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# ON THE SIZE OF MINIMAL SURFACES IN $\mathbb{R}^4$

ARI AIOLFI, MARC SORET AND MARINA VILLE

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## Abstract

The Gauss map  $g$  of a surface  $\Sigma$  in  $\mathbb{R}^4$  takes its values in the Grassmannian of oriented 2-planes of  $\mathbb{R}^4$ :  $G^+(2, 4)$ . We give geometric criteria of stability for minimal surfaces in  $\mathbb{R}^4$  in terms of  $g$ . We show in particular that if the spherical area of the Gauss map  $|g(\Sigma)|$  of a minimal surface is smaller than  $2\pi$  then the surface is stable by deformations which fix the boundary of the surface. This answers the question of [BDC3] in  $\mathbb{R}^4$ .

*Keywords*— Gauss map, Grassmannian, minimal surface, stability

## 1 Introduction

A geometric criterion for the stability of a minimal surface  $\Sigma$  in the Euclidean space  $\mathbb{R}^3$  is the spherical area of its Gauss map image  $|g(\Sigma)|$ . Thus, if this area counted without multiplicity is smaller than  $2\pi$  then the surface is stable (see [BDC] & [BDC2]).

A similar stability criterion was later generalized for simply-connected minimal surfaces in  $\mathbb{R}^n$  where the Gauss map of  $\Sigma$  takes its values in the Grassmannian  $G^+(2, n)$  (see also [HO]):

**Theorem 1** ([BDC3]). *Let  $\Sigma$  be a minimal surface in  $\mathbb{R}^n$ . If  $\Sigma$  is simply-connected and if the area  $|g(\Sigma)|$  counted with multiplicity is smaller than  $\frac{4\pi}{3}$ , then  $\Sigma$  is stable.*

In [BDC3] one asks whether the weaker condition  $|g(\Sigma)| < 2\pi$  ensures stability. We prove in particular that this is true for minimal surfaces in  $\mathbb{R}^4$  even when  $|g(\Sigma)|$  is counted without multiplicity as it is already known for minimal surfaces in  $\mathbb{R}^3$ .

More precisely, we suppose that  $\Sigma$  is a minimal surface of  $\mathbb{R}^4$ . The Gauss map of  $\Sigma$  takes its values in the Grassmannian of oriented 2-planes in  $\mathbb{R}^4$ :  $g : \Sigma \rightarrow G^+(2, 4)$ . Note that  $G^+(2, 4)$  is isometric to the product of spheres  $\mathbb{S}^2\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^2\left(\frac{1}{\sqrt{2}}\right)$  and that  $g$  is the product of two  $\mathbb{S}^2$ -valued maps: the left Gauss map and right Gauss

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map which is denoted by  $g = (\mathfrak{g}_L, \mathfrak{g}_R)$ . As  $\Sigma$  is minimal, it is a critical point for the area functional for deformations with fixed boundary. If it is a local minimum then it is stable.

We begin by studying the relationship between the stability of  $\Sigma$  and the area of each projection of the Gauss map image of  $\Sigma$ . We first show:

**Theorem 2.** *Let  $\Sigma$  be a minimal surface in  $\mathbb{R}^4$ . Let  $\lambda$  (resp.  $\mu$ ) be the first Dirichlet eigenvalues of the Gauss images  $\mathfrak{g}_L(\Sigma)$  (resp.  $\mathfrak{g}_R(\Sigma)$ ). If the harmonic mean of  $\lambda$  and  $\mu$  is larger than 2 then  $\Sigma$  is stable.*

We deduce from this theorem Proposition 3 (p. 11) where the stability condition is expressed in terms of the area of the left and right Gauss map images. This is obtained by replacing the Dirichlet eigenvalues by lower bounds in terms of the area *via* the standard isoperimetric inequality for spherical domains as in [BDC2]. One derives the next corollary which directly implies Theorem 1 of [BDC3] in  $\mathbb{R}^4$ .

**Corollary 1.** *Let  $\Sigma$  be a minimal surface in  $\mathbb{R}^4$ . Suppose that the spherical areas of  $\Sigma$  satisfy  $|\mathfrak{g}_L(\Sigma)| + |\mathfrak{g}_R(\Sigma)| \leq \frac{4\pi}{3}$  then  $\Sigma$  is stable.*

Notice first that there is no condition on the topology of  $\Sigma$  because only the classical isoperimetric inequality on 2-spheres is used (and not an isoperimetric inequality for surfaces in  $G(2, 4)$  as in [BDC3]). Secondly, one always has  $|\mathfrak{g}_L(\Sigma)| + |\mathfrak{g}_R(\Sigma)| \leq |g(\Sigma)|$  (cf. Lemma 3 and Corollary 3), hence the stability condition of Corollary 1 on the sum of the projected area is a weaker hypothesis than the spherical area upper bound of Theorem 1 of [BDC3].

The stability domain obtained in Corollary 1 (resp. in Proposition 3) is represented by Domain 1 (resp. Domain 3) of figure 1.

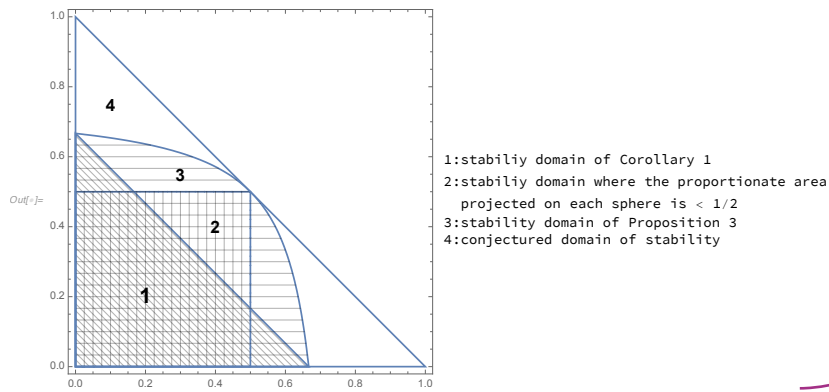


Figure 1.1: Stability wrt to the projected spherical area :  $\Sigma \mapsto \left( \frac{|\mathfrak{g}_L(\Sigma)|}{2\pi}, \frac{|\mathfrak{g}_R(\Sigma)|}{2\pi} \right)$

**Remark 1.** *The spherical areas  $|g(\Sigma)|$  or  $|\mathfrak{g}_L(\Sigma)| + |\mathfrak{g}_R(\Sigma)|$  are counted without multiplicity. In particular  $|\mathfrak{g}_L(\Sigma)| + |\mathfrak{g}_R(\Sigma)|$  is bounded from above by  $4\pi$  contrary to  $|g(\Sigma)|$  which has no a priori upper bound (see Proposition 4). Notice also that the Corollary 1 can be restated in terms of proportionate area by saying that if the proportionate area sum is less than one third - then the minimal surface is stable.*

Complex curves  $\Sigma \subset \mathbb{R}^4$  satisfy  $|\mathfrak{g}_L(\Sigma)| = 0$ ,  $|\mathfrak{g}_R(\Sigma)| \leq 2\pi$ ,  $|g(\Sigma)| \leq 2\pi$ , and are examples of stable minimal surfaces. In light of these examples, one may wonder -as in [BDC3] - if the upper bound  $\frac{4\pi}{3}$  of Theorem 1 can be replaced by  $2\pi$ . Indeed, one shows in the second part of the paper that

**Theorem 3.** *Let  $\Sigma$  be a minimal surface in  $\mathbb{R}^4$ . If  $|g(\Sigma)| < 2\pi$  then  $\Sigma$  is stable.*

The proof relies on Theorem 2 and on next proposition.

