A Lorentz boost as a product of two space-time reflections and some additional results about Clifford algebra

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Abstract. This is a technical clarifying note consisting of two parts. In the first part we derive the expression for a boost in two representations of the homogeneous Lorentz group, viz. the two-dimensional representation SL(2,$\mathbb{C}$) and the four-dimensional Dirac representation in its Cartan-Weyl form. The derivation is purely algebraic. It uses the development of a Clifford algebra for a group of isometries of a vector space, whereby the group is generated by reflections. We prove that a boost can be obtained as a product of two space-time reflections, in perfect analogy with the way a rotation in $\mathbb{R}^3$ can be obtained as a product of two reflections. The derivation does therefore not rely on physical considerations as in Einstein's approach. It is purely based on symmetry arguments. The second part deals with the justification of the definition of a Clifford algebra given in certain mathematical textbooks, which immediately introduce a basis of multi-vectors $\mathbb{1}$, $\mathbf{e}_j$, $\mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2}$, $\mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2} \wedge \mathbf{e}_{j_3}, \cdots$ for this algebra. Rather than as a bemusing "postulate" that descends from heaven, we will present the introduction of this basis as an obvious result of a logical construction of the group representation theory. This will provide the reader with a much better understanding of what is going on behind the scenes of the formalism. We prove that this basis of multi-vectors $\mathbb{1}$, $\mathbf{e}_j$, $\mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2}$, $\mathbf{e}_{j_1} \wedge \mathbf{e}_{j_2} \wedge \mathbf{e}_{j_3}, \cdots$ is orthogonal in terms of a scalar product whose use is very natural in vector spaces of matrices.

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1 Introduction

The idea behind Clifford algebra is to construct a representation for a group of isometries of a vector space by constructing a representation of a larger group G, generated by reflections. The isometry group is the subgroup of the elements of G which are generated by an even number of reflections. The isometries are leaving distances defined by a quadratic form invariant. The group of three-dimensional rotations around the origin $\mathrm{O}(0,0,0) \in \mathbb{R}^3$ are a group of isometries of $\mathbb{R}^3$ which leave the quadratic form $x^2 + y^2 + z^2$ invariant. This group is a three-dimensional curved manifold. That all rotations of $\mathbb{R}^3$ can be constructed as a product of two reflections is illustrated in Fig. 1 of [1]. This is the basis for the construction of SU(2) by Clifford algebra. We will remind shortly below how this construction works. The full details are given in [1] and [2]. SU(2) works on column matrices which are spinors. We will also discuss the representations SO(2) and SO(3). In these representations the column matrices correspond to vectors.

These developments for the rotation groups will serve as models for analogous constructions of two representations of the Lorentz group SO(1,3) by the methods of Clifford algebra, viz. the 2×2 representation SL(2,$\mathbb{C}$) and the 4×4 Dirac representation (in the Cartan-Weyl form). The Lorentz transformations are indeed also isometries of space-time $\mathbb{R}^4$, but now with respect to its pseudo-metric. In this document we will use the pseudo-metric with the signature $(+---)$ based on the quadratic form $c^2t^2 - x^2 - y^2 - z^2$. In all rigour, the representation of the homogeneous Lorentz group based on the signature $(+---)$ for the metric should be noted as SO(1,3), while the notation SO(3,1), which is often used in textbooks corresponds *stricto sensu* to the representation based on the quadratic form $x^2 + y^2 + z^2 - c^2t^2$ with the signature $(-+++)$. But one encounters also the notation SO(3,1) for the Lorentz group while the metric $(+---)$ is being used. This is then sloppy and in principle not correct. However, this error is further immaterial, because it is all only a matter of conventions.

2 The rotation group representations SU(2), SO(3) and SO(2)

2.1 SU(2)

In $\mathbb{R}^3$ we define a plane $\mathscr{A}$ through the origin O by a vector $\mathbf{a} \perp \mathscr{A}$ of unit length, i.e. $|\mathbf{a}| = 1$. As explained in [1], in the construction of SU(2) the reflection with respect to the plane $\mathscr{A}$ defined by the vector $\mathbf{a}$ is given by the matrix:

$$[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,] \hat{=} a_x\sigma_x + a_y\sigma_y + a_z\sigma_z = \begin{bmatrix} a_z & a_x - \imath a_y \\ a_x + \imath a_y & -a_z \end{bmatrix}. \tag{1}$$

Here σ_x, σ_y, and σ_z are the Pauli matrices. [1] We use here the symbol $\hat{=}$ to draw the attention to the fact that the notation $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ is a mere shorthand and should not be confused with a true scalar product.[2] The product of two reflection matrices $[\,\mathbf{b}\cdot\boldsymbol{\sigma}\,][\,\mathbf{a}\cdot\boldsymbol{\sigma}\,] = (\mathbf{a}\cdot\mathbf{b})\,\mathbb{1} - \imath[\,(\mathbf{a}\wedge\mathbf{b})\cdot\boldsymbol{\sigma}\,]$ yields then the Rodriguez formula for a rotation:

$$\mathbf{R}(\mathbf{s},\varphi) = \cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]. \tag{2}$$

Here $\mathbf{s} \parallel \mathbf{a}\wedge\mathbf{b}$ is a unit vector along the rotation axis. [3]. The angle $\varphi/2$ between the reflection planes $\mathscr{A}$ and $\mathscr{B}$ leads to a rotation over an angle φ. The SU(2) matrices are operating on 2×1 column matrices:

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix}, \tag{3}$$

called spinors. These spinors represent rotations as explained in [1,2]. The rotation group of $\mathbb{R}^3$ is a three-dimensional curved manifold, because it is a non-abelian group. Based on its notation as a column matrix we might believe that the spinors $\boldsymbol{\xi}$ are "column vectors" belonging to a complex vector space. This is wrong because the spinors do not build a vector space. They form a curved manifold as discussed in [3]. This has dire consequences for the way we must define infinitesimal generators in the Lie algebra (see Section 4) and and also for the way one must define probabilities in quantum mechanics. [4]

2.2 SO(3)

On the other hand, SO(3) matrices operate on 3×1 column matrices which do represent vectors $(x,y,z) \in \mathbb{R}^3$. As discussed in [1,2], the vectors of $\mathbb{R}^3$ are "squares" of spinors.[5] In SO(3), the 3×3 matrix $\mathbf{A}$ representing the reflection A with respect to a plane $\mathscr{A}$ defined by the normal $\mathbf{a} \perp \mathscr{A}$, is given by (see [1], p.10, Eq. 6):

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[1] For any group G, the application $f \in \mathrm{F}(G,\mathbb{R})$ where $\forall g \in G, f(g) = 1$ is a representation of the group. Here we use the notation $\mathrm{F}(A,B)$ for the set of all mappings whose domain is A and which take their values in B. However such a representation is not bijective because when $g_2 \circ g_1 \neq g_1 \circ g_2$ we will nevertheless obtain $f(g_2 \circ g_1) = 1 = f(g_1 \circ g_2)$. For a non commuting group we must therefore use non-commuting matrices. Now reflections r are certainly not commuting as $(r_2 \circ r_1) \circ (r_1 \circ r_2) = (r_2 \circ r_2)$ is the identity element, such that $(r_2 \circ r_1)$ is not identical to $(r_1 \circ r_2)$ but its inverse. This explains why we need a matrix formalism. Of course also the three-dimensional rotation group is not commuting.

[2] In fact $[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,]$ does not represent a scalar but the vector quantity $\mathbf{a}$. Physics textbooks make the error of considering e.g. $[\,\mathbf{B}\cdot\boldsymbol{\sigma}\,]$, where $\mathbf{B}$ is the magnetic field, as a scalar product when they introduce the expression $\mathbf{B}\cdot\boldsymbol{\mu}$ for the anomalous Zeeman effect into the formalism, thereby considering $\boldsymbol{\sigma}$ as the spin, although the mathematical formalism of SU(2) is purely geometrical and therefore cannot contain a physical concept like spin. In reality, $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ just represents the vector basis $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$, such that $[\,\mathbf{B}\cdot\boldsymbol{\sigma}\,]$ just represents $B_x\mathbf{e}_x + B_y\mathbf{e}_y + B_z\mathbf{e}_z = \mathbf{B}$.

[3] Mathematicians may well point out that the notation used in Eq. 2 is a curse because $\mathbf{a}\wedge\mathbf{b}$ is conceptually a bi-vector rather than a vector $\mathbf{s}$. Similarly, also angular momentum $\mathbf{r}\wedge m\mathbf{v}$ is indeed a bi-vector, which is very illuminating for linking conservation of angular momentum to Kepler's second law, but unlike mathematicians we are also concerned about applying the formalism to physics and in all physics textbooks these bi-vectors are steadfastly presented as axial vectors.

[4] For a Lie group we can define volume elements corresponding to the Haar integral. One may ask why we do not calculate probabilities in quantum mechanics using the Haar integral. The answer is that we enrich the structure of the Lie group when we introduce linear combinations of group elements and that in this extension we must use the Born rule (see e.g. [1] p.15). This is why it is so important to point out that spinors do not form a vector space.

[5] As explained in [1,2] complex 3×1 column vectors $(x,y,z) \in \mathscr{I} = \{(x,y,z) \in \mathbb{C}^3 \parallel x^2+y^2+z^2 = 0\} \subset \mathbb{C}^3$ can be obtained from tensor products $\boldsymbol{\xi}\otimes\boldsymbol{\xi}$. One calls $\mathscr{I}$ the isotropic cone. These tensor products have four components $\xi_0^2, \xi_0\xi_1, \xi_1\xi_0, \xi_1^2$. But as $\xi_0\xi_1 = \xi_1\xi_0$, the tensor product contains only three different components, which explains why we can reduce the 4×1 column matrix to a 3×1 matrix. SO(3) can this way be considered algebraically as a reduced form of the tensor representation SU(2) $\otimes$ SU(2). By this we main that the algebra we carry out on $(x,y,z) \in \mathbb{R}^3$ has the same identical structure as the one we carry out on $(x,y,z) \in \mathscr{I}$. Of course the meaning of (x,y,z) in these two cases is geometrically entirely different. All harmonic polynomials can this way be obtained as tensor products of spinors.

$$\mathbf{A} = \mathbb{1} - 2 \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \otimes \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}. \tag{4}$$

This transforms indeed a vector $\mathbf{r} \in \mathbb{R}^3$ into $\mathbf{r} - 2\,[\mathbf{r} \cdot \mathbf{a}]\,\mathbf{a}$ as is easily checked by operating with Eq. 4 on the 3×1 column matrix corresponding to $\mathbf{r}$. In general, we do not go through the fuss of constructing an SO(3) matrix from two such reflections, because the action of a rotation on a vector is obvious, when the rotation axis is one of the coordinate axes. One can then use Euler's construction based on the three Euler angles (α, β, γ) to obtain the general case. We may note that the expression for a reflection in Eq. 4 is quadratic in the parameters (a_x, a_y, a_z):

$$\mathbf{A} = \begin{bmatrix} a_y^2 + a_z^2 - a_x^2 & -2a_x a_y & -2a_x a_z \\ -2a_x a_y & a_x^2 + a_z^2 - a_y^2 & -2a_y a_z \\ -2a_x a_z & -2a_y a_z & a_x^2 + a_y^2 - a_z^2 \end{bmatrix}, \tag{5}$$

while in SU(2) it is linear in (a_x, a_y, a_z) as evidenced by Eq. 1.

2.3 The group SO(2) $\subset$ SO(3) of rotations in $\mathbb{R}^2$

Let us apply this result in SO(3) to the special case of a reflection in the Oxy plane. This will then be the starting point for a construction of SO(2), which operates on vectors of $\mathbb{R}^2$, in contrast with SU(2) which operates on spinors. A general vector of unit length is parameterized by $(\cos(\varphi), \sin(\varphi))$. Indeed $|(\cos(\varphi), \sin(\varphi))|^2 = \cos^2\varphi + \sin^2\varphi = 1$. Applying Eq. 4 yields then for $\mathbf{A}(\varphi)$:

$$\begin{aligned} \mathbb{1} - 2 \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix} \otimes \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \end{bmatrix} &= \begin{bmatrix} \sin^2\varphi - \cos^2\varphi & -2\cos\varphi\sin\varphi \\ -2\cos\varphi\sin\varphi & \cos^2\varphi - \sin^2\varphi \end{bmatrix} \\ &= \begin{bmatrix} -\cos(2\varphi) & -\sin(2\varphi) \\ -\sin(2\varphi) & +\cos(2\varphi) \end{bmatrix}. \end{aligned} \tag{6}$$

The product $\mathbf{A}(\varphi)\mathbf{A}(0)$ yields then:

$$\begin{bmatrix} -\cos(2\varphi) & -\sin(2\varphi) \\ -\sin(2\varphi) & +\cos(2\varphi) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix} = \begin{bmatrix} +\cos(2\varphi) & -\sin(2\varphi) \\ +\sin(2\varphi) & +\cos(2\varphi) \end{bmatrix}. \tag{7}$$

This is indeed a rotation. We see that starting with an angle φ leads to an angle 2φ in the final result. To obtain a rotation through an angle φ we must therefore use the reflection defined by $(\cos(\varphi/2), \sin(\varphi/2))$. The doubling of the angles is explained in [1], Fig.1.

3 Representations of the Lorentz group

3.1 Methodology

We will derive the expression for a Lorentz boost in space-time by considering it as a product of two space-time reflections. We can no longer represent this kind of geometry visually on a sheet of paper, because the sheet of paper follows the Euclidean metric while the points of space-time follow a hyperbolic pseudo-metric. E.g. lines that are orthogonal in space-time will not look orthogonal in the visual representation of space-time in a Minkowski space-time diagram. The development will therefore be based on algebraic methods and on analogy because it is difficult to get a conceptual grasp on how a boost could be geometrically the product of two space-time reflections. Whereas for rotations we learned the algebra from the geometry, we will now use the dialogue between the geometry and the algebra in the reverse way and learn the geometry from the algebra.

3.2 The abelian group SO(1,1) of boosts along the x-axis

We will first treat the special case of Lorentz boosts along the x-axis by analogy with SO(2). This will lead to the abelian group of boosts SO(1,1). Whereas in the Oxy plane a general vector of unit length is parameterized by $(\cos(\varphi), \sin(\varphi)) = \cos(\varphi)\,\mathbf{e}_x + \sin(\varphi)\,\mathbf{e}_y$, a general vector of unit length in the Otx plane is now parameterized by $(\cosh(\chi/2), \sinh(\chi/2)) = \cosh(\chi/2)\,\mathbf{e}_{ct} + \sinh(\chi/2)\,\mathbf{e}_x$. In fact, $|(\cosh(\chi/2), \sinh(\chi/2))|^2 = \cosh^2(\chi/2) - \sinh^2(\chi/2) = 1$.

