}^N(B_M^c)$ is very small (in particular summable so that Borel-Cantelli's lemma applies). It is enough to show that \tilde{V}_β is also (η', A') -trapping again by Theorem 2.2. (η', A') must not depend on M but can be different from (η, A) (up to take M large enough). We have seen that the commutator is $(0, 0)$ -trapping so we only need to check that the functions \tilde{V} and \tilde{W} are $(\eta/2, A)$ -trapping. This amounts to show that for every real number x ,

$$(39) \quad x\tilde{V}'(x) = \frac{\eta}{2}x^2 + (V'(\varphi_M(x)) - \frac{\eta}{2}\varphi_M(x))x\varphi_M'(x) \geq \frac{\eta}{2}x^2 - A.$$

But our choice of φ_M insures that this is true for $|x| \leq M$. Moreover for all x , $x\varphi_M(x)\varphi'_M(x)$ is non-negative and moreover, since V is (η, A) -trapping

$$\left(V'(\varphi_M(x)) - \frac{\eta}{2}\varphi_M(x) \right) \varphi_M(x) \geq \frac{\eta}{2}\varphi_M(x)^2 - A$$

Hence the LHS is non-negative as soon as $M^2 \geq 2A/\eta$ which shows that (39) is true also for $|x| \geq M$. Assuming this, we conclude that \tilde{V}_β is $(\eta/2, A)$ -trapping so that again up to choose M slightly bigger, we conclude that for M large enough (depending on A and η only), $\mathbb{Q}_{\tilde{V}_\beta}^N(B_M^c)$ goes to zero exponentially fast by (4). Finally, observe that $V(\varphi_M(Y)) - \frac{\eta}{4}\varphi_M(Y)^2$ and $W(\varphi_M(y)) - \frac{\eta}{4}\varphi_M(y)^2$ are uniformly Lipschitz and with finite variation. Hence, we may and shall assume in the sequel that V and W are the sum of a quadratic potential and a Lipschitz function with finite variations.

Lower bound : Let $z_1 < \dots < z_k$ be the minimizers of V . Choose $\eta = \frac{1}{2} \min_{i \neq j} |z_i - z_j|$ and $x_\beta = \min(\frac{1}{\ln \beta}, \eta/2)$. Thus $2x_\beta + \eta \leq \min_{i \neq j} |z_i - z_j|$ and x_β goes to zero when β goes to infinity. We first condition by the number of eigenvalues $(\lambda_i)_{1 \leq i \leq N}$ of Y close to each of these minimizers. For $\mathbf{n} = (n_1, \dots, n_k)$ such that $\sum n_i = N$, we denote by

$$I_{\mathbf{n}} := \cap_r \{ \#\{i : |\lambda_i - z_r| < x_\beta\} = n_r \}.$$

Then for any \mathbf{n} such that $\sum n_i = N$,

$$\left(\frac{2\beta N}{\pi} \right)^{N(N-1)/2} Z_{V_\beta}^N \geq Z_{V_\beta}^{\mathbf{n}}$$

where we have set :

$$Z_{V_\beta}^{\mathbf{n}} := \left(\frac{2\beta N}{\pi} \right)^{N(N-1)/2} \int_{I_{\mathbf{n}}} e^{-N \text{Tr}(-\beta[X, Y]^2 + W(X) + V(Y))} dX dY.$$

Next we write the integral as an integral on the eigenvalues of Y :

$$Z_{V_\beta}^{\mathbf{n}} = \int_{I_{\mathbf{n}}} \prod_{i < j} \left(\frac{2\beta N}{\pi} (\lambda_j - \lambda_i)^2 \right) e^{-N(\beta \sum_{i,j} (\lambda_i - \lambda_j)^2 X_{ij}^2 + \text{Tr} W(X) + \sum V_2(\lambda_i))} dX \prod_i d\lambda_i$$

We denote $i \sim j$ if for some $r \in \{1, \dots, k\}$ we have $\lambda_i, \lambda_j \in [z_r - x_\beta, z_r + x_\beta]$. We write $i \not\sim j$ otherwise. We set

$$\tilde{X}_{ij} = 1_{i \sim j} X_{ij}.$$

\tilde{X} is a block diagonal matrix whose blocks on the diagonal $X(1), \dots, X(k)$ are of size n_1, \dots, n_k . We can also recover directly $X(r)$ by writing $X(r) = \pi_r^* X \pi_r$ with π_r the $N \times n_r$ matrix whose coefficients are $(\pi_r)_{ij} = 1$ iff λ_i is the j th eigenvalue in $[z_r - x_\beta, z_r + x_\beta]$ and 0 otherwise.

Because we assumed that $W(x) = ax^2 + \vartheta(x)$ with some uniformly Lipschitz function ϑ and some $a > 0$, we know by Hoffman-Wielandt's lemma

[2][Lemma 2.1.19] that there exists a finite constant C such that

$$\begin{aligned} \frac{1}{N} \operatorname{Tr} W(X) - \frac{1}{N} \operatorname{Tr} W(\tilde{X}) &\leq a \frac{1}{N} \operatorname{Tr} \left((X)^2 - (\tilde{X})^2 \right) + C \left(\frac{1}{N} \operatorname{Tr} (X - \tilde{X})^2 \right)^{\frac{1}{2}} \\ &\leq \frac{C^2}{\sqrt{\beta} - a/2} + \frac{\sqrt{\beta}}{N} \operatorname{Tr} (X - \tilde{X})^2 \end{aligned}$$

where we finally assumed without loss of generality that $\beta > a^2/4$ and used that $\operatorname{Tr} X^2 - \operatorname{Tr} \tilde{X}^2 = \operatorname{Tr} (X - \tilde{X})^2$. As a consequence, we get

$$\begin{aligned} Z_{V_\beta}^{\mathfrak{n}} &\geq e^{O(\frac{N^2}{\sqrt{\beta}})} \int_{I_{\mathfrak{n}}} \prod_{i < j} \left(\frac{2\beta N(\lambda_j - \lambda_i)^2}{\pi} \right) \\ &\quad \times e^{-N(\beta \sum_{i,j} (\lambda_i - \lambda_j)^2 X_{ij}^2 + \operatorname{Tr}(W(\tilde{X}) + \sqrt{\beta}(X - \tilde{X})^2) + \sum V(\lambda_i))} dX \prod_i d\lambda_i \end{aligned}$$

Therefore, integrating over the entries of $X - \tilde{X}$, we deduce:

$$\begin{aligned} Z_{V_\beta}^{\mathfrak{n}} &\geq e^{O(\frac{N^2}{\sqrt{\beta}})} \int_{I_{\mathfrak{n}}} \prod_{i \sim j} \left(\frac{2\beta N(\lambda_j - \lambda_i)^2}{\pi} \right) \prod_{i \not\sim j} \left(\frac{\beta N(\lambda_j - \lambda_i)^2}{\beta N(\lambda_j - \lambda_i)^2 + N\sqrt{\beta}} \right) \\ &\quad \times \exp \left(-N\beta \sum_{i \sim j} (\lambda_i - \lambda_j)^2 X_{ij}^2 - N \operatorname{Tr}(W(\tilde{X}) + V(Y)) \right) d\tilde{X} \prod_i d\lambda_i \end{aligned}$$

Then we break \tilde{X} in the blocks of size n_r along the matrices $X(r)$ observing that the factor $\frac{\beta N(\lambda_j - \lambda_i)^2}{\beta N(\lambda_j - \lambda_i)^2 + N\sqrt{\beta}}$ is uniformly bounded below by $e^{-\frac{1}{\sqrt{\beta}n^2}}$ when $i \sim j$. Moreover, there is $\varepsilon_\beta = O(x_\beta)$ so that if $c_i(\beta) = (1 + \varepsilon_\beta)c_i$, for any $\lambda \in [z_i - x_\beta, z_i + x_\beta]$, $V(\lambda) \leq c_i(\beta)(z_i - \lambda)^2$. Therefore, we find

$$\begin{aligned} Z_{V_\beta}^{\mathfrak{n}} &\geq e^{O(\frac{N^2}{n\sqrt{\beta}})} \prod_r \int_{[z_r - x_\beta, z_r + x_\beta]^{n_r}} dX(r) \prod_i d\lambda_i \prod_{1 \leq i < j \leq n_r} \left(\frac{2\beta N(\lambda_j - \lambda_i)^2}{\pi} \right) \\ &\quad \times \exp \left\{ -N\beta \sum_{1 \leq i < j \leq n_r} (\lambda_i - \lambda_j)^2 X(r)_{ij}^2 - N \operatorname{Tr} W(X(r)) - N c_r(\beta) \sum_{i=1}^{n_r} (\lambda_i - z_r)^2 \right\}. \end{aligned}$$

Now, we diagonalize the matrices $X(r)$ with eigenvalues $\{\mu_i\}_{1 \leq i \leq n_r}$ and integrate over the matrices $Y(r)$ with eigenvalues $\{\lambda_i - z_r, 1 \leq i \leq n_r\}$ to obtain:

$$\begin{aligned} Z_{V_\beta}^{\mathfrak{n}} &\geq e^{O(\frac{N^2}{\sqrt{\beta}})} \prod_r \int_{\|Y(r)\|_\infty < x_\beta} dY(r) \prod_i d\mu_i \prod_{1 \leq i < j \leq n_r} \left(\frac{2\beta N(\mu_j - \mu_i)^2}{\pi} \right) \\ &\quad \times \exp \left\{ -N\beta \sum_{i,j} (\mu_i - \mu_j)^2 Y(r)_{ij}^2 - N c_r(\beta) \operatorname{Tr} Y(r)^2 - N \sum_i W(\mu_i) \right\} \end{aligned}$$

To compute the RHS note that we can remove the indicator function on the set $\|Y(r)\| < x_\beta$ by Proposition 7.2 (34) as long as $x_\beta \gg \beta^{-\frac{p}{5(2p+1)}}$ with

$p = \max p_i$. This is the case since x_β goes to zero like $1/\ln \beta$. Then, we can compute the integral over $Y(r)$, getting, with the notations of Proposition 7.2,

$$\begin{aligned} Z_{V_\beta}^{\mathbf{n}} &\geq e^{O(\frac{N^2}{\sqrt{\beta}})} \prod_r \int \prod \frac{\beta(\mu_j - \mu_i)^2}{c_r(\beta) + \beta(\mu_j - \mu_i)^2} \exp\left(-N \sum W(\mu_i)\right) \prod_i d\mu_i \\ &= e^{O(\frac{N^2}{\sqrt{\beta}})} \prod_r \int \exp\left(-n_r^2 J_{\beta/c_r(\beta), \neq}^{\frac{N}{n_r}} W\left(\frac{1}{n_r} \sum_{i=1}^{n_r} \delta_{\mu_i}\right)\right) \prod_i d\mu_i \end{aligned}$$

Finally if for all r , $n_r/N \rightarrow \alpha_r \in]0; 1[$, we get by Lemma 8.13.1) that

$$\begin{aligned} (40) \quad Z_{V_\beta}^{\mathbf{n}} &\geq e^{O(\frac{N^2}{\sqrt{\beta}}) + o(N^2)} \prod_r \exp(-n_r^2 c_{\beta/c_r(\beta)}^{W/\alpha_r}) \\ &= e^{O(\frac{N^2}{\sqrt{\beta}}) + o_N N^2} e^{-N^2(1+o_\beta)\beta^{-\frac{1}{3}} c_W \sum_r \alpha_r^{2-\frac{1}{3}} c_r(\beta)^{\frac{1}{3}}}, \end{aligned}$$

with o_β (resp. o_N) going to zero as β (resp. N) goes to infinity. At this stage recall that $c_r(\beta)$ goes to c_r as β goes to infinity. Moreover observe that if u_i and k are positive constant, the minimizer of $\alpha \rightarrow \sum_1^n \alpha_i^{1+k} u_i$ in the simplex is such that the x_i are proportional to $u_i^{-1/k}$ and then the value of the minimum is $(\sum_i u_i^{-1/k})^{-k}$.

