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Monitoring the edges of product networks using distances *

Wen Li^{† ‡}, Ralf Klasing[§], Yaping Mao[¶], Bo Ning^{||}

Abstract

Foucaud *et al.* recently introduced and initiated the study of a new graph-theoretic concept in the area of network monitoring. Let G be a graph with vertex set $V(G)$, M a subset of $V(G)$, and e be an edge in $E(G)$, and let $P(M, e)$ be the set of pairs (x, y) such that $d_G(x, y) \neq d_{G-e}(x, y)$ where $x \in M$ and $y \in V(G)$. M is called a *distance-edge-monitoring set* if every edge e of G is monitored by some vertex of M , that is, the set $P(M, e)$ is nonempty. The *distance-edge-monitoring number* of G , denoted by $\text{dem}(G)$, is defined as the smallest size of distance-edge-monitoring sets of G . For two graphs G, H of order m, n , respectively, in this paper we prove that $\max\{m \text{dem}(H), n \text{dem}(G)\} \leq \text{dem}(G \square H) \leq m \text{dem}(H) + n \text{dem}(G) - \text{dem}(G) \text{dem}(H)$, where \square is the Cartesian product operation. Moreover, we characterize the graphs attaining the upper and lower bounds and show their applications on some known networks. We also obtain the distance-edge-monitoring numbers of join, corona, cluster, and some specific networks.

Keywords: Distance; Distance-edge-monitoring set; Cartesian product; Join; Networks.

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1 Introduction

In the area of network monitoring, Foucaud *et al.* [8] introduced a new graph theoretic concept, called *distance-edge-monitoring*. It aims to monitor some network using distance probes. One wishes to monitor the network when a connection (an edge) fails, that means this failure can be detected. We hope to select a smallest set of vertices of the network, called *probes*, which will monitor all edges of the network. At any given moment, a probe of the network can measure its graph distance to any other vertices of the network. The aim is that, whenever some edge of the network fails, one of the measured distances changes, and then the probes are able to detect the failure of any edge.

In real life networks, using probes to measure distances in a graphs is common. For instance, this is useful in the fundamental task of routing [5,9,10]. It is also usually used for problems concerning network verification [1–3]. In this paper, the networks are modeled by finite undirected simply connected graphs, with vertices representing computers and edges representing connections between computers.

1.1 Distance-edge-monitoring set and number

All graphs considered in this paper are undirected, finite and simple. We refer to the textbook [4] for graph theoretical notation and terminology not defined here. For a graph G , we use $V(G)$ and $E(G)$ to denote the set of vertices and edges, respectively. The number of vertices in G is the *order* of G . For any set $X \subseteq V(G)$, let $G[X]$ denote the subgraph induced by X ; similarly, for any set $F \subseteq E(G)$, let $G[F]$ denote the subgraph induced by F . For an edge set $X \subseteq E(G)$, let $G - X$ denote the subgraph of G obtained by removing all the edges in X from G . If $X = \{x\}$, we may use $G - x$ instead of $G - \{x\}$. For two subsets X and Y of $V(G)$, we denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other one in Y . The set of neighbors of a vertex $v \in G$ is denoted by $N_G(v)$. Moreover, $N_G[v] = N_G(v) \cup \{v\}$. A path of order n is denoted by P_n . The distance between vertices u and v of a graph G , denoted by $d_G(u, v)$, is the length of a shortest u, v -path. Sometimes one can simply write $d(u, v)$ instead of $d_G(u, v)$. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G . Then the *radius* of G , denoted by $rad(G)$, is the smallest eccentricity among the vertices of G .

First, some definitions and useful results about *distance-edge-monitoring* are introduced as follows.

Definition 1 ([8]). *For a set M of vertices and an edge e of a graph G , let $P_G(M, e)$ be the set of pairs (x, y) with x a vertex of M and y a vertex of $V(G)$ such that $d_G(x, y) \neq d_{G-e}(x, y)$.*

Definition 1 means that e belongs to all shortest paths between x and y in G .

Definition 2 ([8]). For a vertex x , let $EM_G(x)$ be the set of edges e such that there exists a vertex v in G with $(x, v) \in P(x, e)$.

If $e \in EM_G(x)$, we say that e is monitored by x . Sometimes one can simply write $P(M, e)$ and $EM(x)$ instead of $P_G(M, e)$ and $EM_G(x)$, respectively.

Definition 3 ([8]). A set M of vertices of a graph G is called a distance-edge-monitoring set if any edge e of G is monitored by some vertex of M .

Definition 3 means that $P(M, e) \neq \emptyset$.

Definition 4 ([8]). The distance-edge-monitoring number $\text{dem}(G)$ of a graph G is defined as the smallest size of distance-edge-monitoring sets.

The vertices of M represent distance probes in a network modeled by G ; when the edge e fails, the distance from x to y increase, and thus we are able to detect the failure. It turns out that not only we can detect it, but we can even correctly locate the failing edge [8]. A detailed discussion of *distance-edge-monitoring sets* and related concepts can be found in [8].

A set C of vertices is a *vertex cover* of G if every edge of G has one of its endpoints in C . The *vertex cover number* of G , denoted by $c(G)$, is the smallest size of a vertex cover of G .

Theorem 1.1 ([8]). Let G be a graph of order n . Any vertex cover C of G is a distance-edge-monitoring set, and hence $\text{dem}(G) \leq c(G) \leq n - 1$. Moreover, $\text{dem}(G) = n - 1$ if and only if $G = K_n$.

Theorem 1.2 ([8]). Let $K_{m,n}$ be the complete bipartite graph with parts of sizes m and n . Then $\text{dem}(K_{m,n}) = c(K_{m,n}) = \min\{m, n\}$.

1.2 Graph products

Product networks were proposed based upon the idea of using the cross product as a tool for “combining” two known graphs with established properties to obtain a new one that inherits properties [6]. In Graph Theory, Cartesian product is one of the main products, and has been studied extensively [12–14]. For more details on graph products, we refer to the monograph [11].

The two binary operations of graphs considered in this paper are defined as follows.

The *join* or *complete product* of two disjoint graphs G and H , denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$.

The *corona* $G * H$ is obtained by taking one copy of G and $|V(G)|$ copies of H , and by joining each vertex of the i -th copy of H with the i -th vertex of G , where $i = 1, 2, \dots, |V(G)|$.

