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Output feedback stabilization of an ODE-Reaction-Diffusion PDE cascade with a long interconnection delay ⋆

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Abstract

This paper addresses the output feedback stabilization of an ODE-PDE cascade. The PDE takes the form of a 1-D reaction-diffusion equation. The output of the ODE enters into the PDE via a Dirichlet/Neumann/Robin boundary condition. It is assumed that both the output of the ODE and the output of the PDE, selected as a boundary Dirichlet trace, are available for feedback control. The proposed control strategy takes the form of a finite-dimensional observer-based controller. Under a suitable structural controllability property, we show that the reported control strategy achieves the exponential stabilization of the plant when the order of the observer is selected large enough. We then demonstrate how such a control strategy can be adapted and augmented with a predictor component in order to achieve the stabilization of the above mentioned PDE-ODE cascade when the output of the ODE enters into the PDE with an arbitrarily long delay.

Key words: ODE-PDE cascade, delayed cascade interconnection, predictor, output feedback

1 Introduction

The topic of feedback stabilization of systems described by an ODE coupled with a PDE has emerged during the last decade. This trend is driven by a number of engineering applications such flexible aircraft [29], flexible cranes [17], power converters connected to transmission lines [13], and solid–gas interaction of heat diffusion and chemical reaction [38]. Coupled PDE and ODE dynamics also arise due to actuator and sensor dynamics employed to implement feedback control strategies [23,36]. For instance, the PDE can represent the open-loop plant while the ODE part gathers actuator/sensor dynamics. Conversely, the PDE can model the sensor dynamics (e.g., to measure heat) of a finite-dimensional plant. This may include the presence of filters (e.g., for noise reduction issues). Moreover, interconnection delays between the ODE and PDE components may appear, e.g., due to network effects or the physical distance between the actuator and the plant; see, e.g., the cases of antennas for nuclear fusion [1] or sprays for surface decontamination [43].

The state-feedback stabilization and the observer design of a diffusion PDE cascaded with an ODE was solved using backstepping design in [23,36]. These control design approaches, which can be seen as the generalization of predictor feedback techniques [19] as they aim at compensating an infinite-dimensional input dynamics [24], have then been extended to the state-feedback stabilization of other systems described by PDEs such as strings [22,36] and linearized Korteweg–de Vries equation [3]. They have also been extended to the state-feedback stabilization and observer design of multi-input–multi-output LTI systems with actuator or sensor dynamics governed by diffusion [7] and wave [6] PDEs, the output feedback stabilization of a diffusion PDE coupled with an ODE [37,38], and the output feedback stabilization of either a diffusion PDE [39] or hyperbolic PDEs [14,41] sandwiched between two ODEs. The adaptive stabilization of a coupled ODE-reaction-diffusion PDE was reported in [30] while the sliding mode control of a heat PDE-ODE cascade was addressed in [42]. The stabilization of PDE-ODE and ODE-PDE cascades using Sylvester equations has been proposed in [31]. Finally, the possible presence of a delay in the ODE-PDE system has been considered in a number of recent works. The state-feedback stabilization of an ODE-heat equation in the presence of a state-delay was studied in [18]. The state-feedback of

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an ODE-linear KdV cascaded system in the presence of a boundary input delay was reported in [4]. The output feedback of a sandwiched ODE–PDE–ODE hyperbolic system in the presence of an output delay for the second ODE was addressed in [40].

This paper is concerned with the output feedback stabilization of an ODE-reaction-diffusion PDE cascade. The output of the ODE enters into the PDE via a Dirichlet/Neumann/Robin boundary condition. The output of the PDE is selected as a boundary Dirichlet trace. We assume that the output of each component of the plant (both ODE and PDE) is available for feedback control. This type of output feedback stabilization problem was solely addressed in [39] for an ODE-diffusion PDE-ODE cascade (see also [18,42] for ODE-heat cascades in a state-feedback setting). In this latter work, only the measurement of the PDE is required but the open loop diffusion PDE is stable (with constant coefficient) and the observer is infinite-dimensional. In contrast, we consider in this work a reaction-diffusion PDE that may be unstable (with spatially varying coefficients) while the adopted control strategy consists of a finite-dimensional observer-based controller [5,11,15,16,20,21,26,27,25,35]. Leveraging spectral reduction methods [9,10,34], we adopt and augment to ODE-PDE cascades the finite-dimensional observer-based controller architecture as introduced in the pioneer work [35] along with a LMI-based strategy as initiated in [20]. More precisely, we adopt the enhanced procedures [26,25] that allow to handle general 1-D reaction-diffusion PDEs with Dirichlet/Neumann/Robin boundary control and Dirichlet/Neumann boundary measurement while performing the control design directly with the actual boundary control input instead of its time derivative. Under a structural controllability property related to the ODE and PDE components, we assess that the proposed finite-dimensional control strategy achieves the output feedback stabilization of the ODE-PDE cascade when the order of the observer is large enough.

In the second part of this paper, we demonstrate that this control strategy can be augmented with a predictor component [2] in order to achieve the stabilization of the above mentioned ODE-PDE cascade when the output of the ODE enters into the PDE with an arbitrarily long delay. Hence, the system to be stabilized can be interpreted as an ODE-(transport PDE)-(reaction-diffusion PDE) cascade. Predictor feedback is a well-known approach for compensating long input/output delays [8,19]. The possibility to couple a finite-dimensional observer and a predictor to compensate an input delay was suggested in [21] in the particular case of a reaction-diffusion equation with Neumann boundary control, a bounded output operator, and with stability of the closed-loop system assessed in L^2-norm. The case of general input delayed 1-D reaction-diffusion PDEs with Dirichlet/Neumann/Robin boundary control and Dirichlet/Neumann boundary measurement was solved in [27] for PDE trajectories in evaluated H^1-norm. The case of a long output delay was addressed in [28]. In contrast, the delay considered in the present work is fully entangled with the system dynamics since the delay occurs in the interconnection of the ODE-PDE cascade. In this context, the adopted control design procedure is inspired by classical approaches for the predictor-based stabilization of finite-dimensional LTI systems [19, Chap. 3]. More precisely, we design a finite-dimensional observer that estimates parts of the state of the ODE-PDE cascade at two distinct times: 1) a delayed version of the state of the ODE; 2) the first modes of the PDE at current time. This observer is coupled with a predictor component that is used to obtain the control input to be applied at current time. Under a suitable structural controllability property and for an arbitrarily given value of the interconnection delay, we show that the reported control strategy achieves the exponential stabilization of the plant provided the order of the observer is selected large enough.

The paper is organized as follows. Definitions and properties of Sturm-Liouville operators are introduced in Section 2. The control design problem of an ODE-PDE cascade is addressed in Section 3. The case of an ODE-PDE cascade with delayed interconnection is then studied in Section 4. A numerical example is provided in Section 5. Finally, concluding remarks are formulated in Section 6.

2 Notation and properties

2.1 Notation

Spaces $\mathbb{R}^n$ are endowed with the Euclidean norm $\|\cdot\|$. The corresponding induced norms of matrices are also denoted by $\|\cdot\|$. For any two vectors X and Y of arbitrary dimensions, $\mathrm{col}(X,Y)$ stands for the vector $[X^\top, Y^\top]^\top$. The space of square integrable functions on $(0,1)$ is denoted by $L^2(0,1)$ and is endowed with the usual inner product $\langle f,g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$ and associated norm denoted by $\|\cdot\|_{L^2}$. For an integer $m \geq 1$, $H^m(0,1)$ denotes the m-order Sobolev space and is equipped with its usual norm $\|\cdot\|_{H^m}$. For a symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succeq 0$ (resp. $P \succ 0$) means that P is positive semi-definite (resp. positive definite).

