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## To cite this version:

Xuyang Chang, Matthieu Vitse, Stéphane Roux. Buckling of a double sector thin elastic plate: Analytical solution. 2022. hal-03862268

HAL Id: hal-03862268

## https://hal.science/hal-03862268

Preprint submitted on 21 Nov 2022

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## Highlights

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- A novel geometry is proposed, which can buckle from an initial planar configuration into a complex 3D shape under tensile load. More importantly, the parametrized closed-form solution of the deformed geometry is obtained analytically.
- The mechanical analysis of the buckling and post-buckling behavior is also provided.
- This closed-form solution can be used to benchmark either a numerical simulation of buckling or a Stereo Digital Image Correlation (Stereo-DIC) code.


# Buckling of a double sector thin elastic plate: Analytical solution 

Xuyang Chang ${ }^{a, *}$, Matthieu Vitse ${ }^{a, b}$ and Stéphane Roux ${ }^{a}$<br>${ }^{a}$ Université Paris-Saclay, CentraleSupélec, ENS Paris-Saclay, CNRS, LMPS - Laboratoire de Mécanique Paris-Saclay, Avenue de Science, Gif-sur-Yvette, 91190, , France<br>${ }^{b}$ SciViz, Paris, France

## ARTICLE INFO

## Keywords:

Elastic Buckling
Analytical solution
Geometrical instability


#### Abstract

Analytical solutions for elastic buckling and post-buckling geometries are scarce. The present work presents such a solution for a sample geometry, initially planar, which buckles into a complex 3D shape under tensile load. The assumptions are that the sample thickness is small, so that flexural stiffness is low compared to in-plane strain stiffness, and hence the sample can be considered inextensible. The initial geometry is composed of two disk angular sectors sharing a common edge. Under tensile loading, the two radial edges of each sector tend to align so that each half turns into two regular cone sectors. As the tensile load increases, the cone angle progressively decreases, but the same generic form holds the sample extension all along the tensile direction. Such a solution may be useful for validating numerical simulation tools, or stereo-vision shape measurement procedures.


## 1. Introduction

Buckling of elastic solids is a fundamental problem of high practical interest as it may limit the load-bearing capacity of engineering structures Godoy (1999), and also because it may trigger the emergence of new shapes and patterns in biology (from plant leaves to brain circumvolutions) Thompson (1942).

A considerable amount of work has been devoted to initially simple geometries such as plates or shells (cylinder, sphere) where its closed-form solutions could be obtained Yamaki (1984). Buckling has also become a paradigm for problems of non-linear complex systems. Notably, the formation of highly generic patterns such as wrinkles, ridges, folds, or singular developable cones has been identified as a ubiquitous phenomenon with which post-buckled thin plate geometries could be understood Ben Amar and Pomeau (1997); Chaïeb, Melo and Géminard (1998); Cerda and Mahadevan (2003); Hamm, Roman and Melo (2004); Ouchi, Yang, Suo and Hayward (2018). Until now, many applications of these instabilities involve surfaces that are patterned into regions with different material properties Genzer and Groenewold (2006); Xu, Chen and Hayward (2014). Nevertheless, the geometrical nature of a buckling instability also gives a wide range of relevance, as this instability may occur at a stage where materials are still in their linear elastic domain, and hence, the buckled geometry is weakly dependent on the material, and the critical load (the onset of buckling) varies in proportion to the elastic modulus.

However, despite the fact that the buckling behavior of elastic solids under load can often be anticipated, analytical solutions are rather scarce. A notable exception is the "tension-field theory" which describes very thin elastic films, which would buckle in compression when subject to a vanishing small compressive stress Reissner (1938); Stein and Hedgepeth (1961); Mansfield (1969). This limit can be incorporated into an effective constitutive law that accounts for this unilaterally Wang and Pipkin (1986); Steigmann (1990); Mahmood, Audoly and Roux (2018). Such a formulation makes the problem of buckling amenable to new analytic or numerical tools. Yet, closed-form solutions are still mostly restricted to elementary geometries.

