



**HAL**  
open science

# Random walks and the symplectic representation of the braid group

Marc Soret, Marina Ville

► **To cite this version:**

Marc Soret, Marina Ville. Random walks and the symplectic representation of the braid group. 2022. hal-03864935

**HAL Id: hal-03864935**

**<https://hal.science/hal-03864935>**

Preprint submitted on 28 Nov 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Random walks and the symplectic representation of the braid groups

Marc Soret and Marina Ville

March 3, 2022

## Abstract

We consider the symplectic representation  $\rho_n$  of a braid group  $B(n)$  in  $Sp(2l, \mathbb{Z})$ , for  $l = \lfloor \frac{n-1}{2} \rfloor$ . If  $P$  is a  $4l^2$  polynomial on the coefficients of the matrices in  $Sp(2l, \mathbb{Z})$ , we show that the set  $\{\beta \in B(n) / P(\rho_n(\beta)) = 0\}$  is transient for non degenerate random walks on  $B(n)$ . If  $n$  is odd, we derive that the  $n$ -braids  $\beta$  verifying  $|\Delta_{\beta}(-1)| \leq C$  for some constant  $C$  form a transient set: here  $\Delta_{\beta}$  denotes the Alexander polynomial of the closure of  $\beta$ .

We also derive that for a random 3-braid, the quasipositive links  $(\beta\sigma_i\beta^{-1}\sigma_j)^p$  have zero signature for every integer  $p$  and  $1 \leq i, j \leq 2$ . As an example of such braids, we investigate the signature of the Lisajous toric knots with 3 strands.

## 1 Preliminaries

### 1.1 Representations of the braid group

If  $B(n)$  is the group of braids with  $n$  strands, we denote by  $\mathcal{B}$  its *reduced Burau representation* which represents braids as  $(n-1) \times (n-1)$  matrices with values in  $\mathbb{Z}[t, t^{-1}]$ . The specialization of  $\mathcal{B}$  for  $t = -1$ , denoted  $\mathcal{B}_{-1}$ , is called the *integral reduced Burau representation*. The representation  $\mathcal{B}_{-1}$  is closely related to the *symplectic representation* of  $B(n)$  which we now recall, using the description of A'Campo ([A'C]).

The braid group  $B(n+1)$  acts on the homology  $H_1(X, \mathbb{Z})$  of a surface  $X$  such that  $H_1(X, \mathbb{Z})$  is freely generated by  $n$  loops. This action is conjugate to the

integral reduced Burau representation. Concretely speaking, this means the existence of a basis of  $H_1(X, \mathbb{Z})$  w.r.t. which the matrices representing the elements of  $B(n+1)$  are the ones given by  $\mathcal{B}_{-1}$ .

This action of  $B(n+1)$  preserves the intersection form  $I$  on  $H_1(X, \mathbb{Z})$ .

1. If  $n$  is even,  $n = 2l$ , the form  $I$  is symplectic nondegenerate on  $H_1(X, \mathbb{Z})$ .
2. If  $n$  is odd,  $n = 2l + 1$ , the kernel of  $I$  is a line  $K$ ;  $\mathcal{B}_{-1}$  restricts to the identity on  $K$  and acts symplectically on the quotient  $H_1(X, \mathbb{Z})/K$

Thus, in both cases, we get a *symplectic representation*

$$\rho_n : B(n+1) \longrightarrow Sp(2l, \mathbb{Z}) \quad (1)$$

## 1.2 Random walks on groups

If  $\mu_1, \mu_2$  are probability measures on a discrete group  $G$  with finite support, their *convolution* is defined for  $g \in G$  as

$$\mu_1 \star \mu_2(g) = \sum_{h \in G} \mu_1(g^{-1}h) \mu_2(h) \quad (2)$$

**Definition 1.** A probability  $\mu$  on a discrete group  $G$  is nondegenerate if  $\text{supp}(\mu)$  generates  $G$  as a semigroup.

Malyutin ([Ma]) defines the *right random  $\mu$ -walk* as the Markov chain which starts at the identity of  $G$  and with transition probability  $P(g, h) = \mu(gh^{-1})$ . If  $X$  is a subset of  $G$ , the probability that this walk hits  $X$  at the  $k$ -th step is  $\mu^{\star k}(X)$ .

## 1.3 The signature of a link

The signature is a classical invariant of links and knots; (see for example [Mu]). If  $\gamma$  is a link on  $\mathbb{S}^3$  which bounds a Seifert surface  $\Sigma$  in  $\mathbb{S}^3$ , we consider the intersection form

$$\begin{aligned} I : H_1(\Sigma) \times H_1(\Sigma) &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\mapsto lk(\hat{\alpha}, \beta) \end{aligned}$$

where  $lk$  is the linking number and  $\hat{\alpha}$  is the link obtained by pushing  $\alpha$  in the direction normal to  $\Sigma$  in  $\mathbb{S}^3$ . Given a base for  $H_1(\Sigma)$ , we define the matrix

$V$  of  $I$ : the signature of  $\gamma$  is the signature of the bilinear form defined by the matrix  $V + {}^t V$ . We point out that the literature contains two different definitions of the signature which coincide up to sign.

The signature of a link can be computed on a braid representing it and Gambodau-Ghys have estimated the signature of a product of two braids in terms of the signatures of the two braids ([G-G]). We recall (cf. [Mu]) that the for a knot  $K$ ,

$$|\text{sign}(K)| \leq 2g_4(K) \quad (3)$$

where  $g_4$  denotes the topological 4-genus. Thus a slice knot has zero signature but the converse is not true and we will see more examples of that here.