**Proposition 1.** *For any minimal surface  $\Sigma$ , there is an associate minimal surface  $\Sigma^*$  isometric to  $\Sigma$ , such that:*

1. *the left and right Gauss area of  $\Sigma^*$  are equal.*
2. *if  $\Sigma^*$  is stable then so is  $\Sigma$ .*

Finally one notices in Section 4.2 that the domain of stability of Proposition 3 and Corollary 1 can be enlarged. Also, the consideration of complex curves show that the domain of stability as illustrated in Figure 1 must contain the left and bottom edge of the square. Hence Domain 4 of Figure 1 may reasonably represent a larger domain of stability and a stronger result than Theorem 3 might then be true.

**Question.** *Let  $\Sigma$  be a minimal surface in  $\mathbb{R}^4$ . If  $|\mathfrak{g}_L(\Sigma)| + |\mathfrak{g}_R(\Sigma)| < 2\pi$  then is  $\Sigma$  stable?*

## 2 Quaternions and Gauss maps

The use of quaternions allows us to combine the classical approach to the Grassmannian  $G^+(2, 4)$  as a quadric in  $\mathbb{C}P^3$  (see for example [HO]) with Eells-Salamon's approach *via* complex structures on  $\mathbb{R}^4$  ([ES]).

We identify  $\mathbb{R}^4$  with  $\mathbb{C}^2$  and with the quaternions  $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}I \oplus \mathbb{R}J \oplus \mathbb{R}K$ ; we choose the identification such that the canonical complex structure on  $\mathbb{C}^2$  will be identified to  $I$ . Let us recall some basic facts. Each quaternion is the sum of a real and a pure quaternion, namely  $X = a + bI + cJ + dK = a + P_X$ . The product of two quaternions is then given by:

$$X \cdot Y = aa' - \langle P_X, P_Y \rangle + aP_Y + a'P_X + P_X \wedge P_Y \quad (1)$$

where  $X = a + P_X$ ,  $Y = a' + P_Y$  and  $\wedge$  is the cross product of  $\mathbb{R}^3$ . The conjugate of  $X$  is denoted by  $\bar{X} = a - P_X$  and  $\Re(X) = a$ . The Euclidean scalar product

on  $\mathbb{H}$  is then given by :  $\langle X, Y \rangle := \Re(X \cdot \bar{Y})$ ,  $X, Y \in \mathbb{H}$ . We identify the set of unit quaternions :  $\mathbb{H}_U$  with the 3-sphere  $\mathbb{S}^3$  of radius 1 around 0. Similarly, we identify the set of pure quaternions  $\mathbb{H}_P$  - ie  $\text{span}(I, J, K)$ - with  $\mathbb{R}^3$ . With these identifications  $\mathbb{H}_{UP} := \mathbb{H}_P \cap \mathbb{H}_U$  is a 2-sphere  $\mathbb{S}^2$  which is an equator of the unit quaternions  $\mathbb{H}_U = \mathbb{S}^3$ .



## 2.1 The Grassmannian $G^+(2, 4)$

For any oriented vector plane  $P \subset \mathbb{H}$  choose any orthogonal positive basis of vectors of same length  $\{T_1, T_2\}$ ; then  $q := T_2 \cdot T_1^{-1}$  doesn't depend on the choice of the direct orthogonal basis of  $P$  and  $q$  is the only pure unit quaternion -or complex orthogonal structure- that leaves the oriented plane invariant when operating on the left ( $qT_1 = T_2T_1^{-1}T_1 = T_2$ , and  $qT_2 = q^2T_1 = -T_1$ . It is also useful to notice that  $X^{-1} = \frac{\bar{X}}{|X|^2}$ ).

Proceeding identically for the right multiplication we define a map from the Grassmannian of oriented planes of  $\mathbb{R}^4$  to  $\mathbb{S}^2 \times \mathbb{S}^2$ :

$$G = (g_L, g_R) : \left( \begin{array}{ccc} G^+(2, 4) & \longrightarrow & \mathbb{S}^2 \times \mathbb{S}^2 \\ P & & \mapsto (T_2 \cdot T_1^{-1}, T_1^{-1} \cdot T_2) \end{array} \right)$$

As  $G$  is onto,  $g$  is 1 to 1 and onto.

We then identify the Grassmannian  $G^+(2, 4)$  with  $\mathbb{S}^2 \times \mathbb{S}^2$  via this 1-1 map  $G$  where  $\mathbb{S}^2$  is the round sphere of radius one.

**Remark 2.** Notice that  $G^+(2, 4)$  provided with its natural metric is isometric to  $\mathbb{S}^2 \left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^2 \left(\frac{1}{\sqrt{2}}\right)$ . Thus, although we will make the computations in  $\mathbb{S}^2 \times \mathbb{S}^2$ , results in final statements will be expressed in terms of the Gauss maps with values in the standard Grassmannian as in the introduction.

## 2.2 Left and right Gauss maps of $T\Sigma$ and $N\Sigma$ in isothermal coordinates

Let  $\Sigma$  be a  $C^2$ -surface. Around any point  $p \in \Sigma$  we will choose local isothermal parameters  $(x, y)$  for the immersion  $X : U \longrightarrow \Sigma \subset \mathbb{H}$ . Whether we consider the tangent bundle  $T\Sigma$  or the normal bundle  $N\Sigma$  then two Gauss maps are generated.

Let us first consider the Gauss map of the tangent bundle. At each tangent plane  $T_p\Sigma$ , we compute, at point  $p = X(x, y)$ , the derivatives  $X_x, X_y \in T_p\Sigma$ ; we define the (tangent) Gauss map of  $\Sigma$  by:

$$G := (g_L, g_R) : \left( \begin{array}{ccc} \Sigma & \longrightarrow & \mathbb{S}^2 \times \mathbb{S}^2 = \mathbb{H}_{PU} \times \mathbb{H}_{PU} \\ p & & \mapsto (X_y \cdot X_x^{-1}, X_x^{-1} \cdot X_y) \end{array} \right)$$

$g_L(p)$  (resp.  $g_R(p)$ ) is the orthogonal complex structure that leaves the tangent plane at  $p$  invariant when acting on the left (resp. on the right).

We then consider the Gauss map of the normal bundle. The useful information here is recalled in a lemma:

**Lemma 1.** *Left (resp. right) multiplication by  $\mathfrak{g}_L$  (resp. by  $\mathfrak{g}_R$ ) on  $\mathbb{H}$  both define rotations of  $\mathbb{H}$  such that the angle between any quaternion and its image is of absolute value  $\pi/2$ . Their respective actions are equal on  $T\Sigma$  and of opposite sign on  $N\Sigma$ .*

*Proof.* Suppose  $q \in \mathbb{H}_U$ . Then left or right multiplication by  $q$  defines an isometry:  $\langle q \cdot x, q \cdot x \rangle = \Re(qx\bar{q}) = |x|^2$ . And  $\langle q \cdot x, x \rangle = \Re(qx\bar{x}) = \Re(q)|x|^2$ . Hence the isometry is a rotation and the absolute angle between any nonzero quaternion and its image is constant. If  $q \in \mathbb{H}_{UP}^*$  then then absolute angle is  $\pi/2$ . By construction, left multiplication by  $\mathfrak{g}_L$  and right multiplication by  $\mathfrak{g}_R$  restricted to  $T\Sigma$  are equal to the same  $\pi/2$ -rotation. Since both are isometries, the normal bundle is stable and the restriction of both rotations to  $N\Sigma$  are  $\pi/2$ -rotations. If they were equal, then  $\mathfrak{g}_L x = x\mathfrak{g}_R$  for any  $x \in \mathbb{H}$ . Hence  $\mathfrak{g}_L = \mathfrak{g}_R$  and  $\mathfrak{g}_L$  commutes with any quaternion, which is impossible since  $\mathfrak{g}_L$  is a nonzero pure quaternion. Hence the rotation of  $\mathfrak{g}_L$  and  $\mathfrak{g}_R$  on  $N\Sigma$  are of opposite angle so  $\mathfrak{g}_L x = x\mathfrak{g}_R^{-1} = -x\mathfrak{g}_R$  for any  $x \in N\Sigma$  since the angle is  $\pi/2$ .  $\square$

The action of  $\mathfrak{g}_L$  on the left is a rotation by the lemma above. Hence it preserves the orientation of  $T_p\Sigma \oplus N_p\Sigma$  and, thus, acts positively on  $N_p\Sigma$ . Hence the left Gauss map  $\mathfrak{g}_{N\Sigma,L}$  of the normal bundle  $N\Sigma$  verifies  $\mathfrak{g}_{N\Sigma,L} = \mathfrak{g}_L$ . Moreover we deduce from the lemma that the right Gauss map  $\mathfrak{g}_{N\Sigma,R}$  of the normal bundle verifies  $\mathfrak{g}_{N\Sigma,R} = \mathfrak{g}_R^{-1}$ . The expression of the normal Gauss map in isothermal coordinates is thus given by:

$$G_{N\Sigma} : \begin{pmatrix} \Sigma & \longrightarrow & \mathbb{S}^2 \times \mathbb{S}^2 \\ p & \longmapsto & (X_y \cdot X_x^{-1}, X_y^{-1} \cdot X_x) \end{pmatrix}$$

and  $G_{N\Sigma} = (Id, -Id) \circ G$ .