2.2 Properties of Sturm-Liouville operators

Let $\theta_1 \in [0,\pi/2]$, $\theta_2 \in [0,\pi/2]$, $p \in \mathcal{C}^1([0,1])$ and $q \in \mathcal{C}^0([0,1])$ with $p > 0$ and $q \geq 0$. Let the Sturm-Liouville operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1)$ be defined by $\mathcal{A}f = -(pf')' + qf$ on the domain $D(\mathcal{A}) = \{f \in H^2(0,1) : c_{\theta_1} f(0) - s_{\theta_1} f'(0) = c_{\theta_2} f(1) + s_{\theta_2} f'(1) = 0\}$ where $c_{\theta_i} = \cos\theta_i$ and $s_{\theta_i} = \sin\theta_i$. The eigenvalues λ_n, $n \geq 1$, of $\mathcal{A}$ are simple, non negative, and form an increasing sequence with $\lambda_n \to +\infty$ as $n \to +\infty$. The corresponding unit

eigenvectors $\phi_n \in L^2(0,1)$ form a Hilbert basis (i.e, an orthonormal basis of the Hilbert space $L^2(0,1)$). The domain of the operator $\mathcal{A}$ is also characterized by $D(\mathcal{A}) = \{f \in L^2(0,1) : \sum_{n\geq 1} |\lambda_n|^2 |\langle f, \phi_n\rangle|^2 < +\infty\}$. Let $p_*, p^*, q^* \in \mathbb{R}$ be such that $0 < p_* \leq p(x) \leq p^*$ and $0 \leq q(x) \leq q^*$ for all $x \in [0,1]$, then it holds $0 \leq \pi^2(n-1)^2 p_* \leq \lambda_n \leq \pi^2 n^2 p^* + q^*$ for all $n \geq 1$ [32]. Moreover if $p \in \mathcal{C}^2([0,1])$, we have (see, e.g., [32]) that $\phi_n(\xi) = O(1)$ and $\phi_n'(\xi) = O(\sqrt{\lambda_n})$ as $n \to +\infty$ for any given $\xi \in [0,1]$. Assuming further that $q > 0$, an integration by parts and the continuous embedding $H^1(0,1) \subset L^\infty(0,1)$ show the existence of constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{H^1}^2 \leq \sum_{n\geq 1} \lambda_n \langle f, \phi_n\rangle^2 = \langle \mathcal{A}f, f\rangle \leq C_2 \|f\|_{H^1}^2 \quad (1)$$

for any $f \in D(\mathcal{A})$. This implies that the series expansion $f = \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n$ holds in $H^2(0,1)$-norm for any $f \in D(\mathcal{A})$. Due to the continuous embedding $H^1(0,1) \subset L^\infty(0,1)$, we obtain that $f(0) = \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n(0)$ and $f'(0) = \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n'(0)$. We finally define, for any integer $N \geq 1$, $\mathcal{R}_N f = \sum_{n\geq N+1} \langle f, \phi_n\rangle \phi_n$.

3 ODE-PDE cascade

3.1 Problem setting

We study first the ODE-PDE cascade described by

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t), \quad y_i(t) = Cx(t) && \text{(2a)}\\
&z_t(t,\xi) = (p(\xi) z_\xi(t,\xi))_\xi - r(\xi) z(t,\xi) && \text{(2b)}\\
&c_{\theta_1} z(t,0) - s_{\theta_1} z_\xi(t,0) = 0 && \text{(2c)}\\
&c_{\theta_2} z(t,1) + s_{\theta_2} z_\xi(t,1) = y_i(t) && \text{(2d)}\\
&y(t) = z(t,0) && \text{(2e)}\\
&x(0) = x_0, \quad z(0,\xi) = z_0(\xi) && \text{(2f)}
\end{aligned}$$

for $t > 0$ and $\xi \in (0,1)$. Here $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x}$, $C \in \mathbb{R}^{1\times n_x}$, $\theta_1 \in (0, \pi/2]$, $\theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $r \in \mathcal{C}^0([0,1])$. The states of the ODE and PDE at time t are denoted by $x(t)$ and $z(t,\cdot)$, respectively. The control input at time t is denoted by $u(t)$. We assume that the interconnection signal $y_i(t) = Cx(t)$ and the PDE measurement $y(t) = z(t,0)$ are available for feedback. Without loss of generality, we introduce $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ such that

$$r(\xi) = q(\xi) - q_c, \quad q(\xi) > 0. \quad (3)$$

Remark 1 *The presentation is focused on the case $\theta_1 \in (0, \pi/2]$ and $\theta_2 \in [0, \pi/2]$ because this setting is the most meaningful from a practical perspective (e.g., the case of a heat equation with convection boundary conditions). Nevertheless, the results derived in this paper readily extend to $\theta_1 \in (0,\pi) \cup (\pi, 2\pi)$ and $\theta_2 \in [0, 2\pi)$ provided q from (3) is selected sufficiently large so that (1) holds.*

3.2 Spectral reduction

Defining the change of variable formula

$$w(t,\xi) = z(t,\xi) - \frac{\xi^2}{c_{\theta_2} + 2s_{\theta_2}} y_i(t), \quad (4)$$

we infer that (2) is equivalently represented by the following homogeneous representation

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t), \quad y_i(t) = Cx(t) && \text{(5a)}\\
&w_t(t,\xi) = (p(\xi) w_\xi(t,\xi))_\xi - r(\xi) w(t,\xi) && \text{(5b)}\\
&\qquad + a(\xi) y_i(t) + b(\xi) \dot{y}_i(t)\\
&c_{\theta_1} w(t,0) - s_{\theta_1} w_\xi(t,0) = 0 && \text{(5c)}\\
&c_{\theta_2} w(t,1) + s_{\theta_2} w_\xi(t,1) = 0 && \text{(5d)}\\
&y(t) = w(t,0) && \text{(5e)}\\
&x(0) = x_0, \quad w(0,\xi) = w_0(\xi) && \text{(5f)}
\end{aligned}$$

where $a(\xi) = \frac{1}{c_{\theta_2}+2s_{\theta_2}}\{2p(\xi) + 2\xi p'(\xi) - \xi^2 r(\xi)\}$, $b(\xi) = -\frac{\xi^2}{c_{\theta_2}+2s_{\theta_2}}$, and $w_0(\xi) = z_0(\xi) - \frac{\xi^2}{c_{\theta_2}+2s_{\theta_2}} y_i(0)$. Define the coefficients of projection $z_n(t) = \langle z(t,\cdot), \phi_n\rangle$, $w_n(t) = \langle w(t,\cdot), \phi_n\rangle$, $a_n = \langle a, \phi_n\rangle$, and $b_n = \langle b, \phi_n\rangle$. This gives

$$w_n(t) = z_n(t) + b_n y_i(t), \quad n \geq 1. \quad (6)$$

The projection of (5) on $(\phi_n)_{n\geq 1}$ gives

$$\dot{w}_n(t) = (-\lambda_n + q_c) w_n(t) + a_n y_i(t) + b_n \dot{y}_i(t), \quad (7)$$

hence

$$\dot{z}_n(t) = (-\lambda_n + q_c) z_n(t) + \beta_n y_i(t) \quad (8)$$

where $\beta_n = a_n + (-\lambda_n + q_c) b_n = p(1)\{-c_{\theta_2}\phi_n'(1) + s_{\theta_2}\phi_n(1)\} = O(\sqrt{\lambda_n})$. Finally, the Dirichlet measurement (5e) can be expressed for classical solutions as:

$$y(t) = w(t,0) = \sum_{n\geq 1} w_n(t)\phi_n(0). \quad (9)$$

3.3 Control strategy

The control strategy adopted in this section is an extension of the controller architecture originally introduced in [35] for parabolic PDEs (and later reused in [5,11,15,16,20,25,35]) to the case of the ODE-parabolic PDE cascade (2). Let $\delta > 0$ be fixed. Let $N_0 \geq 1$ be selected so that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. Let $N \geq N_0 + 1$ be arbitrarily given. The controller dynamics are described by:

$$\begin{aligned}
&\hat{w}_n(t) = \hat{z}_n(t) + b_n y_i(t) && \text{(10a)}\\
&\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L_{xy_i}\{C\hat{x}(t) - y_i(t)\} && \text{(10b)}\\
&\qquad - L_{xy}\left\{\sum_{k=1}^{N} \hat{w}_k(t)\phi_k(0) - y(t)\right\}
\end{aligned}$$

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n y_i(t) - l_{zy_i,n}\{C\hat{x}(t) - y_i(t)\} - l_{zy,n}\left\{\sum_{k=1}^{N} \hat{w}_k(t)\phi_k(0) - y(t)\right\},\ 1 \le n \le N_0 \tag{10c}$$

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n y_i(t),\ N_0 + 1 \le n \le N \tag{10d}$$

$$u(t) = \sum_{k=1}^{N_0} k_k \hat{z}_k(t) + k_x \hat{x}(t) \tag{10e}$$

where $l_{zy_i,n}, l_{zy,n} \in \mathbb{R}$ and $L_{xy_i}, L_{xy} \in \mathbb{R}^{n_x}$ are the observer gains while $k_k \in \mathbb{R}$ and $k_x \in \mathbb{R}^{1\times n_x}$ are the feedback gains.