## 1.4 Positive links and quasipositive braids

A *positive link* is a link which has a diagram with all positive crossings.

**Theorem 1.** ([Pr]) *A positive link has strictly negative signature.*

So it is interesting to look at the signature of quasipositive braids:

**Definition 2.** ([Ru]) *A braid  $\beta$  is quasipositive if it is a product of conjugates of positive braid generators*

$$\beta = \prod_{i=1}^k \gamma_{j_i} \sigma_{j_i} \gamma_{j_i}^{-1} \quad (4)$$

If we close the quasipositive  $N$ -braid (4) in a link  $\hat{\beta}$ , Rudolph proved ([Ru]) that

$$\chi_4(\hat{\beta}) = N - k \quad (5)$$

where  $\chi_4$  denotes the largest Euler characteristic of a *smooth* surface in  $\mathbb{B}^4$  bounded by  $\hat{\beta}$ .

Tanaka ([Ta]) constructed examples of quasipositive braids with zero signature.

## 1.5 Transient sets in the braid groups from quasimorphisms

We recall that a *quasimorphism* on a group  $G$  is a function  $\Phi : G \rightarrow \mathbb{R}$  such that there exists a positive constant  $C$  with

$$\forall a, b \in G, |\Phi(ab) - \Phi(a) - \Phi(b)| \leq D_\Phi \quad (6)$$

where  $D_\Phi$  is called the *defect* of  $\Psi$ .

Quasimorphisms give us examples of transient sets in the braid groups:

**Theorem 2.** ([Ma]) *Let  $G$  be a countable group and  $\mu$  a nondegenerate probability on  $G$  (see Definition 1). If a subset  $S$  of  $G$  has bounded image under an unbounded quasimorphism  $\Phi$ , then the probability that the right random  $\mu$ -walk hits  $S$  at the  $k$ -th step tends to zero as  $k$  tends to infinity.*

We will see below how Gambodau-Ghys's formula for the signature turns it into a quasimorphism. Several other link invariants such as  $\chi_4$  can be used to define quasimorphisms on the braid group ([Br], [B-K]). Other examples of transient sets on the braid groups have been constructed by Ito [It]

## Acknowledgements

We thank Tahl Nowik and Chaim Even-Zohar for reading an earlier draft of this paper, Misha Brandenbursky for very helpful conversation on braids and quasimorphisms and Florence Lecomte for necessary advice on algebraic geometry over finite fields.

## 2 The results

**Theorem 3.** *Let  $P$  be a polynomial in  $(2l)^2$  variables with coefficients in  $\mathbb{Z}$ . If  $M = (m_{ij}) \in Sp(2l, \mathbb{Z})$ , we let  $P(M) = P(m_{11}, \dots, m_{12l}, m_{21}, \dots, m_{22l}, m_{211}, \dots, m_{212l})$ . Suppose that  $P$  does not vanish identically on  $Sp(2l, \mathbb{Z})$ . Then the set*

$$\{\beta \in B(n) / P(\rho_n(\beta)) = 0\}$$

*is transient for the right random  $\mu$ -walk.*

### 2.1 The Alexander polynomial

**Corollary 1.** *Let  $C$  be a real number. If  $2s+1$  is an odd integer, the following set is transient for the right random  $\mu$ -walk on  $B(2s+1)$*

$$\{\beta \in B(2s+1) / |\Delta_{\hat{\beta}}(-1)| < C\} \tag{7}$$

*where  $\Delta_{\hat{\beta}}$  denotes the Alexander polynomial of the closure of  $\beta$ .*

*Proof.* The Alexander polynomial can be expressed in terms of the Burau representation  $\mathcal{B}$  ([Bu]),

$$\Delta_{\hat{\beta}}(t) = \frac{1-t}{1-t^n} \det(\mathcal{B}_{-t}(\beta) - I) \quad (8)$$

If  $n$  is odd,  $\Delta_{\hat{\beta}}(-1) = \det(\rho_n(\beta) - I)$  and the set (7) is a finite union of sets described in Theorem 3.  $\square$

REMARK. If  $n$  is even,  $1 - (-1)^n = 0$  and  $\det(\mathcal{B}_{-1}(\beta) - I) = 0$  since  $\mathcal{B}_{-1}$  acts on  $K$  as the identity; thus the formula (8) is not easy to use. Nevertheless, it seems likely that Theorem 1 is true also for even  $n$ 's.

## 2.2 The signature of 3-braids

**Corollary 2.** *Let  $\mu$  be a nondegenerate probability on the braid group  $B(3)$ . For the right random  $\mu$ -walks on  $B(3)$ , the set*

$$S = \{\beta \in B(3) / \exists n \in \mathbb{N}, \exists i, j \in \{1, 2\} \text{ such that } \text{sign}((\beta\sigma_i\beta^{-1}\sigma_j)^n) \neq 0\} \quad (9)$$

*is transient.*

By contrast, (5) tells us that for every  $n$ ,  $\chi_4((\beta\sigma_1\beta^{-1}\sigma_1)^n) = 3 - 2n$ ; in particular, if  $(\beta\sigma_1\beta^{-1}\sigma_1)^n$  closes into a knot,  $g_4((\beta\sigma_1\beta^{-1}\sigma_1)^n) = n - 1$ .

