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► **To cite this version:**

Parham Shams, Aurélie Beynier, Sylvain Bouveret, Nicolas Maudet. Fair in the Eyes of Others. Journal of Artificial Intelligence Research, 2022, 75, pp.913-951. 10.1613/jair.1.13778 . hal-03863934

**HAL Id: hal-03863934**

**<https://hal.science/hal-03863934>**

Submitted on 21 Nov 2022

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## Fair in the Eyes of Others

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### Abstract

Envy-freeness is a widely studied notion in resource allocation, capturing some aspects of fairness. The notion of envy being inherently subjective though, it might be the case that an agent envies another agent, but that from the other agents' point of view, she has no reason to do so. The difficulty here is to define the notion of objectivity, since no ground-truth can properly serve as a basis of this definition. A natural approach is to consider the judgement of the other agents as a proxy for objectivity. Building on previous work by Parijs (who introduced "unanimous envy") we propose the notion of approval envy: an agent  $a_i$  experiences approval envy towards  $a_j$  if she is envious of  $a_j$ , and sufficiently many agents agree that this should be the case, from their own perspectives. Another thoroughly studied notion in resource allocation is proportionality. The same variant can be studied, opening natural questions regarding the links between these two notions. We exhibit several properties of these notions. Computing the minimal threshold guaranteeing approval envy and approval non-proportionality clearly inherits well-known intractable results from envy-freeness and proportionality, but (i) we identify some tractable cases such as house allocation; and (ii) we provide a general method based on a mixed integer programming encoding of the problem, which proves to be efficient in practice. This allows us in particular to show experimentally that existence of such allocations, with a rather small threshold, is very often observed.

### 1. Introduction

Fair division is an ubiquitous problem in multiagent systems or economics (Steinhaus, 1948; Moulin, 2003; Young, 1994), with applications ranging from allocation of schools, courses or rooms to students (Abraham et al., 2005; Othman et al., 2010), to division of goods in inheritance or divorce settlement (Brams & Taylor, 1996). Envy-freeness (EF), is one of the prominent notions studied in fair division (Foley, 1967; Brams & Fishburn, 2002; Lipton et al., 2004; de Keijzer et al., 2009; Segal-Halevi & Suksompong, 2019). An allocation of items among a set of agents is said to be envy-free if no agent prefers the share of another agent to her own share. Unfortunately, in the indivisible setting, envy-freeness is a pretty demanding notion and an envy-free allocation may not exist. That explains why recent literature has proposed a lot of relaxations of envy-freeness, such as for instance envy-freeness up to one good, EF1 (Budish, 2011), or envy-freeness up to any good, EFX (Gourvès et al., 2014; Caragiannis et al., 2016). When agents interact along a social network, local notions of fairness have been investigated where an agent can only make comparisons with her neighbors in the network. In the divisible setting, local envy-freeness and local proportionality

have been studied for instance by Abebe et al. (2017) and Bei et al. (2017). In the indivisible setting, similar notions have been studied by Aziz et al. (2018) and Chevaleyre et al. (2017), while complexity issues related to local envy-freeness have been investigated in oriented graphs (Bredereck et al., 2018) and non-oriented graphs in house allocation problems (Beynier et al., 2019b).

We propose a slightly different relaxation of envy. To illustrate the notion we introduce, consider a given instance where no envy-free allocation exists. Now suppose that in this instance there exist two allocations  $\pi$  and  $\pi'$  that make a single agent (say,  $a_i$ ) envious of some other agent  $a_j$  (for simplicity). Furthermore, assume that in allocation  $\pi$ , no agent but  $a_i$  thinks that  $a_j$ 's bundle is better than  $a_i$ 's, while in allocation  $\pi'$  all the other agents concur with the assessment that  $a_i$  envies  $a_j$ . According to Parijs (1997),  $\pi'$  exhibits *unanimous envy*, and it seems difficult to justify that  $\pi'$  should be returned in place of  $\pi$ . Inspired by this notion, we define in this paper the notion of  $K$ -approval envy, as a way to introduce a continuum between envy-freeness and unanimous envy. As may be clear from the name, the idea is simply to ask agents to express their own view about envy relations expressed by other agents. The objective will thus be to seek allocations minimizing social support for the expressed envy relations, *i.e.* minimizing the number of agents  $K$  approving the envy. Of course, this approach may be controversial: after all, the notion of preference is inherently subjective. Introducing this flavour of objectivity may lead to undesirable consequences. At the extreme, one may simply replace individual preferences by some unanimous “mean” profile, thus profoundly changing the very nature of the notion. We believe though that there are several justifications to this approach:

- First, note that we only seek the approval of other agents in the case the agent herself explicitly expresses envy: absence of envy thus remains completely subjective. While a symmetrical treatment may also be justifiable in some situations, there is an obvious reason which motivates us to start with the proposed definition, namely the fact that the notion would no longer be a relaxation of envy-freeness.
- Secondly, all other things being equal, we believe that an allocation minimizing  $K$  is socially more desirable. We do not necessarily regard this notion as a compelling choice, but we think this can enrich the picture of fallback allocations when no envy-free allocation exists, as other relaxations do (Amanatidis et al., 2018). In particular, in repeated settings, the fact that agents perceive outcomes as globally fair (not only for themselves, but also for others) may be important as an incentive for participation.
- Finally, one further motivation of our work is that our approach can be seen as providing guidance regarding agents and more specifically agents' preferences, in order to progress towards envy-freeness by helping them revise their utilities for example. In particular, if we envision systems integrating deliberative phases in the collective decision-making process, our model could be used to set the agenda of such deliberations. If a vast majority of agents contradict an agent on her envy towards another agent, it may indicate for instance that she lacks information regarding the actual value of (some items of) her share. Initiating a discussion might help to solve such “objectively unjustified” envies when they occur.

While envy-freeness is a widely studied notion in fair division of indivisible goods, another prominent notion in the literature is *proportionality*. This notion is based on the proportional share: the proportional share of an agent is equal to one of the  $n^{\text{th}}$  of the utility this agent gives to the whole set of objects (with  $n$  the number of agents). An allocation is proportional if and only if each agent

receives at least her proportional share. Note that there are strong links between proportionality and envy-freeness, namely, any envy-free allocation is also proportional, whereas, on the contrary, there are instances for which a proportional allocation exists but no envy-free one (Bouveret & Lemaître, 2016). As is the case for envy-freeness, a proportional allocation is not guaranteed to exist. As a result, there has been a lot of work in recent literature about a relaxation of proportionality called proportionality up to one item (PROP1) (Conitzer et al., 2017). In the same spirit as EF1, PROP1 requires each agent to get her proportional share by obtaining the object of some other agent that she values the most. In light of these remarks and given the strong relationships between envy-freeness and proportionality, we also explore an approval version of this latter notion.

**Outline of the paper.** The remainder of this paper is as follows. Section 2 recalls some basic notions in fair division. Our notion of  $K$ -approval envy is presented in Section 3. Some properties of this notion are then studied in Section 4: it is shown in particular, that if the hypothetical situation of allocation  $\pi$  described at the beginning of the introduction occurs, then an EF allocation must also exist. We also show that our notion inherits from the complexity of related problems. After introducing the approval notion around proportionality in Section 5, some properties are put forward in Section 6 and some links between our two approval notions are studied in Section 7. As we did for approval envy, we show that the problem inherits from the hardness of the classical notion. This hardness results motivate the MIP formulations that we detail in Section 8. We next turn to the *house allocation* setting and we show that if each agent exactly holds a single item, then we can define an efficient algorithm returning an allocation minimizing the value of  $K$  for both our approval notions. One caveat of our notions is that (unlike other relaxations) it is not guaranteed to exist, as intuitively observed in the case of unanimous envy and unanimous non-proportionality. In Section 10, we provide empirical evidences showing that allocations with reasonable values of  $K$  exist under synthetic cultures as well as in real datasets.

## 2. Model and Definitions

We consider MultiAgent Resource Allocation problems (MARA) where we aim at fairly dividing a set of indivisible objects (also called items or goods) among a set of agents. A MARA instance  $I$  is defined as a finite set of *objects*  $\mathcal{O} = \{o_1, \dots, o_m\}$ , a finite set of *agents*  $\mathcal{N} = \{a_1, \dots, a_n\}$  and a preference profile  $\mathcal{P}$  representing the interest of each agent  $a_i \in \mathcal{N}$  towards the objects. An allocation  $\pi$  is a mapping of the objects in  $\mathcal{O}$  to the agents in  $\mathcal{N}$ . In the following,  $\pi_i$  will denote the set of indivisible objects (the share) held by agent  $a_i$ . An allocation is such that  $\forall a_i, \forall a_j$  with  $i \neq j : \pi_i \cap \pi_j = \emptyset$  (a given object cannot be allocated to more than one agent) and  $\bigcup_{a_i \in \mathcal{N}} \pi_i = \mathcal{O}$  (all the objects from  $\mathcal{O}$  are allocated).

In this paper, we consider cardinal preference profiles so, the preferences of an agent  $a_i$  over bundles of objects are defined by a *utility function*  $u_i : 2^{\mathcal{O}} \rightarrow \mathbb{Q}^+$  measuring her satisfaction  $u_i(\pi_i)$  when she obtains share  $\pi_i$ . We make the assumption that utility functions are additive *i.e.* the utility of an agent  $a_i$  over a share  $\pi_i$  is defined as the sum of the utilities over the objects forming  $\pi_i$ :

$$u_i(\pi_i) \stackrel{\text{def}}{=} \sum_{o_k \in \pi_i} u(i, k),$$

where  $u(i, k)$  is the utility given by agent  $a_i$  to object  $o_k$ . This assumption is commonly considered in MARA (Lipton et al., 2004; Procaccia & Wang, 2014; Dickerson et al., 2014; Caragiannis

et al., 2016, for instance) as additive utility functions provide a compact but yet expressive way to represent the preferences of the agents. MARA instances with additive utility functions are called add-MARA instances for short.

Different notions have been proposed in the literature to evaluate the fairness of an allocation. When the agents can compare their shares, the absence of envy (Foley, 1967; Lipton et al., 2004; Chevaleyre et al., 2017) is a particularly relevant notion of fairness. An agent  $a_i$  would envy another agent  $a_j$  if she prefers the share of  $a_j$  over her own share. More formally, an agent  $a_i$  envies an agent  $a_j$  iff

$$u_i(\pi_j) > u_i(\pi_i).$$

A completely fair allocation would thus be an *envy-free allocation* i.e. an allocation where no agent envies another agent. Formally:

$$\forall a_i, a_j \in \mathcal{N}, u_i(\pi_i) \geq u_i(\pi_j).$$

The notion of envy-freeness conveys a natural concept of fairness viewed as social stability: agents are happy with their bundle and hence do not want to swap it with any other agent's (regarding their own preferences). However, as soon as it is required to allocate all the objects in  $\mathcal{O}$ , an envy-free allocation may not exist. An alternative objective may be to minimize a degree of envy of the society (Lipton et al., 2004; Nguyen & Rothe, 2014; Chevaleyre et al., 2017), based on the notion of *pairwise envy*.

**Definition 1** (Pairwise envy). *Let  $\pi$  be an allocation. The pairwise envy  $pe(i, j, \pi)$  of an agent  $a_i$  towards an agent  $a_j$  in  $\pi$  is defined as follows:*

$$pe(i, j, \pi) \stackrel{\text{def}}{=} \max\{0, u_i(\pi_j) - u_i(\pi_i)\}.$$

The pairwise envy can be interpreted as how much agent  $a_i$  envies agent  $a_j$ 's share (this envy being 0 if  $a_i$  does not envy  $a_j$ ). We can derive from this notion a collective measure of envy similar to the one used by Aleksandrov *et al.* (2019):

**Definition 2** (Degree of envy of the society). *The degree of envy  $de(\pi)$  of the society for an allocation  $\pi$  is defined as follows:*

$$de(\pi) \stackrel{\text{def}}{=} \sum_{a_i \in \mathcal{N}} \sum_{a_j \in \mathcal{N}} pe(i, j, \pi).$$

Note that an allocation  $\pi$  is envy-free if and only if  $de(\pi) = 0$ .

To cope with the possible non-existence of an envy-free allocation, another approach is to alleviate the requirements of the fairness notion. Recently, several relaxations of envy-freeness have been proposed such as envy-freeness up to one good (EF1) (Budish, 2011) or envy-freeness up to any good (EFX) (Caragiannis et al., 2016). An allocation is said to be envy-free up to one good (resp. up to any good) if no agent  $a_i$  envies the share  $\pi_j$  of another agent  $a_j$  after removing from  $\pi_j$  *one* (resp. *any*) item. Existence for EF1 is guaranteed, but this is still to the best of our knowledge an open question for EFX in the general case. However, the existence guarantee of an EFX solutions has been proved for few agents (at most 3 agents) and specific utility functions. For instance it has been proved that an EFX allocation always exists for instances with identical valuations and for instances involving two agents with general and possibly distinct valuations (Plaut & Roughgarden, 2018), as well as for three agents with additive valuations (Chaudhury et al., 2020). When the

objects have only two possible valuations, Amanatidis et al. (2020) proved that any allocation maximizing the Nash Social Welfare is EFX. This result provides a polynomial algorithm for computing EFX allocations in the two-agent setting. Amanatidis et al. (2018) studied four fairness notions – envy-freeness up to one good (EF1), envy-freeness up to any good (EFX), maximin share fairness (MMS), and pairwise maximin share fairness (PMMS) – and investigated the relations between these notions and their relaxations. Although PMMS is a stronger notion than EFX, Amanatidis et al. (2018) proved that both notions provide the same worst-case guarantee for MMS. In the same vein, they showed that EFX and EF1 both provide similar approximation for PMMS.

As discussed in Section 1, another widely studied fairness notion is proportionality:

**Definition 3** (Proportional share). *Let  $I$  be an add-MARA instance. The proportional share of an agent  $a_i$  is defined as follows:*

$$Prop_i \stackrel{\text{def}}{=} \frac{\sum_{j=0}^m u(i, j)}{n}.$$

*An allocation  $\pi$  is said to be proportional if and only if every agent gets her proportional share:*

$$\forall a_i \in \mathcal{N}, u_i(\pi_i) \geq Prop_i.$$

As mentioned in Section 1, even though proportionality is a less demanding fairness criterion than envy-freeness (Bouveret & Lemaître, 2016), the existence of a proportional allocation is not guaranteed. For that reason, relaxations of this notion such as proportionality up to one item (PROP1) has been proposed (Conitzer et al., 2017). An allocation satisfies PROP1 if every agent gets at least her proportional share when one item is added to her current bundle. For example, Conitzer et al. (2017) proved the existence of PROP1 allocations for a public decision setting where a decision has to be made on several public issues. Each issue has several possible alternatives and each agent has a utility for each alternative. The decision problem consists in choosing one alternative for each issue. Barman and Krishnamurthy (2019) presented a strongly polynomial-time algorithm to find PROP1 allocations for positive utilities. Note that some of these papers deal with fair division with chores (Brânzei & Sandomirskiy, 2019) or with mixed utilities (Aziz et al., 2019, 2020). It can be noticed that there is a similar kind of link between EF1 and PROP1 as the one that exists between EF and proportionality. Namely, any EF1 allocation is also PROP1 (which we could write  $EF1 \implies PROP1$  for short).