In preparation for stability analysis, we introduce the matrices: $A_0 = \mathrm{diag}(-\lambda_1 + q_c, \ldots, -\lambda_{N_0} + q_c)$, $A_1 = \mathrm{diag}(-\lambda_{N_0+1} + q_c, \ldots, -\lambda_N + q_c)$, $\mathfrak{B}_0 = \begin{bmatrix}\beta_1 & \ldots & \beta_{N_0}\end{bmatrix}^\top$, $\tilde{\mathfrak{B}}_1 = \begin{bmatrix}\frac{\beta_{N_0+1}}{\lambda_{N_0+1}} & \ldots & \frac{\beta_N}{\lambda_N}\end{bmatrix}^\top$, $C_0 = \begin{bmatrix}\phi_1(0) & \ldots & \phi_{N_0}(0)\end{bmatrix}$, $\tilde{C}_1 = \begin{bmatrix}\frac{\phi_{N_0+1}(0)}{\sqrt{\lambda_{N_0+1}}} & \ldots & \frac{\phi_N(0)}{\sqrt{\lambda_N}}\end{bmatrix}$, $C_a = \begin{bmatrix}0 & C\end{bmatrix}$, $\mathfrak{A} = \begin{bmatrix}A_0 & \mathfrak{B}_0 C \\ 0 & A\end{bmatrix}$, $\mathfrak{U} = \begin{bmatrix}A_0 & 0 \\ 0 & A\end{bmatrix}$, $\mathfrak{B} = \begin{bmatrix}0 \\ B\end{bmatrix}$, $\mathfrak{C} = \begin{bmatrix}C_0 & 0 \\ 0 & C\end{bmatrix}$, $K = \begin{bmatrix}k_1 & \ldots & k_{N_0} & k_x\end{bmatrix}$, $L_{zy} = \begin{bmatrix}l_{zy,1} & \ldots & l_{zy,N_0}\end{bmatrix}^\top$, $L_{zy_i} = \begin{bmatrix}l_{zy_i,1} & \ldots & l_{zy_i,N_0}\end{bmatrix}^\top$, $L = \begin{bmatrix}L_{zy} & L_{zy_i} \\ L_{xy} & L_{xy_i}\end{bmatrix}$, $L_y = \begin{bmatrix}L_{zy} \\ L_{xy}\end{bmatrix}$, $\mathfrak{D} = L\mathfrak{C} + \begin{bmatrix}0 & \mathfrak{B}_0 C \\ 0 & 0\end{bmatrix}$,

$$\mathcal{F} = \begin{bmatrix}\mathfrak{A} + \mathfrak{B}K & \mathfrak{D} & 0 & L_y\tilde{C}_1 \\ 0 & \mathfrak{U} - L\mathfrak{C} & 0 & -L_y\tilde{C}_1 \\ \tilde{\mathfrak{B}}_1 C_a & \tilde{\mathfrak{B}}_1 C_a & A_1 & 0 \\ 0 & 0 & 0 & A_1\end{bmatrix},\ \mathcal{L} = \begin{bmatrix}L_y \\ -L_y \\ 0 \\ 0\end{bmatrix},$$

$$E_1 = \begin{bmatrix}C_a & C_a & 0 & 0\end{bmatrix}, E_2 = C_a\begin{bmatrix}\mathfrak{A} + \mathfrak{B}K & \mathfrak{D} + \mathfrak{U} - L\mathfrak{C} & 0 & 0\end{bmatrix}.$$

Remark 2 *By Cauchy uniqueness, it is observed that $\phi_n(0) \neq 0$ and $\beta_n \neq 0$ for all $n \ge 1$. Therefore the Hautus test shows that the pairs $(A_0, \mathfrak{B}_0)$ and (A_0, C_0) satisfy the Kalman rank condition for controllability and observability, respectively. Hence, the pair $(\mathfrak{U}, \mathfrak{C})$ is observable if and only if (A, C) is observable. The case of the pair $(\mathfrak{A}, \mathfrak{B})$ is more involved due to the fact that the control input of the ODE-PDE cascade only appears in input of the ODE. It is easily seen that if (A, B) is not controllable, then $(\mathfrak{A}, \mathfrak{B})$ is not controllable neither. A form of converse result is stated below.*

Lemma 3 *Assume that (A, B) is controllable. Then the pair $(\mathfrak{A}, \mathfrak{B})$ is controllable if and only if $C^\top \notin R\left((\mu I - A^\top)\big|_{\ker(B^\top)}\right)$ for all $\mu \in \mathrm{sp}A_0 = \{-\lambda_n + q_c : 1 \le n \le N_0\}$. Assuming further that $\mathrm{sp}A \cap \mathrm{sp}A_0 = \emptyset$, then $(\mathfrak{A}, \mathfrak{B})$ is controllable if and only if $C(\mu I - A)^{-1}B \neq 0$ for all $\mu \in \mathrm{sp}A_0$.*

Proof. The case $C = 0$ trivially holds true because, in that case, $(\mathfrak{A}, \mathfrak{B})$ is not controllable. Hence we assume for the rest of the proof that $C \neq 0$. Assume that $(\mathfrak{A}, \mathfrak{B})$ is not controllable. The Hautus test shows the existence of $\mu \in \mathbb{C}$ and $(x_1, x_2) \neq 0$ so that $x_1^\top A_0 = \mu x_1^\top$, $x_1^\top \mathfrak{B}_0 C + x_2^\top A = \mu x_2^\top$, and $x_2^\top B = 0$. We observe that $x_1 = 0$ if and only if $x_2 = 0$. Hence $x_1 \neq 0$ and $x_2 \neq 0$, implying that $\mu \in \mathrm{sp}A_0$. Since A_0 is diagonal with simple eigenvalues, we may assume without loss of generality that $x_1 = f_n$ for some $1 \le n \le N_0$ where f_n is the nth element of the canonical basis of $\mathbb{R}^{N_0}$. We deduce that $\beta_n C + x_2^\top A = \mu x_2^\top$. Since $\beta_n \neq 0$, we can define $\tilde{x}_2 = x_2/\beta_n$ which is such that $B^\top \tilde{x}_2 = 0$ and $C^\top = (\mu I - A^\top)\tilde{x}_2$. Conversely, assume that there exists $\mu = -\lambda_n + q_c \in \mathrm{sp}A_0$ and a vector $\tilde{x}_2$ so that $B^\top \tilde{x}_2 = 0$ and $C^\top = (\mu I - A^\top)\tilde{x}_2$. Since $C \neq 0$ we infer that $\tilde{x}_2 \neq 0$. With $\beta_n \neq 0$, this implies that $x_2 = \beta_n \tilde{x}_2 \neq 0$ satisfies $x_2^\top B = 0$ and $\beta_n C + x_2^\top A = \mu x_2^\top$. Defining $x_1 = f_n$, we have $x_1^\top \mathfrak{B}_0 = \beta_n$ hence $x_1^\top A_0 = \mu x_1^\top$, $x_1^\top \mathfrak{B}_0 C + x_2^\top A = \mu x_2^\top$, and $x_2^\top B = 0$. Hence $(\mathfrak{A}, \mathfrak{B})$ is not controllable. To conclude the proof, it is sufficient to notice that if $\mu \notin \mathrm{sp}A$, then the condition $C^\top \in R\left((\mu I - A^\top)\big|_{\ker(B^\top)}\right)$ holds true if and only if there exists a vector $\tilde{x}_2$ so that $\tilde{x}_2^\top B = 0$ and $C = \tilde{x}_2^\top(\mu I - A)$, if and only if $C(\mu I - A)^{-1}B = 0$. □

Remark 4 *In the case of a sole reaction-diffusion PDE, the boundary output feedback stabilization of the plant with Robin boundary control and Dirichlet/Neumann measurement can always be achieved. This is in part due to the fact that the pair $(A_0, \mathfrak{B}_0)$ is always controllable; see [26,25] for details. In sharp contrast, Lemma 3 shows that the possibility to achieve the stabilization of the ODE-PDE cascade (2) heavily depends on structural properties of both the ODE and PDE parts of the system. Note that, in the case that the N_0 first eigenvalues of the reaction-diffusion PDE are not eigenvalues of the ODE plant, the controllability condition reduces to the fact that none of the N_0 first eigenvalues of the reaction-diffusion PDE is a zero of the ODE system.*

3.4 Main stability result

Theorem 5 *Let $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x}$, and $C \in \mathbb{R}^{1\times n_x}$, $\theta_1 \in (0, \pi/2]$, $\theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $r \in \mathcal{C}^0([0,1])$. Let $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ be such that (3) holds. Let $\delta > 0$ and $N_0 \ge 1$ be such that $-\lambda_n + q_c < -\delta$ for all $n \ge N_0 + 1$. Assume that 1) (A, B) is controllable and (A, C) is observable; 2) $C^\top \notin R\left((\mu I - A^\top)\big|_{\ker(B^\top)}\right)$ for all*