In §5, we recall the Lissajous toric knots with 3 strands (cf. [S-V]); they are naturally of the form  $(\beta\sigma_2\beta^{-1}\sigma_1^{\pm 1})^n$  and we prove

**Theorem 4.** *A Lissajous toric knot  $K(3, p, q)$  has zero signature unless it is isotopic to a torus knot.*

We conclude by statistical estimates, done with SAGE.

These results suggest possibles generalizations.

**Question 1.** *Let  $n \in \mathbb{N}$  and let  $\mathcal{I}$  be a numerical link invariant which is unbounded on the closures of the links represented by a  $n$ -braid. If  $C$  is a constant, is the set  $\{\mathcal{I}(\hat{\beta}) \leq C\}$  transient for the right random  $\mu$ -walks?*

**Question 2.** For an integer  $m$ , we set

$$A_m^{(1)} = \prod_{1 \leq 2i+1 \leq m} \sigma_{2i+1} \quad A_m^{(2)} = \prod_{1 \leq 2i \leq m} \sigma_{2i}$$

Is the following set  $S_m$  transient in  $B(m)$  for the right random  $\mu$ -walks?

$$S_m = \{\beta \in B(m) / \exists n \in \mathbb{N}, \exists i, j \in \{1, 2\} \text{ such that } \text{sign}((\beta A_m^{(i)} \beta^{-1} A_m^{(j)})^n) \neq 0\}$$

### 3 Random walks: proof of Theorem 3

Similarly to Rivin in [Ri], we introduce the finite groups  $Sp(2l, \mathbb{Z}_p)$ 's for a prime number  $p > 2$ .

A theorem of A'Campo ([A'C], Theorem 1 (1)) shows that the representation  $B(n) \rightarrow Sp(2l, \mathbb{F}_p)$  is surjective. So we consider the surjective map

$$\Pi_p : B(n) \xrightarrow{\rho_n} Sp(2l, \mathbb{Z}) \rightarrow Sp(2l, \mathbb{F}_p) \xrightarrow{\pi} PSp(2l, \mathbb{F}_p) \quad (10)$$

We consider the image probability  $\Pi_p \mu$  of  $\mu$  via  $\Pi_p$  on  $PSp(2l, \mathbb{F}_p)$ . We recall facts on random walks on finite groups.

**Theorem 5.** (see for exemple [Di]) Let  $G$  be a finite group and  $\mu$  a probability measure on  $G$  such that

1.  $G$  is generated as a semigroup by  $\text{supp}(\mu)$
2.  $\text{supp}(\mu)$  is not included in a coset of  $G$  by a normal subgroup  $H$  of  $G$ .

Then  $\mu^{*m}$  converges to the equidistributed measure on  $G$  as  $m$  tends to infinity.

We know ([As]) that, if  $p > 5$ , the group  $PSp(2n, \mathbb{F}_p)$  is simple so we can apply Theorem 5 and derive

**Lemma 1.** If  $m$  tends to infinity,  $(\Pi_p \mu)^{*m}$  converges to the equidistributed measure on  $PSp(2n, \mathbb{F}_p)$ .

We now investigate the density of the zero set in  $Sp(2l, \mathbb{F}_p)$  of the polynomial  $P$  appearing in Theorem 3.

We write  $P$  in terms of all the multi-indices,  $P(M) = \sum a_I m^I$ , where the  $a_I$ 's belong to  $\mathbb{Z}$ . If  $p$  is an integer, we denote by  $P_p$  its reduction modulo  $p$ .

Since  $P$  is not identically zero, we derive an infinite set  $\mathcal{P}$  of prime numbers  $p$  such that  $P_p$  is not identically zero. To prove the theorem, we need to estimate

$$\frac{|P_p^{-1}(0)|}{|Sp(2l, \mathbb{F}_p)|} \quad (11)$$

We recall (see for example [O'M] or [wi]) that

$$|Sp(2n, \mathbb{Z}_p)| = \prod_{m=1}^n [(p^{2m} - 1)p^{2m-1}] \geq \frac{1}{2^{2n}} \prod_{s=1}^{2l} p^s = \frac{p^{2l^2+l}}{2^{2n}} \quad (12)$$

To estimate  $|P_p^{-1}(0)|$ , we use the following result by Lachaud and Rolland

**Theorem 6.** ([L-R]) *Let  $K$  be the algebraic closure of  $\mathbb{F}_p$ . Let  $X$  be an algebraic variety of  $K^n$  of dimension  $m$  which is the zero set of a family of polynomials  $(f_1, \dots, f_r)$ . Let  $d_i = \deg(f_i)$ . Then*

$$|X \cap \mathbb{F}_p^n| \leq d_1 \dots d_r q^m$$

We view  $P_p$  as a polynomial with coefficients in  $K$  and let  $X = Sp(2l, K) \cap P_p^{-1}(0)$ . We know that for a field  $K$ ,  $Sp(2l, K)$  is irreducible as an algebraic variety ([Mi]), hence  $X$  is of dimension strictly smaller than  $Sp(2l, K)$ , i.e.  $\dim(X) \leq 2l^2 + l - 1$ . Thus, for some constant  $C_1(2l, P)$  depending on  $2l$  and on the degree of  $P$ ,

$$|X \cap \mathbb{F}_p^{4l^2}| \leq C_1(2l, P) p^{2l^2+l-1} \quad (13)$$

Putting (12) and (13) together, we derive that

$$\frac{|P_p^{-1}(0)|}{|Sp(2l, \mathbb{F}_p)|} \leq \frac{C_2(2l, P)}{p} \quad (14)$$

for some constant  $C_2(2l, P)$ .

