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Random walks and the symplectic representation of the braid groups

Marc Soret and Marina Ville

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Abstract

We consider the symplectic representation ρ_n of a braid group $B(n)$ in $Sp(2l,\mathbb{Z})$, for $l = \left[\frac{n-1}{2}\right]$. If P is a $4l^2$ polynomial on the coefficients of the matrices in $Sp(2l,\mathbb{Z})$, we show that the set $\{\beta \in B(n)/ P(\rho_n(\beta)) = 0\}$ is transient for non degenerate random walks on $B(n)$. If n is odd, we derive that the n-braids β verifying $|\Delta_{\hat{\beta}}(-1)| \leq C$ for some constant C form a transient set: here $\Delta_{\hat{\beta}}$ denotes the Alexander polynomial of the closure of β.
We also derive that for a random 3-braid, the quasipositive links $(\beta\sigma_i\beta^{-1}\sigma_j)^p$ have zero signature for every integer p and $1 \leq i,j \leq 2$. As an example of such braids, we investigate the signature of the Lissajous toric knots with 3 strands.

1 Preliminaries

1.1 Representations of the braid group

If $B(n)$ is the group of braids with n strands, we denote by $\mathcal{B}$ its *reduced Burau representation* which represents braids as $(n-1)\times(n-1)$ matrices with values in $\mathbb{Z}[t,t^{-1}]$. The specialization of $\mathcal{B}$ for $t=-1$, denoted $\mathcal{B}_{-1}$, is called the *integral reduced Burau representation*. The representation $\mathcal{B}_{-1}$ is closely related to the *symplectic representation* of $B(n)$ which we now recall, using the description of A'Campo ([A'C]).
The braid group $B(n+1)$ acts on the homology $H_1(X,\mathbb{Z})$ of a surface X such that $H_1(X,\mathbb{Z})$ is freely generated by n loops. This action is conjugate to the

integral reduced Burau representation. Concretely speaking, this means the existence of a basis of $H_1(X,\mathbb{Z})$ w.r.t. which the matrices representing the elements of $B(n+1)$ are the ones given by $\mathcal{B}_{-1}$.
This action of $B(n+1)$ preserves the intersection form I on $H_1(X,\mathbb{Z})$.

1. If n is even, $n=2l$, the form I is symplectic nondegenerate on $H_1(X,\mathbb{Z})$.
2. If n is odd, $n=2l+1$, the kernel of I is a line K; $\mathcal{B}_{-1}$ restricts to the identity on K and acts symplectically on the quotient $H_1(X,\mathbb{Z})/K$

Thus, in both cases, we get a *symplectic representation*

$$\rho_n : B(n+1) \longrightarrow Sp(2l,\mathbb{Z}) \tag{1}$$

1.2 Random walks on groups

If μ_1,μ_2 are probability measures on a discrete group G with finite support, their *convolution* is defined for $g\in G$ as

$$\mu_1 \star \mu_2(g) = \sum_{h\in G} \mu_1(g^{-1}h)\mu_2(h) \tag{2}$$

Definition 1. *A probability μ on a discrete group G is nondegenerate if supp(μ) generates G as a semigroup.*

Malyutin ([Ma]) defines the *right random μ-walk* as the Markov chain which starts at the identity of G and with transition probability $P(g,h)=\mu(gh^{-1})$. If X is a subset of G, the probability that this walk hits X at the k-th step is $\mu^{\star k}(X)$.

1.3 The signature of a link

The signature is a classical invariant of links and knots; (see for example [Mu]). If γ is a link on $\mathbb{S}^3$ which bounds a Seifert surface Σ in $\mathbb{S}^3$, we consider the intersection form

$$I : H_1(\Sigma)\times H_1(\Sigma) \longrightarrow \mathbb{Z}$$

$$(\alpha,\beta)\mapsto lk(\hat{\alpha},\beta)$$

where lk is the linking number and $\hat{\alpha}$ is the link obtained by pushing α in the direction normal to Σ in $\mathbb{S}^3$. Given a base for $H_1(\Sigma)$, we define the matrix

V of I: the signature of γ is the signature of the bilinear form defined by the matrix $V +^t V$. We point out that the literature contains two different definitions of the signature which coincide up to sign.
The signature of a link can be computed on a braid representing it and Gambodau-Ghys have estimated the signature of a product of two braids in terms of the signatures of the two braids ([G-G]). We recall (cf. [Mu]) that the for a knot K,

$$|sign(K)| \leq 2g_4(K) \tag{3}$$

where g_4 denotes the topological 4-genus. Thus a slice knot has zero signature but the converse is not true and we will see more examples of that here.