$\mu \in \{-\lambda_n + q_c : 1 \le n \le N_0\}$. Let $K \in \mathbb{R}^{1\times(N_0+n_x)}$ and $L \in \mathbb{R}^{(N_0+n_x)\times 2}$ be such that $\mathfrak{A} + \mathfrak{B}K$ and $\mathfrak{U} - L\mathfrak{C}$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a given $N \ge N_0 + 1$, assume that there exist $P \succ 0$, $\alpha > 1$, and $\beta, \gamma > 0$ such that

$$\Theta_1 = \begin{bmatrix} \Theta_{1,1} & P\mathcal{L} \\ \mathcal{L}^\top P & -\beta \end{bmatrix} \preceq 0, \quad \Theta_2 \le 0 \tag{11}$$

where $\Theta_{1,1} = \mathcal{F}^\top P + P\mathcal{F} + 2\delta P + \alpha\gamma\|\mathcal{R}_N a\|^2_{L^2} E_1^\top E_1 + \alpha\gamma\|\mathcal{R}_N b\|^2_{L^2} E_2^\top E_2$ and $\Theta_2 = 2\gamma\left\{-\left(1-\frac{1}{\alpha}\right)\lambda_{N+1} + q_c + \delta\right\} + \beta M_\phi$ with $M_\phi = \sum_{n\ge N+1} \frac{|\phi_n(0)|^2}{\lambda_n} < +\infty$. Then there exists a constant $M > 0$ such that for any initial conditions $x_0 \in \mathbb{R}^{n_x}$, $z_0 \in H^2(0,1)$, $\hat{x}(0) \in \mathbb{R}^{n_x}$, $\hat{z}_n(0) \in \mathbb{R}$ so that $c_{\theta_1} z_0(0) - s_{\theta_1} z_0'(0) = 0$ and $c_{\theta_2} z_0(1) + s_{\theta_2} z_0'(1) = Cx_0$, the trajectories of the closed-loop system composed of the plant (2) and the controller (10) satisfy

$$\|x(t)\|^2 + \|z(t,\cdot)\|^2_{H^1} + \|\hat{x}(t)\|^2 + \sum_{n=1}^{N} \hat{z}_n(t)^2 + u(t)^2$$
$$\le Me^{-2\delta t}\left(\|x_0\|^2 + \|z_0\|^2_{H^1} + \|\hat{x}(0)\|^2 + \sum_{n=1}^{N} \hat{z}_n(0)^2\right)$$

for all $t \ge 0$. Moreover, the constraints (11) are always feasible for N selected large enough.

Proof. For initial conditions as in the statement of the theorem, which ensures that $w_0 \in D(\mathcal{A}) \subset D(\mathcal{A}^{1/2})$, the application of [33, Thm. 6.3.1 and 6.3.3] to the closed-loop system formed by (5) and (10) shows the existence and uniqueness of classical solutions, i.e., $w \in \mathcal{C}^0([0,+\infty), L^2(0,1)) \cap \mathcal{C}^1((0,+\infty), L^2(0,1))$ with $w(t,\cdot) \in D(\mathcal{A})$ for all $t \ge 0$. See Appendix A for details. We write a model that captures the dynamics of the ODE and of the N_0 first modes of the PDE. Let the observation errors be defined by $e_x = x - \hat{x}$ and $e_n = z_n - \hat{z}_n$ for all $1 \le n \le N$. Defining the residue of measurement $\zeta = \sum_{n\ge N+1} w_n \phi_n(0)$, the error of estimation of the system outputs appearing in (10) can be written as $C\hat{x}(t) - y_i(t) = -Ce_x(t)$ and $\sum_{k=1}^{N} \hat{w}_k(t)\phi_k(0) - y(t) = -\sum_{k=1}^{N} e_k(t)\phi_k(0) - \zeta(t)$. For $N_0+1 \le n \le N$, we introduce the scaled quantities $\tilde{z}_n = \hat{z}_n/\lambda_n$ and $\tilde{e}_n = \sqrt{\lambda_n} e_n$. With $\hat{Z}^{N_0} = \begin{bmatrix}\hat{z}_1 \ \dots \ \hat{z}_{N_0} \ \hat{x}^\top\end{bmatrix}^\top$, $E^{N_0} = \begin{bmatrix} e_1 \ \dots \ e_{N_0} \ e_x^\top\end{bmatrix}^\top$, $\tilde{Z}^{N-N_0} = \begin{bmatrix}\tilde{z}_{N_0+1} \ \dots \ \tilde{z}_N\end{bmatrix}^\top$, and $\tilde{E}^{N-N_0} = \begin{bmatrix}\tilde{e}_{N_0+1} \ \dots \ \tilde{e}_N\end{bmatrix}^\top$, we infer that

$$u(t) = K\hat{Z}^{N_0}(t) \tag{12a}$$
$$\dot{\hat{Z}}^{N_0}(t) = (\mathfrak{A} + \mathfrak{B}K)\hat{Z}^{N_0}(t) + \mathfrak{D}E^{N_0}(t) + L_y\tilde{C}_1\tilde{E}^{N-N_0}(t) + L_y\zeta(t) \tag{12b}$$
$$\dot{E}^{N_0}(t) = (\mathfrak{U} - L\mathfrak{C})E^{N_0}(t) - L_y\tilde{C}_1\tilde{E}^{N-N_0}(t) - L_y\zeta(t) \tag{12c}$$
$$\dot{\tilde{Z}}^{N-N_0}(t) = A_1\tilde{Z}^{N-N_0}(t) + \tilde{\mathfrak{B}}_1 C_a\hat{Z}^{N_0}(t) + \tilde{\mathfrak{B}}_1 C_a E^{N_0}(t) \tag{12d}$$
$$\dot{\tilde{E}}^{N-N_0}(t) = A_1\tilde{E}^{N-N_0}(t) \tag{12e}$$

Therefore, with $X = \mathrm{col}\left(\hat{Z}^{N_0}, E^{N_0}, \tilde{Z}^{N-N_0}, \tilde{E}^{N-N_0}\right)$, we deduce from (12) that

$$\dot{X} = \mathcal{F}X + \mathcal{L}\zeta. \tag{13}$$

Finally, we observe from $y_i = Cx = C(\hat{x} + e_x) = C_a(\hat{Z}^{N_0} + E^{N_0})$ that $y_i = E_1 X$ and $\dot{y}_i = E_2 X$ where $\|E_1\|$ and $\|E_2\|$ are constants independent of N.

We can now proceed with the stability analysis. To do so, consider the Lyapunov functional candidate defined by $V(t) = X(t)^\top P X(t) + \gamma\sum_{n\ge N+1}\lambda_n w_n(t)^2$. The computation of the time derivative along (7) and (13) gives

$$\dot{V} = X^\top\{\mathcal{F}^\top P + P\mathcal{F}\}X + 2X^\top P\mathcal{L}\zeta + 2\gamma\sum_{n\ge N+1}\lambda_n((-\lambda_n + q_c)w_n + a_n y_i + b_n\dot{y}_i)w_n.$$

Invoking Young's inequality, using $y_i = E_1 X$ and $\dot{y}_i = E_2 X$, and noting that $\zeta^2 \le M_\phi \sum_{n\ge N+1}\lambda_n w_n^2$, we infer that $\dot{V} + 2\delta V \le \tilde{X}^\top\Theta_1\tilde{X} + \sum_{n\ge N+1}\lambda_n\Gamma_n w_n^2$ where $\tilde{X} = \mathrm{col}\,(X,\zeta)$ and $\Gamma_n = 2\gamma\left\{-\left(1-\frac{1}{\alpha}\right)\lambda_n + q_c + \delta\right\} + \beta M_\phi$. With $\alpha > 1$ we note that $\Gamma_n \le \Theta_2$ for all $n \ge N+1$. Hence (11) implies that $\dot{V} + 2\delta V \le 0$. The claimed stability estimate follows from the definition of V, the inequalities (1), the change of variable formula (4), and the definition of the control input (12a).

The feasibility of (11) for N selected large enough is obtained by applying the Lemma reported in Appendix to the matrix[1] $\mathcal{F} + \delta I$ to obtain $P \succ 0$ so that $\mathcal{F}^\top P + P\mathcal{F} + 2\delta P = -I$ with $\|P\| = O(1)$ as $N \to +\infty$ and by setting $\alpha > 1$, $\beta = \sqrt{N}$, and $\gamma = 1/N$. Indeed, this gives $\Theta_2 \to -\infty$ when $N \to +\infty$ while the Schur complement shows that $\Theta_1 \preceq 0$ for N selected large enough. □

4 ODE-PDE cascade with a long interconnection delay

4.1 Problem setting

We now focus on the control design problem of the ODE-PDE cascade with a long interconnection delay described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y_i(t) = Cx(t) \tag{14a}$$

[1] We use in particular the fact that $\|L_y\tilde{C}_1\| = O(1)$ and $\|\tilde{\mathfrak{B}}_1 C_a\| = O(1)$ as $N \to +\infty$

$$z_t(t,\xi) = (p(\xi)z_\xi(t,\xi))_\xi - r(\xi)z(t,\xi) \tag{14b}$$
$$c_{\theta_1} z(t,0) - s_{\theta_1} z_\xi(t,0) = 0 \tag{14c}$$
$$c_{\theta_2} z(t,1) + s_{\theta_2} z_\xi(t,1) = y_{i,h}(t) \triangleq y_i(t-h) \tag{14d}$$
$$y(t) = z(t,0) \tag{14e}$$
$$x(0) = x_0, \quad z(0,\xi) = z_0(\xi) \tag{14f}$$
$$y_i(\tau) = y_{i,0}(\tau),\ \tau \in [-h,0] \tag{14g}$$

for $t > 0$ and $\xi \in (0,1)$. Here $h > 0$ denotes a delay in the cascade interconnection. $y_{i,0} : [-h,0] \to \mathbb{R}$ stands for the initial condition of the cascade interconnection. All other quantities are defined in Subsection 3.1.

