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## Soft elastomers: a playground for guided waves

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1           This paper is part of a special issue on Education in Acoustics.

2           Mechanical waves propagating in soft materials play an important role in physiology.

3           They can be natural such as the cochlear wave in the inner ear of mammals, or

4           controlled such as in elastography in the context of medical imaging. In a recent

5           study<sup>1</sup>, we implemented an experimental tabletop platform that allows direct obser-

6           vation of in-plane guided waves in a soft strip. Here, a detailed description of the

7           set-up and signal processing steps are presented as well as the theoretical framework

8           supporting them. One motivation is to propose a tutorial experiment for visualizing

9           the propagation of guided elastic waves. Last, the versatility of the experimental

10          platform is exploited to illustrate experimentally original features of wave physics

11          such as backward modes, stationary modes and Dirac cones.

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12 **I. INTRODUCTION**

13 The mechanical behaviour of soft materials plays a crucial role in various physiological  
14 processes<sup>2</sup>. For example, the impact of the local stiffness of tissues during their develop-  
15 ment<sup>3</sup>, the stiffening of a tumor cell<sup>4</sup> or the non-linear softening of arteria<sup>5</sup> are customary  
16 mechanisms still under investigation. Most biological tissues are *soft* and nearly *incom-*  
17 *pressible*. Mimicking them requires to fulfil these two mechanical properties. In that regard,  
18 elastomers are interesting candidates. Indeed, they cover a wide range of mechanical prop-  
19 erties and they can be molded into an infinite variety of shapes. In plastic surgery, silicone  
20 rubber have been adopted to reproduce the shapes and mechanical properties of breasts,  
21 lips or noses. In the cinema industry, they have become a standard to conceive skin-masks.  
22 Nowadays, silicone elastomers seem to be promising materials to build artificial organs (such  
23 as vocal folds<sup>6</sup> or hearts<sup>7</sup>), soft robots<sup>8,9</sup> or even baromorph materials<sup>10,11</sup>.

24 Due to their nearly-incompressible nature, these soft materials present interesting dy-  
25 namical properties involved in the propagation of elastic waves: the longitudinal waves are  
26 several orders of magnitude faster than their transverse counterpart ( $V_L \gg V_T$ ). This speci-  
27 ficity has enabled the development of transient elastography<sup>12</sup> which is now clinically used  
28 for liver cirrhosis<sup>13</sup> or tumor detection<sup>14</sup>. However, elastography is not quantitative when  
29 it deals with narrow targets such as artery walls<sup>15-17</sup>, the myocardium<sup>18,19</sup>, Achilles' ten-  
30 don<sup>20</sup> or even biofilms<sup>21</sup>. In these geometries, the edges induce guiding phenomena, thus  
31 resulting in a different apparent wave velocity. Guided elastic waves are also naturally in-  
32 volved in physiological processes. At the cellular scale, pressure pulses are observed in lipid

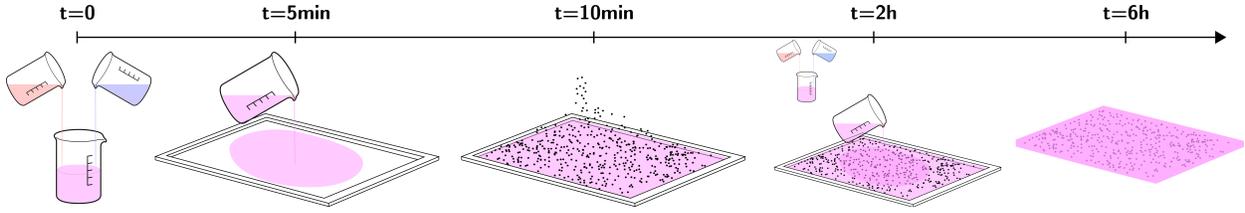


FIG. 1. **Ecoflex<sup>®</sup> sample preparation** – At time  $t = 0$ , the monomer and its cross-linking agent are mixed in equal proportions and a first layer is poured in the sample mould. At  $t = 10$  min, the sample is sprinkled with black carbon grains dedicated to the displacement tracking. At  $t = 2$  hours, a second layer is poured and cures for 6 hours until complete cross-linking.

33 monolayers<sup>22</sup>, and at the macroscopic scale the vocal cords are the support of stationary  
 34 waves<sup>23</sup>. Another compelling example is the sound transduction operated by the inner ear  
 35 of mammals: the *cochlear wave* is a guided mechanical wave that travels along the basilar  
 36 membrane<sup>24,25</sup>.

37 Although, guiding is a universal wave phenomenon, the case of elastic waves is particularly  
 38 fascinating: up to three different polarizations can couple at each reflection<sup>26</sup> and at least two  
 39 distinct wave velocities are involved. Even in a geometry as simple as a plate, elastic guided  
 40 waves present original properties. These waves have been extensively studied, especially  
 41 for non-destructive testing applications<sup>27,28</sup>. Under certain conditions, they display unique  
 42 features such as negative phase velocities<sup>29,30</sup> or zero group velocity (ZGV)<sup>31–34</sup>.

43 This article presents a detailed experimental and theoretical framework for the investi-  
 44 gation of the in-plane motion in soft waveguides. First, an experimental platform designed  
 45 to track the in-plane displacement of a thin plate is proposed. The corresponding theoret-  
 46 ical background (Rayleigh-Lamb equation) is exposed. Then, an equivalence between Lamb

47 modes and the in-plane motion of a thin strip is made. The strip configuration is investi-  
48 gated experimentally as well. Unique wave features such as a backward mode, a zero group  
49 velocity point (ZGV) and a Dirac cone in the  $k \rightarrow 0$  limit are reported<sup>1</sup>. As such, the soft  
50 strip appears as an appropriate tutorial configuration to expose the richness of linear elas-  
51 todynamics. Finally, it is demonstrated how the mode chirality can be exploited to perform  
52 selective excitation.

## 53 II. LOW FREQUENCY IN-PLANE GUIDED WAVES IN A SOFT PLATE

54 This section examines the vibration of a thin plate made of a nearly incompressible mate-  
55 rial. The experimental procedure is detailed and the measured in-plane fields are presented.  
56 Then, the theory supporting these measurements is provided, recalling how the reflections  
57 of bulk elastic waves at a free interface lead to the emergence of shear-horizontal guided  
58 waves and Lamb waves.

### 59 A. Experiments

60 The soft plate preparation and the experimental platform is described. Then, the stro-  
61 boscopic image acquisition and post-processing operations are explained and the resulting  
62 in-plane wave-fields are discussed.

## 63 **1. Sample preparation**

64 All along this article, the selected soft elastomer is the silicone rubber Smooth-On  
65 Ecoflex<sup>®</sup> 00-30, a material that has been widely used in academics in the last few years.  
66 As illustrated in figure 1, the rubber is obtained by mixing a monomer (A) and its cross-  
67 linking agent (B). The liquid can optionally be vacuumed for air bubbles removal. Next,  
68 the mix is poured onto a mould, here consisting of a flat surface with rigid walls forming  
69 a 60-centimeter-side square. For a 3-millimeter-thick Ecoflex<sup>®</sup> plate, one roughly needs  
70 500 mL of each liquid. The mixture is then left for curing at room temperature for several  
71 hours until a translucent soft material is obtained. Here, the monomer and its cross-linking  
72 agent are mixed in equal quantities. Finally, the shear modulus  $\mu$  of the obtained elastomer  
73 is approximately 25 kPa.

74 Anticipating ulterior image processing operations, dark pigments are seeded on the poly-  
75 mer during the curing stage. A good contrast is obtained by using small black carbon powder  
76 from a local art shop. The seeding operation can be performed after pouring half of the  
77 total volume ( $t = 10$  minutes) and before pouring the other half (Fig. 1). In the end, one  
78 gets a single layer of pigment located halfway through the plate. In this study, the grain  
79 density is approximately of one grain per square millimeter.

## 80 **2. Experimental setup**

81 The experiment consists in shaking the plate and imaging its in-plane motion. To this  
82 end, the soft plate is clamped at its top and bottom extremities into a metallic structure

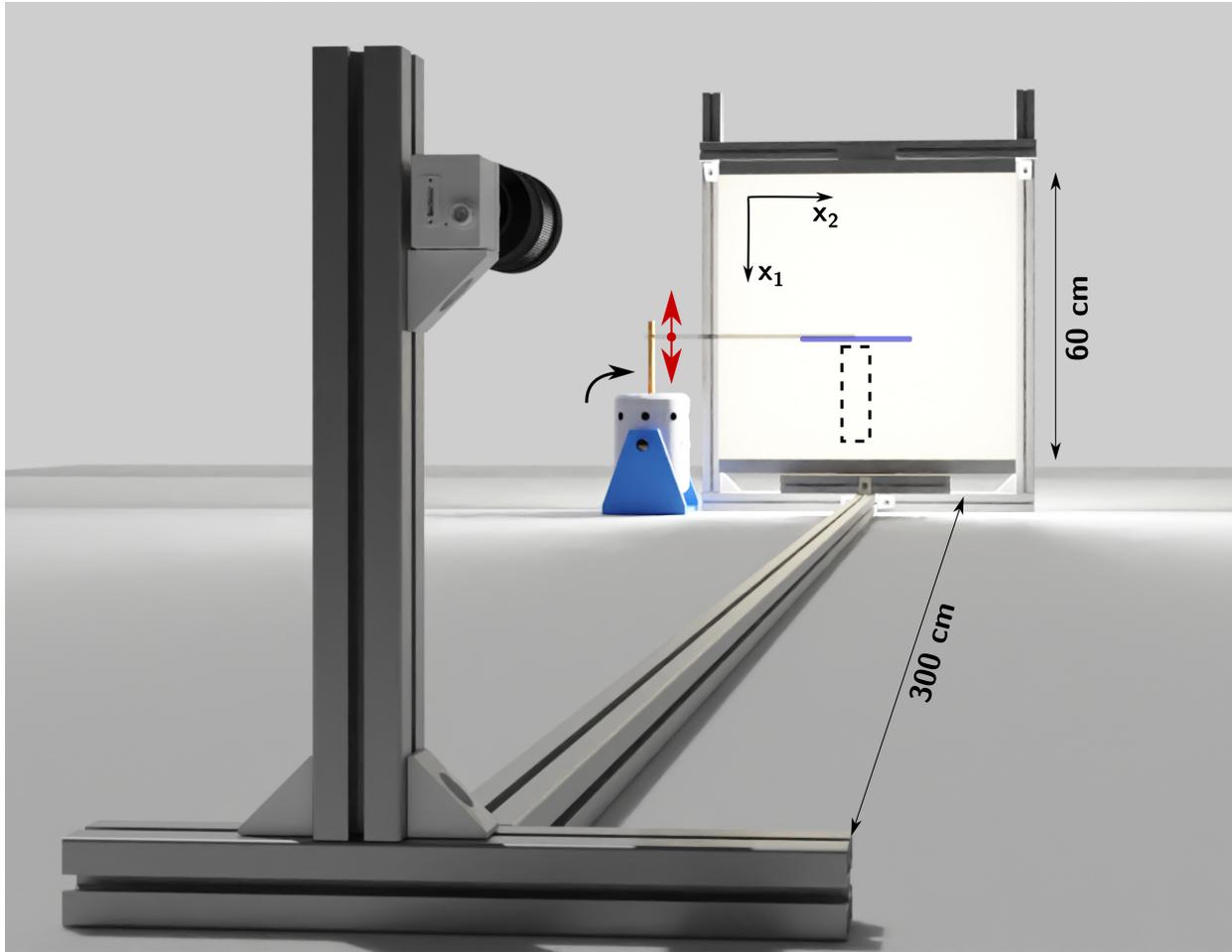


FIG. 2. **Experimental set-up using a line source** – A thin plate of Ecoflex<sup>®</sup> with dimensions 60 cm x 60 cm x 3 mm is clamped to an adjustable frame on its top and bottom edges, and held in a vertical position. Vibrations are generated by a shaker driven monochromatically. Both the vibration direction and the source orientation can be adjusted in order to excite different polarizations. The experiment is recorded using a CCD camera located 3 m away from the plate.

83 (Fig. 2) which dimensions can be adjusted in order to avoid static tension, except from  
 84 gravity.

85 The excitation is performed by a shaker (Tira Vib 51120), driven by an external arbitrary  
 86 wave generator (Keysight AWG 33220) which is itself connected to a power amplifier (Tira

87 Analog Amplifier BAA 500). Typical excitation frequencies span from 1 to 300 Hz. The  
 88 shaker is connected to a plastic clamp holding a pair of aluminum rods placed on both sides  
 89 of the soft plate. The two bars pinch the plate over a 30 cm length ensuring the generation  
 90 of plane-like waves. The shaker and the pinching rods can be rotated to promote specific  
 91 polarizations. The set-up essentially captures displacements parallel to the plate surface. As  
 92 a consequence, it is crucial to carefully align the vibration axis of the shaker with the plate  
 93 to avoid spurious out-of-plane contributions. The motion is captured by a CCD camera  
 94 (Basler acA4112-20um) with a 4112x3008-pixels sensor (Fig. 2). Note that it is necessary  
 95 to use a global shutter: all pixels are exposed simultaneously and capture a full snapshot  
 96 of the scene. A 85-mm zoom lens mounted on the camera and placed 3 meters away from  
 97 the object provides a clean field of view of roughly 30 centimeters wide square. Narrow  
 98 angle lenses drastically reduce optical distortions. For an optimal contrast, the system is  
 99 back-lighted thanks to a wide LED panel placed behind the plate.