We conclude the proof of Theorem 3. Given a  $\epsilon > 0$ , we pick a prime number  $p$  in  $\mathcal{P}$  such that

$$2 \frac{C_2(2l, P)}{p} < \epsilon \quad (15)$$

There exists an integer  $k_0$  such that for every  $k > k_0$ , we have

$$\mu^{\star k}(P^{-1}(0)) \leq (\Pi_p \mu)^{\star k}(P_p^{-1}(0)) < \epsilon.$$

□



## 4 The signature of 3-braids: proof of Corollary 2

### 4.1 Preliminaries: the Gambaudo-Ghys formula for the signature of a 3-braid ([G-G])

#### 4.1.1 The reduced Burau representation for $t = -1$

We let  $B_3$  be the 3-braid group and we denote by

$$\mathcal{B}_{-1} : B(3) \longrightarrow SL(2, \mathbb{Z})$$

the reduced Burau representation of the braid group  $B(3)$  for  $t = -1$ . Because there are different expressions in the literature, we recall the ones [G-G] uses:

$$s_1 = \mathcal{B}_{-1}(\sigma_1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad s_2 = \mathcal{B}_{-1}(\sigma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (16)$$

We recall (see for example [K-T]):

**Fact 1.** *The center of  $B(3)$  is the cyclic group generated by the twist*

$$\Delta = (\sigma_1 \sigma_2 \sigma_1)^2.$$

**Fact 2.** •  $\mathcal{B}_{-1}$  is surjective

- $\mathcal{B}_{-1}(\Delta) = -Id$
- The kernel of  $\mathcal{B}_{-1}$  is the cyclic group generated by  $\Delta^2$ .

#### 4.1.2 The Meyer cocycle and the formula for the signature

If  $\beta$  is a braid, we denote its closure by  $\hat{\beta}$ . For two 3-braids,  $\alpha$  and  $\beta$ , [G-G] proves

$$sign(\widehat{\alpha \cdot \beta}) = sign(\hat{\alpha}) + sign(\hat{\beta}) - Meyer(\mathcal{B}_{-1}(\alpha), \mathcal{B}_{-1}(\beta)) \quad (17)$$

where  $Meyer \in H^2(SL(2, \mathbb{R}))$  is the *Meyer cocycle*. We recall its description by [G-G].

If  $\gamma_1, \gamma_2 \in SL(2, \mathbb{Z})$ , we let

$$E_{\gamma_1, \gamma_2} = Im(\gamma_1^{-1} - Id) \cap Im(\gamma_2 - Id) \quad (18)$$

For a vector  $e$  in  $E_{\gamma_1, \gamma_2}$ , we take  $v_1, v_2$  in  $\mathbb{R}^2$  such that

$$e = \gamma_1^{-1}(v_1) - v_1 = v_2 - \gamma_2(v_2)$$

and define the quadratic form

$$q_{\gamma_1, \gamma_2} = \Omega(e, v_1 + v_2) \tag{19}$$

where  $\Omega$  is the standard symplectic form on  $\mathbb{R}^2$ . Then  $Meyer(\gamma_1, \gamma_2)$  is the signature of  $q_{\gamma_1, \gamma_2}$ .

**Fact 3.** *If  $\gamma \in SL(2, \mathbb{Z})$  is hyperbolic (i.e.  $|\text{tr}(\gamma)| > 2$ ), then for two positive integers  $a, b$ ,*

$$Meyer(\gamma^a, \gamma^b) = 0$$

REMARK. Since a generic element of  $SL(2, \mathbb{Z})$  is hyperbolic, it follows from Fact 3 that for almost every braid  $\beta \in B(3)$ ,  $sign(\beta^n) = nsign(\beta)$ .

## 4.2 Proof of Corollary 2

Since  $\sigma_i$  and  $\beta\sigma_j\beta^{-1}$  close in trivial knots,  $sign(\sigma_i) = sign(\beta\sigma_j\beta^{-1}) = 0$ ; hence

$$sign(\sigma_i\beta\sigma_j\beta^{-1}) = Meyer(\sigma_i, \beta\sigma_j\beta^{-1}) \tag{20}$$

**Lemma 2.** *Let  $i, j \in \{1, 2\}$  and  $\beta \in B(3)$ . If  $Meyer(\sigma_i, \beta\sigma_j\beta^{-1}) \neq 0$ , then one of the coefficients of the matrix  $\mathcal{B}_{-1}(\beta)$  is zero.*

*Proof.* Recall that  $Meyer(\sigma_i, \beta\sigma_j\beta^{-1})$  is the signature of a quadratic form on the space  $E_{\sigma_i, \beta\sigma_j\beta^{-1}}$ , defined in §4.1.2; thus, if  $Meyer(\sigma_i, \beta\sigma_j\beta^{-1})$  is non zero, the space  $E_{\sigma_i, \beta\sigma_j\beta^{-1}}$  contains a non-zero element, i.e. there exist non zero  $v_1, v_2$  such that

$$(s_i^{-1} - Id)(v_i) = \beta \circ (Id - s_j)[\beta^{-1}(v_2)] \tag{21}$$

If we denote by  $e_1, e_2$  the canonical basis of  $\mathbb{R}^2$ , we notice that

$$Im(s_1^{\pm 1} - Id) = \mathbb{R}e_2 \quad Im(s_2^{\pm 1} - Id) = \mathbb{R}e_1 \tag{22}$$