4.2 Spectral reduction

With the change of variable

$$w(t,\xi) = z(t,\xi) - \frac{\xi^2}{c_{\theta_2} + 2s_{\theta_2}} y_{i,h}(t), \tag{15}$$

we obtain from (14) the equivalent homogeneous representation :

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y_i(t) = Cx(t) \tag{16a}$$
$$w_t(t,\xi) = (p(\xi)w_\xi(t,\xi))_\xi - r(\xi)w(t,\xi) \tag{16b}$$
$$+ a(\xi)y_{i,h}(t) + b(\xi)\dot{y}_{i,h}(t) \tag{16c}$$
$$c_{\theta_1} w(t,0) - s_{\theta_1} w_\xi(t,0) = 0 \tag{16d}$$
$$c_{\theta_2} w(t,1) + s_{\theta_2} w_\xi(t,1) = 0 \tag{16e}$$
$$y(t) = w(t,0) \tag{16f}$$
$$x(0) = x_0, \quad w(0,\xi) = w_0(\xi) \tag{16g}$$

where $w_0(\xi) = z_0(\xi) - \frac{\xi^2}{c_{\theta_2}+2s_{\theta_2}} y_{i,0}(-h)$. Hence, we have

$$w_n(t) = z_n(t) + b_n y_{i,h}(t), \quad n \geq 1. \tag{17}$$

The projection of (14) gives

$$\dot{z}_n(t) = (-\lambda_n + q_c)z_n(t) + \beta_n y_{i,h}(t) \tag{18}$$

while the projection of (16) reads

$$\dot{w}_n(t) = (-\lambda_n + q_c)w_n(t) + a_n y_{i,h}(t) + b_n \dot{y}_{i,h}(t). \tag{19}$$

The Dirichlet measurement (16f) is expressed as (9).

4.3 Control strategy

Let $\delta > 0$ be fixed. Let $N_0 \geq 1$ be selected so that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. Let $N \geq N_0 + 1$ be arbitrary. Due to the interconnection delay $h > 0$, the idea is to introduce $\hat{x}(t)$ as the estimate of the state of the ODE at time $t - h$, namely $x(t-h)$. Such an approach follows classical procedures for the predictor-based stabilization of output-delayed finite-dimensional LTI systems [19, Chap. 3] and can be heuristically interpreted as the transfer of the delay from the output of the ODE to the input of the ODE. Regarding the PDE part, we introduce $\hat{z}_n(t)$, for $1 \leq n \leq N$, as the estimate of the n-th mode of the PDE at current time, namely $z_n(t)$. Hence the observer dynamics is described by:

$$\hat{w}_n(t) = \hat{z}_n(t) + b_n y_{i,h}(t) \tag{20a}$$
$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t-h) - L_{xy_i}\{C\hat{x}(t) - y_{i,h}(t)\} \tag{20b}$$
$$- L_{xy}\left\{\sum_{k=1}^{N} \hat{w}_k(t)\phi_k(0) - y(t)\right\}$$
$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n y_{i,h}(t) \tag{20c}$$
$$- l_{zy_i,n}\{C\hat{x}(t) - y_{i,h}(t)\}$$
$$- l_{zy,n}\left\{\sum_{k=1}^{N} \hat{w}_k(t)\phi_k(0) - y(t)\right\},\ 1 \leq n \leq N_0$$
$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n y_{i,h}(t),\ N_0 + 1 \leq n \leq N \tag{20d}$$

where $l_{zy_i,n}, l_{zy,n} \in \mathbb{R}$ and $L_{xy_i}, L_{xy} \in \mathbb{R}^{n_x}$ are the observer gains. We set $u(\tau) = u_0$ for all $\tau \leq 0$ for some $u_0 \in \mathbb{R}$. Note that no control is applied to the system (14) in negative time; this definition is introduced in order to make the dynamics (20) well-defined for all $t \geq 0$.

In order to propose a control strategy that compensates the interconnection delay, which appears as an input delay of the observer dynamics (20), we introduce for $t \geq 0$ the following Artstein transformation [2]:

$$\hat{Z}_A^{N_0}(t) = e^{\mathfrak{A}h}\hat{Z}^{N_0}(t) + \int_{t-h}^{t} e^{\mathfrak{A}(t-s)}\mathfrak{B}u(s)\,\mathrm{d}s \tag{21}$$

where $\hat{Z}^{N_0} = \begin{bmatrix}\hat{z}_1 \ \dots \ \hat{z}_{N_0} \ \hat{x}^\top\end{bmatrix}^\top$ while $\mathfrak{A}, \mathfrak{B}$ are defined in Subsection 3.3. We define the control input as

$$u(t) = K\hat{Z}_A^{N_0}(t) \tag{22}$$

for all $t \geq 0$. Here $K \in \mathbb{R}^{1\times(N_0+n_x)}$ is the feedback gain. The resulting closed-loop system is depicted in Fig 1.

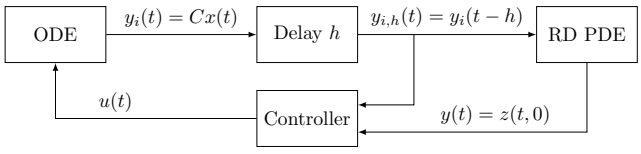


Fig. 1. Diagram of the closed-loop system with ODE = (14a), RD PDE = (14b-14d), Controller = (20-22)

In preparation for stability analysis, we consider the ma-

trices defined in Subsection 3.3 except that:

$$\mathcal{F} = \begin{bmatrix} \mathfrak{A}+\mathfrak{B}K & e^{\mathfrak{A}h}\mathfrak{D} & e^{\mathfrak{A}h}L_y\tilde{C}_1 \\ 0 & \mathfrak{U}-L\mathfrak{C} & -L_y\tilde{C}_1 \\ 0 & 0 & A_1 \end{bmatrix}, \ \mathcal{L} = \begin{bmatrix} e^{\mathfrak{A}h}L_y \\ -L_y \\ 0 \end{bmatrix},$$

$E_1 = \begin{bmatrix} I & 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & I & 0 \end{bmatrix}$, $\tilde{K} = \begin{bmatrix} K & 0 & 0 \end{bmatrix}$, and $E_3 = \begin{bmatrix} e^{-\mathfrak{A}h}(\mathfrak{A}+\mathfrak{B}K) & \mathfrak{U}-L\mathfrak{C}+\mathfrak{D} & 0 \end{bmatrix}$.

4.4 Main stability result

Theorem 6 *Let $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x}$, and $C \in \mathbb{R}^{1\times n_x}$, $\theta_1 \in (0,\pi/2]$, $\theta_2 \in [0,\pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $r \in \mathcal{C}^0([0,1])$. Let $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ be such that (3) holds. Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta$ for all $n \geq N_0+1$. Assume that 1) (A,B) is controllable and (A,C) is observable; 2) $C^\top \notin R\Big((\mu I - A^\top)\big|_{\ker(B^\top)}\Big)$ for all $\mu \in \{-\lambda_n + q_c : 1 \leq n \leq N_0\}$. Let $K \in \mathbb{R}^{1\times(N_0+n_x)}$ and $L \in \mathbb{R}^{(N_0+n_x)\times 2}$ be such that $\mathfrak{A}+\mathfrak{B}K$ and $\mathfrak{U}-L\mathfrak{C}$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. Let $h > 0$ be given. For a given $N \geq N_0+1$, assume that there exist $P \succ 0$, $Q \succeq 0$, $\alpha > 1$, $\beta,\gamma > 0$, and $\rho \geq 0$ such that*

$$\Theta_1 = \begin{bmatrix} \Theta_{1,1} & P\mathcal{L} \\ \mathcal{L}^\top P & -\beta \end{bmatrix} \preceq 0, \quad \Theta_2 \leq 0, \quad R_1 \leq 0, \quad R_2 \preceq 0 \tag{23}$$

where

$$\begin{aligned}
\Theta_{1,1} &= \mathcal{F}^\top P + P\mathcal{F} + 2\delta P + \rho h \tilde{K}^\top \tilde{K} + E_1^\top Q E_1 \\
&\quad + 3\alpha\gamma \|\mathcal{R}_N a\|_{L^2}^2 E_2^\top C_a^\top C_a E_2 \\
&\quad + 3\alpha\gamma \|\mathcal{R}_N a\|_{L^2}^2 \|e^{-\mathfrak{A}h}\|^2 \|C\|^2 E_1^\top E_1 \\
&\quad + 4\alpha\gamma \|\mathcal{R}_N b\|_{L^2}^2 E_3^\top C_a^\top C_a E_3 \\
&\quad + 4\alpha\gamma \|\mathcal{R}_N b\|_{L^2}^2 (C_a e^{-\mathfrak{A}h}\mathfrak{B})^2 \tilde{K}^\top \tilde{K} \\
\Theta_2 &= 2\gamma\left\{-\left(1-\frac{1}{\alpha}\right)\lambda_{N+1} + q_c + \delta\right\} + \beta M_\phi \\
R_1 &= -\rho e^{-2\delta h} + 3\alpha\gamma h \|\mathcal{R}_N a\|_{L^2}^2 m_{\mathfrak{A},h}^2 \|B\|^2 \|C\|^2 \\
&\quad + 4\alpha\gamma h \|\mathcal{R}_N b\|_{L^2}^2 m_{\mathfrak{A},h}^2 \|\mathfrak{A}\|^2 \|B\|^2 \|C\|^2 \\
R_2 &= -e^{-2\delta h} Q + 4\alpha\gamma \|\mathcal{R}_N b\|_{L^2}^2 (CB)^2 K^\top K
\end{aligned}$$

