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Random walks and the symplectic representation of the braid groups

Marc Soret and Marina Ville

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Abstract

We consider the symplectic representation ρ_n of a braid group $B(n)$ in $Sp(2l, \mathbb{Z})$, for $l = \lfloor \frac{n-1}{2} \rfloor$. If P is a $4l^2$ polynomial on the coefficients of the matrices in $Sp(2l, \mathbb{Z})$, we show that the set $\{\beta \in B(n) / P(\rho_n(\beta)) = 0\}$ is transient for non degenerate random walks on $B(n)$. If n is odd, we derive that the n -braids β verifying $|\Delta_{\beta}(-1)| \leq C$ for some constant C form a transient set: here Δ_{β} denotes the Alexander polynomial of the closure of β .

We also derive that for a random 3-braid, the quasipositive links $(\beta\sigma_i\beta^{-1}\sigma_j)^p$ have zero signature for every integer p and $1 \leq i, j \leq 2$. As an example of such braids, we investigate the signature of the Lisajous toric knots with 3 strands.

1 Preliminaries

1.1 Representations of the braid group

If $B(n)$ is the group of braids with n strands, we denote by \mathcal{B} its *reduced Burau representation* which represents braids as $(n-1) \times (n-1)$ matrices with values in $\mathbb{Z}[t, t^{-1}]$. The specialization of \mathcal{B} for $t = -1$, denoted \mathcal{B}_{-1} , is called the *integral reduced Burau representation*. The representation \mathcal{B}_{-1} is closely related to the *symplectic representation* of $B(n)$ which we now recall, using the description of A'Campo ([A'C]).

The braid group $B(n+1)$ acts on the homology $H_1(X, \mathbb{Z})$ of a surface X such that $H_1(X, \mathbb{Z})$ is freely generated by n loops. This action is conjugate to the

integral reduced Burau representation. Concretely speaking, this means the existence of a basis of $H_1(X, \mathbb{Z})$ w.r.t. which the matrices representing the elements of $B(n+1)$ are the ones given by \mathcal{B}_{-1} .

This action of $B(n+1)$ preserves the intersection form I on $H_1(X, \mathbb{Z})$.

1. If n is even, $n = 2l$, the form I is symplectic nondegenerate on $H_1(X, \mathbb{Z})$.
2. If n is odd, $n = 2l + 1$, the kernel of I is a line K ; \mathcal{B}_{-1} restricts to the identity on K and acts symplectically on the quotient $H_1(X, \mathbb{Z})/K$

Thus, in both cases, we get a *symplectic representation*

$$\rho_n : B(n+1) \longrightarrow Sp(2l, \mathbb{Z}) \quad (1)$$

1.2 Random walks on groups

If μ_1, μ_2 are probability measures on a discrete group G with finite support, their *convolution* is defined for $g \in G$ as

$$\mu_1 \star \mu_2(g) = \sum_{h \in G} \mu_1(g^{-1}h) \mu_2(h) \quad (2)$$

Definition 1. A probability μ on a discrete group G is nondegenerate if $\text{supp}(\mu)$ generates G as a semigroup.

Malyutin ([Ma]) defines the *right random μ -walk* as the Markov chain which starts at the identity of G and with transition probability $P(g, h) = \mu(gh^{-1})$. If X is a subset of G , the probability that this walk hits X at the k -th step is $\mu^{\star k}(X)$.

1.3 The signature of a link

The signature is a classical invariant of links and knots; (see for example [Mu]). If γ is a link on \mathbb{S}^3 which bounds a Seifert surface Σ in \mathbb{S}^3 , we consider the intersection form

$$I : H_1(\Sigma) \times H_1(\Sigma) \longrightarrow \mathbb{Z}$$

$$(\alpha, \beta) \mapsto lk(\hat{\alpha}, \beta)$$

where lk is the linking number and $\hat{\alpha}$ is the link obtained by pushing α in the direction normal to Σ in \mathbb{S}^3 . Given a base for $H_1(\Sigma)$, we define the matrix

V of I : the signature of γ is the signature of the bilinear form defined by the matrix $V + {}^t V$. We point out that the literature contains two different definitions of the signature which coincide up to sign.

The signature of a link can be computed on a braid representing it and Gambodau-Ghys have estimated the signature of a product of two braids in terms of the signatures of the two braids ([G-G]). We recall (cf. [Mu]) that the for a knot K ,

$$|\text{sign}(K)| \leq 2g_4(K) \quad (3)$$

where g_4 denotes the topological 4-genus. Thus a slice knot has zero signature but the converse is not true and we will see more examples of that here.

1.4 Positive links and quasipositive braids

A *positive link* is a link which has a diagram with all positive crossings.

