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# Random walks and the symplectic representation of the braid groups

Marc Soret and Marina Ville

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## Abstract

We consider the symplectic representation  $\rho_n$  of a braid group  $B(n)$  in  $Sp(2l, \mathbb{Z})$ , for  $l = \lfloor \frac{n-1}{2} \rfloor$ . If  $P$  is a  $4l^2$  polynomial on the coefficients of the matrices in  $Sp(2l, \mathbb{Z})$ , we show that the set  $\{\beta \in B(n) / P(\rho_n(\beta)) = 0\}$  is transient for non degenerate random walks on  $B(n)$ . If  $n$  is odd, we derive that the  $n$ -braids  $\beta$  verifying  $|\Delta_{\beta}(-1)| \leq C$  for some constant  $C$  form a transient set: here  $\Delta_{\beta}$  denotes the Alexander polynomial of the closure of  $\beta$ .

We also derive that for a random 3-braid, the quasipositive links  $(\beta\sigma_i\beta^{-1}\sigma_j)^p$  have zero signature for every integer  $p$  and  $1 \leq i, j \leq 2$ . As an example of such braids, we investigate the signature of the Lisajous toric knots with 3 strands.

## 1 Preliminaries

### 1.1 Representations of the braid group

If  $B(n)$  is the group of braids with  $n$  strands, we denote by  $\mathcal{B}$  its *reduced Burau representation* which represents braids as  $(n-1) \times (n-1)$  matrices with values in  $\mathbb{Z}[t, t^{-1}]$ . The specialization of  $\mathcal{B}$  for  $t = -1$ , denoted  $\mathcal{B}_{-1}$ , is called the *integral reduced Burau representation*. The representation  $\mathcal{B}_{-1}$  is closely related to the *symplectic representation* of  $B(n)$  which we now recall, using the description of A'Campo ([A'C]).

The braid group  $B(n+1)$  acts on the homology  $H_1(X, \mathbb{Z})$  of a surface  $X$  such that  $H_1(X, \mathbb{Z})$  is freely generated by  $n$  loops. This action is conjugate to the

integral reduced Burau representation. Concretely speaking, this means the existence of a basis of  $H_1(X, \mathbb{Z})$  w.r.t. which the matrices representing the elements of  $B(n+1)$  are the ones given by  $\mathcal{B}_{-1}$ .

This action of  $B(n+1)$  preserves the intersection form  $I$  on  $H_1(X, \mathbb{Z})$ .

1. If  $n$  is even,  $n = 2l$ , the form  $I$  is symplectic nondegenerate on  $H_1(X, \mathbb{Z})$ .
2. If  $n$  is odd,  $n = 2l + 1$ , the kernel of  $I$  is a line  $K$ ;  $\mathcal{B}_{-1}$  restricts to the identity on  $K$  and acts symplectically on the quotient  $H_1(X, \mathbb{Z})/K$

Thus, in both cases, we get a *symplectic representation*

$$\rho_n : B(n+1) \longrightarrow Sp(2l, \mathbb{Z}) \quad (1)$$

## 1.2 Random walks on groups

If  $\mu_1, \mu_2$  are probability measures on a discrete group  $G$  with finite support, their *convolution* is defined for  $g \in G$  as

$$\mu_1 \star \mu_2(g) = \sum_{h \in G} \mu_1(g^{-1}h) \mu_2(h) \quad (2)$$

**Definition 1.** A probability  $\mu$  on a discrete group  $G$  is nondegenerate if  $\text{supp}(\mu)$  generates  $G$  as a semigroup.

Malyutin ([Ma]) defines the *right random  $\mu$ -walk* as the Markov chain which starts at the identity of  $G$  and with transition probability  $P(g, h) = \mu(gh^{-1})$ . If  $X$  is a subset of  $G$ , the probability that this walk hits  $X$  at the  $k$ -th step is  $\mu^{\star k}(X)$ .

## 1.3 The signature of a link

The signature is a classical invariant of links and knots; (see for example [Mu]). If  $\gamma$  is a link on  $\mathbb{S}^3$  which bounds a Seifert surface  $\Sigma$  in  $\mathbb{S}^3$ , we consider the intersection form

$$I : H_1(\Sigma) \times H_1(\Sigma) \longrightarrow \mathbb{Z}$$

$$(\alpha, \beta) \mapsto lk(\hat{\alpha}, \beta)$$

where  $lk$  is the linking number and  $\hat{\alpha}$  is the link obtained by pushing  $\alpha$  in the direction normal to  $\Sigma$  in  $\mathbb{S}^3$ . Given a base for  $H_1(\Sigma)$ , we define the matrix

$V$  of  $I$ : the signature of  $\gamma$  is the signature of the bilinear form defined by the matrix  $V + {}^t V$ . We point out that the literature contains two different definitions of the signature which coincide up to sign.