It follows that the (LHS) of (21) is parallel to  $e_1$  or  $e_2$  and the (RHS) is parallel to  $\beta(e_1)$  or  $\beta(e_2)$ . Thus, for some  $h, k \in \{1, 2\}$ ,  $\beta(e_h)$  is parallel to  $e_k$  which implies that one of the coefficients of the matrix of  $\beta$  is zero.  $\square$

**Lemma 3.** Let  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ . The matrix  $Ms_1M^{-1}s_1$  (resp.  $Ms_2M^{-1}s_2$ ,  $Ms_1M^{-1}s_2$ ,  $Ms_2M^{-1}s_1$ ) is hyperbolic unless

$$b^2 \leq 4 \quad (\text{resp. } c^2 \leq 4, d^2 \leq 4, a^2 \leq 4) \quad (23)$$

*Proof.* Compute the traces, for example  $\text{Trace}(Ms_2M^{-1}s_1) = 2 - a^2$ .  $\square$

We can now conclude the proof of Corollary 2. We let  $\beta$  be a 3-braid such that

$$\forall i, j \in \{1, 2\} \quad |(\mathcal{B}_{-1}(\beta))_{ij}| > 2 \quad (24)$$

Then  $\text{sign}(\sigma_a\beta\sigma_b\beta^{-1}) = 0$  (Lemma 2) and  $\sigma_a\beta\sigma_b\beta^{-1}$  is hyperbolic (Lemma 3). Thus  $\text{Meyer}(\mathcal{B}_{-1}(\sigma_a\beta\sigma_b\beta^{-1}), \mathcal{B}_{-1}((\sigma_a\beta\sigma_b\beta^{-1})^n)) = 0$  (Fact 3 of §4.1.2). It follows that, for every  $n$ ,

$$\text{sign}((\sigma_a\beta\sigma_b\beta^{-1})^n) = 0.$$

On the other hand, Theorem 3 tells us that the braids not verifying (24) form a transient set for the right random  $\mu$ -walks on  $B(3)$ .  $\square$

## 5 The Lissajous toric knots

### 5.1 Description

#### 5.1.1 The knots $K(N, q, p)$

In [S-V], the authors exhibited Lissajous toric knots as one of the classes of boundaries of minimal disks in the 4-ball with a branch point at the origin. Before that, Lamm investigated these knots in connection with billiards in the solid torus ([L-O]). If  $N, q, p$  are integers, with  $(N, q) = (N, p) = 1$ , the  $K(N, q, p)$  Lissajous toric knot is defined in a 3D-cylinder as

$$\begin{aligned} F_{N,q,p} : [0, 2\pi] &\longrightarrow \mathbb{S}^1 \times \mathbb{R}^2 \\ F_{N,q,p} : \theta &\mapsto (e^{Ni\theta}, \sin(q\theta), \cos(p\theta + \alpha)) \end{aligned} \quad (25)$$

for a phase  $\alpha$ . Endow  $\mathbb{R}^3$  with a coordinate system  $(x, y, z)$  and take the Lissajous curve  $C_{p,q} : \theta \mapsto (0, 2 + \sin(q\theta), \cos(p\theta + \alpha))$ ; then the knot  $K(N, q, p)$  is described in  $\mathbb{R}^3$  by a point travelling along  $C_{p,q}$  while  $C_{p,q}$  rotates  $N$  times along the vertical axis  $Oz$ . We recall a few facts

**Fact 4.** ([S-V]) For a finite number of phases  $\alpha$ 's the expression in (25) gives us singular crossing points. Otherwise (25) defines a knot and up to mirror transformation, its knot type does not depend on the phase  $\alpha$ .

Thus we drop the phase  $\alpha$  in (25) and we just talk of a knot  $K(N, q, p)$  defined up to mirror symmetry.

### 5.1.2 The braids $B(N, q, p)$

The knot  $K(N, q, p)$  has a natural  $N$ -braid representation  $B(N, q, p)$ .

**Fact 5.** ([S-V]) Let  $\tilde{p}, \tilde{q}, d$  be three positive integers all coprime with  $N$  and such that  $(\tilde{p}, \tilde{q}) = 1$ . Moreover assume (without loss of generality) that  $\tilde{q}$  is odd. Then

$$B(N, d\tilde{q}, d\tilde{p}) = B(N, \tilde{q}, \tilde{p})^d$$

and there exists a braid  $Q_{N, \tilde{q}, \tilde{p}}$  such that

$$B(N, \tilde{q}, \tilde{p}) = Q_{N, \tilde{q}, \tilde{p}} \sigma_2^{\epsilon(2)} Q_{N, \tilde{q}, \tilde{p}}^{-1} \sigma_1^{\epsilon(1)} \quad (26)$$

with  $\epsilon(1), \epsilon(2) \in \{-1, 1\}$ .

If  $2N$  divides  $\tilde{p} + \tilde{q}$  or  $\tilde{p} - \tilde{q}$ ,  $\epsilon(1) = \epsilon(2) = 1$  so the braid is quasipositive: for  $N = 3$ , this happens if and only if  $\tilde{q}$  and  $\tilde{p}$  are both odd.