Theorem 1. ([Pr]) *A positive link has strictly negative signature.*

So it is interesting to look at the signature of quasipositive braids:

Definition 2. ([Ru]) *A braid β is quasipositive if it is a product of conjugates of positive braid generators*

$$\beta = \prod_{i=1}^k \gamma_{j_i} \sigma_{j_i} \gamma_{j_i}^{-1} \quad (4)$$

If we close the quasipositive N -braid (4) in a link $\hat{\beta}$, Rudolph proved ([Ru]) that

$$\chi_4(\hat{\beta}) = N - k \quad (5)$$

where χ_4 denotes the largest Euler characteristic of a *smooth* surface in \mathbb{B}^4 bounded by $\hat{\beta}$.

Tanaka ([Ta]) constructed examples of quasipositive braids with zero signature.

1.5 Transient sets in the braid groups from quasimorphisms

We recall that a *quasimorphism* on a group G is a function $\Phi : G \rightarrow \mathbb{R}$ such that there exists a positive constant C with

$$\forall a, b \in G, |\Phi(ab) - \Phi(a) - \Phi(b)| \leq D_\Phi \quad (6)$$

where D_Φ is called the *defect* of Ψ .

Quasimorphisms give us examples of transient sets in the braid groups:

Theorem 2. ([Ma]) *Let G be a countable group and μ a nondegenerate probability on G (see Definition 1). If a subset S of G has bounded image under an unbounded quasimorphism Φ , then the probability that the right random μ -walk hits S at the k -th step tends to zero as k tends to infinity.*

We will see below how Gambodau-Ghys's formula for the signature turns it into a quasimorphism. Several other link invariants such as χ_4 can be used to define quasimorphisms on the braid group ([Br], [B-K]). Other examples of transient sets on the braid groups have been constructed by Ito [It]

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2 The results

Theorem 3. *Let P be a polynomial in $(2l)^2$ variables with coefficients in \mathbb{Z} . If $M = (m_{ij}) \in Sp(2l, \mathbb{Z})$, we let $P(M) = P(m_{11}, \dots, m_{12l}, m_{21}, \dots, m_{22l}, m_{211}, \dots, m_{212l})$. Suppose that P does not vanish identically on $Sp(2l, \mathbb{Z})$. Then the set*

$$\{\beta \in B(n) / P(\rho_n(\beta)) = 0\}$$

is transient for the right random μ -walk.

2.1 The Alexander polynomial

Corollary 1. *Let C be a real number. If $2s+1$ is an odd integer, the following set is transient for the right random μ -walk on $B(2s+1)$*

$$\{\beta \in B(2s+1) / |\Delta_{\hat{\beta}}(-1)| < C\} \tag{7}$$

where $\Delta_{\hat{\beta}}$ denotes the Alexander polynomial of the closure of β .

Proof. The Alexander polynomial can be expressed in terms of the Burau representation \mathcal{B} ([Bu]),

$$\Delta_{\hat{\beta}}(t) = \frac{1-t}{1-t^n} \det(\mathcal{B}_{-t}(\beta) - I) \quad (8)$$

If n is odd, $\Delta_{\hat{\beta}}(-1) = \det(\rho_n(\beta) - I)$ and the set (7) is a finite union of sets described in Theorem 3. \square

REMARK. If n is even, $1 - (-1)^n = 0$ and $\det(\mathcal{B}_{-1}(\beta) - I) = 0$ since \mathcal{B}_{-1} acts on K as the identity; thus the formula (8) is not easy to use. Nevertheless, it seems likely that Theorem 1 is true also for even n 's.

2.2 The signature of 3-braids

Corollary 2. *Let μ be a nondegenerate probability on the braid group $B(3)$. For the right random μ -walks on $B(3)$, the set*

$$S = \{\beta \in B(3) / \exists n \in \mathbb{N}, \exists i, j \in \{1, 2\} \text{ such that } \text{sign}((\beta\sigma_i\beta^{-1}\sigma_j)^n) \neq 0\} \quad (9)$$

is transient.

By contrast, (5) tells us that for every n , $\chi_4((\beta\sigma_1\beta^{-1}\sigma_1)^n) = 3 - 2n$; in particular, if $(\beta\sigma_1\beta^{-1}\sigma_1)^n$ closes into a knot, $g_4((\beta\sigma_1\beta^{-1}\sigma_1)^n) = n - 1$.

In §5, we recall the Lissajous toric knots with 3 strands (cf. [S-V]); they are naturally of the form $(\beta\sigma_2\beta^{-1}\sigma_1^{\pm 1})^n$ and we prove

Theorem 4. *A Lissajous toric knot $K(3, p, q)$ has zero signature unless it is isotopic to a torus knot.*

We conclude by statistical estimates, done with SAGE.

These results suggest possibles generalizations.