We apply this with $k = 1 - \frac{1}{3} = \frac{2}{3}$ and $u_r = c_r(\beta)^{\frac{1}{3}} \simeq c_r^{\frac{1}{3}}$ to get that the best lower bound is taken at α_r^* proportional to $c_r^{-\frac{1}{2}}$. To conclude we obtained the lower bound

$$(41) \quad \left(\frac{2\beta N}{\pi}\right)^{N(N-1)/2} Z_{V_\beta}^{\mathbf{n}} \geq \exp\left(-N^2(1+o_\beta(1))\beta^{-\frac{1}{3}} c_W \left(\sum_r c_r^{-1/2}\right)^{-2/3}\right).$$

Upper bound : To get the corresponding upper bound, we first show that, under $\mathbb{Q}_{V_\beta}^N$, the eigenvalues of Y concentrate in a neighborhood of the minimizers of V . We recall that V_β is trapping by Theorem 2.5 and Remark 2.6 so that we can restrict ourselves to the ball $B_M = \{\|X_1\|_\infty \leq M, \|Y\|_\infty \leq M\}$. Next, we take $\kappa > 0$ and choose an interval I such that $V|_I > \kappa > 0$. We fix a positive constant ε and bound from above

$$\Lambda_N(\varepsilon) := \left(\frac{2\beta N}{\pi}\right)^{N(N-1)/2} \int_{\{\hat{\mu}_Y^N(I) \geq \varepsilon\}} e^{-N \text{Tr}(-\beta[X, Y]^2 + W(X) + V(Y))} dX dY$$

by noticing that, since V is non negative, $\text{Tr}V(Y) \geq N\varepsilon\kappa$ on $\{\hat{\mu}_Y(I) \geq \varepsilon\}$. Hence, after diagonalizing X , we find that $\Lambda_N(\varepsilon)$ is bounded from above by

$$e^{-N^2\varepsilon\kappa} \int \prod_{i < j} \left(\frac{2\beta N}{\pi} (\lambda_j - \lambda_i)^2\right) e^{-N\beta \sum_{i,j} (\lambda_i - \lambda_j)^2 |Y_{ij}|^2 - N \sum W(\lambda_i)} dY \prod_i d\lambda_i.$$

Integrating over the Y_{ij} 's gives

$$(42) \quad \Lambda_N(\varepsilon) \leq e^{-N^2\varepsilon\kappa} \int \prod_i d\lambda_i e^{-N \sum W(\lambda_i)} \leq e^{-N^2\varepsilon\kappa + O(N \ln N)}$$

Hence, together with (41), we deduce that $\mathbb{Q}_{V_\beta}^N(\hat{\mu}_Y^N(I) > \varepsilon)$ goes to zero exponentially fast as soon as $\varepsilon\kappa \geq C\beta^{-1/3}$ if C is large enough (and fixed hereafter). We take $\kappa = \kappa(\varepsilon)$ small but fixed so that I^c is included in a ε -neighborhood of the minimizers and assume β small enough $\varepsilon\kappa(\varepsilon) \geq C\beta^{-1/3}$. If there were $o(\beta^{-1/3}N)$ eigenvalues inside I , we could just remove these eigenvalues up to a negligible cost (compared to the partition function) and find again ourselves with eigenvalues in independent wells. To find ourselves in such a situation we shall use a discretization of the set I . Let η be a small positive real number and partition $[z_i + \varepsilon, z_i + 2\varepsilon]$ and $[z_i - 2\varepsilon, z_i - \varepsilon]$ into intervals of width $2\eta \ll \varepsilon$ for $i \in \{1, \dots, k\}$. There are about $k\varepsilon/\eta$ such intervals and their union lies inside I which has at most $C\beta^{-1/3}N$ eigenvalues. Therefore, on the set $\{\hat{\mu}_Y(I) \leq C\beta^{-1/3}\}$, we know that there must exist an interval $J_i^+ = [z_i + \varepsilon + 2\ell_i^+\eta, z_i + \varepsilon + 2(\ell_i^+ + 1)\eta]$ and $J_i^- = [z_i - \varepsilon - 2\ell_i^-\eta, z_i - \varepsilon - 2(\ell_i^- - 1)\eta]$ for some integer ℓ_i^\pm in which there are at most $2C\eta\beta^{-1/3}N$ eigenvalues. Even though ℓ_i^\pm are random, we have at most $(C\varepsilon/2\eta)^{2k}$ choices for these numbers and therefore can just optimize over their choice at the end. We hence consider these intervals fixed. We let $V_i = [z_i - \varepsilon - 2(\ell_i^- - 1)\eta, z_i + \varepsilon + 2\ell_i^+\eta]$ and $F_i = [z_i + \varepsilon + 2(\ell_i^+ + 1)\eta, z_{i+1} - \varepsilon - 2\ell_{i+1}^-\eta]$, $F_0 = (-\infty, z_1 - \varepsilon - 2\ell_1^-]$ and $F_k = [z_k + \varepsilon + 2\ell_k^+\eta, +\infty)$. We let $k \simeq j$ (resp. $k \sim j$) iff there exists i such that both λ_k, λ_j belong to the same V_i or the same F_i (resp. λ_k, λ_j both belong to the same J_i^+ or in J_i^-). Then we set

$$\tilde{X}_{k,\ell} = X_{k\ell} \mathbf{1}_{k \simeq \ell}, \bar{X}_{k\ell} = X_{k\ell} \mathbf{1}_{k \sim \ell}.$$

Then, \bar{X} has at most rank $C\eta\beta^{-1/3}N$ so that since $W(x) = ax^2 + \vartheta(x)$ with ϑ with finite variations,

$$\mathrm{Tr} W(X) - \mathrm{Tr} W(X - \bar{X}) \geq a \mathrm{Tr} \bar{X}^2 - \|\vartheta\|_{TV} N C \eta \beta^{-1/3}.$$

Moreover, because ϑ is Lipschitz,

$$\mathrm{Tr} W(X - \bar{X}) - \mathrm{Tr} W(\tilde{X}) \geq a \mathrm{Tr} (X - \bar{X} - \tilde{X})^2 - \|\vartheta\|_L \sqrt{N} \left(\mathrm{Tr} |X - \bar{X} - \tilde{X}|^2 \right)^{1/2}.$$

Therefore if we let $I_{\mathbf{n}, \mathbf{m}}$ be the set where there are n_i eigenvalues in V_i , m_i eigenvalues in F_i , $\sum (n_i + m_i) \geq N - C\eta\beta^{-1/3}N$, we get after integrating over the entries of \bar{X} (which are very few) and the entries of $X - \bar{X} - \tilde{X}$ (which are such that $|\lambda_i - \lambda_j| \geq 2\eta$. Recall we only imposed that $\varepsilon \gg \eta$ with ε fixed such that $\varepsilon\kappa(\varepsilon) \gg \beta^{-1/3}$. We choose $\eta \gg C\beta^{-1/12}$ so that the contribution of the expectations over the entries of $X - \bar{X} - \tilde{X}$ is negligible (i.e such that $(\sqrt{\beta}\eta^2)^{-1} \ll \beta^{-1/3}$, that is $\eta \gg \beta^{-1/12}$))

$$\begin{aligned}
 Q_{V_\beta}^N(I_{\mathbf{n},\mathbf{m}}) &\leq e^{cN^2\beta^{-1/3}(1+o_{\beta,\eta})} \int_{I_{\mathbf{n},\mathbf{m}}} \prod_{i \simeq j} \left(\frac{2\beta N(\lambda_j - \lambda_i)^2}{\pi} \right) \\
 &\times \exp \left(-N\beta \sum_{i \simeq j} (\lambda_i - \lambda_j)^2 X_{ij}^2 - N\text{Tr}(W(\tilde{X})) - N \sum V(\lambda_i) \right) d\tilde{X} \prod_i d\lambda_i
 \end{aligned}$$

where $o_{\beta,\eta}$ goes to zero when β goes to infinity and η to zero. We can next decompose \tilde{X} in block diagonal matrices corresponding to entries with eigenvalues either both in a V_i or in a F_i and remark that the above integral factorizes and so for $B_r = V_r$ or F_r , we have

$$\begin{aligned}
 Q_{V_\beta}^N(I_{\mathbf{n},\mathbf{m}}) &\leq e^{cN^2\beta^{-1/3}(1+o_{\beta,\eta})} \prod_{1 \leq r \leq k} R_{n_r,r}(V_r) \prod_{0 \leq r \leq k} R_{m_r,r}(F_r) \\
 R_{n_r,r}(B_r) &:= \int_{\text{spec}(Y) \in B_r} \prod_{1 \leq i < j \leq n_r} \left(\frac{2\beta N(\mu_j - \mu_i)^2}{\pi} \right) \\
 &\times \exp \left(-N\beta \sum (\mu_i - \mu_j)^2 Y_{ij}^2 - N \sum W(\mu_i) - N \text{Tr}(V(Y)) \right) dY \prod_i d\mu_i
 \end{aligned}$$

We can bound $R_{n_r,r}(F_r)$ exactly as in (42) as V is lower bounded by $\delta_\varepsilon = \inf_{|x-z_i| \geq \varepsilon} V(x)$ on F_k . On V_r , the eigenvalues are close to z_r and we can therefore approximate V by $c_r(\varepsilon)(x - z_r)^2$ with some $c_r(\varepsilon)$ going to c_r as ε goes to zero and use Proposition 7.2. Hence, we find

$$Q_{V_\beta}^N(I_{\mathbf{n},\mathbf{m}}) \leq e^{cN^2\beta^{-1/3}(1+o_{\beta,\eta})} e^{-\delta_\varepsilon \sum m_i^2} \prod_{1 \leq r \leq k} e^{-n_r^2 c_{2\beta/c_r(\varepsilon)}^{NW/n_r}}$$

For β large enough, the above right hand side is maximized when $m_r = 0$ (as δ_ε is bounded below independently of β) and the probability that $\max m_i \geq 1$ goes exponentially fast to zero. Moreover, thanks to Lemma 8.13.1)

$$Q_{V_\beta}^N(I_{\mathbf{n}}) \leq e^{cN^2\beta^{-1/3}(1+o_{\beta,\eta})} e^{-N^2\beta^{-1/3}c_W \sum_r c_r^{1/3} (\frac{n_r}{N})^{5/3}}$$

from which we see by summing over all possible filling fractions n_r/N that it converges towards the optimizer of $\alpha \rightarrow \sum_r c_r^{1/3} (\alpha_r)^{5/3}$ on $\sum \alpha_i = 1$, which is given by $(c_r^{-1/2} / \sum c_r^{-1/2})_{1 \leq r \leq k}$. \square

8. APPENDIX

8.1. Random matrix inequalities by flow. In this paper, many controls are based on change of variables. The most basic one is the following:

Lemma 8.1 (Inequality by mapping). *Let $f : \mathbb{R}^d \mapsto \mathbb{R}^+$ be a measurable function with finite L^1 norm $\int f(x)dx$. Let X be a \mathbb{R}^d valued random variable with distribution μ absolutely continuous with respect to Lebesgue measure and density proportional to f . If $\phi : A \rightarrow \mathbb{R}^d$ is a C^1 diffeomorphism onto its image then*

$$\mathbb{P}(X \in A) \leq \sup_{x \in A} \frac{f(x)}{f(\phi(x))J_\phi(x)}$$

where J_ϕ is the Jacobian of $\phi : J_\phi(x) = |\det(\partial_i \phi_j(x))|$.