In the present study, a particular geometry is introduced that leads to an analytic solution for buckling, and postbuckling. It concerns thin elastic plates which exhibit a very low (vanishing) bending stiffness, so that in-plane strain can be neglected. A notable property of the proposed geometry is that the initial state is planar, but the buckled and post-buckled configurations are not. The full evolution of the geometry under displacement load (the onset of buckling, and the post-buckling behavior) as well as the force-displacement relation are described analytically.

[^0]

Figure 1: Geometry of the sample

The paper is organized as follows: in Section 2, the design of the initial geometry is provided along with its progressive evolution as a function of the overall extension of the sample. In Section 3, mechanical analysis of the buckling phenomenon is carried out over the proposed geometry to compute the tensile force as a function of the sample extension. In Section 4, conclusions and perspectives are drawn.

## 2. Model

Let us consider a planar sample shown in Fig. 1 that consists of two disk sector ring sharing a common edge along BC. The two sectors BCEA and CBFD are symmetric with respect to the midpoint between $B$ and $C$. The disk radius is denoted $r_{1}$. The midpoint along the outer circle is denoted E (resp. F) for the first (resp. second) sector. The geometry is parametrized by the angle $\alpha=\widehat{A B E}=\widehat{E B C}=\widehat{B C F}=\widehat{F C D}$.

A thin elastic plate having the shape of the geometry shown in Fig. 1 is considered. In practice, to avoid stress singularity at angular points B and C, two small circular holes centered in B and C can be cut out of the sample (for instance, the inner circle radius can be chosen as say one-tenth of the external radius). The presence of such holes has a negligible influence on the buckled shape solution, but limits the risk of plastification or tearing in the vicinity of these points. "Thin" implies that the bending stiffness is small and in-plane stretch can be neglected.
In the following, the plate will be assumed to be in-extensible. The sample is under tensile load between points A and D.

When $\alpha<\pi / 2$, the segment AD lies inside the sample, and hence, because of inextensibility, it is not possible to stretch this segment any further than its end-to-end length. In contrast, when $\alpha>\pi / 2$, it is possible to deform the sample up to the stage where ABCD is aligned onto a straight segment. This configuration is obviously an upper bound on the maximum stretch that can be applied to the structure while violating inextensibility.
It is first assumed that the point-symmetry, with respect to the mid-point between B and C, is not broken in the buckled configuration. Hence, the problem of geometrical description at each given load can be reduced to the upper section ABCE.


Figure 2: Illustration of the deformed arc $\Gamma_{E}$ (black curve). It consists of two parts $A E$ and $E C$ which are two arcs that belong to the sphere centered in $B$, and belong to two cones whose apex is $B$ and which are tangent to each other along the BE line. Both cones and the sphere are drawn with transparent colors.

The arc AEC is initially located at a uniform distance to point B. Because the sample is assumed to be inextensible, the deformed arc AEC, denoted as $\Gamma_{E}$, remains at a fixed distance from center B, i.e., it lies on the sphere of radius $r_{1}$ centered in B. Thus, the deformed (buckled) sample geometry is a conical surface whose apex is point B and ends on the dielectrics $\Gamma_{E}$. Let us emphasize that the elastic energy of the buckled configuration can thus be inferred from the curvature of the deformed arc $\Gamma_{E}$, and the least elastic energy corresponds to the smaller L2 norm of the arc curvature. The arc length of $\Gamma_{E}$ is $2 \ell=2 r_{1} \alpha$ in the initial geometry, and because of inextensibility, it remains constant in the deformed configuration.
Hence the problem reduces to finding an arc $\Gamma_{E}$ along the sphere with a fixed arc length, such that its end-to-end distance achieves a pre-defined value $L$ larger than the initial one (and obviously smaller or equal to $2 r_{1}$ ) with the least L2 norm of its curvature.
Among such curves that would minimize its curvature and elastic energy, one may consider circles whose radius $\rho$ can only be smaller than $r_{1}$. However, a constant arc length $2 \ell$ on such circles brings the end point closer to the starting point and hence is not a feasible solution to the problem. Nevertheless, two such segments of two tangent circles having the same radius $\rho$ allow to form a curve $\Gamma_{E}$ that meets the desired objective of bringing point C closer to the symmetric point of A with respect to $B$. Because the curvature remains constant, it corresponds to a minimum L2 norm of the arc curvature and hence of the deformed plate. These simple observations provide a global and qualitative solution to the buckling problem. To give further insight into the problem, the following gives an analytical characterization of the deformed configuration.