### 100 3. *Monochromatic excitation and stroboscopy*

101 Given the chosen region of interest, the maximum acquisition frame rate of the camera  
 102 is of roughly 130 Hz. This means that Shannon's criterion is not fulfilled for frequencies  
 103 higher than 65 Hz. However, in the linear regime, there is no need for a higher speed camera  
 104 since stroboscopic effect can be exploited. To that end, the acquisition period of the camera  
 105  $T_{\text{cam}}$  is set slightly greater than the excitation period  $T_{\text{excitation}}$ , *i.e.*  $T_{\text{cam}} = T_{\text{excitation}} + \delta t$ .  
 106 Between two successive snapshots, the field undergoes more than a full oscillation period.  
 107 Yet, the accumulated phase shift  $2\pi\delta t/T_{\text{excitation}}$  remains small. The final movie provides the

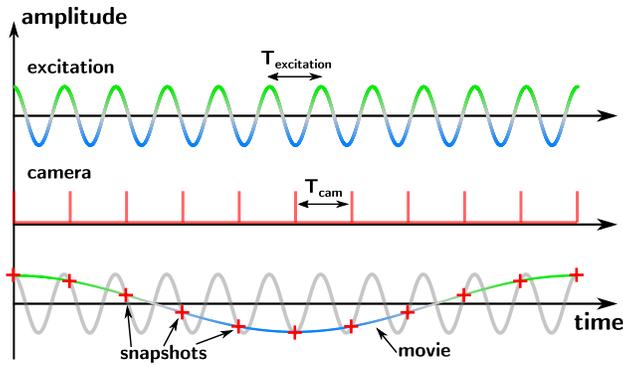


FIG. 3. **Principle of the stroboscopic imaging** – As the recording framerate is lower than the excitation frequency, one full cycle is reconstructed from the measurements, depicted by the red crosses, taken over several cycles of excitation. The sampling rate has to be precisely defined with respect to the driving frequency.

108 illusion than the successive snapshots belong to a single wave period (sketch on figure 3).

109 We refer to this quantity as the pseudo-period.

110 For the following post-processing steps, it is preferable to work with a given amount of  
 111 images per movie. The measurements are performed setting this quantity to  $N = 60$  frames  
 112 over one pseudo-period. This means that the acquisition frame rate has to be determined  
 113 for each different excitation frequency. If the maximum frame rate of the camera is too  
 114 low, one can always reduce the sampling frequency by waiting for several excitation periods  
 115 between successive camera triggers. For example, at 100 Hz, an acquisition sampling rate  
 116 of precisely 24.8963 Hz would yield to 60 frames regularly spaced within one pseudo-period  
 117 (the 61st should be the same as the first image), and successive shots occur roughly every  
 118 4 periods. Note that the exposure time of the camera should always remain much smaller  
 119 than the excitation period. The image would be blurred otherwise. Our measurements are

120 performed with a typical exposure time of  $150 \mu\text{s}$ . The image quality is seriously hampered  
 121 above approximately 300 Hz.

122 In addition to these  $N$  frames, a reference image should be captured as the sample is at  
 123 rest for image processing purposes.

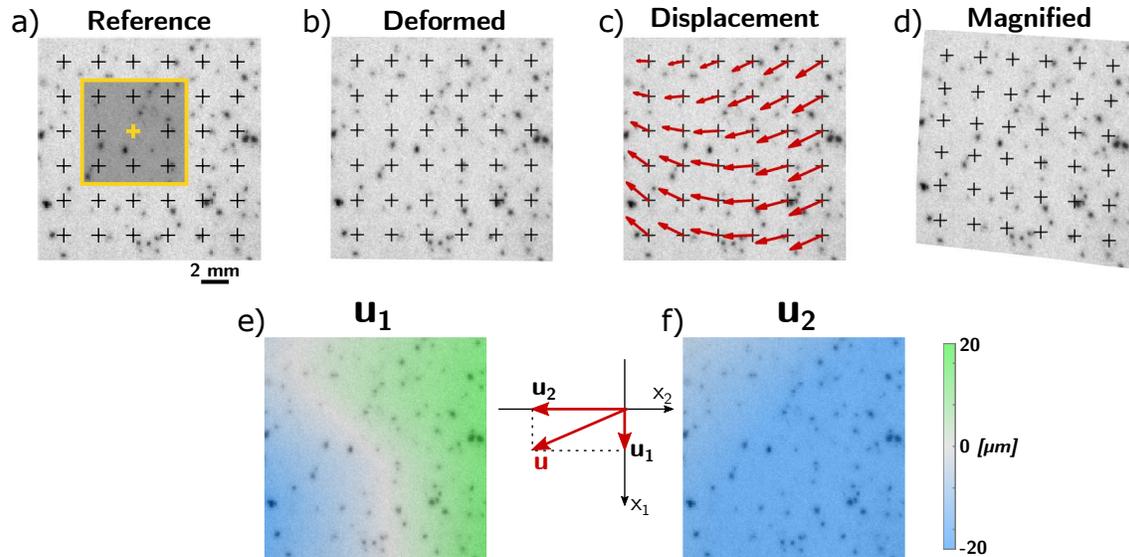


FIG. 4. **Principle of displacement extraction through Digital Image Correlation (DIC)**

**algorithm** – (a) Exemple of a reference image. The black seeds provide a *texture* enabling the DIC analysis. The displacement is computed for each position (crosses) by applying the DIC algorithm over the shaded area. (b) As the shaker is turned on, the image is deformed. The displacements are barely noticeable by eye (typically  $10 \mu\text{m}$ ). The DIC algorithm computes the correlation between the deformed and the reference images. (c) Output of the DIC algorithm. A displacement vector is computed for each macropixel. (d) This displacement is used to build a magnified distorted image where displacements appear clearly. The motion is magnified by a factor 50. (e-f) Vertical (resp. horizontal) displacement maps.

#### 4. *Extraction of the complex displacement maps*

Next, each of the  $N$  frames is compared to the reference thanks to an open source Digital Image Correlation (DIC) algorithm<sup>35,36</sup> which provides the instantaneous displacement (Fig. 4). The correlation is computed on small image regions, called *macropixels*. Each macropixel yields one displacement vector  $(u_1, u_2)$ . By repeating the operation for all the macropixels of a single frame, two displacement maps are obtained (Fig. 4e and f). The macropixel size is set manually. It should be large enough to contain several seeds while remaining smaller than the wavelength. Here, macro-pixels extending over  $25 \text{ pixels} \times 25 \text{ pixels}$  of the original image are chosen. Sometimes, the algorithm fails to find a realistic solution for a given macropixel. In that case, one can always spatially interpolate the missing information or apply spatial convolution filter to smooth the displacement maps. Note that the DIC algorithm enables sub-pixel resolution. For example, displacements down to  $5 \mu\text{m}$  are measured when a single image pixel corresponds to  $100 \mu\text{m}$  on the plate.

The knowledge of the displacement maps gives the opportunity to build magnified version of the deformed image as in Fig 4c. This can be very useful for visualizing the propagation of waves or for teachings. For communication through writing as in this article, a separate representation of the two components of the displacement as a color code (Fig 4d and e) is preferred.

At this stage, for a given excitation pulsation  $\omega$ , a series of  $N$  displacement matrices  $\mathbf{u}^{(n)}(\mathbf{r})$  are obtained, where  $n \in [0, N - 1]$  refers to the frame index. From this series, the

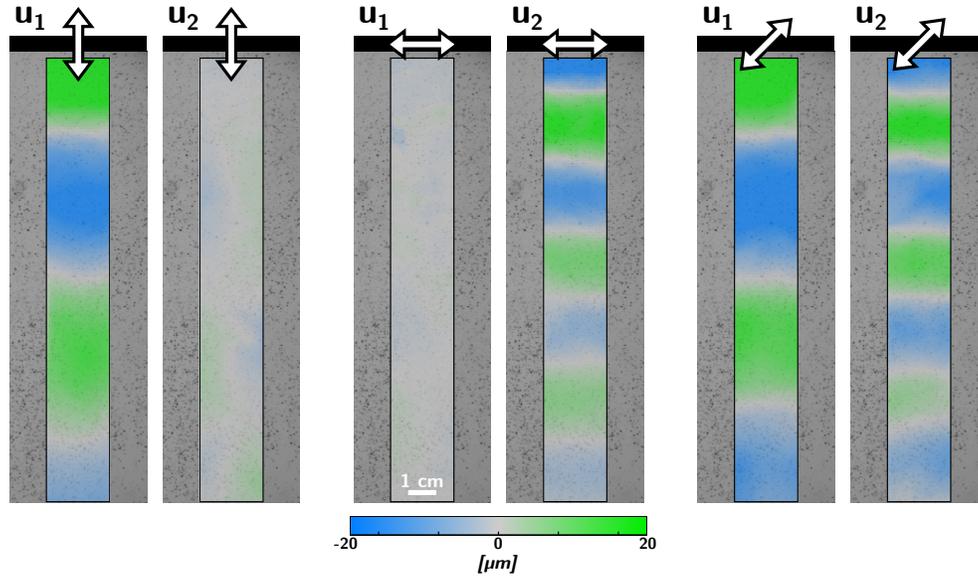


FIG. 5. **Measurement of in-plane waves in a soft plate** – Displacement fields in both vertical ( $u_1$ ) and horizontal ( $u_2$ ) directions measured for three excitation directions with a forcing frequency  $f = 120$  Hz. The source oscillates vertically (left), horizontally (middle) or at  $45^\circ$  (right). The colorbar indicates the magnitude of the in-plane displacements in the elastic plate.

144 complex monochromatic displacement is computed as following:

$$\mathbf{u}(\mathbf{r}, \omega) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{u}^{(n)}(\mathbf{r}) e^{\frac{2in\pi}{N}} \quad (1)$$

145 The data contained in 60 memory-consuming frames of thousands of pixels has been reduced  
 146 to the knowledge of a single complex matrix of a few hundreds points.

### 147 5. Measured in-plane modes

148 With the set-up of figure 2, field maps are acquired in an area of  $17 \text{ cm} \times 2.4 \text{ cm}$  below  
 149 the clamp (dashed area in the same figure). Figure 5 gathers the real parts of the extracted  
 150 displacements for an excitation frequency of 120 Hz. Three different vibration orientations

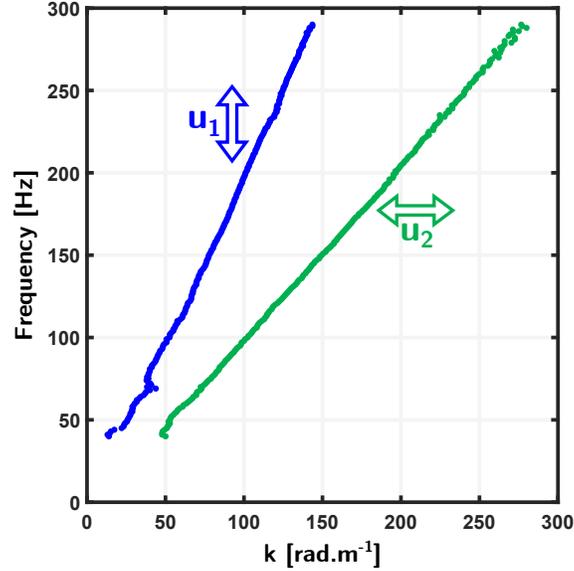


FIG. 6. **Experimental dispersion curves of in-plane modes in a 3-mm-thick soft plate** – A vertically polarized (blue) and a horizontally polarized (green) non dispersive modes are retrieved.

The vertically polarized mode propagates twice faster than the horizontally polarized one.

151 (vertical, horizontal and  $45^\circ$ ) are investigated while the 30-cm-wide clamp is maintained  
 152 horizontal.

153 In the left part of the figure 5, for which the vibration is vertical,  $u_2$  cancels everywhere  
 154 in the measured area: the motion is purely vertical (*ie.* aligned with  $x_1$ ). Also,  $u_1$  exhibits  
 155 a periodic pattern along the  $x_1$  direction and a flat profile along  $x_2$ , and the phase travels  
 156 toward the bottom (not shown here). This measurement corresponds to a plane wave-like  
 157 pattern (with a wavelength  $\lambda$  of roughly 10 cm) with both the displacement and the wavevec-  
 158 tor being parallel to  $x_1$ . The plate thus supports an in-plane guided elastic wave that we can  
 159 qualify as longitudinal. Similarly, the horizontal excitation along  $x_2$  of the clamp (middle  
 160 column in figure 5) generates a plane wave-like propagation with a polarization parallel to

161  $x_2$ , that can be qualified as a transverse wave. Interestingly, its wavelength is exactly half  
162 the wavelength of its longitudinal counterpart. The versatility of this experimental platform  
163 is highlighted in the right panel of figure 5. Indeed, instead of selectively exciting each type  
164 of plane wave, a motion of the clamp along a  $45^\circ$  tilted direction excites simultaneously the  
165 two waves: with one measurement several modes can be retrieved.

166 Finally, a systematic extraction of the two aforementioned plane waves for frequencies  
167 ranging from 50 to roughly 300 Hz is performed. For each frequency, the maps are averaged  
168 along  $x_2$ , meaning projected unto a plane wave. Then, for each solution a wavenumber  
169 (an inverse measure of the wavelength) along the  $x_1$  direction is obtained by keeping the  
170 maximum of the spatial Fourier transform of the profile along  $x_1$ . This way, a dispersion  
171 diagram (*i.e.* frequency as a function of wavenumber) is constructed for the two polarizations  
172 in figure 6. Both the dispersion curves appear to be straight lines passing through the origin.  
173 It corresponds to a non-dispersive propagation, that is to say a propagation at a constant  
174 phase velocity. The factor 2 between the wavelengths here manifests as a factor 2 between  
175 the two slopes: the longitudinal mode travels twice faster than the transverse one.