### 3. Approval Envy

The notion of envy being inherently subjective, it might be the case that an agent envies another agent, but that she has no reason to do so from the point of view of the other agents. The difficulty here is to define the notion of objectivity, since no ground-truth can properly serve as a basis of this definition. In her book, Guibet-Lafaye (2006) recalls the notion of *unanimous envy*, that was initially discussed by Parijs (1997), and that can be defined as follows: an agent  $a_i$  unanimously envies another agent  $a_j$ , if all the agents think that bundle  $\pi_j$  is strictly preferred to  $\pi_i$ . Here, unanimity is used as a proxy for objectivity.

As we can easily imagine, looking for allocations that are free of unanimous envy will be too weak to be interesting: as soon as one agent disagrees with the fact that  $a_i$  envies  $a_j$ , this potential envy will not be taken into account. Here, we propose an intermediate notion between envy-freeness and (unanimous envy)-freeness, based on the notion of approval: an agent  $a_k$  approves of agent  $a_i$

envying agent  $a_j$  if  $a_k$  thinks that bundle  $\pi_i$  is strictly worse than  $\pi_j$ . This allows to define a notion of  $K$ -approval envy :

**Definition 4** ( $K$ -approval envy). *Let  $\pi$  be an allocation,  $a_i, a_j$  be two different agents, and  $1 \leq K \leq n$  be an integer. We say that  $a_i$   $K$ -approval envies ( $K$ -app envies for short)  $a_j$  if there is a subset  $\mathcal{N}_K$  of  $K$  agents including  $a_i$  such that:*

$$\forall a_k \in \mathcal{N}_K, u_k(\pi_i) < u_k(\pi_j).$$

In other words, at least  $K - 1$  agents amongst  $\mathcal{N} \setminus \{a_i\}$  agree with  $a_i$  on the fact that  $\pi_i$  is actually strictly worse than  $\pi_j$ .

**Example 1.** *Let us consider the following add-MARA instance with 3 agents and 6 objects:*

	$o_1$	$o_2$	$o_3$	$o_4$	$o_5$	$o_6$
$a_1$	0	3	3	1	3	2
$a_2$	2	0	7	2	1	0
$a_3$	0	3	5	0	1	3

*Note that there is no envy-free allocation for this instance. In the squared allocation,  $a_1$  is not envious,  $a_2$  envies  $a_3$  and  $a_3$  envies  $a_1$ . Concerning the envy of  $a_2$  towards  $a_3$ ,  $a_1$  disagrees with  $a_2$  being envious of  $a_3$  whereas agent  $a_3$  agrees. Hence, agent  $a_2$  2-app envies agent  $a_3$ . Concerning the envy of  $a_3$  towards  $a_1$ , agent  $a_1$  agrees with  $a_3$  being envious of  $a_1$  whereas agent  $a_2$  does not. Hence,  $a_3$  2-app envies  $a_1$ .*

Note that in the definition, as soon as  $a_i$  does not envy  $a_j$ , then,  $a_i$  does not  $K$ -app envy  $a_j$ , no matter what the value of  $K$  is or how many agents think that  $\pi_i$  is actually worse than  $\pi_j$ . Doing so, we ensure that our approval notion is a relaxation of envy-freeness.

Let us start with an easy observation:

**Observation 1.** *Given an allocation  $\pi$  of an add-MARA instance, if  $a_i$   $K$ -app envies  $a_j$  in  $\pi$ , then  $a_i$   $(K-1)$ -app envies  $a_j$  in  $\pi$ .*

Moreover, if  $a_i$   $n$ -app envies  $a_j$ , we will say that  $a_i$  *unanimously* envies  $a_j$ . Finally, we can observe that  $a_i$  1-app envies  $a_j$  if and only if  $a_i$  envies  $a_j$ .

We can naturally derive from Definition 4 the counterpart of envy-freeness:

**Definition 5** ( $(K$ -approval envy)-free allocation). *An allocation  $\pi$  is said to be  $(K$ -app envy)-free if and only if  $a_i$  does not  $K$ -app envy  $a_j$  in  $\pi$  for all pairs of agents  $(a_i, a_j)$ .*

**Definition 6** ( $(K$ -approval envy)-free instance). *An add-MARA instance  $I$  is said to be  $(K$ -app envy)-free if and only if it accepts a  $(K$ -app envy)-free allocation.*

**Example 2.** *Going back to Example 1, the squared allocation is (3-app envy)-free so the instance is (3-app envy)-free.*

A threshold of special interest is obviously  $\lfloor n/2 \rfloor + 1$ , since it requires a strict majority to approve the envy under inspection. A Strict Majority approval envy-free (SM-app-EF) allocation is a  $(K$ -app envy)-free allocation such that  $K \leq \lfloor n/2 \rfloor + 1$ , translating the fact that every time envy occurs, there is a strict majority of agents that do not agree with that envy.

Going further, it is important to notice that ( $K$ -app envy)-freeness is not guaranteed to exist. Indeed, for all number of agents  $n$  and all number of objects  $m$ , there exist instances for which no ( $K$ -app envy)-free allocation exists, no matter what  $K$  is. Suppose for instance that all the agents rank the same object (say  $o_1$ ) first, and that for all  $a_i$ ,  $u(i, 1) > \sum_{k=2}^m u(i, k)$ . Then obviously, everyone agrees that all the agents envy the one that will receive  $o_1$ . Such instances will be called *unanimous envy instances*:

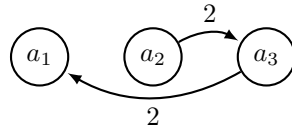
**Definition 7** (Unanimous envy instance). *An add-MARA instance  $I$  is said to exhibit unanimous envy if  $I$  is not ( $K$ -app envy)-free for any value of  $K$ .*

Observe that for an allocation to be ( $K$ -app envy)-free, for all pairs of agents  $(a_i, a_j)$ , either  $a_i$  or at least  $n - K + 1$  agents have to think that  $a_i$  does not envy  $a_j$ . Notice that it is different from requiring that at least  $K$  agents think that this allocation is envy-free. This explains the parenthesis around ( $K$ -app envy): this notion means “free of  $K$ -app envy”, which is different from “ $K$ -app-(envy-free)”.

A useful representation, for a given allocation, is the induced envy graph (Lipton et al., 2004): vertices are agents, and there is a directed edge from  $a_i$  to  $a_j$  if and only if  $a_i$  envies  $a_j$ . An allocation is envy-free if and only if the envy graph has no arc. In our context, we can define a weighted notion of the envy graph.

**Definition 8** (Weighted envy graph). *The weighted envy graph of an allocation  $\pi$  is defined as the weighted graph  $(\mathcal{N}, E)$  where nodes are agents, such that there is an edge  $(a_i, a_j) \in E$  if  $a_i$  envies  $a_j$ , with the weight  $w(a_i, a_j)$  corresponding to the number of agents (including  $a_i$ ) approving this pairwise envy in  $\pi$ .*

**Example 3.** *The induced weighted envy graph of Example 1 is as follows:*



Our notion of  $K$ -approval envy can be interpreted as a vote on envy, that works as follows. For each pair of agents  $(a_i, a_j)$ , if  $a_i$  declares to envy  $a_j$ , we ask the rest of the agents to vote on whether they think that  $a_i$  indeed envies  $a_j$ . Then, a voting procedure is used to determine whether  $a_i$  envies  $a_j$  according to the society of agents. Several voting procedures can be used. However, since there are only two candidates (yes / no), the most reasonable voting rules are based on quotas:  $a_i$  envies  $a_j$  if and only if there is a minimum quota of agents that think so.<sup>1</sup> This makes a connection with a related work of Segal-Halevi and Suksompong (2019) which uses voting to decide upon envy-freeness, but in the context of fair division of resources *jointly* owned by *groups* of agents.

Finally, we want to emphasize that our notion of  $K$ -approval envy is based on *pairwise* envy. Namely, if agent  $a_i$  envies  $a_j$ , we will try to evaluate how many other agents think that this envy is justified. Another possibility<sup>2</sup> would be, for each envious agent  $a_i$ , to evaluate how many other

1. More precisely, these rules exactly characterize the set of anonymous and monotonic voting rules (Perry & Powers, 2010).

2. We warmly thank an anonymous reviewer for pointing out this alternative notion to us.



agents think that  $a_i$  has indeed reasons to be envious, no matter which agent  $a_i$  envies. The difference is subtle. To illustrate this, suppose that  $a_i$  envies  $a_j$  and another agent  $a_k$  disagrees with this particular envy, but thinks that  $a_i$  has indeed reasons to envy  $a_l (\neq a_j)$ . With the first notion (our notion),  $a_k$ 's opinion will be discarded, whereas in the second one, it will be counted.

In practice, we believe that this alternative notion of approval envy will be much less discriminating than ours. The intuition can be explained as follows. Suppose that for some allocation  $\pi$  there is a bundle  $\pi_i$  that is not the top one for any agent. In that case, not only the agent  $a_i$  receiving  $\pi_i$  will envy another agent (since  $\pi_i$  is not  $a_i$ 's top bundle), but every other agent will also agree that  $a_i$  should be envious. Hence, the envy of  $a_i$  in  $\pi$  will be unanimously approved under the alternative notion. Now suppose in the contrary that every bundle of  $\pi$  is the top one for some agent. If preferences are strict on bundles, then by the pigeon hole principle, the top bundle of each agent has to be a different bundle of  $\pi$ , meaning that there is an envy-free allocation in that case. Said otherwise, if preferences are strict on bundles, an instance can only either be envy-free or exhibit unanimous envy under the alternative definition. The only edge case happens when agents can have several tied top bundles, which does not happen very often in practice.

The experiments we ran on the Spliddit instances (see Section 10) tend to confirm this intuition. This is why we decided not to investigate further this alternative notion of approval envy.

#### 4. Some Properties of Approval Envy

There are natural relations between the properties of ( $K$ -app envy)-freeness for different values of  $K$ . The following observation is a direct consequence of Observation 1.

**Observation 2.** *Let  $\pi$  be an allocation, and  $K \leq N$  be an integer. If  $\pi$  is ( $K$ -app envy)-free, then  $\pi$  is also  $((K+1)$ -app envy)-free.*

However, the converse does not hold. More precisely, the following proposition shows that the implication stated in Observation 2 is strict.

**Proposition 1.** *Let  $\pi$  be an allocation, and  $3 \leq K \leq n$  be an integer. If  $\pi$  is ( $K$ -app envy)-free,  $\pi$  is not necessarily  $((K-1)$ -app envy)-free.*

*Proof.* Let  $h \in \{2, \dots, n-1\}$  be an integer, and let us consider the instance with  $n$  agents and  $n$  objects defined as follows:

- $u(1, 1) = 1 - (n-1)\varepsilon$ ;
- $u(i, 1) = u(i, i) = \frac{1-(n-2)\varepsilon}{2}$  for  $i \in \{2, \dots, h-1\}$ ;
- $u(i, i) = 1 - (n-1)\varepsilon$  for  $i \in \{h, n-1\}$ ;
- $u(n, 1) = \frac{2}{n+1}$  and  $u(n, j) = \frac{1}{n+1}$  for  $j > 1$ ;

and  $u(i, j) = \varepsilon$  for other pairs with  $\varepsilon < \frac{1}{n+1}$ .

This construction is illustrated in the general case in Figure 1. Moreover, one instance with  $n = 4$  agents,  $m = 4$  objects and  $h = 3$  is shown in Example 4.

Consider the allocation  $\pi$  where each agent  $a_i$  gets item  $o_i$ . Obviously, the only envy in this allocation concerns  $a_n$  towards  $a_1$ . Moreover, only  $a_1, \dots, a_{h-1}$  agree on this envy. Therefore,  $a_n$   $h$ -app envies  $a_1$ , but does not  $(h+1)$ -app envy her. Moreover,  $\pi$  is  $((h+1)$ -app envy)-free, but not  $(h$ -app envy)-free.  $\square$

	$o_1$	$o_2$	$o_3$	$\dots$	$o_h$	$\dots$	$o_n$
$a_1$	$1 - (n-1)\varepsilon$	$\varepsilon$	$\varepsilon$	$\dots$	$\varepsilon$	$\dots$	$\varepsilon$
$a_2$	$\frac{1-(n-2)\varepsilon}{2}$	$\frac{1-(n-2)\varepsilon}{2}$	$\varepsilon$	$\dots$	$\varepsilon$	$\dots$	$\varepsilon$
$a_3$	$\frac{1-(n-2)\varepsilon}{2}$	$\varepsilon$	$\frac{1-(n-2)\varepsilon}{2}$	$\dots$	$\varepsilon$	$\dots$	$\varepsilon$
$\vdots$							
$a_h$	$\varepsilon$	$\varepsilon$	$\varepsilon$	$\dots$	$1 - (n-1)\varepsilon$	$\dots$	$\varepsilon$
$\vdots$							
$a_n$	$\frac{2}{n+1}$	$\frac{1}{n+1}$	$\frac{1}{n+1}$	$\dots$	$\frac{1}{n+1}$	$\dots$	$\frac{1}{n+1}$

Figure 1: Instance used in the proof of Proposition 1

**Example 4.** In order to illustrate the previous proof, let us consider the following instance with 4 agents, 4 objects (and  $h=3$ ) and the squared allocation  $\pi$ :

	$o_1$	$o_2$	$o_3$	$o_4$
$a_1$	$1 - 3\varepsilon$	$\varepsilon$	$\varepsilon$	$\varepsilon$
$a_2$	$\frac{1}{2} - \varepsilon$	$\frac{1}{2} - \varepsilon$	$\varepsilon$	$\varepsilon$
$a_3$	$\varepsilon$	$\varepsilon$	$1 - 3\varepsilon$	$\varepsilon$
$a_4$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

In this allocation, the only envy concerns  $a_4$  towards  $a_1$ . Moreover, only  $a_1$  and  $a_2$  agree with  $a_4$  on her envy. Hence,  $\pi$  is (4-app envy)-free but is obviously not (3-app envy)-free as we can find 3 agents ( $a_1$ ,  $a_2$  and  $a_4$ ) agreeing on the envy of  $a_4$  towards  $a_1$  (in other words  $a_4$  3-app envies  $a_1$ ).

**Proposition 2.** For any  $K \geq 3$ , there exist instances which are ( $K$ -app envy)-free but not ( $(K-1)$ -app envy)-free.