Proof. If f is the density of μ (up to multiplication by a constant c)

$$\begin{aligned} \mathbb{P}(X \in A) &= \int_A cf(x)dx \leq \sup_{x \in A} \frac{f(x)}{f(\phi(x))J_\phi(x)} \int_A cf(\phi(x))J_\phi(x)dx \\ &\leq \sup_{x \in A} \frac{f(x)}{f(\phi(x))J_\phi(x)} \mathbb{P}(\phi^{-1}(X) \in A) \end{aligned}$$

where we did the change of variable $x \rightarrow \phi(x)$. The lemma follows since the Jacobian and f are non negative. \square

Change of variables were already used to derive concentration of measure, see e.g. [30, Section 6.2], in the context of unitary matrices. However, this strategy may be difficult to apply in non compact manifolds as the reminders may be large. The strategy to use flows allows to bypass this issue.

Lemma 8.2 (Inequality by flow). *Let X be a \mathbb{R}^d valued random variable with distribution $\mu(dx) = ce^{-W(x)}dx$ with $W : \mathbb{R}^d \rightarrow \mathbb{R}$ a C^1 function going to infinity fast enough so that $c^{-1} = \int e^{-W(x)}dx$ is finite. Assume that there exists a flow $(Y_t^x)_{x \in \mathbb{R}^d, t \in \mathbb{R}^+}$ on \mathbb{R}^d such that $Y_0^x = x$ and $\partial_t Y_t^x = Z_{Y_t^x}$ with Z_x a C^1 vector field from \mathbb{R}^d into itself. We fix $t > 0$ and assume that for $s \leq t$, $x \rightarrow Y_s^x$ is a diffeomorphism onto its image. Then, for any finite time $t \geq 0$,*

$$\mathbb{P}(X \in A) \leq \sup_{x \in A} \exp \left(\int_0^t (\langle \nabla_{Y_s^x} W, Z_{Y_s^x} \rangle - \text{Tr Diff}_{Y_s^x} Z) ds \right)$$

where $\langle x, y \rangle$ is the scalar product on \mathbb{R}^d .

Proof. This is a direct application of the previous Lemma with $f(x) = \exp(-W(x))$ and $\phi(x) = Y_t^x$. Because we assumed that $x \rightarrow Y_s^x$ is a diffeomorphism onto its image, for any given x in a bounded subset of \mathbb{R}^d , Y_t^x stays bounded and therefore we may assume without loss of generality that $x \rightarrow Z_x$ is uniformly Lipschitz.

We just need to compute

$$\begin{aligned} \frac{f(x)}{f(\phi(x))} &= \exp(-W(x) + W(Y_t^x)) = \exp\left(\int_0^t \partial_s W(Y_s^x) ds\right) \\ &= \exp\left(\int_0^t \langle \nabla_{Y_s^x} W, Z_{Y_s^x} \rangle ds\right) \end{aligned}$$

Moreover, we can compute the Jacobian of this change of variables by noticing that

$$\partial_t \text{Diff } Y_t^x = \text{Diff } \partial_t Y_t^x = \text{Diff } Z_{Y_t^x} = \text{Diff}_{Y_t^x} Z \text{Diff } Y_t^x$$

Now since for A invertible: $\text{Diff}_A \det .H = \det(A) \text{Tr}(A^{-1}H)$, we deduce that

$$\begin{aligned} \partial_t \det \text{Diff } Y_t^x &= \text{Diff}_{Y_t^x} \det .\partial_t \text{Diff } Y_t^x \\ &= \det \text{Diff } Y_t^x \text{Tr}((\text{Diff}_{Y_t^x})^{-1} \text{Diff}_{Y_t^x} Z \text{Diff } Y_t^x) \\ &= \det \text{Diff } Y_t^x \text{Tr } \text{Diff}_{Y_t^x} Z \end{aligned}$$

This computation is true whenever $\text{Diff } Y_t^x$ is invertible i.e. $\det \text{Diff } Y_t^x \neq 0$, which is true at $t = 0$ and thus for small time s till some time $u > 0$. Thus, we can solve the ODE :

$$\det \text{Diff } Y_u^x = \exp\left(\int_0^u \text{Tr } \text{Diff}_{Y_s^x} Z ds\right).$$

Because the RHS is uniformly bounded below as Y_s^x stays uniformly bounded, we conclude that we can take $u = t$. Finally, putting things together we get:

$$\frac{f(x)}{f \circ \phi(x) J_\phi(x)} = \exp\left(\int_0^t (\langle \nabla_{Y_s^x} W, Z_{Y_s^x} \rangle - \text{Tr } \text{Diff}_{Y_s^x} Z) ds\right)$$

and we can apply the previous Lemma. □

We deduce from this an inequality by flow for matrix models.

Lemma 8.3 (A random matrix inequality). *Let $X^N = (X_1^N, \dots, X_\ell^N)$ be random Hermitian matrices with law \mathbb{P} proportional to*

$$\exp(-N \text{Tr } V(X^N)) dX^N.$$

Let X_t^N be the solution of $X_0^N = X^N$ and $\partial_t(X_i^N)_t = -g_i(X_t^N)$. Then

$$(43) \quad \mathbb{P}(X^N \in A) \leq \sup_{X_0^N \in A} \exp\left(-\int_0^t (N \text{Tr } \mathcal{D}V(X_s^N) \cdot g(X_s^N) - \text{Tr} \otimes \text{Tr}(\partial g(X_s^N))) ds\right).$$

Here $A \cdot B = \sum_{i=1}^\ell A_i B_i$ and $g_i(X) = P_i(f_1(X_1), \dots, f_\ell(X_\ell))$ with self-adjoint non-commutative polynomials P_1, \dots, P_ℓ and C^1 bounded functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$. Moreover, we set for $f \in C^1$, $\partial f(X) = \int_0^1 f'(\alpha X \otimes 1 + (1-\alpha)1 \otimes X) d\alpha$ and with $A \otimes B \# (C \otimes D) = AC \otimes DB$,

$$\partial g(X) = \sum_{i=1}^\ell (\partial_{X_i} P_i)(f(X)) \# \partial f_i(X_i).$$

Proof. This is again a direct application of the previous Lemma since g is uniformly Lipschitz so that the solution of

$$\partial_t X_t^N = Z_{X_t^N}$$

with $Z_X = g(X)$ is such that $X_0^N \rightarrow X_t^N$ is a diffeomorphism onto its image. One notices that

$$\partial_s N \operatorname{Tr} V(X_s^N) = -\nabla_{X_s^N}(N \operatorname{Tr} V) \cdot Z_{X_s^N} = -N \operatorname{Tr} \mathcal{D}V(X_s^N) \cdot g(X_s^N),$$

whereas

$$\begin{aligned} \operatorname{Tr} \operatorname{Diff}_{X_s^N} Z_{X_s^N} &= \sum_{p=1}^{\ell} \sum_{i,j=1}^N 2^{-1_{i \neq j}} (\partial_{\Re(X_p(i,j))} + i \partial_{\Im(X_p(i,j))}) g_p(X)(i,j)(X_s^N) \\ &= \operatorname{Tr} \otimes \operatorname{Tr} \partial g(X_s^N) \end{aligned}$$

where the term $\partial_{\Im(X_p(i,i))}$ should be put to zero. Observing that $(\partial_{\Re(X_p(i,j))} + i \partial_{\Im(X_p(i,j))}) X_{k\ell} = 2^{1_{i \neq j}} 1_{ij=\ell k}$ gives the last equality and completes the proof. \square

Lemma 8.4. *For matrices $X^N = (X_1^N, \dots, X_\ell^N)$ we define the following dynamic in the space of $N \times N$ Hermitian matrices :*

$$\begin{cases} X_i^N(0) &= X_i^N \\ Z^N(t) &= (X_1^N(t))^2 + \dots + (X_\ell^N(t))^2 \\ \partial_t X_i^N(t) &= -((Z^N(t))^k X_i^N(t) + X_i^N(t)(Z^N(t))^k) \end{cases}$$

Then, for any $t > 0$, $X_0^N \rightarrow X_t^N$ is a diffeomorphism onto its image so that (43) extends to the polynomial $g_i(X) = (\sum X_j^2)^k X_i + X_i (\sum X_j^2)^k$. Moreover, for all n , $\frac{1}{N} \operatorname{Tr}(Z^N(t))^n$ is a decreasing function of t . Furthermore, for any positive real number B such that $\frac{1}{N} \operatorname{Tr}(Z^N(0))^{k+1} > B$, for $t \in [0, \frac{1}{N^{\frac{k}{k+1}}}]$,

$$\frac{1}{N} \operatorname{Tr}(Z^N(t))^{k+1} \geq B - 4(k+1)t N^{\frac{k}{k+1}} B^{\frac{2k+1}{k+1}}.$$

Proof. Because g is locally Lipschitz, X_t^N is well defined until a short time from any given initial condition. To show it is globally well-defined it is enough to show that the spectral radius of the solution remains bounded in time. But

$$\begin{aligned} \frac{\partial}{\partial t} \operatorname{Tr}(Z^N(t))^n &= -2n \operatorname{Tr}(Z^N(t))^{n+k} \\ &\quad - 2n \sum_{i=1}^{\ell} \operatorname{Tr} \left(X_i^N(t)(Z^N(t))^k X_i^N(t)(Z^N(t))^{n-1} \right) \end{aligned}$$

which is non-positive since $\operatorname{Tr} X A X B = \operatorname{Tr}(B^{1/2} X A^{1/2})(B^{1/2} X A^{1/2})^*$ is non-negative if A and B are non-negative and X self-adjoint. Therefore $\operatorname{Tr}(Z^N(t))^n$ is decreasing in t , and hence the spectral radius of the solution is bounded by the one of the initial condition. This is enough to guarantee that $X_0^N \rightarrow X_t^N$ is a diffeomorphism onto its image and therefore extend the inequality (43) to this case. Besides, using Lemma 8.5 for the second term in the above inequality, we get

$$\begin{aligned}
 \frac{\partial}{\partial t} \operatorname{Tr}(Z^N(t))^{k+1} &\geq -4(k+1) \operatorname{Tr}(Z^N(t))^{2k+1} \\
 &\geq -4(k+1) (\|Z^N(t)\|_\infty^{k+1})^{\frac{k}{k+1}} \operatorname{Tr}(Z^N(t))^{k+1} \\
 &\geq -4(k+1) (\operatorname{Tr}(Z^N(t))^{k+1})^{\frac{k}{k+1}} \operatorname{Tr}(Z^N(t))^{k+1} \\
 &= -4(k+1) (\operatorname{Tr}(Z^N(t))^{k+1})^{\frac{2k+1}{k+1}}.
 \end{aligned}$$

As a consequence, $y(t) = \operatorname{Tr}(Z^N(t))^{k+1}$ satisfies the one dimensional differential inequality

$$y' \geq -4(k+1)y^{\frac{2k+1}{k+1}}$$

starting from $y(0) \geq NB$. Therefore, solving this differential inequality gives,

$$y(t) \geq (4kt + y(0)^{-\frac{k}{k+1}})^{-\frac{k+1}{k}} = y(0)(1 + 4kt y(0)^{\frac{k}{k+1}})^{-\frac{k+1}{k}}.$$