The signature of a link can be computed on a braid representing it and Gambodau-Ghys have estimated the signature of a product of two braids in terms of the signatures of the two braids ([G-G]). We recall (cf. [Mu]) that the for a knot  $K$ ,

$$|\text{sign}(K)| \leq 2g_4(K) \tag{3}$$

where  $g_4$  denotes the topological 4-genus. Thus a slice knot has zero signature but the converse is not true and we will see more examples of that here.

## 1.4 Positive links and quasipositive braids

A *positive link* is a link which has a diagram with all positive crossings.

**Theorem 1.** ([Pr]) *A positive link has strictly negative signature.*

So it is interesting to look at the signature of quasipositive braids:

**Definition 2.** ([Ru]) *A braid  $\beta$  is quasipositive if it is a product of conjugates of positive braid generators*

$$\beta = \prod_{i=1}^k \gamma_{j_i} \sigma_{j_i} \gamma_{j_i}^{-1} \tag{4}$$

If we close the quasipositive  $N$ -braid (4) in a link  $\hat{\beta}$ , Rudolph proved ([Ru]) that

$$\chi_4(\hat{\beta}) = N - k \tag{5}$$

where  $\chi_4$  denotes the largest Euler characteristic of a *smooth* surface in  $\mathbb{B}^4$  bounded by  $\hat{\beta}$ .

Tanaka ([Ta]) constructed examples of quasipositive braids with zero signature.

## 1.5 Transient sets in the braid groups from quasimorphisms

We recall that a *quasimorphism* on a group  $G$  is a function  $\Phi : G \rightarrow \mathbb{R}$  such that there exists a positive constant  $C$  with

$$\forall a, b \in G, |\Phi(ab) - \Phi(a) - \Phi(b)| \leq D_\Phi \tag{6}$$

where  $D_\Phi$  is called the *defect* of  $\Psi$ .

Quasimorphisms give us examples of transient sets in the braid groups:

**Theorem 2.** ([Ma]) *Let  $G$  be a countable group and  $\mu$  a nondegenerate probability on  $G$  (see Definition 1). If a subset  $S$  of  $G$  has bounded image under an unbounded quasimorphism  $\Phi$ , then the probability that the right random  $\mu$ -walk hits  $S$  at the  $k$ -th step tends to zero as  $k$  tends to infinity.*

We will see below how Gambodau-Ghys's formula for the signature turns it into a quasimorphism. Several other link invariants such as  $\chi_4$  can be used to define quasimorphisms on the braid group ([Br], [B-K]). Other examples of transient sets on the braid groups have been constructed by Ito [It]

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## 2 The results

**Theorem 3.** *Let  $P$  be a polynomial in  $(2l)^2$  variables with coefficients in  $\mathbb{Z}$ . If  $M = (m_{ij}) \in Sp(2l, \mathbb{Z})$ , we let  $P(M) = P(m_{11}, \dots, m_{12l}, m_{21}, \dots, m_{22l}, m_{211}, \dots, m_{212l})$ . Suppose that  $P$  does not vanish identically on  $Sp(2l, \mathbb{Z})$ . Then the set*

$$\{\beta \in B(n) / P(\rho_n(\beta)) = 0\}$$

*is transient for the right random  $\mu$ -walk.*

### 2.1 The Alexander polynomial

**Corollary 1.** *Let  $C$  be a real number. If  $2s+1$  is an odd integer, the following set is transient for the right random  $\mu$ -walk on  $B(2s+1)$*

$$\{\beta \in B(2s+1) / |\Delta_{\hat{\beta}}(-1)| < C\} \tag{7}$$

*where  $\Delta_{\hat{\beta}}$  denotes the Alexander polynomial of the closure of  $\beta$ .*

*Proof.* The Alexander polynomial can be expressed in terms of the Burau representation  $\mathcal{B}$  ([Bu]),

$$\Delta_{\hat{\beta}}(t) = \frac{1-t}{1-t^n} \det(\mathcal{B}_{-t}(\beta) - I) \quad (8)$$

If  $n$  is odd,  $\Delta_{\hat{\beta}}(-1) = \det(\rho_n(\beta) - I)$  and the set (7) is a finite union of sets described in Theorem 3.  $\square$

REMARK. If  $n$  is even,  $1 - (-1)^n = 0$  and  $\det(\mathcal{B}_{-1}(\beta) - I) = 0$  since  $\mathcal{B}_{-1}$  acts on  $K$  as the identity; thus the formula (8) is not easy to use. Nevertheless, it seems likely that Theorem 1 is true also for even  $n$ 's.