**Remark 3.** *We will not need it here but one can prove that  $\mathfrak{g}_L$  (resp.  $\mathfrak{g}_R$ ) is a rotation of angle  $+\frac{\pi}{2}$  (resp.  $-\frac{\pi}{2}$ ) on  $N\Sigma$ .*

### 3 First derivatives of $\mathfrak{g}_L$ and $\mathfrak{g}_R$ for minimal surfaces

In order to prove the results of stability we will need some properties of the left and right Gauss maps. We will use local isothermal coordinates on a Riemann surface  $U$  for the immersion  $X : U \longrightarrow \Sigma \subset \mathbb{H}$  with these notations :  $E := |X_x|^2 = |X_y|^2, \langle X_x, X_y \rangle = 0$ , the induced metric of  $\Sigma$  on  $U$  being  $ds_\Sigma^2 = E|dz|^2$  with  $z = x + iy$ . }

**Proposition 2.** *The left Gauss map  $\mathfrak{g}_L$  and right Gauss map  $\mathfrak{g}_R$  of the Gauss map  $g := (\mathfrak{g}_L, \mathfrak{g}_R)$  of the tangent bundle of a minimal surface in  $\mathbb{R}^4$  satisfy the following properties:*

1. *Left multiplication by  $\mathfrak{g}_L$  defines a complex structure on  $T\Sigma$  and on  $N\Sigma$ . Right multiplication by  $\mathfrak{g}_R$  defines the same complex structure on  $T\Sigma$  but the*

antiholomorphic structure on  $N\Sigma$ . The values of  $\mathfrak{g}_L$  and  $\mathfrak{g}_R$  are pure unitary quaternions and

$$\mathfrak{g}_L^2 = \mathfrak{g}_R^2 = -1.$$

In particular,  $\mathfrak{g}_{L,x}$  and  $\mathfrak{g}_L$  (resp.  $\mathfrak{g}_{R,x}$  and  $\mathfrak{g}_R$ ) anticommute.

2. The Gauss maps  $\mathfrak{g}_L$  and  $\mathfrak{g}_R$  are anti-holomorphic for the  $\mathfrak{g}_L$ - or  $\mathfrak{g}_R$ -complex structure on  $\Sigma$ . More precisely

$$\mathfrak{g}_{L,y} = -\mathfrak{g}_L \mathfrak{g}_{L,x}, \quad \mathfrak{g}_{R,y} = -\mathfrak{g}_R \mathfrak{g}_{R,x}.$$

Notice that the values of the maps  $\mathfrak{g}_{Z,a}$  are pure quaternions where  $Z = L, R$ ,  $a = x, y$ .

3. Let  $B_{ab}$  be the second fundamental form defined by the normal projections of the second derivatives of the position vector  $X_{ab}^N$ , where  $a, b = x$  or  $y$ . We have:

$$\mathfrak{g}_{L,x} = -\mathfrak{g}_L(B_{xx} + \mathfrak{g}_L B_{xy})X_x^{-1}, \quad \mathfrak{g}_{R,x} = -X_x^{-1}(B_{xx} + B_{xy}\mathfrak{g}_R)\mathfrak{g}_R.$$

Notice that left multiplication by  $\mathfrak{g}_{L,x}$  or  $\mathfrak{g}_{L,y}$  ( resp. right multiplication by  $\mathfrak{g}_{R,x}$  or  $\mathfrak{g}_{R,y}$ ) permutes  $T\Sigma$  and  $N\Sigma$  and are also elements of  $TS^2$ .

4. The quaternion fields

$$B_{xx} + \mathfrak{g}_L B_{xy} (= \mathfrak{g}_{L,y} X_x), \quad B_{xx} + B_{xy}\mathfrak{g}_R (= X_x \mathfrak{g}_{R,y})$$

are anti-holomorphic sections of the normal bundle of  $\Sigma$ .

5. We have:

$$\begin{cases} E|\mathfrak{g}_{L,x}|^2 = \frac{1}{2}|B|^2 + \delta \\ E|\mathfrak{g}_{R,x}|^2 = \frac{1}{2}|B|^2 - \delta \end{cases} \quad (2)$$

where  $|B|^2 := 2|B_{xx}|^2 + 2|B_{xy}|^2$ ,  $\delta = 2\langle B_{xx}, \mathfrak{g}_L B_{xy} \rangle$ .

*Proof.* 1. is clear.

2. We first show that the right Gauss map  $\mathfrak{g}_R$  is anti-conformal.

Recall that  $X : U \rightarrow \Sigma \subset \mathbb{H}$  be a minimal immersion then for isothermal coordinates we have  $|X_x| = |X_y|$ ,  $X_x \perp X_y$ . If  $X$  minimal then  $\Delta X = X_{xx} + X_{yy} = 0$ .

In order to prove

$$\mathfrak{g}_{R,y} = -\mathfrak{g}_{R,x}\mathfrak{g}_R$$

one first computes:

$$\mathfrak{g}_{R,x} = (X_x^{-1} X_y)_x = X_x^{-1}(-X_{xx} X_x^{-1} X_y + X_{xy}) = X_x^{-1}(-X_{xx}\mathfrak{g}_R + X_{xy})$$

and

$$\mathfrak{g}_{R,y} = (X_x^{-1} X_y)_y = X_x^{-1}(-X_{xy} X_x^{-1} X_y + X_{yy}) = -X_x^{-1}(X_{xy} - X_{xx}\mathfrak{g}_R)\mathfrak{g}_R = -\mathfrak{g}_{R,x}\mathfrak{g}_R.$$

Similarly one proves that

$$\mathfrak{g}_{L,y} = -\mathfrak{g}_L \mathfrak{g}_{L,x}. \quad (3)$$

From which one deduces the conformal property of the map :

$$|\mathfrak{g}_{L,x}| = |\mathfrak{g}_{L,y}|, \langle \mathfrak{g}_{L,x}, \mathfrak{g}_{L,y} \rangle = 0. \quad (4)$$

3. A direct computation gives:

$$\mathfrak{g}_{L,x} = (X_y X_x^{-1})_x = (X_{xy} - \mathfrak{g}_L X_{xx}) X_x^{-1} = -\mathfrak{g}_L (X_{xx} + \mathfrak{g}_L X_{xy}) X_x^{-1}.$$

Similarly, for the right Gauss map

$$\mathfrak{g}_{R,x} = (X_x^{-1} X_y)_x = X_x^{-1} (X_{xy} - X_{xx} \mathfrak{g}_R) = -X_x^{-1} (X_{xx} + X_{xy} \mathfrak{g}_R) \mathfrak{g}_R.$$

Thus from (3)

$$\mathfrak{g}_{L,y} X_x = -(B_{xx} + \mathfrak{g}_L B_{xy}).$$

4. We need the following lemma:

**Lemma 2.**  $X_{xx} + \mathfrak{g}_L X_{xy}$  (resp.  $X_{xx} + X_{xy} \mathfrak{g}_R$ ) is a holomorphic normal section of  $\Sigma$  equal to  $B_{xx} + \mathfrak{g}_L B_{xy}$  (resp. to  $B_{xx} + B_{xy} \mathfrak{g}_R$ ).