The braid (26) is a symmetric union as defined by [La] thus:

**Fact 6.** If  $N, q, p$  are all mutually prime,  $K(N, q, p)$  is a ribbon knot.

See also [L-O] and [S-V].

## 5.2 The signature of the Lissajous toric knots

For  $N = 3$ , a Lissajous toric knot has zero signature unless it is a torus knot. More precisely,

**Theorem 7.** Let  $\tilde{q}, \tilde{p}, n$  be three positive integers, none of them divisible by 3 and  $\tilde{q}, \tilde{p}$  mutually prime. Let  $K(3, n\tilde{q}, n\tilde{p})$  be the corresponding Lissajous toric knot.

1. If  $\tilde{q}$  and  $\tilde{p}$  are both odd, up to mirror image,  $K(3, n\tilde{q}, n\tilde{p})$  is a quasipositive knot with

$$g_4(K(3, n\tilde{q}, n\tilde{p})) = d - 1 \quad (27)$$

and verifying one of the following

- (a) the signature of  $K(3, n\tilde{q}, n\tilde{p})$  is zero
  - (b)  $B(3, \tilde{q}, \tilde{p}) = \sigma_2\sigma_1$  so  $K(3, \tilde{q}, \tilde{p})$  is a trivial knot and  $K(3, n\tilde{q}, n\tilde{p})$  is a  $(3, n)$ -torus knot.
2. If  $\tilde{q}$  and  $\tilde{p}$  have different parities,  $K(3, n\tilde{q}, n\tilde{p})$  is isotopic to its mirror image, so it has zero signature.

REMARK 1. The Lissajous toric knots  $K(3, n, n)$  are just the  $(3, n)$  torus knots so clearly they are in the case 1 (b) of Theorem 7, but they are not the only ones. For example if  $\tilde{p} = \tilde{q} + 6$  the knot  $K(3, \tilde{q}, \tilde{q} + 6)$  (e.g.  $K(3, 7, 13)$ ) is represented by the braid

$$B(3, \tilde{q}, \tilde{q} + 6) = \sigma_2\sigma_1$$

so it is trivial and, for a positive integer  $n$ ,

$$B(3, n\tilde{q}, n(\tilde{q} + 6)) = (\sigma_2\sigma_1)^n$$

so  $K(3, n\tilde{q}, n(\tilde{q} + 6))$  is a  $(3, n)$ -torus knot.

In §6.2, we see that at least for  $\tilde{q}$ 's up to 100, the majority of knots of 1. in Theorem 7 1. are in the case 1. (a) of that Theorem.

An example is  $K(3, 5, 7)$  which is, up to mirror image, the knot  $10_{155}$  in the Rolfsen classification and its braid is

$$B(3, 5, 7) = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1 \quad (28)$$

### 5.3 Proof of Theorem 7

The proof depends on the parities of  $\tilde{q}$  and  $\tilde{p}$ ; since these numbers are mutually prime, either they have the same parity or they are both odd.

#### 5.3.1 1st case: $\tilde{q}$ and $\tilde{p}$ are both odd

Up to mirror symmetry, the braid  $B(3, \tilde{q}, \tilde{p})$  is of the form (Fact 5)

$$B(3, \tilde{q}, \tilde{p}) = Q_{3, \tilde{q}, \tilde{p}}\sigma_2Q_{3, \tilde{q}, \tilde{p}}^{-1}\sigma_1$$

with ([S-V])

$$Q_{3, \tilde{q}, \tilde{p}} = \sigma_2^{\lambda(1)}\sigma_1^{\lambda(2)}\sigma_2^{\lambda(3)}\sigma_1^{\lambda(4)}\dots\sigma_2^{\lambda(\tilde{q}-2)}\sigma_1^{\lambda(\tilde{q}-1)} \quad (29)$$

where  $\lambda$  is an expression with values in  $\{-1, 1\}$  verifying

$$\lambda(k) = -\lambda(\tilde{q} - k)$$

so we rewrite  $Q_{3, \tilde{q}, \tilde{p}}$  and introduce the braid  $P$ :

$$Q_{3, \tilde{q}, \tilde{p}} = \underbrace{\sigma_2^{\lambda(1)} \sigma_1^{\lambda(2)} \sigma_2^{\lambda(\frac{\tilde{q}-3}{2})} \sigma_1^{\lambda(\frac{\tilde{q}-1}{2})}}_P \sigma_2^{-\lambda(\frac{\tilde{q}-1}{2})} \sigma_1^{-\lambda(\frac{\tilde{q}-3}{2})} \dots \sigma_2^{-\lambda(2)} \sigma_1^{-\lambda(1)} \quad (30)$$

We have defined braids and we now turn to their images under the Burau representation  $B_{-1}$ . We let

$$\mathcal{Q} = \mathcal{B}_{-1}(Q_{3, \tilde{q}, \tilde{p}}) \quad \mathcal{P} = \mathcal{B}_{-1}(P).$$

We notice that the  $s_i$ 's of (16) verify

$$s_2 = {}^t s_1^{-1} \quad (31)$$

and derive

$$\mathcal{Q} = \mathcal{P}^t \mathcal{P} \quad (32)$$

**Lemma 4.**  $\mathcal{B}_{-1}(B(3, \tilde{q}, \tilde{p}))$  is hyperbolic except if  $\mathcal{P}$  is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{with } a, b \in \{-1, 0, 1\} \quad (33)$$