## 2.2 The signature of 3-braids

**Corollary 2.** *Let  $\mu$  be a nondegenerate probability on the braid group  $B(3)$ . For the right random  $\mu$ -walks on  $B(3)$ , the set*

$$S = \{\beta \in B(3) / \exists n \in \mathbb{N}, \exists i, j \in \{1, 2\} \text{ such that } \text{sign}((\beta\sigma_i\beta^{-1}\sigma_j)^n) \neq 0\} \quad (9)$$

*is transient.*

By contrast, (5) tells us that for every  $n$ ,  $\chi_4((\beta\sigma_1\beta^{-1}\sigma_1)^n) = 3 - 2n$ ; in particular, if  $(\beta\sigma_1\beta^{-1}\sigma_1)^n$  closes into a knot,  $g_4((\beta\sigma_1\beta^{-1}\sigma_1)^n) = n - 1$ .

In §5, we recall the Lissajous toric knots with 3 strands (cf. [S-V]); they are naturally of the form  $(\beta\sigma_2\beta^{-1}\sigma_1^{\pm 1})^n$  and we prove

**Theorem 4.** *A Lissajous toric knot  $K(3, p, q)$  has zero signature unless it is isotopic to a torus knot.*

We conclude by statistical estimates, done with SAGE.

These results suggest possible generalizations.

**Question 1.** *Let  $n \in \mathbb{N}$  and let  $\mathcal{I}$  be a numerical link invariant which is unbounded on the closures of the links represented by a  $n$ -braid. If  $C$  is a constant, is the set  $\{\mathcal{I}(\hat{\beta}) \leq C\}$  transient for the right random  $\mu$ -walks?*

**Question 2.** For an integer  $m$ , we set

$$A_m^{(1)} = \prod_{1 \leq 2i+1 \leq m} \sigma_{2i+1} \quad A_m^{(2)} = \prod_{1 \leq 2i \leq m} \sigma_{2i}$$

Is the following set  $S_m$  transient in  $B(m)$  for the right random  $\mu$ -walks?

$$S_m = \{\beta \in B(m) / \exists n \in \mathbb{N}, \exists i, j \in \{1, 2\} \text{ such that } \text{sign}((\beta A_m^{(i)} \beta^{-1} A_m^{(j)})^n) \neq 0\}$$

### 3 Random walks: proof of Theorem 3

Similarly to Rivin in [Ri], we introduce the finite groups  $Sp(2l, \mathbb{Z}_p)$ 's for a prime number  $p > 2$ .

A theorem of A'Campo ([A'C], Theorem 1 (1)) shows that the representation  $B(n) \rightarrow Sp(2l, \mathbb{F}_p)$  is surjective. So we consider the surjective map

$$\Pi_p : B(n) \xrightarrow{\rho_n} Sp(2l, \mathbb{Z}) \rightarrow Sp(2l, \mathbb{F}_p) \xrightarrow{\pi} PSp(2l, \mathbb{F}_p) \quad (10)$$

We consider the image probability  $\Pi_p \mu$  of  $\mu$  via  $\Pi_p$  on  $PSp(2l, \mathbb{F}_p)$ . We recall facts on random walks on finite groups.

**Theorem 5.** (see for exemple [Di]) Let  $G$  be a finite group and  $\mu$  a probability measure on  $G$  such that

1.  $G$  is generated as a semigroup by  $\text{supp}(\mu)$
2.  $\text{supp}(\mu)$  is not included in a coset of  $G$  by a normal subgroup  $H$  of  $G$ .

Then  $\mu^{*m}$  converges to the equidistributed measure on  $G$  as  $m$  tends to infinity.

We know ([As]) that, if  $p > 5$ , the group  $PSp(2n, \mathbb{F}_p)$  is simple so we can apply Theorem 5 and derive

**Lemma 1.** If  $m$  tends to infinity,  $(\Pi_p \mu)^{*m}$  converges to the equidistributed measure on  $PSp(2n, \mathbb{F}_p)$ .

We now investigate the density of the zero set in  $Sp(2l, \mathbb{F}_p)$  of the polynomial  $P$  appearing in Theorem 3.

We write  $P$  in terms of all the multi-indices,  $P(M) = \sum a_I m^I$ , where the  $a_I$ 's belong to  $\mathbb{Z}$ . If  $p$  is an integer, we denote by  $P_p$  its reduction modulo  $p$ .

Since  $P$  is not identically zero, we derive an infinite set  $\mathcal{P}$  of prime numbers  $p$  such that  $P_p$  is not identically zero. To prove the theorem, we need to estimate

$$\frac{|P_p^{-1}(0)|}{|Sp(2l, \mathbb{F}_p)|} \quad (11)$$

We recall (see for example [O'M] or [wi]) that

$$|Sp(2n, \mathbb{Z}_p)| = \prod_{m=1}^n [(p^{2m} - 1)p^{2m-1}] \geq \frac{1}{2^{2n}} \prod_{s=1}^{2l} p^s = \frac{p^{2l^2+l}}{2^{2n}} \quad (12)$$

To estimate  $|P_p^{-1}(0)|$ , we use the following result by Lachaud and Rolland