with $M_\phi = \sum_{n\geq N+1} \frac{|\phi_n(0)|^2}{\lambda_n} < +\infty$, and $m_{\mathfrak{A},h} = \max_{\tau\in[0,h]} \|e^{-\mathfrak{A}\tau}\|$. Then there exists a constant $M > 0$ such that for any initial conditions $x_0 \in \mathbb{R}^{n_x}$, $z_0 \in H^2(0,1)$, $\hat{x}(0) \in \mathbb{R}^{n_x}$, $\hat{z}_n(0) \in \mathbb{R}$, $u_0 \in \mathbb{R}$, and $y_{i,0} \in \mathcal{C}^2([-h,0])$ so that $c_{\theta_1} z_0(0) - s_{\theta_1} z_0'(0) = 0$, $c_{\theta_2} z_0(1) + s_{\theta_2} z_0'(1) = y_{i,0}(-h)$, and $y_{i,0}(0) = Cx_0$, the trajectories of the closed-loop system composed of the plant (14) and the controller (20-22) satisfy

$$\begin{aligned}
&\|x(t)\|^2 + \|z(t,\cdot)\|_{H^1}^2 + \|\hat{x}(t)\|^2 + \sum_{n=1}^N \hat{z}_n(t)^2 + u(t)^2 \\
&\leq M e^{-2\delta t}\Big(\|x_0\|^2 + \|z_0\|_{H^1}^2 + \|\hat{x}(0)\|^2 \\
&\qquad + \sum_{n=1}^N \hat{z}_n(0)^2 + u_0^2 + \|y_{i,0}\|_\infty^2 + \|\dot{y}_{i,0}\|_\infty^2\Big)
\end{aligned} \tag{24}$$

for all $t \geq 0$. Moreover, the constraints (23) are always feasible for N selected large enough.

Proof. For initial conditions as in the statement of the theorem, the well-posedness in terms of classical solutions follows as in [28, Rem. 3.3] from [33, Thm. 6.3.1 and 6.3.3] and an induction argument for handling the interconnection delay. We reuse the notations of the proof of Theorem 6 except that the error of observation of the ODE is now defined for $t \geq h$ by $e_x(t) = x(t-h) - \hat{x}(t)$. Then the $\hat{Z}^{N_0}$ dynamics reads for $t \geq h$

$$\begin{aligned}
\dot{\hat{Z}}^{N_0}(t) &= \mathfrak{A}\hat{Z}^{N_0}(t) + \mathfrak{B}u(t-h) + \mathfrak{D}E^{N_0}(t) \\
&\quad + L_y\tilde{C}_1\tilde{E}^{N-N_0}(t) + L_y\zeta(t).
\end{aligned}$$

We have used that $y_{i,h}(t) = C(\hat{x}(t) + e_x(t))$ for all $t \geq h$. In view of the Artstein transformation (21) and the feedback (22), we infer that

$$\begin{aligned}
\dot{\hat{Z}}_A^{N_0}(t) &= (\mathfrak{A}+\mathfrak{B}K)\hat{Z}_A^{N_0}(t) + e^{\mathfrak{A}h}\mathfrak{D}E^{N_0}(t) \\
&\quad + e^{\mathfrak{A}h}L_y\tilde{C}_1\tilde{E}^{N-N_0}(t) + e^{\mathfrak{A}h}L_y\zeta(t)
\end{aligned} \tag{25}$$

for $t \geq h$. The dynamics of $\tilde{Z}^{N-N_0}$ reads

$$\begin{aligned}
\dot{\tilde{Z}}^{N-N_0}(t) &= A_1\tilde{Z}^{N-N_0}(t) + \tilde{\mathfrak{B}}_1 y_{i,h}(t), \ \forall t \geq 0 \\
&= A_1\tilde{Z}^{N-N_0}(t) + \tilde{\mathfrak{B}}_1 C_a\hat{Z}^{N_0}(t) + \tilde{\mathfrak{B}}_1 C_a E^{N_0}(t), \ \forall t \geq h
\end{aligned} \tag{26}$$

Finally, the dynamics of the estimation errors E^{N_0} and $\tilde{E}^{N-N_0}$ are given by (12c) for $t \geq h$ and (12e) for $t \geq 0$, respectively. Hence, introducing the vector defined by

$$X = \mathrm{col}\left(\hat{Z}_A^{N_0}, E^{N_0}, \tilde{E}^{N-N_0}\right), \tag{27}$$

we obtain from (12c), (12e), and (25) that

$$\dot{X}(t) = \mathcal{F}X(t) + \mathcal{L}\zeta(t) \tag{28}$$

for all $t \geq h$. We also have

$$\hat{Z}_A^{N_0} = E_1 X, \quad E^{N_0} = E_2 X, \quad u = \tilde{K}X. \tag{29}$$

We can now proceed with the stability analysis. Let $V(t) = V_0(t) + V_1(t) + V_2(t)$ be defined for $t \geq h$ by

$$V_0(t) = X(t)^\top P X(t) + \gamma \sum_{n\geq N+1} \lambda_n w_n(t)^2, \tag{30a}$$

$$V_1(t) = \rho \int_{-h}^{0} \int_{t+\theta}^{t} e^{-2\delta(t-s)} |u(s)|^2 \, \mathrm{d}s \, \mathrm{d}\theta, \tag{30b}$$

$$V_2(t) = \int_{t-h}^{t} e^{-2\delta(t-s)} \hat{Z}_A^{N_0}(s)^\top Q \hat{Z}_A^{N_0}(s) \, \mathrm{d}s. \tag{30c}$$

Introducing $\tilde{X} = \mathrm{col}\,(X, \zeta)$, the computation of the time derivative of V along the system trajectories (19) and (28), and the use of Young's inequality, give

$$\begin{aligned}
\dot{V}_0 &= X^\top \{\mathcal{F}^\top P + P\mathcal{F}\} X + 2X^\top P \mathcal{L} \zeta \\
&\quad + 2\gamma \sum_{n \geq N+1} \lambda_n ((-\lambda_n + q_c) w_n + a_n y_{i,h} + b_n \dot{y}_{i,h}) w_n \\
&\leq \tilde{X}^\top \begin{bmatrix} \mathcal{F}^\top P + P\mathcal{F} & P\mathcal{L} \\ \mathcal{L}^\top P & 0 \end{bmatrix} \tilde{X} \qquad (31) \\
&\quad + 2\gamma \sum_{n \geq N+1} \lambda_n \left\{ -\left(1 - \frac{1}{\alpha}\right) \lambda_n + q_c \right\} w_n^2 \\
&\quad + \alpha\gamma \|\mathcal{R}_N a\|_{L^2}^2 |y_{i,h}|^2 + \alpha\gamma \|\mathcal{R}_N b\|_{L^2}^2 |\dot{y}_{i,h}|^2, \\
\dot{V}_1(t) &= -2\delta V_1(t) + \rho h |u(t)|^2 - \rho \int_{-h}^{0} e^{2\delta\theta} |u(t+\theta)|^2 \, \mathrm{d}\theta \\
&\leq -2\delta V_1(t) + \rho h X(t)^\top \tilde{K}^\top \tilde{K} X(t) \qquad (32) \\
&\quad - \rho e^{-2\delta h} \int_{t-h}^{t} |u(s)|^2 \, \mathrm{d}s, \\
\dot{V}_2(t) &= -2\delta V_2(t) + \hat{Z}_A^{N_0}(t)^\top Q \hat{Z}_A^{N_0}(t) \qquad (33) \\
&\quad - e^{-2\delta h} \hat{Z}_A^{N_0}(t-h)^\top Q \hat{Z}_A^{N_0}(t-h).
\end{aligned}$$