*Proof.* Consider the same instance as in Proposition 1. We have already shown that we have an allocation  $\pi$  that is  $((h+1)$ -app envy)-free which means that the instance is  $((h+1)$ -app envy)-free. We just have to show that there is no ( $h$ -app envy)-free allocation in order to conclude. For that purpose, we first note that each agent has to get one and exactly one object. Indeed, if it is not the case at least one agent  $a_i$  will have no object and will thus be envious of any agent  $a_j$  that has an object. Moreover, as all agents value the empty bundle with utility 0 and every object is valued with a strictly positive utility, this envy will be unanimous. Hence, each agent has to get one and exactly one object in order to minimize the ( $K$ -app envy)-freeness. Now consider objects  $o_j$  for  $j \in \{h, n\}$ . The agents  $a_j$  that receive an object  $o_j$  and that are envious will  $h$ -app envy the agent that received  $o_1$ . Indeed, agents  $a_i$  for  $i \in \{h, n-1\}$  value objects  $o_j$  with a utility higher than (or equal to) the one of  $o_1$  (and thus do not approve the envy) while it is the opposite for the other agents who are exactly  $h$  hence the  $h$ -app envy. So if we want to avoid that envy, we have to give the objects  $o_j$  to agents so that they do not experience envy at all but it is not possible as such agents are agents  $a_p$  for  $p \in \{h, n-1\}$ . It means that we have  $n-1-h+1$  agents that have to receive the  $n-h+1$

objects which is obviously impossible. This means that we cannot avoid  $h$ -app envy which implies that no allocation can be ( $h$ -app envy)-free.  $\square$

Proposition 2 proves that the hierarchy of ( $K$ -app) envy-free instances is strict for  $K \geq 3$ . Rather surprisingly, we will see that it is not the case for  $K = 2$ .

In order to show this result, we will resort to a tool that has been proved to be useful and powerful in many contexts dealing with envy (Biswas & Barman, 2018; Amanatidis et al., 2020; Beynier et al., 2019a): the *bundle reallocation cycle technique*. This technique, originating from the seminal work of Lipton *et al.* (2004), consists in performing a cyclic reallocation of *bundles* so that every agent is strictly better in the new allocation. Thus, such a reallocation corresponds to a cycle in the opposite direction of the edges in the — weighted — envy graph introduced in Definition 8. It is known that performing a reallocation cycle decreases the degree of envy (Lipton et al., 2004). Unfortunately, our first remark is that it does not necessarily decrease the level of  $K$ -app envy. Worse than that, it can actually increase it:

**Proposition 3.** *Let  $\pi$  be a ( $K$ -app envy)-free allocation, for  $3 \leq K \leq n - 1$ . After performing an improving bundle reallocation cycle (even between two agents), there can be an integer  $K' > K$  such that the resulting allocation is ( $K'$ -app envy)-free (and not ( $K$ -app envy)-free).*

*Proof.* Let  $h \in \{0, \dots, n - 4\}$  be an integer, and let us consider the instance with  $n$  agents and  $n$  objects defined by the following utility functions:

- $a_1$ :  $u(1, 1) = \varepsilon$ ,  $u(1, 2) = 2\varepsilon$ ,  $u(1, 3) = 1 - 3\varepsilon$ ;
- $a_2$ :  $u(2, 1) = 1 - \varepsilon$ ,  $u(2, 2) = \varepsilon$ ;
- $a_3$ :  $u(3, 3) = 1$ ;
- $a_l$  for  $l \in \{4, h + 3\}$ :  $u(l, 1) = u(l, 3) = \varepsilon$ ,  $u(l, j) = \frac{1-2\varepsilon}{n-3}$  for  $j \geq 4$ ;
- $a_m$  for  $m \in \{h + 4, n\}$ :  $u(m, 2) = \varepsilon$ ,  $u(m, 3) = 2\varepsilon$ ,  $u(m, i) = \frac{1-3\varepsilon}{n-3}$  for  $i \geq 4$ ;

and  $u(i, j) = 0$  for other pairs. We assume in this construction that  $\varepsilon \leq \frac{1}{2n-1}$ .

This construction is illustrated in Figure 2.

Consider the allocation  $\pi$  where each agent  $a_i$  gets item  $o_i$  (corresponding to the squared allocation in Figure 2). Obviously, in this allocation, there is no envy, except:

- $a_1$  envying  $a_2$  (agents  $a_{h+4} \dots a_n$  agree on that);
- $a_1$  envying  $a_3$  (agents  $a_3$  and  $a_{h+4} \dots a_n$  agree on that);
- $a_2$  envying  $a_1$  (agents  $a_4 \dots a_{h+3}$  agree on that).

Hence the allocation is  $((\max\{n - h, h + 2\})$ -app envy)-free. We now consider the allocation  $\pi'$  resulting from the improving bundle reallocation cycle between  $a_1$  and  $a_2$  (circled allocation in Figure 2). Observe that the only envy in  $\pi'$  is the one of  $a_1$  towards  $a_3$ , which is approved by everyone except  $a_2$ . This allocation is thus  $(n$ -app envy)-free and not  $((n - 1)$ -app envy)-free. If  $h > 0$ , then  $\max\{n - h, h + 2\} < n$ , which proves the proposition.  $\square$

	$o_1$	$o_2$	$o_3$	$o_4$	$\dots$	$o_{h+3}$	$o_{h+4}$	$\dots$	$o_n$
$a_1$	$\boxed{\varepsilon}$	$\boxed{2\varepsilon}$	$1 - 3\varepsilon$	0	$\dots$	0	0	$\dots$	0
$a_2$	$\boxed{1 - \varepsilon}$	$\boxed{\varepsilon}$	0	0	$\dots$	0	0	$\dots$	0
$a_3$	0	0	$\boxed{1}$	0	$\dots$	0	0	$\dots$	0
$a_4$	$\varepsilon$	0	$\varepsilon$	$\boxed{\frac{1-2\varepsilon}{n-3}}$	$\dots$	$\frac{1-2\varepsilon}{n-3}$	$\frac{1-2\varepsilon}{n-3}$	$\dots$	$\frac{1-2\varepsilon}{n-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{h+3}$	$\varepsilon$	0	$\varepsilon$	$\frac{1-2\varepsilon}{n-3}$	$\dots$	$\boxed{\frac{1-2\varepsilon}{n-3}}$	$\frac{1-2\varepsilon}{n-3}$	$\dots$	$\frac{1-2\varepsilon}{n-3}$
$a_{h+4}$	0	$\varepsilon$	$2\varepsilon$	$\frac{1-3\varepsilon}{n-3}$	$\dots$	$\frac{1-3\varepsilon}{n-3}$	$\boxed{\frac{1-3\varepsilon}{n-3}}$	$\dots$	$\frac{1-3\varepsilon}{n-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_n$	0	$\varepsilon$	$2\varepsilon$	$\frac{1-3\varepsilon}{n-3}$	$\dots$	$\frac{1-3\varepsilon}{n-3}$	$\frac{1-3\varepsilon}{n-3}$	$\dots$	$\boxed{\frac{1-3\varepsilon}{n-3}}$

Figure 2: The instance used in the proof of Proposition 3.

Now consider a slight generalization of Lipton’s cycles, *weakly improving cycles* (WIC), that correspond to a reallocation of bundles where all the agents in the cycle receive a bundle they like at least as much as the one they held, with one agent at least being strictly happier. Of course, our example of Proposition 3 still applies. On the other hand, this notion suffices to guarantee the decrease of the degree of envy (note that *identifying* the cycles themselves may not be easy any longer, but this is irrelevant for our purpose). The proof follows directly from the arguments of Lipton *et al.* (2004) (proof of Lemma 2).

**Observation 3.** *Let  $\pi$  be an allocation, and  $\pi'$  the allocation obtained after performing a weakly improving cycle. It holds that  $de(\pi') < de(\pi)$ .*

*Proof.* Let us consider an allocation  $\pi'$  obtained after performing a WIC on an allocation  $\pi$ . First note that the envies agents who are not involved in the WIC stay unchanged. By definition of a WIC, all the agents get at least as much in  $\pi'$  as they had in  $\pi$ . Thus basically  $de(\pi') \leq de(\pi)$ . Moreover, at least one agent gets a strictly better bundle so her envy strictly decreases. We finally get that  $de(\pi') < de(\pi)$ .  $\square$

We now show that (2-app envy)-freeness exhibits a special behaviour: in contrast with Proposition 3, improving cycles (in fact, even weakly improving cycles) enjoy the property of preserving the (2-app envy)-freeness level of an allocation. We provide this result for *swaps* (cycles involving two agents only) as this is sufficient to establish our main result.

**Lemma 1.** *Let  $\pi$  be a (2-app envy)-free allocation that is not EF. There always exists a WIC (that we can identify) between two agents such that the resulting allocation is ( $K'$ -app envy)-free, with  $K' \leq 2$ .*

*Proof.* Let  $a_i$  be an envious agent (there is at least one). We identify the agent that  $a_i$  envies the most and call her  $a_j$  (if there are several agents that  $a_i$  envies the most, we can pick randomly one of them). As  $a_i$  envies  $a_j$  and  $a_j$  necessarily does not agree on this envy because otherwise it would

contradict (2-app envy)-freeness of  $\pi$ , swapping the bundle of  $a_i$  and  $a_j$  is a WIC. Let us call  $\pi'$  this new allocation. We will now show that  $\pi'$  is a ( $K'$ -app envy)-free allocation with  $K' \leq 2$ .

In  $\pi'$ , all the agents except  $a_i$  and  $a_j$  have the same approval envy. Moreover,  $a_i$  is now EF in  $\pi'$  as she has received her preferred bundle. Suppose for contradiction that  $\pi'$  is ( $K'$ -app envy)-free with  $K' > 2$ . Then necessarily, this is due to  $a_j$  2-app envying (at least) some other agent  $a_h$  (that can obviously not be  $a_i$ ). For this to be the case,  $a_j$  has to envy  $a_h$  and another agent  $a_l$  has to approve this envy: (1)  $u_j(\pi'_j) < u_j(\pi'_h)$ , (2)  $u_l(\pi'_j) < u_l(\pi'_h)$ . However, as  $a_i$  envies  $a_j$  in  $\pi$  then (3)  $u_i(\pi_i) < u_i(\pi_j)$  and as  $\pi$  is (2-app envy)-free and (3) holds, every agent  $a_l$  (except  $a_i$  of course) verifies (4)  $u_l(\pi_i) \geq u_l(\pi_j)$ .

Besides,  $\pi'$  is obtained after swapping the bundles of  $a_i$  and  $a_j$  in  $\pi$  so  $\pi'_j = \pi_i$ ,  $\pi'_i = \pi_j$  and  $\pi'_h = \pi_h$ ; and from (2) we get: (5)  $u_l(\pi_i) < u_l(\pi_h)$ . By transitivity with (5) and (4), we obtain: (6)  $u_l(\pi_j) < u_l(\pi_h)$ . However, we know that  $a_j$  has the same utility in  $\pi$  and  $\pi'$  so  $u_j(\pi'_j) = u_j(\pi_j)$ . The latter combined with (1) (and the fact that  $\pi'_h = \pi_h$ ) gives: (7)  $u_j(\pi_j) < u_j(\pi_h)$ . Finally, note that (6) and (7) translate the fact that  $a_j$  2-app envies  $a_h$  in  $\pi$  which contradicts the fact that  $\pi$  is (2-app envy)-free.  $\square$

Putting Lemma 1 and Observation 3 together allows us to prove that (2-app envy)-freeness is essentially a vacuous notion, in the sense that any instance enjoying an allocation with this property will have an EF allocation as well.

**Proposition 4.** *If an add-MARA instance is (2-app envy)-free then it is also envy-free.*

*Proof.* Take  $\pi$  as being an arbitrary (2-app envy)-free allocation. First note that if there is no envious agent in  $\pi$  then, by definition,  $\pi$  is envy-free and the proposition holds. We perform a WIC leading to  $\pi'$  that is still (2-app envy)-free (see Lemma 1). If  $\pi'$  is envy-free then we are done. Otherwise, from Observation 3 we know the degree of envy has strictly decreased and that the resulting allocation is still (2-app envy)-free by Lemma 1. Hence we can repeat this process until the current allocation is EF. The process is guaranteed to stop because the degree of envy of the society is bounded below by zero and the degree of envy of the society strictly decreases at each step until it reaches zero (which corresponds to an envy-free allocation).  $\square$

Another consequence is that, for two agents, instances fall either in the envy-free or unanimous envy category:

**Corollary 1.** *Let  $I$  be an add-MARA instance with  $n = 2$ , if there is no envy-free allocation in  $I$  then  $I$  is a unanimous envy instance.*

**Complexity** We conclude this section with a few considerations on the computational complexity of the problems mentioned so far. First of all, as envy-freeness is (1-app envy)-freeness, the problem of finding the minimum  $K$  for which there exists a ( $K$ -app envy)-free allocation is at least as hard as determining whether an envy-free allocation exists.

One may also wonder how hard the problem is to determine whether a given instance exhibits unanimous envy or not, *i.e.* whether a ( $K$ -app envy)-free allocation exists for *some* value of  $K$ . For this question, instances where agents all have the same preferences provide insights.

**Proposition 5.** *For any add-MARA instance, if all the agents have the same preferences then the notions of (1-app envy)-freeness and ( $n$ -app envy)-freeness coincide.*

*Proof.* We already know from Observation 2 that (1-app envy)-freeness implies ( $n$ -app envy)-freeness for any add-MARA instance. So we just have to prove that if all the agents have the same preferences then ( $n$ -app envy)-freeness implies (1-app envy)-freeness. If an allocation  $\pi$  is ( $n$ -app envy)-free then it means that for any pair  $(a_i, a_j)$  of agents,  $a_i$  does not envy  $a_j$  or there is at least one agent  $a_h$  that disagrees on the envy of  $a_i$  towards  $a_j$ . Obviously, if for every pair of agents  $(a_i, a_j)$  we have  $a_i$  envy-free towards  $a_j$  then the allocation  $\pi$  is envy-free and the proof concludes. Besides, for every pair of envious/envied agents there is at least one agent disagreeing on the envy. But all the agents have the same preferences so it means that every agent should agree with each other. Hence, no envied agent can exist and we have (1-app envy)-freeness of allocation  $\pi$ .  $\square$

From Proposition 5 we get that the problem of deciding the existence of unanimous envy is at least as hard as deciding the existence of an EF allocation when agents have similar preferences which is known to be NP-hard (Lipton et al., 2004). As membership in NP is direct, we thus get as a corollary that:

**Corollary 2.** *Deciding whether an allocation exhibits unanimous envy is NP-Complete.*

## 5. Approval Non-Proportionality

As there is a clear hierarchy in the notions of fairness deriving from envy-freeness, it can be natural to consider how the different notions of this hierarchy would behave in an approval setting as we studied in Sections 3 and 4. Indeed, it has been shown (Bouveret & Lemaître, 2016) that envy-freeness implies proportionality. Moreover, some relaxations of proportionality have been studied such as PROP1 in very recent works (Aziz et al., 2020; Barman & Krishnamurthy, 2019; Brânzei & Sandomirskiy, 2019; Conitzer et al., 2017). This motivates us to investigate how we can derive an approval notion of proportionality.