Such a unit vector will now serve as reflection normal. This time we use the half of the argument χ right from the start, based on what we learned from the derivation in the rotation group. The form analogous to Eq. 4 for the reflection matrix is now:

$$\mathbf{A}(\chi) = \mathbb{1} - 2 \begin{bmatrix} \cosh(\chi/2) \\ \sinh(\chi/2) \end{bmatrix} \otimes \begin{bmatrix} \cosh(\chi/2) & -\sinh(\chi/2) \end{bmatrix}. \tag{8}$$

The minus sign $-\sinh(\chi/2)$ in the 1×2 line matrix is due to the signature of the pseudo-metric $c^2t^2 - x^2$ in Minkowski space time, such that the scalar product of (ct_1, x_1) and (ct_2, x_2) is now $c^2t_1t_2 - x_1x_2$. This intervenes when we apply $\mathbf{A}(\chi)$ to the column vector representing $\mathbf{r} = (ct, x)$. Writing $\mathbb{1}$ as $(\cosh^2(\chi/2) - \sin^2(\chi/2))\mathbb{1}$, the calculation yields:

$$\begin{aligned} \mathbf{A}(\chi) &= (\cosh^2(\chi/2) - \sin^2(\chi/2))\mathbb{1} - \begin{bmatrix} 2\cosh^2(\chi/2) & -2\cosh(\chi/2)\sinh(\chi/2) \\ 2\sinh(\chi/2)\cosh(\chi/2) & -2\sinh^2(\chi/2) \end{bmatrix} \\ &= \begin{bmatrix} -\cosh^2(\chi/2) - \sinh^2(\chi/2) & +2\cosh(\chi/2)\sinh(\chi/2) \\ -2\sinh(\chi/2)\cosh(\chi/2) & \cosh^2(\chi/2) + \sinh^2(\chi/2) \end{bmatrix} \\ &= \begin{bmatrix} -\cosh(\chi) & +\sinh(\chi) \\ -\sinh(\chi) & +\cosh(\chi) \end{bmatrix}. \end{aligned} \tag{9}$$

The transformation:

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} -\cosh(\chi) & +\sinh(\chi) \\ -\sinh(\chi) & +\cosh(\chi) \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \quad \text{implies:} \quad ct' = -\cosh(\chi)\, ct + \sinh(\chi)\, x, \tag{10}$$

such that in the new frame the time runs backwards. To obtain a Lorentz boost with the time running forwards, we must therefore combine $\mathbf{A}(\chi)$ with the pure time reversal $\mathbf{A}(0)$. We then obtain an orthochronous transformation:

$$\mathbf{A}(\chi)\mathbf{A}(0) = \begin{bmatrix} -\cosh(\chi) & +\sinh(\chi) \\ -\sinh(\chi) & +\cosh(\chi) \end{bmatrix} \begin{bmatrix} -1 & \\ & +1 \end{bmatrix} = \begin{bmatrix} +\cosh(\chi) & +\sinh(\chi) \\ +\sinh(\chi) & +\cosh(\chi) \end{bmatrix}. \tag{11}$$

We can actually take the opposite sign with $+\sinh(\chi)$ in Eq. 8 for a motion along the positive x-axis. We obtain then exactly the textbook equation for a Lorentz boost along the positive x-axis in the Oxt plane. In all this we can make the usual identifications $\cosh(\chi) = \gamma$ and $\sinh(\chi) = \gamma v/c$, which are compatible with the identity $\cosh^2\chi - \sinh^2\chi = 1$. We have then $[\cosh(\chi), \sinh(\chi)] = \cosh(\chi)\,\mathbf{e}_{ct} + \sinh(\chi)\,\mathbf{e}_x$, whereby $\mathbf{e}_x = \mathbf{v}/v$. We obtain then $x' = \gamma(x - vt)$, which is like the Galileo transformation, but with the four-vector $(1, -v/c)$ renormalized to 1. We may note that $\det \mathbf{A}(\chi) = -1$, such that $\det[\,\mathbf{A}(\chi)\mathbf{A}(0)\,] = \det \mathbf{A}(\chi) \det \mathbf{A}(0) = 1$.

The we way we introduce here the identifications $\cosh(\chi) = \gamma$ and $\sinh(\chi) = \gamma v/c$ is actually a kind of a cheat based on using information obtained from the traditional textbook derivation, rather than deriving everything completely from the framework of the Clifford algebra. To obtain the identification entirely within the framework of the Clifford algebra we can write $x' = \cosh\chi(x - \tanh\chi\, ct)$ and consider the point vt. As for this point $x' = 0$ must be true at any moment t, it follows then from $x' = \cosh\chi(v - c\tanh\chi)t$ that we must identify $\tanh\chi = v/c$. From this result one obtains then easily $\cosh\chi = \gamma$. This way we have derived the algebraic expression for a boost of SO(1,1) by considering a boost as a product of two space-time reflections, just like we obtained a rotation as the product of two reflections in SU(2). Of course this derivation follows a logic that is different from the one used in the more intuitive approach presented in physics textbooks. We have derived this proof to show that there is always an all-out equivalence between the intuitive geometrical approach and the more formal algebraic approach. All geometry can be translated into algebra and vice versa. This insight is important because it indicates that the whole algebraic formalism of quantum mechanics has a natural geometrical meaning, such that all attempts to find a physical interpretation of the algebra must just stick to this geometrical meaning. All alternative attempts are undue parallel interpretations that will *a priori* be wrong.

3.3 The full homogeneous Lorentz group SO(1,3)

Let us now generalize this to a Lorentz transformation in $\mathbb{R}^4$ within the Cartan-Weyl representation, defined by:

$$\gamma_x = \begin{bmatrix} & \sigma_x \\ -\sigma_x & \end{bmatrix}, \quad \gamma_y = \begin{bmatrix} & \sigma_y \\ -\sigma_y & \end{bmatrix}, \quad \gamma_z = \begin{bmatrix} & \sigma_z \\ -\sigma_z & \end{bmatrix}, \quad \gamma_{ct} = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}. \tag{12}$$

This will be done in analogy with the derivation of SU(2) but not with the same complete generality. In fact, the homogeneous Lorentz group is a six-dimensional curved manifold and a general Lorentz transformation is a product

RB or BR of a rotation R and a boost B. We will therefore derive the expressions for the boosts and for the rotations separately. For the rotations this is easy enough such that the real issue is to derive the expression for a general boost. To derive this expression we carry over the idea from Subsection 3.2 to obtain the boost as a product $\mathbf{A}(\chi)\mathbf{A}(0)$ whereby we now replace the direction $\mathbf{e}_x$ of the boost vector $\mathbf{v}$ by a general unit vector $\mathbf{u}$.[6]

We define therefore $\mathbf{u} = \mathbf{v}/v$ and the reflection normal will now be $\cosh(\chi/2)\,\mathbf{e}_{ct} + \sinh(\chi/2)\,\mathbf{u}$, represented by $\cosh(\chi/2)\,\gamma_{ct} + \sinh(\chi/2)\,[\,\mathbf{u}\cdot\boldsymbol{\gamma}\,]$. In analogy with the derivation in SU(2), the pendant of Eq. 11 becomes then:

$$\boldsymbol{A}(\chi)\boldsymbol{A}(0) = \begin{bmatrix} \cosh(\chi/2)\,\mathbb{1} - \sinh(\chi/2)[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,] & \cosh(\chi/2)\,\mathbb{1} + \sinh(\chi/2)[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,] \end{bmatrix} \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}. \tag{13}$$

This is of the form:

$$\boldsymbol{B}(\mathbf{v}) = \begin{bmatrix} \mathbf{B}(\mathbf{v}) & \\ & \mathbf{B}(-\mathbf{v}) \end{bmatrix}, \tag{14}$$

where the 2×2 boost matrices $\mathbf{B}(\mathbf{v})$ and $\mathbf{B}(-\mathbf{v})$ can be used as 2×2 representations of boosts $B(\pm\mathbf{v})$. We may note that now $\det[\![\,\cosh(\chi/2)\,\mathbb{1} \pm \sinh(\chi/2)[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]\,]\!] = 1$ and $\det\mathbb{1} = 1$ such that $\det\mathbf{B}(\pm\mathbf{v}) = 1$. These 2×2 representations are both SL(2,$\mathbb{C}$) representations. We can figure out how we must write $\cosh(\chi/2)$ and $\sinh(\chi/2)$ as follows. As $\cosh(\chi) = \gamma, \sinh(\chi) = \gamma v/c$ we have $1 + \cosh(\chi) = 1 + \gamma = 2\cosh^2(\chi/2)$, $\cosh(\chi) - 1 = \gamma - 1 = 2\sinh^2(\chi/2)$, such that:

$$\begin{aligned} \cosh(\chi/2) &= \sqrt{\frac{\gamma+1}{2}}, \\ \sinh(\chi/2) &= \sqrt{\frac{\gamma-1}{2}}. \end{aligned} \tag{15}$$

We have this way proved that in SL(2,$\mathbb{C}$) a boost $\mathbf{B}(\mathbf{v})$ with velocity parameter $\mathbf{v}$ is given by:

$$\mathbf{B}(\mathbf{v}) = \sqrt{\frac{\gamma+1}{2}}\mathbb{1} - \sqrt{\frac{\gamma-1}{2}}\,[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,], \tag{16}$$

where $\mathbf{v} = v\mathbf{u}$. This has now been derived in a purely algebraic way. In [4]. The general reflection normal $\mathbf{a}$ in space-time has then the form:

$$\mathbf{a} = \cosh(\chi/2)\,\mathbb{1} \pm \sinh(\chi/2)[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,] = (\cos(\chi/2), \sin(\chi/2)\sin(\theta)\cos(\phi), \sin(\chi/2)\sin(\theta)\sin(\phi), \sin(\chi/2)\cos(\theta)), \tag{17}$$

where the unit vector $\mathbf{u}$ is defined by its spherical coordinates (θ, ϕ). Note that we use here ϕ and φ as different symbols. In the Cartan-Weyl representation we can rewrite Eq. 14 as:

$$\boldsymbol{B}(\mathbf{v}) = \begin{bmatrix} \mathbf{B}(\mathbf{v}) & \\ & [\,\mathbf{B}(\mathbf{v})\,]^{\dagger-1} \end{bmatrix}. \tag{18}$$

[6] If we wanted to use the analogy with SU(2) in its full generality, we might think that we would just have to consider products $\mathbf{A}_2\mathbf{A}_1$ of two reflections, and that this will then lead to a general Lorentz transformation of the form RB or BR, which in principle depends on six independent real parameters whereas both pure boosts and pure rotations depend on only three independent real parameters. But it is not God-given that in $\mathbb{R}^4$ a general Lorentz transformation could be obtained as the product of just two reflections, in strict analogy with what we can do in SU(2). In fact, each reflection A in $\mathbb{R}^4$ has a three-dimensional invariant subspace $\{(ct,x,y,z) \in \mathbb{R}^4 \parallel A(ct,x,y,z) = (ct,x,y,z)\}$, whose points remain invariant under the transformation (the reflection hyperplane). The composition of two reflections would then have a two-dimensional invariant subspace, viz. the intersection of the two invariant three-dimensional subspaces. But the invariant subspace of a boost B along the x-axis is the subspace $\{(ct,x,y,z) \in \mathbb{R}^4 \parallel B(ct,x,y,z) = (ct,x,y,z)\} = \{(0,0,y,z) \parallel (x,y) \in \mathbb{R}^2\}$. The invariant subspace of a rotation R in the Oyz-plane is the subspace $\{(ct,x,y,z) \in \mathbb{R}^4 \parallel R(ct,x,y,z) = (ct,x,y,z)\} = \{(ct,x,0,0) \parallel (ct,x) \in \mathbb{R}^2\}$. These two invariant subspaces do not have a two-dimensional intersection. The intersection of the sets of invariant points of RB is just the singleton $\{(0,0,0,0)\}$, which proves that RB cannot be obtained as the product of two reflections. However, RB can obviously be obtained as the product of four reflections. Hence the Lorentz group is generated by reflections but generating a general Lorentz transformation takes more than two reflections, in marked contrast with what happens in SU(2). This shows that we can no longer determine the composition of two Lorentz transformations geometrically in the same way as we did for the product of two rotations in SO(3), described in the legend of Fig. 3.3 of [4]. It also shows that a rotation of SO(4) will in general not have a "rotation axis". On the other hand, as explained above, a pure rotation in SO(4) does not have a single axis but a whole plane as invariant subspace and the same applies for a pure boost.

Here $[\,\mathbf{B}(\mathbf{v})\,]^{\dagger-1} = \mathbf{B}(-\mathbf{v})$, because the Pauli matrices are Hermitian and $[\,\mathbf{B}(\mathbf{v})\,]^{-1} = \mathbf{B}(-\mathbf{v})$. The expression in Eq. 18 is a generalization to take into account that for rotations $\mathbf{R}$ we have $\mathbf{R}^{\dagger} = \mathbf{R}^{-1}$. A rotation is represented by:

$$\boldsymbol{BA} = \begin{bmatrix} & [\,\mathbf{b}\cdot\boldsymbol{\sigma}\,] \\ -[\,\mathbf{b}\cdot\boldsymbol{\sigma}\,] & \end{bmatrix} \begin{bmatrix} & [\,\mathbf{a}\cdot\boldsymbol{\sigma}\,] \\ -[\,\mathbf{a}\cdot\boldsymbol{\sigma}\,] & \end{bmatrix} = \begin{bmatrix} -\mathbf{R}(\mathbf{s},\varphi) & \\ & -\mathbf{R}(\mathbf{s},\varphi) \end{bmatrix}. \tag{19}$$

As SU(2) is a double covering of SO(3) we can drop the minus signs. A rotation is thus represented by:

$$\boldsymbol{R}(\mathbf{s},\varphi) = \begin{bmatrix} \mathbf{R}(\mathbf{s},\varphi) & \\ & \mathbf{R}(\mathbf{s},\varphi) \end{bmatrix} = \begin{bmatrix} \mathbf{R}(\mathbf{s},\varphi) & \\ & [\,\mathbf{R}(\mathbf{s},\varphi)\,]^{\dagger-1} \end{bmatrix}. \tag{20}$$

This way we can see that for a general Lorentz transformation, which is the product of a rotation and a boost, we have:

$$\boldsymbol{L}(\mathbf{v},\mathbf{s},\varphi) = \begin{bmatrix} \mathbf{L}(\mathbf{v},\mathbf{s},\varphi) & \\ & [\,\mathbf{L}(\mathbf{v},\mathbf{s},\varphi)\,]^{\dagger-1} \end{bmatrix}. \tag{21}$$

The 4×4 matrices $\boldsymbol{L}(\mathbf{v},\mathbf{s},\varphi)$ are the Cartan-Weyl form of the Dirac representation of the Lorentz group, which is a six-dimensional manifold. The 2×2 matrices $\mathbf{L}(\mathbf{v},\mathbf{s},\varphi)$ build the representation SL(2,$\mathbb{C}$), characterized by $\det\mathbf{L}(\mathbf{v},\mathbf{s},\varphi) = 1$. The rotations are obtained from reflections with respect to hyperplanes defined by the unit vectors:

$$\mathbf{a} = (0, \sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)). \tag{22}$$

In [4] on p. 354 we have derived the expression for $\mathbf{B}(\mathbf{v})$ in SL(2,$\mathbb{C}$) in a rather clumsy *ad hoc* fashion. by using the fact that a general element $\mathbf{L} \in$ SL(2,$\mathbb{C}$) operates on a four-vector (ct, x, y, z) represented by:

$$\mathbf{X} = \begin{bmatrix} ct+z & x-\imath y \\ x+\imath y & ct-z \end{bmatrix}, \tag{23}$$

according to:

$$\mathbf{X} \rightarrow \mathbf{L}\mathbf{X}\mathbf{L}^{\dagger}, \tag{24}$$

thereby using our prior knowledge for the expression of the Lorentz transformation and working backwards. We may note that Eq. 24 leaves $\det\mathbf{X} = c^2t^2 - x^2 - y^2 - z^2$ invariant because $\det\mathbf{L} = \det\mathbf{L}^{\dagger} = 1$ by definition. In many textbooks the expression for a boost along the x-axis is only derived for points that are reached by a light ray emitted at $x = 0$ at $t = 0$, which is even not general. But all this is of course not deriving the expression for the Lorentz transformation from the Clifford algebra.

4 Derivation of the expression for a boost by using the Lie algebra

4.1 The textbook algebra

4.1.1 Context

Our motivation for the derivations given above has been that in [5] it is claimed that spinor calculus in SL(2,$\mathbb{C}$) can be obtained by generating the Lorentz group from reflections while the calculus developed afterwards does not follow this principle. It rather uses the relation between the Lie algebra and the Lie group, based on infinitesimal transformations in the representation SL(2,$\mathbb{C}$). It inverts in a sense the procedure of deriving the infinitesimal generators of the Lie algebra from the expressions for the group elements of the Lie group. The aim of the present Section 4 is to raise a number of red flags about pitfalls the reader may not be aware of. Most of it concerns the use we can make of the number $\imath$ in the representation theory. Some of it is quite meticulous and requires a very good understanding of the group representation theory. Consulting [6] beforehand may be compulsory in order to avoid that some less knowledgeable readers will think that what we will present here is far-fetched moonshine. The reader who is in a hurry might want to skip the reading of this Section and take up the argument again in Section 5.