**Theorem 6.** ([L-R]) *Let  $K$  be the algebraic closure of  $\mathbb{F}_p$ . Let  $X$  be an algebraic variety of  $K^n$  of dimension  $m$  which is the zero set of a family of polynomials  $(f_1, \dots, f_r)$ . Let  $d_i = \deg(f_i)$ . Then*

$$|X \cap \mathbb{F}_p^n| \leq d_1 \dots d_r q^m$$

We view  $P_p$  as a polynomial with coefficients in  $K$  and let  $X = Sp(2l, K) \cap P_p^{-1}(0)$ . We know that for a field  $K$ ,  $Sp(2l, K)$  is irreducible as an algebraic variety ([Mi]), hence  $X$  is of dimension strictly smaller than  $Sp(2l, K)$ , i.e.  $\dim(X) \leq 2l^2 + l - 1$ . Thus, for some constant  $C_1(2l, P)$  depending on  $2l$  and on the degree of  $P$ ,

$$|X \cap \mathbb{F}_p^{4l^2}| \leq C_1(2l, P) p^{2l^2+l-1} \quad (13)$$

Putting (12) and (13) together, we derive that

$$\frac{|P_p^{-1}(0)|}{|Sp(2l, \mathbb{F}_p)|} \leq \frac{C_2(2l, P)}{p} \quad (14)$$

for some constant  $C_2(2l, P)$ .

We conclude the proof of Theorem 3. Given a  $\epsilon > 0$ , we pick a prime number  $p$  in  $\mathcal{P}$  such that

$$2 \frac{C_2(2l, P)}{p} < \epsilon \quad (15)$$

There exists an integer  $k_0$  such that for every  $k > k_0$ , we have

$$\mu^{\star k}(P^{-1}(0)) \leq (\Pi_p \mu)^{\star k}(P_p^{-1}(0)) < \epsilon.$$

□



## 4 The signature of 3-braids: proof of Corollary 2

### 4.1 Preliminaries: the Gambaudo-Ghys formula for the signature of a 3-braid ([G-G])

#### 4.1.1 The reduced Burau representation for $t = -1$

We let  $B_3$  be the 3-braid group and we denote by

$$\mathcal{B}_{-1} : B(3) \longrightarrow SL(2, \mathbb{Z})$$

the reduced Burau representation of the braid group  $B(3)$  for  $t = -1$ . Because there are different expressions in the literature, we recall the ones [G-G] uses:

$$s_1 = \mathcal{B}_{-1}(\sigma_1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad s_2 = \mathcal{B}_{-1}(\sigma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (16)$$

We recall (see for example [K-T]):

**Fact 1.** *The center of  $B(3)$  is the cyclic group generated by the twist*

$$\Delta = (\sigma_1 \sigma_2 \sigma_1)^2.$$

**Fact 2.** •  $\mathcal{B}_{-1}$  is surjective

- $\mathcal{B}_{-1}(\Delta) = -Id$
- The kernel of  $\mathcal{B}_{-1}$  is the cyclic group generated by  $\Delta^2$ .

#### 4.1.2 The Meyer cocycle and the formula for the signature

If  $\beta$  is a braid, we denote its closure by  $\hat{\beta}$ . For two 3-braids,  $\alpha$  and  $\beta$ , [G-G] proves

$$sign(\widehat{\alpha \cdot \beta}) = sign(\hat{\alpha}) + sign(\hat{\beta}) - Meyer(\mathcal{B}_{-1}(\alpha), \mathcal{B}_{-1}(\beta)) \quad (17)$$

where  $Meyer \in H^2(SL(2, \mathbb{R}))$  is the *Meyer cocycle*. We recall its description by [G-G].

If  $\gamma_1, \gamma_2 \in SL(2, \mathbb{Z})$ , we let

$$E_{\gamma_1, \gamma_2} = Im(\gamma_1^{-1} - Id) \cap Im(\gamma_2 - Id) \quad (18)$$

For a vector  $e$  in  $E_{\gamma_1, \gamma_2}$ , we take  $v_1, v_2$  in  $\mathbb{R}^2$  such that

$$e = \gamma_1^{-1}(v_1) - v_1 = v_2 - \gamma_2(v_2)$$

and define the quadratic form

$$q_{\gamma_1, \gamma_2} = \Omega(e, v_1 + v_2) \tag{19}$$

where  $\Omega$  is the standard symplectic form on  $\mathbb{R}^2$ . Then  $Meyer(\gamma_1, \gamma_2)$  is the signature of  $q_{\gamma_1, \gamma_2}$ .

**Fact 3.** *If  $\gamma \in SL(2, \mathbb{Z})$  is hyperbolic (i.e.  $|\text{tr}(\gamma)| > 2$ ), then for two positive integers  $a, b$ ,*

$$Meyer(\gamma^a, \gamma^b) = 0$$

REMARK. Since a generic element of  $SL(2, \mathbb{Z})$  is hyperbolic, it follows from Fact 3 that for almost every braid  $\beta \in B(3)$ ,  $sign(\beta^n) = nsign(\beta)$ .