1.4 Positive links and quasipositive braids

A *positive link* is a link which has a diagram with all positive crossings.

Theorem 1. *([Pr]) A positive link has strictly negative signature.*

So it is interesting to look at the signature of quasipositive braids:

Definition 2. *([Ru]) A braid β is quasipositive if it is a product of conjugates of positive braid generators*

$$\beta = \prod_{i=1}^{k} \gamma_{j_i} \sigma_{j_i} \gamma_{j_i}^{-1} \tag{4}$$

If we close the quasipositive N-braid (4) in a link $\hat{\beta}$, Rudolph proved ([Ru]) that

$$\chi_4(\hat{\beta}) = N - k \tag{5}$$

where χ_4 denotes the largest Euler characteristic of a *smooth* surface in $\mathbb{B}^4$ bounded by $\hat{\beta}$.
Tanaka ([Ta]) constructed examples of quasipositive braids with zero signature.

1.5 Transient sets in the braid groups from quasimorphisms

We recall that a *quasimorphism* on a group G is a function $\Phi : G \longrightarrow \mathbb{R}$ such that there exists a positive constant C with

$$\forall a, b \in G,\ |\Phi(ab) - \Phi(a) - \Phi(b)| \leq D_\Phi \tag{6}$$

where D_Φ is called the *defect* of Ψ.
Quasimorphisms give us exemples of transient sets in the braid groups:

Theorem 2. *([Ma]) Let G be a countable group and μ a nondegerate probability on G (see Definition 1). If a subset S of G has bounded image under an unbounded quasimorphism Φ, then the probability that the right random μ-walk hits S at the k-th step tends to zero as k tends to infinity.*

We will see below how Gambodau-Ghys's formula for the signature turns it into a quasimorphism. Several other link invariants such as χ_4 can be used to define quasimorphisms on the braid group ([Br], [B-K]).
Other exemples of transient sets on the braid groups have been constructed by Ito [It]

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2 The results

Theorem 3. *Let P be a polynomial in $(2l)^2$ variables with coefficients in $\mathbb{Z}$. If $M = (m_{ij}) \in Sp(2l, \mathbb{Z})$, we let $P(M) = P(m_{11}, ..., m_{12l}, m_{21}, ..., m_{22l}, m_{2l1}, ..., m_{2l2l})$. Suppose that P does not vanish identically on $Sp(2l, \mathbb{Z})$. Then the set*

$$\{\beta \in B(n)/P(\rho_n(\beta)) = 0\}$$

is transient for the right random μ-walk.

2.1 The Alexander polynomial

Corollary 1. *Let C be a real number. If $2s+1$ is an odd integer, the following set is transient for the right random μ-walk on $B(2s+1)$*

$$\{\beta \in B(2s+1)/|\Delta_{\hat{\beta}}(-1)| < C\} \tag{7}$$

where $\Delta_{\hat{\beta}}$ denotes the Alexander polynomial of the closure of β.

Proof. The Alexander polynomial can be expressed in terms of the Burau representation $\mathcal{B}$ ([Bu]),

$$\Delta_{\hat{\beta}}(t) = \frac{1-t}{1-t^n} det(\mathcal{B}_{-t}(\beta) - I) \tag{8}$$

If n is odd, $\Delta_{\hat{\beta}}(-1) = det(\rho_n(\beta) - I)$ and the set (7) is a finite union of sets described in Theorem 3. □

REMARK. If n is even, $1-(-1)^n = 0$ and $det(\mathcal{B}_{-1}(\beta) - I) = 0$ since $\mathcal{B}_{-t}$ acts on K as the identity; thus the formula (8) is not easy to use. Nevertheless, it seems likely that Theorem 1 is true also for even n's.

2.2 The signature of 3-braids

Corollary 2. *Let μ be a nondegenerate probability on the braid group $B(3)$. For the right random μ-walks on $B(3)$, the set*

$$S = \{\beta \in B(3)/\exists n \in \mathbb{N}, \exists i, j \in \{1,2\} \quad such\ that \quad sign\big((\beta\sigma_i\beta^{-1}\sigma_j)^n\big) \neq 0\} \tag{9}$$

is transient.

By contrast, (5) tells us that for every n, $\chi_4\big((\beta\sigma_1\beta^{-1}\sigma_1)^n\big) = 3-2n$; in particular, if $(\beta\sigma_1\beta^{-1}\sigma_1)^n$ closes into a knot, $g_4((\beta\sigma_1\beta^{-1}\sigma_1)^n) = n-1$.