4.1.2 Rotations

Several textbooks also derive the expression for a rotation in SU(2) by this reverse method from the Lie algebra. This runs as follows. The eigenvalue equation for the matrix $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ leads to the following polynomial :

$$\det\begin{bmatrix} s_z-\lambda & s_x-\imath s_y \\ s_x+\imath s_y & -s_z-\lambda \end{bmatrix} = \lambda^2 - s_x^2 - s_y^2 - s_z^2. \tag{25}$$

The eigenvalues of $[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ are therefore given by $\lambda \in \{-1, 1\}$. Even without making this calculation, we actually know that:

$$\exists \mathbf{R} \in \mathrm{SU}(2): \quad [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,] = \mathbf{R}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\mathbf{R}^{-1} = \mathbf{R}\,[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}^{-1}, \tag{26}$$

because $\mathbf{s}$ and $\mathbf{e}_z$ are vectors and vectors transform by similarity transformations. Now:

$$e^{-\imath\,[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]\,\varphi/2} = \exp\begin{bmatrix} -\imath\varphi/2 & \\ & +\imath\varphi/2 \end{bmatrix} = \begin{bmatrix} e^{-\imath\varphi/2} & \\ & e^{+\imath\varphi/2} \end{bmatrix} = \cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]. \tag{27}$$

Therefore:

$$\begin{aligned} e^{-\imath\,[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\varphi/2} &= e^{-\imath\,\mathbf{R}\,[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}^{-1}\,\varphi/2} \\ &= \mathbf{R}\,[\,e^{-\imath\,[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]\,\varphi/2}\,]\,\mathbf{R}^{-1} \\ &= \mathbf{R}\,\{\cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]\,\}\,\mathbf{R}^{-1} \\ &= \cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]. \end{aligned} \tag{28}$$

In conclusion, we have now derived the Rodriguez formula from the Lie algebra by exponentiating:

$$e^{-\imath\,[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\varphi/2} = \cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]. \tag{29}$$

4.1.3 Boosts

We can use the same methods for Lorentz boosts. First we calculate a boost along the z-axis according to:

$$e^{-[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]\,\chi/2} = \begin{bmatrix} \exp(-\chi/2) & \\ & \exp(+\chi/2) \end{bmatrix} = \cosh(\chi/2)\mathbb{1} - \sinh(\chi/2)\,[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,] = \sqrt{\frac{\gamma+1}{2}}\mathbb{1} - \sqrt{\frac{\gamma-1}{2}}\,[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]. \tag{30}$$

We have here used Eq. 15. Now we use:

$$\exists \mathbf{R} \in \mathrm{SU}(2) : [\,\mathbf{u}\cdot\boldsymbol{\sigma}\,] = \mathbf{R}\,[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}^{-1}. \tag{31}$$

This leads then to:

$$e^{-[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]\,\chi/2} = \sqrt{\frac{\gamma+1}{2}}\mathbb{1} - \sqrt{\frac{\gamma-1}{2}}\,[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]. \tag{32}$$

However, the calculations in Eqs. 29, 30 and 32 must be considered as a mere cheat. They are elegant identities, but they do not explain why the exponentiation procedures proposed are supposed to lead to correct results in the first place. We can note the relation between Eq. 29 and the differential equation:

$$\frac{d\mathbf{R}}{d\varphi} = -\imath\,(\varphi/2)\,[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\mathbf{R}. \tag{33}$$

And we have a similar relation between Eq. 32 and the differential equation:

$$\frac{d\mathbf{B}}{d\chi} = -(\chi/2)\,[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]\,\mathbf{B}. \tag{34}$$

These differential equations lead to the infinitesimal generators:

$$\imath\left[\frac{\partial \mathbf{R}}{\partial \varphi}\right]_{\varphi=0} = \frac{1}{2}\,[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]. \tag{35}$$

$$\imath\left[\frac{\partial \mathbf{B}}{\partial \chi}\right]_{\chi=0} = -\frac{\imath}{2}\,[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]. \tag{36}$$

Therefore the calculations presented in the Eqs. 29 and 32 are just a reversal of the calculations of the infinitesimal generators, as we pointed out above. The true logical path of the derivation is $p \Rightarrow q \Rightarrow r$. The reversal presents this under the form: $r \Rightarrow q$. Here $p \Rightarrow q$ is the derivation of the expressions for a boost and a rotation given in Sections 2 and 3, and $q \Rightarrow r$ the ensuing calculation of the infinitesimal generators. The logical proposition r is the end point of the reasoning. The reversed calculation $r \Rightarrow q$ uses r as the starting point, which it just introduces out of the blue without any motivation for it. This is not a complete, logical build-up like $p \Rightarrow q \Rightarrow r$. It is only a leading argument that overwhelms the reader with a loose "plausible" narrative. Now $\cosh\chi = \gamma$ such that:

$$e^{+\chi} + e^{-\chi} = 2\gamma. \tag{37}$$

Putting $e^{+\chi} = h$ we obtain the quadratic equation

$$h^2 - 2\gamma h + 1 = 0, \tag{38}$$

with two solutions:

$$h = +\gamma \pm \sqrt{\gamma^2 - 1}, \tag{39}$$

such that:

$$\chi = \ln[\,\gamma \pm \sqrt{\gamma^2 - 1}\,], \tag{40}$$

and:

$$\chi/2 = \ln\sqrt{\gamma \pm \sqrt{\gamma^2 - 1}}\,. \tag{41}$$

We may wonder why we are obtaining two solutions rather than just one. To answer this question we may start noting that the product of these two solutions for χ is:

$$[\,\gamma + \sqrt{\gamma^2 - 1}\,]\,[\,\gamma - \sqrt{\gamma^2 - 1}\,] = 1. \tag{42}$$

We have thus:

$$\chi = \pm\ln[\,\gamma + \sqrt{\gamma^2 - 1}\,], \tag{43}$$

This conveniently illustrates the fact that Eq. 37 defines χ only up to the sign, because the substitution $\chi| - \chi$ leaves the equation invariant. If we adopt the convention $\chi \geq 0$, then we can drop the minus sign. We can elaborate the expressions:

$$\gamma^2 - 1 = \frac{1}{1 - \frac{v^2}{c^2}} - 1 = \frac{\gamma^2 v^2}{c^2}. \tag{44}$$

Therefore:

$$\chi = \ln[\,\gamma(1 + \frac{v}{c})\,] = \ln\sqrt{\frac{c+v}{c-v}} \geq 0. \tag{45}$$

The opposite sign convention for χ leads to:

$$\chi = \ln[\,\gamma(1 - \frac{v}{c})\,] = \ln\sqrt{\frac{c-v}{c+v}} \leq 0. \tag{46}$$

4.2 Caveat

In [5], the representation SL(2,$\mathbb{C}$) is presented as the result of a generalization of SU(2) according to:

$$[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\varphi/2 \rightarrow [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\varphi/2 + \imath[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]\,\chi/2. \tag{47}$$

This may look rather natural. But a generalization is a method of inductive reasoning. It is a heuristic tool permitting to guess results but does not constitute by any means a form of valid proof. At the very best we can validate afterwards that the result is correct, but obtained this way the good result must be qualified as a fluke, because it stems from an approach based on trial and error. There is actually no real proof given here that the result obtained is correct except *a posteriori* verification. It is only tacitly assumed to be correct. We have shown above that the whole presentation in [5] rests on replacing the correct derivation $p \Rightarrow q \Rightarrow r$ by a surrogate derivation $r \Rightarrow q$.

The development in [5] is therefore misleading. We may note that the boosts in the Dirac equation are by construction expressed in the quantities $\mathbb{1}, \gamma_x\gamma_t, \gamma_y\gamma_t, \gamma_z\gamma_t$ while the rotations are expressed in the quantities $\mathbb{1}, \gamma_x\gamma_y, \gamma_y\gamma_z, \gamma_z\gamma_x$. This suggests that the six bi-vectors $\gamma_x\gamma_t, \gamma_y\gamma_t, \gamma_z\gamma_t, \gamma_x\gamma_y, \gamma_y\gamma_z, \gamma_z\gamma_x$ constitute a basis for the six-dimensional homogeneous Lorentz group (see below). However, the complexification procedure proposed in [5] creates the illusion that we have a three dimensional basis. This is only an illusion, as conceptually, the vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ that occur in $\mathbf{s}$ cannot be considered to be parallel to the vectors $\imath\mathbf{e}_x, \imath\mathbf{e}_y, \imath\mathbf{e}_z$ that occur in $\imath\mathbf{u}$. They must be seen as an extension of a basis for $\mathbb{R}^3$ to a basis of $\mathbb{R}^6$. In other words, the step $\mathbf{e}_x \rightarrow \imath\mathbf{e}_x$ is not a multiplication with a scalar, but an expedient to introduce new basis vectors. In fact, in the 4×4 formalism the matrices $\gamma_y\gamma_z$ and $\gamma_x\gamma_t$ are different because $\gamma_y\gamma_z$ contains two identical blocks $-\sigma_y\sigma_z = -\imath\sigma_x$, while $\gamma_x\gamma_t$ contains two blocks of opposite signs $\pm\sigma_x$. Due to these opposite signs in its blocks $\gamma_x\gamma_t$ is not proportional to $\gamma_y\gamma_z$ by a scalar factor $\imath$, because $\gamma_y\gamma_z$ has identical signs in its blocks. In fact, $\mathbf{s}$ is an axial vector, while $\mathbf{u}$ is a true vector. This transpires also from the following calculation:

$$\begin{bmatrix} & \frac{\partial}{\partial ct}\mathbb{1} - \nabla\cdot\boldsymbol{\sigma} \\ \frac{\partial}{\partial ct}\mathbb{1} + \nabla\cdot\boldsymbol{\sigma} & \end{bmatrix} \begin{bmatrix} & \frac{V}{c}\mathbb{1} + \mathbf{A}\cdot\boldsymbol{\sigma} \\ \frac{V}{c}\mathbb{1} - \mathbf{A}\cdot\boldsymbol{\sigma} & \end{bmatrix} =$$
$$\begin{bmatrix} \frac{1}{c^2}\frac{\partial V}{\partial t}\,\mathbb{1} - \frac{\partial}{\partial ct}\mathbf{A}\cdot\boldsymbol{\sigma} - \frac{1}{c}\nabla V\cdot\boldsymbol{\sigma} + (\nabla\cdot\mathbf{A})\,\mathbb{1} + \imath(\nabla\wedge\mathbf{A})\cdot\boldsymbol{\sigma} & \\ & \frac{1}{c^2}\frac{\partial V}{\partial t}\,\mathbb{1} + \frac{\partial}{\partial ct}\mathbf{A}\cdot\boldsymbol{\sigma} + \frac{1}{c}\nabla V\cdot\boldsymbol{\sigma} + (\nabla\cdot\mathbf{A})\,\mathbb{1} + \imath(\nabla\wedge\mathbf{A})\cdot\boldsymbol{\sigma} \end{bmatrix} =$$
$$\begin{bmatrix} \left[\frac{1}{c^2}\frac{\partial V}{\partial t} + (\nabla\cdot\mathbf{A})\right]\mathbb{1} - \frac{1}{c}[(\nabla V + \frac{\partial\mathbf{A}}{\partial t})\cdot\boldsymbol{\sigma}] + \imath[\,(\nabla\wedge\mathbf{A})\cdot\boldsymbol{\sigma}\,] & \\ & \left[\frac{1}{c^2}\frac{\partial V}{\partial t} + (\nabla\cdot\mathbf{A})\right]\mathbb{1} + \frac{1}{c}[(\nabla V + \frac{\partial\mathbf{A}}{\partial t})\cdot\boldsymbol{\sigma}] + \imath[(\nabla\wedge\mathbf{A})\cdot\boldsymbol{\sigma}] \end{bmatrix}. \tag{48}$$

Here:

$$\frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla\cdot\mathbf{A} = 0, \tag{49}$$

corresponds to the Lorentz condition, while:

$$-\,[\,\nabla V + \frac{\partial\mathbf{A}}{\partial t}\,] = \mathbf{E}, \tag{50}$$

and:

$$\nabla\wedge\mathbf{A} = \mathbf{B}, \tag{51}$$

such that we obtain:

$$\frac{1}{c}\begin{bmatrix} [\,\mathbf{E}\cdot\boldsymbol{\sigma}\,] + \imath c[\,\mathbf{B}\cdot\boldsymbol{\sigma}\,] & \\ & -[\,\mathbf{E}\cdot\boldsymbol{\sigma}\,] + \imath c[\,\mathbf{B}\cdot\boldsymbol{\sigma}\,] \end{bmatrix}. \tag{52}$$

We see that $\mathbf{B}$ does not change sign under a parity transformation, because it takes the same expression in both SL(2,$\mathbb{C}$) blocks. It is therefore an axial vector. On the other hand $\mathbf{E}$ is a true vector because it changes sign under the parity transformation. We see also that $\mathbf{B}$ and $\mathbf{E}$ which look parallel in SL(2,$\mathbb{C}$) are not parallel in the Dirac formalism. They are not three-dimensional vector quantities but build a six-component tensor with Lorentz symmetry. The vector $\mathbf{E}$ is expressed in a basis of matrices of the type $\gamma_z\gamma_t$ while the axial vector $c\mathbf{B}$ is expressed in a basis of matrices $\gamma_y\gamma_z$. The four Maxwell equations can be written as a single matrix equation in this representation, illustrating their Lorentz symmetry.[7] We may further illustrate the absence of a true proportionality by noting that for $v > c$ we obtain:

$$\chi = \ln\imath\sqrt{\frac{c+v}{v-c}} = \imath(\pi/2) + \ln\sqrt{\frac{v+c}{v-c}} = \imath(\pi/2) + \ln\sqrt{\frac{vw+cw}{vw-cw}} = \imath(\pi/2) + \ln\sqrt{\frac{c+w}{c-w}}, \tag{53}$$

[7] These calculations have not been carried out correctly in [4]. There are errors in the pre-factors.

where we have introduced $wv = c^2$, such that $w < 0$. Based on the presentation in [5] one might start to suspect that this parameter would correspond to the product of a boost and a rotation. For sure, we all know about the pitfall:

$$e^{-\imath[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\varphi/2-[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]\,\chi/2} \neq [\,e^{-\imath[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]\,\varphi/2}\,]\,[\,e^{-[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]\,\chi/2}\,], \tag{54}$$

but one might still be tempted to wonder if it could be a product of a different boost and a different rotation. The surprise contained in Eq. 53 shows that the "self-evident" easy-going complexification of $\varphi\,[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ as is proposed in [5] is too fast and too furious a procedure.

If we took the example of Eq. 53 literally, it would indicate that not all linear combinations of the infinitesimal generators need to correspond to elements of the Lie group. In fact SL(2,$\mathbb{C}$) implies that the determinant of the matrix must be 1, which is not the case for $v > c$. We observe then that both $\imath(\pi/2)[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$ and $[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]\,\chi/2$ give rise to elements of the Lie group, while their sum apparently does not. That seems to contradict the proof given in Schuller's lecture [6]. The solution is that Schuller considers only charts on the manifold that are based on a real basis of $\mathbb{R}^n$ such that all the basis vectors are real. In the Lorentz group, the basis vectors $\mathbf{e}_j$ and $\imath\mathbf{e}_j$ are conceptually not parallel even if they look parallel due to the proportionality. This is why it is important that we can do the Clifford algebra with only real numbers [7]. This renders the confusion impossible. We can then define the correspondence between the real and the complex formalism.