In this section, we will introduce the approval version of proportionality. Observe first that our approval version of envy-freeness was based on a pairwise notion that we do not have in proportionality. This is why we slightly adapt the approval notion to this property.

**Definition 9** ( $K$ -approval non-proportionality). *Let  $\pi$  be an allocation,  $a_i$  be an agent, and  $1 \leq K \leq n$  be an integer. We say that  $\pi_i$  is  $K$ -approval non-proportional ( $K$ -app non-prop for short) in  $\pi$  if there is a subset  $\mathcal{N}_K$  of  $K$  agents including  $a_i$  such that:*

$$\forall a_k \in \mathcal{N}_K, u_k(\pi_i) < Prop_k.$$

In other words, at least  $K - 1$  agents amongst  $\mathcal{N} \setminus \{a_i\}$  agree with  $a_i$  on the fact that she does not have her proportional share. We emphasize that we chose to focus on non-proportionality rather than on proportionality, to be consistent with our definition of  $K$ -app envy. The other related notions are defined accordingly as follows.

**Definition 10** ( $(K$ -approval non-proportional)-free allocation). *An allocation  $\pi$  is said to be ( $K$ -app non-proportional)-free if and only if no  $\pi_i$  is  $K$ -app non-proportional.*

Once again, observe that the interpretation of this property is that an allocation is free of  $K$ -app non-prop: each agent  $a_i$  either thinks she receives a proportional share, or, if it is not the case, no more than  $K - 2$  agents agree with  $a_i$ .

**Definition 11** ( $(K$ -approval non-proportional)-free instance). *An add-MARA instance  $I$  is said to be  $(K$ -approval non-proportional)-free if and only if it accepts a  $(K$ -approval non-proportional)-free allocation.*

**Definition 12** (Unanimous non-proportional allocation). *An add-MARA allocation  $\pi$  is said to exhibit unanimous non-proportionality if  $\pi$  is not  $(K$ -approval non-proportional)-free for any value of  $K$ .*

**Definition 13** (Unanimous non-proportional instance). *An add-MARA instance  $I$  is said to exhibit unanimous non-proportionality if  $I$  is not  $(K$ -approval non-proportional)-free for any value of  $K$ .*

**Example 5.** *Let us consider the add-MARA instance introduced in Example 1:*

	$o_1$	$o_2$	$o_3$	$o_4$	$o_5$	$o_6$
$a_1$	0	3*	3	1	3	2*
$a_2$	2*	0	7	2*	1*	0
$a_3$	0	3	5*	0	1	3

*It is easy to notice that the proportional share of all the agents is the same and is worth 4. In the squared allocation,  $\pi_1$  is not proportional as she values her bundle 3. Moreover  $a_2$  and  $a_3$  agree on the non-proportionality of her bundle. Hence  $\pi_1$  is unanimous non-proportional. Besides  $\pi_2$  is not proportional either. However neither  $a_1$  nor  $a_3$  agree on this non-proportionality as they value  $a_2$ 's bundle with respective utilities 6 and 4. So  $\pi_2$  is 1-app non-prop. Finally,  $\pi_3$  is proportional. Consequently, we can say that the squared allocation is unanimous non-proportional because of  $a_1$ . However, the instance itself is not unanimous non-proportional as we can easily notice that the star allocation is proportional and hence (1-app non-prop)-free.*

## 6. Some Properties of Approval Non-Proportionality

In this section, we will present some properties about the notion of  $(K$ -app non-prop)-freeness for different values of  $K$ . We will also present some complexity results.

We start with an easy observation which is the counterpart for  $(K$ -app non-prop)-freeness of Observation 2:

**Observation 4.** *Given an allocation  $\pi$  of an add-MARA instance, if  $\pi_i$  is  $K$ -app non-proportional in  $\pi$ , then  $\pi_i$  is  $(K-1)$ -app non-proportional in  $\pi$ .*

The following observation is a direct consequence of Observation 4.

**Observation 5.** *Let  $\pi$  be an allocation, and  $K \leq N$  be an integer. If  $\pi$  is  $(K$ -app non-prop)-free, then  $\pi$  is also  $((K+1)$ -app non-prop)-free.*

However, the converse does not hold. More precisely, the following proposition shows that the implication stated in Observation 5 is strict.

**Proposition 6.** *Let  $\pi$  be an allocation, and  $3 \leq K \leq n$  be an integer. If  $\pi$  is  $(K$ -app non-prop)-free,  $\pi$  is not necessarily  $((K-1)$ -app non-prop)-free.*

*Proof.* Let us consider the following instance with 3 agents and 3 objects and the squared allocation  $\pi$ . Recall that  $Prop_i$  denotes the proportional share of  $a_i$  as stated in Definition 3:

	$o_1$	$o_2$	$o_3$
$a_1$	$Prop_1 + 1$	$Prop_1 + 1$	$Prop_1 - 2$
$a_2$	$Prop_2 - 1$	$Prop_2 - 1$	$Prop_2 + 2$
$a_3$	$Prop_3 - 1$	$Prop_3 - 1$	$Prop_3 + 2$

In this allocation, the only agent that does not hold her proportional share is  $a_2$ . Moreover, we can easily see that  $a_3$  agrees with this non-proportionality whereas  $a_1$  does not. So  $a_2$  experiences 2-app non-prop and thus  $\pi$  is a (3-app non-prop)-free allocation but not (2-app non-prop)-free.  $\square$

**Proposition 7.** *For any  $K \geq 3$ , there exists instances which are ( $K$ -app non-prop)-free but not ( $(K-1)$ -app non-prop)-free.*

*Proof.* Consider the same instance as in Proposition 6. We have already shown that we have an allocation  $\pi$  that is (3-app non-prop)-free which means that the instance is (3-app non-prop)-free. We just have to show that there is no (2-app non-prop)-free allocation in order to conclude. For that purpose, we first note that each agent has to get one and exactly one object. Indeed, if it is not the case at least one agent  $a_i$  will have no object and will thus not obtain her proportional share. Moreover, as all agents value the empty bundle with utility 0 this non-proportionality will be unanimous. Hence, each agent has to get one and exactly one object in order to minimize the ( $K$ -app non-prop)-freeness. Moreover, as  $a_2$  and  $a_3$  have the same preferences and only  $o_3$  fulfils their proportional share then there is obviously no proportional allocation. Finally, this means that one of them will get either  $o_1$  or  $o_2$ , and the non-proportionality of their bundles will be approved by the other, leading to a 2-app non-prop. Thus there is no (2-app non-prop)-free allocation and so the instance is (3-app non-prop)-free and not (2-app non-prop)-free.  $\square$

Proposition 7 proves that the hierarchy of  $K$ -app non-prop instances is strict for  $K \geq 3$ . As it was the case for the approval notion derived from envy-freeness we will see that it is not the case for  $K = 2$  by show that (2-app non-prop)-freeness exhibits a special behaviour. For that, we start with a simple result.

**Lemma 2.** *Let  $\pi$  be an allocation. For each agent  $a_i$ , there is at least one bundle  $\pi_j$  such that  $u_i(\pi_j) \geq Prop_i$ .*

*Proof.* Let us consider for the sake of contradiction that there exists one allocation  $\pi$  in which an agent  $a_i$  cannot find any bundle that fulfils her proportional share. This means that every bundle is valued strictly less than  $Prop_i = \frac{\sum_{j=0}^m u(i,j)}{n}$ . By adding all the bundles (there are by definition  $n$  bundles in any allocation) we get that  $a_i$  values all the bundles strictly less than  $n \times Prop_i = n \times \frac{\sum_{j=0}^m u(i,j)}{n} = \sum_{j=0}^m u(i,j)$  which is an obvious contradiction.  $\square$

We now establish a result similar to Lemma 1:

**Lemma 3.** *Let  $\pi$  be a (2-app non-prop)-free allocation that is not proportional. There always exists a bundle exchange between two agents (swap), that is not necessarily improving, such that the resulting allocation is ( $K'$ -app non-prop)-free (with  $K' \leq 2$ ) and such that the number of agents with a non-proportional bundle has strictly decreased.*



*Proof.* Let  $\pi$  be a (2-app non-prop)-free allocation that is not proportional. Let  $a_i$  be an agent whose  $\pi_i$  is non-proportional in  $\pi$  (there is at least one). According to Lemma 2, there is (at least) one share  $\pi_j$  such that  $u_i(\pi_j) \geq Prop_i$ . Let  $\pi'$  be the allocation resulting from swapping  $a_i$ 's and  $a_j$ 's bundles in  $\pi$ . In  $\pi'$ , all the agents except  $a_i$  and  $a_j$  have bundles with the same approval non-proportionality. Moreover,  $\pi'_i$  is now proportional in  $\pi'$  by definition of the swap we chose. Finally,  $\pi'_j$  is also proportional: suppose for contradiction that it is not the case. Then it would mean that  $u_j(\pi'_j) < Prop_j$ , which in turns implies  $u_j(\pi_i) < Prop_j$ . In other words, in  $\pi$ ,  $\pi_i$  was not proportional and  $a_j$  agreed, which contradicts the fact that  $\pi$  was (2-app non-prop)-free. Hence,  $\pi'_j$  is proportional, and as a result,  $\pi'$  is still (2-app non-prop)-free, and the number of agents with a non-proportional bundle has increased by at least 1 ( $a_i$  is the new agent with a proportional bundle).  $\square$

Putting together Lemma 2 and Lemma 3 allows us to prove that (2-app non-prop)-freeness is essentially a vacuous notion, in the same sense as it is for (2-app envy)-freeness (Proposition 4):

**Proposition 8.** *If an add-MARA instance is (2-app non-prop)-free then it is also proportional.*

*Proof.* Let  $\pi$  be an arbitrary (2-app non-prop)-free allocation. First note that if all the agents have proportional bundles in  $\pi$  then, by definition,  $\pi$  is proportional and the proposition holds. Otherwise, we perform a swap leading to  $\pi'$  that is still (2-app non-prop)-free (see Lemma 3). If  $\pi'$  is proportional then we are done. Otherwise, thanks to the second part of Lemma 3 we know the number of agents with a non-proportional bundle has strictly decreased. We can repeat this process until the current allocation is proportional. The process is guaranteed to stop because the number of agents with a non-proportional bundle is bounded below by zero and decreases at each step until it equals zero (which corresponds to a proportional allocation).  $\square$

Another consequence is that, for two agents, instances are either proportional or unanimous non-proportional:

**Corollary 3.** *Let  $I$  be an add-MARA instance with  $n = 2$ , if there is no proportional allocation in  $I$  then  $I$  is an unanimous non-proportional instance.*

*Proof.* For any add-MARA instance involving exactly 2 agents, we can (by definition) only find (1-app non-prop)-free allocations or (2-app non-prop)-free allocations (as  $1 \leq K \leq n$  for any add-MARA instance). By the contraposition of Proposition 8 we conclude the proof.  $\square$

We also note that, as it was the case for  $K$ -app envy, performing a reallocation cycle can increase the level of  $K$ -app non-prop:

**Proposition 9.** *Let  $\pi$  be a ( $K$ -app non-prop)-free allocation, for  $3 \leq K \leq n - 1$ . After performing an improving bundle reallocation cycle (even between two agents), the resulting allocation may be ( $K'$ -app non-prop)-free (and not ( $K$ -app non-prop)-free) such that  $K' > K$ .*

*Proof.* Let us consider the following instance with 3 agents and 3 objects:

	$o_1$	$o_2$	$o_3$
$a_1$	$Prop_1 - 1$	$Prop_1 - 2$	$Prop_1 + 3$
$a_2$	$Prop_2$	$Prop_2 + 3$	$Prop_2 - 3$
$a_3$	$Prop_3 - 1$	$Prop_3$	$Prop_3 + 1$

First consider the squared allocation that is (2-app non-prop)-free as only  $a_1$  does not hold her proportional share and that it is not approved by any other agent. Let us now consider the underlined allocation  $\pi$  that is the result of the improving bundle reallocation between  $a_1$  and  $a_2$ . We can see that only  $a_1$  does not hold her proportional share and that this time  $a_3$  approves it leading to a 2-app non-prop and thus a (3-app non-prop)-free allocation.  $\square$

**Complexity** We conclude this section with a few considerations on the computational complexity of the problems mentioned so far around the approval notion of proportionality. First of all, as proportionality is equivalent to (1-app non-prop)-freeness, the problem of finding the minimum  $K$  for which there exists a ( $K$ -app non-prop)-free allocation is at least as hard as determining whether a proportional allocation exists which is known to be NP-complete.

One may also wonder how hard the problem of determining whether a given instance exhibits unanimous non proportionality or not is, *i.e.* whether a ( $K$ -app non-prop)-free allocation exists for *some* value of  $K$ . For this question, as in Proposition 5, instances where agents all have the same preferences provide the answer.

**Proposition 10.** *For any add-MARA instance, if all the agents have the same preferences then the notions of (1-app non-prop)-freeness and ( $n$ -app non-prop)-freeness coincide.*

*Proof.* We already know from Observation 5 that (1-app non-prop)-freeness implies ( $n$ -app non-prop)-freeness for any add-MARA instance. So we just have to prove that if all the agents have the same preferences then ( $n$ -app non-prop)-freeness implies (1-app non-prop)-freeness. Let  $\pi$  be an ( $n$ -app non-prop)-free allocation. Then for any agent  $a_i$ , either  $u_i(\pi_i) \geq Prop_i$ , or there exists an agent  $a_j$  such that  $u_j(\pi_i) \geq Prop_j$ . Since all the agents have identical preferences, the last inequality reduces to  $u_i(\pi_i) \geq Prop_i$ , showing that  $a_i$  receives her proportional share. Hence in this case,  $\pi$  is proportional.  $\square$

From Proposition 10 we get that the problem of deciding the existence of a unanimous non-proportional allocation is at least as hard as deciding the existence of a proportional allocation when agents have similar preferences which is known to be NP-hard (see for instance Bouveret and Lemaître (2016)). As membership in NP is direct, we thus get as a corollary that:

**Corollary 4.** *Deciding whether an allocation exhibits unanimous non-proportionality is NP-Complete.*

## 7. Link Between Approval Envy-Freeness and Approval Non-Proportionality

After having introduced some properties of approval non-proportionality, we will now investigate the relationships between this notion and approval envy-freeness introduced earlier.

We first recall that envy-freeness implies proportionality and that this implication is still valid for EF1 and PROP1. It is thus natural to wonder whether it is also the case for our approval notions. As we will see, the answer is negative.