In §5, we recall the Lissajous toric knots with 3 strands (cf. [S-V]); they are naturally of the form $(\beta\sigma_2\beta^{-1}\sigma_1^{\pm 1})^n$ and we prove

Theorem 4. *A Lissajous toric knot $K(3,p,q)$ has zero signature unless it is isotopic to a torus knot.*

We conclude by statistical estimates, done with SAGE.

These results suggest possibles generalizations.

Question 1. *Let $n \in \mathbb{N}$ and let $\mathcal{I}$ be a numerical link invariant which is unbounded on the closures of the links represented by a n-braid. If C is a constant, is the set $\{\mathcal{I}(\hat{\beta}) \leq C\}$ transient for the right random μ-walks?*

Question 2. *For an integer m, we set*

$$A_m^{(1)} = \prod_{1 \leq 2i+1 \leq m} \sigma_{2i+1} \qquad A_m^{(2)} = \prod_{1 \leq 2i \leq m} \sigma_{2i}$$

Is the following set S_m transient in $B(m)$ for the right random μ-walks?

$S_m = \{\beta \in B(m) / \exists n \in \mathbb{N}, \exists i, j \in \{1, 2\}$ *such that* $sign\big((\beta A_m^{(i)} \beta^{-1} A_m^{(j)})^n\big) \neq 0\}$

3 Random walks: proof of Theorem 3

Similarly to Rivin in [Ri], we introduce the finite groups $Sp(2l, \mathbb{Z}_p)$'s for a prime number $p > 2$.
A theorem of A'Campo ([A'C], Theorem 1 (1)) shows that the representation $B(n) \longrightarrow Sp(2l, \mathbb{F}_p)$ is surjective. So we consider the surjective map

$$\Pi_p : B(n) \xrightarrow{\rho_n} Sp(2l, \mathbb{Z}) \longrightarrow Sp(2l, \mathbb{F}_p) \xrightarrow{\pi} PSp(2l, \mathbb{F}_p) \tag{10}$$

We consider the image probability $\Pi_p \mu$ of μ via Π_p on $PSp(2l, \mathbb{F}_p)$. We recall facts on random walks on finite groups.

Theorem 5. *(see for exemple [Di]) Let G be a finite group and μ a probability measure on G such that*

1. *G is generated as a semigroup by $supp(\mu)$*
2. *$supp(\mu)$ is not included in a coset of G by a normal subgroup H of G.*

Then $\mu^{\star m}$ converges to the equidistributed measure on G as m tends to infinity.

We know ([As]) that, if $p > 5$, the group $PSp(2n, \mathbb{F}_p)$ is simple so we can apply Theorem 5 and derive

Lemma 1. *If m tends to infinity, $(\Pi_p \mu)^{\star m}$ converges to the equidistributed measure on $PSp(2n, \mathbb{F}_p)$.*

We now investigate the density of the zero set in $Sp(2l, \mathbb{F}_p)$ of the polynomial P appearng in Theorem 3.
We write P in terms of all the multi-indices, $P(M) = \sum a_I m^I$, where the a_I's belong to $\mathbb{Z}$. If p is an integer, we denote by P_p its reduction modulo p.

Since P is not identically zero, we derive an infinite set $\mathcal{P}$ of prime numbers p such that P_p is not identically zero. To prove the theorem, we need to estimate

$$\frac{|P_p^{-1}(0)|}{|Sp(2l,\mathbb{F}_p)|} \tag{11}$$

We recall (see for example [O'M] or [wi]) that

$$|Sp(2n,\mathbb{Z}_p)| = \prod_{m=1}^{n}\left[(p^{2m}-1)p^{2m-1}\right] \geq \frac{1}{2^{2n}}\prod_{s=1}^{2l} p^s = \frac{p^{2l^2+l}}{2^{2n}} \tag{12}$$