The point is therefore that $\imath[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]\pi/2$ in Eq. 53 does not define a rotation like $\imath(\pi/2)[\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]$. The multiplication by the number $\imath$ in $\imath[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]\pi/2$ in Eq. 53 induces truly a multiplication and not a change of basis vector. We see that the restriction of the Dirac representation to the representation SL(2,$\mathbb{C}$) introduces ambiguity. SL(2,$\mathbb{C}$) is also not able to distinguish between $\mathbf{e}_t$ and $\mathbb{1}$.

5 Observation: The bi-vectors $\gamma_t\gamma_z$ and $\gamma_x\gamma_y$ are not parallel

In the Clifford algebra we have expressed orthogonality of vectors $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ by: $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\mathbb{1}$. Based on this orthogonality criterion, we can qualify γ_μ as a basis. We will see now that we cannot use this orthogonality criterion for bi-vectors. Nevertheless we would like to state that $\gamma_\mu\gamma_\nu$ with $\mu \neq \nu$ form an orthogonal basis $\mathbf{e}_\mu \wedge \mathbf{e}_\nu$ for the bi-vectors. This will be the motivation for introducing a new criterion for orthogonality. Let us remind that:

$$\sigma_x\sigma_y = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} & -\imath \\ \imath & \end{bmatrix} = \begin{bmatrix} \imath & \\ & -\imath \end{bmatrix} = \imath\sigma_z \text{ (cycl)}. \tag{55}$$

Let us now calculate the products of the gamma matrices:

$$\gamma_x\gamma_y = \begin{bmatrix} & \sigma_x \\ -\sigma_x & \end{bmatrix}\begin{bmatrix} & \sigma_y \\ -\sigma_y & \end{bmatrix} = \begin{bmatrix} -\sigma_x\sigma_y & \\ & -\sigma_x\sigma_y \end{bmatrix} = \begin{bmatrix} -\imath\sigma_z & \\ & -\imath\sigma_z \end{bmatrix}, \tag{56}$$

where we have used Eq. 55 and:

$$\gamma_t\gamma_z = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix}\begin{bmatrix} & \sigma_z \\ -\sigma_z & \end{bmatrix} = \begin{bmatrix} -\sigma_z & \\ & \sigma_z \end{bmatrix}. \tag{57}$$

We can notice that $\neg\exists\lambda \in \mathbb{C} \parallel \gamma_x\gamma_y = \lambda\gamma_t\gamma_z$. In fact we have: $\gamma_x\gamma_y = \imath\gamma_5\gamma_t\gamma_z$, where:

$$\gamma_5 = \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}, \tag{58}$$

such that $\gamma_x\gamma_y$ and $\gamma_t\gamma_z$ are not parallel. However, when we restrict the formalism to SL(2,$\mathbb{C}$), e.g. by selecting the 2×2 block matrix with indices $(1,1)$, then $\gamma_5 \curvearrowright \mathbb{1}$, such that the restriction of $\imath\gamma_5$ to SL(2,$\mathbb{C}$) becomes a proportionality constant: $\imath\gamma_5 \curvearrowright \imath$. Here we use the notation $A \curvearrowright B$ to indicate that A reduces to B in the restriction. The restrictions to SL(2,$\mathbb{C}$) are then $\gamma_x\gamma_y \curvearrowright -\imath\sigma_z$ and $\gamma_t\gamma_z \curvearrowright -\sigma_z$ such that the restrictions of the matrices $\gamma_x\gamma_y$ and $\gamma_t\gamma_z$ to SL(2,$\mathbb{C}$) look proportional with proportionality constant $\imath$. But in reality they are not, as shown by the equation $\gamma_x\gamma_y = \imath\gamma_5\gamma_t\gamma_z$. The matrices $\gamma_x\gamma_y$ and $\gamma_t\gamma_z$ are commuting, but they are not parallel. They are also not orthogonal according to the criteria of the Clifford algebra:

$$\gamma_t\gamma_z\gamma_x\gamma_y + \gamma_x\gamma_y\gamma_t\gamma_z = \begin{bmatrix} -\sigma_z & \\ & \sigma_z \end{bmatrix}\begin{bmatrix} -\imath\sigma_z & \\ & -\imath\sigma_z \end{bmatrix} + \begin{bmatrix} -\imath\sigma_z & \\ & -\imath\sigma_z \end{bmatrix}\begin{bmatrix} -\sigma_z & \\ & \sigma_z \end{bmatrix} = 2\imath\gamma_5, \tag{59}$$

with $\gamma_t\gamma_z\gamma_x\gamma_y - \gamma_x\gamma_y\gamma_t\gamma_z = 0$. It is therefore not obvious that $\gamma_\mu\gamma_{nu}$ constitute a basis for the bi-vectors as we have not yet proved their linear independence.

6 The orthogonality criterion for the Lie algebra and for matrices

We see that $\gamma_t\gamma_z$ is not orthogonal to $\gamma_x\gamma_y$ according to the criteria used in the build-up of the Clifford algebra. In fact, we must always have $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\mathbb{1}$. Therefore:

$$[\,\gamma_t\gamma_z\,]\,[\,\gamma_x\gamma_y\,] = -\gamma_t\gamma_x\gamma_z\gamma_y = \gamma_t\gamma_x\gamma_y\gamma_z = -\gamma_x\gamma_t\gamma_y\gamma_z = [\,\gamma_x\gamma_y\,]\,[\,\gamma_t\gamma_z\,]. \tag{60}$$

Hence:

$$[\,\gamma_t\gamma_z\,]\,[\,\gamma_x\gamma_y\,] + [\,\gamma_x\gamma_y\,]\,[\,\gamma_t\gamma_z\,] \neq 0. \tag{61}$$

In fact, we have $[\,\gamma_t\gamma_z\,]\,[\,\gamma_x\gamma_y\,] - [\,\gamma_x\gamma_y\,]\,[\,\gamma_t\gamma_z\,] = 0$ such that $[\,\gamma_t\gamma_z\,]\,[\,\gamma_x\gamma_y\,] + [\,\gamma_x\gamma_y\,]\,[\,\gamma_t\gamma_z\,] = 0$ would imply that $[\,\gamma_t\gamma_z\,] = 0 \vee [\,\gamma_x\gamma_y\,] = 0$. But the later is meaningless. In fact, assume e.g. $[\,\gamma_x\gamma_y\,] = 0$. Then $[\,\gamma_y\gamma_x\,] = -[\,\gamma_x\gamma_y\,] = 0$. As $\gamma_y^2 = -\mathbb{1}$, $[\,\gamma_x\gamma_y\,][\,\gamma_y\gamma_x\,] = 0$ would imply $\gamma_x^2 = 0$ while we must have $\gamma_x^2 = -\mathbb{1} \neq 0$. However:

$$[\,\gamma_t\gamma_z\,]\,[\,\gamma_z\gamma_y\,] + [\,\gamma_z\gamma_y\,]\,[\,\gamma_t\gamma_z\,] = -\gamma_t\gamma_y - \gamma_y\gamma_t = 0. \tag{62}$$

According to the orthogonality criteria of the Clifford algebra we have therefore $\gamma_t\gamma_z \perp \gamma_z\gamma_y$ but $\neg(\gamma_t\gamma_z \perp \gamma_x\gamma_y)$. Hence, if the six bi-vectors constitute a basis then this basis is not orthogonal according to the criteria we used up to now. Now we may think that we can identify $\gamma_\mu\gamma_\nu$ with $\mathbf{e}_\mu \wedge \mathbf{e}_\nu$ based on the anti-symmetry. But if we tacitly imply that this automatically entails that $\mathbf{e}_\mu \wedge \mathbf{e}_\nu$ form an orthogonal basis for the volume elements, then we are wrong.

As orthogonality is a convenient property for a basis, which permits to check linear independence, we will introduce a new definition of orthogonality. According to this new criterion the six bi-vectors $\gamma_\mu\gamma_\nu$, with $\mu \neq \nu$ will now be all mutually orthogonal, which proves then that they constitute an orthogonal basis for the Lorentz group, and that we can identify $\gamma_\mu\gamma_\nu$ with $\mathbf{e}_\mu \wedge \mathbf{e}_\nu$.

To calculate the norm of an $n \times n$ matrix $\mathbf{A}$ we can use:

$$|\mathbf{A}|^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}^* a_{jk}. \tag{63}$$

This leads to the orthogonality condition:

$$\mathbf{A} \perp \mathbf{B} \quad \Leftrightarrow \quad \mathbf{A} \star \mathbf{B} = \sum_{j=1}^{n}\sum_{k=1}^{n} (a_{jk}^* b_{jk} + a_{jk} b_{jk}^*) = 0. \tag{64}$$

This follows from defining the scalar product by:

$$\mathbf{A} \star \mathbf{B} = \frac{1}{2}\,\left[\; |\mathbf{A}+\mathbf{B}|^2 - |\mathbf{A}|^2 - |\mathbf{B}|^2 \;\right]. \tag{65}$$

When $b_{jk} = \imath a_{jk}$ with $a_{jk} \in \mathbb{R}$ this leads to: $a_{jk}^* \imath a_{jk} + a_{jk}(-\imath)a_{jk}^* = 0$. We have then e.g.:

$$\begin{aligned}
\sigma_x \star\ \sigma_y &= \begin{bmatrix} & +1 \\ +1 & \end{bmatrix} \star \begin{bmatrix} & -\imath \\ +\imath & \end{bmatrix} = +1(-\imath) + 1(+\imath) + 1(+\imath) + 1(-\imath) = 0, \\
\sigma_x \star \imath\sigma_y &= \begin{bmatrix} & +1 \\ +1 & \end{bmatrix} \star \begin{bmatrix} & +1 \\ -1 & \end{bmatrix} = +1(+1) + 1(-1) + 1(+1) + 1(-1) = 0, \\
\sigma_y \star \imath\sigma_y &= \begin{bmatrix} & -\imath \\ +\imath & \end{bmatrix} \star \begin{bmatrix} & +1 \\ -1 & \end{bmatrix} = +\imath(+1) - \imath(-1) - \imath(+1) + \imath(-1) = 0.
\end{aligned} \tag{66}$$

In summary:

$$\begin{array}{rrr}
\sigma_x \star \sigma_y = 0, & \sigma_y \star \sigma_z = 0, & \sigma_z \star \sigma_x = 0, \\
\imath\sigma_x \star \imath\sigma_y = 0, & \imath\sigma_y \star \imath\sigma_z = 0, & \imath\sigma_z \star \imath\sigma_x = 0, \\
\sigma_x \star \imath\sigma_x = 0, & \sigma_x \star \imath\sigma_y = 0, & \sigma_x \star \imath\sigma_z = 0, \\
\sigma_y \star \imath\sigma_x = 0, & \sigma_y \star \imath\sigma_y = 0, & \sigma_y \star \imath\sigma_z = 0, \\
\sigma_z \star \imath\sigma_x = 0, & \sigma_z \star \imath\sigma_y = 0, & \sigma_z \star \imath\sigma_z = 0, \\
\sigma_x \star \mathbb{1} = 0, & \sigma_y \star \mathbb{1} = 0, & \sigma_z \star \mathbb{1} = 0, \\
\imath\sigma_x \star \mathbb{1} = 0, & \imath\sigma_y \star \mathbb{1} = 0, & \imath\sigma_z \star \mathbb{1} = 0.
\end{array} \tag{67}$$

We can then use $\sum \lambda_j \mathbf{A}_j$ with $\lambda_j \in \mathbb{R}$ as vectors of a real vector space with basis $\mathbf{A}_j$. We see then that the six basis vectors σ_j and $\imath\sigma_j$ of SL(2,$\mathbb{C}$) are all orthogonal and that it is not allowed to have $\sum \lambda_j\sigma_j + \imath \sum \mu_j\sigma_j$ with $\lambda_j \in \mathbb{C}\backslash\mathbb{R} \vee$

$\mu_j \in \mathbb{C}\backslash\mathbb{R}$. The coefficients λ_j and μ_j must belong to $\mathbb{R}$. In other words we have a set of vectors $\sum \lambda_j \mathbf{e}_j + \imath \sum \mu_j \mathbf{e}_{j+3}$, where $\lambda_j \in \mathbb{R}$, $\mu_j \in \mathbb{R}$, which constitutes a real six-dimensional vector space $\mathbb{R}^6$ despite the fact that the basis vectors $\mathbf{e}_j$ and $\mathbf{e}_{j+3}$ are represented by complex matrices. Complex coefficients λ_j, μ_j are then by definition not allowed. This is the reason why we cannot obtain a boost with $v > c$.

7 Orthogonality of vectors and bi-vectors according to the scalar product $\star$

We will show that orthogonality according to the Clifford algebra implies orthogonality according to the scalar product $\star$. However, the reverse is not true, and the definition of orthogonality according to the scalar product $\star$ is more general. It is important to point out the difference between the definitions of orthogonality for a Lie group and a Lie algebra. It explains e.g. why we cannot derive the eight infinitesimal generators g_j for SU(3) by postulating $g_j g_k + g_k g_j = 2\delta_{jk}\mathbb{1}$. It just will not work. For SU(2) not pointing out this difference can lead to confusion because the infinitesimal generators coincide accidentally with the reflection operators σ_j introduced by the Clifford algebra. The Lorentz group shows that this equality is truly accidental because the Clifford algebra defines four basic reflections γ_j while there are six infinitesimal generators. For the vectors we have:

$$\gamma_x \star \gamma_y = \begin{bmatrix} & \sigma_x \\ -\sigma_x & \end{bmatrix} \star \begin{bmatrix} & \sigma_y \\ -\sigma_y & \end{bmatrix} = \sigma_x \star \sigma_y + \sigma_x \star \sigma_y = 0. \tag{68}$$

$$\gamma_x \star \gamma_z = \begin{bmatrix} & \sigma_x \\ -\sigma_x & \end{bmatrix} \star \begin{bmatrix} & \sigma_z \\ -\sigma_z & \end{bmatrix} = \sigma_x \star \sigma_z + \sigma_x \star \sigma_z = 0. \tag{69}$$

$$\gamma_y \star \gamma_z = \begin{bmatrix} & \sigma_y \\ -\sigma_y & \end{bmatrix} \star \begin{bmatrix} & \sigma_z \\ -\sigma_z & \end{bmatrix} = \sigma_y \star \sigma_z + \sigma_y \star \sigma_z = 0. \tag{70}$$

Note that we will have $\sigma_j \star \sigma_k = 0 \Leftrightarrow \sigma_k \star \sigma_j = 0$ when $j \neq k$ and therefore also $\gamma_j \star \gamma_k = 0 \Leftrightarrow \gamma_k \star \gamma_j = 0$, when $j \neq k$. This follows from the fact that the definition of $\mathbf{A} \star \mathbf{B}$ in Eq. 63 is symmetrical. We will use Roman indices for the x, y, z and Greek indices for x, y, z, ct and 5. As far as the Greek indices are concerned, we also have:

$$\gamma_x \star \gamma_t = \begin{bmatrix} & \sigma_x \\ -\sigma_x & \end{bmatrix} \star \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \sigma_x \star \mathbb{1} - \sigma_x \star \mathbb{1} = 0. \tag{71}$$

$$\gamma_y \star \gamma_t = \begin{bmatrix} & \sigma_y \\ -\sigma_y & \end{bmatrix} \star \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \sigma_y \star \mathbb{1} - \sigma_y \star \mathbb{1} = 0. \tag{72}$$

$$\gamma_z \star \gamma_t = \begin{bmatrix} & \sigma_z \\ -\sigma_z & \end{bmatrix} \star \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \sigma_z \star \mathbb{1} - \sigma_z \star \mathbb{1} = 0. \tag{73}$$