The *cluster* or *rooted product* $G \odot H$ is obtained by taking one copy of G and $|V(G)|$ copies of a rooted graph H , and by identifying the root of the i -th copy of H with the i -th vertex of G , where $i = 1, 2, \dots, |V(G)|$.

The *Cartesian product* of two graphs G and H , written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$.

Foucaud *et al.* [8] introduced and initiated the study of distance-edge-monitoring sets. They showed that for a nontrivial connected graph G of order n , $1 \leq \text{dem}(G) \leq n - 1$ with $\text{dem}(G) = 1$ if and only if G is a tree, and $\text{dem}(G) = n - 1$ if and only if it is a complete graph. They derived the exact value of dem for grids, hypercubes, and complete bipartite graphs. Meantime, they related dem to other standard graph parameters, and showed that $\text{dem}(G)$ is lower-bounded by the arboricity of the graph, and upper-bounded by its vertex cover number. It is also upper-bounded by twice its feedback edge set number. Moreover, they characterized connected graphs G with $\text{dem}(G) = 2$. For the aspects of algorithm, they showed that determining $\text{dem}(G)$ for an input graph G is an NP-complete problem, even for apex graphs. There exists a polynomial-time logarithmic-factor approximation algorithm, however it is NP-hard to compute an asymptotically better approximation, even for bipartite graphs of small diameter and for bipartite subcubic graphs. For such instances, the problem is also unlikely to be fixed parameter tractable when parameterized by the solution size.

In this paper, we study the *distance-edge-monitoring numbers* of join, corona, cluster, and Cartesian products, and obtain the exact values of some specific networks.

2 Join, corona, and cluster operations

In this section, we obtain some results about distance-edge-monitoring numbers on the binary operations given above. Firstly, some theorems of [8] are given below is very helpful for our proof.

Theorem 2.1 ([8]). *Let G be a connected graph and $x \in V(G)$. The following two conditions are equivalent.*

- (1) $EM(x)$ is the set of edges incident with x .

(2) For every vertex y of G with $y \in V(G) - N_G[x]$, there exist two shortest paths from x to y sharing at most one edge (the one incident with x).

Theorem 2.2 ([8]). *If $EM(v)$ consists exactly of the edges incident with any vertex v of a graph G , then a set M is a distance-edge-monitoring set of G if and only if it is a vertex cover of G .*

Theorem 2.3 ([8]). *For any graph G , we have $c(G) \leq \text{dem}(G \vee K_1) \leq c(G) + 1$. Moreover, if G has radius at least 4, then $\text{dem}(G \vee K_1) = c(G)$.*

Next we study the distance-edge-monitoring numbers on the join operation of two graphs.

Theorem 2.4. *Let G and H be two graphs with $|V(G)|, |V(H)| \geq 2$. Then*

$$\text{dem}(G \vee H) = c(G \vee H) = \min\{c(G) + |V(H)|, c(H) + |V(G)|\}.$$

Proof. Since the distance between any two non-adjacent vertices of $G \vee H$ is two, it follows that there are at least two internally disjoint shortest paths between any two non-adjacent vertices. By Theorems 2.1 and 2.2, for any vertex $x \in V(G \vee H)$, $EM(x)$ is the set of edges incident with x . That is, a minimum vertex cover of $G \vee H$ is a distance-edge-monitoring set of $G \vee H$. Hence, we have $\text{dem}(G \vee H) = c(G \vee H)$.

Let $|V(G)| = m$ and $|V(H)| = n$. Without loss of generality, let $m \leq n$. For any vertex $x \in V(G \vee H)$, we have $EM(x) = \{xy \mid y \in N_{G \vee H}(x)\}$. Since all the edges in $\{uv \in E(G \vee H) \mid u \in V(G), v \in V(H)\}$ need all the m monitoring vertices in G , it follows that the remaining edges in $E(H)$ need at least $c(H)$ monitoring vertices, and hence $\text{dem}(G \vee H) \geq c(H) + |V(G)|$. To show $\text{dem}(G \vee H) \leq c(H) + |V(G)|$, we suppose that C is a vertex cover of H . Let $M = V(G) \cup C$. Clearly, M is a distance-edge-monitoring set of $G \vee H$, and so $\text{dem}(G \vee H) \leq c(H) + |V(G)|$, which implies $\text{dem}(G \vee H) = c(H) + |V(G)|$. The case $m \geq n$ can be discussed similarly. \square

Then we study the distance-edge-monitoring number on the corona of two graphs G and H . Let $V(G) = \{u_1, u_2, \dots, u_m\}$ and $V(H) = \{v_1, v_2, \dots, v_n\}$. Then $V(G * H) = V(G) \cup \{(u_i, v_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$, where $*$ denotes the corona product operation. For $u_i \in V(G)$, we use $H(u_i)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u_i, v_j) \mid 1 \leq j \leq n\}$. For a fixed integer i ($1 \leq i \leq m$), these edges $u_i(u_i, v_j) \in E(G * H)$ for each j ($1 \leq j \leq n$) are denoted by $E_i = \{u_i(u_i, v_j) \mid j = 1, 2, \dots, n\}$. Then $V(G * H) = V(G) \cup (\bigcup_{i=1}^m V(H(u_i)))$, and $E(G * H) = E(G) \cup (\bigcup_{i=1}^m E(H(u_i))) \cup (\bigcup_{i=1}^m E_i)$.

Theorem 2.5. *Let G and H be two graphs. Then $\text{dem}(G * H) = |V(G)| \text{dem}(H)$.*

Proof. Let $V(G) = \{u_1, u_2, \dots, u_m\}$ and $V(H) = \{v_1, v_2, \dots, v_n\}$. Let M_H be a distance-edge-monitoring set of H with $|M_H| = \text{dem}(H)$. Then $M_{H(u_i)}$ is the distance-edge-monitoring set of $H(u_i)$ corresponding to M_H in H , where $1 \leq i \leq m$.