To estimate $|P_p^{-1}(0)|$, we use the following result by Lachaud and Rolland

Theorem 6. *([L-R]) Let K be the algebraic closure of $\mathbb{F}_p$. Let X be an algebraic variety of K^n of dimension m which is the zero set of a family of polynomials $(f_1,...,f_r)$. Let $d_i = deg(f_i)$. Then*

$$|X \cap \mathbb{F}_p^n| \leq d_1...d_r q^m$$

We view P_p as a polynomial with coefficients in K and let $X = Sp(2l,K) \cap P_p^{-1}(0)$. We know that for a field K, $Sp(2l,K)$ is irreducible as an algebraic variety ([Mi]), hence X is of dimension strictly smaller that $Sp(2l,K)$, i.e. $dim(X) \leq 2l^2+l-1$. Thus, for some constant $C_1(2l,P)$ depending on $2l$ and on the degree of P,

$$|X \cap \mathbb{F}_p^{4l^2}| \leq C_1(2l,P)p^{2l^2+l-1} \tag{13}$$

Putting (12) and (13) together, we derive that

$$\frac{|P_p^{-1}(0)|}{|Sp(2l,\mathbb{F}_p)|} \leq \frac{C_2(2l,P)}{p} \tag{14}$$

for some constant $C_2(2l,P)$.
We conclude the proof of Theorem 3. Given a $\epsilon > 0$, we pick a prime number p in $\mathcal{P}$ such that

$$2\frac{C_2(2l,P)}{p} < \epsilon \tag{15}$$

There exists an integer k_0 such that for every $k > k_0$, we have

$$\mu^{\star k}(P^{-1}(0)) \leq (\Pi_p\mu)^{\star k}(P_p^{-1}(0)) < \epsilon.$$

□

4 The signature of 3-braids: proof of Corollary 2

4.1 Preliminaries: the Gambaudo-Ghys formula for the signature of a 3-braid ([G-G])

4.1.1 The reduced Burau representation for $t = -1$

We let B_3 be the 3-braid group and we denote by

$$\mathcal{B}_{-1} : B(3) \longrightarrow SL(2, \mathbb{Z})$$

the reduced Burau representation of the braid group $B(3)$ for $t = -1$. Because there are different expressions in the literature, we recall the ones [G-G] uses:

$$s_1 = \mathcal{B}_{-1}(\sigma_1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad s_2 = \mathcal{B}_{-1}(\sigma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{16}$$

We recall (see for example [K-T]):

Fact 1. *The center of $B(3)$ is the cyclic group generated by the twist*

$$\Delta = (\sigma_1\sigma_2\sigma_1)^2.$$

Fact 2.

- $\mathcal{B}_{-1}$ *is surjective*
- $\mathcal{B}_{-1}(\Delta) = -Id$
- *The kernel of $\mathcal{B}_{-1}$ is the cyclic group generated by Δ^2.*

4.1.2 The Meyer cocycle and the formula for the signature

If β is a braid, we denote its closure by $\hat{\beta}$. For two 3-braids, α and β, [G-G] proves

$$sign(\widehat{\alpha.\beta}) = sign(\hat{\alpha}) + sign(\hat{\beta}) - Meyer(\mathcal{B}_{-1}(\alpha), \mathcal{B}_{-1}(\beta)) \tag{17}$$

where $Meyer \in H^2(SL(2, \mathbb{R}))$ is the *Meyer cocycle*. We recall its description by [G-G].
If $\gamma_1, \gamma_2 \in SL(2, \mathbb{Z})$, we let

$$E_{\gamma_1,\gamma_2} = Im(\gamma_1^{-1} - Id) \cap Im(\gamma_2 - Id) \tag{18}$$

For a vector e in E_{γ_1,γ_2}, we take v_1, v_2 in $\mathbb{R}^2$ such that

$$e = \gamma_1^{-1}(v_1) - v_1 = v_2 - \gamma_2(v_2)$$

and define the quadratic form

$$q_{\gamma_1,\gamma_2} = \Omega(e, v_1 + v_2) \tag{19}$$

where Ω is the standard symplectic form on $\mathbb{R}^2$. Then $Meyer(\gamma_1, \gamma_2)$ is the signature of q_{γ_1,γ_2}.

Fact 3. *If $\gamma \in SL(2,\mathbb{Z})$ is hyperbolic (i.e. $|tr(\gamma)| > 2$), then for two positive integers a, b,*

$$Meyer(\gamma^a, \gamma^b) = 0$$

REMARK. Since a generic element of $SL(2,\mathbb{Z})$ is hyperbolic, it follows from Fact 3 that for almost every braid $\beta \in B(3)$, $sign(\beta^n) = nsign(\beta)$.