176 In the higher part of the measured frequency range the experimental points slightly move  
177 off the linear behaviour. The rheology of the polymer is at the origin of this deviation, as  
178 explained in section III C, but it remains anecdotal at this stage.

179 **B. Theoretical background**

180 These observations can be explained thanks to a simple theoretical description. The  
 181 propagation of elastic waves in isotropic solids is a well documented topic. Comprehensive  
 182 developments can be found in the following textbooks<sup>26,37</sup>.

183 **1. Bulk waves**

184 The elastodynamics of a homogeneous isotropic solid requires the knowledge of, at least,  
 185 two elastic moduli. For historical reasons, people usually refer to the Lamé constants  $\lambda$  and  
 186  $\mu$ <sup>1</sup>. But, these two constants can be substituted by any other couple of elastic coefficients  
 187 such as the Young modulus  $E$ , the bulk modulus  $K$  or the Poisson's ratio  $\nu$ . The case of  
 188 nearly incompressible media corresponds to the limit when  $\nu \rightarrow \frac{1}{2}$ . Knowing that  $\nu = \frac{\lambda}{2(\lambda + \mu)}$ ,  
 189 this amounts to  $\lambda \gg \mu$  in terms of the Lamé constants.

190 By combining the Newton's law of motion and the Hooke's law applied to an infinitesimal  
 191 volume, one finds that the local displacement field  $\mathbf{u}(\mathbf{r}, t)$  obeys the following vector wave  
 192 equation:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u}) + \mu \Delta \mathbf{u} \quad (2)$$

193 where  $\rho$  stands for the mass density of the solid and  $\nabla$  is the gradient operator. In this  
 194 equation the three components of the displacement field are coupled. In order to decouple  
 195 the equations it is common to introduce the scalar potential  $\phi$  and the vector potential  $\Psi$

196 as:

$$\mathbf{u} = \nabla\phi + \nabla \times \Psi \quad (3)$$

197 The component  $\nabla\phi$  corresponds to an irrotational vector field while the component  $\nabla \times \Psi$   
 198 is associated to a divergence free field, that is a deformation without any volume change.  
 199 These two potentials are independent and satisfy the following decoupled wave equations:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\lambda + 2\mu}{\rho}\Delta\phi = 0 \quad (4)$$

$$\frac{\partial^2\Psi}{\partial t^2} - \frac{\mu}{\rho}\Delta\Psi = 0 \quad (5)$$

200 These d'Alembert equations confirm the propagation of two different types of waves with  
 201 distinct polarizations and velocities. On the one hand, equation (4) corresponds to a longi-  
 202 tudinal wave propagating at velocity  $V_L = \sqrt{(\lambda + 2\mu)/\rho}$  and with a displacement parallel  
 203 to the propagation direction. On the other hand, equation (5) stands for transverse (or  
 204 shear) waves propagating at the velocity  $V_T = \sqrt{\mu/\rho}$  with displacements perpendicular to  
 205 the propagation direction. Ecoflex<sup>®</sup> 00-30 has a measured longitudinal velocity of approxi-  
 206 mately 1000 m/s while the shear wave velocity is about 5 m/s. This high contrast between  
 207 the two velocities ( $V_L \gg V_T$ ) confirms its incompressible nature as:

$$\nu = \frac{V_L^2 - 2V_T^2}{2(V_L^2 - V_T^2)} \approx \frac{1}{2} \quad (6)$$

## 208 **2. Reflection at a free interface**

209 The problem of reflection at an interface reveals the richness of the elastodynamics.  
 210 Consider an incident plane wave propagating in the plane  $(x_1, x_3)$  impinging on a medium

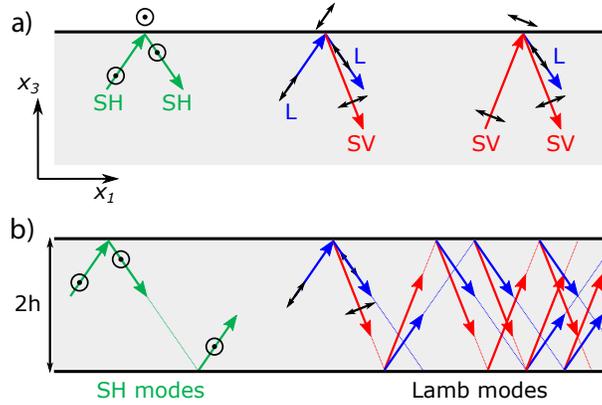


FIG. 7. **Reflection at a free interface and mode coupling** – (a) Reflection of a shear horizontal ( $SH$ ) wave at an interface does not generate out-of-plane displacements while longitudinal ( $L$ ) and shear vertical ( $SV$ ) waves couple. (b) When multiple reflections occur,  $SH$  waves remain independent while  $L$  and  $SV$  waves couple leading to a new family of modes: namely Lamb waves.

211 interface at  $x_3 = 0$  (Fig. 7a). As elastic waves have three polarizations, the reflection at the  
 212 interface gives rise to three different plane waves. However, the so-called shear horizontal  
 213 ( $SH$ ) wave with displacement along  $x_2$  ( $u_1 = u_3 = 0$ ) can only be generated as a reflection  
 214 of a shear horizontal wave as sketched in figure 7a. On the contrary, longitudinal and shear  
 215 vertical waves (by opposition to the shear horizontal ones) with displacements in the plane  
 216 ( $x_1, x_3$ ) are coupled through reflections at the interface.

217 Reflection at one interface is the entrance door to more complicated wave phenomena,  
 218 and notably waveguiding which occurs as a second interface, parallel to the first one, is  
 219 added. As sketched in figure 7b, the separation between the  $SH$  waves and the two others  
 220 remains valid in this configuration. The two following sections describe the two families of  
 221 modes that can propagate in a soft plate of thickness  $2h$ .

222 **3. *SH guided waves***

223 The case of shear-horizontal guided waves is relatively simple because its dispersion curves  
 224 map those of the well known acoustic waveguides. Indeed, as all displacements occur along  
 225 the  $x_2$  direction, the problem becomes a scalar wave problem. Applying the translational  
 226 invariance along the  $x_1$  direction one seeks for monochromatic solutions of the form:

$$\mathbf{u}(\mathbf{r}, \omega) = \begin{pmatrix} 0 \\ u_2(x_3, \omega) \\ 0 \end{pmatrix} e^{ikx_1} \quad (7)$$

227 Assuming that the interfaces at  $x_3 = \pm h$  are free to move, the stress component  $T_{23}$  at these  
 228 interfaces vanishes:

$$T_{23}(x_3 = \pm h) = \mu \left. \frac{\partial u_2}{\partial x_3} \right|_{x_3 = \pm h} = 0 \quad (8)$$

229 Solving the wave equation (2) for shear waves together with these boundary conditions  
 230 provides the solutions for the guided shear horizontal waves inside the plate:

$$u_2(x_3, \omega) = C \cos\left(\frac{n\pi}{2h}(x_3 - h)\right) \quad (9)$$

231 where  $C$  is a scalar constant. And the dispersion relation simply writes:

$$k^2 = \left(\frac{\omega}{V_T}\right)^2 - \left(\frac{n\pi}{2h}\right)^2 \quad (10)$$

233 Such a dispersion relation (figure 8) exhibits a non-dispersive mode, denoted  $SH_0$ , propa-  
 234 gating at all frequencies at the shear velocity  $V_T$ , as well as dispersive propagating modes  
 235 above their respective cut-off frequencies of  $f_{c_n} = nV_T/4h$ . For a thickness of 3 mm and

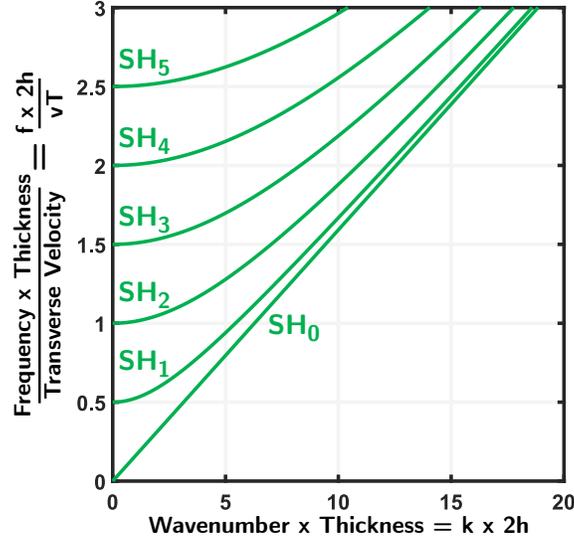


FIG. 8. **Theoretical dispersion curves of SH waves** – The  $SH_0$  mode is non dispersive with a velocity  $V_T$  while higher modes appear at cut-off frequencies corresponding to every multiple of  $\frac{V_T}{4h}$ .

236 a shear velocity of roughly 5 m/s, the first cut-off frequency is at 833 Hz, far above the  
 237 measured frequencies in the experimental part. Thus, the displacement field displayed in  
 238 figure 5 polarized along  $x_2$  corresponds to this  $SH_0$  mode. This transverse mode has already  
 239 experimentally demonstrated its non-dispersive nature in figure 6 (green line).

#### 240 4. *Lamb waves*

241 Due to the coupling at each reflection, the case of longitudinal waves and shear vertical  
 242 ones is more complicated. However, the calculation steps to establish the dispersion relation  
 243 and the solutions remain similar. Here, it is preferable to start back from the scalar and  
 244 vector potentials  $\phi$  and  $\Psi$ . Applying some geometrical arguments, their expressions can  
 245 be simplified. First, the invariance by translation along  $x_1$  implies the dependence on  $x_1$

246 to be on the form  $e^{ikx_1}$ . Second, the component of the displacement  $u_2$  is zero and the  
 247 other components should not depend on  $x_2$ . Third, the plane  $x_3 = 0$  is a symmetry plane,  
 248 so the solutions should either be symmetric or anti-symmetric. Considering all of these  
 249 simplifications and solving the wave equations (4) and (5), their analytical formulations  
 250 write:

$$\begin{cases} \phi(\mathbf{r}, \omega) = \phi_0 \cos(px_3 + \alpha) e^{ikx_1} \\ \Psi(\mathbf{r}, \omega) = \psi_2 \sin(qx_3 + \alpha) e^{ikx_1} \mathbf{x}_2 \end{cases} \quad (11)$$

251 with  $p^2 = (\omega/V_L)^2 - k^2$  and  $q^2 = (\omega/V_T)^2 - k^2$ . Symmetrical solutions correspond to  $\alpha = 0$   
 252 and anti-symmetrical ones to  $\alpha = \pi/2$ . From these potentials, the displacements now write:

$$\mathbf{u}(\mathbf{r}, \omega) = \begin{pmatrix} u_1(x_3, \omega) \\ 0 \\ u_3(x_3, \omega) \end{pmatrix} e^{ikx_1} \quad (12)$$

253 with the two non-zero components being:

$$\begin{cases} u_1(x_3, \omega) = ik\phi_0 \cos(px_3 + \alpha) - q\psi_2 \cos(qx_3 + \alpha) \\ u_3(x_3, \omega) = -p\phi_0 \sin(px_3 + \alpha) + ik\psi_2 \sin(qx_3 + \alpha) \end{cases} \quad (13)$$

254 The dispersion relation of these modes is deduced from the boundary conditions. Assum-  
 255 ing free boundaries at both interfaces  $x_3 = \pm h$  the stresses  $T_{13}$  and  $T_{33}$  must cancel, which  
 256 imply:

$$\begin{cases} (k^2 - q^2)\phi_0 \cos(ph + \alpha) = 2ikq\psi_2 \cos(qh + \alpha) \\ (k^2 - q^2)\psi_2 \sin(qh + \alpha) = 2ikp\phi_0 \sin(ph + \alpha) \end{cases} \quad (14)$$



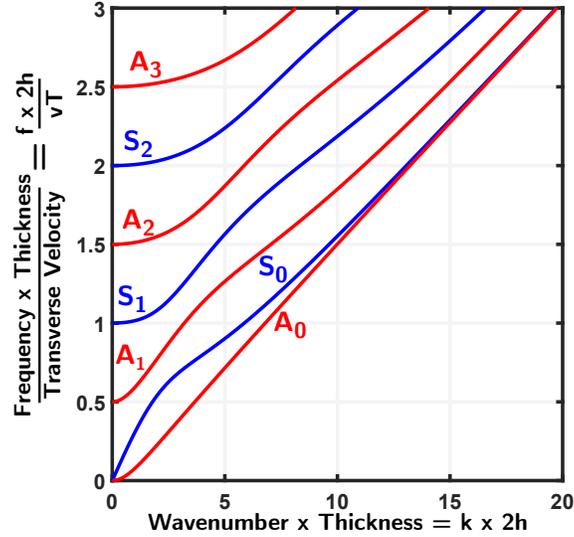


FIG. 9. **Theoretical dispersion curves of Lamb waves in a nearly incompressible material** – Dispersion curves of the symmetric (blue) and anti-symmetric (red) Lamb modes. In the low frequency range, the  $S_0$  mode is non dispersive with a velocity  $V_P = 2V_T$  while the  $A_0$  dispersion curve is parabolic. Higher modes exhibit cut-off frequencies every multiple of  $\frac{V_T}{4h}$ .