Question 1. *Let $n \in \mathbb{N}$ and let \mathcal{I} be a numerical link invariant which is unbounded on the closures of the links represented by a n -braid. If C is a constant, is the set $\{\mathcal{I}(\hat{\beta}) \leq C\}$ transient for the right random μ -walks?*

Question 2. For an integer m , we set

$$A_m^{(1)} = \prod_{1 \leq 2i+1 \leq m} \sigma_{2i+1} \quad A_m^{(2)} = \prod_{1 \leq 2i \leq m} \sigma_{2i}$$

Is the following set S_m transient in $B(m)$ for the right random μ -walks?

$$S_m = \{\beta \in B(m) / \exists n \in \mathbb{N}, \exists i, j \in \{1, 2\} \text{ such that } \text{sign}((\beta A_m^{(i)} \beta^{-1} A_m^{(j)})^n) \neq 0\}$$

3 Random walks: proof of Theorem 3

Similarly to Rivin in [Ri], we introduce the finite groups $Sp(2l, \mathbb{Z}_p)$'s for a prime number $p > 2$.

A theorem of A'Campo ([A'C], Theorem 1 (1)) shows that the representation $B(n) \rightarrow Sp(2l, \mathbb{F}_p)$ is surjective. So we consider the surjective map

$$\Pi_p : B(n) \xrightarrow{\rho_n} Sp(2l, \mathbb{Z}) \rightarrow Sp(2l, \mathbb{F}_p) \xrightarrow{\pi} PSp(2l, \mathbb{F}_p) \quad (10)$$

We consider the image probability $\Pi_p \mu$ of μ via Π_p on $PSp(2l, \mathbb{F}_p)$. We recall facts on random walks on finite groups.

Theorem 5. (see for exemple [Di]) Let G be a finite group and μ a probability measure on G such that

1. G is generated as a semigroup by $\text{supp}(\mu)$
2. $\text{supp}(\mu)$ is not included in a coset of G by a normal subgroup H of G .

Then μ^{*m} converges to the equidistributed measure on G as m tends to infinity.

We know ([As]) that, if $p > 5$, the group $PSp(2n, \mathbb{F}_p)$ is simple so we can apply Theorem 5 and derive

Lemma 1. If m tends to infinity, $(\Pi_p \mu)^{*m}$ converges to the equidistributed measure on $PSp(2n, \mathbb{F}_p)$.

We now investigate the density of the zero set in $Sp(2l, \mathbb{F}_p)$ of the polynomial P appearing in Theorem 3.

We write P in terms of all the multi-indices, $P(M) = \sum a_I m^I$, where the a_I 's belong to \mathbb{Z} . If p is an integer, we denote by P_p its reduction modulo p .

Since P is not identically zero, we derive an infinite set \mathcal{P} of prime numbers p such that P_p is not identically zero. To prove the theorem, we need to estimate

$$\frac{|P_p^{-1}(0)|}{|Sp(2l, \mathbb{F}_p)|} \quad (11)$$

We recall (see for example [O'M] or [wi]) that

$$|Sp(2n, \mathbb{Z}_p)| = \prod_{m=1}^n [(p^{2m} - 1)p^{2m-1}] \geq \frac{1}{2^{2n}} \prod_{s=1}^{2l} p^s = \frac{p^{2l^2+l}}{2^{2n}} \quad (12)$$

To estimate $|P_p^{-1}(0)|$, we use the following result by Lachaud and Rolland

Theorem 6. ([L-R]) *Let K be the algebraic closure of \mathbb{F}_p . Let X be an algebraic variety of K^n of dimension m which is the zero set of a family of polynomials (f_1, \dots, f_r) . Let $d_i = \deg(f_i)$. Then*

$$|X \cap \mathbb{F}_p^n| \leq d_1 \dots d_r q^m$$

We view P_p as a polynomial with coefficients in K and let $X = Sp(2l, K) \cap P_p^{-1}(0)$. We know that for a field K , $Sp(2l, K)$ is irreducible as an algebraic variety ([Mi]), hence X is of dimension strictly smaller than $Sp(2l, K)$, i.e. $\dim(X) \leq 2l^2 + l - 1$. Thus, for some constant $C_1(2l, P)$ depending on $2l$ and on the degree of P ,

$$|X \cap \mathbb{F}_p^{4l^2}| \leq C_1(2l, P) p^{2l^2+l-1} \quad (13)$$

Putting (12) and (13) together, we derive that

$$\frac{|P_p^{-1}(0)|}{|Sp(2l, \mathbb{F}_p)|} \leq \frac{C_2(2l, P)}{p} \quad (14)$$

for some constant $C_2(2l, P)$.