4.2 Proof of Corollary 2

Since σ_i and $\beta\sigma_j\beta^{-1}$ close in trivial knots, $sign(\sigma_i) = sign(\beta\sigma_j\beta^{-1}) = 0$; hence

$$sign(\sigma_i\beta\sigma_j\beta^{-1}) = Meyer(\sigma_i, \beta\sigma_j\beta^{-1}) \tag{20}$$

Lemma 2. *Let $i, j \in \{1, 2\}$ and $\beta \in B(3)$. If $Meyer(\sigma_i, \beta\sigma_j\beta^{-1}) \neq 0$, then one of the coefficients of the matrix $\mathcal{B}_{-1}(\beta)$ is zero.*

Proof. Recall that $Meyer(\sigma_i, \beta\sigma_j\beta^{-1})$ is the signature of a quadratic form on the space $E_{\sigma_i,\beta\sigma_j\beta^{-1}}$, defined in §4.1.2; thus, if $Meyer(\sigma_i, \beta\sigma_j\beta^{-1})$ is non zero, the space $E_{\sigma_i,\beta\sigma_j\beta^{-1}}$ contains a non-zero element, i.e. there exist non zero v_1, v_2 such that

$$(s_i^{-1} - Id)(v_i) = \beta \circ (Id - s_j)[\beta^{-1}(v_2)] \tag{21}$$

If we denote by e_1, e_2 the canonical basis of $\mathbb{R}^2$, we notice that

$$Im(s_1^{\pm 1} - Id) = \mathbb{R}e_2 \qquad Im(s_2^{\pm 1} - Id) = \mathbb{R}e_1 \tag{22}$$

It follows that the (LHS) of (21) is parallel to e_1 or e_2 and the (RHS) is parallel to $\beta(e_1)$ or $\beta(e_2)$. Thus, for some $h, k \in \{1, 2\}$, $\beta(e_h)$ is parallel to e_k which implies that one of the coefficients of the matrix of β is zero. $\square$

Lemma 3. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. *The matrix* $Ms_1M^{-1}s_1$ *(resp.* $Ms_2M^{-1}s_2$, $Ms_1M^{-1}s_2$, $Ms_2M^{-1}s_1$*) is hyperbolic unless*

$$b^2 \leq 4 \quad (\textit{resp. } c^2 \leq 4,\ d^2 \leq 4,\ a^2 \leq 4) \tag{23}$$

Proof. Compute the traces, for example $Trace(Ms_2M^{-1}s_1) = 2 - a^2$. □

We can now conclude the proof of Corollary 2. We let β be a 3-braid such that

$$\forall i,j \in \{1,2\} \quad |\big(\mathcal{B}_{-1}(\beta)\big)_{ij}| > 2 \tag{24}$$

Then $sign(\sigma_a\beta\sigma_b\beta^{-1}) = 0$ (Lemma 2) and $\sigma_a\beta\sigma_b\beta^{-1}$ is hyperbolic (Lemma 3). Thus $Meyer(\mathcal{B}_{-1}(\sigma_a\beta\sigma_b\beta^{-1}), \mathcal{B}_{-1}((\sigma_a\beta\sigma_b\beta^{-1})^n) = 0$ (Fact 3 of §4.1.2). It follows that, for every n,

$$sign((\sigma_a\beta\sigma_b\beta^{-1})^n) = 0.$$

On the other hand, Theorem 3 tells us that the braids not verifying (24) form a transient set for the right random μ-walks on $B(3)$. □

5 The Lissajous toric knots

5.1 Description

5.1.1 The knots $K(N,q,p)$

In [S-V], the authors exhibited Lissajous toric knots as one of the classes of boundaries of minimal disks in the 4-ball with a branch point at the origin. Before that, Lamm investigated these knots in connection with billiards in the solid torus ([L-O]). If N, q, p are integers, with $(N,q) = (N,p) = 1$, the $K(N,q,p)$ Lissajous toric knot is defined in a $3D$-cylinder as

$$F_{N,q,p} : [0, 2\pi] \longrightarrow \mathbb{S}^1 \times \mathbb{R}^2$$

$$F_{N,q,p} : \theta \mapsto (e^{Ni\theta}, \sin(q\theta), \cos(p\theta + \alpha)) \tag{25}$$

for a phase α. Endow $\mathbb{R}^3$ with a coordinate system (x,y,z) and take the Lissajous curve $C_{p,q} : \theta \mapsto \big(0, 2 + \sin(q\theta), \cos(p\theta + \alpha)\big)$; then the knot $K(N,q,p)$ is described in $\mathbb{R}^3$ by a point travelling along $C_{p,q}$ while $C_{p,q}$ rotates N times along the vertical axis Oz. We recall a few facts

Fact 4. *([S-V]) For a finite number of phases α's the expression in (25) gives us singular crossing points. Otherwise (25) defines a knot and up to mirror transformation, its knot type does not depend on the phase α.*

Thus we drop the phase α in (25) and we just talk of a knot $K(N, q, p)$ defined up to mirror symmetry.