Fact 1. *For any vertex $(u_s, v_t) \in M_{H(u_s)}$ where $1 \leq s \leq m$, we have that*

$$\left(\bigcup_{i=1, i \neq s}^m E_i \right) \cup E[\{u_s\}, N_G(u_s)] \subseteq EM_{G*H}((u_s, v_t)).$$

Proof. First, we want to show that $\bigcup_{i=1, i \neq s}^m E_i \subseteq EM_{G*H}((u_s, v_t))$. For any edge of $\bigcup_{i=1, i \neq s}^m E_i$, say $u_i(u_i, v_j)$, since $d_{G*H}((u_s, v_t), (u_i, v_j)) = d_{G*H}((u_s, v_t), u_i) + 1$, where $i \neq s$, it follows that

$$\begin{aligned} d_{G*H-u_i(u_i, v_j)}((u_s, v_t), (u_i, v_j)) &\geq d_{G*H}((u_s, v_t), u_i) + 2 \\ &> d_{G*H}((u_s, v_t), u_i) + 1 = d_{G*H}((u_s, v_t), (u_i, v_j)). \end{aligned}$$

Thus $u_i(u_i, v_j) \in EM_{G*H}((u_s, v_t))$, and so $\bigcup_{i=1, i \neq s}^m E_i \subseteq EM_{G*H}((u_s, v_t))$.

Next we need to prove that $E[\{u_s\}, N_G(u_s)] \subseteq EM_{G*H}((u_s, v_t))$. For any edge $u_s u_k \in E[\{u_s\}, N_G(u_s)]$, since $d_{G*H}((u_s, v_t), u_k) = d_{G*H}((u_s, v_t), u_s) + 1$, it follows that

$$d_{G*H-u_s u_k}((u_s, v_t), u_k) \geq d_{G*H}((u_s, v_t), u_s) + 2 > d_{G*H}((u_s, v_t), u_s) + 1 = d_{G*H}((u_s, v_t), u_k).$$

Therefore $u_s u_k \in EM_{G*H}((u_s, v_t))$, and so $E[\{u_s\}, N_G(u_s)] \subseteq EM_{G*H}((u_s, v_t))$. Hence we have

$$\left(\bigcup_{i=1, i \neq s}^m E_i \right) \cup E[\{u_s\}, N_G(u_s)] \subseteq EM_{G*H}((u_s, v_t)).$$

□

It follows from Fact 1 that $M_{H(u_s)}$ not only monitor all the edges of $H(u_s)$, but also monitor the edges in $\bigcup_{i=1, i \neq s}^m E_i$ and $E[\{u_s\}, N_G(u_s)]$. Then $\bigcup_{i=1}^m M_{H(u_i)}$ is a distance-edge-monitoring set of $G * H$, and hence $\text{dem}(G * H) \leq m \text{dem}(H)$.

In the end, we want to show that $\text{dem}(G * H) \geq m \text{dem}(H)$. Suppose that M' is a distance-edge-monitoring set of $G * H$ with $|M'| = m \text{dem}(H) - 1$. There exists at least a copy of $H(u_k)$ such that $|M' \cap V(H(u_k))| \leq \text{dem}(H) - 1$. Then there exists at least one edge e of $H(u_k)$ cannot be monitored by $M' \cap V(H(u_k))$. Since all the edges of $H(u_k)$ cannot be monitored by any vertex of $V(G * H) - V(H(u_k))$, it follows that e cannot be monitored by M' , a contradiction. Hence $\text{dem}(G * H) \geq m \text{dem}(H)$.

□

In the end, we study the distance-edge-monitoring number on the cluster of two graphs G and H . Recall the definition of cluster, we have $V(G \odot H) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, where \odot denotes the cluster product operation. For $u \in V(G)$, we let $H(u)$

denote the subgraph of $G \odot H$ induced by the vertex set $\{(u, v_j) \mid 1 \leq j \leq m\}$. Without loss of generality, we assume (u_i, v_1) is the root of $H(u_i)$ for each $u_i \in V(G)$. Let $G(v_1)$ be the graph induced by the vertices in $\{(u_i, v_1) \mid 1 \leq i \leq n\}$. Clearly, $G(v_1) \simeq G$, and $V(G \odot H) = V(H(u_1)) \cup V(H(u_2)) \cup \dots \cup V(H(u_n))$.

Foucaud et al. [8] studied distance-edge-monitoring number of tree and proved the following.

Theorem 2.6 ([8]). *Let G be a connected graph with at least one edge. Then $\text{dem}(G) = 1$ if and only if G is a tree.*

Let G be a graph. The *base graph* G_b of G is the graph obtained from G by iteratively deleting pendant edges together with vertices of degree 1. Then Foucaud et al. [8] derived following.

Theorem 2.7. [8] *Let G_b be the base graph of a graph G . Then we have $\text{dem}(G) = \text{dem}(G_b)$.*

In this paper, the distance-edge-monitoring number of cluster is determined below.

Theorem 2.8. *Let G and H be two graphs. Then*

$$\text{dem}(G) \leq \text{dem}(G \odot H) \leq |V(G)| \text{dem}(H).$$

Moreover, $\text{dem}(G \odot H) = \text{dem}(G)$ if and only if H is a tree.

Proof. Let $|V(G)| = m$, $|V(H)| = n$, $\text{dem}(G) = d_1$, and $\text{dem}(H) = d_2$. First, if H is a tree, it follows from Theorem 2.7 that $\text{dem}(G \odot H) = d_1$. Conversely, we want to show that if $\text{dem}(G \odot H) = d_1$, then H is a tree. Assume that H is not a tree, it follows from Theorem 2.6 that $d_2 \geq 2$. Suppose that M is a distance-edge-monitoring set of $G \odot H$ with $|M| = d_1$. If $M' \subset V(G)$, then there is an edge e of some copy of H which cannot be monitored by M' as $d_2 \geq 2$, a contradiction. If $|M' \cap V(G)| = d_1 - a$ ($1 \leq a \leq d_1 \leq m - 1$), then $M'' = M' - V(G) = \{(u_i, v_j) \mid 1 \leq i \leq m, 2 \leq j \leq n\}$, and $|M''| = a$. Since the a vertices in M'' are only allowed in the m copies of H , it follows that there exists a copy, say $H(u_t)$, such that $M'' \cap V(H(u_t)) = \emptyset$. It is clearly that $EM_{H(u_t)}((u_i, v_j)) = EM_{H(u_t)}((u_t, v_1))$, where $i \neq t$ and $1 \leq j \leq m$. Since $\text{dem}(H(u_t)) \geq 2$, it follows that there is an edge $e \in E(H(u_t))$ which cannot be monitored by M' , a contradiction. Hence H is a tree.