We conclude the proof of Theorem 3. Given a $\epsilon > 0$, we pick a prime number p in \mathcal{P} such that

$$2 \frac{C_2(2l, P)}{p} < \epsilon \quad (15)$$

There exists an integer k_0 such that for every $k > k_0$, we have

$$\mu^{\star k}(P^{-1}(0)) \leq (\Pi_p \mu)^{\star k}(P_p^{-1}(0)) < \epsilon.$$

□

4 The signature of 3-braids: proof of Corollary 2

4.1 Preliminaries: the Gambaudo-Ghys formula for the signature of a 3-braid ([G-G])

4.1.1 The reduced Burau representation for $t = -1$

We let B_3 be the 3-braid group and we denote by

$$\mathcal{B}_{-1} : B(3) \longrightarrow SL(2, \mathbb{Z})$$

the reduced Burau representation of the braid group $B(3)$ for $t = -1$. Because there are different expressions in the literature, we recall the ones [G-G] uses:

$$s_1 = \mathcal{B}_{-1}(\sigma_1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad s_2 = \mathcal{B}_{-1}(\sigma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (16)$$

We recall (see for example [K-T]):

Fact 1. *The center of $B(3)$ is the cyclic group generated by the twist*

$$\Delta = (\sigma_1 \sigma_2 \sigma_1)^2.$$

Fact 2. • \mathcal{B}_{-1} is surjective

- $\mathcal{B}_{-1}(\Delta) = -Id$
- The kernel of \mathcal{B}_{-1} is the cyclic group generated by Δ^2 .

4.1.2 The Meyer cocycle and the formula for the signature

If β is a braid, we denote its closure by $\hat{\beta}$. For two 3-braids, α and β , [G-G] proves

$$sign(\widehat{\alpha \cdot \beta}) = sign(\hat{\alpha}) + sign(\hat{\beta}) - Meyer(\mathcal{B}_{-1}(\alpha), \mathcal{B}_{-1}(\beta)) \quad (17)$$

where $Meyer \in H^2(SL(2, \mathbb{R}))$ is the *Meyer cocycle*. We recall its description by [G-G].

If $\gamma_1, \gamma_2 \in SL(2, \mathbb{Z})$, we let

$$E_{\gamma_1, \gamma_2} = Im(\gamma_1^{-1} - Id) \cap Im(\gamma_2 - Id) \quad (18)$$

For a vector e in E_{γ_1, γ_2} , we take v_1, v_2 in \mathbb{R}^2 such that

$$e = \gamma_1^{-1}(v_1) - v_1 = v_2 - \gamma_2(v_2)$$

and define the quadratic form

$$q_{\gamma_1, \gamma_2} = \Omega(e, v_1 + v_2) \tag{19}$$

where Ω is the standard symplectic form on \mathbb{R}^2 . Then $Meyer(\gamma_1, \gamma_2)$ is the signature of q_{γ_1, γ_2} .

Fact 3. *If $\gamma \in SL(2, \mathbb{Z})$ is hyperbolic (i.e. $|\text{tr}(\gamma)| > 2$), then for two positive integers a, b ,*

$$Meyer(\gamma^a, \gamma^b) = 0$$

REMARK. Since a generic element of $SL(2, \mathbb{Z})$ is hyperbolic, it follows from Fact 3 that for almost every braid $\beta \in B(3)$, $sign(\beta^n) = nsign(\beta)$.

4.2 Proof of Corollary 2

Since σ_i and $\beta\sigma_j\beta^{-1}$ close in trivial knots, $sign(\sigma_i) = sign(\beta\sigma_j\beta^{-1}) = 0$; hence

$$sign(\sigma_i\beta\sigma_j\beta^{-1}) = Meyer(\sigma_i, \beta\sigma_j\beta^{-1}) \tag{20}$$

Lemma 2. *Let $i, j \in \{1, 2\}$ and $\beta \in B(3)$. If $Meyer(\sigma_i, \beta\sigma_j\beta^{-1}) \neq 0$, then one of the coefficients of the matrix $\mathcal{B}_{-1}(\beta)$ is zero.*

Proof. Recall that $Meyer(\sigma_i, \beta\sigma_j\beta^{-1})$ is the signature of a quadratic form on the space $E_{\sigma_i, \beta\sigma_j\beta^{-1}}$, defined in §4.1.2; thus, if $Meyer(\sigma_i, \beta\sigma_j\beta^{-1})$ is non zero, the space $E_{\sigma_i, \beta\sigma_j\beta^{-1}}$ contains a non-zero element, i.e. there exist non zero v_1, v_2 such that

$$(s_i^{-1} - Id)(v_i) = \beta \circ (Id - s_j)[\beta^{-1}(v_2)] \tag{21}$$

If we denote by e_1, e_2 the canonical basis of \mathbb{R}^2 , we notice that

$$Im(s_1^{\pm 1} - Id) = \mathbb{R}e_2 \quad Im(s_2^{\pm 1} - Id) = \mathbb{R}e_1 \tag{22}$$

It follows that the (LHS) of (21) is parallel to e_1 or e_2 and the (RHS) is parallel to $\beta(e_1)$ or $\beta(e_2)$. Thus, for some $h, k \in \{1, 2\}$, $\beta(e_h)$ is parallel to e_k which implies that one of the coefficients of the matrix of β is zero. \square