As a consequence, if $y(0) \geq BN$ and $t \leq 1/N^{\frac{k}{k+1}}$, using that for $x \geq 0$, $(1+x)^{-\frac{k+1}{k}} \geq 1 - \frac{k+1}{k}x$, we get

$$\begin{aligned}
 \operatorname{Tr}(Z^N(t))^{k+1} &= y(t) \geq BN(1 + 4kt(BN)^{\frac{k}{k+1}})^{-\frac{k+1}{k}} \\
 &\geq BN(1 - 4(k+1)t(BN)^{\frac{k}{k+1}}).
 \end{aligned}$$

□

8.2. Matrix inequalities.

Lemma 8.5. *If X, Y are two $N \times N$ hermitian matrices and u and v are integers then if either $u + v$ is even or X is positive.*

$$\operatorname{Tr}(X^{u+v}Y^2) \geq \operatorname{Tr}(X^uYX^vY).$$

Proof. To prove this, we may assume that X is a diagonal matrix. If the x_a 's are the eigenvalues of X then,

$$\begin{aligned}
 \operatorname{Tr}(X^{u+v}Y^2) - \operatorname{Tr}(X^uYX^vY) &= \sum_{1 \leq a, b \leq N} |Y_{ab}|^2 x_a^{u+v} - \sum_{1 \leq a, b \leq N} |Y_{ab}|^2 x_a^u x_b^v \\
 &= \frac{1}{2} \sum_{1 \leq a, b \leq N} |Y_{ab}|^2 (x_a^u - x_b^u)(x_a^v - x_b^v)
 \end{aligned}$$

which is non-negative if the x_a 's are positive or if u and v have the same parity. □

Lemma 8.6. *For all $D \in \mathbb{N}$, $\ell \in \mathbb{N}^*$ for any Hermitian matrices, X_1^N, \dots, X_ℓ^N if $Z^N = (X_1^N)^2 + \dots + (X_\ell^N)^2$,*

$$\operatorname{Tr}((Z^N)^{k+D}) \leq \ell^D \operatorname{Tr}((Z^N)^k \sum_{i=1}^{\ell} (X_i^N)^{2D}).$$

Proof. A difficulty of this result is that in general $(X + Y)^{2D}$ is not smaller than $4^D(X^{2D} + Y^{2D})$ as in the commutative case.

First step : $k = 0$. We first prove the case $k = 0$ by using the non-commutative Hölder's inequality: for any choice of $i_1, \dots, i_{2D} \in \{1, \dots, \ell\}^{2D}$,

$$\mathrm{Tr}(X_{i_1}^N \dots X_{i_{2D}}^N) \leq \max_i \|X_i^N\|_{2D}^{2D}.$$

Therefore after expanding the product we get :

$$\mathrm{Tr}(Z^N)^D \leq \ell^D \max_i \mathrm{Tr}(X_i^N)^{2D}.$$

Second step : general case We apply the case $k = 0$ to the new matrices

$$\tilde{X}_i^N = ((Z^N)^{\frac{k}{2D}} (X_i^N)^2 (Z^N)^{\frac{k}{2D}})^{\frac{1}{2}}$$

$$\tilde{Z}^N = (Z^N)^{\frac{k}{2D}} ((X_1^N)^2 + \dots + (X_\ell^N)^2) (Z^N)^{\frac{k}{2D}} = (Z^N)^{\frac{k+D}{D}}$$

Therefore,

$$\mathrm{Tr}((Z^N)^{k+D}) = \mathrm{Tr}((\tilde{Z}^N)^D) \leq \ell^D \sum_{i=1}^{\ell} \mathrm{Tr}((Z^N)^{\frac{k}{2D}} (X_i^N)^2 (Z^N)^{\frac{k}{2D}})^D.$$

Finally we use the Araki-Lieb-Thirring inequality [38] on the right hand side

$$\mathrm{Tr}((Z^N)^{k+D}) \leq \ell^D \sum_{i=1}^{\ell} \mathrm{Tr}((Z^N)^{\frac{k}{2}} (X_i^N)^{2D} (Z^N)^{\frac{k}{2}}).$$

□

Lemma 8.7. *If X^N is an Hermitian matrix with Gaussian independent centered entries with variance uniformly bounded by K and such that $X_{ij}^N = 0$ if $|i - j| > uN$, then with probability greater than $1 - \frac{2}{u}e^{-uN}$*

$$\|X^N\|_{\infty} \leq CK\sqrt{uN}$$

for some universal constant C .

Proof. Define $a_k^N = \lfloor kuN \rfloor$ and by induction (with convention $a_{-1}^N = -\infty$) define for $k \geq 0$,

$$A_{ij}^{N,k} = X_{ij}^N (1_{a_k^N \leq i, j < a_{k+2}^N} - 1_{a_{k-1}^N \leq i, j < a_{k+1}^N})_+$$

Since $A_{ij}^{N,k}$ has only non zero coefficients in a block of size at most $2uN + 1$ we deduce from [44, Corollary 4.4.8] that

$$\mathbb{P}(\|A^{N,k}\| \geq CK(\sqrt{2uN} + t)) \leq 4e^{-t^2}$$

Now

$$X^N = A^N + B^N$$

with

$$A^N = A^{N,0} + A^{N,2} + \dots + A^{N,2\lfloor(2u)^{-1}\rfloor+2}, B^N = A^{N,1} + A^{N,3} + \dots + A^{N,2\lfloor(2u)^{-1}\rfloor+3}$$

Moreover A^N has independent blocks coming from each $A^{N,2k}$

$$\begin{aligned} & P(\|A^N\|_\infty \geq CK(\sqrt{2uN} + t)) \\ &= 1 - \prod_k (1 - \mathbb{P}(\|A^{N,2k}\|_\infty \geq CK(\sqrt{2uN} + t))) \\ &\leq 1 - (1 - \sup_k \mathbb{P}(\|A^{N,2k}\|_\infty \geq CK(\sqrt{2uN} + t)))^{\lfloor (2u)^{-1} \rfloor} \leq \frac{1}{u} e^{-t^2} \end{aligned}$$

The same result holds for B_N and therefore the result follows for $X^N = A^N + B^N$. Taking $t = \sqrt{uN}$ allows to conclude. \square

8.3. Change of variables in one-matrix models. We compute the Jacobian of a change of variables acting only on the eigenvalues of matrices.

Lemma 8.8 (Jacobian of a function of hermitian matrices). *Let f be a C^1 function from \mathbb{R} into \mathbb{R} . Let ϕ be a function from $\mathcal{H}^N(\mathbb{C})$ into $\mathcal{H}^N(\mathbb{C})$ acting on the eigenvalues $\lambda = (\lambda_1 > \dots > \lambda_N)$ so that if $\text{Diag}(\lambda)$ is the diagonal matrix with entries λ and U is any unitary matrix,*

$$\phi(U^* \text{Diag}(\lambda) U) = U^* \text{Diag}(f(\lambda)) U.$$

Then if $\Delta(x) = \prod_{i < j} (x_j - x_i)$ is the VanderMonde determinant, then the Jacobian of ϕ is

$$J_\phi = \left(\frac{\Delta(f(\lambda))}{\Delta(\lambda)} \right)^2 e^{\text{Tr} \ln f'(X)}.$$

Remark 8.9. *In fact we only need f to be C^1 in a neighborhood of the spectrum.*

Proof. We decompose the map ϕ in three parts : first the reduction of X into its eigenvectors, eigenvalues. The Jacobian of this change of variable is $\Delta(\lambda)^{-2}$. Then f is applied to each eigenvalue which creates the factor $\prod_i f'(\lambda_i)$. Finally the last factor comes from the map $(U, f(\Lambda)) \rightarrow U^* f(\Lambda) U$. Roughly, if $W_X \subset V_X$ is the subset of hermitian matrices where f is still C^1 in each eigenvalue,

$$\begin{aligned} \int_{W_X} h(\phi(X)) J_\phi dX &= \int_{\phi(W_X)} h(X) dX = \int_{\phi(W_X)} h(U^* \Lambda U) \Delta(\Lambda)^2 d\Lambda dU \\ &= \int_{W_X} h(U^* f(\Lambda) U) \Delta(f(\Lambda))^2 \prod f'(\lambda_i) d\Lambda dU \\ &= \int_{W_X} h(f(U^* \Lambda) U) \Delta(f(\Lambda))^2 \prod f'(\lambda_i) d\Lambda dU \\ &= \int_{W_X} h(f(X)) \Delta(f(\Lambda))^2 \prod f'(\lambda_i) / \Delta(\Lambda)^2 dX \end{aligned}$$

And we identify J_ϕ by comparing the first and last integral. \square

The main result of this section is to construct a change of variables mapping the eigenvalues outside some interval I to the neighborhood of some point and bound its Jacobian from below.

Lemma 8.10. *Let I be an open set. For $n \in \{0, \dots, N\}$ and $(x_1, \dots, x_n) \in I^n$, let $\mathcal{H}_L^I(x_1, \dots, x_n)$ be the set of $X \in \mathcal{H}^N(\mathbb{C})$ with eigenvalues in the interior of I^c except for n of them which are equal to (x_1, \dots, x_n) . Let α be a real number and fix $\varepsilon \in (0, 4)$. Then, there exists an injective map ϕ from $\mathcal{H}_L^I(x_1, \dots, x_n)$ into $\mathcal{H}^N(\mathbb{C})$ such that if $X = U^* \text{Diag}(x_1, \dots, x_n, \lambda_{n+1}, \dots, \lambda_N)U$ with U unitary and $\lambda_{n+1}, \dots, \lambda_N$ belongs to the interior of I^c then*

$$\phi(X) = U^* \text{Diag}(x_1, \dots, x_n, \phi_{n+1}, \dots, \phi_N)U$$

with the ϕ_i in $] \alpha - \varepsilon; \alpha + \varepsilon [$ and

$$J_\phi \geq \left(\frac{\varepsilon}{4e} \right)^{3N(N-n)} e^{-2N \text{Tr} \ln(1+|X|1_{X \notin I}) - 2(N-n) \text{Tr} \ln(1+|X|1_{X \in I})}$$

As an immediate application with $I = \emptyset$ we see that there is an injective map $\phi : \mathcal{H}^N(\mathbb{C}) \rightarrow \{X \in \mathcal{H}^N(\mathbb{C}) : \|X - \alpha\| < \varepsilon\}$ such that

$$J_\phi \geq \left(\frac{\varepsilon}{4e} \right)^{3N^2} e^{-2N \text{Tr} \ln(1+|X|)}$$

A key step of the proof is the next lemma.

Lemma 8.11. *Let I be a bounded close interval and x_1, \dots, x_n in \mathbb{R} be fixed. Let $0 < \eta < 1$. Then, there exists y in I at distance greater than $\eta \frac{|I|}{2n}$ from $\{x_1, \dots, x_n\}$ such that*

$$\sum_{i=1}^n \ln \left(|x_i - y| - \eta \frac{|I|}{2n} \right) \geq n \ln \frac{|I|(1-\eta)}{2e}.$$

More generally, for any integer number $N \geq n+1$, there exists (x_{n+1}, \dots, x_N) at distance greater than $\eta \frac{|I|}{2N}$ from $\{x_1, \dots, x_n\}$ such that for all $j \in [n+1, N]$,

$$\sum_{i=1}^{j-1} \ln \left(|x_i - x_j| - \eta \frac{|I|}{2N} \right) \geq (j-1) \ln \frac{|I|(1-\eta)}{2e}.$$

Proof. Let $\varepsilon = \eta \frac{|I|}{2n}$ and define

$$V_{I,x,\varepsilon} = \sup_{y \in I} \sum_{i=1}^n \ln(|x_i - y| - \varepsilon).$$

Observe that by a simple dilatation of the space by a factor $\alpha > 0$ we have $V_{\alpha I, \alpha x, \alpha \varepsilon} = V_{I,x,\varepsilon} + n \ln \alpha$.