Next, we want to prove that $d_1 + 1 \leq \text{dem}(G \odot H) \leq md_2$, where H is not a tree. Let M_H be a distance-edge-monitoring set of H with $|M_H| = d_2 \geq 2$. Without loss of generality, let $M_{H(u_i)} = \{(u_i, v_j) \mid 1 \leq j \leq n\}$ be the distance-edge-monitoring sets of $H(u_i)$ corresponding to M_H in H . where $1 \leq i \leq m$. For any vertex $(u_s, v_1) \in N_G((u_i, v_1))$, and for any vertex

$(u_i, v_j) \in M_{H(u_i)}$, it is clearly that

$$\begin{aligned} & d_{G \odot H}((u_i, v_j), (u_s, v_1)) \\ &= d_{H(u_i)}((u_i, v_j), (u_i, v_1)) + d_{G \odot H}((u_i, v_1), (u_s, v_1)) \\ &= d_{H(u_i)}((u_i, v_j), (u_i, v_1)) + 1. \end{aligned}$$

Since

$$d_{G \odot H}((u_i, v_1), (u_s, v_1)) < d_{G \odot H - (u_i, v_1)(u_s, v_1)}((u_i, v_1), (u_s, v_1)),$$

it follows that

$$\begin{aligned} & d_{G \odot H - (u_i, v_1)(u_s, v_1)}((u_i, v_j), (u_s, v_1)) \\ &= d_{H(u_i)}((u_i, v_j), (u_i, v_1)) + d_{G \odot H - (u_i, v_1)(u_s, v_1)}((u_i, v_1), (u_s, v_1)) \\ &> d_{H(u_i)}((u_i, v_j), (u_i, v_1)) + d_{G \odot H}((u_i, v_1), (u_s, v_1)) \\ &= d_{G \odot H}((u_i, v_j), (u_s, v_1)). \end{aligned}$$

This means that $(u_i, v_1)(u_s, v_1) \in EM_{G \odot H}((u_i, v_j))$. Hence, we have that $M = \bigcup_{i=1}^m M_{H(u_i)}$ is a distance-edge-monitoring set of $G \odot H$, and so $\text{dem}(G \odot H) \leq md_2$.

In the end, we shall to prove that $\text{dem}(G \odot H) \geq d_1 + 1$. For any vertex subset $M' \subset V(G \odot H)$ such that $|M'| = d_1$, we have the fact that M' is not a distance-edge-monitoring set of $G \odot H$. If $M' \subset V(G)$, then there is an edge e of some copy of H which cannot be monitored by M' . Therefore, M' is not a distance-edge-monitoring set of $G \odot H$. If $|M' \cap V(G)| = d_1 - a$ ($1 \leq a \leq d_1 \leq m - 1$), then

$$M'' = M' - V(G) = \{(u_i, v_j) \mid 1 \leq i \leq m, 2 \leq j \leq n\},$$

and $|M''| = a$. Since the a vertices in M'' are only allowed in the m copies of H , it follows that there exists a copy, say $H(u_t)$, such that $M'' \cap V(H(u_t)) = \emptyset$. It is clearly that $EM_{H(u_i)}((u_i, v_j)) = EM_{H(u_t)}((u_t, v_1))$, where $i \neq t$ and $1 \leq j \leq m$. Since $\text{dem}(H(u_t)) \geq 2$, it follows that there is an edge $e \in E(H(u_t))$ which cannot be monitored by M' . Therefore, M' is not a distance-edge-monitoring set of $G \odot H$. We have proved $\text{dem}(G \odot H) \geq d_1$. \square

3 Results for Cartesian product

In this section, we study $P(M, e)$, $EM(x)$ and distance-edge-monitoring number of Cartesian product of two general graphs. Furthermore, we determine distance-edge-monitoring numbers of some specific networks.

For any vertex $v \in V(H)$, the subgraph of $G \square H$ induced by the vertex set $\{(u_i, v) \mid 1 \leq i \leq |V(G)|\}$ is denoted by $G(v)$. Similarly, for any vertex $u \in V(G)$, the subgraph of $G \square H$ induced by the vertex set $\{(u, v_j) \mid 1 \leq j \leq |V(H)|\}$ is denoted by $H(u)$.

3.1 General results

In this section, we firstly study $P(M, e)$ and $EM(x)$ of the Cartesian product of two general graphs. We also determine distance-edge-monitoring number of the Cartesian product of two general graphs. Finally, we characterize the graphs attaining the upper and lower bounds.

The following result is from the book [12].

Theorem 3.1 ([12]). *Let (u, v) and (u', v') are two vertices of $G \square H$. Then*

$$d_{G \square H}((u, v), (u', v')) = d_G(u, u') + d_H(v, v').$$

From the definition of the Cartesian product, an edge of $G \square H$ is either in the form $(u, v)(u, v')$ or $(u, v)(u', v)$. So we have the following.

Theorem 3.2. *For any two edges $(u, v)(u, v')$ and $(u, v)(u', v)$ of $G \square H$, we have that*

- (i) $P_{G \square H}(M, (u, v)(u, v')) = P_{H(u)}(M \cap V(H(u)), (u, v)(u, v'))$;
- (ii) $P_{G \square H}(M, (u, v)(u', v)) = P_{G(v)}(M \cap V(G(v)), (u, v)(u', v))$.