5.1.2 The braids $B(N, q, p)$

The knot $K(N, q, p)$ has a natural N-braid representation $B(N, q, p)$.

Fact 5. *([S-V]) Let $\tilde{p}, \tilde{q}, d$ be three positive integers all coprime with N and such that $(\tilde{p}, \tilde{q}) = 1$. Moreover assume (without loss of generality) that $\tilde{q}$ is odd. Then*

$$B(N, d\tilde{q}, d\tilde{p}) = B(N, \tilde{q}, \tilde{p})^d$$

and there exists a braid $Q_{N,\tilde{q},\tilde{p}}$ such that

$$B(N, \tilde{q}, \tilde{p}) = Q_{N,\tilde{q},\tilde{p}} \sigma_2^{\epsilon(2)} Q_{N,\tilde{q},\tilde{p}}^{-1} \sigma_1^{\epsilon(1)} \tag{26}$$

with $\epsilon(1), \epsilon(2) \in \{-1, 1\}$.
If $2N$ divides $\tilde{p} + \tilde{q}$ or $\tilde{p} - \tilde{q}$, $\epsilon(1) = \epsilon(2) = 1$ so the braid is quasipositive: for $N = 3$, this happens if and only if $\tilde{q}$ and $\tilde{p}$ are both odd.

The braid (26) is a symmetric union as defined by [La] thus:

Fact 6. *If N, q, p are all mutually prime, $K(N, q, p)$ is a ribbon knot.*

See also [L-O] and [S-V].

5.2 The signature of the Lissajous toric knots

For $N = 3$, a Lissajous toric knot has zero signature unless it is a torus knot. More precisely,

Theorem 7. *Let $\tilde{q}, \tilde{p}, n$ be three positive integers, none of them divisible by 3 and $\tilde{q}, \tilde{p}$ mutually prime. Let $K(3, n\tilde{q}, n\tilde{p})$ be the corresponding Lissajous toric knot.*

1. *If $\tilde{q}$ and $\tilde{p}$ are both odd, up to mirror image, $K(3, n\tilde{q}, n\tilde{p})$ is a quasipositive knot with*

$$g_4(K(3, n\tilde{q}, n\tilde{p})) = d - 1 \tag{27}$$

and verifying one of the following

(a) *the signature of* $K(3, n\tilde{q}, n\tilde{p})$ *is zero*

(b) $B(3, \tilde{q}, \tilde{p}) = \sigma_2\sigma_1$ *so* $K(3, \tilde{q}, \tilde{p})$ *is a trivial knot and* $K(3, n\tilde{q}, n\tilde{p})$ *is a* $(3, n)$*-torus knot.*

2. *If* $\tilde{q}$ *and* $\tilde{p}$ *have different parities,* $K(3, n\tilde{q}, n\tilde{p})$ *is isotopic to its mirror image, so it has zero signature.*

REMARK 1. The Lissajous toric knots $K(3, n, n)$ are just the $(3, n)$ torus knots so clearly they are in the case 1 (b) of Theorem 7, but they are not the only ones. For example if $\tilde{p} = \tilde{q} + 6$ the knot $K(3, \tilde{q}, \tilde{q} + 6)$ (e.g. $K(3, 7, 13)$) is represented by the braid

$$B(3, \tilde{q}, \tilde{q} + 6) = \sigma_2\sigma_1$$

so it is trivial and, for a positive integer n,

$$B(3, n\tilde{q}, n(\tilde{q} + 6)) = (\sigma_2\sigma_1)^n$$

so $K(3, n\tilde{q}, n(\tilde{q} + 6))$ is a $(3, n)$-torus knot.
In §6.2, we see that at least for $\tilde{q}$'s up to 100, the majority of knots of 1. in Theorem 7 1. are in the case 1. (a) of that Theorem.

An example is $K(3, 5, 7)$ which is, up to mirror image, the knot 10_{155} in the Rolfsen classification and its braid is

$$B(3, 5, 7) = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1 \tag{28}$$

5.3 Proof of Theorem 7

The proof depends on the parities of $\tilde{q}$ and $\tilde{p}$; since these numbers are mutually prime, either they have the same parity or they are both odd.