Lemma 3. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. The matrix $Ms_1M^{-1}s_1$ (resp. $Ms_2M^{-1}s_2$, $Ms_1M^{-1}s_2$, $Ms_2M^{-1}s_1$) is hyperbolic unless

$$b^2 \leq 4 \quad (\text{resp. } c^2 \leq 4, d^2 \leq 4, a^2 \leq 4) \quad (23)$$

Proof. Compute the traces, for example $\text{Trace}(Ms_2M^{-1}s_1) = 2 - a^2$. \square

We can now conclude the proof of Corollary 2. We let β be a 3-braid such that

$$\forall i, j \in \{1, 2\} \quad |(\mathcal{B}_{-1}(\beta))_{ij}| > 2 \quad (24)$$

Then $\text{sign}(\sigma_a\beta\sigma_b\beta^{-1}) = 0$ (Lemma 2) and $\sigma_a\beta\sigma_b\beta^{-1}$ is hyperbolic (Lemma 3). Thus $\text{Meyer}(\mathcal{B}_{-1}(\sigma_a\beta\sigma_b\beta^{-1}), \mathcal{B}_{-1}((\sigma_a\beta\sigma_b\beta^{-1})^n)) = 0$ (Fact 3 of §4.1.2). It follows that, for every n ,

$$\text{sign}((\sigma_a\beta\sigma_b\beta^{-1})^n) = 0.$$

On the other hand, Theorem 3 tells us that the braids not verifying (24) form a transient set for the right random μ -walks on $B(3)$. \square

5 The Lissajous toric knots

5.1 Description

5.1.1 The knots $K(N, q, p)$

In [S-V], the authors exhibited Lissajous toric knots as one of the classes of boundaries of minimal disks in the 4-ball with a branch point at the origin. Before that, Lamm investigated these knots in connection with billiards in the solid torus ([L-O]). If N, q, p are integers, with $(N, q) = (N, p) = 1$, the $K(N, q, p)$ Lissajous toric knot is defined in a 3D-cylinder as

$$\begin{aligned} F_{N,q,p} : [0, 2\pi] &\longrightarrow \mathbb{S}^1 \times \mathbb{R}^2 \\ F_{N,q,p} : \theta &\mapsto (e^{Ni\theta}, \sin(q\theta), \cos(p\theta + \alpha)) \end{aligned} \quad (25)$$

for a phase α . Endow \mathbb{R}^3 with a coordinate system (x, y, z) and take the Lissajous curve $C_{p,q} : \theta \mapsto (0, 2 + \sin(q\theta), \cos(p\theta + \alpha))$; then the knot $K(N, q, p)$ is described in \mathbb{R}^3 by a point travelling along $C_{p,q}$ while $C_{p,q}$ rotates N times along the vertical axis Oz . We recall a few facts

Fact 4. ([S-V]) For a finite number of phases α 's the expression in (25) gives us singular crossing points. Otherwise (25) defines a knot and up to mirror transformation, its knot type does not depend on the phase α .

Thus we drop the phase α in (25) and we just talk of a knot $K(N, q, p)$ defined up to mirror symmetry.

5.1.2 The braids $B(N, q, p)$

The knot $K(N, q, p)$ has a natural N -braid representation $B(N, q, p)$.

Fact 5. ([S-V]) Let \tilde{p}, \tilde{q}, d be three positive integers all coprime with N and such that $(\tilde{p}, \tilde{q}) = 1$. Moreover assume (without loss of generality) that \tilde{q} is odd. Then

$$B(N, d\tilde{q}, d\tilde{p}) = B(N, \tilde{q}, \tilde{p})^d$$

and there exists a braid $Q_{N, \tilde{q}, \tilde{p}}$ such that

$$B(N, \tilde{q}, \tilde{p}) = Q_{N, \tilde{q}, \tilde{p}} \sigma_2^{\epsilon(2)} Q_{N, \tilde{q}, \tilde{p}}^{-1} \sigma_1^{\epsilon(1)} \quad (26)$$

with $\epsilon(1), \epsilon(2) \in \{-1, 1\}$.

If $2N$ divides $\tilde{p} + \tilde{q}$ or $\tilde{p} - \tilde{q}$, $\epsilon(1) = \epsilon(2) = 1$ so the braid is quasipositive: for $N = 3$, this happens if and only if \tilde{q} and \tilde{p} are both odd.

The braid (26) is a symmetric union as defined by [La] thus:

Fact 6. If N, q, p are all mutually prime, $K(N, q, p)$ is a ribbon knot.

See also [L-O] and [S-V].