Proof. For some vertex $(a, b) \in M$ and $a \neq u$, we claim that there is no vertex (x, y) of $G \square H$ such that $d_{G \square H}((a, b), (x, y)) \neq d_{G \square H - (u, v)(u, v')}((a, b), (x, y))$. If there is a vertex (x, y) of $G \square H$ such that $d_{G \square H}((a, b), (x, y)) \neq d_{G \square H - (u, v)(u, v')}((a, b), (x, y))$, in other words, $(u, v)(u, v')$ belongs to all shortest paths between (a, b) and (x, y) in $G \square H$. This is a contradiction as we can find two shortest paths, say

$$P_1 = (a, b) \cdots (u, b) \cdots (u, v)(u, v') \cdots (x, y),$$

and

$$P_2 = (a, b) \cdots (a, v)(a, v') \cdots (a, y) \cdots (x, y),$$

such that the edge $(u, v)(u, v')$ is not in one of them. From Theorem 3.1, we can see P_1 and P_2 are the shortest paths from (a, b) to (x, y) . But $(u, v)(u, v') \in E(P_1)$ and $(u, v)(u, v') \notin E(P_2)$. Therefore, we only need to consider the vertices of $M \cap V(H(u))$, that is $P_{G \square H}(M, (u, v)(u, v')) = P_{G \square H}(M \cap V(H(u)), (u, v)(u, v'))$.

Next we want to show $P_{G \square H}(M \cap H(u), (u, v)(u, v')) = P_{H(u)}(M \cap V(H(u)), (u, v)(u, v'))$. According to the above proof, we can claim that for any vertex $(a, b) = (u, b) \in M \cap V(H(u))$, if there is a vertex (x, y) of $G \square H$ such that $d_{G \square H}((a, b), (x, y)) \neq d_{G \square H - (u, v)(u, v')}((a, b), (x, y))$, then $(x, y) = (u, y) \in V(H(u))$, that is

$$P_{G \square H}(M, (u, v)(u, v')) = P_{G \square H}(M \cap V(H(u)), (u, v)(u, v')) = P_{H(u)}(M \cap V(H(u)), (u, v)(u, v')).$$

By the symmetry, we can also prove that

$$P_{G \square H}(M, (u, v)(u', v)) = P_{G(v)}(M \cap V(G(v)), (u, v)(u', v)).$$

This proves the theorem. □

Example 1. Let P_m and P_n are two paths. For any vertex set $M \subset V(P_m \square P_n)$, we have $P_{P_m \square P_n}(M, (u, v)(u, v')) = P_{P_n(u)}(M \cap V(P_n(u)), (u, v)(u, v'))$ and $P_{P_m \square P_n}(M, (u, v)(u', v)) = P_{P_m(v)}(M \cap V(P_m(v)), (u, v)(u', v))$.

Note that if any vertex pair $(a, b) \in M$ and $a \neq u$, then $P_{G \square H}(M, (u, v)(u, v')) = \emptyset$. If any vertex pair $(a, b) \in M$ and $b \neq v$, then $P_{G \square H}(M, (u, v)(u', v)) = \emptyset$.

From Theorem 3.2, we have the following immediately.

Corollary 3.3. For any vertex (u_i, v_j) of $G \square H$, we have

$$EM_{G \square H}((u_i, v_j)) = EM_{H(u_i)}(u_i) \cup EM_{G(v_j)}(v_j).$$

Theorem 3.4. Let G and H be two graphs of order m and n , respectively. Then

$$\max\{m \text{dem}(H), n \text{dem}(G)\} \leq \text{dem}(G \square H) \leq m \text{dem}(H) + n \text{dem}(G) - \text{dem}(G) \text{dem}(H).$$

Moreover, the bounds are sharp.

Proof. Let $\text{dem}(G) = d_1$ and $\text{dem}(H) = d_2$. Then we have the following claim.

Claim 1. $\text{dem}(G \square H) \geq \max\{md_2, nd_1\}$.

Proof. Assume, to the contrary, that $\text{dem}(G \square H) \leq \max\{md_2, nd_1\} - 1$. Without loss of generality, let $md_2 \leq nd_1$. Then $\text{dem}(G \square H) \leq nd_1 - 1$. Suppose that M is a distance-edge-monitoring set of size $nd_1 - 1$ of $G \square H$. Then there exists some $G(v_j)$, where $1 \leq j \leq n$, such that $|V(G(v_j)) \cap M| \leq d_1 - 1$. Since $\text{dem}(G) = \text{dem}(G(v_j)) = d_1$, it follows that there exists some edge $e_j \in E(G(v_j))$ such that e_j cannot be monitored by $V(G(v_j)) \cap M$. Additionally, for any vertex (u_i, v_s) of $G \square H$, where $s \neq j$, from Corollary 3.3, we have $e_j \notin EM_{G \square H}((u_i, v_s)) = EM_{H(v_j)}(u_i) \cup EM_{G(u_i)}(v_s)$. Hence, the edge e_j cannot be monitored by M in $G \square H$, a contradiction. This proves Claim 1. \square

From Claim 1, we have $\text{dem}(G \square H) \geq \max\{md_2, nd_1\}$.

It suffices to show that $\text{dem}(G \square H) \leq md_2 + nd_1 - d_1d_2$. Without loss of generality, let $M_H^i = \{(u_i, v_1), (u_i, v_2), \dots, (u_i, v_{d_2})\}$ be a monitor set of $H(u_i)$, where $1 \leq i \leq m$. Let $M_G^j = \{(u_1, v_j), (u_2, v_j), \dots, (u_{d_1}, v_j)\}$ be a monitor set of $G(v_j)$, where $1 \leq j \leq n$. From Corollary 3.3, $\bigcup_{i=1}^m M_H^i$ can monitor all edges in $\bigcup_{i=1}^m E(H(u_i))$ and $\bigcup_{j=1}^{d_2} E(G(v_j))$, and $\bigcup_{j=d_2+1}^n M_G^j$ can monitor all edges in $\bigcup_{j=d_2+1}^n E(G(v_j))$. Hence, $M = (\bigcup_{i=1}^m M_H^i) \cup (\bigcup_{j=d_2+1}^n M_G^j)$ is a distance-edge-monitoring set of $G \square H$ with size $md_2 + (n - d_2)d_1 = md_2 + nd_1 - d_1d_2$, and so $\text{dem}(G \square H) \leq md_2 + nd_1 - d_1d_2$. \square

Next, the necessary and sufficient condition for the upper bound of Theorem 3.4 is given below.

Theorem 3.5. *Let G and H be two graphs of order m and n , respectively. Then*

$$\text{dem}(G \square H) = m \text{dem}(H) + n \text{dem}(G) - \text{dem}(G) \text{dem}(H)$$

if and only if there is only one distance-edge-monitoring set in G or H .