## 4.2 Proof of Corollary 2

Since  $\sigma_i$  and  $\beta\sigma_j\beta^{-1}$  close in trivial knots,  $sign(\sigma_i) = sign(\beta\sigma_j\beta^{-1}) = 0$ ; hence

$$sign(\sigma_i\beta\sigma_j\beta^{-1}) = Meyer(\sigma_i, \beta\sigma_j\beta^{-1}) \tag{20}$$

**Lemma 2.** *Let  $i, j \in \{1, 2\}$  and  $\beta \in B(3)$ . If  $Meyer(\sigma_i, \beta\sigma_j\beta^{-1}) \neq 0$ , then one of the coefficients of the matrix  $\mathcal{B}_{-1}(\beta)$  is zero.*

*Proof.* Recall that  $Meyer(\sigma_i, \beta\sigma_j\beta^{-1})$  is the signature of a quadratic form on the space  $E_{\sigma_i, \beta\sigma_j\beta^{-1}}$ , defined in §4.1.2; thus, if  $Meyer(\sigma_i, \beta\sigma_j\beta^{-1})$  is non zero, the space  $E_{\sigma_i, \beta\sigma_j\beta^{-1}}$  contains a non-zero element, i.e. there exist non zero  $v_1, v_2$  such that

$$(s_i^{-1} - Id)(v_i) = \beta \circ (Id - s_j)[\beta^{-1}(v_2)] \tag{21}$$

If we denote by  $e_1, e_2$  the canonical basis of  $\mathbb{R}^2$ , we notice that

$$Im(s_1^{\pm 1} - Id) = \mathbb{R}e_2 \quad Im(s_2^{\pm 1} - Id) = \mathbb{R}e_1 \tag{22}$$

It follows that the (LHS) of (21) is parallel to  $e_1$  or  $e_2$  and the (RHS) is parallel to  $\beta(e_1)$  or  $\beta(e_2)$ . Thus, for some  $h, k \in \{1, 2\}$ ,  $\beta(e_h)$  is parallel to  $e_k$  which implies that one of the coefficients of the matrix of  $\beta$  is zero.  $\square$

**Lemma 3.** Let  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ . The matrix  $Ms_1M^{-1}s_1$  (resp.  $Ms_2M^{-1}s_2$ ,  $Ms_1M^{-1}s_2$ ,  $Ms_2M^{-1}s_1$ ) is hyperbolic unless

$$b^2 \leq 4 \quad (\text{resp. } c^2 \leq 4, d^2 \leq 4, a^2 \leq 4) \quad (23)$$

*Proof.* Compute the traces, for example  $\text{Trace}(Ms_2M^{-1}s_1) = 2 - a^2$ .  $\square$

We can now conclude the proof of Corollary 2. We let  $\beta$  be a 3-braid such that

$$\forall i, j \in \{1, 2\} \quad |(\mathcal{B}_{-1}(\beta))_{ij}| > 2 \quad (24)$$

Then  $\text{sign}(\sigma_a\beta\sigma_b\beta^{-1}) = 0$  (Lemma 2) and  $\sigma_a\beta\sigma_b\beta^{-1}$  is hyperbolic (Lemma 3). Thus  $\text{Meyer}(\mathcal{B}_{-1}(\sigma_a\beta\sigma_b\beta^{-1}), \mathcal{B}_{-1}((\sigma_a\beta\sigma_b\beta^{-1})^n)) = 0$  (Fact 3 of §4.1.2). It follows that, for every  $n$ ,

$$\text{sign}((\sigma_a\beta\sigma_b\beta^{-1})^n) = 0.$$

On the other hand, Theorem 3 tells us that the braids not verifying (24) form a transient set for the right random  $\mu$ -walks on  $B(3)$ .  $\square$

## 5 The Lissajous toric knots

### 5.1 Description

#### 5.1.1 The knots $K(N, q, p)$

In [S-V], the authors exhibited Lissajous toric knots as one of the classes of boundaries of minimal disks in the 4-ball with a branch point at the origin. Before that, Lamm investigated these knots in connection with billiards in the solid torus ([L-O]). If  $N, q, p$  are integers, with  $(N, q) = (N, p) = 1$ , the  $K(N, q, p)$  Lissajous toric knot is defined in a 3D-cylinder as

$$\begin{aligned} F_{N,q,p} : [0, 2\pi] &\longrightarrow \mathbb{S}^1 \times \mathbb{R}^2 \\ F_{N,q,p} : \theta &\mapsto (e^{Ni\theta}, \sin(q\theta), \cos(p\theta + \alpha)) \end{aligned} \quad (25)$$

for a phase  $\alpha$ . Endow  $\mathbb{R}^3$  with a coordinate system  $(x, y, z)$  and take the Lissajous curve  $C_{p,q} : \theta \mapsto (0, 2 + \sin(q\theta), \cos(p\theta + \alpha))$ ; then the knot  $K(N, q, p)$  is described in  $\mathbb{R}^3$  by a point travelling along  $C_{p,q}$  while  $C_{p,q}$  rotates  $N$  times along the vertical axis  $Oz$ . We recall a few facts

**Fact 4.** ([S-V]) For a finite number of phases  $\alpha$ 's the expression in (25) gives us singular crossing points. Otherwise (25) defines a knot and up to mirror transformation, its knot type does not depend on the phase  $\alpha$ .