*Proof.* Let us prove first that there is no component along  $X_x$ :

$$\langle \mathfrak{g}_L X_{xy}, X_x \rangle = -\langle X_{xy}, X_y \rangle = \langle X_x, X_{yy} \rangle = -\langle X_x, X_{xx} \rangle.$$

Hence

$$\langle X_{xx} + \mathfrak{g}_L X_{xy}, X_x \rangle = 0.$$

Similarly  $\langle X_{xx} + \mathfrak{g}_L X_{xy}, X_y \rangle = 0$ . Hence  $X_{xx} + \mathfrak{g}_L X_{xy}$  is a normal section hence equal to  $B_{xx} + \mathfrak{g}_L B_{xy}$ .

In order to prove the holomorphy, we write Codazzi equations in the isothermal coordinates  $x, y$ :

$$\begin{cases} B_{xx,y} - B_{xy,x} = 0 \\ B_{xy,y} - B_{yy,x} = 0. \end{cases} \quad (5)$$

Hence, using the Codazzi equations:

$$(B_{xx} + \mathfrak{g}_L B_{xy})_y = B_{xx,y} + \mathfrak{g}_{L,y} B_{xy} + \mathfrak{g}_L B_{xy,y} = B_{xy,x} - \mathfrak{g}_L \mathfrak{g}_{L,x} B_{xy} + \mathfrak{g}_L B_{yy,x}.$$

Since  $\Sigma$  is minimal:  $B_{xx} = -B_{yy}$  and

$$(B_{xx} + \mathfrak{g}_L B_{xy})_y = -\mathfrak{g}_L (B_{xx} + \mathfrak{g}_L B_{xy})_x.$$

□



**Remark 4.** The Gauss map  $\mathfrak{g}_{L,\cdot}$  is of the form  $v.w^{-1}$  where  $v \in N\Sigma$  and  $w \in T\Sigma$  from which it is clear that  $\mathfrak{g}_{L,\cdot}(T\Sigma) = N\Sigma$  and from which one deduces also that  $\mathfrak{g}_{L,\cdot}(N\Sigma) = T\Sigma$ .

5. Let  $E = |X_x|^2 = |X_y|^2$  then

$$|\mathfrak{g}_{L,x}|^2 = \frac{1}{E}|X_{xx} + \mathfrak{g}_L X_{xy}|^2 \text{ and } |\mathfrak{g}_{R,x}|^2 = \frac{1}{E}|X_{xx} + X_{xy}\mathfrak{g}_R|^2. \quad (6)$$

Thus

$$E|\mathfrak{g}_{L,x}|^2 = |B_{xx} + \mathfrak{g}_L B_{xy}|^2 = |B_{xx}|^2 + |B_{xy}|^2 + 2\langle B_{xx}, \mathfrak{g}_L B_{xy} \rangle.$$

And

$$E|\mathfrak{g}_{R,x}|^2 = |B_{xx}|^2 + |B_{xy}|^2 + 2\langle B_{xx}, B_{xy}\mathfrak{g}_R \rangle.$$

As  $\langle B_{xx}, B_{xy}\mathfrak{g}_R \rangle = -\langle B_{xx}, \mathfrak{g}_L B_{xy} \rangle$  we obtain:

$$\begin{cases} E|\mathfrak{g}_{L,x}|^2 = \frac{1}{2}|B|^2 + \delta \\ E|\mathfrak{g}_{R,x}|^2 = \frac{1}{2}|B|^2 - \delta \end{cases} \quad (7)$$

where  $|B|^2 := 2|B_{xx}|^2 + 2|B_{xy}|^2$  and  $\delta = 2\langle B_{xx}, \mathfrak{g}_L B_{xy} \rangle$ . □

**Remark 5.** One can express the curvatures of the tangent and normal bundle of a minimal surface in terms of the derivatives of the left and right Gauss maps in isothermal coordinates.

From the Gauss equation we deduce that the curvature tensor of the tangent bundle is equal to

$$R(X_x, X_y, X_y, X_x) := R_{xyyx} = \langle B_{xx}, B_{yy} \rangle - \langle B_{xy}, B_{yx} \rangle = -|B_{xx}|^2 - |B_{xy}|^2.$$

Hence the tangent Gaussian curvature computed in the chosen isothermal coordinates is equal to:  $K^T := \frac{R_{xyyx}}{|X_x|^2|X_y|^2 - \langle X_x, X_y \rangle^2} = -\frac{|B|^2}{2E^2}$ . Similarly, the Ricci equation

gives for the normal curvature in terms of  $B$ :  $\langle R^N(X_x, X_y)\xi, \eta \rangle = \langle [A_\xi, A_\eta] X_x, X_y \rangle$  where  $\xi, \eta$  are normal vectors and  $A_\xi$  is the shape operator of the second form  $B$  in the direction  $\xi$  so that  $\langle A_\xi X, Y \rangle = \langle B(X, Y), \xi \rangle$ . Choose  $\xi = e_1, \eta = e_2$  such that  $\{X_x, X_y, e_1, e_2\}$  is a conformal frame in  $\mathbb{H}$ . Let  $B_{ab}^i := \langle B_{ab}, e_i \rangle$  ( $i = 1, 2, a, b = x, y$ ). Then  $K^N := \frac{R_{xy12}^N}{E^2} = \frac{\langle R^N(X_x, X_y)e_1, e_2 \rangle}{E^2} = -\frac{2}{E}(B_{xx}^1 B_{xy}^2 - B_{xx}^2 B_{xy}^1) = \frac{2}{E^2} \langle B_{xx}, \mathfrak{g}_L B_{xy} \rangle = \frac{1}{E^2} \delta$ . From equations (7) we deduce that

$$\begin{cases} K^T &= -\frac{1}{2E} (|\mathfrak{g}_{L,x}|^2 + |\mathfrak{g}_{R,x}|^2) \\ K^N &= -\frac{1}{2E} (|\mathfrak{g}_{L,x}|^2 - |\mathfrak{g}_{R,x}|^2) \end{cases} \quad (8)$$

We see in particular that  $|K^T| \geq |K^N|$ .

## 4 Proof of Theorem 2 and corollaries

### 4.1 Proof of Theorem 2

Let us recall [BDC]'s stability condition for minimal surfaces in  $\mathbb{R}^3$ .

**Theorem 4.** [BDC] *Let  $\Sigma$  be a minimal surface. If the spherical area  $g(\Sigma) \subset \mathbb{S}^2$  in the round sphere of radius one is smaller than  $2\pi$  then  $\Sigma$  is stable.*

We first follow the proof of [BDC] applied to the right and left Gauss maps.

*Proof.* Let  $X : U \rightarrow \Sigma \subset \mathbb{R}^4$  be the minimal immersion. Suppose the first Dirichlet eigenvalue of the spherical image  $D := \mathbf{g}_L(U)$  is  $\lambda$ ; then from Rayleigh's formula

$$\int_D |\nabla f|_{\mathbb{S}^2}^2 da_{\mathbb{S}^2} - \lambda \int_D f^2 da_{\mathbb{S}^2} \geq 0 \quad (9)$$

for all compact support functions  $f : D \rightarrow \mathbb{R}$ .