Proof. Let $\text{dem}(G) = d_1$ and $\text{dem}(H) = d_2$. We first show that if $\text{dem}(G \square H) = md_2 + nd_1 - d_1d_2$, then there is only one distance-edge-monitoring set in G or H . Assume that M_G^1 and M_G^2 are two distance-edge-monitoring sets of G , and M_H^1 and M_H^2 are two distance-edge-monitoring sets of H , respectively. Let $M_{G(v_j)}^1 = \{(u_s, v_j) \mid 1 \leq s \leq d_1\}$ and $M_{H(u_i)}^1 = \{(u_i, v_t) \mid 1 \leq t \leq d_2\}$, where $1 \leq i \leq m$, $1 \leq j \leq n$, be the distance-edge-monitoring sets of $G(v_j)$ and $H(u_i)$ corresponding to M_G^1 and M_H^1 in G and H , respectively. Let $x = |M_{G(v_j)}^1 - M_{G(v_j)}^2|$ and $y = |M_{H(u_i)}^1 - M_{H(u_i)}^2|$. Then $1 \leq x \leq |M_{G(v_j)}^1|$ and $1 \leq y \leq |M_{H(u_i)}^1|$.

From Corollary 3.3, $\bigcup_{j=1}^{d_2} M_{G(v_j)}^1$ can monitor all the edges in

$$\{E(G(v_j)) \mid 1 \leq j \leq d_2\} \cup \{E(H(u_i)) \mid 1 \leq i \leq d_1\},$$

and $\bigcup_{j=d_2+1}^n M_{G(v_j)}^2$ can monitor all the edges in $\{E(G(v_j)) \mid d_2 + 1 \leq j \leq n\}$.

If $M_{H(u_i)}^1 \cap M_{H(u_i)}^2 = \emptyset$, then all the edges in $\{E(H(u_i)) \mid d_1 + 1 \leq i \leq d_1 + x\}$ can be monitored by $\bigcup_{j=d_2+1}^n M_{G(v_j)}^2$, and all the edges in $\{E(H(u_i)) \mid d_1 + x + 1 \leq i \leq m\}$ can be monitored by $\bigcup_{i=d_1+x+1}^m M_{H(u_i)}^2$. This means that

$$M = \left(\bigcup_{j=1}^{d_2} M_{G(v_j)}^1 \right) \cup \left(\bigcup_{j=d_2+1}^n M_{G(v_j)}^2 \right) \cup \left(\bigcup_{i=d_1+x+1}^m M_{H(u_i)}^2 \right)$$

is a distance-edge-monitoring set of $G \square H$. However, we have

$$|M| = nd_1 + (m - d_1 - x)d_2 = nd_1 + md_2 - d_1d_2 - xd_2 < nd_1 + md_2 - d_1d_2,$$

a contradiction.

If $M_{H(u_i)}^1 \cap M_{H(u_i)}^2 \neq \emptyset$, then all edges in $\{E(H(u_i)) \mid d_1 + 1 \leq i \leq m\}$ can be monitored by $\bigcup_{i=d_1+1}^m M_{H(u_i)}^2$, and hence

$$M = \left(\bigcup_{j=1}^{d_2} M_{G(v_j)}^1 \right) \cup \left(\bigcup_{j=d_2+1}^n M_{G(v_j)}^2 \right) \cup \left(\bigcup_{i=d_1}^m M_{H(u_i)}^2 \right)$$

is a distance-edge-monitoring set of $G \square H$. Since $\left| \left(\bigcup_{j=d_2+1}^n M_{G(v_j)}^2 \right) \cap \left(\bigcup_{i=d_1}^m M_{H(u_i)}^2 \right) \right| = xy$, it follows that

$$|M| = nd_1 + (m - d_1)d_2 - xy = nd_1 + md_2 - d_1d_2 - xy \leq nd_1 + md_2 - d_1d_2 - 1 < nd_1 + md_2 - d_1d_2,$$

a contradiction.

By the above arguments, there is only one distance-edge-monitoring set in G or H .

Conversely, we suppose that there is only one distance-edge-monitoring set in G or H . Without loss of generality, we suppose that there is only one distance-edge-monitoring set in G , say $M_G = \{u_1, u_2, \dots, u_{d_1}\}$. For each j ($1 \leq j \leq n$), $M_{G(v_j)} = \{(u_1, v_j), (u_2, v_j), \dots, (u_{d_1}, v_j)\}$ is the distance-edge-monitoring set of $G(v_j)$ corresponding to M_G in G .

Claim 1. $\text{dem}(G \square H) \geq md_2 + nd_1 - d_1d_2$.

Proof. Assume, to the contrary, that $\text{dem}(G \square H) \leq md_2 + nd_1 - d_1d_2 - 1$. Let M be a distance-edge-monitoring set of size $md_2 + nd_1 - d_1d_2 - 1$ of $G \square H$. Let M_H be a distance-edge-monitoring set of H . For each i ($1 \leq i \leq m$), $M_{H(u_i)} = \{(u_i, v_1), (u_i, v_2), \dots, (u_i, v_{d_2})\}$ is the distance-edge-monitoring set of $H(u_i)$ corresponding to M_H in H . From Corollary 3.3, each $M_{G(v_j)}$ can only monitor all the edges in $E(G(v_j))$, and hence $\bigcup_{j=1}^n M_{G(v_j)}$ can monitor all the edges in

$$\{E(G(v_j)) \mid 1 \leq j \leq n\} \cup \{E(H(u_i)) \mid 1 \leq i \leq d_1\},$$

and so the vertex set $M' = M - \bigcup_{j=1}^n M_{G(v_j)}$ with size

$$(md_2 + nd_1 - d_1d_2 - 1) - nd_1 = md_2 - d_1d_2 - 1 = (m - d_1)d_2 - 1$$

monitors all the edges in $\{E(H(u_i)) \mid d_1 + 1 \leq i \leq m\}$. Hence, there exists some $H(u_i)$ such that $|H(u_i) \cap M'| < d_2 - 1$. In other words, there is some edge $e \in E(H(u_i))$ which cannot be monitored by M in $G \square H$, a contradiction. \square

From Claim 1, we have $\text{dem}(G \square H) \geq md_2 + nd_1 - d_1d_2$. From Theorem 3.4, we have $\text{dem}(G \square H) \leq md_2 + nd_1 - d_1d_2$, and hence $\text{dem}(G \square H) = md_2 + nd_1 - d_1d_2$. \square

To show the upper bound of Theorem 3.4 is sharp, we recall the *book graph* initially introduced by Erdős [7] in 1962. A book graph B_q of size q consists of q triangles sharing a common edge. In the following, the distance-edge-monitoring set of B_n is determined.