271 ( $kx_3 \rightarrow 0$ ). This mode, generally called flexural mode, is therefore mostly a transverse  
 272 vertical mode. It is not measured in the previous experiment since the shaker is aligned in  
 273 order to avoid out-of-plane displacements.

274 Taking the limit for the  $S_0$  mode gives:

$$\begin{cases} u_1(x_3, \omega) = C' + o(k) \\ u_3(x_3, \omega) = iC'kx_3 + o(k) \end{cases} \quad (18)$$

275 This time, the displacement  $u_1$  is homogeneous across the plate and is far greater than the  
 276 displacement  $u_3$ . In a sense, in this low frequency limit and long wavelength approximation  
 277 (compared to the thickness) the  $S_0$  mode is seen as a longitudinal mode. Its phase velocity,

278 known as the plate velocity, is given by:

$$V_P = 2V_T \sqrt{1 - \left(\frac{V_T}{V_L}\right)^2} = \sqrt{\frac{2}{1-\nu}} V_T \quad (19)$$

279 The surprising feature is that, in the incompressible limit, the phase velocity of  $S_0$  simplifies  
 280 to  $V_P = 2V_T$ . It is thus independent of the longitudinal velocity  $V_L$  despite its apparent  
 281 longitudinal polarization. This mode corresponds to the measured displacement  $u_1$  presented  
 282 in figure 5, which has twice the wavelength of the  $SH_0$  mode. This analytical derivation now  
 283 explains the ratio of 2 observed in the experimental dispersion curves in figure 6.

### 284 III. IN-PLANE GUIDED WAVES IN A SOFT STRIP

285 In this section, a different geometry is considered: a thin rectangular waveguide made of  
 286 the same nearly incompressible material. It is made by a parallel cutting of the previous  
 287 plate. First, an analogy is made between this geometry and the plate geometry already  
 288 described. Notably, the dispersion of in-plane modes propagating in this strip is shown to  
 289 be similar to the one of Lamb waves propagating in an isotropic plate with a longitudinal  
 290 wave velocity being exactly twice the shear wave velocity. Then, the experimental results  
 291 already reported in Lanoy *et al.*<sup>1</sup> are presented. The procedure used to separate the modes  
 292 in order to obtain their profiles as well as their phase velocities is thoroughly described.

#### 293 A. Theoretical framework

294 The theory of elastic modes propagating in rectangular waveguide is not straightforward.  
 295 As this geometry involves three coupled polarizations, obtaining the full dispersion diagram

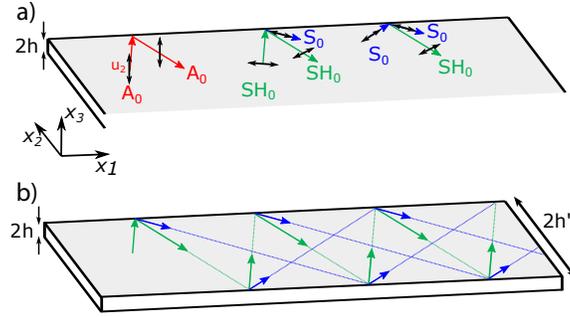


FIG. 10. **Mode coupling in a strip** – (a) The edge reflection of the  $A_0$  mode only generates  $A_0$  while  $S_0$  and  $SH_0$  waves couple. (b) Multiple reflections lead to  $S_0$  and  $SH_0$  in-plane mode coupling in a similar manner to shear and longitudinal waves couple in an infinite plate.

296 can be challenging<sup>40</sup>. Thanks to the Rayleigh-Lamb approximation<sup>41,42</sup>, the problem dras-  
 297 tically simplifies as one deals with the in-plane modes of a strip with a large aspect ratio.  
 298 This part treats this problem in the specific case of a soft solid.

### 299 1. Analogy with Lamb waves in a plate

300 As explained in the previous section, at low frequencies, only three modes propagate in a  
 301 plate: the first shear horizontal mode  $SH_0$  (Fig. 8), and the symmetric  $S_0$  and anti-symmetric  
 302  $A_0$  Lamb modes (Fig. 9). They have uniform profiles across the plate and can roughly be  
 303 considered as linearly polarized. In particular,  $S_0$  can be seen as a pseudo-longitudinal wave  
 304 propagating at the constant “plate” velocity  $V_P$ . Besides, as shown in equation (13), for  
 305 nearly incompressible materials the plate velocity is  $V_P = 2V_T$ .

306 These observations enable to build an analogy between Lamb waves in a plate and in-plane  
 307 guided waves in a thin strip as sketched in figure 10. The polarization of  $A_0$  is essentially  
 308 parallel to the strip edge. As a consequence, adding a second edge to form a ribbon of width

309  $2h'$  (figure 10b),  $SH_0$  and  $S_0$  give rise to guided waves whose displacements stay in the main  
 310 plane of the ribbon. They are called in-plane guided modes. As shown in the section III of  
 311 reference<sup>41</sup>, this coupling is similar to the one of shear and compression bulk waves in a plate.  
 312 In other words, the low frequency dispersion diagram for in-plane guided waves in a strip is  
 313 equivalent to the one for guided waves in a plate. In the following descriptions, the symbol  
 314 “ ’ ” will be added to the notations when dealing with the strip configuration. Typically,  
 315 the plate thickness  $2h$  is replaced by the strip width  $2h'$ , the longitudinal wave propagating  
 316 at  $V_L$  is replaced by the linearly polarized in-plane wave  $S_0$  propagating at velocity  $V_P$  (that  
 317 is  $V'_L = V_P$ ), and the  $SV$  wave propagating at  $V_T$  is replaced by the transversely polarized  
 318 in-plane wave  $SH_0$  propagating at  $V_T$  (that is  $V'_T = V_T$ ). This amounts to solving the Lamb  
 319 problem for a material of equivalent Poisson's ratio:

$$\nu' = \frac{\nu}{1 + \nu} \quad (20)$$

320 where  $\nu$  is the Poisson's ratio of the strip material. For incompressible materials, the equiv-  
 321 alent Poisson's ratio is  $\nu' = 1/3$ , and the knowledge of  $V_T$  is sufficient to obtain the full  
 322 dispersion diagram of the in-plane guided waves in the low frequency range.

## 323 2. *Dispersion relation: key physical features*

324 The dispersion curves of the in-plane modes propagating in a soft strip are thus obtained  
 325 by finding the roots of the Rayleigh-Lamb equations (16). The solutions are displayed in  
 326 figure 11 in normalized units. Several interesting properties deserve to be highlighted.

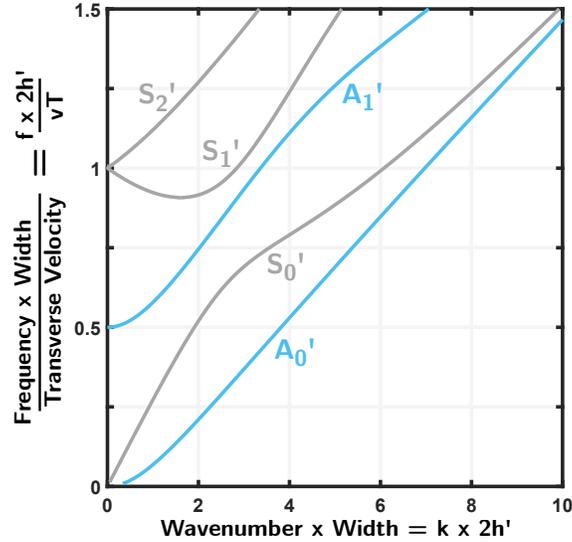


FIG. 11. **Theoretical dispersion curves of in-plane modes in a soft strip** – Dispersion curves of symmetric (gray and labeled  $S'$ ) and anti-symmetric (blue,  $A'$ ) modes without damping. From these curves, one can extract the bar velocity (mode  $S'_0$  at low frequencies), evidence a Zero Group Velocity point ( $S'_1$ ) and a backward branch, as well as a Dirac cone with a finite group velocity at  $k = 0$  (at  $f = V_T/2h'$ )

327 *a. Bar velocity.* The first symmetrical mode, denoted  $S'_0$ , is non dispersive for frequen-  
 328 cies below the first cut-off frequency. Similarly, to the first symmetric  $S_0$  Lamb mode,  $S'_0$   
 329 can be seen as a longitudinally polarized mode since it corresponds to a pure compression  
 330 mode of the ribbon. Its phase velocity can be calculated as a pseudo-plate velocity  $V'_P$ . This  
 331 velocity deduced from equation (19) has a remarkably simple formulation:

$$V'_P = \sqrt{\frac{2}{1-\nu'}} V_T = \sqrt{2(1+\nu')} V_T \quad (21)$$

332 In the incompressible limit, it simplifies to  $V'_P = \sqrt{3} V_T$ . As a consequence,  $S'_0$  roughly travels  
 333 at the transverse velocity, despite being longitudinally polarized mode. This is all the more

334 striking as, in the incompressible limit, the transverse velocity happens to be several orders  
 335 of magnitude smaller than the longitudinal velocity.

336 Furthermore,  $V_P'$  also corresponds to the well known bar velocity, associated to the prop-  
 337 agation of compression waves along any bar or rod regardless of their cross-section. It can  
 338 be obtained from an intuitive reasoning. As it corresponds to a longitudinal extension of  
 339 the waveguide, the relevant elastic modulus is the Young's modulus  $E$  and the associated  
 340 velocity is  $\sqrt{E/\rho}$ . After injecting the expression  $E = 2(1 + \nu)\mu$ , one immediately gets  
 341 equation (21). For a soft material, the Young's modulus simplifies to  $E = 3\mu$  and the bar  
 342 velocity again appears independent of  $V_L$ .

343 *b. Zero Group Velocity and Negative Phase Velocity.* Like for Lamb modes, the second  
 344 symmetrical mode  $S_1'$  has a remarkable behavior. Indeed, the corresponding branch exhibits  
 345 a local minimum for a finite wavenumber. At this specific location, the group velocity  
 346  $V_g = d\omega/dk$  vanishes. This is the signature of a Zero Group Velocity (ZGV) point. For  
 347 small wave numbers, the  $S_1'$  branch has a negative slope. This indicates that the group  
 348 velocity is opposite to the phase velocity. Causality imposes that the energy travels from  
 349 the source to the receiver. As a consequence the group velocity should always remain  
 350 positive. In practice, this negative slope section cannot be measured. The experimentalist  
 351 rather accesses its symmetric with respect to the  $k=0$  axis. This is further discussed in the  
 352 following sections.

353 *c. Dirac cones: finite group velocity at  $k \rightarrow 0$ .* In the small wavenumber limit ( $k \rightarrow 0$ ),  
 354 the branches of the dispersion curve become horizontal (for example the Lamb modes in the  
 355 plate of figure 9). The dispersion relation  $\omega(k)$  is quadratic around the cut-off pulsation

356  $\omega_c$ . As shown by Mindlin<sup>43</sup>, this expansion does not hold for Lamb modes when there is a  
 357 coincidence between a shear and a longitudinal cut-off frequency of the same symmetry. In  
 358 these particular cases, the dispersion law is linear in the limit  $k \rightarrow 0$  and writes to the first  
 359 order in  $k$  as:

$$\omega(k) = \omega_c + V_g k + o(k) \quad (22)$$

360 Such coincidences occur for symmetrical modes  $S_{2m+1}$  and  $S_{2n}$ , when the bulk velocity  
 361 ratio  $V_L/V_T$  is equal to  $2n/(2m+1)$ , and for anti-symmetrical modes  $A_{2m+1}$  and  $A_{2n}$  when  
 362  $V_L/V_T = (2m+1)/2n$ . For example, recent experiments conducted in a cooled aluminum  
 363 plate ( $V_L/V_T = 2$ ) by Stobbe and Murray<sup>44</sup> illustrate this linear dispersion near  $k = 0$ .  
 364 For modes  $S_1$  and  $S_2$ , the linear slopes of the curve  $\omega(k)$  can be derived by developing  
 365 equation (16) to the first order and it was found<sup>43</sup> to be  $V_g = \pm 2V_T/\pi$ .

366 The Lamb wave approximation for in-plane modes in a thin soft strip (*ie.*  $\nu' = 1/3$ ) reveals  
 367 a coincidence frequency for the symmetrical modes  $S'_1, S'_2$ . As a result these two modes cross  
 368 linearly at the normalized frequency  $f2h'/V_T = 1$  in figure 11. This linear crossing is also  
 369 referred to as a Dirac cone<sup>45,46</sup>.

370 *d. Displacement near the Dirac cone.* From equations (15) one can determine the dis-  
 371 placements close to the cut off frequencies. For the strip configuration, index 3 must be  
 372 replaced by 2,  $h$  by  $h'$  and  $p$  by  $p'$  such that  $p'^2 = (\omega/V_P)^2 - k^2$ . The Taylor expansion of  
 373 coefficients  $p'$  and  $q$  near the value  $k=0$  are:

$$\begin{cases} p' = \frac{\pi}{2h'} + \frac{V_g}{2V_T} k + o(k) \\ q = \frac{\pi}{h'} + \frac{V_g}{V_T} k + o(k) \end{cases} \quad (23)$$

Using  $s$  as the sign of the group velocity, the Taylor expansion of the displacement components at the coordinate point  $x_2$  becomes:

$$\begin{cases} u_1(x_2, \omega) = C \frac{\pi}{h'} \cos\left(\frac{\pi}{h'} x_2\right) + O(k) \\ u_2(x_2, \omega) = -isC \frac{\pi}{h'} \sin\left(\frac{\pi}{2h'} x_2\right) + O(k) \end{cases}$$

374 For ordinary cut-offs, the displacement is either purely longitudinal or purely transverse.  
 375 Instead, when there is a coincidence, both polarizations are involved. The factor  $i$  between  
 376 the two components denotes an elliptical polarization. At the specific location  $x_2 = \pm h'$ , the  
 377 polarization becomes purely circular. The factor  $s$  indicates that modes of opposite group  
 378 velocities are associated to opposite rotation directions.