From the conformality of  $\mathbf{g}_L$  and from equations (4), the left Gauss map  $\mathbf{g}_L$  induces on  $U$  a spherical area and a spherical pseudo-metric which in terms of the pulled-back metric element  $ds_{\Sigma}^2$  or area element  $da_{\Sigma}$  of  $\Sigma$  on  $U$  are -using equations (8)- given by

$$\mathbf{g}_L^*(da_{\mathbb{S}^2}) = \frac{|\mathbf{g}_{L,x}|^2}{E} da_{\Sigma} = -(K^T + K^N) da_{\Sigma} \quad \mathbf{g}_L^*(ds_{\mathbb{S}^2}^2) = -(K^T + K^N) ds_{\Sigma}^2. \quad (10)$$

On a subdomain  $U' \subset U$  where  $\mathbf{g}_L$  is a diffeomorphism,  $|\mathbf{g}_{L,x}|^2 ds^2$  is a metric element and  $|\nabla(f \circ \mathbf{g}_L)|_{\Sigma}^2 da_{\Sigma} = |\nabla f|_{\mathbb{S}^2}^2 da_{\mathbb{S}^2}$ . Thus

$$\int_{\mathbf{g}_L(U')} (|\nabla f|_{\mathbb{S}^2}^2 - \lambda f^2) da_{\mathbb{S}^2} = \int_{U'} (|\nabla(f \circ \mathbf{g}_L)|_{\Sigma}^2 + \lambda(K^T + K^N)(f \circ \mathbf{g}_L)^2) da_{\Sigma}. \quad (11)$$

The functional defined by the RHS of (11) extends to  $U$  and [BDC2] show that for all compact support functions  $f : U \rightarrow \mathbb{R}$

$$\int_U (|\nabla f|_{\Sigma}^2 + \lambda(K^T + K^N)f^2) da_{\Sigma} \geq 0 \quad (12)$$

(if the LHS were negative for a compact support function  $f$  on  $U$ , then the function on  $D$  defined by  $\tilde{f}(x) := \sum_{\mathbf{g}_L(y)=x} f(y)$  would not satisfy inequality (9); for more details see [BDC2]).

The same considerations for the right Gauss map yield also - for the first Dirichlet eigenvalue  $\mu$  of the spherical domain  $\mathbf{g}_R(U)$ :

$$\int_U |\nabla f|_{\Sigma}^2 da_{\Sigma} + \mu \int_U (K^T - K^N) f^2 da_{\Sigma}$$

and combining linearly the former two inequalities and applying equations (8):

$$\int_U |\nabla f|_{\Sigma}^2 da_{\Sigma} + \frac{2\lambda\mu}{\lambda + \mu} \int_U K^T f^2 da_{\Sigma} \geq 0 \quad (13)$$

On the other hand recall that the surface  $\Sigma$  is stable in  $\mathbb{R}^4$  if for any normal section  $\xi$  with compact support which vanishes at the boundary (cf. for example [L]):

$$\int_U \left( |\nabla^N \xi|_\Sigma^2 - \sum_{i,j=1,2} \langle B_{ij}, \xi \rangle^2 \right) da_\Sigma \geq 0 \quad (14)$$

where  $\nabla^N$  is the induced connection on the normal bundle of  $\Sigma$  and where  $B_{ij}$  are the coordinates of the second form wrt to an orthonormal basis of  $T_p\Sigma$  - for example wrt  $\{\frac{X_x}{\sqrt{E}}, \frac{X_y}{\sqrt{E}}\}$ . Hence:

$$\int_U \left( |\nabla^N \xi|_\Sigma^2 - \sum_{i,j=x,y} \frac{1}{E^2} \langle B_{ij}, \xi \rangle^2 \right) da_\Sigma \geq 0 \quad (15)$$

Since

$$\frac{1}{E^2} \sum_{i,j=x,y} \langle B_{ij}, \xi \rangle^2 \leq -2K^T |\xi|^2,$$

the LHS of inequality (15) is larger than

$$\int_U |\nabla^N \xi|_\Sigma^2 da_\Sigma + 2 \int_U K^T |\xi|^2 da_\Sigma. \quad (16)$$

Choose  $\xi = f\zeta$  where  $f$  is of compact support and where  $\zeta$  is a unitary vector field. Using the same notations as in Proposition 2 then

$$\int_U |\nabla^N \xi|_\Sigma^2 da_\Sigma = \int_U |\nabla^N \zeta|_\Sigma^2 f^2 da_\Sigma + \int_U |\nabla f|_\Sigma^2 da_\Sigma. \quad (17)$$

Replacing in (16) the first term by (17) and using (13), we obtain

$$\int |\nabla^N \zeta|_\Sigma^2 f^2 ds + \left( 2 - \frac{2\lambda\mu}{\lambda + \mu} \right) \int K^T f^2 ds \quad (18)$$

which is positive if

$$\frac{\lambda\mu}{\lambda + \mu} > 1 \quad (19)$$

that is if the harmonic mean of the eigenvalues  $\lambda$  and  $\mu$  is larger than 2.  $\square$

## 4.2 Stability in terms of the spherical area of the Gauss maps

One can deduce from the isoperimetric inequality in the round sphere a lower bound for the first Dirichlet eigenvalue of the spherical domain  $D$  of area  $A$  as in [BDC2]:

$$\lambda_1(A) \geq \frac{2(4\pi - A)}{A} = 2\left(\frac{1}{a} - 1\right) \quad (20)$$

where  $a := \frac{A}{4\pi}$  is the proportionate area of  $D$ . Hence Inequality (19) is true if

$$\frac{1}{\lambda} + \frac{1}{\mu} \leq \frac{a}{2(1-a)} + \frac{b}{2(1-b)} \leq 1$$

Therefore:

**Proposition 3.** *If the proportionate areas  $a = \frac{|\mathfrak{g}_L(\Sigma)|}{2\pi}$  and  $b = \frac{|\mathfrak{g}_R(\Sigma)|}{2\pi}$  satisfy  $a \leq \frac{2}{3}$ ,  $b \leq \frac{2-3a}{3-4a}$  then  $\Sigma$  is stable.*

The set  $\{(a, b) : 0 \leq b \leq \frac{2-3a}{3-4a}, 0 \leq a, b \leq \frac{2}{3}\}$  is a domain in the square  $[0, 1] \times [0, 1]$  bounded by the edges  $x = 0$ ,  $y = 0$  and the equilateral hyperbola passing through the points  $(2/3, 0)$ ,  $(0, 2/3)$ ,  $(1/2, 1/2)$  which corresponds to Domain 3 in Figure 1 and Domain 1 in Figure 4.2.

The lower bound in inequality (20) is not optimal. Using Sato's second approximation in [S], we obtain a larger domain of stability bounded by the equilateral hyperbola passing through the points  $(.737, 0)$ ,  $(0, .737)$ ,  $(1/2, 1/2)$  (illustrated by Domain 2 of Figure 4.2). In a similar fashion the stability domain of Corollary 1 can be enlarged.

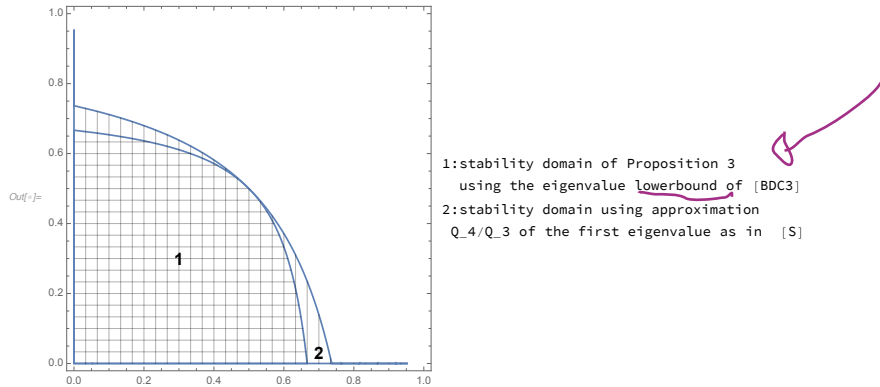


Figure 4.1:  $\Sigma \mapsto \left( \frac{|\mathfrak{g}_L(\Sigma)|}{2\pi}, \frac{|\mathfrak{g}_R(\Sigma)|}{2\pi} \right)$

### 4.3 Local parametrization of a minimal surface

For the proof of Theorem 3 and for explicit examples of minimal surfaces it is useful to introduce Weierstrass-type coordinates.