Proposition 3.1. *For a book graph B_n with n pages, we have*

$$\text{dem}(B_n) = 2.$$

Moreover, the two vertices of the common edge in B_n form the unique distance-edge-monitoring set of B_n .

Proof. Let uv be the common edge of B_n , and let $V(B_n) - u - v = \{u_i \mid 1 \leq i \leq n\}$. Each triangle of B_n induced by the vertex set $\{u, v, u_i\}$ is denoted by K_3^i , where $1 \leq i \leq n$. For each K_3^i , we need at least two vertices to monitor all the three edges, and so $\text{dem}(B_n) \geq 2$.

Let $M = \{u, v\}$. Then M can monitor all the edges of B_n , and hence $\text{dem}(B_n) \leq 2$, and so $\text{dem}(B_n) = 2$.

We have already proved that $M = \{u, v\}$ is a distance-edge-monitoring set of B_n . It suffices to show that M is a unique distance-edge-monitoring set of B_n . Suppose that M' is another distance-edge-monitoring set of B_n . Then we have $0 \leq |M \cap M'| \leq 2$. Suppose that $|M \cap M'| = 0$. Without loss of generality, let $M' = \{u_a, u_b\}$, where $1 \leq a, b \leq n$ and $a \neq b$. Since $|M' \cap K_3^a| = |M' \cap K_3^b| = 2$, it follows that M' can monitor all the edges of K_3^a and K_3^b . However, $|M' \cap K_3^i| = 0$, and so M' cannot monitor all the edges of K_3^i , where $i = \{1, 2, \dots, n\} - a - b$, a contradiction. Suppose that $|M \cap M'| = 1$. Without loss of generality, let $M' = \{u, u_a\}$, where $1 \leq a \leq n$. Clearly, since $|M' \cap K_3^a| = 2$, it follows that M' can monitor all the edges of K_3^a . However, $|M' \cap K_3^j| = 1$, and so M' cannot monitor all the edges of K_3^j , where $j = \{1, 2, \dots, n\} - a$, a contradiction. If $|M \cap M'| = 2$ then $M = M'$, and hence $M = \{u, v\}$ is the unique distance-edge-monitoring set of B_n . \square

From Theorem 3.5 and Proposition 3.1, the following results can show that the upper bound of Theorem 3.4 is sharp.

Corollary 3.6. *For two book graphs B_n and B_m , we have*

$$\text{dem}(B_n \square B_m) = 2m + 2n + 4.$$

Corollary 3.7. *For a connected graph G with order m and a book graph B_n , we have*

$$\text{dem}(G \square B_n) = 2m + (n + 2) \text{dem}(G) - 2 \text{dem}(G).$$

Then the necessary and sufficient condition for the lower bound of Theorem 3.4 is given below.

Theorem 3.8. *Let G and H be two graphs of order m and n ($m \leq n$), respectively, such that $\text{dem}(G) \geq \text{dem}(H)$. Then $\text{dem}(G \square H) = n \text{dem}(G)$ if and only if all of the following hold.*

- (1) *For any vertex $x \in V(G)$, there exists a distance-edge-monitoring set M_G with $x \in M_G$.*
- (2) *H has at least k distance-edge-monitoring sets, say M_H^j ($j = 1, 2, \dots, k$), such that $M_H^p \cap M_H^q = \emptyset$ ($1 \leq p, q \leq k$, $p \neq q$) for any two sets M_H^p and M_H^q where $k = \min\{s \mid \bigcup_{i=1}^s M_G^i = V(G)\} \geq 2$.*

Proof. Let $\text{dem}(G) = d_1$ and $\text{dem}(H) = d_2$. Without loss of generality, let $d_1 \geq d_2$. Suppose that $\text{dem}(G \square H) = nd_1$. Let M be the distance-edge-monitoring set of $G \square H$. Then $|M| = nd_1$, and each $G(v_j)$ has d_1 vertices to monitor all edges in $G(v_j)$, where $1 \leq j \leq n$.

Claim 1. For any vertex $x \in V(G)$, there exists a distance-edge-monitoring set M_G with $x \in M_G$.

Proof. Assume, to the contrary, that there exists a vertex $x \in V(G)$ such that x is not in any distance-edge-monitoring set of G . Without loss of generality, let (u_a, v_j) be the vertex of $G(v_j)$ corresponding to x in G , where $1 \leq j \leq n$. Then all vertices of M are not in $H(u_a)$. From Corollary 3.3, M cannot monitor the edges of $H(u_a)$, a contradiction. \square

By Claim 1, (1) holds. By (1), there exist some distance-edge-monitoring sets, say $M_G^1, M_G^2, \dots, M_G^s$, such that $\bigcup_{i=1}^s M_G^i = V(G)$, where $s \geq 2$. Let $k = \min\{s \mid \bigcup_{i=1}^s M_G^i = V(G)\}$. So $k \geq 2$.

Claim 2. H has at least k distance-edge-monitoring sets.

Proof. Assume, to the contrary, that H has at most $k - 1$ distance-edge-monitoring sets, say M_H^j ($j = 1, 2, \dots, k - 1$). Let $M_H^j = \{v_j^1, v_j^2, \dots, v_j^{d_2}\}$, $M_G^i = \{u_i^1, u_i^2, \dots, u_i^{d_1}\}$, and let $M^i = M_G^i \times M_H^i$. Let $M_{H(u_i)}^j$ ($j = 1, 2, \dots, k - 1$) be the distance-edge-monitoring sets of $H(u_i)$ corresponding to M_H^j in H .

If $\bigcup_{t=1}^{k-1} M_H^t = V(H)$, then $M' = \bigcup_{i=1}^{k-1} M^i$ can monitor at most all the edges in

$$\{E(G(v_i)), E(H(u_j)) \mid 1 \leq i \leq n, 1 \leq j \leq d - 1\},$$

where $|V(G) - \bigcup_{i=1}^{k-1} M_G^i| = d$. Since the edges in $\{E(H(u_j)) \mid d \leq j \leq m\}$ cannot be monitored by M' , it follows that $\text{dem}(G \square H) \geq nd_1 + 1$, a contradiction.