### 2.1. First circle arc

Its length is half that of the initial $\Gamma_{E}$, hence $\ell=r_{1} \alpha$. In the deformed configuration, its radius is $\rho$, and hence the polar angle around the cone axis is $\beta$ such that $\rho \beta=r_{1} \alpha$.

The cone semi-angle is denoted as $\gamma$, and the associated stretching parameter, $p$, reads as

$$
\begin{equation*}
p=\frac{2 \gamma}{\pi} \tag{1}
\end{equation*}
$$

Because the $z$-axis is part of the cone, the cone axis is inclined at an angle $\gamma$ from the $z$-axis. This leads to the relation

$$
\begin{equation*}
\rho=r_{1} \sin (\gamma) \tag{2}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\beta=\frac{\alpha}{\sin (\gamma)} \tag{3}
\end{equation*}
$$

Conventionally, the axis can be oriented in the $(x, z)$ plane, and hence its direction vector, $\boldsymbol{n}_{\mathbf{1}}$, is written

$$
\boldsymbol{n}_{1}=\left(\begin{array}{c}
\sin (\gamma)  \tag{4}\\
0 \\
\cos (\gamma)
\end{array}\right)
$$

The projection of the first circle on its cone axis is the vector $r_{c} \boldsymbol{n}_{1}$, where $r_{c}=r_{1} \cos (\gamma)=\sqrt{r_{1}^{2}-\rho^{2}}$. Thus, choosing the origin of the frame in B , the AE arc is parameterized with $0 \leq \theta \leq \beta$, reads as

$$
\left\{\begin{align*}
x & =\left(r_{1} / 2\right) \sin (2 \gamma)(1-\cos (\theta))  \tag{5}\\
y & =r_{1} \sin (\gamma) \sin (\theta) \\
z & =\left(r_{1} / 2\right)[(1+\cos (2 \gamma))+(1-\cos (2 \gamma)) \cos (\theta)]
\end{align*}\right.
$$

The end point E is obtained from the previous expression with $\theta=\beta$.
More compact writing of such an arc is obtained via quaternions to encode the 3D rotation. Indeed, the rotation about the axis $\boldsymbol{n}_{1}$, by an angle $\theta$, is written $\mathbf{q}_{1}(\theta)=\left(\cos (\theta / 2), \sin (\theta / 2) \boldsymbol{n}_{1}\right)$. Vectors in 3 D , such as $\boldsymbol{n}_{1}$, are 3 component elements denoted with bold and italic fonts. Quaternions are written as 4 D vectors, and denoted by bold but nonitalicised fonts. A vector $\boldsymbol{n}_{1}$ can be associated to the quaternion $\mathbf{n}_{1}=\left(0, \boldsymbol{n}_{1}\right)=(0, \sin (\gamma), 0, \cos (\gamma))$ with a null real part (the first component of the quaternion). The rotation of the BA vector (written as a quaternion $\mathbf{a}_{0}=r_{1}(0,0,0,1)$ ), about the axis $\boldsymbol{n}_{1}$ by an angle $\theta$ is obtained from

$$
\begin{equation*}
\mathbf{a}_{\theta}=\mathbf{q}_{1}(\theta) \cdot \mathbf{a}_{0} \cdot \overline{\mathbf{q}_{1}}(\theta) \tag{6}
\end{equation*}
$$

using the standard multiplication rules for quaternions, and where $\overline{\mathbf{q}}$ denotes the conjugate of $\mathbf{q}$. Using the above presentation, the BE vector is obtained as the quaternion $\mathbf{a}_{\beta}$.