**Proposition 11.** *A unanimous envy instance can be proportional.*

*Proof.* Let us consider the following generic add-MARA instance (here,  $\varepsilon \leq 1/n$ ):

	$o_1$	$o_2$	$\dots$	$o_{n-1}$	$o_n$
$a_1$	$\boxed{1/n}$	$1/n - \varepsilon$	$\dots$	$1/n - \varepsilon$	$1/n + (n - 2)\varepsilon$
$a_2$	$1/n - \varepsilon$	$\boxed{1/n}$	$\dots$	$1/n - \varepsilon$	$1/n + (n - 2)\varepsilon$
$\vdots$					
$a_{n-1}$	$1/n - \varepsilon$	$1/n - \varepsilon$	$\dots$	$\boxed{1/n}$	$1/n + (n - 2)\varepsilon$
$a_n$	$1/n - \varepsilon$	$1/n - \varepsilon$	$\dots$	$1/n$	$\boxed{1/n + (n - 2)\varepsilon}$

In this instance, the squared allocation is proportional (and so (1-app non-prop)-free) whereas it is easy to see that the instance is a unanimous envy one as  $o_n$  is the top object of every agent. Hence the agent that gets  $o_n$  will be envied and this envy will be approved by everyone.  $\square$

From this result, we can generalize the statement to any level of  $K$ -app envy and any level of ( $L$ -app non-prop)-freeness. First of all, from Observation 5, it is clear that the counter-example of Proposition 11 establishes that a unanimous envy instance can be ( $L$ -app non-prop)-free, for any  $L \geq 1$ . But note also that if (counterfactually) it was the case that proportionality (or indeed any level of ( $L$ -app non-prop)-freeness) would imply some level of ( $K$ -app)-envy freeness, then by invoking Observation 2 this would also imply (unanimous envy)-freeness, a contradiction with Proposition 11. Putting all these remarks together allows us to state the following result.

**Corollary 5.** *For any  $K \geq 1$  and any  $L \geq 1$ , an allocation exhibiting  $K$ -app envy can be ( $L$ -app non-prop)-free.*

Since proportionality is a weaker notion than envy-freeness, the previous result may not come as a surprise. It seems much more likely to obtain a positive result in the other direction, that is, that some level of ( $K$ -app envy)-freeness actually implies some level of ( $L$ -app non-prop)-freeness. It turns out that this is not the case.

**Proposition 12.** *An instance that exhibits unanimous non-proportionality can be (3-app envy)-free.*

*Proof.* Let us consider the following add-MARA instance for which  $Prop_i = \frac{1}{n}$  for all  $i$ :

	$o_1$	$o_2$	$o_3$	$\dots$	$o_{n-1}$	$o_n$
$a_1$	$\boxed{\varepsilon}$	$\varepsilon$	$\varepsilon$	$\dots$	$\varepsilon$	$1 - (n - 1)\varepsilon$
$a_2$	$\varepsilon$	$\varepsilon$	$\varepsilon$	$\dots$	$\varepsilon$	$\boxed{1 - (n - 1)\varepsilon}$
$a_3$	$\varepsilon$	$\varepsilon$	$\boxed{1 - (n - 1)\varepsilon}$	$\dots$	$\varepsilon$	$\varepsilon$
$\vdots$						
$a_n$	$\varepsilon$	$\varepsilon$	$\varepsilon$	$\dots$	$\boxed{1 - (n - 1)\varepsilon}$	$\varepsilon$

It is obvious that in any allocation the agent that gets  $o_1$  will not get her proportional share and that this non-proportionality will be approved by everyone. However, the squared allocation is (3-app envy)-free since the only envy in this allocation is  $a_1$ 's towards  $a_2$ , and only  $a_2$  approves this envy.  $\square$

Again, this allows us to state a more general result. First of all, it is direct from Observation 2 that the counter-example of Proposition 12 establishes that an unanimous non-proportional instance can be ( $K$ -app envy)-free, for any  $K \geq 3$ . But note also that if (counterfactually) it was the case that (3-app envy)-freeness (or indeed any level  $L \geq 3$  of ( $L$ -app envy)-freeness) would imply some level of ( $K$ -app)-non-prop freeness, then by invoking Observation 5 this would also imply (unanimous non-prop)-freeness, a contradiction with Proposition 12. Putting all these remarks together allows us to state the following result.

**Corollary 6.** *For any  $K \geq 3$  and any  $L \geq 1$ , a ( $K$ -app envy)-free instance can exhibit ( $L$ -app) non-proportionality.*

Note that this is the best we can do, since by Observation 2, Observation 5 and the well-known implication between envy-freeness and proportionality, we have an implication from (2-app envy)-freeness and any level of ( $L$ -app non-prop)-freeness.

Now in principle, and even if counter-intuitive at first sight, it could still be that exhibiting unanimous envy could imply proportionality; or that exhibiting unanimous non-proportionality could imply (3-app)-envy-free. The following result shows that both implications do not hold.

**Proposition 13.** *An instance can exhibit at the same time unanimous envy and unanimous non-proportionality.*

*Proof.* Let us consider the following instance with  $n$  agents and commensurable utilities ( $Prop_i = \frac{1}{n}$  for all  $i$  and we assume that  $\varepsilon < \frac{1}{n}$ ):

	$o_1$	$o_2$	$\dots$	$o_n$
$a_1$	$\varepsilon$	$\varepsilon$	$\dots$	$1 - (n - 1)\varepsilon$
$a_2$	$\varepsilon$	$\varepsilon$	$\dots$	$1 - (n - 1)\varepsilon$
$\vdots$				
$a_n$	$\varepsilon$	$\varepsilon$	$\dots$	$1 - (n - 1)\varepsilon$

It is obvious to see that any agent getting an object different from  $o_n$  (say w.l.o.g  $o_1$ ) will not be proportional and will envy the agent receiving  $o_n$ . Moreover, since all the agents have the same preferences, they will all agree with this non-proportionality and envy. □

We have summed up the relations between approval envy notions and approval non-proportionality ones in Figure 3.

## 8. Computation

We have seen at the end of Section 4 (respectively Section 6) that the problem of determining, for a given instance  $I$ , the minimal value of  $K$  such that a ( $K$ -app envy)-free (respectively a ( $K$ -app non-prop)-free) allocation exists inherited from the high complexity of determining whether an envy-free (respectively a proportional) allocation exists.

To address this problem, we present in this section two Mixed Integer Linear Programs that return, for a given add-MARA instance  $I$ , a ( $K$ -app envy)-free (respectively ( $K$ -app non-prop)-free) allocation with the minimal  $K$  and no solution when  $I$  is an unanimous envy (respectively

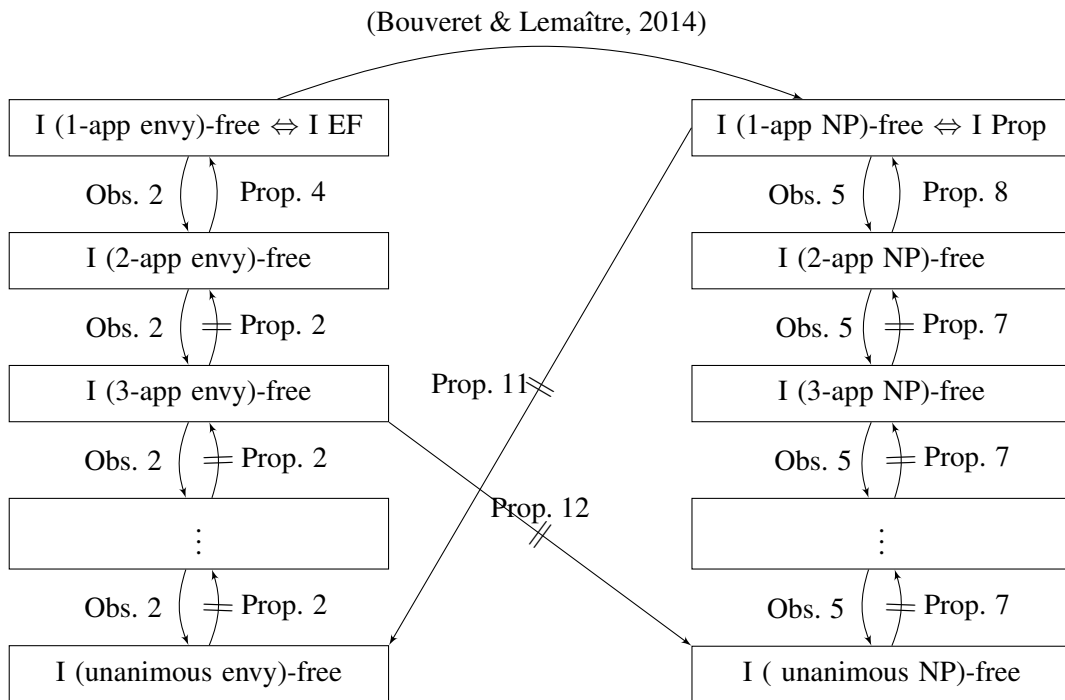


Figure 3: Hierarchy among instance properties. *I* stipulates that the relationships are drawn among instances. A simple edge denotes an implication relation. A striked out edge has been drawn when we have found a counter-example showing that this implication is not valid. Edges obtained by transitivity are not shown. All the remaining missing arcs are non-implication edges which can be obtained thanks to Corollaries 5 and 6.

non-prop) instance. We will first introduce and thoroughly explain the MIP for  $K$ -app envy. Then, we will show how to adapt it to  $K$ -app non-prop.

In this section, we assume that all the utilities are integers. If they are not (recall that they are assumed to be in  $\mathbb{Q}^+$ ) we can transform the instance at stake into a new one only involving integer utilities by multiplying them by the least common multiple of their denominators.

### 8.1 A MILP Formulation for $K$ -Approval Envy

In this MILP, we use  $n \times m$  Boolean variables  $\mathbf{z}_i^j$  (we use bold letters to denote variables) to encode an allocation:  $\mathbf{z}_i^j = 1$  if and only if  $a_i$  gets item  $o_j$ . We also introduce  $n^3$  Boolean variables  $\mathbf{e}_{kih}$  such that  $\mathbf{e}_{kih} = 1$  if and only if  $u_k(\pi_i) < u_k(\pi_h)$ . We also need to add  $n^2$  Boolean variables  $\mathbf{x}_{ih}$  used to linearize the constraints on  $\mathbf{e}_{kih}$ . Finally, we use an integer variable  $\mathbf{K}$  corresponding to the  $K$ -app envy we seek to minimize.

We first need to write the constraints preventing an item from being allocated to several agents:

$$\sum_{i=1}^n \mathbf{z}_i^j = 1 \quad \forall j \in \llbracket 1, m \rrbracket \quad (1)$$

By adding these constraints we also guarantee completeness of the returned allocation (all the items have to be allocated to an agent).

Secondly, we have to write the constraints that link the variables  $\mathbf{e}_{kih}$  with the allocation variables  $\mathbf{z}_i^j$ :

$$\sum_{j=1}^m u(k, j)(\mathbf{z}_h^j - \mathbf{z}_i^j) > 0 \iff \mathbf{e}_{kih} = 1 \quad \forall k, i, h \in \llbracket 1, n \rrbracket$$

As the utilities are integers, we can replace  $> 0$  by  $\geq 1$ . In order to linearize the equivalence between the two constraints we introduce a number  $M$  that can be arbitrarily chosen such that  $M > \max_{a_k \in \mathcal{N}} \sum_{j=1}^m u(k, j)$ :

$$M\mathbf{e}_{kih} \geq \sum_{j=1}^m u(k, j)(\mathbf{z}_h^j - \mathbf{z}_i^j) \quad \forall k, i, h \in \llbracket 1, n \rrbracket \quad (2)$$

$$\sum_{j=1}^m u(k, j)(\mathbf{z}_h^j - \mathbf{z}_i^j) \geq 1 - M(1 - \mathbf{e}_{kih}) \quad \forall k, i, h \in \llbracket 1, n \rrbracket \quad (3)$$

Finally, we have to write the constraints that convey the fact that the allocation we look for is ( $K$ -app envy)-free:

$$\mathbf{e}_{iih} = 0 \vee \sum_{k=1}^n \mathbf{e}_{kih} \leq \mathbf{K} - 1 \quad \forall i, h \in \llbracket 1, n \rrbracket$$

Since  $\mathbf{e}_{iih}$  are Boolean variables, we can replace  $\mathbf{e}_{iih} = 0$  by  $\mathbf{e}_{iih} \leq 0$ . Now, these logical constraints are linearized as follows:

$$\mathbf{e}_{i\mathbf{h}} \leq \mathbf{x}_{i\mathbf{h}} \quad \forall i, h \in \llbracket 1, n \rrbracket \quad (4)$$

$$\sum_{k=1}^n \mathbf{e}_{k\mathbf{ih}} \leq \mathbf{K} - 1 + n(1 - \mathbf{x}_{i\mathbf{h}}) \quad \forall i, h \in \llbracket 1, n \rrbracket \quad (5)$$

We can now put things together. Let  $I$  be an instance. Then, we will denote by  $\mathcal{M}_1(I)$  the MIP defined as:

$$\begin{aligned} & \text{minimize } \mathbf{K} \\ & \text{such that } \mathbf{z}_i^j, \mathbf{e}_{k\mathbf{ih}}, \mathbf{x}_{i\mathbf{h}} \in \{0, 1\} \quad \forall k, i, h \in \llbracket 1, n \rrbracket, j \in \llbracket 1, m \rrbracket \\ & \quad \mathbf{K} \in \llbracket 1, N \rrbracket \\ & \quad + \text{Constraints (1, 2, 3, 4, 5)} \end{aligned}$$

**Proposition 14.** *Let  $I$  be an add-MARA instance. Then, there is an optimal solution with  $\mathbf{K} = L$  to  $\mathcal{M}_1(I)$  if and only if  $I$  is an  $(L\text{-app envy})$ -free instance and not an  $((L - 1)\text{-app envy})$ -free one. Moreover,  $\mathcal{M}_1(I)$  does not admit any solution if and only if  $I$  is an unanimous envy instance.*

The proof of this result can be found in the Appendix.

## 8.2 A MILP Formulation for $K$ -Approval Non-Proportionality

In the previous subsection, we have introduced a Mixed Integer Linear Program that returns a  $(K\text{-app envy})$ -free allocation with the minimal  $K$  and no solution when  $I$  is an unanimous envy (respectively non-prop) instance. We will now explain how to adapt it to  $K$ -app non-proportionality.

In this adapted MIP, we use the same Boolean variables  $\mathbf{z}_i^j$ . We also introduce  $n^2$  Boolean variables  $\mathbf{p}_{k\mathbf{i}}$  such that  $\mathbf{p}_{k\mathbf{i}} = 1$  if and only if according to  $a_k$ 's preferences  $a_i$ 's bundle is worth strictly less than the proportional share of  $a_k$ . We also need to add  $n$  Boolean variables  $\mathbf{x}_i$  used to linearize the constraints on  $\mathbf{p}_{k\mathbf{i}}$ . Finally, we use an integer variable  $\mathbf{K}$  corresponding to the  $K$ -app non-proportionality we seek to minimize.