The set of minimal surfaces in  $\mathbb{R}^4$  are locally easy to describe. The data defining a minimal immersion of a disk is a quadruple of holomorphic function  $(e, f, g, h)$  on  $U$  such that

$$e' f' + g' h' = 0. \quad (21)$$

The minimal immersion is then given by the map

$$X : \begin{pmatrix} U & \longrightarrow & \mathbb{H} \\ z & \longmapsto & e(z) + \bar{f}(z) + (g(z) + \bar{h}(z))J \end{pmatrix} \quad (22)$$

the coordinates of the immersion are automatically harmonic. In this algebraic setting, the conformality condition of  $X$  - given by equation (21) - is best understood in the complexified ambient space.

In Section 6 we will show that the Gauss maps  $(\mathfrak{g}_L, \mathfrak{g}_R)$  are given in terms of these holomorphic coordinates as  $(g_L, g_R)$  where

$$g_L(z) = \frac{e'(z)}{h'(z)} = -\frac{g'(z)}{f'(z)}, \quad g_R(z) = -\frac{e'(z)}{g'(z)} = \frac{h'(z)}{f'(z)}. \quad (23)$$

### 4.4 Stability domain for minimal surfaces with flat normal bundle

Note from (2) and (8) that  $|\mathfrak{g}_{L,x}|^2$  and  $|\mathfrak{g}_{R,x}|^2$  are identical iff  $\delta = K^N = 0$ . The normal bundle is then flat which means  $B_{xx}$  and  $B_{xy}$  are colinear from the expression of  $\delta$  in Equation (2). One deduces that the surface lies in a hyperplane (see for example [A]). Hence minimal surfaces have a flat normal bundle iff they lie in  $\mathbb{R}^3$ . Furthermore the Weierstrass representation of such minimal surface can be chosen of the form  $X = (e + \bar{f}, g + \bar{g})$  in the coordinate system of (22); we see from (23) that the first eigenvalues  $\lambda$  and  $\mu$  of  $\mathfrak{g}_L$  and  $\mathfrak{g}_R$  are identical. Stability is then obtained if the first eigenvalue  $\lambda$  of the right or left spherical domain is larger than 2. The corresponding domain of stability is the diagonal of the square represented by Domain 2 of Figure 1 .

#### 4.4.1 Unstability

If the surface lies in a hyperplane then there is a constant normal vector field  $\zeta$ . Then plug  $\xi = f\zeta$  into LHS of (15) and using equality (17) we obtain:

$$\int_U |\nabla f|_{\Sigma}^2 da_{\Sigma} - \int_U \sum_{i,j=x,y} \frac{1}{E^2} \langle B_{ij}, \zeta \rangle^2 f^2 da_{\Sigma}. \quad (24)$$

Then replace in expression (24) the normal field  $\zeta$  by the normal fields  $e^{gt}\zeta = (\cos t + \mathfrak{g} \sin t)\zeta$  and average wrt  $t \in [0, 2\pi]$ . Choose  $f \in C_0(U)$  such that

$$\int |\nabla f|^2 ds = \frac{\lambda}{2} \int \frac{1}{E^2} (|B_{xx}|^2 + |B_{xy}|^2) f^2 ds.$$

Then (24) is negative if  $\lambda < 2$ . Hence

**Corollary 2.** *A minimal surface of  $\mathbb{R}^4$  with flat normal bundle is stable if the first eigenvalue  $\lambda$  of the left (or right) spherical domain  $\mathfrak{g}_L(\Sigma)$  satisfies  $\lambda > 2$  and unstable if  $\lambda < 2$ .*

#### 4.4.2 An example

Following the notations defined in (22), define on the slab  $U = \{z \in \mathbb{C} : 0 \leq \Im(z) < 2\pi\}$  a minimal immersion  $X : U \rightarrow \mathbb{R}^4$  of Weierstrass representation:  $f_1 = e^{-z}, f_2 = e^z, f_3 = z = f_4$  where  $X(z) = (f_1(z) + \bar{f}_2(z), f_3(z) + \bar{f}_3(z)) = 2(\cosh xe^{iy} + xJ)$ .

From (27)  $g_L = -g_R$  and

$$X_x = e' + \bar{f}' + (g' + \bar{g}')J = 2(\sinh xe^{iy} + J), X_y = i(e' - \bar{f}' + (g' - \bar{g}')J) = 2i \cosh xe^{iy}$$

$$g_L = \frac{1}{E} X_y \bar{X}_x = \frac{i}{E} (|e^z|^2 - |e^{-z}|^2 + 2(e^{-z} + \bar{e}^z)J)$$

$$g_R = \frac{1}{E} X_y \bar{X}_x = \frac{i}{E} (|e^z|^2 - |e^{-z}|^2 + 2(e^z + \bar{e}^{-z})J)$$

$$g_R - g_L = \alpha k.$$

Hence

$$\langle g_L - g_R, X \rangle = 0.$$

The catenoid  $X(U)$  is contained in  $\text{span}(1, I, J)$  and  $\xi = g_L - g_R$  is a normal section of the normal bundle of  $X(U)$ .

## 5 The area of a holomorphic curve in $\mathbb{S}^2 \times \mathbb{S}^2$

Kählerian geometry on  $\mathbb{S}^2 \times \mathbb{S}^2$  provided with the product of complex structures on each  $\mathbb{S}^2$  shows that the area of a holomorphic curve in  $\mathbb{S}^2 \times \mathbb{S}^2$  is the sum of the projected areas on each sphere- counted with multiplicity (see Remark 1). More precisely

**Lemma 3.** *Let  $G$  be a holomorphic curve in  $\mathbb{S}^2 \times \mathbb{S}^2$  and let  $\pi_i, i = 1, 2$  be the projections maps of  $G$  on each sphere, then*

$$|G| = |\pi_1(G)|_{\pi_1^*(\mathbb{S}^2)} + |\pi_2(G)|_{\pi_2^*(\mathbb{S}^2)}$$

where  $|\pi_i(G)|_{\pi_i^*(\mathbb{S}^2)}$  denotes the spherical area on  $G$  pulled-back by  $\pi_i, i = 1, 2$ .

*Proof.* The Kähler form of  $\mathbb{S}^2 \times \mathbb{S}^2$  (provided with the metric which is the direct product of round sphere metrics) is the sum of the Kähler form on each factor. Choose local isothermal coordinates of the curve  $G$ ; then the Kähler-area form -  $\omega$  in the ambient space is equal to:

$$\omega(X_x, X_y) = \langle iX_x, X_y \rangle = \langle iX_x^1 + iX_x^2, X_y^1 + X_y^2 \rangle.$$

Hence

$$\omega(X_x, X_y) = \langle iX_x^1, X_y^1 \rangle + \langle iX_x^2, X_y^2 \rangle = \omega_{\mathbb{S}^2}(X_x^1, X_y^1) + \omega_{\mathbb{S}^2}(X_x^2, X_y^2).$$

And

$$|G| = \int_G \omega = \int_G \pi_1^* \omega_{\mathbb{S}^2} + \int_G \pi_2^* \omega_{\mathbb{S}^2}$$

□

Comparing the projected area on each sphere with the projected area counted with multiplicity, we have

**Corollary 3.** *Let  $\Sigma$  be a minimal surface in  $\mathbb{R}^4$  then*

$$|g(\Sigma)| \geq |\mathfrak{g}_L(\Sigma)| + |\mathfrak{g}_R(\Sigma)|$$

where  $g = (\mathfrak{g}_L, \mathfrak{g}_R)$  is the Gauss map of  $\Sigma$ .

Hence we see that an upper bound on the sum  $|\mathfrak{g}_L(\Sigma)| + |\mathfrak{g}_R(\Sigma)|$  as in Corollary 1 does not impose an upper bound on  $|g(\Sigma)|$  as in Theorem 3. Furthermore there is no a priori upper bound for  $|g(\Sigma)|$ .