The proofs for $\gamma_t \star \gamma_x = 0$, $\gamma_t \star \gamma_y = 0$ and $\gamma_t \star \gamma_z = 0$ are similar. This can be proved by detailed calculations but we can use also the fact that the definition of $\mathbf{A} \star \mathbf{B}$ in Eq. 63 is symmetrical. Finally, we have:

$$\gamma_j \star \gamma_5 = \begin{bmatrix} & \sigma_x \\ -\sigma_x & \end{bmatrix} \star \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} = 0. \tag{74}$$

$$\gamma_t \star \gamma_5 = \begin{bmatrix} & \mathbb{1} \\ \mathbb{1} & \end{bmatrix} \star \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} = 0. \tag{75}$$

Consequently, we have also $\gamma_5 \star \gamma_j = 0$ and $\gamma_5 \star \gamma_t = 0$. For the bi-vectors we have:

$$\gamma_t\gamma_z \star \gamma_x\gamma_y = \begin{bmatrix} -\sigma_z & \\ & \sigma_z \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_z & \\ & -\imath\sigma_z \end{bmatrix} = -\sigma_z \star (-\imath\sigma_z) + \sigma_z \star (-\imath\sigma_z) = 0. \tag{76}$$

$$\gamma_t\gamma_z \star \gamma_y\gamma_z = \begin{bmatrix} -\sigma_z & \\ & \sigma_z \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_x & \\ & -\imath\sigma_x \end{bmatrix} = -\sigma_z \star (-\imath\sigma_x) + \sigma_z \star (-\imath\sigma_x) = 0. \tag{77}$$

$$\gamma_t\gamma_z \star \gamma_z\gamma_x = \begin{bmatrix} -\sigma_z & \\ & \sigma_z \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_y & \\ & -\imath\sigma_y \end{bmatrix} = -\sigma_z \star (-\imath\sigma_y) + \sigma_z \star (-\imath\sigma_y) = 0. \tag{78}$$

$$\gamma_t\gamma_x \star \gamma_x\gamma_y = \begin{bmatrix} -\sigma_x & \\ & \sigma_x \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_z & \\ & -\imath\sigma_z \end{bmatrix} = -\sigma_x \star (-\imath\sigma_z) + \sigma_x \star (-\imath\sigma_z) = 0. \tag{79}$$

$$\gamma_t\gamma_x \star \gamma_y\gamma_z = \begin{bmatrix} -\sigma_x & \\ & \sigma_x \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_x & \\ & -\imath\sigma_x \end{bmatrix} = -\sigma_x \star (-\imath\sigma_x) + \sigma_x \star (-\imath\sigma_x) = 0. \tag{80}$$

$$\gamma_t\gamma_x \star \gamma_z\gamma_x = \begin{bmatrix} -\sigma_x & \\ & \sigma_x \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_y & \\ & -\imath\sigma_y \end{bmatrix} = -\sigma_x \star (-\imath\sigma_y) + \sigma_x \star (-\imath\sigma_y) = 0. \tag{81}$$

$$\gamma_t\gamma_y \star \gamma_x\gamma_y = \begin{bmatrix} -\sigma_y & \\ & \sigma_y \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_z & \\ & -\imath\sigma_z \end{bmatrix} = -\sigma_y \star (-\imath\sigma_z) + \sigma_y \star (-\imath\sigma_z) = 0. \tag{82}$$

$$\gamma_t\gamma_y \star \gamma_y\gamma_z = \begin{bmatrix} -\sigma_y & \\ & \sigma_y \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_x & \\ & -\imath\sigma_x \end{bmatrix} = -\sigma_y \star (-\imath\sigma_x) + \sigma_y \star (-\imath\sigma_x) = 0. \tag{83}$$

$$\gamma_t\gamma_y \star \gamma_z\gamma_x = \begin{bmatrix} -\sigma_y & \\ & \sigma_y \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_y & \\ & -\imath\sigma_y \end{bmatrix} = -\sigma_y \star (-\imath\sigma_y) + \sigma_y \star (-\imath\sigma_y) = 0. \tag{84}$$

$$\gamma_x\gamma_y \star \gamma_y\gamma_z = \begin{bmatrix} -\imath\sigma_z & \\ & -\imath\sigma_z \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_x & \\ & -\imath\sigma_x \end{bmatrix} = (-\imath\sigma_z) \star (-\imath\sigma_x) + (-\imath\sigma_z) \star (-\imath\sigma_x) = 0. \tag{85}$$

$$\gamma_y\gamma_z \star \gamma_z\gamma_x = \begin{bmatrix} -\imath\sigma_x & \\ & -\imath\sigma_x \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_y & \\ & -\imath\sigma_y \end{bmatrix} = (-\imath\sigma_x) \star (-\imath\sigma_y) + (-\imath\sigma_x) \star (-\imath\sigma_y) = 0. \tag{86}$$

$$\gamma_z\gamma_x \star \gamma_x\gamma_y = \begin{bmatrix} -\imath\sigma_y & \\ & -\imath\sigma_y \end{bmatrix} \star \begin{bmatrix} -\imath\sigma_z & \\ & -\imath\sigma_z \end{bmatrix} = (-\imath\sigma_y) \star (-\imath\sigma_z) + (-\imath\sigma_y) \star (-\imath\sigma_z) = 0. \tag{87}$$

The proofs for $\gamma_5\gamma_j \star \gamma_k\gamma_\ell$, $\gamma_5\gamma_t \star \gamma_j\gamma_k$, $\gamma_5\gamma_j \star \gamma_t\gamma_k$ are trivial because these products do not contain non-zero terms, because the matrix products which do not contain γ_5 are block diagonal while the ones that contain γ_5 are block off-diagonal.

8 Rotations and boosts in the Cartan-Weyl representation

We have:

$$\exp(\gamma_x\gamma_y\varphi/2) = \begin{bmatrix} e^{-\imath\sigma_z\varphi/2} & \\ & e^{-\imath\sigma_z\varphi/2} \end{bmatrix} = \begin{bmatrix} \cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,] & \\ & \cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,] \end{bmatrix}. \tag{88}$$

$$\exp(\gamma_t\gamma_z\chi/2) = \begin{bmatrix} e^{-\sigma_z\chi/2} & \\ & e^{+\sigma_z\chi/2} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\gamma+1}{2}}\mathbb{1} - \sqrt{\frac{\gamma-1}{2}}\,[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,] & \\ & \sqrt{\frac{\gamma+1}{2}}\mathbb{1} + \sqrt{\frac{\gamma-1}{2}}\,[\,\mathbf{e}_z\cdot\boldsymbol{\sigma}\,] \end{bmatrix}. \tag{89}$$

We see that there is no "factor" $\imath$ needed to obtain the matrices for the rotations and the boosts. They are just obtained both by exponentiating, which clearly shows that $\imath$ is not a factor but a means to define new basis vectors in the restricted context of SL(2,$\mathbb{C}$). We recover Eq. 14 and we recover Eq. 19 up to a sign. Introducing $\imath$ nevertheless, we would obtain:

$$\exp(\imath\gamma_t\gamma_z\varphi/2) = \begin{bmatrix} e^{-\imath\sigma_z\varphi/2} & \\ & e^{+\imath\sigma_z\varphi/2} \end{bmatrix} = \begin{bmatrix} \cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2)\sigma_z & \\ & \cos(\varphi/2)\mathbb{1} + \imath\sin(\varphi/2)\sigma_z \end{bmatrix}. \tag{90}$$

We could rewrite this as:

$$\cos(\varphi/2)\mathbb{1} - \imath\sin(\varphi/2) \begin{bmatrix} \sigma_z & \\ & -\sigma_z \end{bmatrix}, \tag{91}$$

whereby:

$$\begin{bmatrix} \sigma_z & \\ & -\sigma_z \end{bmatrix} \star \begin{bmatrix} \sigma_z & \\ & \sigma_z \end{bmatrix} = 0, \tag{92}$$

such that this looks almost as defining SO(4). At first sight this cannot be right because it does not seem to correspond to the right metric. But in reality $\imath\gamma_t$ would square to $-\mathbb{1}$ just like γ_x, γ_y, and γ_z. We would then have the quadratic form $-c^2t^2-x^2-y^2-z^2$. To obtain the right quadratic form $c^2t^2+x^2+y^2+z^2$, the better approach would consist in keeping γ_t and replacing $\gamma_x|\imath\gamma_x$, $\gamma_y|\imath\gamma_y$, and $\gamma_z|\imath\gamma_z$. This would result in the substitutions: $\gamma_x\gamma_y|-\gamma_x\gamma_y$ and $\gamma_t\gamma_x|\imath\gamma_t\gamma_x$, which leads to a correct formalism for SO(4). Moreover, we obtain then Eq. 19 with the right sign. We may finally note that the infinitesimal generators g corresponding to Eqs. 88 and 89 would be:

$$\frac{1}{2}\begin{bmatrix} \sigma_z & \\ & \sigma_z \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} -\imath\sigma_z & \\ & +\imath\sigma_z \end{bmatrix}. \tag{93}$$

This shows that the infinitesimal generators are not basis vectors taken from $\mathbb{R}^4$ as the SL(2,$\mathbb{C}$) formalism might suggest. They constitute a basis of bi-vectors. In general, the infinitesimal generators are:

$$\begin{bmatrix} [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]/2 & \\ & [\,\mathbf{s}\cdot\boldsymbol{\sigma}\,]/2 \end{bmatrix}, \quad \begin{bmatrix} -\imath[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]/2 & \\ & +\imath[\,\mathbf{u}\cdot\boldsymbol{\sigma}\,]/2 \end{bmatrix}, \tag{94}$$

for pure rotations and pure boosts. All transformations are then of the type $e^{-\imath g\varphi}$ or $e^{-\imath g\chi}$.

9 The multi-vectors build an orthogonal basis for the Clifford algebra

9.1 Motivation

Certain textbooks introduce the Clifford algebra as the linear combinations of quantities:

$$\begin{gathered} 1, \quad \mathbf{e}_j, \cdots \mathbf{e}_j, \cdots \mathbf{e}_N, \quad \mathbf{e}_1\wedge\mathbf{e}_2, \cdots \mathbf{e}_j\wedge\mathbf{e}_k, \cdots \mathbf{e}_{N-1}\wedge\mathbf{e}_N, \\ \mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3, \cdots \mathbf{e}_j\wedge\mathbf{e}_k\wedge\mathbf{e}_\ell, \cdots \mathbf{e}_{N-2}\wedge\mathbf{e}_{N-1}\wedge\mathbf{e}_N, \quad \cdots \quad \mathbf{e}_1\wedge\mathbf{e}_2\wedge\cdots\wedge\mathbf{e}_j\wedge\cdots\mathbf{e}_N. \end{gathered} \tag{95}$$

whereby $\wedge$ is anti-symmetrical and $\mathbf{e}_j$ are a basis of $\mathbb{R}^N$. The quantities are multi-vectors of rank 0, 1, 2, $\cdots$ N. In principle $N=2n$ or $N=2n+1$, such that $\lfloor\frac{N}{2}\rfloor = n$. The wedge products correspond then to oriented hyper-volumes in $\mathbb{R}^N$, as explained in [8]. By using this formalism one can derive the group theory for rotations in $\mathbb{R}^N$. The results can be obtained with almost no effort. All one needs is the slight of hand that consists in considering the completely different quantities $\mathbf{e}_{j_1}\wedge\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_p}$ with $p\in[\,0,n\,]\cap\mathbb{Z}$ as belonging to a vector space. In textbooks the vector space spanned by the multi-vectors is often written as (see e.g. [8], p.41):

$$\mathbb{R}\oplus\mathbb{R}^N\oplus\bigwedge^2\mathbb{R}^N\oplus\bigwedge^3\mathbb{R}^N\oplus\cdots\oplus\bigwedge^N\mathbb{R}^N. \tag{96}$$

To paraphrase it ironically we are adding here kiwis and bananas, something we have been taught we should never do in mathematics. Well, apparently once in a blue Moon it is nevertheless allowed. That the whole group theory with all its impressive results can be obtained without any effort looks like magic and this raises the suspicion that the reason for this might be concealed behind the slight of hand of representing Eq. 96 as self-evident. The definition just descends from heaven without any explanation, and there is a passage in the work of Hestenes [9] where he ridicules rather aggressively persons who are expressing their bewilderment with these definitions, only to hide the fact that he does not know to justify this puzzling procedure himself. The justification for these definitions resides in the way one builds up the Clifford algebra logically in the construction of the group representation theory, where all these multi-vectors become represented by matrices of the same dimensions $2^{\lfloor N/2\rfloor}\times 2^{\lfloor N/2\rfloor}$ where N is the dimension of the vector space $\mathbb{R}^N$ and its multi-vectors the isometries are acting on. It is this logical construction which is being bluntly glossed over in the rather cavalier, abrupt introduction of the definition of the Clifford algebra described above. Sweeping these prior justifications under the rug leaves the reader clueless behind in the dark.

In this respect, arguing that a presentation of the group representation theory for the isometries of a vector space can be done much more elegantly in terms of Clifford algebra based on the postulate in Eq. 96 than by our approach is just a hype whereby the price to pay for the sacred cow of elegance is a significant loss of insight.[8] The ever so elegant development reads then like a book of which the first chapters have gone missing. Mathematicians may have very good

[8] In April 2014 an anonymous Springer referee mentioning Kähler's work stipulated that I should rewrite my whole monograph [4] building it on Clifford algebra, thereby censoring my attempts to clarify the construction of Clifford algebra itself and the surprizing postulate formulated in Eq. 96. He stated that "it is remarkable that the author has seen things that others do not see" but that their publication could wait. All this happened after the editor Aldo Rampioni had already stalled the monograph for more than a year. As true professionals we are expected to accept such tactics tongue in cheek because it is all about gold standards and quality control.

reasons to present their results the way they do, but they should write them down in less cryptic ways, because some of their results will have to be used by physicists. We are convinced that the stilted elitist presentation of spinors in Cartan's monograph [10] has endowed it with a rocksolid impenetrability that renders it totally unaccessible to almost all physicists, while it holds a crucial key to understanding quantum mechanics, as we have tried to convey by our work. If my presentations look unconventional it can only illustrate how I have been forced by this impenetrability to figure it out all by myself.

Expressing one's dismay about such expeditious procedures is all but ridiculous. Quantum mechanics is written in group representation theory. The impression that this group representation theory seems to churn out its results by magic can only amplify one's feelings of alienation and the conviction that quantum mechanics is beyond human understanding. In reality, there is no magic at work in the formalism, but one can only become aware of this or start suspecting this, after completely demystifying the group representation theory in a prior stage of the development. Purely mathematical facts one has failed to understand cannot be qualified as quantum mysteries in physics. This is why we try to clarify here further what is going on behind the scenes of the group representation theory. Just pointing out that geometrical algebra magically reproduces the results of quantum mechanics, does not explain it as long as we have not understood in minute detail the whole machinery of the algebra itself.