We now need to evaluate $|y_{i,h}(t)|^2$ and $|\dot{y}_{i,h}(t)|^2$ appearing in (31). We have $y_{i,h}(t) = Cx(t-h) = C(\hat{x}(t) + e_x(t)) = C_a(\hat{Z}^{N_0}(t) + E^{N_0}(t)) = C_a \hat{Z}^{N_0}(t) + C_a E_2 X(t)$ for all $t \geq h$, where we have used (29). Recalling that $m_{\mathfrak{A},h} = \max_{\tau \in [0,h]} \|e^{-\mathfrak{A}\tau}\|$, we infer from (21) and using Cauchy-Schwarz inequality that $\|\hat{Z}^{N_0}(t)\| \leq \|e^{-\mathfrak{A}h}\| \|\hat{Z}_A^{N_0}(t)\| + m_{\mathfrak{A},h} \|B\| \sqrt{h} \left\{ \int_{t-h}^{t} |u(s)|^2 \, \mathrm{d}s \right\}^{1/2}$. Using again (29) we infer that

$$\begin{aligned}
|y_{i,h}(t)|^2 &\leq 3 \|e^{-\mathfrak{A}h}\|^2 \|C\|^2 X(t)^\top E_1^\top E_1 X(t) \qquad (34) \\
&\quad + 3h m_{\mathfrak{A},h}^2 \|B\|^2 \|C\|^2 \int_{t-h}^{t} |u(s)|^2 \, \mathrm{d}s \\
&\quad + 3X(t)^\top E_2^\top C_a^\top C_a E_2 X(t)
\end{aligned}$$

for all $t \geq h$. We also have $\dot{y}_{i,h}(t) = C_a(\dot{\hat{Z}}^{N_0}(t) + \dot{E}^{N_0}(t))$ for $t > h$. In view of the Artstein transformation (21), we obtain that $\dot{\hat{Z}}^{N_0}(t) = e^{-\mathfrak{A}h} \dot{\hat{Z}}_A^{N_0}(t) - \mathfrak{A} \int_{t-h}^{t} e^{\mathfrak{A}(t-s-h)} \mathfrak{B} u(s) \, \mathrm{d}s - e^{-\mathfrak{A}h} \mathfrak{B} u(t) + \mathfrak{B} u(t-h)$. Based on (22) and (29), and using $e^{-\mathfrak{A}h} \dot{\hat{Z}}_A^{N_0}(t) + \dot{E}^{N_0}(t) = E_3 X(t)$ for all $t > h$, we deduce that

$$\begin{aligned}
|\dot{y}_{i,h}(t)|^2 &\leq 4X(t)^\top E_3^\top C_a^\top C_a E_3 X(t) \qquad (35) \\
&\quad + 4h m_{\mathfrak{A},h}^2 \|\mathfrak{A}\|^2 \|B\|^2 \|C\|^2 \int_{t-h}^{t} |u(s)|^2 \, \mathrm{d}s \\
&\quad + 4(C_a e^{-\mathfrak{A}h} \mathfrak{B})^2 X(t)^\top \tilde{K}^\top \tilde{K} X(t) \\
&\quad + 4(CB)^2 \hat{Z}_A^{N_0}(t-h)^\top K^\top K \hat{Z}_A^{N_0}(t-h)
\end{aligned}$$

for all $t > h$. Combining now (31-35) and using the estimate $\zeta^2 \leq M_\phi \sum_{n \geq N+1} \lambda_n w_n^2$, we infer that

$$\begin{aligned}
\dot{V}(t) + 2\delta V(t) &\leq \tilde{X}(t)^\top \Theta_1 \tilde{X}(t) + \sum_{n \geq N+1} \lambda_n \Gamma_n w_n(t)^2 \\
&\quad + R_1 \int_{t-h}^{t} |u(s)|^2 \, \mathrm{d}s + \hat{Z}_A^{N_0}(t-h)^\top R_2 \hat{Z}_A^{N_0}(t-h)
\end{aligned}$$

where $\Gamma_n = 2\gamma \left\{ -\left(1 - \frac{1}{\alpha}\right) \lambda_n + q_c + \delta \right\} + \beta M_\phi$ for $t > h$. With $\alpha > 1$ we note that $\Gamma_n \leq \Theta_2$ for all $n \geq N+1$. Hence (23) implies that $\dot{V}(t) + 2\delta V(t) \leq 0$ for all $t > h$. This shows that $V(t) \leq e^{-2\delta(t-h)} V(h)$ for all $t \geq h$. Denoting by $\mathrm{CI} = \|x_0\|^2 + \sum_{n \geq 1} \lambda_n w_n(0)^2 + \|\hat{x}(0)\|^2 + \sum_{n=1}^{N} \hat{z}_n(0)^2 + u_0^2 + \|y_{i,0}\|_\infty^2 + \|\dot{y}_{i,0}\|_\infty^2$, standard results [33, Thm. 6.3.3] show the existence of a constant $c_1 > 0$, independent of the initial conditions, so that $\|x(t)\|^2 + \sum_{n \geq 1} \lambda_n w_n(t)^2 + \|\hat{x}(t)\|^2 + \sum_{n=1}^{N} \hat{z}_n(t)^2 \leq c_1 \mathrm{CI}$ for all $t \in [0, h]$. Based on the definition of V given by (30) and (21-22), we obtain the existence of a constant $c_2 > 0$, independent of the initial conditions, so that $V(h) \leq c_2 \mathrm{CI}$ hence $V(t) \leq c_2 e^{-2\delta(t-h)} \mathrm{CI}$ for all $t \geq h$. From (27) and using the definition of the Artstein transformation (21) along with the input (22), we obtain the existence of a constant $c_3 > 0$, independent of the initial conditions, so that $\|\hat{Z}^{N_0}(t)\|^2 + \|E^{N_0}(t)\|^2 \leq c_3 e^{-2\delta(t-h)} \mathrm{CI}$ for all $t \geq h$. In view of (26) and recalling that the eigenvalues of A_1 are given by $-\lambda_n + q_c < -\delta < 0$ for $N_0 + 1 \leq n \leq N$, we infer the existence of a constant $c_4 > 0$, independent of the initial conditions, so that $\|\tilde{Z}^{N-N_0}(t)\|^2 \leq c_4 e^{-2\delta t} \mathrm{CI}$ for all $t \geq h$. Combining together all the above estimates, the stability estimate (24) is now a direct consequence of the inequalities (1), the change of variable (15), and the control input (22).

It remains to show that (23) is feasible provided the order N of the observer is selected large enough. Let $\alpha > 1$ be arbitrarily. For any given $N \geq N_0 + 1$ we fix $\gamma = 1/N$ and $\beta = \sqrt{N}$. Due to the upper triangular nature of $\mathcal{F}$ and since $\|e^{\mathfrak{A}h} L_y \tilde{C}_1\| = O(1)$ and $\|L_y \tilde{C}_1\| = O(1)$ as $N \to +\infty$, an argument similar to the Lemma reported in the Appendix applied to the matrix $\mathcal{F} + \delta I$ shows the existence of $P \succ 0$ such that $\mathcal{F}^\top P + P\mathcal{F} + 2\delta P = -I$ with $\|P\| = O(1)$ as $N \to +\infty$. We finally define $Q = 4e^{2\delta h} \alpha\gamma \|\mathcal{R}_N b\|_{L^2}^2 (CB)^2 K^\top K \succeq 0$ and $\rho = e^{2\delta h} \alpha\gamma h \left(3\|\mathcal{R}_N a\|_{L^2}^2 + 4\|\mathcal{R}_N b\|_{L^2}^2 \|\mathfrak{A}\|^2\right) m_{\mathfrak{A},h}^2 \|B\|^2 \|C\|^2 \geq 0$. Hence, we have $R_1 = 0$ and $R_2 = 0$ for all $N \geq N_0 + 1$. Moreover we have $\Theta_2 \to -\infty$ as $N \to +\infty$. Finally, noting that all the matrices involved in the definition of Θ_{11} are bounded in norm as $N \to +\infty$, we observe that $\Theta_{11} \preceq -\frac{1}{2} I$ for N selected large enough. Since P and $\mathcal{L}$

are bounded in norm as $N \to +\infty$, the Schur complement gives $\Theta_1 \preceq 0$ for N selected large enough. $\square$

Remark 7 *The $\tilde{Z}^{N-N_0}$ dynamics (26) is not included in reduced model (28). The reason is that this dynamics involves $\hat{Z}^{N_0}(t)$ which cannot be expressed based on the sole vector $\hat{Z}_A^{N_0}(t)$ at time t. Hence (26) is considered separately in the stability analysis. Note that a similar procedure could have been applied in the proof of Theorem 5. However, the approach developed in the proof of Theorem 5 is more general as it provides a Lyapunov functional for the full system dynamics (it can be used, e.g., to study robustness properties).*