**Proposition 4.** *There is a family of minimal surfaces  $\Sigma_n$  such that  $\lim |g(\Sigma_n)| = +\infty$ .*

*Proof.* Consider the minimal surface  $\Sigma_n$  conformal to  $\mathbb{C}$  and given by the Weierstrass representation (see expression (22) in Section 4.3)

$$e' = z^n \quad f' = 1 \quad g' = z^p \quad h' = -z^q$$

with  $p + q = n$ ,  $(p, q) = 1$ .

The holomorphic coordinates of the Gauss maps - defined as in equations (23) - are  $g_L = z^p$  and  $g_R = -z^q$ . Let us show that the map  $(\mathfrak{g}_L, \mathfrak{g}_R)$  is injective. Let  $(z_1, z_2)$  be a double point of  $(g_L, g_R)$ . Then  $z_1^p = z_2^p$  and  $z_1^q = z_2^q$ . Since  $(p, q) = 1$ , this implies that  $z_1 = z_2$ . Thus from Lemma 3,  $|g(\Sigma)|$  is equal to  $2\pi p + 2\pi q = 2\pi n$ . □

Still, the area of the Gauss map image of complex curves is bounded above by  $2\pi$ .

## 6 Proof of Theorem 3

We denote the complexified quaternions by:  $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ . The Euclidean scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{H}$  extends to  $\mathbb{H}_{\mathbb{C}}$  either as the complexified scalar product  $\langle \cdot, \cdot \rangle_{\mathbb{C}}$  or as the hermitian metric  $((\cdot, \cdot))_{\mathbb{C}}$ .

The immersion  $X$  is conformal wrt  $z \in U$  iff the complexified vector  $(X_x - iX_y)(z)$  is a null vector i.e. iff  $\langle X_x - iX_y, X_x - iX_y \rangle_{\mathbb{C}} = 0$ . The tangent Gauss map can be identified to

$$G' : \begin{pmatrix} U & \longrightarrow & Q'_2 \subset P(\mathbb{H}_{\mathbb{C}}) \\ z & \longmapsto & [X_x(z) - iX_y(z)] \end{pmatrix} \quad (25)$$

where  $Q'_2$  is the complex 2-dimensional quadric of the complex 3-dimensional projective space  $P(\mathbb{H}_{\mathbb{C}})$  - identified to the Grassmanian  $G^+(2,4)$  - which is defined as the zeroes of the following quadratic form

$$Q'_2 := \{[Z] \in P(\mathbb{H}_{\mathbb{C}}) : q'_2(Z) := z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \text{ for } Z = z_1 + z_2I + z_3J + z_4K\}.$$

If we plug the holomorphic coordinates of (22) into  $G'$ , we obtain an equivalent Gauss map

$$G : \left( \begin{array}{l} U \longrightarrow Q_2 \subset P(\mathbb{H}_{\mathbb{C}}) \\ z \longmapsto [e'(z), f'(z), g'(z), h'(z)] \end{array} \right) \quad (26)$$

where

$$Q_2 := \{[Z] \in P(\mathbb{H}_{\mathbb{C}}) : q_2(Z) := z_1z_2 + z_3z_4 = 0 \text{ for } Z = z_1 + z_2I + z_3J + z_4K\}.$$

(The Weierstrass coordinates whose Gauss map takes its values in  $Q'_2$  appeared in [E] and those adapted to a parametrization of  $Q_2$  appeared in [MW]). The passage from this algebraic description of the Gauss map to the geometric one is summarized in the following diagrams; the second diagram gives the expressions of the maps of the first diagram.

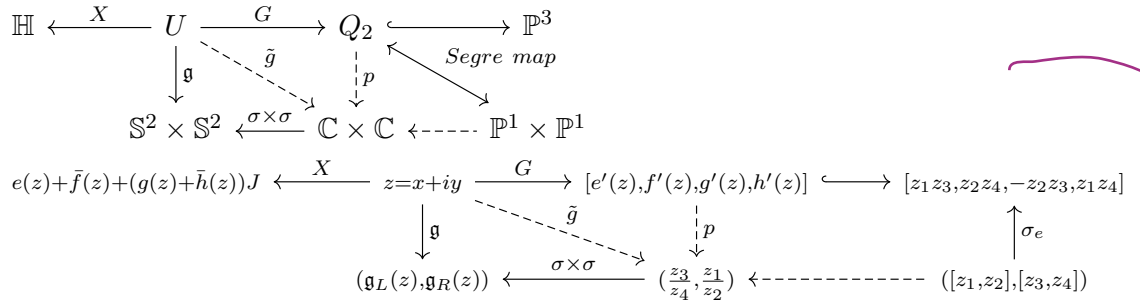


Figure 6.1: Map diagram of the Gauss map in Weierstrass coordinates

where  $G(z) = X_x - iX_y$ ,  $\sigma : \mathbb{C} \rightarrow \mathbb{S}^2$  is the stereographic projection,  $\mathfrak{g} := (\mathfrak{g}_L, \mathfrak{g}_R)$ ,  $\tilde{g} := (g_L, g_R)$ ,  $p([\zeta_1, \zeta_2, \zeta_3, \zeta_4]) = (\frac{\zeta_1}{\zeta_4}, \frac{\zeta_4}{\zeta_2})$ . The dotted arrows represent rational maps. The holomorphic Weierstrass data of the Gauss maps are then:

$$g_L(z) = \frac{z_3}{z_4} = \frac{e'(z)}{h'(z)} = -\frac{g'(z)}{f'(z)}, \quad g_R(z) = \frac{z_1}{z_2} = -\frac{e'(z)}{g'(z)} = \frac{h'(z)}{f'(z)}. \quad (27)$$

Moreover  $P(\mathbb{H}_{\mathbb{C}})$  is provided with the Fubini metric  $ds_F^2$  so that the projection  $\pi : (\mathbb{H}_{\mathbb{C}} \setminus \{0\}, (\cdot, \cdot)_{\mathbb{C}}) \rightarrow (P(\mathbb{H}_{\mathbb{C}}), ds_F^2)$  is a Riemannian submersion. Thus the unitary group  $U(4)$  action on  $\mathbb{H}_{\mathbb{C}}$  descends to isometries on  $P(\mathbb{H}_{\mathbb{C}})$ .

The quadrics  $Q_2$  and  $Q'_2$  are each provided with the metric induced by the ambient Fubini metric. One can easily check that  $Q_2$  is isometric to  $Q'_2$  via the



group element  $A \in U(4)$  defined by :

$$A = \begin{pmatrix} A_1 & O \\ O & A_1 \end{pmatrix} \text{ where } A_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \quad (28)$$

i.e. :  $\forall v \in \mathbb{H}_{\mathbb{C}} \quad q_2(Av) = q'_2(v)$ .

Moreover it is clear that  $Q'_2$  is stable by  $O(4, \mathbb{C})$ . We then deduce that  $Q_2$  is stable by the conjugate group  $A \cdot O(4, \mathbb{C}) \cdot A^{-1}$ . In particular  $Q_2$  is stabilized by the subgroup  $A \cdot SO(4, \mathbb{R}) \cdot A^{-1} \subset SU(4)$ .

We introduce now the following permutation of the left and right sphere:

$$\pi : \begin{pmatrix} \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ ([a, b], [c, d]) & \mapsto ([d, c], [a, b]) \end{pmatrix} \quad (29)$$

One checks that  $\pi = \sigma_e^{-1} \circ S \circ \sigma_e$  where  $\sigma_e$  is defined in the diagram of Figure 6.1 and  $S$  is the ambient linear map  $S$  of  $\mathbb{H}_{\mathbb{C}}$ :

$$S = \begin{pmatrix} O & S_1 \\ S_2 & O \end{pmatrix}, \text{ where } S_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } S_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (30)$$

By construction  $Q_2$  is stable by the action of  $S$  and by its expression one sees that  $S \in SO(4, \mathbb{R}) \subset SU(4)$ .