### 379 **B. Measurements in a soft strip**

380 The soft plate is now replaced with a soft strip. The edges are cut using a laser cutter  
 381 (Trotec Engraver Speedy 100). The final strip is 60 cm long, 3 mm thick and its width is  
 382  $h' = 4$  cm. The wide source is replaced with a point-like clamp, obtained by putting two  
 383 half spheres in contact. The clamp is slightly off-centered and vibrated vertically in order to  
 384 feed the system in a non-selective manner. A picture of the setup<sup>1</sup> is reported in figure 12.

385 Here again, the strip is shaken monochromatically for frequencies ranging from 1 to  
 386 200 Hz. The camera captures the motion by following the stroboscopic sketch pictured in  
 387 figure 3. Finally, the displacement field is extracted by applying the DIC algorithm to the  
 388 successive snapshots. For example, the obtained field maps at 110 Hz are represented in  
 389 figure 13. The wave pattern is rather different from the one obtained in the plate experiment.

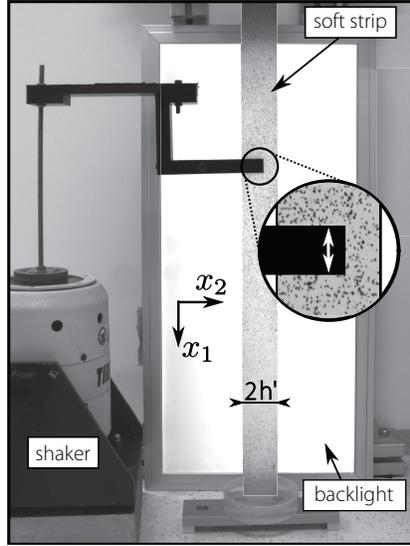


FIG. 12. **Experimental set-up using a point-like source** – A thin strip ( $L = 60$  cm,  $2h' = 40$  mm,  $2h = 3$  mm) is held vertically. A shaker excites in-plane displacements propagating in the strip.

390 This comes from a superposition between several modes with different propagation constants  
 391 and spatial profiles.

392 Separating and identifying them requires two additional post-processing steps schematized  
 393 in figure 13. First, the symmetrical and anti-symmetrical parts are extracted by  
 394 respectively summing or subtracting the displacement map with its flipped (along the  $x_2$   
 395 direction) counterpart. The concatenation of these field maps yields to the construction of  
 396 two bigger matrices  $\mathbf{U}$  (top of Fig. 13), one for each type of symmetry. Then, a Singular  
 397 Value Decomposition (SVD) is performed on each matrix. This amounts to the following  
 398 matrix decomposition:

$$\mathbf{U} = \mathbf{V}\Sigma\mathbf{W} \quad (24)$$

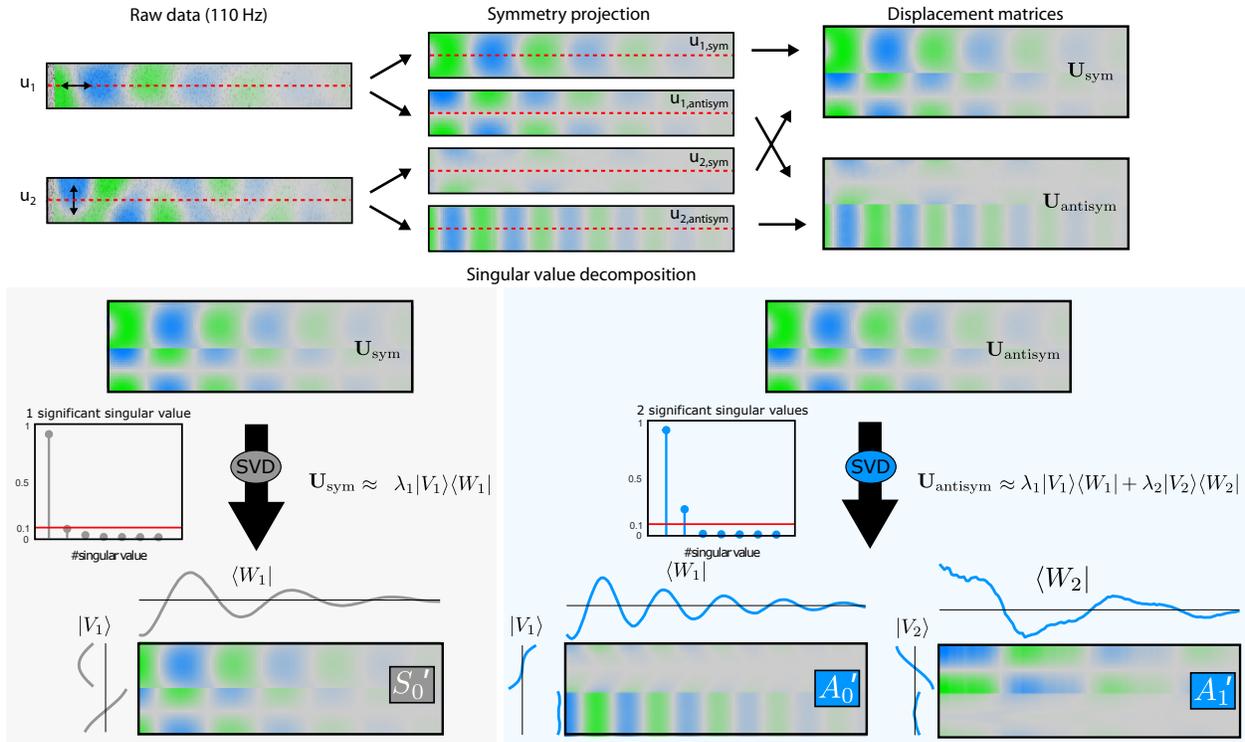


FIG. 13. **Mode separation via Singular Value Decomposition (SVD)** – The in-plane displacement components  $u_1$  and  $u_2$  (top left) are projected onto their symmetrical and anti-symmetrical parts (top right). Data are then concatenated into a single complex matrix  $U_{\text{sym}}$  (resp.  $U_{\text{antisym}}$ ) on which the SVD is directly applied. After extracting the most significant modes (singular values above a 10% threshold), we obtain (here at 110 Hz) one symmetrical ( $S'_0$ ) and two anti-symmetrical ( $A'_0$  and  $A'_1$ ) modes.

399 where  $\mathbf{V}$  and  $\mathbf{W}$  are unitary matrices providing the displacement profile along  $x_2$  and  $x_1$   
 400 respectively and  $\mathbf{\Sigma}$  is a diagonal matrix providing the singular values (*i.e.* the mode promi-  
 401 nence in the overall measurement). As a selection criterion, all modes associated with singu-  
 402 lar values of at least 10% of the maximum singular value are considered as meaningful. The  
 403 other ones are rejected. At 110 Hz (see figure 13), three modes have a relevant contribution:

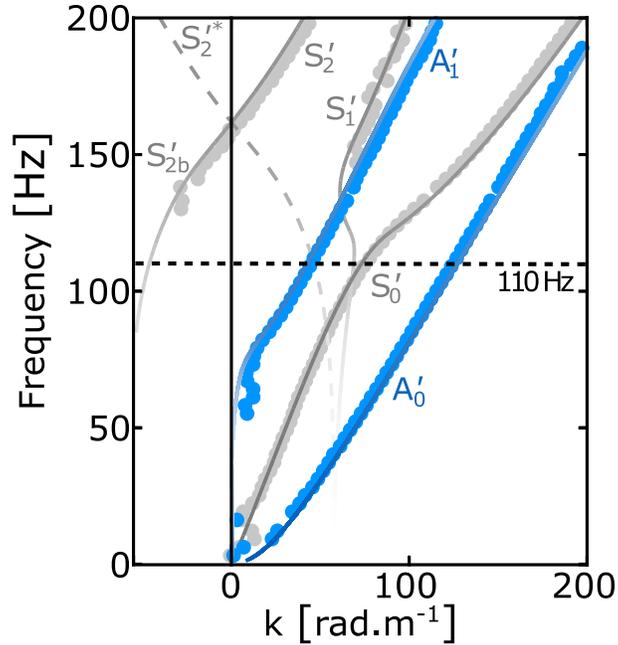


FIG. 14. **Dispersion curves of in-plane modes in a free strip** ( $2h' = 39 \text{ mm}$ ) – Experimental (symbols) and theoretical (lines) dispersion curves with damping (the more transparent the curve the more attenuated the mode). The Dirac cone (linear crossing of the  $k = 0$  axis) and the backward modes (negative wavenumbers) are unambiguously evidenced, while the ZGV point has disappeared. The dashed line denoted by a “\*” is drawn by symmetry and corresponds to a mode propagating in the direction  $-\mathbf{x}_1$ .

404 two symmetrical modes and one anti-symmetrical. Since  $\mathbf{W}$  gives the displacement along  
 405 the propagation direction, its Fourier transform yields the wavenumbers of the contributing  
 406 modes.

407 These steps are repeated for all frequencies and the dispersion curves represented as  
 408 symbols in figure 14 are constructed. As stated earlier, in the experiments, one measures  
 409 negative phase velocities rather than negative group velocities. This is the reason why the  
 410 horizontal axis covers negative values.

411 **C. Discussion**

412 Overall experimental dispersion curves in figure 14 relatively resembles the theoretical one  
 413 in figure 11 and most of the discussed key features are visible. Indeed, the bar velocity of the  
 414 strip (mode  $S'_0$  at low frequency) matches the expected value of  $\sqrt{3}V_T$  (where  $V_T$  is deduced  
 415 from the  $SH_0$  velocity measured in the plate of figure 5). At 150 Hz,  $S'_2$  crosses the axis  $k = 0$   
 416 with a linear slope: this is a Dirac cone. Note that, below the Dirac frequency, the measured  
 417 points have negative wave-numbers : this is a signature of a negative phase velocity. The  
 418 continuity in the measured points naturally leads to label this backward branch  $S'_{2b}$  (“b”  
 419 for backward). This may appear in contradiction with the dispersion curves for lossless  
 420 material shown in Fig. 11 where the backward mode belongs to the  $S'_1$  branch. However,  
 421 when the complex wave numbers are displayed as done by Mindlin for Lamb modes<sup>47</sup>, it  
 422 clearly appears that the backward branch is connected to  $S'_2$  mode even when the cone does  
 423 not exist<sup>43</sup>, thus this notation is adopted in several papers<sup>30,33,42,48</sup>.

425 However, there are two main differences between the theory in figure 11 and the exper-  
 426 iment. First, the Dirac cone should be at exactly  $2h'/V_T$  but it does not match the value  
 427 found from the bar velocity neither the value deduced from the asymptotic behaviour at  
 428 high frequencies of  $A'_0$  and  $S'_0$ . Second, the ZGV point is not visible in the experiment.  
 429 These two differences both originate from the complex rheology of the elastomer<sup>2</sup>.

430 It is independently measured with a conventional rheometer (Anton-Paar MCR501) which  
 431 operates in the plate-plate configuration. To this end, a different sample of Ecoflex<sup>®</sup> 00-30  
 432 is cured in the rheometer itself. Both the real and the imaginary parts of the measured shear

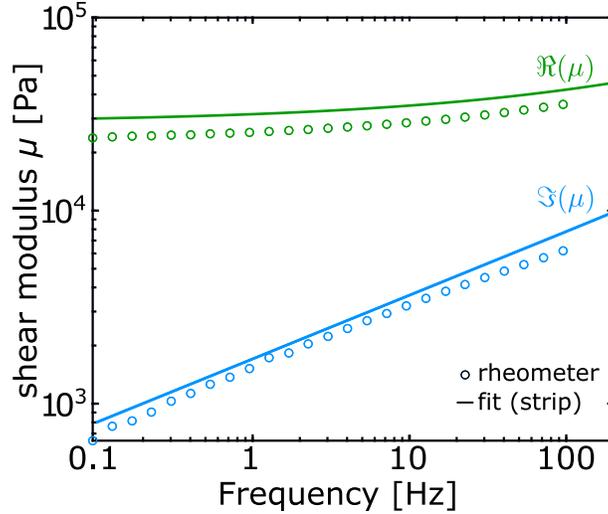


FIG. 15. **Rheology of Ecoflex<sup>®</sup>** – Measurement of the complex shear modulus of Ecoflex<sup>®</sup> in the range 0.1 – 100 Hz with a conventional plane-plane rheometer (circles).  $\Re(\mu)$  is the storage modulus of the rubber while  $\Im(\mu)$  is its loss modulus. Lines correspond to the values extracted from the dispersion curves (see text).