Thus we drop the phase  $\alpha$  in (25) and we just talk of a knot  $K(N, q, p)$  defined up to mirror symmetry.

### 5.1.2 The braids $B(N, q, p)$

The knot  $K(N, q, p)$  has a natural  $N$ -braid representation  $B(N, q, p)$ .

**Fact 5.** ([S-V]) Let  $\tilde{p}, \tilde{q}, d$  be three positive integers all coprime with  $N$  and such that  $(\tilde{p}, \tilde{q}) = 1$ . Moreover assume (without loss of generality) that  $\tilde{q}$  is odd. Then

$$B(N, d\tilde{q}, d\tilde{p}) = B(N, \tilde{q}, \tilde{p})^d$$

and there exists a braid  $Q_{N, \tilde{q}, \tilde{p}}$  such that

$$B(N, \tilde{q}, \tilde{p}) = Q_{N, \tilde{q}, \tilde{p}} \sigma_2^{\epsilon(2)} Q_{N, \tilde{q}, \tilde{p}}^{-1} \sigma_1^{\epsilon(1)} \quad (26)$$

with  $\epsilon(1), \epsilon(2) \in \{-1, 1\}$ .

If  $2N$  divides  $\tilde{p} + \tilde{q}$  or  $\tilde{p} - \tilde{q}$ ,  $\epsilon(1) = \epsilon(2) = 1$  so the braid is quasipositive: for  $N = 3$ , this happens if and only if  $\tilde{q}$  and  $\tilde{p}$  are both odd.

The braid (26) is a symmetric union as defined by [La] thus:

**Fact 6.** If  $N, q, p$  are all mutually prime,  $K(N, q, p)$  is a ribbon knot.

See also [L-O] and [S-V].

## 5.2 The signature of the Lissajous toric knots

For  $N = 3$ , a Lissajous toric knot has zero signature unless it is a torus knot. More precisely,

**Theorem 7.** Let  $\tilde{q}, \tilde{p}, n$  be three positive integers, none of them divisible by 3 and  $\tilde{q}, \tilde{p}$  mutually prime. Let  $K(3, n\tilde{q}, n\tilde{p})$  be the corresponding Lissajous toric knot.

1. If  $\tilde{q}$  and  $\tilde{p}$  are both odd, up to mirror image,  $K(3, n\tilde{q}, n\tilde{p})$  is a quasipositive knot with

$$g_4(K(3, n\tilde{q}, n\tilde{p})) = d - 1 \quad (27)$$

and verifying one of the following

- (a) the signature of  $K(3, n\tilde{q}, n\tilde{p})$  is zero
  - (b)  $B(3, \tilde{q}, \tilde{p}) = \sigma_2\sigma_1$  so  $K(3, \tilde{q}, \tilde{p})$  is a trivial knot and  $K(3, n\tilde{q}, n\tilde{p})$  is a  $(3, n)$ -torus knot.
2. If  $\tilde{q}$  and  $\tilde{p}$  have different parities,  $K(3, n\tilde{q}, n\tilde{p})$  is isotopic to its mirror image, so it has zero signature.

REMARK 1. The Lissajous toric knots  $K(3, n, n)$  are just the  $(3, n)$  torus knots so clearly they are in the case 1 (b) of Theorem 7, but they are not the only ones. For example if  $\tilde{p} = \tilde{q} + 6$  the knot  $K(3, \tilde{q}, \tilde{q} + 6)$  (e.g.  $K(3, 7, 13)$ ) is represented by the braid

$$B(3, \tilde{q}, \tilde{q} + 6) = \sigma_2\sigma_1$$

so it is trivial and, for a positive integer  $n$ ,

$$B(3, n\tilde{q}, n(\tilde{q} + 6)) = (\sigma_2\sigma_1)^n$$

so  $K(3, n\tilde{q}, n(\tilde{q} + 6))$  is a  $(3, n)$ -torus knot.

In §6.2, we see that at least for  $\tilde{q}$ 's up to 100, the majority of knots of 1. in Theorem 7 1. are in the case 1. (a) of that Theorem.

An example is  $K(3, 5, 7)$  which is, up to mirror image, the knot  $10_{155}$  in the Rolfsen classification and its braid is

$$B(3, 5, 7) = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1 \quad (28)$$

### 5.3 Proof of Theorem 7

The proof depends on the parities of  $\tilde{q}$  and  $\tilde{p}$ ; since these numbers are mutually prime, either they have the same parity or they are both odd.