### 2.2. Second circle arc

The second part of the arc, is also a cone of similar semi-angle $\gamma$, and axis along $\boldsymbol{n}_{2}$. The point E is part of this second cone, and the two circles are tangent at this point. Thus $\boldsymbol{n}_{1}$ and $\boldsymbol{n}_{2}$ are symmetric with respect to the BE line. The projection of $\boldsymbol{n}_{1}$ along the BE direction has a length $\boldsymbol{r}_{1} \cos (\gamma)^{2}$, and hence $\boldsymbol{n}_{1}+\boldsymbol{n}_{2}=\boldsymbol{B} \boldsymbol{E} \cos (\gamma)^{2}$.

Quaternions offer again a convenient expression for $\mathbf{n}_{2}$ as the rotation of $\mathbf{n}_{2}$ about the BE vector by an angle $\pi$. This rotation is written $\mathbf{r}_{E}=\left(1 / r_{1}\right) \mathbf{a}_{\beta}$, and hence

$$
\begin{equation*}
\mathbf{n}_{2}=\mathbf{r}_{E} \cdot \mathbf{n}_{1} \cdot \overline{\mathbf{r}_{E}} \tag{7}
\end{equation*}
$$

The second part of the arc, segment EC, is obtained from BE by a rotation about $\boldsymbol{n}_{2}$ by an angle $-\theta$, for $0 \leq \theta \leq \beta$. This rotation is written $\mathbf{q}_{2}(\theta)=\left(\cos (\theta / 2),-\sin (\theta / 2) \boldsymbol{n}_{2}\right)$, and the arc is described by the quaternion

$$
\begin{equation*}
\mathbf{b}_{\theta}=\mathbf{q}_{2}(\theta) \cdot \mathbf{a}_{\beta} \cdot \overline{\mathbf{q}_{2}}(\theta) \tag{8}
\end{equation*}
$$

and hence vector BC is $\mathbf{b}_{\beta}$.
Now that the first half of the sample in its deformed configuration has been described as a function of the single parameter $\gamma$, the second half is deduced by a mere point symmetry about the midpoint between B and C . This completes the description of the full deformed geometry. As an illustration, ${ }^{1}$ Figure 3 shows respectively the initial reference geometry, the intermediary deformed configuration, the maximum deformed geometry (with a corresponding parameter $p=1,1.2,1.45$ ). Let us note that at the maximum deformed configuration (Figure 3 and Figure 4), points $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D are all aligned with the $z$-axis.

[^1]

Figure 3: Initial configuration ( $p=1$ ), intermediary deformation ( $p=1.2$ ) and maximum stretch ( $p=1.485$ ). The angle $\alpha$ has been chosen to be $5 \pi / 8$.


Figure 4: Another point of view of the maximum stretch configuration ( $p=1.485$ ). The angle $\alpha$ has been chosen to be $5 \pi / 8$.

## 3. Mechanical analysis

The above analysis of the deformed state has been reduced to a mere geometrical problem, and no mechanical information has been obtained so far.

Let us first note that the parameterization with $\gamma$ was convenient but difficult to assess experimentally. However, the above description provides easy access to the extension $\Delta U$ as a function of $\gamma$. It can also be noted that the maximum stretch displacement, $\Delta U_{\max }$ is easily obtained from the initial length $|A D|$ and the maximum one equal to $3 r_{1}$, so that the dimensionless ratio $\Delta U / \Delta U_{\max }$ will be used in the following to pair with the aforementioned stretching parameter $p$. Figure 5 a shows the correspondence between $p$ and the elongation during the tensile test, $\Delta U / \Delta U_{\max }$, scaled by the maximum stretch.