Recall that we assume in this section that the utilities are integers. We will further assume that  $Prop_k = \sum_{j=1}^m u(k, j)/n$  is also an integer for each  $k$ . If it is not the case, they all the utilities can be multiplied by  $n$  without changing the result.

We first need Constraint (1) to ensure the correctness of the allocation.

Secondly, we have to write the constraints that link variables  $\mathbf{p}_{k\mathbf{i}}$  with the allocation variables  $\mathbf{z}_i^j$ :

$$\sum_{j=1}^m u(k, j) \cdot \mathbf{z}_i^j < \frac{\sum_{j=1}^m u(k, j)}{n} (= Prop_k) \iff \mathbf{p}_{k\mathbf{i}} = 1 \quad \forall k, i \in \llbracket 1, n \rrbracket$$

As the utilities are integers, we can replace  $> 0$  by  $\geq 1$ . In order to linearize the equivalence between the two constraints we introduce a number  $M$  that can be once again arbitrarily chosen such that  $M > \max_{a_k \in \mathcal{N}} \sum_{j=1}^m u(k, j)$ :

$$M \mathbf{p}_{ki} \geq Prop_k - \sum_{j=1}^m u(k, j) \mathbf{z}_i^j \quad \forall k, i \in \llbracket 1, n \rrbracket \quad (6)$$

$$Prop_k - \sum_{j=1}^m u(k, j) \mathbf{z}_i^j \geq 1 - M(1 - \mathbf{p}_{ki}) \quad \forall k, i \in \llbracket 1, n \rrbracket \quad (7)$$

Finally, we have to write the constraints that convey the fact that the allocation we look for is ( $K$ -app non-prop)-free:

$$\mathbf{p}_{ii} = 0 \vee \sum_{k=1}^n \mathbf{p}_{ki} \leq \mathbf{K} - 1 \quad \forall i \in \llbracket 1, n \rrbracket$$

Since  $\mathbf{p}_{ii}$  is a Boolean variable for each  $i$ , we can replace  $\mathbf{p}_{ii} = 0$  by  $\mathbf{p}_{ii} \leq 0$ . Now, this logical constraint is linearized as follows:

$$\mathbf{p}_{ii} \leq \mathbf{x}_i \quad \forall i \in \llbracket 1, n \rrbracket \quad (8)$$

$$\sum_{k=1}^n \mathbf{p}_{ki} \leq \mathbf{K} - 1 + n(1 - \mathbf{x}_i) \quad \forall i \in \llbracket 1, n \rrbracket \quad (9)$$

We can now put things together. Let  $I$  be an instance. Then, we will denote by  $\mathcal{M}_2(I)$  the MIP defined as:

$$\begin{aligned} & \text{minimize} && \mathbf{K} \\ & \text{such that} && \mathbf{z}_i^j, \mathbf{p}_{ki}, \mathbf{x}_i \in \{0, 1\} \quad \forall k, i \in \llbracket 1, n \rrbracket, j \in \llbracket 1, m \rrbracket \\ & && \mathbf{K} \in \llbracket 1, N \rrbracket \\ & && + \text{Constraints (1, 6, 7, 8, 9)} \end{aligned}$$

**Proposition 15.** *Let  $I$  be an instance. Then, there is an optimal solution with  $\mathbf{K} = L$  to  $\mathcal{M}_2(I)$  if and only if  $I$  is an ( $L$ -app non-prop)-free instance and not an  $((L - 1)$ -app non-prop)-free one. Moreover,  $\mathcal{M}_2(I)$  does not admit any solution if and only if  $I$  is an unanimous non-proportional instance.*

The proof of this result can be found in the Appendix.

## 9. House Allocation Setting

We have seen in Corollaries 2 and 4 the problems of finding the minimal level  $K$  for which there exists a ( $K$ -app envy)-free or a ( $K$ -app non-prop)-free allocation are difficult in the general case. A natural way to tackle this difficulty is to look for particular restrictions where these problems can be solved efficiently. In this section, we will deal with the House Allocation setting.

The House Allocation Problem (HAP for short) is a standard setting where there are exactly as many items as agents, and each agent receives exactly one item. This setting is relevant in many situations and has been extensively studied (Shapley & Scarf, 1974; Roth & Sotomayor, 1992; Abraham et al., 2005, to cite a few of them). In House Allocation Problems, computing an envy-free



allocation and a proportional allocation reduces to the problem of finding a matching in a bipartite graph, which can be done in  $O(n^3)$  (Gondran & Minoux, 1984). Indeed, an envy-free allocation exists if and only if all the agents get (one of) their top item(s) and a proportional allocation exists if and only if each agent  $a_i$  gets an item whose value is greater than  $Prop_i$ . It is therefore natural to wonder whether an allocation minimizing  $K$ -app envy or  $K$ -app non-proportionality could also be computed efficiently.

Our first observation hints in that direction. Indeed, characterizing unanimous envy becomes easy in house allocation problems.

**Proposition 16.** *Let  $I$  be an instance of HAP.  $I$  is an unanimous envy instance if and only if there exists at least one pair of items  $(o_i, o_j)$  such that all the agents unanimously strictly prefer  $o_i$  to  $o_j$ .*

*Proof.* ( $\Rightarrow$ ) Suppose that for any pair of items  $(o_i, o_j)$ , there are two agents  $(a_k, a_l)$  such that  $u(k, i) \geq u(k, j)$  and  $u(l, i) \leq u(l, j)$ . Let  $\pi$  be an allocation, and suppose w.l.o.g that  $\pi_i = \{o_i\}$ . Then for any pair of agents  $(a_i, a_j)$ , either (i)  $u(i, i) \geq u(i, j)$ , in which case  $a_i$  does not envy  $a_j$ , or (ii)  $u(i, i) < u(i, j)$ , in which case  $a_i$  envies  $a_j$ , but there is another agent  $a_k$  such that  $u(k, i) \geq u(k, j)$ . In the latter case,  $a_k$  disagrees with  $a_i$ 's envy towards  $a_j$ . Hence  $a_i$  does not unanimously envy  $a_j$ . Therefore  $I$  is not an unanimous envy instance.

( $\Leftarrow$ ) Suppose now that there is a pair of items  $(o_i, o_j)$  such that  $u(k, i) > u(k, j)$  for all agents  $a_k$ . In any allocation one agent (say  $a_i$ ) holds  $o_i$  while another agent (say  $a_j$ ) holds  $o_j$ :  $a_j$  envies  $a_i$  and all the agents approve this envy. Therefore  $I$  is an unanimous envy instance.  $\square$

Incidentally, we get as a corollary:

**Corollary 7.** *One can check in  $O(n^3)$  whether an instance  $I$  of HAP is a unanimous envy instance or not.*

From this characterization we can also derive a result on the likelihood that unanimous envy exists when the utilities are uniformly distributed (that is, for each agent  $a_i$  and object  $o_j$ , utilities are drawn i.i.d. following the uniform distribution on some interval  $[x, y]$ ). The interested reader can find this result in the Appendix.

We will now investigate the case of approval non-proportionality in the context of HAP. Interestingly, it is also possible to exactly characterize the set of unanimous non-proportional instances.

**Proposition 17.** *Let  $I$  be an HAP instance.  $I$  is an unanimous non-proportional instance if and only if there exists at least an item  $o_p$  such that  $u(k, p) < Prop_k$  for all agents  $a_k$ .*

*Proof.* ( $\Rightarrow$ ) Suppose no such item  $o_p$  exists. Let  $\pi$  be any allocation giving to each agent  $a_i$  an item (say  $o_i$  w.l.o.g). Then either  $u(i, i) \geq Prop_i$ , in which case  $a_i$  receives her proportional share, or  $u(i, i) < Prop_i$ , in which case there is another agent  $a_k$  such that  $u(k, i) > Prop_k$ .  $a_k$  thus disagrees with  $\pi_i$  being non-proportional. Hence the instance is not unanimous non-proportional.

( $\Leftarrow$ ) Now suppose that there is an item  $o_p$  such that  $u(k, p) < Prop_k$  for all agents  $a_k$ . In any allocation one agent (say  $a_p$ ) holds  $o_p$ . By definition,  $a_p$  does not get her proportional share, and all the agents agree with that. Therefore, the instance is unanimous non-proportional.  $\square$

As for approval envy-freeness, this result yields an efficient way of checking whether an instance is unanimous non-proportional or not:

**Corollary 8.** *One can check in  $O(n^2)$  whether an instance  $I$  of HAP is an unanimous non-proportional instance or not.*

Like in the approval envy case, we can derive from this characterization an upper bound on the probability for an instance to be unanimous non-proportional (see Appendix).

We will now show that finding an allocation minimizing ( $K$ -app envy)-freeness can be done in polynomial time. Before introducing the idea, we need an additional notation. For any pair of objects  $(o_j, o_{j'})$ , let  $\#_{\prec}(o_j, o_{j'})$  denote the number of agents strictly preferring  $o_{j'}$  to  $o_j$ . For any agent  $a_i$  and object  $o_j$ , we will also define  $\maxEnvy(i, j)$  as follows:

$$\maxEnvy(i, j) = \max_{o_{j'} \text{ s.t. } u(i, j') > u(i, j)} \#_{\prec}(o_j, o_{j'})$$

In other words,  $\maxEnvy(i, j)$  denotes the maximal value of  $\#_{\prec}(o_j, o_{j'})$  among the objects that are strictly preferred to  $o_j$  by  $a_i$ . As we can imagine, this will exactly be the value of the  $K$ -app envy experienced by  $a_i$  if she gets item  $o_j$  (note that if  $o_j$  is among  $a_i$ 's top objects, this value will be 0).

The key to the algorithm is to see that for a given  $K$ , determining whether a ( $K$ -app envy)-free allocation exists can be done in polynomial time by solving a matching problem. Namely, for each  $K$ , we build the following bipartite graph:  $\mathcal{N} \cup \mathcal{O}$  is the set of nodes, and we add an edge  $(a_i, o_j) \in \mathcal{N} \times \mathcal{O}$  if and only if  $\maxEnvy(i, j)$  is lower than or equal to  $K$ . We can observe that any perfect matching in this graph corresponds to a  $((K + 1)$ -app envy)-free allocation. More precisely, if there exists a perfect matching, that means that the allocation  $\pi$  resulting from the perfect matching is  $((K + 1)$ -app envy)-free but there could exist another allocation with lower (approval envy)-freeness. If there is no perfect matching, then there could exist a ( $h$ -app envy)-free allocation with  $h > K + 1$ . The only thing that remains to do is to run through all possible values of  $K$ , which can be done by dichotomous search between 0 and  $n$ . This is formalized in Algorithm 9.1.

**Proposition 18.** *For any HAP instance, we can find (one of) its optimal ( $K$ -app envy)-free allocations in  $O(n^3 \log(n))$ .*

*Proof.* First, the computation of the matrix  $\maxEnvy$  runs in  $O(n^3)$ . Indeed, to compute  $\maxEnvy(i, j)$  we first need to compute  $\#_{\prec}(o_j, o_{j'})$  which already runs in  $O(n^3)$  as we have to ask for each couple of objects ( $n^2$  in total) the point of view of all the agents ( $n$  in total). From that, since

$$\maxEnvy(i, j) = \max_{o_{j'} \text{ s.t. } u(i, j') > u(i, j)} \#_{\prec}(o_j, o_{j'})$$

we can compute  $\maxEnvy(i, j)$  in  $O(n)$ . As there are  $n^2$  different pairs  $(a_i, o_j)$  we have the final  $O(n^3)$  complexity of computing  $\maxEnvy$ .

Due to the dichotomous search, the algorithm needs to solve  $\log(n)$  perfect matching problems, that can be solved in  $O(n^3)$  (Gondran & Minoux, 1984). The overall complexity of Algorithm 9.1 is thus  $O(n^3 \log(n))$ .  $\square$

Following the same idea, we can propose an algorithm that returns an allocation minimizing ( $K$ -app non-prop)-freeness in polynomial time. For this case, we no longer need the matrix  $\maxEnvy$ , but we have to replace it by some vector  $\#_{\text{nonProp}}$  that tells for each object  $o_j$  how many agents think this object is not worth their proportional share:

**Algorithm 9.1:** Minimizing ( $K$ -app envy)-freeness in the HAP

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**input :**  $I = \langle \mathcal{N}, \mathcal{O}, w \rangle$  a HAP instance  
**output:** Allocation  $\pi$  and its level minimizing the ( $K$ -app envy)-freeness or **None** if  $I$  is a unanimous envy instance

```

1  $maxEnvy \leftarrow \text{computeMaxEnvy}()$ ;
2  $res \leftarrow \text{None}$ ;
3  $low \leftarrow 0, high \leftarrow n$ ;
4 while  $low \leq high$  do
5    $K \leftarrow \lfloor (low + high)/2 \rfloor$ ;
6    $G \leftarrow \text{buildBipartiteGraph}(maxEnvy, K)$ ;
7    $\pi \leftarrow \text{perfectMatching}(G)$ ;
8   if  $\pi$  is not None then
9      $res \leftarrow \pi, K + 1$ ;
10     $high \leftarrow K - 1$ ;
11  else
12     $low \leftarrow K + 1$ ;
13 return  $res$ 

```

---

$$\#nonProp(j) = |\{a_i \text{ s.t. } u(i, j) < Prop_i\}|$$

In Algorithm 9.1 we then replace Line 9.1 by an instruction computing  $\#nonProp$  for each  $o_j$ . Then, we replace the bipartite graph computed at Line 5 by the graph defined as follows:  $\mathcal{N} \cup \mathcal{O}$  is still the set of nodes, and we add an edge  $(a_i, o_j) \in \mathcal{N} \times \mathcal{O}$  if and only if  $u(i, j) < Prop_i$  or  $\#nonProp(j)$  is lower than or equal to  $K$ .

**Proposition 19.** *For any HAP instance, we can find (one of) its optimal  $K$ -app non-prop-free allocations in  $O(n^3 \log(n))$ .*

*Proof.* We know from the proof of Proposition 18 that the algorithm runs in at least  $O(n^3 \log(n))$  due to the dichotomous search associated with the perfect matching problem resolutions. But the complexity could be worse because of the computation of  $\#nonProp$  and the construction of the bipartite graph. To compute  $\#nonProp$ , it is enough for each object  $o_j$  to run through all the agents and count how many of them think  $o_j$  is not worth their proportional share. This can be done in  $O(n^2)$  steps, provided that we have pre-computed the values  $Prop_i$  first (which can be done in  $O(n)$  for each agent, that is,  $O(n^2)$  in total). Computing the bipartite graph does not take longer than before, since we just have to check for each pair  $(a_i, o_j)$  whether  $u(i, j) < Prop_i$  or  $\#nonProp(j) \leq K$  (which can be made in constant time if the values  $Prop_i$  and  $\#nonProp(j)$  have been pre-computed). Thus in total, the adaptation of the algorithm does not cause any added complexity, so the global complexity is still  $O(n^3 \log(n))$ .  $\square$

## 10. Experimental Results

We conducted an experimental evaluation of our approval notions and solving methods. These experiments serve two purposes: (i) evaluate the behaviour of the MIPs we presented in Section 8

and of the polynomial algorithms described in Section 9 for the HAP setting, and (ii) observe the relevance of our two approval notions when varying the number of agents, of items, and the type of preferences. All the tests presented in this section have been run on an Intel(R) Core(TM) i7-2600K CPU with 16GB of RAM and using the Gurobi solver to solve the Mixed Integer Program.