**Lemma 4.** *There exists an isotopy  $H : ([0, 1] \longrightarrow \mathcal{J} := U(4) \cap A \cdot SO(4, \mathbb{C}) \cdot A^{-1})$  of isometries acting on  $Q_2 \subset P(\mathbb{H}_{\mathbb{C}})$  such that  $H(0) = Id$  and  $H(1) = S$ .*

*Proof.*  $Id$  and  $S$  belong to  $\mathcal{J}$  which is path-connected. Indeed, its conjugate by  $A \in U(4)$  equals  $A^{-1} \mathcal{J} A = U(4) \cap SO(4, \mathbb{C}) = SO(4, \mathbb{R})$  which is path-connected.  $\square$

## 6.1 Deformations by associate minimal surfaces

We start with a minimal surface whose Gauss map in Weierstrass coordinates - following the notations of (22) - are of the form

$$G : \begin{pmatrix} U & \longrightarrow G(U) \subset Q_2 \subset P(\mathbb{H} \otimes \mathbb{C}) \\ z & \longrightarrow [e'(z), f'(z), g'(z), h'(z)] \end{pmatrix} \quad (31)$$

**Remark 6.** *If the Riemann surface  $U$  is not simply-connected then we replace it by its universal cover; the image by the Gauss map of  $U$  will not be affected.*

From Lemma 4 , there exists a continuous path  $\gamma : [0, 1] \longrightarrow \mathcal{J}$  with  $\gamma_0 = Id$  and  $\gamma_1 = S$ .

This generates a continuous family of maps  $G_t := \gamma_t \cdot G$ . Since  $\gamma_t \in \mathcal{J}$  then  $\gamma_t G(U) \subset Q_2$ . In terms of Weierstrass coordinates, let  $G_t := [e_{t,1}, e_{t,2}, e_{t,3}, e_{t,4}]$  then  $e_{t,1} \cdot e_{t,2} + e_{t,3} \cdot e_{t,4} = 0$ . The  $G_t$  are then the Gauss maps of a family of new minimal surfaces - so called associate minimal surfaces to  $\Sigma$ .

$$X_t : \begin{pmatrix} U & \longrightarrow \Sigma_t \subset \mathbb{H} \\ z & \mapsto \int_z e_{1,t} + \overline{\int_z e_{2,t}} + (\int_z e_{3,t} + \overline{\int_z e_{4,t}})J \end{pmatrix} \quad (32)$$

where  $\Sigma_0 = \Sigma$  and by the definition of (29) such that the Gauss map of  $\Sigma_1$  is given by :

$$(\mathfrak{g}_{L,1}, \mathfrak{g}_{R,1}) = (\sigma_a \circ \mathfrak{g}_R, \mathfrak{g}_L) \quad (33)$$

where  $\sigma_a$  is the antipodal symmetry (see Figure 6.2 below).

Let us describe some invariants of this family of minimal surfaces.

**Lemma 5.** *For any minimal surface  $\Sigma$  there is an associate family of minimal surfaces  $\{\Sigma_t\}_{t \in [0,1]}$  defined by (32) such that the following conditions are satisfied:  $\forall p \in U$  and  $\forall t \in [0, 1]$  :*

1.  $\Sigma_t$  is locally isometric to  $\Sigma_0 = \Sigma$ :

$$ds_t^2(p) := \lambda_t^2(p)|dz|^2 = \lambda^2(p)|dz|^2$$

2. The tangent curvature is invariant by deformation:

$$K_t^T(p) = K^T(p)$$

3. The images of the Gauss maps have the same area:

$$|g_t(\Sigma_t)| = |g(\Sigma)|$$

4. If the operator  $\Delta_{\Sigma_{t_0}} - 2K_{t_0}^T$  is positive for some  $t_0 \in [0, 1]$  then  $\Delta_{\Sigma_t} - 2K_t^T$  is positive for all  $t \in [0, 1]$ .

*Proof.* 1. We consider the deformation of  $\Sigma$  defined in (32). In local isothermal coordinates  $z$ , in a neighborhood of some  $p \in U$ ,  $ds^2 = \lambda|dz|^2$ . where  $\lambda = |e'|^2 + |f'|^2 + |g'|^2 + |h'|^2 = \langle\langle G, G \rangle\rangle$  where  $\langle\langle \cdot, \cdot \rangle\rangle$  is the hermitian metric in  $\mathbb{H}_{\mathbb{C}}$ . As  $\gamma_t \in U(n)$ , the metrics  $ds_t^2 = \lambda_t|dz|^2$  of  $\Sigma_t$  are all locally isometric to  $\Sigma$  since

$$\lambda_t = \langle\langle G_t, G_t \rangle\rangle = |e_{1,t}|^2 + |e_{2,t}|^2 + |e_{1,2}|^2 + |e_{3,t}|^2 + |e_{4,t}|^2 = \langle\langle \gamma_t G, \gamma_t G \rangle\rangle = \langle\langle G, G \rangle\rangle.$$

2. The metric is invariant, so is the tangent curvature.

3.  $|g_t(\Sigma)| = |\gamma_t g(\Sigma)| = |g(\Sigma)|$

4. If for some  $t_0$ :

$$\int_U |\nabla \phi|_{\Sigma_{t_0}}^2 da_{\Sigma_{t_0}} \geq -2 \int_U K_{t_0}^T \phi^2 da_{\Sigma_{t_0}} \quad \forall \phi \in C_0(U), \quad (34)$$

then, as the  $\Sigma_t$  are all isometric,  $|\nabla \phi|_{\Sigma_{t_0}}^2 = |\nabla \phi|_{\Sigma_t}^2$  and  $da_{\Sigma_{t_0}} = da_{\Sigma_t}$  and by the second point, the stability inequality (34) is true for any  $t \in [0, 1]$ .  $\square$

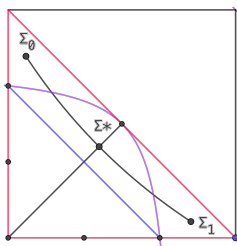


Figure 6.2:  $t \mapsto \left( \frac{|\mathfrak{g}_L(\Sigma_t)|}{2\pi}, \frac{|\mathfrak{g}_R(\Sigma_t)|}{2\pi} \right)$

Let us conclude the proof of Theorem 3.

Let  $\Sigma$  be a minimal surface such that the Gauss map area  $|g(\Sigma)| < 2\pi$ .

Then the projected area verifies  $|\mathfrak{g}_L(\Sigma)| + |\mathfrak{g}_R(\Sigma)| < 2\pi$  by Corollary 3.

Suppose  $|\mathfrak{g}_L(\Sigma)| = |\mathfrak{g}_R(\Sigma)|$  then by the previous inequality,  $|\mathfrak{g}_L(\Sigma_{t_0})|$  and  $|\mathfrak{g}_R(\Sigma_{t_0})|$  are each less than  $\pi$  ie the proportionate area of the left and right Gauss map are less than  $\frac{1}{2}$  and by Proposition 3,  $\Sigma$  is stable.

Suppose that  $|\mathfrak{g}_L(\Sigma)| < |\mathfrak{g}_R(\Sigma)|$ , then by Equation (33):

$$|\mathfrak{g}_L(\Sigma_1)| = |\mathfrak{g}_R(\Sigma)| > |\mathfrak{g}_L(\Sigma)| = |\mathfrak{g}_R(\Sigma_1)|.$$

By continuity there is necessarily a  $t_0 \in [0, 1[$  such that the left and right spherical area are equal:  $|\mathfrak{g}_L(\Sigma_{t_0})| = |\mathfrak{g}_R(\Sigma_{t_0})|$ . By Lemma 5-3 and Corollary 3 each area is less than  $\pi$  hence  $\Sigma_{t_0}$  is stable. By Lemma 5-4  $\Sigma$  is stable.  $\square$

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