5.2 The signature of the Lissajous toric knots

For $N = 3$, a Lissajous toric knot has zero signature unless it is a torus knot. More precisely,

Theorem 7. Let \tilde{q}, \tilde{p}, n be three positive integers, none of them divisible by 3 and \tilde{q}, \tilde{p} mutually prime. Let $K(3, n\tilde{q}, n\tilde{p})$ be the corresponding Lissajous toric knot.

1. If \tilde{q} and \tilde{p} are both odd, up to mirror image, $K(3, n\tilde{q}, n\tilde{p})$ is a quasipositive knot with

$$g_4(K(3, n\tilde{q}, n\tilde{p})) = d - 1 \quad (27)$$

and verifying one of the following

- (a) the signature of $K(3, n\tilde{q}, n\tilde{p})$ is zero
 - (b) $B(3, \tilde{q}, \tilde{p}) = \sigma_2\sigma_1$ so $K(3, \tilde{q}, \tilde{p})$ is a trivial knot and $K(3, n\tilde{q}, n\tilde{p})$ is a $(3, n)$ -torus knot.
2. If \tilde{q} and \tilde{p} have different parities, $K(3, n\tilde{q}, n\tilde{p})$ is isotopic to its mirror image, so it has zero signature.

REMARK 1. The Lissajous toric knots $K(3, n, n)$ are just the $(3, n)$ torus knots so clearly they are in the case 1 (b) of Theorem 7, but they are not the only ones. For example if $\tilde{p} = \tilde{q} + 6$ the knot $K(3, \tilde{q}, \tilde{q} + 6)$ (e.g. $K(3, 7, 13)$) is represented by the braid

$$B(3, \tilde{q}, \tilde{q} + 6) = \sigma_2\sigma_1$$

so it is trivial and, for a positive integer n ,

$$B(3, n\tilde{q}, n(\tilde{q} + 6)) = (\sigma_2\sigma_1)^n$$

so $K(3, n\tilde{q}, n(\tilde{q} + 6))$ is a $(3, n)$ -torus knot.

In §6.2, we see that at least for \tilde{q} 's up to 100, the majority of knots of 1. in Theorem 7 1. are in the case 1. (a) of that Theorem.

An example is $K(3, 5, 7)$ which is, up to mirror image, the knot 10_{155} in the Rolfsen classification and its braid is

$$B(3, 5, 7) = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1 \quad (28)$$

5.3 Proof of Theorem 7

The proof depends on the parities of \tilde{q} and \tilde{p} ; since these numbers are mutually prime, either they have the same parity or they are both odd.

5.3.1 1st case: \tilde{q} and \tilde{p} are both odd

Up to mirror symmetry, the braid $B(3, \tilde{q}, \tilde{p})$ is of the form (Fact 5)

$$B(3, \tilde{q}, \tilde{p}) = Q_{3, \tilde{q}, \tilde{p}}\sigma_2Q_{3, \tilde{q}, \tilde{p}}^{-1}\sigma_1$$

with ([S-V])

$$Q_{3, \tilde{q}, \tilde{p}} = \sigma_2^{\lambda(1)}\sigma_1^{\lambda(2)}\sigma_2^{\lambda(3)}\sigma_1^{\lambda(4)}\dots\sigma_2^{\lambda(\tilde{q}-2)}\sigma_1^{\lambda(\tilde{q}-1)} \quad (29)$$

where λ is an expression with values in $\{-1, 1\}$ verifying

$$\lambda(k) = -\lambda(\tilde{q} - k)$$

so we rewrite $Q_{3, \tilde{q}, \tilde{p}}$ and introduce the braid P :

$$Q_{3, \tilde{q}, \tilde{p}} = \underbrace{\sigma_2^{\lambda(1)} \sigma_1^{\lambda(2)} \sigma_2^{\lambda(\frac{\tilde{q}-3}{2})} \sigma_1^{\lambda(\frac{\tilde{q}-1}{2})}}_P \sigma_2^{-\lambda(\frac{\tilde{q}-1}{2})} \sigma_1^{-\lambda(\frac{\tilde{q}-3}{2})} \dots \sigma_2^{-\lambda(2)} \sigma_1^{-\lambda(1)} \quad (30)$$

We have defined braids and we now turn to their images under the Burau representation B_{-1} . We let

$$\mathcal{Q} = \mathcal{B}_{-1}(Q_{3, \tilde{q}, \tilde{p}}) \quad \mathcal{P} = \mathcal{B}_{-1}(P).$$

We notice that the s_i 's of (16) verify

$$s_2 = {}^t s_1^{-1} \quad (31)$$

and derive

$$\mathcal{Q} = \mathcal{P}^t \mathcal{P} \quad (32)$$

Lemma 4. $\mathcal{B}_{-1}(B(3, \tilde{q}, \tilde{p}))$ is hyperbolic except if \mathcal{P} is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{with } a, b \in \{-1, 0, 1\} \quad (33)$$

Proof. Compute $\text{Trace}(\mathcal{B}_{-1}(B(3, \tilde{q}, \tilde{p}))) =$

$$\text{Trace}[(\mathcal{B}_{-1}(\mathcal{P})^t \mathcal{B}_{-1}(\mathcal{P})) s_2 (\mathcal{B}_{-1}(\mathcal{P})^t \mathcal{B}_{-1}(\mathcal{P}))^{-1} s_1] = 2 - (a^2 + b^2)^2.$$

hence if it is hyperbolic, $a^2 + b^2 = 1$ or $a^2 + b^2 = 2$ \square

If \mathcal{P} verifies (33), it has one of the following forms, for some integer h .