We have tested our methods on three types of instances: Spliddit instances (Goldman & Procaccia, 2015), instances under uniformly distributed preferences and instances under an adaptation of Mallows distributions to cardinal utilities (Durand et al., 2016).

## 10.1 Spliddit Instances

We have first experimented our MIPs on real-world data from the fair division website Spliddit (Goldman & Procaccia, 2015). There is a total of 3535 instances from 2 agents to 15 agents and up to 96 items. Note that 1849 of these instances involve 3 agents and 6 objects. The program we ran for Spliddit instances proceeds as follows. It first checks whether the instance is HAP. If it is the case, it runs Algorithm 9.1 to compute the optimal level of approval envy. If this level is 1, it means that the instance is EF, and hence proportional (Bouveret & Lemaître, 2016). We stop there in this case. Otherwise, we run the adaptation of Algorithm 9.1 to compute the level of approval non-proportionality. If the instance is not HAP, we proceed the same way, replacing Algorithm 9.1 and its adaptation by MIPs  $\mathcal{M}_1$  and  $\mathcal{M}_2$ .

**Approval Envy** Concerning approval envy, by setting a timeout of 1 minute, the program was able to solve all but 6 instances optimally. By extending the timeout to 10 minutes, we were able to solve 4 additional instances. We were however unable to solve the last 2 remaining instances optimally within 5 hours. Those instances respectively concern 6 agents and 15 objects, and 4 agents and 29 objects. However, by examining this latter instance, we could notice that all the agents had the same preferences. Running MIP  $\mathcal{M}_2$  on this instance lead us to find an allocation that is proportional, meaning that this allocation is also envy-free in that case. Hence, in the end, only one instance still resists to our attempts. Among the 3534 instances that have been solved optimally, 63.8% admit an EF allocation, while 24.6% exhibit unanimous envy. Moreover, 29% of the 83 instances with more than 5 agents are Strict Majority-app EF (SM-app EF).

We have also implemented the alternative notion of ( $K$ -app envy)-freeness mentioned at the end of Section 3 and computed the optimal  $K$  for the 3469 easiest Spliddit instances (we removed those that timed out after 20 seconds). Among these instances, only 47 were found to be neither EF nor unanimous-envy, that is, about 1.4%, which confirms our intuition that this alternative notion is much less discriminating than the notion of  $K$ -app envy we use in this paper.

**Approval Non-Proportionality** Concerning approval non-proportionality, all Spliddit instances have been solved optimally within 1 minute. 69.3% of the instances turn out to be proportional, while 25.4% exhibit unanimous non-proportionality. Note that since we know (Bouveret & Lemaître, 2016) that envy-freeness implies proportionality, we knew from the previous experiments that the percentage of proportional instances would be greater than 63.8%. So we can notice that around 5.3% of the instances actually are neither proportional nor unanimous non-proportional against the 11.7% we had for the approval envy notion.

$(n, m)$	(2,3)	(3,4)	(4,5)	(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,13)
% EF	86	59	42	58	36	29	11	9	14

Table 1: Percentage of envy-free instances as a function of the number of (agents,objects).

## 10.2 Uniformly Distributed Preferences: General Setting

We also ran tests on instances under uniformly distributed preferences, with  $n$  varying from 2 to 10 and  $m$  such that we produce appropriate settings to study our notions of approval envy-freeness and approval non-proportionality. Under Impartial Culture, all preference profiles are equally likely. It is a commonly studied in computational social choice (Black et al., 1958; Gehrlein & Fishburn, 1976) as a limit case, also providing an easy way to get syntactic instances without knowledge on preference characteristics from a particular concrete problem.

**Approval Envy** We first studied the notion of approval envy and thus considered settings where few EF allocations exist (Dickerson et al., 2014). More precisely we took  $m$  almost equal to  $n$ , for example 2 agents with 3 objects, 5 agents with 7 objects and 10 agents with 13 objects. As shown by (Dickerson et al., 2014), the percentage of EF instances is tightly related to the ratio between the number of agents and the number of objects. The probability of EF instances is small when the number of objects is not much larger than the number of agents. For each couple  $(n, m)$ , Table 1 reports the percentage of envy-free instances obtained over 1000 randomly generated instances. It can be noticed that the number of EF instances decreases as the numbers of agents and objects increase. The worst-case in Table 1 is obtained for 9 agents and 11 objects where only 90 over 1000 instances are envy-free. For each couple  $(n, m)$ , we randomly picked 60 instances over the instances not EF that were randomly generated. Indeed, we wanted to investigate the behavior of our notion when no EF allocation exists (we know that if an EF allocation exists it will be returned by our methods). As we are in the general setting we solved the instances via the MIP  $\mathcal{M}_1$  with a timeout of 10 minutes. Experimental results are depicted in Table 2.

The first three rows of Table 2 respectively report the percentage of instances that have been solved to optimal (a solution has been returned before the timeout), the percentage of unanimous envy instances and the percentage of Strict Majority-app-EF instances (SM-app-EF instances). The mean value of  $K/n$  gives a good insight on how many agents agree on the fairness notion (in Table 2, on the envy of an agent). Moreover, as it is a normalised measure it allows us to compare the level of approval non-proportionality and envy for instances with different number of agents. Finally, we store the mean computation time (in seconds) of the instances (solved to optimal).

First note that considering 2 agents is a special case as shown in Corollary 1. Indeed, as we have removed the EF instances, all the remaining instances are unanimous envy ones (denoted by - in the tables). Moreover, we observe that the percentage of SM-app-EF allocations is zero for up to 4 agents, which can be easily explained. Indeed, for 3 or 4 agents, being SM-app-EF means that there exists a  $(K$ -app envy)-free allocation with  $K \leq 2$ , which comes down (by Proposition 4) to say that there exists an envy-free allocation. Since all the EF instances have been removed, we cannot find an SM-app-EF allocation for  $n \leq 4$ .

Besides, without any surprise, the computation time rapidly increases while the percentage of instances solved to optimal (under a timeout of 10 minutes) starts decreasing for 7 agents.

$n$	2	3	4	5	6	7	8	9	10
% optimal	100	100	100	100	100	68.3	1.7	1.7	0
% UE I	100	21.7	5	0	0	0	0	0	0
% SM-app EF I	0	0	0	50	50	75	40	33.3	6.7
mean( $K/n$ )	-	1	0.85	0.72	0.61	0.57	0.59	0.63	0.66
time(s)	$\varepsilon$	0.008	0.04	0.21	1.97	21.29	50.09	56.16	-

Table 2: Performances of MIP  $\mathcal{M}_1$  on randomly generated no EF instances

$n$	2	3	4	5	6	7	8	9	10
% optimal	100	100	100	100	100	100	100	100	100
% proportional	86.94	83.77	98.41	99.98	99.76	99.95	100	100	100
% UNPI	100	79.2	69.8	100	70.8	100	-	-	-
mean( $K/n$ )	-	1.0	0.96	-	0.95	-	-	-	-
time(s)	0.002	0.03	0.1	1.8	1.5	8.6	-	-	-

Table 3: Performances of MIP  $\mathcal{M}_2$  on randomly generated instances.

Finally, positive results can be pinpointed. The percentage of unanimous envy instances is very low. This highlights the relevance of the  $K$ -approval envy-free notion. Indeed, in most instances, there exists allocation where we can find a subset of the agents supporting the absence of envy. Minimizing the number of agents approving the envy is thus relevant in almost all instances. Moreover, the experiments show that the percentage of SM-app-EF instances is higher than 30% except for 10 agents. Such instances are desirable as it means that the absence of envy is supported by more than half the agents: from the point of view of the social acceptance, it is thus possible to find an allocation where the fairness is supported by a majority of agents.

**Approval Non-Proportionality** First note that a proportional allocation is likely to exist as soon as  $m \geq n$  (Suksompong, 2016). As we do not want to be in the House Allocation setting yet, we considered instances for which  $m = n + 1$ . We have tested our MIP  $\mathcal{M}_2$  described in Section 8.2 on such instances with a timeout of 10 minutes. For each couple  $(n, m)$ , we generated 10 000 instances.

The first four rows of Table 3 respectively represent the percentage of instances that have been solved to optimal (a solution has been returned before the timeout), the percentage of proportional instances, the percentage of unanimous non-proportional instances (among the ones that are not proportional) and the mean value of  $K/n$  that gives a insight on how many agents agree on the non-proportionality of an agent. Finally, we store the mean computation time (in seconds) of the instances that are not proportional.

We can first notice that all the instances have been solved to optimal and the number of proportional instances remains very high even if we considered a favourable context with  $m = n + 1$ . Notably, for more than 8 agents, all the instances were proportional leaving no space for our relaxation to be useful.

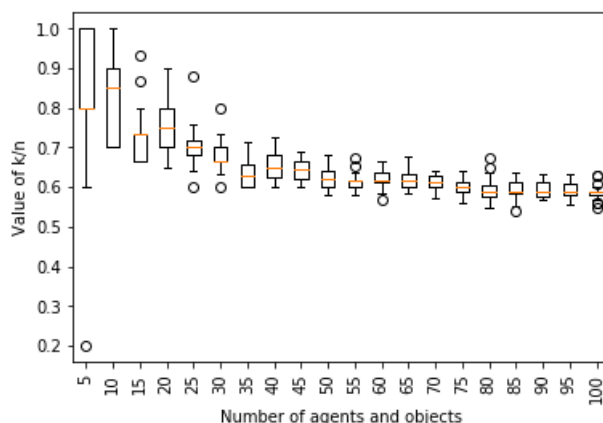


Figure 4: Optimal  $K/n$  (envy) in the HAP as a function of  $n$

Note that considering 2 agents is a special case as shown in Corollary 3. Indeed, as we do not consider the proportional instances, all the remaining instances are unanimous non-proportional ones. For more than 2 agents, we can see that the percentage of unanimous non-proportional instances is almost 70% among non-proportional instances. Besides, we can notice that when it is relevant to look at the mean  $K/n$  metric, it tells us that the level of approval is very high. In light of these results, we could conclude that while proportionality is a much less demanding notion, it turns out that when it is not satisfied it is extremely often unanimously not satisfied.

### 10.3 Uniformly Distributed Preferences: House Allocation Problems

We have also tested our polynomial algorithms on HAP instances under uniformly distributed preferences. We have generated 20 instances for each number of agents from 5 to 100 agents (and objects) by steps of 5.

**Approval Envy** Figure 4 shows the evolution of  $K/n$  as a function of the number of agents  $n$  (and hence also as a function of  $m$  as  $n = m$ ) when minimizing the  $K$ -approval envy. First, note that we have only found 5 unanimous envy instances and all of them involved 5 agents. Indeed the probability of unanimous envy instance can be shown to quickly converge to 0 –see Proposition 22 in Appendix. In HAP, agents are very likely to be envious as an agent envies someone as soon as she does not obtain her most preferred object. Let consider an agent  $a_j$  that holds  $o_j$  and that envies another agent  $a_k$  holding  $o_k$ . This envy is approved by all the agents that rank  $o_k$  over  $o_j$ . This envy is likely to be approved but it is also unlikely that all agents agree on this envy. In such contexts where the agents are likely to have mixed opinions, the  $K$ -approval envy-free notion and our related algorithm allow for computing allocations where the envy is supported by the smallest subset of agents. As shown in Figure 4, even if the optimal  $K/n$  value is high for small problems, it slightly decreases as the size of the instances increases.

Note that the algorithm runs, without any surprise (in light of Proposition 18) much faster than our MIP  $\mathcal{M}_1$ . Indeed, the mean runtime for 100 objects and agents is still around 2 seconds whereas we already observed that our MIP cannot solve easier problems within 10 minutes.

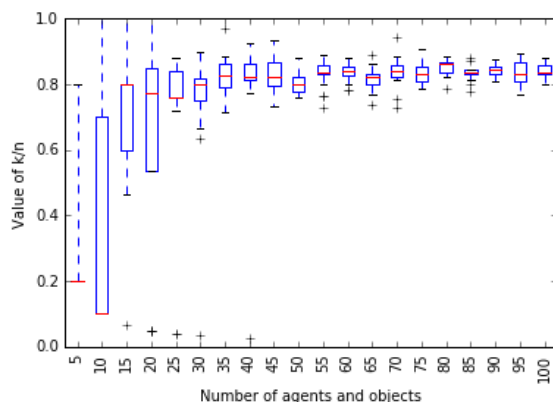


Figure 5: Optimal  $K/n$  (non-proportionality) in the HAP as a function of  $n(= m)$

**Approval Non-Proportionality** We have also tested our polynomial algorithm to find an optimal  $K$ -approval non-proportional-free allocation. Although the algorithm was running very fast even for 100 agents and objects (confirming what we showed in Proposition 19), we almost only obtained proportional instances. We thus decided to test other instances: by using Borda utilities for each agent and randomly choosing one object per agent whose utility has been multiplied by the number of agents, we built a instances where only one object per agent fulfills the proportional share. We can see in Figure 5 that the value of  $K/n$  is stabilising around 0.8 meaning that around 80% of the agents agree with (at least) one agent’s non-proportionality.

#### 10.4 Correlated Preferences

As Impartial Culture may not reflect realistic preference profiles, we also generated instances where the preferences of the different agents may have similarities. In strict ordinal settings, a classical way to capture correlated preferences is to use Mallows distributions (Mallows, 1957) allowing us to measure the impact of the similarity of the preferences between agents. In these experiments, we used a generalization of the Mallows distribution to cardinal preferences based on Von Mises–Fisher distributions (Durand et al., 2016). Like the dispersion parameter in Mallows distributions, the similarity between the preferences of the agents is tuned by a *concentration* parameter: when the concentration is zero the agents’ preferences are uniformly distributed, whereas when the concentration is infinite all the agents have the same preferences. The concentration can be viewed as the degree of conflicts among the resources. High concentration values lead to similar preferences among the agents for a given item.

We expected that the more similar the preferences between the agents are, the higher the degrees of  $K$ -app envy and non-proportionality would get and the more likely unanimous envy and non proportionality would occur. The results of our experiments both in the general setting and in HAP support this: the number of envy-free and proportional instances is decreasing along with the concentration value, and from a given threshold, all the instances exhibit unanimous envy and unanimous non-proportionality. We can see it for example through Figure 6.