Now since $\varepsilon < \frac{|I|}{2n}$, we can consider Y to be the random variable uniformly distributed on

$$E = I \setminus \bigcup_{i=1}^n]x_i - \varepsilon; x_i + \varepsilon[.$$

and $U = \sum_{i=1}^n \ln(|x_i - Y| - \varepsilon)$. Since $\ln|x|$ is locally integrable, U is L^1 and its expectation is bounded from below

$$\mathbb{E}[U] = \frac{1}{|E|} \int_E \sum_{i=1}^n \ln(|x_i - y| - \varepsilon) dy \geq \frac{n}{|E|} \int_{-1}^1 \ln(|y|) dy \geq \frac{-2n}{|I| - 2n\varepsilon}$$

As a consequence

$$V_{I,x,\varepsilon} \geq \mathbb{E}[U] \geq \frac{-2n}{|I| - 2n\varepsilon}$$

But due to the scaling argument we see that

$$V_{I,x,\varepsilon} \geq \sup_{\alpha > 0} V_{\alpha^{-1}I, \alpha^{-1}x, \alpha^{-1}\varepsilon} + n \ln \alpha \geq \sup_{\alpha > 0} \left\{ \frac{-2n\alpha}{|I| - 2n\varepsilon} + n \ln \alpha \right\}$$

Taking the optimal value $\alpha = (1/2)(|I| - 2n\varepsilon)$ shows that

$$V_{I,x,\varepsilon} = \sup_{y \in I} \sum_{i=1}^n \ln(|x_i - y| - \varepsilon) \geq n \ln \frac{|I|(1 - \eta)}{2e}.$$

Finally, since $y \mapsto \sum_{i=1}^n \ln(|x_i - y| - \varepsilon)$ is continuous and concave on each of the interval when it is not equal to $-\infty$, it achieves its maximal value on the closed set I . We take y to be such a maximizer.

The generalization is straightforward if we assume that the logarithm in the right hand side is negative (because in that case we can add the points one by one using the above argument). To conclude we again use the scale invariance which allow us to go back to the case where the right hand side is negative. □

Proof of Lemma 8.10. In the following, (x_1, \dots, x_n) are fixed and we wish to map $(\lambda_{n+1}, \dots, \lambda_N)$ to some neighborhood of α at sufficient distance from the x_i 's to bound from below the associated Jacobian. We first choose a family of points $\{x_{n+1}, \dots, x_N\}$ as in Lemma 8.11 with $I =]\alpha - \varepsilon; \alpha + \varepsilon[$ and fix them: they are some measurable functions of $\{x_1, \dots, x_n\}$. We construct ϕ_{n+k} close to x_{n+k} for $k \in [1, N - n]$ by putting

$$\phi_{n+k} = x_{n+k} + \frac{\varepsilon}{8N} \frac{\lambda_{n+k}}{1 + |\lambda_{n+k}|} = x_{n+k} + \frac{\varepsilon}{8N} \psi(\lambda_{n+k})$$

with $\phi(x) = x(1 + |x|)^{-1}$. Since ϕ is a diffeomorphism from \mathbb{R} into $[-1, 1]$, $\psi : (x_1, \dots, x_n, \lambda_{n+1}, \dots, \lambda_N) \mapsto (x_1, \dots, x_n, \phi_{n+1}, \dots, \phi_N)$ is also a diffeomorphism. Now we use Lemma 8.8 to compute the Jacobian J_ϕ of ϕ which is equal to

$$\left(\prod_{\substack{1 \leq i \leq n \\ 1 \leq k \leq N-n}} \frac{|\phi_{n+k} - x_i|}{|\lambda_{n+k} - x_i|} \prod_{1 \leq k \leq k' \leq N-n} \frac{|\phi_{n+k} - \phi_{n+k'}|}{|\lambda_{n+k} - \lambda_{n+k'}|} \right)^2 e^{\sum \text{Tr} \ln \frac{\varepsilon}{8N(1+|\lambda_{n+k}|)^2}}.$$

To bound the denominator, we use the inequality $\ln|x-y| \leq \ln(1+|x-\gamma|) + \ln(1+|y-\gamma|)$ so that

$$\left(\prod_{\substack{1 \leq i \leq n \\ 1 \leq k \leq N-n}} |\lambda_{n+k} - x_i| \prod_{1 \leq k \leq k' \leq N-n} |\lambda_{n+k} - \lambda_{n+k'}| \right)^2 \leq e^{2(N-1) \operatorname{Tr} \ln(1+|X|1_{X \notin I})}.$$

To bound the numerator we use our construction of the x_{n+k} , since $|\phi_{n+k} - x_{n+k}| < \frac{\varepsilon}{8N}$ and Lemma 8.11 holds, to find that for all $k \in [1, N-n]$:

$$\prod_{i \leq n} |\phi_{n+k} - x_i| \prod_{k' < k} |\phi_{n+k} - \phi_{n+k'}| \geq \prod_{i \leq n+k-1} (|x_{n+k} - x_i| - \frac{\varepsilon}{4N}) \geq \left(\frac{\varepsilon}{4e}\right)^{(n+k-1)}$$

Putting everything together we get,

$$J_\phi \geq e^{(-2N \operatorname{Tr} \ln(1+|X|1_{X \notin I}) + \frac{1}{2}(N-n)(N+n+1) \ln \frac{\varepsilon}{4e} + (N-n) \ln \frac{N}{4e})}$$

Finally the bound follows from $\frac{1}{2}(N+n-1) \ln \frac{\varepsilon}{4e} + \ln \frac{N}{4e} \leq 3N$. \square

8.4. Study of the energy functional related to the commutator model.

Lemma 8.12 (Energy problem without a potential). *Let β be a positive real number. Define for a measure μ on the real line its energy:*

$$J_\beta^0(\mu) = - \int \int \ln \frac{\beta|x-y|^2}{1+\beta|x-y|^2} d\mu(x)d\mu(y).$$

- (1) J_β^0 is a lower continuous strictly convex function on the set of probability measures on \mathbb{R} equipped with the weak topology.
- (2) Let $I = [a, b]$, $a < b$, be a closed non empty bounded interval. Then, there is a unique probability measure μ_β^I supported on I such that

$$J_\beta^0(\mu_\beta^I) = \inf\{J_\beta^0(\mu) : \mu(I) = 1\} =: c_\beta^{|I|}.$$

μ_β^I has no atoms.

- (3) For every positive real numbers α, a , we have $c_{\alpha^2\beta}^\alpha = c_\beta^{\alpha a}$ so that $c_\beta^\alpha = c_{\alpha^2\beta}^1$. Moreover, for every real number b and any measurable subset B of I

$$\mu_{\alpha^2\beta}^I(B) = \mu_\beta^{\alpha I+b}(\alpha(B+b)).$$

- (4) The application $\beta \rightarrow c_\beta^1$ is a strictly positive decreasing continuous function on $(0, \infty)$ and

$$2\pi - O(\beta^{-\frac{1}{4}}) \leq \sqrt{\beta}c_\beta^1 \leq 2\pi$$

In particular, for any positive real number a ,

$$c_\beta^a \sim_{\beta \rightarrow +\infty} \frac{2\pi a}{\sqrt{\beta}}.$$

Proof. The first point was proved in the proof of Proposition 7.2 and the second point is clear as a simple change of variable $h_{\alpha,b}(x) = \alpha x + b$ shows that $J_{\beta}^0(h_{\alpha,b}\#\mu) = J_{\beta\alpha^2}^0(\mu)$ if $h\#\mu$ denotes the push-forward of μ by h . This insures by uniqueness of the minimizers that $h_{\alpha,b}\#\mu_{\alpha^2\beta}^I = \mu_{\beta}^{\alpha I+b}$ and $c_{\alpha^2\beta}^a = c_{\beta}^{\alpha a}$. Hereafter we denote by

$$\phi_{\beta}(t) := -\ln \frac{\beta|t|^2}{1 + \beta|t|^2} = \ln\left(1 + \frac{1}{\beta|t|^2}\right)$$

and observe that $\phi_{\beta} \geq 0$ is continuous except at 0 where it blows up. Moreover $\beta \rightarrow \phi_{\beta}(t)$ is decreasing for every real number t . As a consequence,

$$\beta \rightarrow c_{\beta}^1 = \inf_{\mu \in \mathcal{P}([0,1])} J_{\beta}^0(\mu) = \inf_{\mu \in \mathcal{P}([0,1])} \int \int \phi_{\beta}(x-y) d\mu(x) d\mu(y)$$

is a decreasing function from $(0, \infty)$ into itself. c^1 is moreover continuous. Indeed, it is lower semi-continuous since it is decreasing. The upper semi-continuity can be checked by noticing that for every $\varepsilon \in [0, \beta)$

$$\begin{aligned} J_{\beta}^0(\mu) &= J_{\beta-\varepsilon}^0(\mu) + \int \ln \left(\frac{(1 + \beta(x-y)^2)(\beta - \varepsilon)}{(1 + (\beta - \varepsilon)(x-y)^2)\beta} \right) d\mu(x) d\mu(y) \\ &\geq J_{\beta-\varepsilon}^0(\mu) + \ln\left(1 - \frac{\varepsilon}{\beta}\right) \end{aligned}$$

which implies

$$c_{\beta}^1 = J_{\beta}^0(\mu_{\beta}^{[0;1]}) \geq J_{\beta-\varepsilon}^0(\mu_{\beta}^{[0;1]}) + \ln\left(1 - \frac{\varepsilon}{\beta}\right) \geq c_{\beta-\varepsilon}^1 + O\left(\frac{\varepsilon}{\beta}\right).$$

We next show that the function $\psi(\beta) := (c_{\beta^2}^1)^{-1}$ is subadditive. To this end, we cut the interval $[0, 1]$ in two subintervals $[0; u]$ and $[u; 1]$. Let us denote $\gamma := \mu_{\beta}^{[0;1]}([0; u])$, since $\mu_{\beta}^{[0;1]}$ has no atoms and we integrate a non-negative function,

$$\begin{aligned} c_{\beta}^1 &= J_{\beta}^0(\mu_{\beta}^{[0;1]}) \geq J_{\beta}^0(\mu_{\beta}^{[0;1]}|_{[0;u]}) + J_{\beta}^0(\mu_{\beta}^{[0;1]}|_{[u;1]}) \\ (44) \quad &\geq \gamma^2 c_{\beta}^u + (1 - \gamma)^2 c_{\beta}^{1-u} \\ &\geq \inf_{0 \leq \delta \leq 1} \{\delta^2 c_{u^2\beta}^1 + (1 - \delta)^2 c_{(1-u)^{-2}\beta}^1\} = ((c_{u^2\beta}^1)^{-1} + (c_{(1-u)^2\beta}^1)^{-1})^{-1}. \end{aligned}$$

Therefore ψ is subadditive since for all $u \in [0, 1]$

$$\psi(\beta) \leq \psi(\beta u) + \psi(\beta(1 - u)).$$

As a consequence there exists a constant c such that $c_{\beta^2}^1 \sim c/\beta$, as β goes to infinity. The aim of the rest of the proof is to show that $c = 2\pi$, which requires extra work. As in (44), we find

$$c_{\beta}^1 \geq J_{\beta}^0(\mu_{\beta}^{[0;1]}|_I) \geq \mu_{\beta}^{[0;1]}(I)^2 c_{\beta}^{|I|} \geq \mu_{\beta}^{[0;1]}(I)^2 c_{|I|^2\beta}^1.$$

Therefore, we deduce that if $|I_\beta|^2\beta \rightarrow \infty$ then

$$(45) \quad \limsup_{\beta} \frac{\mu_\beta^{[0;1]}(I_\beta)}{\sqrt{|I_\beta|}} \leq \limsup_{\beta} \sqrt{\frac{c_\beta^1}{|I_\beta|c_{|I_\beta|^2\beta}^1}} = 1$$

As a consequence, for every measurable decreasing function $f : [0; 1] \rightarrow \mathbb{R}^+$, we have:

$$(46) \quad \limsup_{\beta} \int_0^1 f d\mu_\beta^{[0;1]} \leq 2 \int_0^1 f(x) \frac{1}{\sqrt{x}} dx$$

Indeed it is trivially true for $f(x) = 1_{x < t}$ and then one can extend the result to all decreasing f by linearity and monotone limit.