433 modulus for frequencies ranging from 0.1 to 100 Hz are displayed as symbols in figure 15.  
 434 In such a logarithmic scale the imaginary part of the shear modulus appears linear with a  
 435 slope of almost 1/3, while the real part seems to increase slowly. Among all the available  
 436 models, as the slope is not an integer it balances for a fractional derivative model. One of the  
 437 simplest model which also satisfies the Kramers-Kronig relations is the fractional derivative  
 438 Kelvin-Voigt model<sup>50–52</sup> which takes the form:

$$\mu(\omega) = \mu_0 [1 + (i\omega\tau)^n] \quad (25)$$

439 This frequency dependent complex shear modulus is injected in the Rayleigh-Lamb equa-  
 440 tion (16). Again, solving this transcendental equation is not an easy task and its roots are  
 441 found with the help of an internally developed numerical Muller’s algorithm. The latter

442 is ran with several sets of parameters of the rheological model until a satisfying agreement  
 443 between theory and experiment is reached. The final set of parameters is  $\mu_0 = 26$  kPa,  
 444  $\tau = 260$   $\mu$ s and  $n = 0.33$ . It almost corresponds to the measured rheology (Fig. 15) but  
 445 slightly overestimates  $\Re(\mu)$ . This discrepancy can be attributed to temperature changes<sup>53</sup>  
 446 or to differences (preparation or ageing) between the two samples.

447 The theoretical curves in figure 14 are calculated with these parameters. The wavenumber  
 448 is complex and its imaginary part is rendered by the transparency of the theoretical lines.  
 449 The frequency dependence of  $\Re(\mu)$  induces a frequency dependence of the velocity  $V_T$  which  
 450 allows to fit the entire  $S'_0$ ,  $A'_0$  and  $A'_1$  branches. Adding the imaginary part of the shear  
 451 modulus also explains the lowered Dirac frequency. As for the absence of ZGV points, it  
 452 is solely due to the viscous damping. While for a lossless material, the  $S'_1$  branch and the  
 453  $S'_{2b}$ \* (symmetrical to  $S'_{2b}$  with respect to the  $k=0$  axis) connect at the ZGV point, here, the  
 454 losses separate those two branches.

#### 455 IV. GOING FURTHER

456 The fundamental aspects of the system have now been identified. In this section, the  
 457 experiment is altered in order to investigate the role of the boundary conditions. In the  
 458 Dirichlet configuration, the dispersion is found to simplify. After examining the modes  
 459 polarization, selective excitation are performed by designing specific chiral sources.

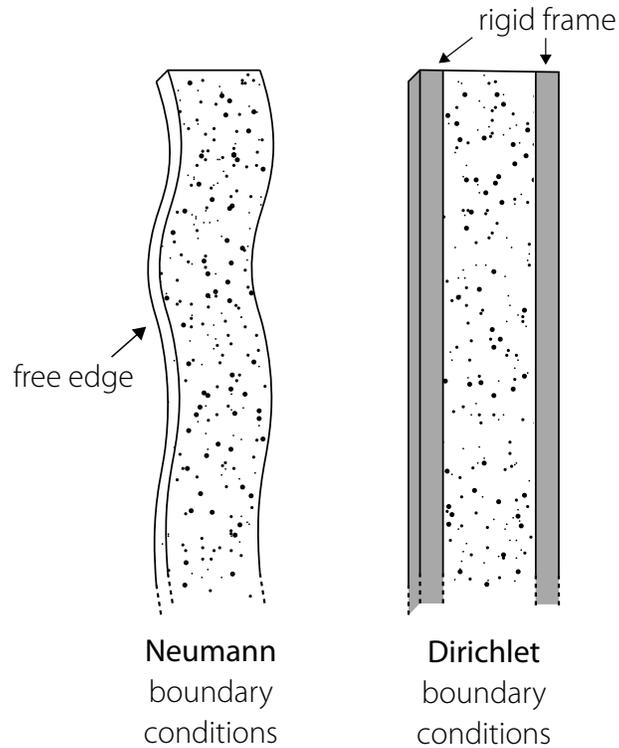


FIG. 16. **Boundary conditions** – We study the Neumann (free edges) and Dirichlet (fixed edges) boundary conditions.

#### 460 **A. Investigating Dirichlet boundary conditions**

461 In section [II B 4](#), the analytical Lamb problem is derived assuming free boundary condi-  
 462 tions (Neumann configuration). Here, the case of fixed boundaries (Dirichlet configuration)  
 463 is investigated. In practice, these conditions can be implemented by clamping the strip in a  
 464 rigid frame (Fig. [16](#)).

##### 465 **1. Theory**

466 *a. Dispersion relation.* From a theoretical point of view, switching from Neumann to  
 467 Dirichlet boundaries amounts to replacing the strain cancellation condition of equations [\(14\)](#)

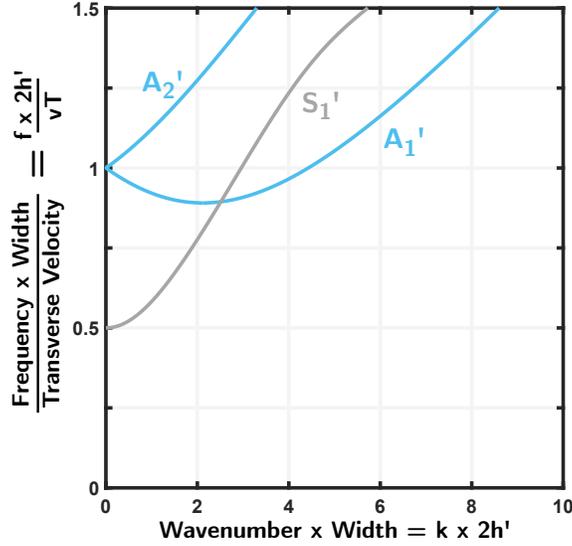


FIG. 17. **Theoretical dispersion curves of in-plane modes in a clamped soft strip** – Dispersion curves of symmetric (gray and labeled  $S'$ ) and anti-symmetric (blue,  $A'$ ) modes without damping. These curves evidence a Zero Group Velocity point ( $A'_1$ ) and a backward branch, as well as a Dirac cone with a finite group velocity at  $k=0$  and  $f=V_T/2h'$ .

468 by a displacement cancellation condition as follows:

$$\begin{cases} ik\phi_0 \cos(p'h' + \alpha) + q\psi_2 \cos(qh' + \alpha) = 0 \\ -p'\phi_0 \sin(p'h' + \alpha) + ik\psi_2 \sin(qh' + \alpha) = 0 \end{cases} \quad (26)$$

469 The equivalent of the Rayleigh-Lamb equation for rigid boundaries is then:

$$k^2 \sin(qh' + \alpha) \cos(p'h' + \alpha) + qp' \sin(p'h' + \alpha) \cos(qh' + \alpha) = 0 \quad (27)$$

470 The dispersion curves (figure 17) are obtained by searching for the roots of this equation.

471 Compared to the Neumann case (figure 11), one important feature is the absence of prop-

472 agation at low frequency ( $A'_0$  and  $S'_0$  have disappeared). Indeed, the rigid walls imply that

473 no static in-plane deformation can be solution to the problem. However, the cut-off modes

474 ( $A'_1, S'_1, A'_2$  etc) still exist. Note that the negative sloped branch, the Dirac cone and the  
 475 ZGV are still visible but for anti-symmetric modes rather than symmetric ones.

476 Finally, the displacements can be obtained by eliminating the coefficients  $\phi_0$  and  $\psi_2$  in  
 477 the boundary conditions of equation 26:

$$\left\{ \begin{array}{l} u_1(x_2, \omega) = Cq \left[ \cos(p'h' + \alpha) \cos(qx_2 + \alpha) \right. \\ \qquad \qquad \qquad \left. + \cos(qh' + \alpha) \cos(p'x_2 + \alpha) \right] \\ u_3(x_2, \omega) = -iC \left[ k \cos(p'h' + \alpha) \sin(qx_2 + \alpha) \right. \\ \qquad \qquad \qquad \left. + p'q \cos(qh' + \alpha) \sin(p'x_2 + \alpha) \right] \end{array} \right. \quad (28)$$

478 Here again, the  $\pi/2$  phase shift between the two components implies that the motion is  
 479 elliptically polarized.

480 *b. Dirac cone.* The Dirac cone appears here for the anti-symmetric modes at the same  
 481 frequency  $f_c = V_T/2h$  as for the Neumann configuration. At this frequency, the Taylor  
 482 expansion of  $p$  and  $q$  are also given by the equation (23). These expressions are substituted  
 483 into the dispersion relation (27):

$$k^2 - \frac{\pi}{h'} \frac{\pi}{2h'} \frac{V_g h'}{2V_T} k \frac{V_g h'}{V_T} k = 0 \quad (29)$$

484 Leading to the same expression for the group velocity  $V_g = \pm \frac{2}{\pi} V_T$ .

485 Also, equation (28) provides the Taylor expansion of the displacements:

$$\left\{ \begin{array}{l} u_1(x_2, \omega) = -\frac{\pi}{h'} C \sin\left(\frac{\pi}{h'} x_2\right) \\ u_2(x_2, \omega) = i s \frac{\pi}{h'} C \cos\left(\frac{\pi}{2h'} x_2\right) \end{array} \right. \quad (30)$$

486 where  $s$  indicates the sign of the group velocity. As  $\cos(\pi/6) = \sin(\pi/3)$ , a circular polar-  
 487 ization occurs for  $x_2 = \pm h'/3$  while it appears at  $x_2 = \pm h'$  in the Neumann configuration.

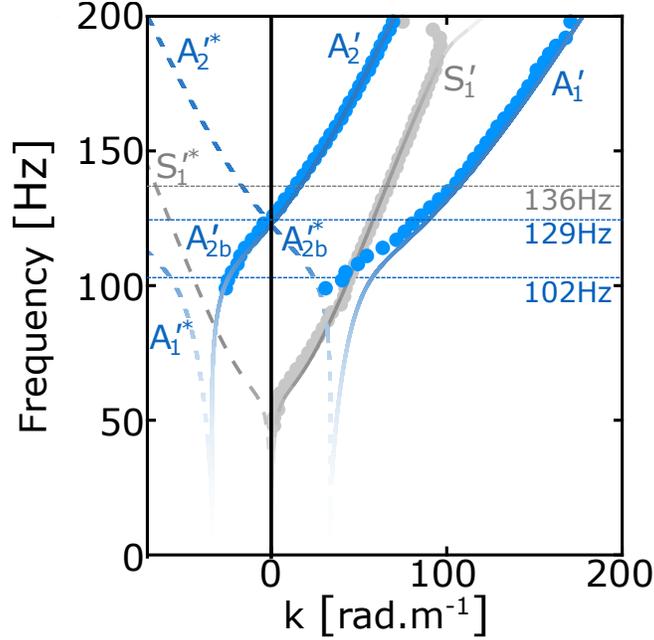


FIG. 18. **Dispersion curves of in-plane modes in the clamped strip** ( $2h' = 50.6$  mm) – Experimental (symbols) and theoretical (lines) dispersion with damping (the more transparent the curve the more attenuated the mode). The  $A'_2$  Dirac cone (linear crossing of the  $k=0$  axis) and the backward  $A'_{2b}$  modes (negative wavenumbers) are unambiguously evidenced. The dashed lines labelled with a “\*” correspond to modes propagating in the direction  $-\mathbf{x}_1$  and are not measured.

## 488 2. Measurements in a clamped soft strip

489 *a. Dispersion.* The experiment is performed under the same conditions as before. The  
 490 strip is held along its edges between two steel plates; and the separation between the edges is  
 491 adjusted in order to avoid buckling or static tension. The excitation clamp is again slightly  
 492 off-centered and vibrated vertically from 40 to 200 Hz. The same image analysis as in the  
 493 free edges configuration allows to extract the experimental dispersion curves represented  
 494 as symbols in figure 18. The three modes expected in the measured frequency range are

retrieved, and the same observations as in the free strip configuration can be made. First, the data points around 129 Hz show a linear crossing of the axis  $k=0$  which evidences the existence of a Dirac cone for the anti-symmetric modes. Second, points measured below this cut-off frequency correspond to negative wavenumbers, which is the signature of a backward mode. Here again, the continuity of the points across the Dirac cone logically leads to attribute the backward modes to the branch  $A'_2$ , unlike what is indicated for the lossless medium theoretical curves (Fig. 17). This part of the curve is thus referred to as  $A'_{2b}$ .

Just like for the Neumann configuration, the theory provides a convincing agreement on the condition that the complex rheology of the material is taken into account. The value of  $\Re(\mu)$  has an effect on the asymptotic slope of the branches. The value of  $\Im(\mu)$  affects the Dirac frequency. In addition, the ZGV point is accurately defined only when  $\Im(\mu) = 0$ . In this lossy material, two modes with almost opposite wavenumbers coexist, which corresponds to a quasi ZGV point. The absence of actual ZGV point is evidenced by the disconnection between branches  $A'_1$  and  $A'_{2b}$ . This is a direct consequence of the increase in losses near this point as rendered by the transparency of the lines.

*b. Tracking the displacement at the Dirac point.* Shaking the strip at 129 Hz (Dirac frequency), the in-plane motion over a complete period was extracted for 25 positions regularly spaced across the strip and located at a distance  $x_1 = 18$  cm from the source (far enough to avoid evanescent contributions). After removing the symmetric contribution and specifically selecting the  $A'_2$  mode thanks to the SVD algorithm, one can reconstruct the full trajectories as shown in figure 19a. They appear to be essentially elliptical and nearly circular at the location  $x_2 = \pm h'/3$  (red dashed lines) which is in agreement with equation (30).