Furthermore, when we are defining the $2^n \times 2^n$ representation matrices γ_j corresponding to $\mathbf{e}_j$, the products $\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_p}$ supposed to correspond to $\mathbf{e}_{j_1} \wedge \mathbf{e}_{j_1} \wedge \cdots \mathbf{e}_{j_p}$ for some $p \in [\,0, n\,] \cap \mathbb{Z}$ will be indeed anti-symmetrical. But the question arises if $\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_p}$ truly corresponds to $\mathbf{e}_{j_1} \wedge \mathbf{e}_{j_1} \wedge \cdots \mathbf{e}_{j_p}$ and what kind of basis the matrices $\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_p}$, of rank $p \in [\,0, n\,] \cap \mathbb{Z}$ form for the group of the $2^n \times 2^n$ matrices of the Clifford algebra, if they constitute a basis in the first place. We will show below that they form an orthogonal basis when we define the scalar product of two matrices $\mathbf{A}$ and $\mathbf{B}$ as $\mathbf{A} \star \mathbf{B}$ according to Eqs. 64-65, and we limit the rank p to $p \in [\,0, n\,] \cap \mathbb{Z}$ rather than $p \in [\,0, N\,] \cap \mathbb{Z}$. The proof based on Peano induction is painstaking and lengthy due to the fact that one must consider all different possible ranks for the multi-vectors. It requires considering a profusion of tedious details. Of course, the basis can be rendered orthonormal by a straightforward normalization procedure. We are not aware of a previous derivation of our result in the existing literature, especially not in physics textbooks. The theorem we prove here is not mentioned in [8]

9.2 General proof by Peano induction - First steps

It is annoying to make general proofs with the pseudo-metric, because for each basis vector we will add we will have to decide if its square is 1 or -1. It is therefore more convenient not to bother about the metric and to consider the three matrices:

$$\lambda_x = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad \lambda_y = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \quad \lambda_z = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{97}$$

We have then $\lambda_j\lambda_k + \lambda_k\lambda_j = 2g_{jk}\mathbb{1}$, with the pseudo-metric $g_{xx} = 1$, $g_{yy} = -1$ and $g_{zz} = 1$, implying $\lambda_x^2 = \mathbb{1}$, $\lambda_y^2 = -\mathbb{1}$, and $\lambda_z^2 = \mathbb{1}$. To obtain $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\mathbb{1}$ with the Euclidean metric, we must take $\sigma_x = \lambda_x$, $\sigma_y = \imath\lambda_y$ and $\sigma_z = \lambda_z$. We have now $\lambda_j \star \lambda_k = 0$, for $j \neq k$, and $\lambda_j \star \mathbb{1} = 0$, for all j because λ_x and λ_y have no overlap with $\mathbb{1}$. Multiplying λ_y by $\imath$ will not change the orthogonality relations involving σ_y. In fact, if λ_y is orthogonal to some matrix, then *a fortiori* σ_y will be orthogonal to it. We also do not introduce fake orthogonalities because σ_y and λ_y do not occur simultaneously. It is easy to check that $\lambda_x \star \lambda_x = 2$, $\lambda_y \star \lambda_y = 2$, $\lambda_z \star \lambda_z = 2$, and that $\sigma_y \star \sigma_y = 2$. We have:

$$\lambda_x\lambda_y = \lambda_z, \quad \lambda_y\lambda_z = \lambda_x, \quad \lambda_z\lambda_x = -\lambda_y. \tag{98}$$

We see thus that the bi-vectors are mutually orthogonal, because their scalar products can be expressed as scalar products of vectors and the vectors are mutually orthogonal. We have also the identity:

$$\lambda_x\lambda_y\lambda_z = \mathbb{1}, \quad \lambda_y\lambda_z\lambda_x = \mathbb{1}, \quad \lambda_z\lambda_x\lambda_y = \mathbb{1}. \tag{99}$$

From these identities we can actually derive the ones in Eq. 98. E.g. from $\lambda_x(\lambda_x\lambda_y\lambda_z)\lambda_x = \lambda_x^2\mathbb{1}$ we obtain: $\lambda_y\lambda_z\lambda_x = \mathbb{1}$ and $\lambda_y\lambda_z = \lambda_y\lambda_z\lambda_x\lambda_x = \lambda_x$. We see that the bi-vectors are not necessarily orthogonal to the vectors, because:

$$\lambda_x\lambda_y \star \lambda_z = \lambda_z \star \lambda_z = 2. \tag{100}$$

In fact, due to the identities in Eq. 99 there is some redundancy in using bi-vectors and the vectors simultaneously. In general, there will be a redundancy in using simultaneously multi-vectors of rank p and rank $2n + 1 - p$, where $p \in [\,0, 2n+1\,] \cap \mathbb{Z}$. In other words, the basis vectors for the vector space will only include multi-vectors up to rank n.

Let us now explore the Peano induction scheme in going from $\mathbb{R}^3$ to $\mathbb{R}^5$ in order to figure out how we can generalize it. We expect here an equivalence between bi-vectors and tri-vectors and between vectors and quadri-vectors. We define:

$$\gamma_x = \lambda_x \Lambda_x = \begin{bmatrix} & \lambda_x \\ \lambda_x & \end{bmatrix}, \quad \gamma_y = \lambda_y \Lambda_x = \begin{bmatrix} & \lambda_y \\ \lambda_y & \end{bmatrix}, \quad \gamma_z = \lambda_z \Lambda_x = \begin{bmatrix} & \lambda_z \\ \lambda_z & \end{bmatrix},$$

$$\gamma_4 = \Lambda_y = \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix}, \quad \gamma_5 = \Lambda_z = \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}. \tag{101}$$

We may note that if we call $\gamma_x = \gamma_1$, $\gamma_y = \gamma_2$ and $\gamma_z = \gamma_3$ then we will have $\gamma_2^2 = -\mathbb{1}$, $\gamma_4^2 = -\mathbb{1}$ and $\gamma_1^2 = \mathbb{1}$, $\gamma_3^2 = \mathbb{1}$, $\gamma_5^2 = \mathbb{1}$. In other words, the matrices with even indices square to $-\mathbb{1}$, while the matrices with odd indices square to $\mathbb{1}$. This will be general by induction. We have now by pure analogy:

$$\Lambda_x \Lambda_y = \Lambda_z, \quad \Lambda_y \Lambda_z = \Lambda_x, \quad \Lambda_z \Lambda_x = -\Lambda_y. \tag{102}$$

and:

$$\Lambda_x \Lambda_y \Lambda_z = \mathbb{1}, \quad \Lambda_y \Lambda_z \Lambda_x = \mathbb{1}, \quad \Lambda_z \Lambda_x \Lambda_y = \mathbb{1}. \tag{103}$$

This leads to:

$$\gamma_x \gamma_y \gamma_z \gamma_4 \gamma_5 = \mathbb{1} \text{ (cycl.)}. \tag{104}$$

and identities of the type:

$$\gamma_z \gamma_4 \gamma_5 = -\gamma_y \gamma_x \mathbb{1} = \gamma_x \gamma_y \mathbb{1}, \tag{105}$$

confirming the redundance between bi-vectors and tri-vectors anticipated. It also illustrates the redundance between multi-vectors of rank 0 and rank 5. We can already see that these properties will be generalized in the Peano induction scheme. For the vectors we have now $\gamma_x \star \mathbb{1} = 0$ due to absence of overlap, $\gamma_x \star \gamma_y = 2\lambda_x \star \lambda_y = 0$, $\gamma_x \star \gamma_4 = \lambda_x \star \mathbb{1} - \lambda_x \star \mathbb{1} = 0$, $\gamma_x \star \gamma_5 = 0$ and $\gamma_4 \star \gamma_5 = 0$ due to absence of overlap. We have further $\gamma_\mu \star \gamma_\mu = 4$. Generalizing this to $\mathbb{R}^N$ we will have $\gamma_\mu \star \gamma_\mu = 2^n$.

Now we see that there is a single term with no gamma matrices, and there is a single term with all gamma matrices. These numbers 1 correspond to:

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = 1, \quad \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 1. \tag{106}$$

We have further:

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} = 5, \quad \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5, \quad \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10, \quad \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10. \tag{107}$$

Therefore the number of possibly orthogonal terms will be:

$$\sum_{j=0}^{\lfloor 5/2 \rfloor} \begin{pmatrix} 5 \\ j \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = (1/2)2^5. \tag{108}$$

That is in general:

$$\sum_{j=0}^{\lfloor \frac{2n+1}{2} \rfloor} \begin{pmatrix} 2n+1 \\ j \end{pmatrix} = 2^{2n+1}/2 = 2^{2n} = 2^n \times 2^n = 2^{\lfloor \frac{2n+1}{2} \rfloor} \times 2^{\lfloor \frac{2n+1}{2} \rfloor}, \tag{109}$$

which is exactly the number of entries in the $2^{\lfloor \frac{2n+1}{2} \rfloor} \times 2^{\lfloor \frac{2n+1}{2} \rfloor}$ matrix. The idea is thus, that there could be a theorem that the multi-vectors of dimension 0 to $\lfloor \frac{2n+1}{2} \rfloor$ form an orthogonal basis for the $2^{\lfloor \frac{2n+1}{2} \rfloor} \times 2^{\lfloor \frac{2n+1}{2} \rfloor}$ matrices. We can expect that this will work because $\mathbb{1} = \mathbf{e}_{11} + \mathbf{e}_{22}$ and $\lambda_z = \mathbf{e}_{11} - \mathbf{e}_{22}$ are replacing $\mathbf{e}_{11}$ and $\mathbf{e}_{22}$ while $\lambda_x = \mathbf{e}_{12} + \mathbf{e}_{21}$ and $\lambda_y = \mathbf{e}_{12} - \mathbf{e}_{21}$ are replacing $\mathbf{e}_{12}$ and $\mathbf{e}_{21}$. They are filling thus the whole 2×2 square. The same happens now with $\mathbb{1}$ and Λ_z and Λ_x and Λ_y for the 2×2 square of block matrices. And by combining the products up to the multi-vectors of rank 2 we will fill the whole 4×4 square of simple matrices. Adding a rank will fill up the whole 8×8 square, etc... In this reasoning we find again the rank $n = \lfloor (2n+1)/2 \rfloor$.

9.3 Defining the setting for the general step

We start from the representation matrices that occur in the Clifford algebra associated with the group representation theory of the isometries of $\mathbb{R}^{2n+1}$. We know that this algebra contains multi-vectors up to rank n, because the multi-vectors of rank $2n+1-p$ are redundant with those of rank p, for $p \in [\,0,n\,] \cap \mathbb{Z}$. Here the trivial rank 0 corresponds a multi-vector constructed by picking zero vectors from the set of $2n+1$ basis vectors. It corresponds to a scalar quantity. We consider a general multi-vector of rank $p \in [\,1,n\,] \cap \mathbb{N}$:

$$V^{(p)} = \mathbf{e}_{j_1} \wedge \mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_k} \wedge \cdots \wedge \mathbf{e}_{j_{p-1}} \wedge \mathbf{e}_{j_p}, \tag{110}$$

where $\forall p \in [\,1,n\,] \cap \mathbb{N} : j_p \in [\,1,2n+1\,] \cap \mathbb{N}$. The number of different multi-vectors of rank p is:

$$\left(\begin{array}{c} 2n+1 \\ p \end{array} \right) = \left(\begin{array}{c} 2n+1 \\ 2n+1-p \end{array} \right), \tag{111}$$

in agreement with the fact that the multi-vectors of rank p and $2n+1-p$ are equal up to a parity factor $(-1)^P$. We assume also that $V^{(p)}$ has been ordered by transpositions such that:

$$j_1 < j_2 < \cdots < j_k < \cdots < j_{p-1} < j_p. \tag{112}$$

Ordering will just introduce a parity term $(-1)^P$ due to the anti-symmetry. We define further $V^{(0)} = 1$. The corresponding matrices are:

$$\mathbf{V}^{(0)} = \mathbb{1}, \quad \text{and:} \quad (\forall p \in [\,1,n\,] \cap \mathbb{N})\,(\mathbf{V}^{(p)} = \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_k}\cdots\lambda_{j_{p-1}}\lambda_{j_p}). \tag{113}$$

In the extension from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+2}$ and $\mathbb{R}^{2n+3}$, the new representation matrices are given by:

$$\forall \ell \in [1,2n+1] :\ \gamma_\ell = \left[\begin{array}{cc} & \lambda_\ell \\ \lambda_\ell & \end{array} \right], \quad \gamma_{2n+2} = \left[\begin{array}{cc} & -\mathbb{1} \\ \mathbb{1} & \end{array} \right], \quad \gamma_{2n+3} = \left[\begin{array}{cc} \mathbb{1} & \\ & -\mathbb{1} \end{array} \right]. \tag{114}$$

When in the Peano induction scheme we extend the formalism from $\mathbb{R}^{2n+3}$ to $\mathbb{R}^{2n+5}$, the matrices γ_j, $j \in [\,1,2n+3\,] \cap \mathbb{N}$ will then take up the rôle in $\mathbb{R}^{2n+3}$ that was played by λ_ℓ, $\ell \in [\,1,2n+1\,] \cap \mathbb{N}$ in $\mathbb{R}^{2n+1}$. We will have to break down the proof by Peano induction into two steps, because the extension from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+2}$ involves a matrix γ_{2n+2} while the extension from $\mathbb{R}^{2n+2}$ to $\mathbb{R}^{2n+3}$ involves a different matrix γ_{2n+3}.

From $\lambda_1\lambda_2\cdots\lambda_\ell\cdots\lambda_{2n+1} = \mathbb{1}$, it follows that $\gamma_1\gamma_2\cdots\gamma_\ell\cdots\gamma_{2n+1}\gamma_{2n+2}\gamma_{2n+3} = \mathbb{1}$. In fact:

$$\gamma_1\gamma_2\cdots\gamma_\ell\cdots\gamma_{2n+1}\gamma_{2n+2}\gamma_{2n+3} = (\lambda_1\Lambda_x)(\lambda_2\Lambda_x)\cdots(\lambda_\ell\Lambda_x)\cdots(\lambda_{2n+1}\Lambda_x)\Lambda_y\Lambda_z = \lambda_1\lambda_2\cdots\lambda_\ell\cdots\lambda_{2n+1}\mathbb{1} = \mathbb{1}, \tag{115}$$

where we have used $\Lambda_x^{2n+1} = \Lambda_x$ and $\Lambda_x\Lambda_y\Lambda_z = \mathbb{1}$. The fact that the product of all the vector representation matrices γ_ℓ is $\mathbb{1}$ is this way established by Peano induction. In the extension to $\mathbb{R}^{2n+2}$ and $\mathbb{R}^{2n+3}$ the matrix $\mathbf{V}^{(p)}$ is replaced by:

$$\forall p \equiv 1\,(\text{mod } 2):\ \mathbf{W}^{(p)} = \left[\begin{array}{cc} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{array} \right], \quad \forall p \equiv 0\,(\text{mod } 2):\ \mathbf{W}^{(p)} = \left[\begin{array}{cc} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{array} \right]. \tag{116}$$

In the Peano induction scheme it is assumed that $(\forall p \in [\,0,n\,] \cap \mathbb{Z})\,(\forall q \in [\,0,n\,] \cap \mathbb{Z})\,(p \neq q \Rightarrow \mathbf{V}^{(p)} \star \mathbf{V}^{(q)} = 0)$. For $p = q$ it is further assumed that $\mathbf{V}_1^{(p)} \neq \mathbf{V}_2^{(p)} \Rightarrow \mathbf{V}_1^{(p)} \star \mathbf{V}_2^{(p)} = 0$. From this it follows automatically that: $(\forall p \in [\,0,n\,] \cap \mathbb{Z})\,(\forall q \in [\,0,n\,] \cap \mathbb{Z})\,(p \neq q \Rightarrow \mathbf{W}^{(p)} \star \mathbf{W}^{(q)} = 0)$. In fact, when $p-q \equiv 1 \pmod 2$, then $\mathbf{W}^{(p)}$ and $\mathbf{W}^{(q)}$ have no overlap. When $p-q \equiv 0 \pmod 2$, then $\mathbf{W}^{(p)} \star \mathbf{W}^{(q)} = 2(\mathbf{V}^{(p)} \star \mathbf{V}^{(q)}) = 0$. For $p = q$, it follows automatically that: $\mathbf{W}_1^{(p)} \neq \mathbf{W}_2^{(p)} \Rightarrow \mathbf{W}_1^{(p)} \star \mathbf{W}_2^{(p)} = 0$, again because then $\mathbf{W}_1^{(p)} \star \mathbf{W}_2^{(p)} = 2(\mathbf{V}_1^{(p)} \star \mathbf{V}_2^{(p)}) = 0$. In other words, in the extension the orthogonality relations remain preserved. We must now still prove that the new terms we introduce by an extension are all mutually orthogonal and also orthogonal to all the old terms. As we already mentioned, we must break this extension down into two steps $\mathbb{R}^{2n+1} \to \mathbb{R}^{2n+2}$ and $\mathbb{R}^{2n+2} \to \mathbb{R}^{2n+3}$. In all the discussions we will never mention the multi-vector $\mathbb{1}$ of rank zero because its orthogonality to all other multi-vectors is trivial.