#### 5.3.1 1st case: $\tilde{q}$ and $\tilde{p}$ are both odd

Up to mirror symmetry, the braid  $B(3, \tilde{q}, \tilde{p})$  is of the form (Fact 5)

$$B(3, \tilde{q}, \tilde{p}) = Q_{3, \tilde{q}, \tilde{p}}\sigma_2Q_{3, \tilde{q}, \tilde{p}}^{-1}\sigma_1$$

with ([S-V])

$$Q_{3, \tilde{q}, \tilde{p}} = \sigma_2^{\lambda(1)}\sigma_1^{\lambda(2)}\sigma_2^{\lambda(3)}\sigma_1^{\lambda(4)}\dots\sigma_2^{\lambda(\tilde{q}-2)}\sigma_1^{\lambda(\tilde{q}-1)} \quad (29)$$

where  $\lambda$  is an expression with values in  $\{-1, 1\}$  verifying

$$\lambda(k) = -\lambda(\tilde{q} - k)$$

so we rewrite  $Q_{3, \tilde{q}, \tilde{p}}$  and introduce the braid  $P$ :

$$Q_{3, \tilde{q}, \tilde{p}} = \underbrace{\sigma_2^{\lambda(1)} \sigma_1^{\lambda(2)} \sigma_2^{\lambda(\frac{\tilde{q}-3}{2})} \sigma_1^{\lambda(\frac{\tilde{q}-1}{2})}}_P \sigma_2^{-\lambda(\frac{\tilde{q}-1}{2})} \sigma_1^{-\lambda(\frac{\tilde{q}-3}{2})} \dots \sigma_2^{-\lambda(2)} \sigma_1^{-\lambda(1)} \quad (30)$$

We have defined braids and we now turn to their images under the Burau representation  $B_{-1}$ . We let

$$\mathcal{Q} = \mathcal{B}_{-1}(Q_{3, \tilde{q}, \tilde{p}}) \quad \mathcal{P} = \mathcal{B}_{-1}(P).$$

We notice that the  $s_i$ 's of (16) verify

$$s_2 = {}^t s_1^{-1} \quad (31)$$

and derive

$$\mathcal{Q} = \mathcal{P}^t \mathcal{P} \quad (32)$$

**Lemma 4.**  $\mathcal{B}_{-1}(B(3, \tilde{q}, \tilde{p}))$  is hyperbolic except if  $\mathcal{P}$  is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{with } a, b \in \{-1, 0, 1\} \quad (33)$$

*Proof.* Compute  $\text{Trace}(\mathcal{B}_{-1}(B(3, \tilde{q}, \tilde{p}))) =$

$$\text{Trace}[(\mathcal{B}_{-1}(\mathcal{P})^t \mathcal{B}_{-1}(\mathcal{P})) s_2 (\mathcal{B}_{-1}(\mathcal{P})^t \mathcal{B}_{-1}(\mathcal{P}))^{-1} s_1] = 2 - (a^2 + b^2)^2.$$

hence if it is hyperbolic,  $a^2 + b^2 = 1$  or  $a^2 + b^2 = 2$   $\square$

If  $\mathcal{P}$  verifies (33), it has one of the following forms, for some integer  $h$ .

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} = (\pm 1) s_1^{-h}$

Thus  $\mathcal{Q} = \mathcal{P}^t \mathcal{P} = s_1^{-h} s_2^h$  and, for some integer  $k$ ,

$$\beta = (\sigma_1^{-h} \sigma_2^h \Delta^k) \sigma_2 (\Delta^{-k} \sigma_2^{-h} \sigma_1^h) \sigma_1 = \sigma_1^{-h} (\sigma_2 \sigma_1) \sigma_1^h$$

so  $\beta$  is conjugate to  $\sigma_2 \sigma_1$  and the knot is as in 1. (b) of Theorem 7.

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 0 & 1 \\ -1 & h \end{pmatrix} = (\pm 1) s_1^{1-h} s_2 s_1$

Thus  $\mathcal{Q} = s_1^{-h} s_2^{-h}$  and similarly to above,

$$\beta = \sigma_1^{-h} (\sigma_2 \sigma_1) \sigma_1^h.$$

Again we are in Case 1. (b) of Theorem 7.

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 1 & 1 \\ h & h+1 \end{pmatrix} = (\pm 1) s_1^{-h} s_2$

Then  $\mathcal{Q} = s_1^{-h} s_2 s_1^{-1} s_2^h$  and

$$\beta = \sigma_1^{-h} (\sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1) \sigma_1^{-h}$$

This braid  $\beta$  closes into a link with 3 components, thus it cannot be the braid of a knot.