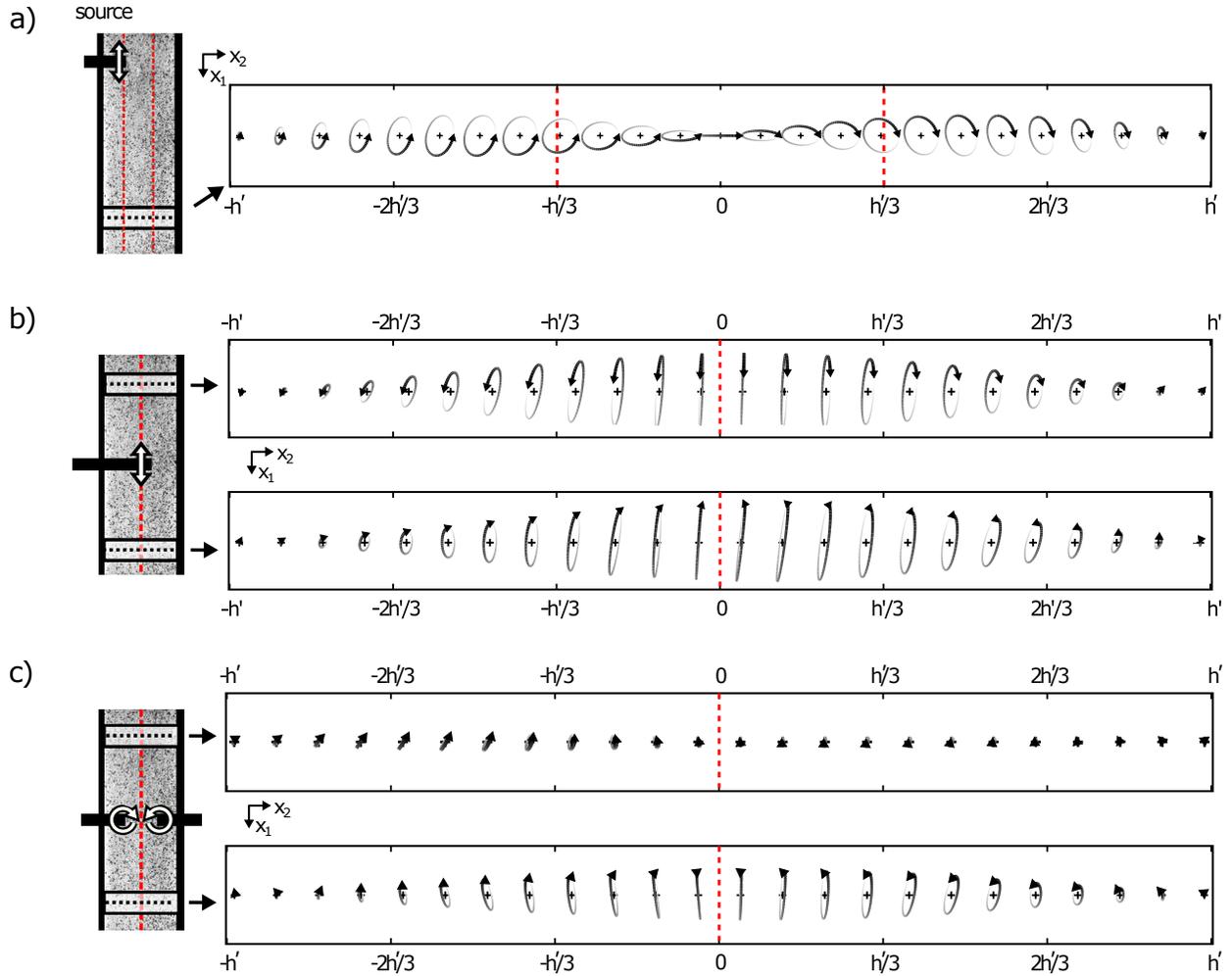


FIG. 19. **Trajectories** – From the instantaneous motion, one can reconstruct the trajectory of a solid element over a full wave period. Here, we specifically display the trajectories of elements distributed along the width of the strip at a distance  $x_1 = 18$  cm (a),  $\pm 10.5$  cm (b and c) from the source. The motion is magnified by a factor 80 (a), 130 (b) and 180 (c). (a) The SVD allows to separate the  $A_2$  contribution and retrieve the displacement at the Dirac cone (at 129 Hz): a circular motion at positions  $\pm h/3$  is observed. (b) For a symmetrical input (at 136 Hz), only the symmetrical mode  $S_1$  and its reciprocal counterpart  $S_1^*$  which propagates in the opposite direction are fed. (c) A chiral excitation, *i.e.* two sources placed at roughly  $\pm 2h/3$  and driven circularly in a symmetric manner, permits to only feed  $S_1$  and no displacement is measured on the top.

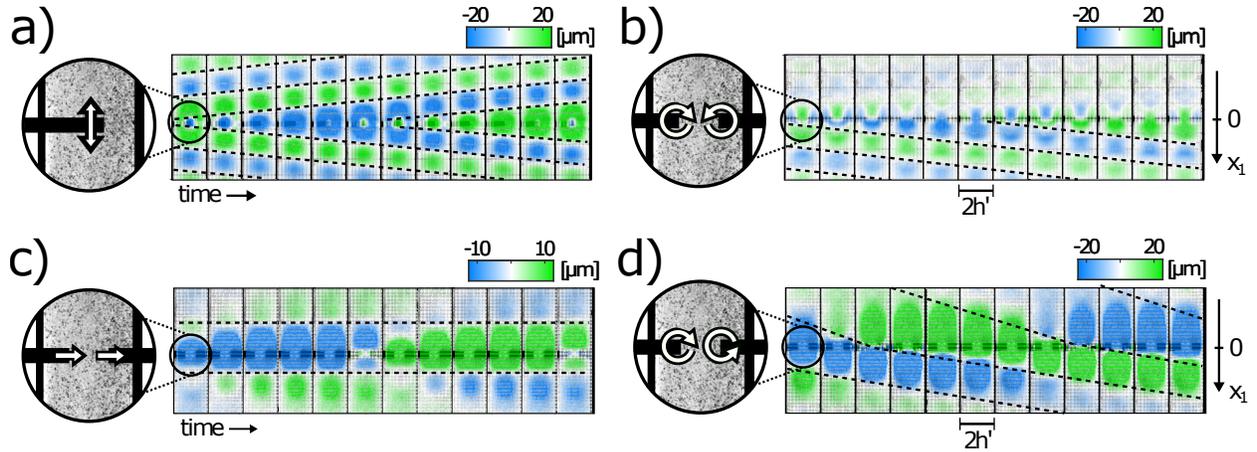


FIG. 20. **Selective excitation at 136 Hz (a and b) and 102 Hz (c and d)** – *Figure reproduced from ref<sup>1</sup>*. The strip is excited in a symmetrical manner (a and b) and in an anti-symmetrical manner (c and d). The black dashed lines are visual guides highlighting the zeroes of displacement and the sketches indicate the source geometry and motion. For the sake of clarity, only  $u_1$  (a and b) or  $u_2$  (c and d) is reported here. (a) Linear excitation. The source is placed in the centre of the strip and shaken vertically:  $S_1$  and  $S_1^*$  are excited and respectively travel to the bottom and the top of the strip. (b) Chiral excitation. Two sources facing each other are rotated in a symmetrical way directions: the energy is directed toward the bottom, meaning that  $S_1$  is selected. (c) Linear excitation. Two sources are shaken horizontally: a stationary wave is observed. This is the signature of the quasi ZGV mode. (d) The two sources are rotated and the wave is propagative. The phase velocity is negative in the top region ( $x_1 < 0$ ) and positive in the bottom region ( $x_1 > 0$ ).

## 517 B. Selective excitation

518 *a. One way excitation.* In the previous paragraph, the  $A'_2$  mode can be isolated during  
 519 the post-processing stage (thanks to the SVD). Here, the selection is directly performed  
 520 during the generation stage. Curie's principle states that the resulting wavefield should at

521 least have the same symmetries as the source. As a consequence, a vertically driven source  
 522 placed in the middle of the strip only feeds the symmetrical modes. The displacements over  
 523 one cycle for such an excitation at 136 Hz are represented in figure 19b. The only available  
 524 symmetrical modes at 136 Hz is  $S'_1$  for the bottom part and  $S_1'^*$  for the top one. The motion  
 525 is vertical in the center while it remains elliptical everywhere else as a consequence of the  $\pi/2$   
 526 phase shift between the two components (Equations (28)). However, the rotation directions  
 527 are different on either side of the source. This is consistent with the fact that  $S'_1$  and  $S_1'^*$   
 528 are phase-conjugate partners.

529 In order to exploit this specific polarization, the single vertical source is now replaced by  
 530 two chiral sources (see figure 19c) which are rotated in a symmetric fashion. The rotating  
 531 sources are designed by connecting a clamp to two perpendicular speakers, and the phase  
 532 quadrature excitation is controlled thanks to a 4 channels soundboard (Presonus Audiobox  
 533 44VSL). The resulting trajectories (figure 19c) demonstrate that  $S'_1$  is fed but not  $S_1'^*$ : the  
 534 top part of the strip remains still. Indeed the rotation direction of the source corresponds to  
 535 that of  $S'_1$ ; it demonstrates how chirality can be used to perform selective excitation. One  
 536 can get a clearer picture of the phenomenon by examining the field maps. The 12 successive  
 537 snapshots of the strip over a full wave period are represented next to each other in figure 20.  
 538 The color scale here only indicates the displacement  $u_1$ . Just like for figure 19b, when the  
 539 excitation is purely vertical,  $S'_1$  and  $S_1'^*$  are fed and the whole strip is excited (Fig. 20a). As  
 540 the source becomes chiral, only  $S'_1$  is selected: waves travel in the bottom part, while the  
 541 top part is not excited (Fig. 20b).

542 *b. Mode separation.* Chiral selection can also be performed with anti-symmetric modes.  
 543 The strip is now shaken horizontally by two clamps driven simultaneously at 102 Hz (near  
 544 the quasi ZGV frequency) following an anti-symmetric scheme. The field pattern in fig-  
 545 ure 20c (which here corresponds to the displacement  $u_2$ ) reveals stationary: the zeroes of  
 546 displacement remain at the same position over a full excitation period (dashed lines). This  
 547 is possible only if two waves propagating in opposite directions interfere on both sides of  
 548 the strip. According to figure 18, at this frequency, there are precisely two coexisting modes  
 549 on either side of the strip:  $A'_1$  and  $A'_{2b}$  on the bottom part, and  $A'^*_1$  and  $A'^*_{2b}$  on the top  
 550 part. Near the quasi ZGV point, their wavenumber magnitudes become similar, meaning  
 551 that their interferences can give birth to a standing wave.

552 As depicted in figure 20d, when the clamps are rotated in an anti-symmetrical manner,  
 553 the wavefield returns to propagative on both parts. And notably, the zeroes travel toward  
 554 the bottom on both sides (dashed lines). On the upper part, the wave-fronts are backward,  
 555 *i.e.* they move toward the source, and therefore corresponds to  $A'^*_{2b}$ . In the bottom part,  
 556 only  $A'_1$  is fed and the wave-fronts travel away from the source. The chiral excitation has  
 557 allowed here to separate the two components of a quasi ZGV point.

## 558 V. CONCLUSION

559 This article introduces a new “playground” to study wave guiding of elastic waves. It  
 560 relies on the use of a commercial silicon elastomer. Soft elastomers enable large displacements  
 561 and slow propagation, which drastically facilitates the experimental procedure. With a few  
 562 different configurations, we show how this highly visual tool is adequate to explore wave

563 physics phenomena. Furthermore, their quasi-incompressible nature enables the observation  
564 of original dispersion effects such as a Dirac cone<sup>1</sup>. Starting with simple experiments of  
565 linearly polarized plane-waves propagating in a thin plate, it ends with complex chiral mode  
566 selection near a quasi zero-group velocity mode in a strip with clamped edges. This allows  
567 to go over the simple theory of a scalar field guided by two interfaces, namely *SH* modes,  
568 to more complex waveguides where two waves with different velocities and polarizations are  
569 coupled at each reflection, namely Lamb modes.

570 The work is not over and many other complex guiding geometries can be envisioned.  
571 The nearly incompressible nature of the medium being a property shared with most of the  
572 biological tissues, analogies with elastic waves existing in the living world can be made. At  
573 least three types of wave guides can be identified in the human body. The cochlear wave  
574 inside the inner ear of mammals is supported by the basilar membrane which resembles  
575 the clamped strip studied here. The vocal cords, whose vibrations are responsible of sound  
576 control, could be the support of complex stationary fields. Last, arteries or neuronal axons  
577 are fluid filled circular soft waveguides also hosting interesting wave phenomena.