5.3.1 1st case: $\tilde{q}$ and $\tilde{p}$ are both odd

Up to mirror symmetry, the braid $B(3, \tilde{q}, \tilde{p})$ is of the form (Fact 5)

$$B(3, \tilde{q}, \tilde{p}) = Q_{3,\tilde{q},\tilde{p}}\sigma_2 Q_{3,\tilde{q},\tilde{p}}^{-1}\sigma_1$$

with ([S-V])

$$Q_{3,\tilde{q},\tilde{p}} = \sigma_2^{\lambda(1)}\sigma_1^{\lambda(2)}\sigma_2^{\lambda(3)}\sigma_1^{\lambda(4)}...\sigma_2^{\lambda(\tilde{q}-2)}\sigma_1^{\lambda(\tilde{q}-1)} \tag{29}$$

where λ is an expression with values in $\{-1,1\}$ verifying

$$\lambda(k) = -\lambda(\tilde{q}-k)$$

so we rewrite $Q_{3,\tilde{q},\tilde{p}}$ and introduce the braid P:

$$Q_{3,\tilde{q},\tilde{p}} = \underbrace{\sigma_2^{\lambda(1)}\sigma_1^{\lambda(2)}\sigma_2^{\lambda(\frac{\tilde{q}-3}{2})}\sigma_1^{\lambda(\frac{\tilde{q}-1}{2})}}_{P}\sigma_2^{-\lambda(\frac{\tilde{q}-1}{2})}\sigma_1^{-\lambda(\frac{\tilde{q}-3}{2})}...\sigma_2^{-\lambda(2)}\sigma_1^{-\lambda(1)} \tag{30}$$

We have defined braids and we now turn to their images under the Burau representation B_{-1}. We let

$$\mathcal{Q} = \mathcal{B}_{-1}(Q_{3,\tilde{q},\tilde{p}}) \qquad\qquad \mathcal{P} = \mathcal{B}_{-1}(P).$$

We notice that the s_i's of (16) verify

$$s_2 = {}^t s_1^{-1} \tag{31}$$

and derive

$$\mathcal{Q} = \mathcal{P}^t\mathcal{P} \tag{32}$$

Lemma 4. *$\mathcal{B}_{-1}(B(3,\tilde{q},\tilde{p}))$ is hyperbolic except if $\mathcal{P}$ is of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\textit{with} \quad a,b \in \{-1,0,1\} \tag{33}$$

Proof. Compute $Trace(\mathcal{B}_{-1}(B(3,\tilde{q},\tilde{p}))) =$

$$Trace[(\mathcal{B}_{-1}(\mathcal{P})^t\mathcal{B}_{-1}(\mathcal{P}))s_2(\mathcal{B}_{-1}(\mathcal{P})^t\mathcal{B}_{-1}(\mathcal{P}))^{-1}s_1] = 2-(a^2+b^2)^2.$$

hence if it is hyperbolic, $a^2+b^2=1$ or $a^2+b^2=2$ □

If $\mathcal{P}$ verifies (33), it has one of the following forms, for some integer h.

- $\mathcal{P} = (\pm 1)\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} = (\pm 1)s_1^{-h}$
  Thus $\mathcal{Q} = \mathcal{P}^t\mathcal{P} = s_1^{-h}s_2^h$ and, for some integer k,

  $$\beta = (\sigma_1^{-h}\sigma_2^h\Delta^k)\sigma_2(\Delta^{-k}\sigma_2^{-h}\sigma_1^h)\sigma_1 = \sigma_1^{-h}(\sigma_2\sigma_1)\sigma_1^h$$

  so β is conjugate to $\sigma_2\sigma_1$ and the knot is as in 1. (b) of Theorem 7.

- $\mathcal{P} = (\pm 1)\begin{pmatrix} 0 & 1 \\ -1 & h \end{pmatrix} = (\pm 1) s_1^{1-h} s_2 s_1$
  Thus $\mathcal{Q} = s_1^{-h} s_2^{-h}$ and similarly to above,
  $$\beta = \sigma_1^{-h}(\sigma_2\sigma_1)\sigma_1^h.$$
  Again we are in Case 1. (b) of Theorem 7.

- $\mathcal{P} = (\pm 1)\begin{pmatrix} 1 & 1 \\ h & h+1 \end{pmatrix} = (\pm 1) s_1^{-h} s_2$
  Then $\mathcal{Q} = s_1^{-h} s_2 s_1^{-1} s_2^h$ and
  $$\beta = \sigma_1^{-h}(\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1)\sigma_1^{-h}$$
  This braid β closes into a link with 3 components, thus it cannot be the braid of a knot.