In terms of elastic energy, the deformed surface consists of four portions of cones with the same semi-angle. The curvature of the cone is proportional to $\cot (\gamma) / r$, where $r$ designates the distance to either B or C , for the upper (respectively lower) half of the specimen. The singularity in B or C is susceptible to create either damage or plastic yielding, and hence in practice, drilling a small hole in B and in C will avoid these difficulties and still preserve the validity of the solution. Along the outer edge, the curvature $\kappa=\cot (\gamma) / r$ is spatially uniform. Its variation with the relative extension in shown in Figure 5 b. It allows to construct the total elastic energy as proportionnal to $\kappa^{2}$, as shown in Figure 6a. It is to be observed that for a small stretch of the specimen, the curvature $\kappa$ behaves as $(\Delta U)^{1 / 2}$, and hence the elastic energy grows linearly with $\Delta U$. Its derivative provides a finite force at vanishing stretch which can be interpreted as a buckling force, $F_{c}$, below which the sample remains non-deformed. At maximum stretch, the tensile force diverges, $F \propto\left(\Delta U_{\max }-\Delta U\right)^{-1 / 2}$. The tensile force versus stretch is shown in Figure 6 b .


Figure 5: (a) Stretching parameter, $p=2 \gamma / \pi$, and (b) curvature, $\kappa$, are shown as a function of the dimensionless ratio $\Delta U / \Delta U_{\text {max }}$


Figure 6: (a) Elastic energy and (b) relative tensile force versus scaled elongation

## 4. Conclusion

This paper has introduced a simple sample planar geometry that buckles into a non-trivial shape in three dimensions and whose evolution under load can be followed analytically up to a maximum elongation.

Although the problem was addressed in the framework of linear elasticity, it is worth emphasizing that the deformed geometry would be identical for a wide category of hyper-elastic laws. Indeed, in the deformed geometry all elementary angular sectors (centered either in $B$ or in $C$ ) are subjected to the very same deformation as the segment of the cone (with an identical cone semi-angle). Therefore whatever the hyper-elastic law, the elastic energy stored in such sectors is identical. Thus, the relevance of the proposed deformed geometry for a more general constitutive law resumes to its stability, or essentially the convexity of the hyper-elastic law.

One may fear that the high (singular) strains expected at the apex of the cones in the deformed geometry (close to either B or C), may induce plastic strain, or even tearing. This is indeed very likely to occur. However, in practice, cutting out two small disks centered in B and C from the sample would limit the most intense stresses, and still preserve the validity of the solution.

Thus, the obtained solution appears to be quite robust, even with respect to the constitutive law of the material. As such, it constitutes an interesting benchmark solution for numerical simulations of buckling, challenging large 3D displacements, as well as stereo-vision systems for shape characterization and displacement measurements Strecha, von Hansen, Van Gool, Fua and Thoennessen (2008); Keller, Enzweiler and Gavrila (2011); Hu and Mordohai (2012). For the latter case, an experimental study is underway and will be reported in a forthcoming study.

## CRediT authorship contribution statement

Xuyang Chang: Conceptualization of this study, Visualization,Writing - Original draft preparation. Matthieu Vitse: Software, Visualization. Stéphane Roux: Conceptualization of this study, Methodology, Writing - Original draft preparation, Supervision.

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[^0]:    ${ }^{\star}$ This work has been supported by EikoSim, and the Ecole Normale Supérieure Paris-Saclay.
    *Corresponding author
    ®uyang.chang@ens-paris-saclay.fr (X. Chang); stephane.roux@ens-paris-saclay.fr (.S. Roux)

    - https://gitlab.com/sciviz/blendic (M. Vitse)

    ORCID(s): 0000-0002-8239-4714 (X. Chang); 0000-0002-4230-6396 (M. Vitse); 0000-0003-4885-6732 (.S. Roux)

[^1]:    ${ }^{1}$ Using the provided closed-form solution, the visual rendering is achieved via the BlenDIC add-on for Blender software Vitse