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583 **REFERENCES**

- 584 <sup>1</sup>M. Lanoy, F. Lemoult, A. Eddi, and C. Prada, “Dirac cones and chiral selection of elastic  
585 waves in a soft strip,” *Proc Natl Acad Sci USA* **117**(48), 30186–30190 (2020) doi: [10.  
586 1073/pnas.2010812117](https://doi.org/10.1073/pnas.2010812117).
- 587 <sup>2</sup>I. Levental, P. C. Georges, and P. A. Janmey, “Soft biological materials and their impact  
588 on cell function,” *Soft Matter* **3**(3), 299–306 (2007).
- 589 <sup>3</sup>M. A. Wozniak and C. S. Chen, “Mechanotransduction in development: a growing role for  
590 contractility,” *Nature reviews Molecular cell biology* **10**(1), 34 (2009).
- 591 <sup>4</sup>S. Kumar and V. M. Weaver, “Mechanics, malignancy, and metastasis: the force journey  
592 of a tumor cell,” *Cancer and Metastasis Reviews* **28**(1-2), 113–127 (2009).
- 593 <sup>5</sup>P. Kalita and R. Schaefer, “Mechanical models of artery walls,” *Archives of Computational*  
594 *Methods in Engineering* **15**(1), 1–36 (2008).
- 595 <sup>6</sup>P. R. Murray and S. L. Thomson, “Synthetic, multi-layer, self-oscillating vocal fold model  
596 fabrication,” *JoVE (Journal of Visualized Experiments)* (58), e3498 (2011).
- 597 <sup>7</sup>C. Vannelli, J. Moore, J. McLeod, D. Ceh, and T. Peters, “Dynamic heart phantom with  
598 functional mitral and aortic valves,” in *Medical Imaging 2015: Image-Guided Procedures,  
599 Robotic Interventions, and Modeling*, International Society for Optics and Photonics (2015),  
600 Vol. 9415, p. 941503.
- 601 <sup>8</sup>G. R. Gossweiler, C. L. Brown, G. B. Hewage, E. Sapiro-Gheiler, W. J. Trautman, G. W.  
602 Welshofer, and S. L. Craig, “Mechanochemically Active Soft Robots,” *ACS Appl. Mater.*  
603 *Interfaces* **7**(40), 22431–22435 (2015) doi: [10.1021/acsami.5b06440](https://doi.org/10.1021/acsami.5b06440).

- 604 <sup>9</sup>L. Marechal, P. Balland, L. Lindenroth, F. Petrou, C. Kontovounisios, and F. Bello, “To-  
605 ward a common framework and database of materials for soft robotics,” *Soft Robotics* **8**(3),  
606 284–297 (2021).
- 607 <sup>10</sup>E. Siéfert, E. Reyssat, J. Bico, and B. Roman, “Bio-inspired pneumatic shape-morphing  
608 elastomers,” *Nature Mater* **18**(1), 24–28 (2019) doi: [10.1038/s41563-018-0219-x](https://doi.org/10.1038/s41563-018-0219-x).
- 609 <sup>11</sup>S. Y. Kim, R. Baines, J. Booth, N. Vasios, K. Bertoldi, and R. Kramer-Bottiglio, “Recon-  
610 figurable soft body trajectories using unidirectionally stretchable composite laminae,” *Nat*  
611 *Commun* **10**(1), 3464 (2019) doi: [10.1038/s41467-019-11294-7](https://doi.org/10.1038/s41467-019-11294-7).
- 612 <sup>12</sup>L. Sandrin, M. Tanter, J.-L. Gennisson, S. Catheline, and M. Fink, “Shear elasticity  
613 probe for soft tissues with 1-d transient elastography,” *IEEE transactions on ultrasonics,*  
614 *ferroelectrics, and frequency control* **49**(4), 436–446 (2002).
- 615 <sup>13</sup>J. Foucher, E. Chanteloup, J. Vergniol, L. Castera, B. Le Bail, X. Adhoute, J. Bertet,  
616 P. Couzigou, and V. de Ledinghen, “Diagnosis of cirrhosis by transient elastography (fi-  
617 broscan): a prospective study,” *Gut* **55**(3), 403–408 (2006).
- 618 <sup>14</sup>J.-L. Gennisson, T. Deffieux, M. Fink, and M. Tanter, “Ultrasound elastography: princi-  
619 ples and techniques,” *Diagnostic and interventional imaging* **94**(5), 487–495 (2013).
- 620 <sup>15</sup>M. Couade, M. Pernot, C. Prada, E. Messas, J. Emmerich, P. Bruneval, A. Criton, M. Fink,  
621 and M. Tanter, “Quantitative assessment of arterial wall biomechanical properties using  
622 shear wave imaging,” *Ultrasound in medicine & biology* **36**(10), 1662–1676 (2010).
- 623 <sup>16</sup>A. V. Astaneh, M. W. Urban, W. Aquino, J. F. Greenleaf, and M. N. Guddati, “Arterial  
624 waveguide model for shear wave elastography: implementation and in vitro validation,”

- 625 Physics in Medicine & Biology **62**(13), 5473 (2017).
- 626 <sup>17</sup>E. Maksuti, F. Bini, S. Fiorentini, G. Blasi, M. W. Urban, F. Marinozzi, and M. Larsson,  
627 “Influence of wall thickness and diameter on arterial shear wave elastography: a phantom  
628 and finite element study,” Physics in Medicine & Biology **62**(7), 2694 (2017).
- 629 <sup>18</sup>I. Z. Nenadic, M. W. Urban, S. A. Mitchell, and J. F. Greenleaf, “Lamb wave dispersion  
630 ultrasound vibrometry (lduv) method for quantifying mechanical properties of viscoelastic  
631 solids,” Physics in Medicine & Biology **56**(7), 2245 (2011).
- 632 <sup>19</sup>I. Z. Nenadic, M. W. Urban, S. Aristizabal, S. A. Mitchell, T. C. Humphrey, and J. F.  
633 Greenleaf, “On lamb and rayleigh wave convergence in viscoelastic tissues,” Physics in  
634 Medicine & Biology **56**(20), 6723 (2011).
- 635 <sup>20</sup>J. Brum, M. Bernal, J. Gennisson, and M. Tanter, “In vivo evaluation of the elastic  
636 anisotropy of the human achilles tendon using shear wave dispersion analysis,” Physics in  
637 Medicine & Biology **59**(3), 505 (2014).
- 638 <sup>21</sup>H.-C. Liou, F. Sabba, A. I. Packman, G. Wells, and O. Balogun, “Nondestructive charac-  
639 terization of soft materials and biofilms by measurement of guided elastic wave propagation  
640 using optical coherence elastography,” Soft matter **15**(4), 575–586 (2019).
- 641 <sup>22</sup>J. Griesbauer, S. Bössinger, A. Wixforth, and M. F. Schneider, “Propagation of 2d pressure  
642 pulses in lipid monolayers and its possible implications for biology,” Phys. Rev. Lett. **108**,  
643 198103 (2012) doi: [10.1103/PhysRevLett.108.198103](https://doi.org/10.1103/PhysRevLett.108.198103).
- 644 <sup>23</sup>M. Hirano, “Morphological structure of the vocal cord as a vibrator and its variations,”  
645 Folia Phoniatica et Logopaedica **26**(2), 89–94 (1974).

- 646 <sup>24</sup>L. Robles and M. A. Ruggero, “Mechanics of the mammalian cochlea,” *Physiological*  
647 *reviews* **81**(3), 1305–1352 (2001).
- 648 <sup>25</sup>T. Reichenbach and A. Hudspeth, “The physics of hearing: fluid mechanics and the active  
649 process of the inner ear,” *Reports on Progress in Physics* **77**(7), 076601 (2014).
- 650 <sup>26</sup>D. Royer and E. Dieulesaint, *Elastic waves in solids I: Free and guided propagation*  
651 (Springer Science & Business Media, 1999).
- 652 <sup>27</sup>D. N. Alleyne and P. Cawley, “The interaction of lamb waves with defects,” *IEEE trans-*  
653 *actions on ultrasonics, ferroelectrics, and frequency control* **39**(3), 381–397 (1992).
- 654 <sup>28</sup>Z. Su, L. Ye, and Y. Lu, “Guided lamb waves for identification of damage in composite  
655 structures: A review,” *Journal of sound and vibration* **295**(3-5), 753–780 (2006).
- 656 <sup>29</sup>S. Bramhavar, C. Prada, A. A. Maznev, A. G. Every, T. B. Norris, and T. W. Murray,  
657 “Negative refraction and focusing of elastic lamb waves at an interface,” *Physical Review*  
658 *B* **83**(1), 014106 (2011).
- 659 <sup>30</sup>F. D. Philippe, T. W. Murray, and C. Prada, “Focusing on plates: controlling guided  
660 waves using negative refraction,” *Scientific reports* **5**, 11112 (2015).
- 661 <sup>31</sup>I. Tolstoy and E. Usdin, “Wave propagation in elastic plates: low and high mode disper-  
662 sion,” *The Journal of the Acoustical Society of America* **29**(1), 37–42 (1957).
- 663 <sup>32</sup>S. D. Holland and D. E. Chimenti, “Air-coupled acoustic imaging with zero-group-velocity  
664 lamb modes,” *Applied physics letters* **83**(13), 2704–2706 (2003).
- 665 <sup>33</sup>C. Prada, O. Balogun, and T. Murray, “Laser-based ultrasonic generation and detection  
666 of zero-group velocity lamb waves in thin plates,” *Applied Physics Letters* **87**(19), 194109

667 (2005).

668 <sup>34</sup>C. Prada, D. Clorennec, and D. Royer, “Local vibration of an elastic plate and zero-group  
669 velocity lamb modes,” *The Journal of the Acoustical Society of America* **124**(1), 203–212  
670 (2008).

671 <sup>35</sup>S. Wildeman, “Real-time quantitative schlieren imaging by fast fourier demodulation of a  
672 checkered backdrop,” *Experiments in Fluids* **59**(6), 97 (2018).

673 <sup>36</sup>S. Wildeman (2021), <https://github.com/swildeman/dicflow>.

674 <sup>37</sup>B. A. Auld, *Acoustic Fields and Waves in Solids* (John Wiley & Sons Inc., 1973).

675 <sup>38</sup>In the polymer or rheology communities the Lamé coefficient  $\mu$  is noted  $G$ .

676 <sup>39</sup>D. E. Muller, “A method for solving algebraic equations using an automatic computer,”  
677 *Mathematics of Computation* **10**, 208–215 (1956).

678 <sup>40</sup>A. A. Krushynska and V. V. Meleshko, “Normal waves in elastic bars of rectangular cross  
679 section,” *The Journal of the Acoustical Society of America* **129**(3), 1324–1335 (2011) doi:  
680 [10.1121/1.3531800](https://doi.org/10.1121/1.3531800).

681 <sup>41</sup>M. Cross and R. Lifshitz, “Elastic wave transmission at an abrupt junction in a thin plate  
682 with application to heat transport and vibrations in mesoscopic systems,” *Physical Review*  
683 *B* **64**, 085324 (2001) doi: [10.1103/PhysRevB.64.085324](https://doi.org/10.1103/PhysRevB.64.085324).

684 <sup>42</sup>J. Laurent, D. Royer, and C. Prada, “In-plane backward and zero-group-velocity guided  
685 modes in rigid and soft strips,” *The Journal of the Acoustical Society of America* **147**  
686 (2020) doi: [10.1121/10.0000760](https://doi.org/10.1121/10.0000760).

- 687 <sup>43</sup>R. D. Mindlin, *An Introduction to the Mathematical Theory of Vibrations of Elastic Plates*  
688 (J. Yang (Ed.),, World Scientific, Singapore, 2006).
- 689 <sup>44</sup>D. M. Stobbe and T. W. Murray, “Conical dispersion of lamb waves in elastic plates,”  
690 *Physical Review B* **96**(14), 144101 (2017).
- 691 <sup>45</sup>A. Maznev, “Dirac cone dispersion of acoustic waves in plates without phononic crystals,”  
692 *The Journal of the Acoustical Society of America* **135**(2), 577–580 (2014).
- 693 <sup>46</sup>X. Huang, Y. Lai, Z. H. Hang, H. Zheng, and C. Chan, “Dirac cones induced by acciden-  
694 tal degeneracy in photonic crystals and zero-refractive-index materials,” *Nature materials*  
695 **10**(8), 582 (2011).
- 696 <sup>47</sup>R. D. Mindlin and M. A. Medick, “Extensional Vibrations of Elastic Plates,” *Journal of*  
697 *Applied Mechanics* **26**(4), 561–569 (2021) doi: [10.1115/1.4012112](https://doi.org/10.1115/1.4012112).
- 698 <sup>48</sup>B. Gérardin, J. Laurent, C. Prada, and A. Aubry, “Negative reflection of lamb waves  
699 at a free edge: Tunable focusing and mimicking phase conjugation,” *The Journal of the*  
700 *Acoustical Society of America* **140**(1), 591–600 (2016).
- 701 <sup>49</sup>Again, in the polymer or rheology communities the Lamé coefficient  $\mu$  is noted  $G$  which  
702 leads to the real part  $G'$  and the imaginary part  $G''$ .
- 703 <sup>50</sup>F. C. Meral, T. J. Royston, and R. L. Magin, “Surface response of a fractional order  
704 viscoelastic halfspace to surface and subsurface sources,” *The Journal of the Acoustical*  
705 *Society of America* **126**(6), 3278–3285 (2009).
- 706 <sup>51</sup>S. P. Kearney, A. Khan, Z. Dai, and T. J. Royston, “Dynamic viscoelastic models of human  
707 skin using optical elastography,” *Physics in Medicine & Biology* **60**(17), 6975 (2015).

708 <sup>52</sup>E. Rolley, J. H. Snoeijer, and B. Andreotti, “A flexible rheometer design to measure the  
709 visco-elastic response of soft solids over a wide range of frequency,” *Review of scientific  
710 instruments* **90**(2), 023906 (2019).

711 <sup>53</sup>Y. Wan, Y. Xiong, and S. Zhang, “Temperature dependent dynamic mechanical properties  
712 of magnetorheological elastomers: Experiment and modeling,” *Composite Structures* **202**,  
713 768–773 (2018).

714 **REFERENCES**

715 <sup>1</sup>In the polymer or rheology communities the Lamé coefficient  $\mu$  is noted  $G$ .

716 <sup>2</sup>Again, in the polymer or rheology communities the Lamé coefficient  $\mu$  is noted  $G$  which  
717 leads to the real part  $G'$  and the imaginary part  $G''$ .