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 1 & -1 \\ h & 1-h \end{pmatrix} = s_1^{-h} s_2^{-1}$  and

$$\beta = \sigma_1^{-h} (\sigma_2^{-1} \sigma_1 \sigma_2) (\sigma_1^{-1} \sigma_2 \sigma_1) \sigma_1^h$$

Again  $\beta$  closes in a link with 3 components. □

### 5.3.2 2nd case: $\tilde{q}$ and $\tilde{p}$ have different parities

The knot is preserved by an isometry which is a rotation in one plane and a symmetry in its orthogonal plane, thus the knot is isotopic to its mirror image and has zero signature.

To spell this out, we assume that  $\tilde{q}$  is odd and  $\tilde{p}$  is even. The knot  $K(N, d\tilde{q}, d\tilde{p})$  is defined by the function  $F_{N, d\tilde{q}, d\tilde{p}}$  (see (25) above), so we can also define it by the function

$$\theta \mapsto F_{N, d\tilde{q}, d\tilde{p}}(\theta + \frac{\pi}{d}) = (e^{iN\theta} e^{iN\frac{\pi}{d}}, -\sin(d\tilde{q}\theta), \cos(d\tilde{p}\theta + \alpha))$$

thus  $K(N, d\tilde{q}, d\tilde{p})$  is invariant under the transformation

$$\begin{pmatrix} \cos(\frac{N\pi}{d}) & -\sin(\frac{N\pi}{d}) & 0 & 0 \\ \sin(\frac{N\pi}{d}) & \cos(\frac{N\pi}{d}) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (34)$$

which reverses the orientation on  $\mathbb{R}^4$  and on  $\mathbb{S}^3$ . □

## 6 Statistics

### 6.1 Random quasipositive braids

The following estimates were done with SAGE.

We define

$$Z = \{\beta \in B(3) : |(\mathcal{B}_{-1}(\beta))_{11}| > 2\}.$$

It follows from the previous discussion that, if a 3-braid  $\beta$  belongs to  $Z$ , then for all positive integers  $n$ , we have

$$\text{sign}\left((\beta\sigma_1\beta^{-1}\sigma_2)^n\right) = 0.$$

Let  $W$  be the  $\mu$ -random walk on  $B(3)$  given by the probability  $\mu$  on  $B(3)$  where

$$\mu(\sigma_1) = \mu(\sigma_2) = \mu(\sigma_1^{-1}) = \mu(\sigma_2^{-1}) = \frac{1}{4}$$

Here are the probabilities  $p(n)$  that  $W$  hits  $Z$  at the  $n$ -th step, for  $n \leq 12$ .

$n$	1	2	3	4	5	6	7	8	9	10	11	12
$p(n)$	0	0	0.06	0.11	0.17	0.22	0.27	0.32	0.36	0.41	0.45	0.48

### 6.2 Lissajous toric knots with 3 strands

We investigate here the proportion of knots which verify (a) or (b) of 1. of Theorem 7. 1. First we prove

**Lemma 5.** *To get a full list of the  $K(3, \tilde{q}, \tilde{p})$ 's for a given  $\tilde{q}$ , it is enough to take*

$$\tilde{p} \in [\tilde{q} + 1, 2\tilde{q}]$$

*Proof.* If  $\tilde{q}$  and  $\tilde{p}$  are odd, mutually prime and not divisible by 3, we use the representation of  $K(3, \tilde{q}, \tilde{p})$  from (5.3.1). The expression of  $\lambda$  is based on the Bezout identity:

$$6A + Bq = 1 \tag{35}$$

for two integers  $A$  and  $B$ . Then, up to mirror transformation

$$\lambda(k) = (-1)^{\left\lfloor \frac{2Apk}{q} \right\rfloor} \tag{36}$$



It follows from (36) that, if we replace  $p$  by  $p + q$ , the expression of  $\lambda$  remains the same.  $\square$

For a given  $\tilde{q}$ , we let

$$P(\tilde{q}) = \{\tilde{p} \in [\tilde{q} + 1, 2\tilde{q}], \text{ such that } (2, \tilde{p}) = (3, \tilde{p}) = 1\}$$

$$S(\tilde{q}) = \{\tilde{p} \in P(\tilde{q}), \text{ such that } K(3, \tilde{q}, \tilde{p}) \text{ verifies 1) (a) of Theorem 7}\}$$

The table below gives us, for a given  $\tilde{q}$ , the percentage of elements of  $P(\tilde{q})$  which belong to  $S(\tilde{q})$ : i.e. the percentage of  $K(3, \tilde{q}, \tilde{p})$ 's for which every  $K(3, n\tilde{q}, n\tilde{p})$ ,  $n \in \mathbb{N}$ , has zero signature. The percentages have been rounded to the integer part.

$\tilde{q}$	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49
%	40	57	54	61	70	63	69	64	68	77	62	75	78	74	80	69

$\tilde{q}$	53	55	59	61	65	67	71	73	77	79	83	85	89	91	95	97	101
%	79	72	77	78	76	80	67	82	75	83	84	77	80	74	80	82	83

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