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} = (\pm 1) s_1^{-h}$

Thus $\mathcal{Q} = \mathcal{P}^t \mathcal{P} = s_1^{-h} s_2^h$ and, for some integer k ,

$$\beta = (\sigma_1^{-h} \sigma_2^h \Delta^k) \sigma_2 (\Delta^{-k} \sigma_2^{-h} \sigma_1^h) \sigma_1 = \sigma_1^{-h} (\sigma_2 \sigma_1) \sigma_1^h$$

so β is conjugate to $\sigma_2 \sigma_1$ and the knot is as in 1. (b) of Theorem 7.

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 0 & 1 \\ -1 & h \end{pmatrix} = (\pm 1) s_1^{1-h} s_2 s_1$

Thus $\mathcal{Q} = s_1^{-h} s_2^{-h}$ and similarly to above,

$$\beta = \sigma_1^{-h} (\sigma_2 \sigma_1) \sigma_1^h.$$

Again we are in Case 1. (b) of Theorem 7.

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 1 & 1 \\ h & h+1 \end{pmatrix} = (\pm 1) s_1^{-h} s_2$

Then $\mathcal{Q} = s_1^{-h} s_2 s_1^{-1} s_2^h$ and

$$\beta = \sigma_1^{-h} (\sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1) \sigma_1^{-h}$$

This braid β closes into a link with 3 components, thus it cannot be the braid of a knot.

- $\mathcal{P} = (\pm 1) \begin{pmatrix} 1 & -1 \\ h & 1-h \end{pmatrix} = s_1^{-h} s_2^{-1}$ and

$$\beta = \sigma_1^{-h} (\sigma_2^{-1} \sigma_1 \sigma_2) (\sigma_1^{-1} \sigma_2 \sigma_1) \sigma_1^h$$

Again β closes in a link with 3 components. □

5.3.2 2nd case: \tilde{q} and \tilde{p} have different parities

The knot is preserved by an isometry which is a rotation in one plane and a symmetry in its orthogonal plane, thus the knot is isotopic to its mirror image and has zero signature.

To spell this out, we assume that \tilde{q} is odd and \tilde{p} is even. The knot $K(N, d\tilde{q}, d\tilde{p})$ is defined by the function $F_{N, d\tilde{q}, d\tilde{p}}$ (see (25) above), so we can also define it by the function

$$\theta \mapsto F_{N, d\tilde{q}, d\tilde{p}}(\theta + \frac{\pi}{d}) = (e^{iN\theta} e^{iN\frac{\pi}{d}}, -\sin(d\tilde{q}\theta), \cos(d\tilde{p}\theta + \alpha))$$

thus $K(N, d\tilde{q}, d\tilde{p})$ is invariant under the transformation

$$\begin{pmatrix} \cos(\frac{N\pi}{d}) & -\sin(\frac{N\pi}{d}) & 0 & 0 \\ \sin(\frac{N\pi}{d}) & \cos(\frac{N\pi}{d}) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (34)$$

which reverses the orientation on \mathbb{R}^4 and on \mathbb{S}^3 . □

6 Statistics

6.1 Random quasipositive braids

The following estimates were done with SAGE.

We define

$$Z = \{\beta \in B(3) : |(\mathcal{B}_{-1}(\beta))_{11}| > 2\}.$$

It follows from the previous discussion that, if a 3-braid β belongs to Z , then for all positive integers n , we have

$$\text{sign}\left((\beta\sigma_1\beta^{-1}\sigma_2)^n\right) = 0.$$

Let W be the μ -random walk on $B(3)$ given by the probability μ on $B(3)$ where

$$\mu(\sigma_1) = \mu(\sigma_2) = \mu(\sigma_1^{-1}) = \mu(\sigma_2^{-1}) = \frac{1}{4}$$

Here are the probabilities $p(n)$ that W hits Z at the n -th step, for $n \leq 12$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$p(n)$	0	0	0.06	0.11	0.17	0.22	0.27	0.32	0.36	0.41	0.45	0.48

6.2 Lissajous toric knots with 3 strands

We investigate here the proportion of knots which verify (a) or (b) of 1. of Theorem 7. 1. First we prove