Even though at the extreme (when all agents have the same preferences) all notions become unanimous, one may still wonder whether some degree of correlation among preferences may help to find large majorities of agents that contradict the envy of an agent. We thus studied how the



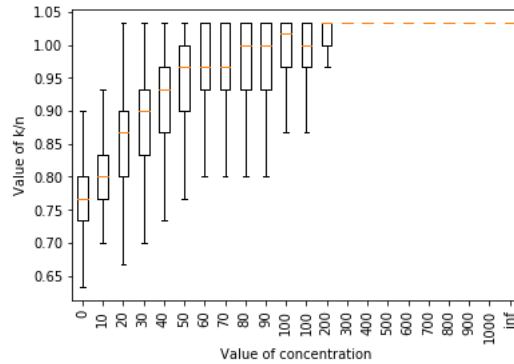


Figure 6: Optimal  $K/n$  (approval envy)-freeness in the HAP as a function of the value of concentration for  $n = 30$  and  $m = 30$

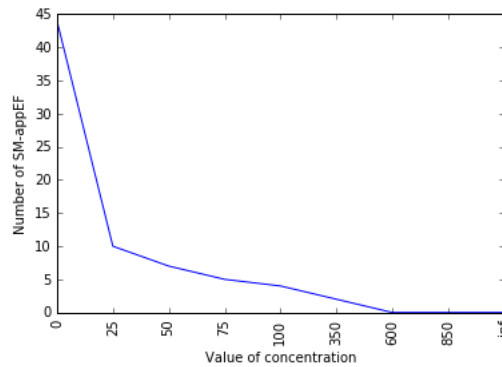


Figure 7: Number of SM-app EF instances as a function of the value of concentration for  $n = 7$  and  $m = 9$

number of SM-app EF instances varies as a function of the concentration. We considered instances involving 7 agents and 9 objects as we had previously noticed that under uniformly distributed preferences (which is equivalent to a value of concentration of 0), it was very likely to find SM-app EF instances. We then varied the concentration value. For each value, we generated 100 instances and counted the number of SM-app EF instances. As shown in Figure 7, the higher the concentration (and hence the more similar the preferences), the less SM-app EF instances are found, contradicting our hypothesis that correlation might make large majorities of agents contradicting envy more likely to occur.

## 11. Conclusion

In this paper, we have introduced a new relaxation of envy-freeness and proportionality. These relaxations use a consensus notion, approval envy or non-proportionality, as a proxy for an idealized notion of envy between pairs of agents or proportionality of an agent. We have proposed algo-

gorithms to compute an allocation minimizing the approval envy or non-proportionality, and we have experimentally studied how these notions behave on real world data, as well as on instances with uniformly distributed or correlated preferences; more particularly in situations where no envy-free allocation exists and where no proportional allocation exists. We have shown that our notion of approval envy (less so for approval non-proportionality) strikes an interesting balance allowing to discriminate in practice among instances depending on the social support envy relations experience. In comparison, using consensus to determine whether a given agent should be envious or not in general proves to be of limited interest: except in rare cases, instances will either be envy-free or unanimous envy.

This work also opens up to a more general study of consensus-based notions of envy. One could for instance look for allocations that are judged envy-free by a given quota of agents. Restrictions of the approval notions such as an underlying social graph constraining the agents that can approve or disapprove –those agents you deem legitimate to express her view about a specific envy relation– could also be of interest for future work. Other domain restrictions, beyond house allocation, could be studied. For instance, the domain of binary additive preferences, with a cap on the number of items that an agent can like, may offer other tractable cases for our problem. Besides, the approval notions introduced in this article also call for a study of the manipulation that could arise from it. Indeed, asking the opinion of the agents gives birth to new ways of manipulating. More generally, an axiomatic study of the notions proposed here could nicely complement the results obtained.

Finally, it could also be interesting to propose extensions to the case where some items can be shared. Indeed, the approval concepts are a way to mix voting concepts with fair division and shared items is another way of building a continuum between voting and fair division. There may exist a potential link between both approaches. We leave the study of these notions for future work.

## Appendix

**Proposition 20.** *Let  $I$  be an add-MARA instance. Then, there is an optimal solution with  $\mathbf{K} = L$  to  $\mathcal{M}_1(I)$  if and only if  $I$  is an  $(L\text{-app envy})$ -free instance and not an  $((L - 1)\text{-app envy})$ -free one. Moreover,  $\mathcal{M}_1(I)$  does not admit any solution if and only if  $I$  is an unanimous envy instance.*

*Proof.* To prove the proposition, we show that there is an  $(L\text{-app envy})$ -free allocation in  $I$  if and only if there is a solution to the MIP  $\mathcal{M}_1(I)$  such that  $\mathbf{K} = L$ .

( $\Rightarrow$ ) Let  $I$  be an instance, and  $\pi$  be an  $(L\text{-app envy})$ -free allocation. Then, consider the partial instantiation of the variables such that  $\mathbf{z}_i^j = 1$  if and only if  $o_j \in \pi_i$ . We prove that this partial instantiation extends to a solution of  $\mathcal{M}_1(I)$  such that  $\mathbf{K} = L$ .

First observe that Constraint 1 is directly satisfied.

Now, consider any triple of agents  $(a_k, a_i, a_h)$ . Suppose that agent  $a_k$  thinks  $a_i$  should envy  $a_h$ . Then in this case, we have  $\sum_{j \in \pi_h} u(k, j) > \sum_{j \in \pi_i} u(k, j)$ . In other words,  $\sum_{j=1}^m u(k, j)(\mathbf{z}_h^j - \mathbf{z}_i^j) > 0$  which is in turn equivalent to  $\sum_{j=1}^m u(k, j)(\mathbf{z}_h^j - \mathbf{z}_i^j) \geq 1$  since all utilities are integers. By Constraint 2, we thus have that  $\mathbf{e}_{kih} = 1$  which implies that Constraint 3 is satisfied as well.

Conversely, suppose that agent  $a_k$  thinks  $a_i$  should not envy  $a_h$ . Then, we have  $\sum_{j \in \pi_h} u(k, j) \leq \sum_{j \in \pi_i} u(k, j)$ . In other words,  $\sum_{j=1}^m u(k, j)(\mathbf{z}_h^j - \mathbf{z}_i^j) \leq 0$ . By Constraint 3, we thus have that  $\mathbf{e}_{kih} = 0$  in this case, which in turns implies that Constraint 2 is satisfied as well. Hence, we have that  $\mathbf{e}_{kih} = 1$  if and only if  $a_k$  thinks  $a_i$  should envy  $a_h$  in  $\pi$ .

Finally, consider any pair of agents  $(a_i, a_h)$ . If  $a_i$  does not envy  $a_h$  then  $e_{iih} = 0$ . As a consequence,  $x_{ih}$  can be null and still satisfy Constraints 4 and 5 (no matter the value of  $\mathbf{K}$  is).

Now suppose that  $a_i$  does envy  $a_h$  (hence  $e_{iih} = 1$ ). Then, we should have  $x_{ih} = 1$  to satisfy Constraint 4. Since  $\pi$  is  $(L\text{-app envy})$ -free, then at most  $L - 1$  agents (including  $a_i$  herself) think that  $a_i$  should indeed envy  $a_h$ , which means that  $\sum_{k=1}^n e_{kih} \leq L - 1$ . Instantiating  $\mathbf{K}$  to  $L$  is hence enough to satisfy Constraint 5.

( $\Leftarrow$ ) Now suppose that there is a solution to  $\mathcal{M}_1(I)$  such that  $\mathbf{K} = L$ . Then we will prove that the allocation  $\pi$  such that  $o_j \in \pi_i$  if and only if  $z_{ij} = 1$  is a valid  $(L\text{-app envy})$ -free allocation.

First, according to Constraints 1,  $\pi$  is indeed a valid allocation.

Secondly, Constraint 2 ensures that if  $e_{kih} = 0$  then  $\sum_{j=1}^m u(k, j)(z_h^j - z_i^j) \leq 0$ , in turn meaning that agent  $a_k$  thinks that  $a_i$  should not envy  $a_h$ . Conversely, Constraint 3 ensures that if  $e_{kih} = 1$  then  $\sum_{j=1}^m u(k, j)(z_h^j - z_i^j) > 0$ , in turn meaning that agent  $a_k$  thinks that  $a_i$  should envy  $a_h$ . It also obviously implies that  $e_{iih} = 1$  if and only if  $a_i$  envies  $a_h$ .

Now consider any pair of agents  $(a_i, a_h)$  such that  $a_i$  envies  $a_h$ . From what precedes,  $e_{iih} = 1$ . By Constraint 4,  $x_{ih} = 1$ . Hence, by Constraint 5,  $\sum_{k=1}^h e_{kih} \leq L - 1$ . This implies that the total number of agents agreeing with the fact that  $a_i$  envies  $a_h$  is strictly lower than  $L$ . In other words,  $\pi$  is  $(L\text{-app envy})$ -free.  $\square$

**Proposition 21.** *Let  $I$  be an instance. Then, there is an optimal solution with  $\mathbf{K} = L$  to  $\mathcal{M}_2(I)$  if and only if  $I$  is an  $(L\text{-app non-prop})$ -free instance and not an  $((L - 1)\text{-app non-prop})$ -free one. Moreover,  $\mathcal{M}_2(I)$  does not admit any solution if and only if  $I$  is an unanimous non-proportional instance.*

*Proof.* The key here is to show that there is a solution to the MIP  $\mathcal{M}_2(I)$  such that  $\mathbf{K} = L$  iff the corresponding allocation  $\pi$  such that  $z_i^j = 1$  if and only if  $o_j \in \pi_i$  is  $(L\text{-app non-prop})$ -free. However this is done in the proof of Proposition 20. We also have to show that Constraints 6 and 7 are indeed a valid translation of the logical equivalence, and that Constraints 8 and 9 correctly encode the logical OR. The same type of linearization is also done in the proof of Proposition 20.  $\square$

**Proposition 22.** *The probability for an instance being randomly generated under uniformly distributed preferences to exhibit unanimous envy is upper bounded by  $n(n - 1)/2^n$ .*

*Proof.* The probability of the event  $o_i$  is strictly preferred to  $o_j$  by one agent is  $1/2$  if preferences are strict. As preferences are not strict, this probability becomes an upper bound (think for instance if the agent values all the objects the same then the probability to have strict preference between two objects is zero). Hence, the probability of the event  $o_i$  is strictly preferred to  $o_j$  by all agents is upper bounded by  $1/2^{n-1}$  as the preferences between the agents are independent. Assuming, for all pairs of items, these events to be independent (which is not the case, hence an upper bound of the upper bound), we derive our result by summing up over the  $n(n - 1)/2$  possible pairs.  $\square$

Note that this value quickly tends towards 0: for instance, for 10 agents, the probability for an instance to exhibit unanimous envy is upper-bounded by 0.088.

**Proposition 23.** *The probability for an instance being randomly generated under uniformly distributed preferences to exhibit unanimous non-proportionality is upper bounded by  $n/2^n$ .*

To prove this property, we will need a small lemma:<sup>3</sup>

**Lemma 4.** *Let  $U_1, \dots, U_n$  be  $n$  independent random variables having a uniform distribution over real interval  $[a, b]$ , and let us denote by  $\bar{U}_n$  the empiric mean of  $U_1, \dots, U_n$ :  $\bar{U}_n = \frac{1}{n} \sum_{i=1}^n U_i$ .*

*Then we have:*

$$P(U_1 < \bar{U}_n) = \frac{1}{2}.$$

*Proof.* The probability we seek can be reformulated as follows:

$$P(U_1 < \bar{U}_n) = P\left(U_1 < \frac{1}{n} \sum_{i=1}^n U_i\right) = P\left(U_1 < \frac{1}{n-1} \sum_{i=2}^n U_i\right) = P(U_1 < \bar{U}_{n-1}). \quad (10)$$

We can notice that the two latter variables,  $U_1$  and  $\bar{U}_{n-1}$  are independent. For any two independent variables  $X$  and  $Y$ , we have that  $P(X < Y) = \mathbb{E}[F_X(Y)]$ , where  $F_X$  is the cumulative distribution function of  $X$ . To see this, we can consider the following steps:

$$\begin{aligned} P(X < Y) &= \int P(X < Y | Y = y) f_Y(y) dy = \int P(X < y) f_Y(y) dy \\ &= \int F_X(y) f_Y(y) dy \\ &= \mathbb{E}[F_X(Y)], \end{aligned} \quad (11)$$

where the first step is obtained using the law of total probability, and the last step is obtained using the law of unconscious statistician.

Putting Equations (10) and (11) together, we obtain:

$$P(U_1 < \bar{U}_n) = \mathbb{E}[F_{U_1}(\bar{U}_{n-1})] = \mathbb{E}\left(\frac{\bar{U}_{n-1} - a}{b - a}\right) = \frac{\mathbb{E}(\bar{U}_{n-1}) - a}{b - a} \quad (12)$$

Observing that  $\bar{U}_{n-1}$  has the same (uniform) law as  $U_1$ , we have that  $\mathbb{E}(\bar{U}_{n-1}) = \frac{a+b}{2}$ . Injecting this to Equation (12) yields:

$$P(U_1 < \bar{U}_n) = \frac{1}{2} \quad (13)$$

as expected.  $\square$

We are now ready to prove Proposition 23.

*Proof (Proposition 23).* Let  $I$  be a random instance generated under uniformly distributed preferences. According to Proposition 17,  $I$  is unanimous non-proportional if and only if there exists at least an item  $o_p$  such that  $u(k, p) < Prop_k$  for all agents  $a_k$ . In what follows, we will denote by  $U_{k,p}$  the random variable corresponding to  $u(k, p)$ .

For any item  $o_p$  and any agent  $a_k$ ,  $P(U_{k,p} < \frac{1}{n} \sum_{j=1}^n U_{k,j}) = \frac{1}{2}$  by Lemma 4. All the variables  $U_{k,p}$  being independent, we have that:

$$P\left(\bigcap_{k=1}^n \left(U_{k,p} < \frac{1}{n} \sum_{j=1}^n U_{k,j}\right)\right) = \frac{1}{2^n} \text{ for all } o_p.$$

3. We warmly thank Olivier François for this result.

The events  $U_{k,p} < \frac{1}{n} \sum_{j=1}^n U_{k,j}$  not being independent, we can only derive an upper bound on the probability for  $I$  to have at least one object  $o_k$  such that  $u(k,p) < Prop_k$  for all agents  $a_k$ . Namely:

$$P \left( \exists k \in \llbracket 1, n \rrbracket \mid \bigcap_{k=1}^n \left( U_{k,p} < \frac{1}{n} \sum_{j=1}^n U_{k,j} \right) \right) \leq \frac{n}{2^n},$$

which concludes the proof.  $\square$

Note that once again this value quickly tends towards 0: for instance, for 10 agents, the probability for an instance to exhibit unanimous non-proportionality is upper-bounded by 0.00977.

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