9.4 From $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+2}$

Let us first handle the extension $\mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+2}$. We must then consider four different cases.

□ Case 1. First we consider the case that the rank $p < n$ and $p \equiv 1 \pmod 2$. We have then:

$$\mathbf{W}^{(p)} = \begin{bmatrix} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix}. \tag{117}$$

We can now obtain a term $\mathbf{W}^{(p+1)}$ of rank $p+1$ from a term $\mathbf{W}^{(p)}$ of rank p as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix} \begin{bmatrix} & \lambda_{j_{p+1}} \\ \lambda_{j_{p+1}} & \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(p)}\lambda_{j_{p+1}} & \\ & \mathbf{V}^{(p)}\lambda_{j_{p+1}} \end{bmatrix}, \tag{118}$$

where: $j_{p+1} \in [\,1, 2n+1\,] \cap \mathbb{N}$ & $j_{p+1} > j_p$, or as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & -\mathbf{V}^{(p)} \end{bmatrix}. \tag{119}$$

In the first case (Eq. 118), we just generate a term that is not new. As $p+1 \leq n$ we already generated it in $\mathbb{R}^{2n+1}$. All orthogonality relations involving $\mathbf{W}^{(p+1)}$ and other multi-vectors of $\mathbb{R}^{2n+1}$ are then already established by assumption in the Peano induction scheme. In the second case (Eq. 119) we obtain a result that is orthogonal to all multi-vectors we generated in $\mathbb{R}^{2n+1}$ because the matrices corresponding to these multi-vectors can only be of the form:

$$\mathbf{G}_1 = \begin{bmatrix} \mathbf{V}^{(q)} & \\ & \mathbf{V}^{(q)} \end{bmatrix}, \quad \text{or:} \quad \mathbf{G}_2 = \begin{bmatrix} & \mathbf{V}^{(q)} \\ \mathbf{V}^{(q)} & \end{bmatrix}. \tag{120}$$

Now $\mathbf{G}_2 \star \mathbf{W}^{(p+1)} = 0$ due to an absence of overlap and $\mathbf{G}_1 \star \mathbf{W}^{(p+1)} = \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} - \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} = 0$.

□ Case 2. Next we consider the case that the rank $p < n$ and $p \equiv 0 \pmod 2$. The matrix $\mathbf{W}^{(p)}$ has then a different block structure than in Eq.117. It is now block diagonal instead of block off-diagonal:

$$\mathbf{W}^{(p)} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{bmatrix}. \tag{121}$$

We can now obtain a term $\mathbf{W}^{(p+1)}$ of rank $p+1$ from a term $\mathbf{W}^{(p)}$ of rank p as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{bmatrix} \begin{bmatrix} & \lambda_{j_{p+1}} \\ \lambda_{j_{p+1}} & \end{bmatrix} = \begin{bmatrix} & \mathbf{V}^{(p)}\lambda_{j_{p+1}} \\ \mathbf{V}^{(p)}\lambda_{j_{p+1}} & \end{bmatrix}, \tag{122}$$

where $j_{p+1} \in [\,1, 2n+1\,] \cap \mathbb{N}$ & $j_{p+1} > j_p$, or as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \begin{bmatrix} & -\mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix}. \tag{123}$$

In the first case (Eq. 122), we just generate a term that is not new. As $p+1 \leq n$ we already generated it in $\mathbb{R}^{2n+1}$. All orthogonality relations involving $\mathbf{W}^{(p+1)}$ and other multi-vectors of $\mathbb{R}^{2n+1}$ are then already established by assumption in the Peano induction scheme. In the second case (Eq. 123), we obtain a result that is orthogonal to all terms we generated in $\mathbb{R}^{2n+1}$ because these terms can only be of the form, given by Eq. 120. Now $\mathbf{G}_1 \star \mathbf{W}^{(p+1)} = 0$ due to an absence of overlap while $\mathbf{G}_2 \star \mathbf{W}^{(p+1)} = -\mathbf{V}^{(q)} \star \mathbf{V}^{(p)} + \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} = 0$.

□ Case 3. Next we consider the case that the rank $p = n$ and $p \equiv 1 \pmod 2$. What changes with respect to Case 1 is that now:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} & \mathbf{V}^{(n)} \\ \mathbf{V}^{(n)} & \end{bmatrix} \begin{bmatrix} & \lambda_{j_{n+1}} \\ \lambda_{j_{n+1}} & \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(n)}\lambda_{j_{n+1}} & \\ & \mathbf{V}^{(n)}\lambda_{j_{n+1}} \end{bmatrix}, \tag{124}$$

where $j_{n+1} \in [\,1, 2n+1\,] \cap \mathbb{N}$ & $j_{n+1} > j_n$, is no longer a term that already existed in $\mathbb{R}^{2n+1}$. In fact, it is now of rank $n+1$. Now we use the result $\lambda_1\lambda_2\cdots\lambda_\ell\cdots\lambda_{2n+1} = \mathbb{1}$ to replace $\mathbf{V}^{(n)}\lambda_{j_{n+1}}$ by a term $(-1)^P\mathbf{G}^{(n)}$. This term is obtained by removing all $n+1$ factors that occur in $\mathbf{V}^{(n)}\lambda_{j_{n+1}}$ from $\lambda_1\lambda_2\cdots\lambda_\ell\cdots\lambda_{2n+1}$. The result will contain $(2n+1)-(n+1) = n$ factors instead of $n+1$. Here $(-1)^P$ is a parity term. It follows from this that:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} (-1)^P\mathbf{G}^{(n)} & \\ & (-1)^P\mathbf{G}^{(n)} \end{bmatrix} \quad \text{and:} \quad \mathbf{W}^{(n+1)} \star \mathbf{H}^{(n)} = \begin{bmatrix} \mathbf{G}^{(n)} & \\ & \mathbf{G}^{(n)} \end{bmatrix} \star \begin{bmatrix} & \mathbf{G}^{(n)} \\ \mathbf{G}^{(n)} & \end{bmatrix} = 0. \quad (125)$$

Here $\mathbf{H}^{(n)}$ corresponds to $\mathbf{G}^{(n)}$ in the extension from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+2}$ or $\mathbb{R}^{2n+3}$. We can use the same algebra to show that $\mathbf{W}^{(n+1)}$ will also be orthogonal to all other terms $\mathbf{H}_1^{(n)}$ of rank n due to absence of overlap. Furthermore, all different terms $\mathbf{W}^{(n+1)}$ of rank $n+1$ will be mutually orthogonal because the corresponding terms $\mathbf{G}^{(n)}$ are mutually orthogonal. Finally, all other terms $\mathbf{W}^{(q)}$ of rank $q < n$ will be of the forms $\mathbf{G}_1$ or $\mathbf{G}_2$ defined in Eq. 120, and we will have $\mathbf{G}^{(n)} \star \mathbf{V}^{(q)} = 0$, such that $.\mathbf{W}^{(n+1)} \star \mathbf{G}_1 = 0$ and $\mathbf{W}^{(n+1)} \star \mathbf{G}_2 = 0$, either by absence of overlap or due to $\mathbf{G}^{(n)} \star \mathbf{V}^{(q)} = 0$. This way all terms $\mathbf{W}^{(n+1)}$ are mutually orthogonal (when they are different) and orthogonal to all terms of lower rank. The other possibility to obtain a term $\mathbf{W}^{(n+1)}$ of rank $n+1$ is:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} & \mathbf{V}^{(n)} \\ \mathbf{V}^{(n)} & \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(n)} & \\ & -\mathbf{V}^{(n)} \end{bmatrix}. \quad (126)$$

These terms are orthogonal to all other terms of rank $n+1$ discussed in Eq. 125, because $(-1)^P\mathbf{G}^{(n)} \star \mathbf{V}^{(n)} - (-1)^P\mathbf{G}^{(n)} \star \mathbf{V}^{(n)} = 0$. They are also mutually orthogonal because the corresponding terms $\mathbf{V}^{(n)}$ are mutually orthogonal. Finally, they are also orthogonal to terms $\mathbf{W}^{(q)}$ of rank $q \leq n$ either by absence of overlap, or due to the fact that $\mathbf{V}^{(q)} \star \mathbf{V}^{(n)} - \mathbf{V}^{(q)} \star \mathbf{V}^{(n)} = 0$.

□ Case 4. Next we must consider the case that the rank $p = n$ and $p \equiv 0 \pmod 2$. What changes with respect to Case 2 is that now:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} \mathbf{V}^{(n)} & \\ & \mathbf{V}^{(n)} \end{bmatrix} \begin{bmatrix} & \lambda_{j_{n+1}} \\ \lambda_{j_{n+1}} & \end{bmatrix} = \begin{bmatrix} & \mathbf{V}^{(n)}\lambda_{j_{n+1}} \\ \mathbf{V}^{(n)}\lambda_{j_{n+1}} & \end{bmatrix}, \quad (127)$$

where $j_{n+1} \in [\,1, 2n+1\,] \cap \mathbb{N}$ & $j_{n+1} > j_n$, is no longer a term that already existed in $\mathbb{R}^{2n+1}$. In fact, it is now of rank $n+1$. Again we use the result $\lambda_1\lambda_2\cdots\lambda_\ell\cdots\lambda_{2n+1} = \mathbb{1}$ to replace $\mathbf{V}^{(n)}\lambda_{j_{n+1}}$ by a term $(-1)^P\mathbf{G}^{(n)}$. This term is obtained by removing all $n+1$ factors that occur in $\mathbf{V}^{(n)}\lambda_{j_{n+1}}$ from $\lambda_1\lambda_2\cdots\lambda_\ell\cdots\lambda_{2n+1}$. The result will contain $(2n+1) - (n+1) = n$ factors instead of $n+1$. Here $(-1)^P$ is a parity term. It follows from this that:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} & (-1)^P\mathbf{G}^{(n)} \\ (-1)^P\mathbf{G}^{(n)} & \end{bmatrix} \quad \text{and:} \quad \mathbf{W}^{(n+1)} \star \mathbf{H}^{(n)} = \begin{bmatrix} & \mathbf{G}^{(n)} \\ \mathbf{G}^{(n)} & \end{bmatrix} \star \begin{bmatrix} \mathbf{G}^{(n)} & \\ & \mathbf{G}^{(n)} \end{bmatrix} = 0. \quad (128)$$

Here $\mathbf{H}^{(n)}$ corresponds to $\mathbf{G}^{(n)}$ in the extension from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+2}$ or $\mathbb{R}^{2n+3}$. We can use the same algebra to show that $\mathbf{W}^{(n+1)}$ will also be orthogonal to all other terms $\mathbf{H}_1^{(n)}$ of rank n due to absence of overlap. Furthermore, all different terms $\mathbf{W}^{(n+1)}$ of rank $n+1$ will be mutually orthogonal because the corresponding terms $\mathbf{G}^{(n)}$ are mutually orthogonal. Finally, all other terms $\mathbf{W}^{(q)}$ of rank $q < n$ will be of the form $\mathbf{G}_1$ or $\mathbf{G}_2$, and we will have $\mathbf{G}^{(n)} \star \mathbf{V}^{(q)} = 0$, such that $.\mathbf{W}^{(n+1)} \star \mathbf{G}_1 = 0$ and $\mathbf{W}^{(n+1)} \star \mathbf{G}_2 = 0$, either by absence of overlap or due to $\mathbf{G}^{(n)} \star \mathbf{V}^{(q)} = 0$. This way all terms $\mathbf{W}^{(n+1)}$ are mutually orthogonal (when they are different) and orthogonal to all terms of lower rank. The other possibility to obtain a term $\mathbf{W}^{(n+1)}$ of rank $n+1$ is:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} \mathbf{V}^{(n)} & \\ & \mathbf{V}^{(n)} \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \begin{bmatrix} & \mathbf{V}^{(n)} \\ -\mathbf{V}^{(n)} & \end{bmatrix}. \quad (129)$$

These terms are orthogonal to all other terms of rank $n+1$ discussed in Eq. 128, because $(-1)^P\mathbf{G}^{(n)} \star \mathbf{V}^{(n)} - (-1)^P\mathbf{G}^{(n)} \star \mathbf{V}^{(n)} = 0$. They are also mutually orthogonal because the corresponding terms $\mathbf{V}^{(n)}$ are mutually orthogonal. Finally, they are also orthogonal to terms $\mathbf{W}^{(q)}$ of rank $q \leq n$ either by absence of overlap, or due to the fact that $\mathbf{V}^{(q)} \star \mathbf{V}^{(n)} - \mathbf{V}^{(q)} \star \mathbf{V}^{(n)} = 0$.

9.5 From $\mathbb{R}^{2n+2}$ to $\mathbb{R}^{2n+3}$

Let us now handle the extension $\mathbb{R}^{2n+2} \to \mathbb{R}^{2n+3}$. We must then consider several different cases. First of all, the maximum rank p of a matrix $\mathbf{W}^{(p)}$ in $\mathbb{R}^{2n+2}$ can now be $n+1$. Contrary to what happened in Subsection 9.4, the two blocks in a matrix $\mathbf{W}^{(p)}$ can now also have opposite signs. This happens then because $\mathbf{W}^{(p)}$ contains a factor γ_{2n+2}. As both γ_ℓ and γ_{2n+2} are block off-diagonal, the discussion about the block structure of $\mathbf{W}^{(p)}$ in terms of $p \equiv 1 \pmod$

2) or $p \equiv 0 \pmod 2$ remains the same. We will first consider the cases where $\mathbf{W}^{(p)}$ does not contain γ_{2n+2}.

□ Case 5. First we consider the case that the rank $p < n+1$ and $p \equiv 1 \pmod 2$, while the multi-vector dies not contain a factor γ_{2n+2}. We have then:

$$\mathbf{W}^{(p)} = \begin{bmatrix} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix}. \tag{130}$$

We can now obtain a term $\mathbf{W}^{(p+1)}$ of rank $p+1$ from a term $\mathbf{W}^{(p)}$ of rank p as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix} \begin{bmatrix} & \lambda_{j_{p+1}} \\ \lambda_{j_{p+1}} & \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(p)}\lambda_{j_{p+1}} & \\ & \mathbf{V}^{(p)}\lambda_{j_{p+1}} \end{bmatrix}, \tag{131}$$

where: $j_{p+1} \in [\,1, 2n+1\,] \cap \mathbb{N}$ & $j_{p+1} > j_p$, or as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & -\mathbf{V}^{(p)} \end{bmatrix}. \tag{132}$$

or as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix} \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} = \begin{bmatrix} & -\mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix}. \tag{133}$$

In the first case (Eq. 131), we just generate a term that is not new. As $p+1 \leq n+1$ we already generated it in $\mathbb{R}^{2n+2}$. All orthogonality relations involving $\mathbf{W}^{(p+1)}$ and other multi-vectors of $\mathbb{R}^{2n+2}$ are then already established by assumption in the Peano induction scheme. In the second case (Eq. 132), we obtain a result that is orthogonal to all multi-vectors we generated in $\mathbb{R}^{2n+2}$ because the matrices corresponding to these multi-vectors can only be of the forms $\mathbf{G}_1$ or $\mathbf{G}_1$ defined in Eq. 120. Now $\mathbf{G}_2 \star \mathbf{W}^{(p+1)} = 0$ due to absence of overlap and $\mathbf{G}_1 \star \mathbf{W}^{(p+1)} = \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} - \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} = 0$. In the third case (Eq. 133), we obtain a result that is orthogonal to all multi-vectors we generated in $\mathbb{R}^{2n+2}$ because the matrices corresponding to these multi-vectors can only be of the form given in Eq. 120. Now $\mathbf{G}_1 \star \mathbf{W}^{(p+1)} = 0$ due to absence of overlap and $\mathbf{G}_2 \star \mathbf{W}^{(p+1)} = \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} - \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} = 0$.