*Proof.* Compute  $\text{Trace}(\mathcal{B}_{-1}(B(3, \tilde{q}, \tilde{p}))) =$

$$\text{Trace}[(\mathcal{B}_{-1}(\mathcal{P})^t \mathcal{B}_{-1}(\mathcal{P})) s_2 (\mathcal{B}_{-1}(\mathcal{P})^t \mathcal{B}_{-1}(\mathcal{P}))^{-1} s_1] = 2 - (a^2 + b^2)^2.$$

hence if it is hyperbolic,  $a^2 + b^2 = 1$  or  $a^2 + b^2 = 2$   $\square$

If  $\mathcal{P}$  verifies (33), it has one of the following forms, for some integer  $h$ .

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} = (\pm 1) s_1^{-h}$

Thus  $\mathcal{Q} = \mathcal{P}^t \mathcal{P} = s_1^{-h} s_2^h$  and, for some integer  $k$ ,

$$\beta = (\sigma_1^{-h} \sigma_2^h \Delta^k) \sigma_2 (\Delta^{-k} \sigma_2^{-h} \sigma_1^h) \sigma_1 = \sigma_1^{-h} (\sigma_2 \sigma_1) \sigma_1^h$$

so  $\beta$  is conjugate to  $\sigma_2 \sigma_1$  and the knot is as in 1. (b) of Theorem 7.

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 0 & 1 \\ -1 & h \end{pmatrix} = (\pm 1) s_1^{1-h} s_2 s_1$

Thus  $\mathcal{Q} = s_1^{-h} s_2^{-h}$  and similarly to above,

$$\beta = \sigma_1^{-h} (\sigma_2 \sigma_1) \sigma_1^h.$$

Again we are in Case 1. (b) of Theorem 7.

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 1 & 1 \\ h & h+1 \end{pmatrix} = (\pm 1) s_1^{-h} s_2$

Then  $\mathcal{Q} = s_1^{-h} s_2 s_1^{-1} s_2^h$  and

$$\beta = \sigma_1^{-h} (\sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1) \sigma_1^{-h}$$

This braid  $\beta$  closes into a link with 3 components, thus it cannot be the braid of a knot.

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 1 & -1 \\ h & 1-h \end{pmatrix} = s_1^{-h} s_2^{-1}$  and

$$\beta = \sigma_1^{-h} (\sigma_2^{-1} \sigma_1 \sigma_2) (\sigma_1^{-1} \sigma_2 \sigma_1) \sigma_1^h$$

Again  $\beta$  closes in a link with 3 components. □

### 5.3.2 2nd case: $\tilde{q}$ and $\tilde{p}$ have different parities

The knot is preserved by an isometry which is a rotation in one plane and a symmetry in its orthogonal plane, thus the knot is isotopic to its mirror image and has zero signature.

To spell this out, we assume that  $\tilde{q}$  is odd and  $\tilde{p}$  is even. The knot  $K(N, d\tilde{q}, d\tilde{p})$  is defined by the function  $F_{N, d\tilde{q}, d\tilde{p}}$  (see (25) above), so we can also define it by the function

$$\theta \mapsto F_{N, d\tilde{q}, d\tilde{p}}\left(\theta + \frac{\pi}{d}\right) = (e^{iN\theta} e^{iN\frac{\pi}{d}}, -\sin(d\tilde{q}\theta), \cos(d\tilde{p}\theta + \alpha))$$

thus  $K(N, d\tilde{q}, d\tilde{p})$  is invariant under the transformation

$$\begin{pmatrix} \cos\left(\frac{N\pi}{d}\right) & -\sin\left(\frac{N\pi}{d}\right) & 0 & 0 \\ \sin\left(\frac{N\pi}{d}\right) & \cos\left(\frac{N\pi}{d}\right) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (34)$$

which reverses the orientation on  $\mathbb{R}^4$  and on  $\mathbb{S}^3$ . □

## 6 Statistics

### 6.1 Random quasipositive braids

The following estimates were done with SAGE.

We define

$$Z = \{\beta \in B(3) : |(\mathcal{B}_{-1}(\beta))_{11}| > 2\}.$$

It follows from the previous discussion that, if a 3-braid  $\beta$  belongs to  $Z$ , then for all positive integers  $n$ , we have

$$\text{sign}\left((\beta\sigma_1\beta^{-1}\sigma_2)^n\right) = 0.$$

Let  $W$  be the  $\mu$ -random walk on  $B(3)$  given by the probability  $\mu$  on  $B(3)$  where

$$\mu(\sigma_1) = \mu(\sigma_2) = \mu(\sigma_1^{-1}) = \mu(\sigma_2^{-1}) = \frac{1}{4}$$

Here are the probabilities  $p(n)$  that  $W$  hits  $Z$  at the  $n$ -th step, for  $n \leq 12$ .