Now let us prove that the support of $\mu_\beta^{[0;1]}$ is the whole interval $[0; 1]$. Since J_β^0 is left constant under translation of the measure and decreases by dilatation we already know that the convex closure of the support of $\mu_\beta^{[0;1]}$ is $[0; 1]$. We therefore only need to make sure that there is no hole in the support. We know that there exists a constant C_β such that $\mu_\beta^{[0;1]}$ p.p. :

$$U(x) = 2 \int \ln \frac{1 + \beta|x - y|^2}{\beta|x - y|^2} d\mu_\beta^{[0;1]}(y) = 2 \int \phi_\beta(x - y) d\mu_\beta^{[0;1]}(y) = C_\beta$$

and everywhere on $[0; 1]$, $U(x) \geq C_\beta$. Indeed, this can be deduced from the fact that $J_\beta^0(\mu_\beta^{[0;1]} + \delta\nu) \geq J_\beta^0(\mu_\beta^{[0;1]})$ for any measure ν on $[0, 1]$ such that $\mu_\beta^{[0;1]} + \delta\nu$ is a probability measure. But $U(x)$ is strictly convex on the complementary of the support of $\mu_\beta^{[0;1]}$ (since ϕ_β is strictly convex) and therefore the support of $\mu_\beta^{[0;1]}$ is connected and therefore equals $[0, 1]$. So U is constant equal to C_β on $[0; 1]$. Therefore its integral with respect to $d\mu_\beta^{[0;1]}$ is the same than its integral with respect to the uniform measure on I , from there we get the new formula for the energy:

$$J_\beta^0(\mu_\beta^{[0;1]}) = \int \int_0^1 \ln \frac{1 + \beta|x - y|^2}{\beta|x - y|^2} dy d\mu_\beta^{[0;1]}(x) = \int \left(\int_0^1 \phi_\beta(x - y) dy \right) d\mu_\beta^{[0;1]}(x)$$

Now for any $0 < x < 1$, because ϕ_β is non negative,

$$(47) \quad \begin{aligned} \sqrt{\beta} \int_0^1 \phi_\beta(x - y) dy &\leq \sqrt{\beta} \int_{\mathbb{R}} \phi_\beta(z) dz = \int_{\mathbb{R}} \phi_1(z) dz \\ &= [z \ln \frac{z^2}{1 + z^2} + 2 \arctan(z)]_{-\infty}^{\infty} = 2\pi \end{aligned}$$

Therefore $c_\beta^1 \leq \frac{2\pi}{\sqrt{\beta}}$ and we have the upper bound in the fourth point of the proposition. We finally prove the converse inequality and notice that for all real number x ,

$$\begin{aligned}
 0 &\leq 2\pi + \sqrt{\beta} \int_0^1 \ln \frac{\beta|x-y|^2}{1+\beta|x-y|^2} dy = \sqrt{\beta} \int_{[0,1]^c} \phi_\beta(x-y) dy \\
 &\leq \int_{-\infty}^{-\sqrt{\beta}x} \phi_1(y) dy + \int_{-\sqrt{\beta}x+\sqrt{\beta}}^{+\infty} \phi_1(y) dy \\
 &\leq 2\pi \left(1 \wedge \left(\frac{1}{\sqrt{\beta}x} + \frac{1}{\sqrt{\beta}(1-x)}\right)\right)
 \end{aligned}$$

where we used that the left hand side is bounded by 2π and $\phi_1(x) \leq x^{-2}$. From this we deduce, using (46) that

$$\begin{aligned}
 \limsup_{\beta} \beta^{\frac{1}{4}} (2\pi - \sqrt{\beta}c_\beta^1) &\leq \limsup_{\beta} \beta^{\frac{1}{4}} 4\pi \int_0^{\frac{1}{2}} \left(1 \wedge \frac{1}{\sqrt{\beta}x}\right) d\mu_\beta^{[0;1]} \\
 &\leq \limsup_{\beta} \beta^{\frac{1}{4}} 8\pi \int_0^{\frac{1}{2}} \left(1 \wedge \frac{1}{\sqrt{\beta}x}\right) \frac{1}{\sqrt{x}} dx = 32\pi.
 \end{aligned}$$

□

We next consider the case where the energy functional to minimize contains a non trivial potential.

Lemma 8.13 (Energy problem with a potential). *Let $V : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous functions such that $\lim_{|x| \rightarrow \infty} V(x) = +\infty$ and V achieves its minimum value (equal to zero) exactly on the finite set x_1, \dots, x_r with $V(x) = c_i(x - x_i)^{2p_i}(1 + O(|x - x_i|))$ near x_i for each $i \in \{1, \dots, r\}$ for some positive integers p_i . We set $p = \max p_i$ and for a measure μ , we recall that*

$$J_\beta^V(\mu) = \int \left(\frac{1}{2}(V(x) + V(y)) - \ln \frac{\beta|x-y|^2}{1+\beta|x-y|^2} \right) d\mu(x)d\mu(y).$$

We define $\beta_p = \beta^{\frac{p}{2p+1}}$ and $\gamma_p = \beta^{-\frac{1}{2(2p+1)}}$. Then:

- (1) J_β^V is a lower continuous strictly convex function on the set $\mathcal{P}(\mathbb{R})$ of probability measures on the real line. It achieves its minimum value at a unique probability measure μ_β^V and there exists a constant $0 < c_V < \infty$ such that

$$c_\beta^V := \inf J_\beta^V = \beta_p^{-1} c_V + o(\beta_p^{-1})$$

where $o(\beta_p^{-1})$ goes to zero as β goes to infinity. The constant c_V satisfies the scaling property $c_{\alpha V} = \alpha^{\frac{1}{2p+1}} c_V$ for every $\alpha > 0$. Besides we have an explicit expression for c_V :

$$c_V = \left(1 + \frac{1}{2p}\right)^{\frac{4p+1}{2p+1}} \pi^{1-\frac{1}{2p+1}} \left(\sum_{j:p_j=p} c_j^{-1/(2p)}\right)^{\frac{1}{2p+1}-1}$$

- (2) *There exists a constant $L_V < \infty$ such that for all $\beta > 1$, μ_β^V is compactly supported in*

$$S(L_V) := \bigcup_{i=1}^r \{x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V]\}$$

Moreover, for every interval I_β of length $|I_\beta|$ such that $|I_\beta|^2 \beta \rightarrow \infty$ then

$$\mu_\beta^V(I_\beta) = O(\beta^{\frac{1}{4(2p+1)}}) \sqrt{|I_\beta|}.$$

- (3) *There exists a constant $0 < C < \infty$ such that for all i :*

$$\lim_{\beta \rightarrow \infty} \mu_\beta^V(x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V]) = 1_{p_i=p} C^{-1} c_i^{-\frac{1}{2p}}$$

- (4) *Recall that $h_\alpha(x) = \alpha x$. If V has a single minimum at $x_1 = 0$ and $V(x) \simeq cx^{2p}$ in a neighborhood of the origin, then*

$$h_{\beta^{-\frac{1}{4p+2}}} \# \mu_\beta^V \Big|_{\{\beta^{\frac{-1}{2(2p+1)}} [-L_V; L_V]\}} \rightarrow d\mu(z) := \frac{1}{2\pi} (A - cz^{2p})_+ dz$$

where A is the only constant such that the RHS is a probability measure. More precisely, the Wasserstein distance W_1 between these two measures is bounded by $\beta_p^{-\frac{1}{11}}$.

In other words, the energy is of order β_p^{-1} and γ_p is the typical length of the support of the minimizer around the minima.

Remark 8.14. *As a special case of interest we get that if all global minima of V are quadratic ($p = p_i = 1$) then the minimizer has its support with endpoints at distance of order $\beta^{-\frac{1}{6}}$ of the minima of V and $\inf J_\beta^V$ is of order $\beta^{-\frac{1}{3}}$.*

Proof. The strict convexity of J_β^V is a direct consequence of the strict convexity of J_β^0 as they differ only by a linear term. We next estimate c_β^V . We will assume without loss of generality that 0 minimizes V and that near 0, $V(x) = cx^{2p}(1 + o(1))$ with $c > 0$. Recall that μ_β^I denotes the minimizer of J_β on $\mathcal{P}(I)$. We have:

$$c_\beta^V \leq \inf_{\gamma > 0} J_\beta^V(\mu_\beta^{[-\gamma; \gamma]}) \leq \inf_{\gamma} \{c\gamma^{2p}(1 + o(\gamma)) + c_\beta^{2\gamma}\} = \inf_{\gamma} \{c\gamma^{2p}(1 + o(\gamma)) + c_{4\gamma^2\beta}^1\}$$

where we used Lemma 8.12. Recalling that $c_{4\gamma^2\beta}^1 \simeq \pi/\gamma\sqrt{\beta}$, and taking $\gamma = \gamma_p$, we get

$$(48) \quad c_\beta^V \leq O(\beta_p^{-1}).$$

where β_p, γ_p are defined in the statement of the Theorem.