- $\mathcal{P} = (\pm 1)\begin{pmatrix} 1 & -1 \\ h & 1-h \end{pmatrix} = s_1^{-h} s_2^{-1}$ and
  $$\beta = \sigma_1^{-h}(\sigma_2^{-1}\sigma_1\sigma_2)(\sigma_1^{-1}\sigma_2\sigma_1)\sigma_1^h$$
  Again β closes in a link with 3 components. □

5.3.2 2nd case: $\tilde{q}$ and $\tilde{p}$ have different parities

The knot is preserved by an isometry which is a rotation in one plane and a symmetry in its orthogonal plane, thus the knot is isotopic to its mirror image and has zero signature.
To spell this out, we assume that $\tilde{q}$ is odd and $\tilde{p}$ is even. The knot $K(N, d\tilde{q}, d\tilde{p})$ is defined by the function $F_{N,d\tilde{q},d\tilde{p}}$ (see (25) above), so we can also define it by the function

$$\theta \mapsto F_{N,d\tilde{q},d\tilde{p}}(\theta + \frac{\pi}{d}) = (e^{iN\theta}e^{iN\frac{\pi}{d}}, -\sin(d\tilde{q}\theta), \cos(d\tilde{p}\theta + \alpha))$$

thus $K(N, d\tilde{q}, d\tilde{p})$ is invariant under the transformation

$$\begin{pmatrix} \cos(\frac{N\pi}{d}) & -\sin(\frac{N\pi}{d}) & 0 & 0 \\ \sin(\frac{N\pi}{d}) & \cos(\frac{N\pi}{d}) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{34}$$

which reverses the orientation on $\mathbb{R}^4$ and on $\mathbb{S}^3$. □

6 Statistics

6.1 Random quasipositive braids

The following estimates were done with SAGE.

We define

$$Z = \{\beta \in B(3) : |\big(\mathcal{B}_{-1}(\beta)\big)_{11}| > 2\}.$$

It follows from the previous discussion that, if a 3-braid β belongs to Z, then for all positive integers n, we have

$$sign\Big((\beta\sigma_1\beta^{-1}\sigma_2)^n\Big) = 0.$$

Let W be the μ-random walk on $B(3)$ given by the probability μ on $B(3)$ where

$$\mu(\sigma_1) = \mu(\sigma_2) = \mu(\sigma_1^{-1}) = \mu(\sigma_2^{-1}) = \frac{1}{4}$$

Here are the probabilities $p(n)$ that W hits Z at the n-th step, for $n \leq 12$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$p(n)$	0	0	0.06	0.11	0.17	0.22	0.27	0.32	0.36	0.41	0.45	0.48

.

6.2 Lissajous toric knots with 3 strands

We investigate here the proportion of knots which verify (a) or (b) of 1. of Theorem 7. 1. First we prove

Lemma 5. *To get a full list of the $K(3, \tilde{q}, \tilde{p})$'s for a given $\tilde{q}$, it is enough to take*

$$\tilde{p} \in [\tilde{q} + 1, 2\tilde{q}]$$

Proof. If $\tilde{q}$ and $\tilde{p}$ are odd, mutually prime and not divisible by 3, we use the representation of $K(3, \tilde{q}, \tilde{p})$ from (5.3.1). The expression of λ is based on the Bezout identity:

$$6A + Bq = 1 \tag{35}$$

for two integers A and B. Then, up to mirror transformation

$$\lambda(k) = (-1)^{\left[\frac{2Apk}{q}\right]} \tag{36}$$

It follows from (36) that, if we replace p by $p+q$, the expression of λ remains the same. $\square$

For a given $\tilde{q}$, we let

$$P(\tilde{q}) = \{\tilde{p} \in [\tilde{q}+1, 2\tilde{q}], \text{ such that } (2,\tilde{p}) = (3,\tilde{p}) = 1\}$$

$$S(\tilde{q}) = \{\tilde{p} \in P(\tilde{q}), \text{ such that } K(3,\tilde{q},\tilde{p}) \text{ verifies 1) (a) of Theorem 7}\}$$

The table below gives us, for a given $\tilde{q}$, the percentage of elements of $P(\tilde{q})$ which belong to $S(\tilde{q})$: i.e. the percentage of $K(3,\tilde{q},\tilde{p})$'s for which every $K(3,n\tilde{q},n\tilde{p})$, $n \in \mathbb{N}$, has zero signature. The percentages have been rounded to the integer part.

$\tilde{q}$	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49
%	40	57	54	61	70	63	69	64	68	77	62	75	78	74	80	69

$\tilde{q}$	53	55	59	61	65	67	71	73	77	79	83	85	89	91	95	97	101
%	79	72	77	78	76	80	67	82	75	83	84	77	80	74	80	82	83

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