5 Numerical example

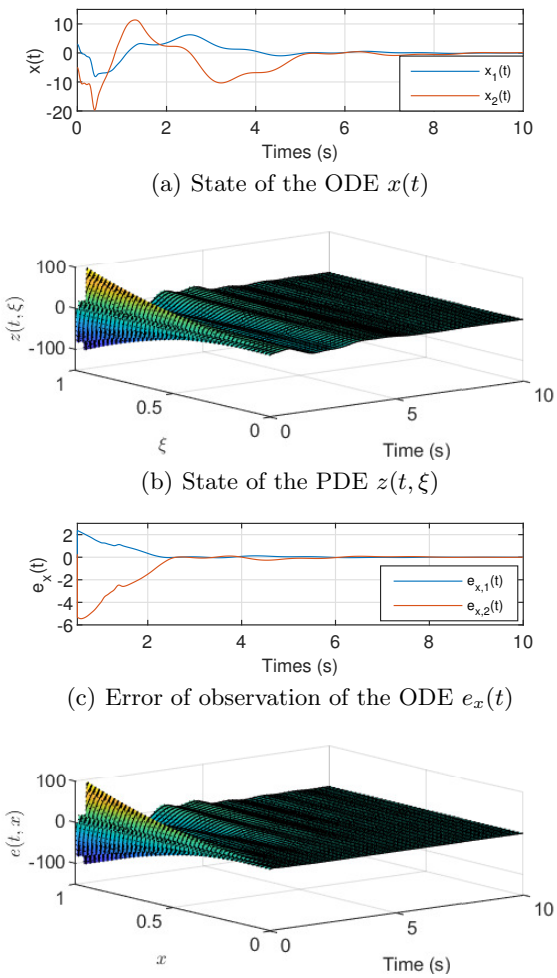
We consider the delayed ODE-reaction-diffusion PDE cascade described by either (2) or (14) with $A = \begin{bmatrix} 3/2 & 1 \\ 1/2 & 2/3 \end{bmatrix}$, $B = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 3/4 & 1/3 \end{bmatrix}$, $p = 1$, $r = -5$, $\theta_1 = \pi/3$, and $\theta_2 = 0$. With these parameters, both open-loop ODE and PDE are open-loop unstable. We set the feedback gains $K = \begin{bmatrix} -21.2455 & -5.9072 & -8.2153 \end{bmatrix}^\top$ and the observer gain $L = \begin{bmatrix} 3.1778 & -0.0946 & 0.4047 \\ 0.0239 & 0.6971 & 19.9290 \end{bmatrix}$ in order to guarantee the exponential decay rate $\delta = 0.5$.

For the ODE-PDE cascade (2), the constraints (11) of Theorem 5 are found feasible for $N = 3$ using MATLAB LMI solvers. This ensures the exponential stability of the closed-loop system composed of the plant (2) and the controller given by (10). Considering now the ODE-PDE cascade with interconnection delay $h = 0.5\,\mathrm{s}$ described by (14), the constraints (23) are found feasible for $N = 7$. Hence Theorem 6 ensures the exponential stability of the closed-loop system composed of the plant (14) and the controller described by (20-22) in the sense of (24).

We illustrate with numerical simulations the case of the delayed interconnection (14). The adopted numerical scheme consists of the modal approximation of the PDE using its first 20 dominant modes while the predictor feedback (21) is implemented using a Riemann sum approximation of the integral with a time-step of 1ms. We set $x_0 = \begin{bmatrix} 3 & -5 \end{bmatrix}^\top$, $z_0(x) = 50x^2(x-1)^2 + c_0 x^2$ with $c_0 = Cx_0 = 7/12$, $y_{i,0}(\tau) = 3\cos(10\pi(h+\tau))\sin(3\pi\tau) + c_0$ for $\tau \leq 0$, and zero initial condition for the observer. The evolution of the closed-loop system is depicted in Fig. 2.

6 Conclusion

This paper has addressed the output feedback stabilization of an ODE-Reaction-Diffusion PDE cascade. Using the output of the both ODE and PDE parts, we designed a finite-dimensional control strategy achieving the exponential stabilization of the plant. This control strategy was augmented with a predictor component to embrace the case of an arbitrarily long interconnection delay between the ODE and the PDE. While we considered the case of a Dirichlet boundary measurement, our approach extends to the case of a Neumann boundary measurement by adapting the arguments from [25]. Future works may deal with the extension of the approach to the case where only the output of the PDE is available for feedback control or to the case of time-varying delays.



(a) State of the ODE $x(t)$

(b) State of the PDE $z(t, \xi)$

(c) Error of observation of the ODE $e_x(t)$

(d) Error of observation of the PDE $e(t, \xi)$

Fig. 2. Closed-loop system composed of the plant (14) and the controller (20-22) with interconnection delay $h = 0.5\,\mathrm{s}$

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A Well-posedness assessment

With $X_c = \begin{bmatrix}\hat{x}^\top & \hat{z}_1 & \hat{z}_2 & \dots & \hat{z}_N\end{bmatrix}^\top \in \mathbb{R}^{n_x+N}$, the controller dynamics (10) takes the form: $\dot{X}_c = A_c X_c + B_{c,1}u + B_{c,2}y_i + B_{c,3}y$ and $u = C_c X_c$ for suitable real matrices $A_c, B_{c,1}, B_{c,2}, B_{c,3}, C_c$. Using (3), the plant (5) can be written as $\dot{x} = Ax + Bu$ and $w_t = -\mathcal{A}w + q_c w + ay_i + b\dot{y}_i$ where $y_i = Cx$ and $\dot{y}_i = C\dot{x} = CAx + CBu$. Let the Hilbert space $\mathcal{H} = L^2(0,1) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x+N}$ and the state vector $X = (w, x, X_c)$. Then, the closed-loop system formed by (5) and (10) reads $\frac{\mathrm{d}X}{\mathrm{d}t} = -\mathcal{A}_e X + L_e X$ where $\mathcal{A}_e = \mathrm{diag}(\mathcal{A}, I, I)$ with $D(\mathcal{A}_e) = D(\mathcal{A}) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x+N}$ while $L_e = L_{e,1} + L_{e,2}$ is a linear operator with $L_{e,1} : \mathcal{H} \to \mathcal{H}$ a bounded linear operator defined by

$$L_{e,1}(w, x, X_c) = \begin{pmatrix} q_c w + aCx + bCAx + bCBC_c X_c \\ (I + A)x + BC_c X_c \\ (I + A_c)X_c + B_{c,1}C_c X_c + B_{c,2}Cx \end{pmatrix}$$

and $L_{e,2} : \left(D(\mathcal{A}^{1/2}) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x+N}\right) \to \mathcal{H}$ defined by $L_{e,2}(w, x, X_c) = (0, 0, B_{c,3}w(0))$. From Subsection 2.2, it is seen that $-\mathcal{A}_e$ is the generator a diagonal semigroup $T_e(t)$ which is analytic and satisfies $\|T_e(t)\| \le 1$ for all $t \ge 0$ while $0 \in \rho(-\mathcal{A}_e)$; see, e.g., [12, Sec. 2.3]. We infer from $w(0) = \sum_{n\ge1} \langle w, \phi_n \rangle \phi_n(0)$ and Cauchy-Schwarz inequality the existence of a constant $c > 0$ so that $\|L_{e,2}(w, x, X_c)\| \le c\|\mathcal{A}^{1/2}w\|$, hence $\|L_e(w, x, X_c)\| \le \left(\|L_{e,1}\|\|\mathcal{A}_e^{-1/2}\| + c\right) \|\mathcal{A}_e^{1/2}(w, x, X_c)\|$. This allows the application of [33, Thm. 6.3.1 and 6.3.3] for initial conditions $w_0 \in D(\mathcal{A}^{1/2})$, $x_0 \in \mathbb{R}^{n_x}$, and $X_c(0) \in \mathbb{R}^{n_x+N}$.

B Technical lemma

The following Lemma is a direct extension of [25, Annex A] to the case $M_{32}^N \neq 0$.

Lemma 8 *Let $n, m, N \ge 1$, $M_{11} \in \mathbb{R}^{n\times n}$ and $M_{22} \in \mathbb{R}^{m\times m}$ Hurwitz, $M_{12} \in \mathbb{R}^{n\times m}$, $M_{14}^N \in \mathbb{R}^{n\times N}$, $M_{24}^N \in \mathbb{R}^{m\times N}$, $M_{31}^N \in \mathbb{R}^{N\times n}$, $M_{32}^N \in \mathbb{R}^{N\times m}$, $M_{33}^N, M_{44}^N \in \mathbb{R}^{N\times N}$, and*

$$F^N = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{14}^N \\ 0 & M_{22} & 0 & M_{24}^N \\ M_{31}^N & M_{32}^N & M_{33}^N & 0 \\ 0 & 0 & 0 & M_{44}^N \end{bmatrix}.$$

We assume that there exist constants $C_0, \kappa_0 > 0$ such that $\|e^{M_{33}^N t}\| \le C_0 e^{-\kappa_0 t}$ and $\|e^{M_{44}^N t}\| \le C_0 e^{-\kappa_0 t}$ for all $t \ge 0$ and all $N \ge 1$. Moreover, we assume that there exists a constant $C_1 > 0$ such that $\|M_{14}^N\| \le C_1$, $\|M_{24}^N\| \le C_1$, $\|M_{31}^N\| \le C_1$, and $\|M_{32}^N\| \le C_1$ for all $N \ge 1$. Then there exists a constant $C_2 > 0$ such that, for any $N \ge 1$, there exists a symmetric matrix $P^N \in \mathbb{R}^{n+m+2N}$ with $P^N \succ 0$ such that $(F^N)^\top P^N + P^N F^N = -I$ and $\|P^N\| \le C_2$.