Let us now prove that the support of a minimizer of J_β^V is concentrated in a neighborhood of the minima of V . Let μ be a measure such that there exists z in the support of μ such that $V(z) > 2L\beta_p^{-1}$ for some constant L to

be chosen later. Choose ε such that $V > L\beta_p^{-1}$ on $I_z^\varepsilon := [z - \varepsilon, z + \varepsilon]$ and define $\mu_z = \mu(I_z^\varepsilon)^{-1}\mu|_{I_z^\varepsilon}$. Now we can define:

$$\mu_t = \mu + t(\lambda_{\gamma_p} - \mu_z)$$

with λ_{γ_p} the uniform measure on $[-\gamma_p, \gamma_p]$. μ_t is a probability measure for $t \in [0, \mu(I_z^\varepsilon)]$ and $\mu_0 = \mu$. Besides

$$\begin{aligned} \partial_t J_\beta^V(\mu_t)|_{t=0} &= \int V d(\lambda_{\gamma_p} - \mu_z) + 2 \int \phi_\beta(x - y) d\mu(x) d(\lambda_{\gamma_p} - \mu_z)(y) \\ &\leq \sup_{[-\gamma_p, \gamma_p]} V - L\beta_p^{-1} + 2 \sup_u \int \phi_\beta(u - y) d\lambda_{\gamma_p}(y) \end{aligned}$$

where we again used that ϕ_β is non-negative. It is easy to see that the supremum in the last term is achieved at $u = 0$ and then by similar computations as in (47) to see that this term is at most of order $O(\beta_p^{-1})$. Since the first term is of the same order of magnitude, we conclude that there exists a finite constant L_V such that if $L \geq L_V$ the above right hand side is non-negative. Then, μ can not be the minimizer of J_β^V . Hence, the support of μ_β^V is included in $S(L_V) := \bigcup_{i=1}^r \{x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V]\}$. For β sufficiently large, the restriction of V to each of the interval $x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V]$ is convex (and strictly convex except at the point $\{x_i\}$) and an argument identical to the one for $\mu_\beta^{[0;1]}$ in the proof of the previous Lemma 8.12 shows that the restriction of the support of μ_β^V in $x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V]$ is an interval. Let us call γ_β^i the length of this interval and define $\alpha_\beta^i = \mu_\beta^V(x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V])$. Then, we get the complementary lower bound to (48) by using again that ϕ_β and V are non-negative:

$$\begin{aligned} c_\beta^V &\geq \sum_i \inf_{\mu \in \mathcal{P}(x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V])} J_\beta^0(\alpha_\beta^i \mu) \\ (49) \quad &= (1 + o(1)) \sum_i (\alpha_\beta^i)^2 \frac{2\pi}{L_V \sqrt{\beta} \beta^{-\frac{p}{2p_i(2p+1)}}} \end{aligned}$$

where we finally used Lemma 8.13. We notice that for i such that $p_i = p$

$$\sqrt{\beta} \beta^{-\frac{p}{2p_i(2p+1)}} = \beta^{\frac{p}{2p+1}} = \beta_p$$

whereas for i such that $p_i < p$, $\sqrt{\beta} \beta^{-\frac{p}{2p_i(2p+1)}} \ll \beta_p$. Since we have seen in (48) that c_β^V is at most of order β_p^{-1} we conclude that for i such that $p_i < p$, α_β^i goes to zero with β (is in fact bounded by $\beta^{\frac{(p_i-1)p}{4(2p+1)}}$) and also that $0 < \liminf \beta_p c_\beta^V \leq \limsup \beta_p c_\beta^V < \infty$, i.e.

$$(50) \quad c_\beta^V \asymp \beta_p^{-1}.$$

This allows us to complete the proof of Lemma 8.13 (2) and show that μ_β^V can not concentrate too much on small intervals. Indeed, let I_β be a sequence of intervals of Lebesgue measure $|I_\beta|$ such that $|I_\beta|^2\beta \rightarrow \infty$, then, since V and ϕ_β are non-negative

$$c_\beta^V \geq J_\beta^0(\mu_\beta^V|_{I_\beta}) = \mu_\beta^V(I_\beta)^2 c_\beta^{|I_\beta|}$$

and therefore by Lemma 8.12(4) we get

$$\mu_\beta^V(I_\beta) \leq \sqrt{\frac{c_\beta^V}{c_{|I_\beta|^2\beta}^1}} = \sqrt{O(\beta^{-\frac{p}{2p+1}} \sqrt{\beta}|I_\beta|)} = O(\beta^{\frac{1}{4(2p+1)}}) \sqrt{|I_\beta|}.$$

We next complete the proof of the first point of Lemma 8.13, as well as its third and fourth points. First notice that we can remove the interaction between wells since they are at distance of order one, up to an error of order $1/\beta$:

$$J_\beta^V(\mu_\beta) = \sum_i J_\beta^V(\alpha_i^\beta \mu_i^\beta) + O(1/\beta)$$

where μ_i^β is the restriction of μ_β^V to the neighborhood $x_i + \beta^{-\frac{p}{2p_i(2p+1)}} [-L_V; L_V]$ of the minimizer $\{x_i\}$. We estimate now the energy in each of these wells.

The energy problem with one well

Assume that $V(x) = cx^{2p}$. With the previous discussion in mind we look at the rescaled energy functional for a probability measure μ on $[-L_V; L_V]$:

$$\begin{aligned} \tilde{J}_\beta^V(\mu) &:= \beta_p J_\beta^V(h_{\beta^{-\frac{1}{2(2p+1)}}} \# \mu) \\ &= \int cx^{2p} d\mu(x) + \beta_p \int \int \ln \phi_{\beta^2}^2(x-y) d\mu(x) d\mu(y). \end{aligned}$$

We are going to show that the infimum of \tilde{J}_β^V over the set of probability measures on $[-L_V; L_V]$ is converging to a constant depending only on p and c . More precisely, with $a_p = \left(1 + \frac{1}{2p}\right)^{\frac{4p+1}{2p+1}} \pi^{1-\frac{1}{2p+1}}$, we show:

$$(51) \quad \inf \tilde{J}_\beta^{cx^{2p}} = a_p c^{\frac{1}{2p+1}} + O(\beta_p^{-\frac{1}{11}}).$$

To this end, let m be a large integer (to be chosen later) and cover $[-L_V; L_V]$ by the union of segments $[z_i; z_{i+1}]$ of width m^{-1} with $z_i = -L_V + (i+1)/m$, $-1 \leq i \leq \lfloor 2L_V \rfloor$.

We first lower bound the infimum of \tilde{J}_β^V . Again this infimum is achieved at a unique probability measure that we denote ν_β^V . We let $\alpha_i^m(\beta) =$

$\nu_\beta^V([z_i; z_{i+1}])$ and notice that, because $\phi_{\beta_p^2}$ is non-negative,

$$\begin{aligned}
\tilde{J}_\beta^V(\nu_\beta^V) &\geq \sum_{i=-1}^m \tilde{J}_\beta^V(\nu_\beta^V|_{[z_i; z_{i+1}]}) \\
&\geq \sum_{i=-1}^{\lfloor 2L_V \rfloor} \left\{ \alpha_i^m(\beta) c \inf_{[z_i; z_{i+1}]} z^{2p} + (\alpha_i^m(\beta))^2 \tilde{J}_\beta^0((\alpha_i^m(\beta))^{-1} \mu|_{[z_i; z_{i+1}]))} \right\} \\
&\geq (1 - O(\frac{1}{m})) \sum_{i=-1}^{\lfloor 2L_V \rfloor} \left\{ \alpha_i^m(\beta) c z_i^{2p} + (\alpha_i^m(\beta))^2 \beta_p c_{\beta_p^2}^{1/m} \right\} \\
&= (1 - O(\frac{1}{m})) \sum_{i=-1}^{\lfloor 2L_V \rfloor} \left\{ \alpha_i^m(\beta) c z_i^{2p} + (\alpha_i^m(\beta))^2 m 2\pi (1 - O((\frac{\beta_p}{m})^{-1/2})) \right\} \\
(52) \quad &= (1 - O(\frac{1}{m}) - O((\frac{\beta_p}{m})^{-1/2})) \phi_m(\alpha^m(\beta))
\end{aligned}$$

where

$$\phi_m(\alpha) = \sum_{i=-1}^{\lfloor 2L_V \rfloor} \{ \alpha_i c z_i^{2p} + 2\pi m (\alpha_i)^2 \}$$

and we assumed that β_p/m is large to use the fourth point of Lemma 8.12. Now let $\bar{\alpha}^m = (\bar{\alpha}_i^m)_{i=1, \dots, r}$ be the value which achieves the minimum of the strictly convex functional ϕ_m on the convex set $\{\alpha_i \in [0, 1] : \sum \alpha_i = 1\}$ and $\mu_{\beta_p^2}^{[z_i; z_{i+1}]}$ the probability on $[z_i; z_{i+1}]$ minimizing \tilde{J}_β^0 . From the above we deduce that

$$\begin{aligned}
\inf \tilde{J}_\beta^{cx^{2p}} &\geq (1 - O(\frac{1}{m}) - O((\frac{\beta_p}{m})^{-1/2})) \inf \phi_m(\alpha^m(\beta)) \\
(53) \quad &= (1 - O(\frac{1}{m}) - O((\frac{\beta_p}{m})^{-1/2})) \tilde{J}_\beta^{cx^{2p}}(\mu^m)
\end{aligned}$$

where we define μ^m to be the probability measure

$$\mu^m =: \sum_i \bar{\alpha}_i^m \mu_{\beta_p^2}^{[z_i; z_{i+1}]}$$

We next prove the complimentary upper bound. In fact:

$$\begin{aligned}
\tilde{J}_\beta^V(\nu_\beta^V) &\leq \tilde{J}_\beta^V(\mu^m) \\
&\leq \sum_{i=1}^m \bar{\alpha}_i^m c \sup_{[z_i; z_{i+1}]} z^{2p} + (\bar{\alpha}_i^m)^2 \tilde{J}_\beta^0(\mu_{\beta_p^2}^{[z_i; z_{i+1}]}) \\
&\quad - \sum_{i \neq j} \bar{\alpha}_i^m \bar{\alpha}_j^m \beta_p \int \int \ln \frac{\beta_p^2 |x - y|^2}{1 + \beta_p^2 |x - y|^2} d\mu_{\beta_p^2}^{[z_i; z_{i+1}]}(x) d\mu_{\beta_p^2}^{[z_j; z_{j+1}]}(y)
\end{aligned}$$

Our goal is to get rid of the last term. Define the function :

$$g_\beta^i(y) = -\beta_p \int \ln \frac{\beta_p^2 |x - y|^2}{1 + \beta_p^2 |x - y|^2} d\mu_{\beta_p^2}^{[z_i; z_{i+1}]}(x).$$

The function g_β^i is positive and decreases when y goes away from $[z_i; z_{i+1}]$. Moreover, for $y \in [z_i; z_{i+1}]$, $g_\beta^i(y)$ is constant and its value is

$$g_\beta^i(y) = a_m^p(\beta) := \beta_p J_{\beta_p^2}^0(\mu_{\beta_p^2}^{[z_i; z_{i+1}]}) = \beta_p c_{\beta_p^2}^{\frac{1}{m}} = 2m\pi(1 + O(\frac{\beta_p}{m})^{-1/2}),$$

where we again assumed that β_p/m is large. This is sufficient to get a bound on our sum. Indeed, for $j > i$, we get for all $\eta \in [0, \frac{2}{m}]$,

$$\begin{aligned} \int g_\beta^i(y) d\mu_{\beta_p^2}^{[z_j; z_{j+1}]}(y) &\leq \int g_\beta^i(y) d\mu_{\beta_p^2}^{[z_{i+1}; z_{i+2}]}(y) \\ &\leq \int_{z_{i+1}}^{z_{i+1}+\eta} g_\beta^i(z_{i+1}) d\mu_{\beta_p^2}^{[z_{i+1}; z_{i+2}]}(y) + g_\beta^i(z_{i+1} + \eta) \\ &\leq \mu_{\beta_p^2}^{[0; 1/m]}([0; \eta]) g_\beta^0(z_1) - \beta_p \ln \frac{\beta_p^2 \eta^2}{1 + \beta_p^2 \eta^2}. \end{aligned}$$

Now according to (45), $\mu_{\beta_p^2}^{[0; 1/m]}([0; \eta]) = \mu_{\beta_p^2/m^2}^{[0; 1]}([0; m\eta]) = O((m\eta)^{1/2})$ as long as $\eta\beta_p \rightarrow \infty$. Therefore, we deduce that if $\eta\beta_p \rightarrow \infty$,

$$\sum_{i \neq j} \bar{\alpha}_i^m \bar{\alpha}_j^m \int g_\beta^i(y) d\mu_{\beta_p^2}^{[z_j; z_{j+1}]}(y) = O(m^{3/2} \eta^{1/2} (1 + (\frac{\beta_p}{m})^{1/4})) + \beta_p^{-1} \eta^{-2}$$