$n$	1	2	3	4	5	6	7	8	9	10	11	12
$p(n)$	0	0	0.06	0.11	0.17	0.22	0.27	0.32	0.36	0.41	0.45	0.48

### 6.2 Lissajous toric knots with 3 strands

We investigate here the proportion of knots which verify (a) or (b) of 1. of Theorem 7. 1. First we prove

**Lemma 5.** *To get a full list of the  $K(3, \tilde{q}, \tilde{p})$ 's for a given  $\tilde{q}$ , it is enough to take*

$$\tilde{p} \in [\tilde{q} + 1, 2\tilde{q}]$$

*Proof.* If  $\tilde{q}$  and  $\tilde{p}$  are odd, mutually prime and not divisible by 3, we use the representation of  $K(3, \tilde{q}, \tilde{p})$  from (5.3.1). The expression of  $\lambda$  is based on the Bezout identity:

$$6A + Bq = 1 \tag{35}$$

for two integers  $A$  and  $B$ . Then, up to mirror transformation

$$\lambda(k) = (-1)^{\left\lfloor \frac{2Apk}{q} \right\rfloor} \tag{36}$$



It follows from (36) that, if we replace  $p$  by  $p + q$ , the expression of  $\lambda$  remains the same.  $\square$

For a given  $\tilde{q}$ , we let

$$P(\tilde{q}) = \{\tilde{p} \in [\tilde{q} + 1, 2\tilde{q}], \text{ such that } (2, \tilde{p}) = (3, \tilde{p}) = 1\}$$

$$S(\tilde{q}) = \{\tilde{p} \in P(\tilde{q}), \text{ such that } K(3, \tilde{q}, \tilde{p}) \text{ verifies 1) (a) of Theorem 7}\}$$

The table below gives us, for a given  $\tilde{q}$ , the percentage of elements of  $P(\tilde{q})$  which belong to  $S(\tilde{q})$ : i.e. the percentage of  $K(3, \tilde{q}, \tilde{p})$ 's for which every  $K(3, n\tilde{q}, n\tilde{p})$ ,  $n \in \mathbb{N}$ , has zero signature. The percentages have been rounded to the integer part.

$\tilde{q}$	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49
%	40	57	54	61	70	63	69	64	68	77	62	75	78	74	80	69

$\tilde{q}$	53	55	59	61	65	67	71	73	77	79	83	85	89	91	95	97	101
%	79	72	77	78	76	80	67	82	75	83	84	77	80	74	80	82	83

## References

- [A'C] N. A'Campo *Tresses, monodromie et le groupe symplectique*, Comm. Math. Helvetici (54) 318-327, (1979)
- [As] M. Aschbacher *Finite group theory*, Cambridge Univ. Press, London 1986
- [B-K] M. Brandenbursky, J. Kedra *Concordance group and stable commutator length in braid groups*, Algebraic & Geometric Topology (5), 2861-2886, (2015).
- [Br] M. Brandenbursky, *On quasi-morphisms from knot and braid invariants*, Jour. of Knot Theory and its Ramifications, vol. 20, No. 10 (2011), 1397-1417.
- [Bu] W. Burau, *Über Zopfgruppen und gleichsinnig verdrillte Verkettungen* Abh. Math. Sem. Univ. Hamburg, 11(1) (1935) 179–186
- [Di] P. Diaconis, *Representations in probability and statistics*, IMS, Hayward, 1986

- [G-G] J.-M. Gambaudo, E. Ghys *Braids and signatures*, Bull. SMF 133 (2005), 541-579
- [It] T. Ito, *On a structure of random open books and closed braids*, Proc. Japan Acad. 91, Ser. A (2015)
- [K-T] C. Kassel, V. Turaev *Braid groups*, Graduate texts in maths, Springer-Verlag New York (2008)
- [La] C. Lamm *Symmetric unions and ribbons knots*, Osaka J. Math. 37 (2000), 537-550
- [L-O] C. Lamm, D. Obermeyer *Billiard knots in a cylinder* J. Knot Theory and its Ramifications 8(3) (1999) Vol. 353-366.
- [L-R] G. Lachaud, R. Rolland *On the number of points of algebraic sets over finite fields*, Journal of Pure and Applied Algebra 219 (11) , 5117-5136 (2015) On the number of points of algebraic sets over finite fields
- [Ma] A. V. Mal'jutin, *Quasimorphisms, random walks, and transient subsets in countable groups*, Jour. of Math. Sciences 181(6) 871–885 (2012)
- [Mi] J.S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics (170), CUP 2017
- [Mu] K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. 117 (1965), 387-422.
- [O'M] O. T. O'Meara *Symplectic groups*, AMS, Providence (1978)
- [Pr] J. H. Przytycki *Positive knots have negative signature*, Bull. Ac. Pol.: Math. 37, 1989, 559-562.
- [Ri] I. Rivin *Geometric phenomena in groups in Thin groups and superstrong approximation* MSRI Publications, Volume 61 (2013)
- [Ru] L. Rudolph *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. 29 (1993), 51-59.
- [S-V] M. Soret, M. Ville *Lissajous-toric knots*, Jour. of Knot Theory and Its Ramifications 29(1) (2020)

[Ta] T. Tanaka, *Four-genera of quasipositive knots*, Topology and its applications 83 (1998) 187-192

[wi] [https://groupprops.subwiki.org/wiki/Order\\_formulas\\_for\\_symplectic\\_groups](https://groupprops.subwiki.org/wiki/Order_formulas_for_symplectic_groups)

Marc Soret: Université F. Rabelais, Dép. de Mathématiques, 37000 Tours, France,  
Marc.Soret@lmpt.univ-tours.fr

Marina Ville: Univ Paris Est Creteil, CNRS, LAMA, F-94010 Creteil, France  
villemarina@yahoo.fr