□ Case 6. Next we consider the case that the rank $p < n+1$ and $p \equiv 0 \pmod 2$, while the multi-vector dies not contain a factor γ_{2n+2}.The matrix $\mathbf{W}^{(p)}$ has then a different block structure than in Eq.130. It is now block diagonal instead of block off-diagonal:

$$\mathbf{W}^{(p)} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{bmatrix}. \tag{134}$$

We can now obtain a term $\mathbf{W}^{(p+1)}$ of rank $p+1$ from a term $\mathbf{W}^{(p)}$ of rank p as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{bmatrix} \begin{bmatrix} & \lambda_{j_{p+1}} \\ \lambda_{j_{p+1}} & \end{bmatrix} = \begin{bmatrix} & \mathbf{V}^{(p)}\lambda_{j_{p+1}} \\ \mathbf{V}^{(p)}\lambda_{j_{p+1}} & \end{bmatrix}, \tag{135}$$

where $j_{p+1} \in [\,1, 2n+1\,] \cap \mathbb{N}$ & $j_{p+1} > j_p$, or as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix} = \begin{bmatrix} & -\mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix}. \tag{136}$$

or as:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & -\mathbf{V}^{(p)} \end{bmatrix}. \tag{137}$$

In the first case (Eq. 135), we just generate a term that is not new. As $p+1 \leq n+1$ we already generated it in $\mathbb{R}^{2n+2}$. All orthogonality relations involving $\mathbf{W}^{(p+1)}$ and other multi-vectors of $\mathbb{R}^{2n+2}$ are then already established by assumption in the Peano induction scheme. In the second case (Eq. 136), we obtain a result that is orthogonal to all multi-vectors we generated in $\mathbb{R}^{2n+2}$ because the matrices corresponding to these multi-vectors can only be of the form given by Eq. 120. Now $\mathbf{G}_1 \star \mathbf{W}^{(p+1)} = 0$ due to absence of overlap and $\mathbf{G}_2 \star \mathbf{W}^{(p+1)} = \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} - \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} = 0$. In the third case (Eq. 137), we obtain a result that is orthogonal to all multi-vectors we generated in $\mathbb{R}^{2n+2}$ because the matrices corresponding to these multi-vectors can only be of the form given in Eq. 120. Now $\mathbf{G}_2 \star \mathbf{W}^{(p+1)} = 0$ due to absence of overlap and $\mathbf{G}_1 \star \mathbf{W}^{(p+1)} = \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} - \mathbf{V}^{(q)} \star \mathbf{V}^{(p)} = 0$.

□ Case 7. Next we might think of considering that the rank $p = n+1$ and $p \equiv 1 \pmod 2$. We have then $n \equiv 0 \pmod 2$. And $\mathbf{W}^{(p)}$ would then be given by:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} & \mathbf{V}^{(n+1)} \\ \mathbf{V}^{(n+1)} & \end{bmatrix}. \tag{138}$$

We could then obtain:

$$\mathbf{W}^{(n+2)} = \begin{bmatrix} & \mathbf{V}^{(n+1)} \\ \mathbf{V}^{(n+1)} & \end{bmatrix} \begin{bmatrix} & \lambda_{n_{j+2}} \\ \lambda_{n_{j+2}} & \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(n+1)}\lambda_{n_{j+2}} & \\ & \mathbf{V}^{(n+1)}\lambda_{n_{j+2}} \end{bmatrix}. \tag{139}$$

Here $\mathbf{W}^{(n+1)}$ is itself obtained as:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} \mathbf{V}^{(n)} & \\ & \mathbf{V}^{(n)} \end{bmatrix} \begin{bmatrix} & \lambda_{n_{j+1}} \\ \lambda_{n_{j+1}} & \end{bmatrix} = \begin{bmatrix} & \mathbf{V}^{(n)}\lambda_{n_{j+1}} \\ \mathbf{V}^{(n)}\lambda_{n_{j+1}} & \end{bmatrix}, \tag{140}$$

such that:

$$\mathbf{W}^{(n+2)} = \begin{bmatrix} \mathbf{V}^{(n)} & \\ & \mathbf{V}^{(n)} \end{bmatrix} \begin{bmatrix} & \lambda_{n_{j+1}} \\ \lambda_{n_{j+1}} & \end{bmatrix} \begin{bmatrix} & \lambda_{n_{j+2}} \\ \lambda_{n_{j+2}} & \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(n)}\lambda_{n_{j+1}}\lambda_{n_{j+2}} & \\ & \mathbf{V}^{(n)}\lambda_{n_{j+1}}\lambda_{n_{j+2}} \end{bmatrix}. \tag{141}$$

But these cases do not exist. We have shown that in the transition from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+2}$, $\mathbf{W}^{(n+1)}$ must be reduced to:

$$\begin{bmatrix} & \mathbf{G}^{(n)} \\ \mathbf{G}^{(n)} & \end{bmatrix}. \tag{142}$$

This way we are falling back onto Case 3. We might think that we can get then $\mathbf{W}^{(n+2)}$ as a multi-vector of rank $n+1$:

$$\begin{bmatrix} \mathbf{G}^{(n)}\lambda_{n_{j+2}} & \\ & \mathbf{G}^{(n)}\lambda_{n_{j+2}} \end{bmatrix}. \tag{143}$$

This would then again be reduced to a multi-vector of rank n. But this is wrong because it overlooks some details in the reduction schemes. As a matter of fact, by an alternative reasoning based on Eq. 141 we see that the reduction scheme will yield in reality a multi-vector of rank $2n+1-(n+2) = n-1$. This, rather than n is the correct value for the rank. Let us show how this works on an exemple. Imagine that we work in $\mathbb{R}^7$ and that we start from $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. In a first step we add $\mathbf{e}_4$. But $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$ must now be reduced to $\mathbf{e}_5 \wedge \mathbf{e}_6 \wedge \mathbf{e}_7$. When we further add $\mathbf{e}_5$ we can argue that $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5$ must be reduced to $\mathbf{e}_6 \wedge \mathbf{e}_7$. But we can also reason on $\mathbf{e}_5 \wedge (\mathbf{e}_5 \wedge \mathbf{e}_6 \wedge \mathbf{e}_7)$ which will also yield $\mathbf{e}_6 \wedge \mathbf{e}_7$. We see that $\mathbf{e}_5$ is eliminated in both reasonings. In one reasoning by a reduction of the rank from $n+2$ to $n-1$. In the other reasoning because $\mathbf{e}_5$ occurs twice, such that we do not extend the rank again from n to $n+1$ but further reduce it from n to $n-1$. Now reasoning on $\mathbf{e}_5 \wedge \mathbf{e}_6 \wedge \mathbf{e}_7$ we could also think of adding $\mathbf{e}_1$ rather than $\mathbf{e}_5$. This would yield $\mathbf{e}_1 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 \wedge \mathbf{e}_7$ which must be reduced and yields then $\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$. Again, this yields the same result as reasoning on $\mathbf{e}_1 \wedge (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4)$. In other words, we really obtain a multi-vector of rank $n-1$ rather than n and the calculations are self-consistent.

However, there are now extra possibilities to obtain multi-vectors. First of all, we can obtain now a multi-vector of rank $n+1$ from the multi-vector whose rank has been reduced from $n+1$ to n:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} & \mathbf{G}^{(n)} \\ \mathbf{G}^{(n)} & \end{bmatrix} \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} = \begin{bmatrix} & -\mathbf{G}^{(n)} \\ \mathbf{G}^{(n)} & \end{bmatrix}, \tag{144}$$

These terms are orthogonal to all other terms of rank n:

$$\mathbf{W}^{(n)} = \begin{bmatrix} & \mathbf{H}^{(n)} \\ \mathbf{H}^{(n)} & \end{bmatrix}. \tag{145}$$

discussed in Eq. 142, because $\mathbf{G}^{(n)} \star \mathbf{H}^{(n)} - \mathbf{G}^{(n)} \star \mathbf{H}^{(n)} = 0$. They are also mutually orthogonal because the corresponding terms $\mathbf{G}^{(n)}$ are mutually orthogonal. Finally, they are also orthogonal to terms $\mathbf{W}^{(q)}$ of rank $q \leq n$ either by absence of overlap, or due to the fact that $\mathbf{V}^{(q)} \star \mathbf{G}^{(n)} - \mathbf{V}^{(q)} \star \mathbf{G}^{(n)} = 0$. The second extra case is:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} & \mathbf{G}^{(n)} \\ \mathbf{G}^{(n)} & \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ +\mathbb{1} & \end{bmatrix} = \begin{bmatrix} \mathbf{G}^{(n)} & \\ & -\mathbf{G}^{(n)} \end{bmatrix}. \tag{146}$$

These terms are orthogonal to all other terms of rank n given by Eq. 145 or discussed in Eq. 142, due to absence of overlap. They are also mutually orthogonal because the corresponding terms $\mathbf{G}^{(n)}$ are mutually orthogonal. Finally, they are also orthogonal to terms $\mathbf{W}^{(q)}$ of rank $q \leq n$ either by absence of overlap, or due to the fact that $\mathbf{V}^{(q)} \star \mathbf{G}^{(n)} - \mathbf{V}^{(q)} \star \mathbf{G}^{(n)} = 0$.

□ Case 8. Next we might think of considering that the rank $p = n + 1$ and $p \equiv 0 \pmod 2$. Just as in Case 7 this leads to considering expressions that do not exist. However, there now also extra cases. A first one is:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} \mathbf{G}^{(n)} & \\ & \mathbf{G}^{(n)} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbf{G}^{(n)} & \\ & -\mathbf{G}^{(n)} \end{bmatrix}. \tag{147}$$

These terms are orthogonal to all other terms of rank n:

$$\mathbf{W}^{(n)} = \begin{bmatrix} \mathbf{H}^{(n)} & \\ & \mathbf{H}^{(n)} \end{bmatrix}. \tag{148}$$

because $\mathbf{G}^{(n)} \star \mathbf{H}^{(n)} - \mathbf{G}^{(n)} \star \mathbf{H}^{(n)} = 0$. They are also mutually orthogonal because the corresponding terms $\mathbf{G}^{(n)}$ are mutually orthogonal. Finally, they are also orthogonal to terms $\mathbf{W}^{(q)}$ of rank $q \leq n$ either by absence of overlap, or due to the fact that $\mathbf{V}^{(q)} \star \mathbf{G}^{(n)} - \mathbf{V}^{(q)} \star \mathbf{G}^{(n)} = 0$. The second extra case is:

$$\mathbf{W}^{(n+1)} = \begin{bmatrix} \mathbf{G}^{(n)} & \\ & \mathbf{G}^{(n)} \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ +\mathbb{1} & \end{bmatrix} = \begin{bmatrix} & -\mathbf{G}^{(n)} \\ \mathbf{G}^{(n)} & \end{bmatrix}. \tag{149}$$

These terms are orthogonal to all other terms of rank n given by Eq. 148, due to absence of overlap. They are also mutually orthogonal because the corresponding terms $\mathbf{G}^{(n)}$ are mutually orthogonal. Finally, they are also orthogonal to terms $\mathbf{W}^{(q)}$ of rank $q \leq n$ either by absence of overlap, or due to the fact that $\mathbf{V}^{(q)} \star \mathbf{G}^{(n)} - \mathbf{V}^{(q)} \star \mathbf{G}^{(n)} = 0$.

We must finally treat the cases where $\mathbf{W}^{(p)}$ contains γ_{2n+2}. In all cases Case 1-Case 8 we had to consider sub-cases. We could multiply $\mathbf{W}^{(p)}$ by terms of the type $\Lambda_x \lambda$, γ_{2n+2} or by γ_{2n+3}. Here we no longer have to consider multiplying with a term of the type $\Lambda_x \lambda$, because after reordering that will be equivalent with a previous case where we multiplied by γ_{2n+2}. We also do not have to consider multiplying with a term γ_{2n+2} because that term is already present. The only case we must consider is therefore multiplying by γ_{2n+3}.

□ Case 9. First we consider the case that the effective rank $p < n + 1$ and $p \equiv 0 \pmod 2$. We have then:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} & -\mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix}. \tag{150}$$

The condition $p < n + 1$ stems from the fact that in $\mathbb{R}^{2n+1}$ the maximum rank of the multi-vector was n and we have multiplied this multi-vector by γ_{2n+2}, which does not change its effective rank. By the effective rank p we mean here the number of matrices λ_ℓ we have used in the construction of the multi-vector. We note this multi-vector as $\mathbf{W}^{(p+1)}$ to signal that we have multiplied $p + 1$ matrices, such that its true rank is $p + 1$. We obtain then:

$$\mathbf{W}^{(p+2)} = \begin{bmatrix} & -\mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix} \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} = \begin{bmatrix} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix}. \tag{151}$$

As $p \equiv 0 \pmod 2$, the blocks $\mathbf{V}^{(p)}$ of rank p in $\mathbf{W}^{(p+2)}$ do not occur in the natural off-diagonal positions for a multi-vector of rank p. The multi-vector $\mathbf{W}^{(p+2)}$ of effective rank p is therefore orthogonal to all multi-vectors of rank p due to absence of overlap. It is also orthogonal to all other multi-vectors $\mathbf{W}^{(q)}$ of rank q because $\mathbf{V}^{(p)} \star \mathbf{V}^{(q)} = 0$ or due to absence of overlap.

□ Case 10. Finally we consider the case that the effective rank $p < n + 1$ and $p \equiv 1 \pmod 2$. We have then:

$$\mathbf{W}^{(p+1)} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & -\mathbf{V}^{(p)} \end{bmatrix} = \begin{bmatrix} & \mathbf{V}^{(p)} \\ \mathbf{V}^{(p)} & \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix}. \tag{152}$$

For the terminology, the condition $p < n+1$ and the notation $\mathbf{W}^{(p+1)}$ the same remarks apply as in Case 9. We obtain then:

$$\mathbf{W}^{(p+2)} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & -\mathbf{V}^{(p)} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{(p)} & \\ & \mathbf{V}^{(p)} \end{bmatrix}. \tag{153}$$

As $p \equiv 1$ (mod2), the blocks $\mathbf{V}^{(p)}$ of rank p in $\mathbf{W}^{(p+2)}$ do not occur in the natural off-diagonal positions for a multi-vector of rank p. The multi-vector $\mathbf{W}^{(p+2)}$ of effective rank p is therefore orthogonal to all multi-vectors of rank p due to absence of overlap. It is also orthogonal to all other multi-vectors $\mathbf{W}^{(q)}$ of rank q because $\mathbf{V}^{(p)} \star \mathbf{V}^{(q)} = 0$ or due to absence of overlap. This concludes the proof.

References

1. Coddens, G.: The geometrical meaning of spinors lights the way to make sense of quantum mechanics, Symmetry **13**, 659 (2021).
2. Coddens, G.: Spinors for everyone, https://hal-cea.archives-ouvertes.fr/cea-01572342.
3. Coddens, G.: Why spinors do not form a vector space, https://hal.archives-ouvertes.fr/hal-03289828.
4. Coddens, G., in *From Spinors to Quantum Mechanics*, Imperial College Press: London, UK, (2015).
5. Misner, C.W., Thorne, K.S., and Wheeler, J.A., in *Gravitation*, Freeman, San Francisco (1970).
6. Schuller, F.: Differential structures: the pivotal concept of tangent vector spaces - Lec 09, https://www.youtube.com/watch?v=UPGoXBfm6Js.
7. Coddens, G.: Quantum mechanics with only real numbers, https://hal.archives-ouvertes.fr/hal-03559591v2.
8. Lounesto, P., in *Clifford Algebras and Spinors*, Cambridge University Press, Cambridge, UK (2009).
9. Hestenes, D.: Zitterbewegung in Radiative Processes. In The Electron, New Theory and Experiment; Hestenes, D., Weingartshofer, A., Eds.; Fundamental Theories of Physics; Springer, Berlin/Heidelberg, Germany, (1991), pp. 21-36.
10. Cartan, E., in *The Theory of Spinors*, Dover, New York, NY, USA, (1981).