We finally choose m, η so that $\eta = \beta_p^{-\frac{1}{10}}, m = \lfloor \beta_p^{\frac{1}{11}} \rfloor$. With this choice, together with (52), we have proved

$$(54) \quad \phi_m(\alpha^m(\beta)) + O(\beta_p^{-\frac{1}{11}}) \leq \tilde{J}_\beta^V(\nu_\beta^V) \leq \phi_m(\bar{\alpha}^m) + O(\beta_p^{-\frac{1}{11}})$$

Because $\phi_m(\bar{\alpha}^m) \leq \phi_m(\alpha^m(\beta))$ we conclude that

$$(55) \quad \liminf_{\beta \rightarrow \infty} \tilde{J}_\beta^V = \lim_{m \rightarrow \infty} \inf \phi_m$$

if these limits exists. We finally show that the above limits are positive and finite, and moreover that $\alpha_\beta^m - \bar{\alpha}^m$ goes to zero. Looking at the critical points $\bar{\alpha}^m$ of ϕ_m on $\{\alpha_i \in [0, 1] : \sum_i \alpha_i = 1\}$ we see that for the indices in $S_m := \{i : \bar{\alpha}_i^m > 0\}$ we must have

$$cz_i^{2p} + 2\pi m \bar{\alpha}_i^m = A_m$$

where A_m is a constant. Hence, the minimizer of ϕ_m is given by

$$\bar{\alpha}_i^m = \frac{1}{2\pi m} (A_m - cz_i^{2p})_+$$

where A_m is the only positive solution to

$$\sum_i \frac{1}{2\pi m} (A_m - cz_i^{2p})_+ = 1.$$

The continuum limit of this minimizer is given by the probability measure

$$(56) \quad \nu_c(dx) = \frac{1}{2\pi} (A - cx^{2p})_+ dx$$

where A is the only positive solution to $\int \frac{1}{2\pi}(A - cz^{2p})_+ dz = 1$. This can be solved directly and gives :

$$A = c^{\frac{1}{2p+1}} \left(\frac{(2p+1)\pi}{2p} \right)^{\frac{2p}{2p+1}}.$$

Plugging this formula in ϕ_m we see that

$$\begin{aligned} \inf \phi_m &= \sum_{i=1}^m \left\{ \frac{1}{2\pi m} (A_m - cz_i^{2p})_+ cz_i^{2p} + \frac{1}{4\pi m} (A_m - cz_i^{2p})_+^2 \right\} \\ (57) \quad &\rightarrow \int cx^{2p} \nu_c(dx) + \pi \int \frac{d\nu_c(x)}{dx} d\nu_c(x) =: F(cx^{2p}) \end{aligned}$$

A direct computation proves that

$$F(cx^{2p}) = \left(1 + \frac{1}{2p} \right)^{\frac{4p+1}{2p+1}} \pi^{1-\frac{1}{2p+1}} c^{\frac{1}{2p+1}}.$$

This completes the proof of (51). We can also see that $A_m = A + O(1/m)$ and $W_1(\nu, \sum_i \bar{\alpha}_i^m \delta_{z_i}) = O(1/m)$.

We next can quantify this convergence as follows. Indeed, for any sequence α_i^m on the simplex:

$$\begin{aligned} &\phi_m(\alpha^m) - \phi_m(\bar{\alpha}^m) \\ &= \sum_i (\alpha_i^m - \bar{\alpha}_i^m) (cz_i^{2p} + 2\pi m \bar{\alpha}_i^m) + \sum_i \pi m (\alpha_i^m - \bar{\alpha}_i^m)^2 \\ &= \sum_i (\alpha_i^m - \bar{\alpha}_i^m) A_m + \sum_{i \notin S_m} \alpha_i^m (cz_i^{2p} - A_m) + \sum_i \pi m (\alpha_i^m - \bar{\alpha}_i^m)^2 \\ &\geq \sum_i \pi m (\alpha_i^m - \bar{\alpha}_i^m)^2 = \pi m \|\alpha^m - \bar{\alpha}^m\|_2^2 \end{aligned}$$

Therefore (54) implies that

$$\pi m \|\alpha^m(\beta) - \bar{\alpha}^m\|_2^2 \leq O(\beta_p^{-\frac{1}{11}}).$$

This allows us to bound the Wasserstein distance between the corresponding measures:

$$\begin{aligned} W_1\left(\sum_i \bar{\alpha}_i^m \delta_{z_i}, \sum_i \alpha_i^m(\beta) \delta_{z_i}\right) &\leq 2d_{TV}\left(\sum_i \bar{\alpha}_i^m \delta_{z_i}, \sum_i \alpha_i^m(\beta) \delta_{z_i}\right) \\ &= 2\|\alpha^m(\beta) - \bar{\alpha}^m\|_1 \leq 2\|\alpha^m(\beta) - \bar{\alpha}^m\|_2 \\ &\leq 2\sqrt{O(m^{-1}\beta_p^{-\frac{1}{11}})} = O(\beta_p^{-\frac{1}{11}}). \end{aligned}$$

Finally :

$$\begin{aligned} W_1(\nu_c, \nu_\beta^V) &\leq W_1(\nu_c, \sum_i \bar{\alpha}_i^m \delta_{z_i}) + W_1\left(\sum_i \bar{\alpha}_i^m \delta_{z_i}, \sum_i \alpha_i^m(\beta) \delta_{z_i}\right) \\ &\quad + W_1\left(\sum_i \alpha_i^m(\beta) \delta_{z_i}, \nu_\beta^V\right) \end{aligned}$$

and the first and last term are bounded by $O(m^{-1}) = O(\beta_p^{-\frac{1}{11}})$. So we conclude that $W_1(\nu_c, \nu_\beta^V) = O(\beta_p^{-\frac{1}{11}})$.

End of the proof of the Lemma. We have already proved that:

$$\inf J_\beta^V = \inf \{ J_\beta(\mu) : \mu(\bigcup_{i=1}^r \{x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V]\}) = 1 \}.$$

But on $\{ \mu : \mu(\bigcup_{i=1}^r \{x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V]\}) = 1 \}$,

$$\begin{aligned} & \sup_{\mu} \left| J_\beta^V(\mu) - \sum_i J_\beta^{c_i(x-x_i)^{2p_i}}(\mu|_{x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V]}) \right| \\ & \leq O\left(\frac{1}{\beta}\right) + O\left(\max_i \beta^{\frac{-p(2p_i+1)}{(2p_i)(2p+1)}}\right) = O(\beta^{-1/2}) \end{aligned}$$

Indeed neglecting the interaction term gives an error of order β^{-1} and replacing in the i th well V by $c_i(x-x_i)^{2p_i}$ gives an error of order $\beta^{\frac{-p(2p_i+1)}{(2p_i)(2p+1)}}$ whose maximal value is obtained for $p_i = p$. Therefore,

$$\inf J_\beta^V = \inf \left\{ \sum_i J_\beta^{c_i(x-x_i)^{2p_i}}(\mu_i) : \sum_i \alpha_i = 1 \right\} + O(\beta^{-1/2})$$

where in the above sum we take the infimum on α in the simplex and μ_i is a measure of mass α_i (The minimisation problem forces the optimal μ_i to have a support in $x_i + \beta^{\frac{-p}{(2p_i)(2p+1)}} [-L_V; L_V]$ if L_V was taken sufficiently large). We next notice that we can remove the terms with small α'_i s. Indeed, if for $\epsilon > 0$ we set $T_\epsilon = \{i : \alpha_i \geq \epsilon\}$, we have

$$\begin{aligned} \inf \left\{ \sum_{i \in T_\epsilon} J_\beta^{c_i(x-x_i)^{2p_i}}(\mu_i) : \sum_i \alpha_i = 1 \right\} & \leq \inf \left\{ \sum_i J_\beta^{c_i(x-x_i)^{2p_i}}(\mu_i) : \sum_i \alpha_i = 1 \right\} \\ & \leq \inf \left\{ \sum_{i \in T_\epsilon} J_\beta^{c_i(x-x_i)^{2p_i}}(\mu_i) : \sum_{i \in T_\epsilon} \alpha_i = 1 \right\} \end{aligned}$$

where in the left hand side we just used that $J_\beta^{c_i(x-x_i)^{2p_i}}$ is non-negative and in the right hand side that we restrict the set of measures where we take the infimum. We set $\eta(\epsilon) = \sum_{i \in T_\epsilon^c} \alpha_i \leq r\epsilon$. Rescaling in the left hand side $\{\alpha_i, i \in T_\epsilon\}$ by $\{\alpha_i(1-\eta(\epsilon))^{-1}, i \in T_\epsilon\}$ while using that $J_\beta^{c_i(x-x_i)^{2p_i}}(\alpha\mu_i) \geq \alpha^2 J_\beta^{c_i(x-x_i)^{2p_i}}(\mu_i)$ for all $\alpha \in [0, 1]$ since the potential is non negative, we conclude

$$\begin{aligned} & (1-r\epsilon)^{-2} \inf \left\{ \sum_{i \in T_\epsilon} J_\beta^{c_i(x-x_i)^{2p_i}}(\mu_i) : \sum_{i \in T_\epsilon} \alpha_i = 1 \right\} \\ & \leq \inf \left\{ \sum_i J_\beta^{c_i(x-x_i)^{2p_i}}(\mu_i) : \sum_i \alpha_i = 1 \right\} \end{aligned}$$

We have seen that the masses α_i for i such that $p_i < p$ go to zero with β like $\beta^{\frac{p}{p_i-1} \frac{p}{4(2p+1)}}$ and so we may assume that they are in T_ϵ . Then, factorizing

the masses α_i gives for β large,

$$\begin{aligned} \inf J_\beta^V &= (1 + o(\epsilon)) \left(\inf \left\{ \sum_{i \in T_\epsilon} \alpha_i^2 J_\beta^{\frac{c_i}{\alpha_i}(x-x_i)^{2p_i}}(\mu_i) : \sum_{i \in T} \alpha_i = 1 \right\} + O(\beta^{-1/2}) \right) \\ &= (1 + o(\epsilon)) \beta_p^{-1} \inf \left\{ \sum_{i \in T_\epsilon} \alpha_i^{2 - \frac{1}{2p_i+1}} (a_{p_i} c_i^{\frac{1}{2p_i+1}} + O(\beta_{p_i}^{-\frac{1}{11}})) : \sum_i \alpha_i = 1 \right\} \end{aligned}$$

where we finally used (51) assuming that c_i/α_i does not blow up, namely α_i does not go to zero. Finally, we see that the above infimum is taken at $\alpha_i^* = c_i^{-\frac{1}{2p}} / \sum_i c_i^{-\frac{1}{2p}}$ yielding

$$\inf J_\beta^V = (1 + o(1)) \left(\beta_p^{-1} a_p \left(\sum_{j:p_j=p} c_j^{-1/(2p)} \right)^{\frac{1}{2p+1}-1} + O(\beta^{-1/2}) \right)$$

The expression for c_V and its scaling property $c_{\alpha V} = \alpha^{\frac{1}{2p+1}} c_V$ is a trivial consequence from this formula. Moreover, our proof shows that $\{\alpha_i^*\}_{1 \leq i \leq r}$ is the limit of the mass of the neighborhood of the minimizers $\{x_i\}_{1 \leq i \leq r}$ since it is the unique minimizer of the above approximation of the energy. This proves the third point of the Lemma. \square

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UMPA, CNRS UMR 5669, ENS LYON, 46 ALLÉE D’ITALIE, 69007 LYON, FRANCE
Email address: Alice.Guionnet@umpa.ens-lyon.fr

UNIVERSITÉ PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D’ORSAY, 91405, ORSAY, FRANCE
Email address: edouard.maurel-segala@universite-paris-saclay.fr