Lemma 5. *To get a full list of the $K(3, \tilde{q}, \tilde{p})$'s for a given \tilde{q} , it is enough to take*

$$\tilde{p} \in [\tilde{q} + 1, 2\tilde{q}]$$

Proof. If \tilde{q} and \tilde{p} are odd, mutually prime and not divisible by 3, we use the representation of $K(3, \tilde{q}, \tilde{p})$ from (5.3.1). The expression of λ is based on the Bezout identity:

$$6A + Bq = 1 \tag{35}$$

for two integers A and B . Then, up to mirror transformation

$$\lambda(k) = (-1)^{\left\lfloor \frac{2Apk}{q} \right\rfloor} \tag{36}$$

It follows from (36) that, if we replace p by $p + q$, the expression of λ remains the same. \square

For a given \tilde{q} , we let

$$P(\tilde{q}) = \{\tilde{p} \in [\tilde{q} + 1, 2\tilde{q}], \text{ such that } (2, \tilde{p}) = (3, \tilde{p}) = 1\}$$

$$S(\tilde{q}) = \{\tilde{p} \in P(\tilde{q}), \text{ such that } K(3, \tilde{q}, \tilde{p}) \text{ verifies 1) (a) of Theorem 7}\}$$

The table below gives us, for a given \tilde{q} , the percentage of elements of $P(\tilde{q})$ which belong to $S(\tilde{q})$: i.e. the percentage of $K(3, \tilde{q}, \tilde{p})$'s for which every $K(3, n\tilde{q}, n\tilde{p})$, $n \in \mathbb{N}$, has zero signature. The percentages have been rounded to the integer part.

\tilde{q}	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49
%	40	57	54	61	70	63	69	64	68	77	62	75	78	74	80	69

\tilde{q}	53	55	59	61	65	67	71	73	77	79	83	85	89	91	95	97	101
%	79	72	77	78	76	80	67	82	75	83	84	77	80	74	80	82	83

References

- [A'C] N. A'Campo *Tresses, monodromie et le groupe symplectique*, Comm. Math. Helvetici (54) 318-327, (1979)
- [As] M. Aschbacher *Finite group theory*, Cambridge Univ. Press, London 1986
- [B-K] M. Brandenbursky, J. Kedra *Concordance group and stable commutator length in braid groups*, Algebraic & Geometric Topology (5), 2861-2886, (2015).
- [Br] M. Brandenbursky, *On quasi-morphisms from knot and braid invariants*, Jour. of Knot Theory and its Ramifications, vol. 20, No. 10 (2011), 1397-1417.
- [Bu] W. Burau, *Über Zopfgruppen und gleichsinnig verdrillte Verkettungen* Abh. Math. Sem. Univ. Hamburg, 11(1) (1935) 179–186
- [Di] P. Diaconis, *Representations in probability and statistics*, IMS, Hayward, 1986

- [G-G] J.-M. Gambaudo, E. Ghys *Braids and signatures*, Bull. SMF 133 (2005), 541-579
- [It] T. Ito, *On a structure of random open books and closed braids*, Proc. Japan Acad. 91, Ser. A (2015)
- [K-T] C. Kassel, V. Turaev *Braid groups*, Graduate texts in maths, Springer-Verlag New York (2008)
- [La] C. Lamm *Symmetric unions and ribbons knots*, Osaka J. Math. 37 (2000), 537-550
- [L-O] C. Lamm, D. Obermeyer *Billiard knots in a cylinder* J. Knot Theory and its Ramifications 8(3) (1999) Vol. 353-366.
- [L-R] G. Lachaud, R. Rolland *On the number of points of algebraic sets over finite fields*, Journal of Pure and Applied Algebra 219 (11) , 5117-5136 (2015) On the number of points of algebraic sets over finite fields
- [Ma] A. V. Mal'jutin, *Quasimorphisms, random walks, and transient subsets in countable groups*, Jour. of Math. Sciences 181(6) 871–885 (2012)
- [Mi] J.S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics (170), CUP 2017
- [Mu] K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. 117 (1965), 387-422.
- [O'M] O. T. O'Meara *Symplectic groups*, AMS, Providence (1978)
- [Pr] J. H. Przytycki *Positive knots have negative signature*, Bull. Ac. Pol.: Math. 37, 1989, 559-562.
- [Ri] I. Rivin *Geometric phenomena in groups in Thin groups and superstrong approximation* MSRI Publications, Volume 61 (2013)
- [Ru] L. Rudolph *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. 29 (1993), 51-59.
- [S-V] M. Soret, M. Ville *Lissajous-toric knots*, Jour. of Knot Theory and Its Ramifications 29(1) (2020)

[Ta] T. Tanaka, *Four-genera of quasipositive knots*, Topology and its applications 83 (1998) 187-192

[wi] https://groupprops.subwiki.org/wiki/Order_formulas_for_symplectic_groups

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