If $|\bigcup_{t=1}^k M_H^t - V(H)| = r \geq 1$, then $M' = (\bigcup_{i=1}^{k-1} M^i) \cup M_G^k$ can monitor at most all the edges in $\{E(G(v_i)), E(H(u_j)) \mid 1 \leq i \leq n, 1 \leq j \leq d - 1\}$, where $|V(G) - \bigcup_{i=1}^{k-1} M_G^i| = d$. Since the edges in $\{E(H(u_j)) \mid d \leq j \leq m\}$ cannot be monitored by M' , it follows that $\text{dem}(G \square H) \geq nd_1 + 1$, a contradiction. \square

From Claim 2, let M_H^j ($j = 1, 2, \dots, k$) be the distance-edge-monitoring sets of H .

Claim 3. $M_H^p \cap M_H^q = \emptyset$ for each $1 \leq p \neq q \leq k$.

Proof. Assume, to the contrary, that there exist two distance-edge-monitoring sets M_H^p, M_H^q such that $M_H^p \cap M_H^q \neq \emptyset$. Without loss of generality, let $|M_H^1 \cap M_H^2| = f$ ($1 \leq f \leq d_2 - 1$), and let $M_H^1 = \{v_1, v_2, \dots, v_{d_2}\}$, $M_H^2 = \{v_{d_2-f}, v_{d_2-f+1}, \dots, v_{d_2}, v_{d_2+1}, \dots, v_{2d_2-f}\}$. Similarly, let $|M_G^1 \cap M_G^2| = l$ ($0 \leq l \leq d_1 - 1$), and let $M_G^1 = \{u_1, u_2, \dots, u_{d_1}\}$, $M_G^2 = \{u_{d_1-l}, u_{d_1-l+1}, \dots, u_{d_1}, u_{d_1+1}, \dots, u_{2d_1-l}\}$. Then $M_{G(v_i)}^1$ and $M_{G(v_i)}^2$ are the distance-edge-monitoring sets of $G(v_i)$ corresponding to M_G^1 and M_G^2 in G . Then $M_{H(u_i)}^1$ and $M_{H(u_i)}^2$ are the distance-edge-monitoring sets of $H(u_i)$ corresponding to M_H^1 and M_H^2 in H , respectively. For any distance-edge-monitoring set M with size nd_1 of $G \square H$, since $\text{dem}(G \square H) = nd_1$,

it follows from Corollary 3.3 that, $|M \cap G(v_j)| = d_1$, where $j = 1, 2, \dots, n$. For all the $H(u_i), i = 1, 2, \dots, 2d_1 - l$, since $|M_{H(u_{d_1-l+1})}^1 \cap M_{H(u_{d_1})}^2| = f$, it follows that there are at least $d_1 - l$ copy of H , say $H(u_j)$, such that $|M \cap H(u_j)| < d_2$, and hence M cannot monitor those edges of $H(u_j)$, a contradiction. \square

Conversely, we suppose that (1) and (2) hold. Since (1) holds, it follows that G has k distance-edge-monitoring sets M_G^i , where $k = \min\{s \mid \bigcup_{i=1}^s M_G^i = V(G)\} \geq 2$. Since (2) holds, it follows that H has k distance-edge-monitoring sets M_H^j such that for any two sets M_H^p and M_H^q , $M_H^p \cap M_H^q = \emptyset$ ($1 \leq i, j \leq k, i \neq j$). Without loss of generality, let $M_H^j = \{v_j^1, v_j^2, \dots, v_j^{d_2}\}$, and $M_G^i = \{u_i^1, u_i^2, \dots, u_i^{d_1}\}$. Then $M = \bigcup_{i=1}^k (M_G^i \times M_H^i)$ is a distance-edge-monitoring set of $G \square H$ and $|M| = nd_1$, and hence $\text{dem}(G \square H) \leq nd_1$. It follows from Theorem 3.4 that $\text{dem}(G \square H) \geq nd_1$, and hence $\text{dem}(G \square H) = nd_1$. \square

3.2 Results for some specific networks

In this section, we determine distance-edge-monitoring numbers of some specific networks. For any two integers $m, n \geq 2$, it is clear that the Cartesian product $P_m \square P_n \cong G_{m,n}$, where the grid of dimension $m \times n$ is denoted by $G_{m,n}$. Foucaud et al. [8] got a result for $G_{m,n}$.

Theorem 3.9. [8] *For any integers $m, n \geq 2$, we have $\text{dem}(P_m \square P_n) = \max\{m, n\}$.*

From Theorem 3.9, we can directly obtain the following.

Corollary 3.10. *For any integers $m, n \geq 2$, we have $\text{dem}(T_m \square T_n) = \max\{m, n\}$.*

Now, let us consider the Cartesian product of the tree T_m and the circle C_n . For any integers $m \geq 2, n \geq 3$, the Cartesian product of T_m and C_n is denoted by $T_m \square C_n$. Next, we study $\text{dem}(T_m \square C_n)$.

Theorem 3.11. *Let two integers $m \geq 2, n \geq 3$. Then*

$$\text{dem}(T_m \square C_n) = \begin{cases} n & \text{if } n \geq 2m + 1, \\ 2m & \text{if } n \leq 2m. \end{cases}$$

Proof. In $T_m \square C_n$, there exist m disjoint copies of C_n , and n disjoint copies of T_m . Note that each cycle C_n needs 2 monitoring vertices, and each path T_m needs one monitoring vertex. Hence, for any two vertices $(u_i, v_j), (u_i, v'_j)$ of $T_m \square C_n$, where v_j and v'_j are not adjacent in $C_n(u_i)$, we have $EM((u_i, v_j)) \cup EM((u_i, v'_j)) = E(T_m(v_j)) \cup E(T_m(v'_j)) \cup E(C_n(u_i))$.

If $n \geq 2m + 1$, then any distance-edge-monitoring set M contains not only two vertices of each copy of C_n , but also a vertex of each copy of T_m . Hence $|M| \geq n$, otherwise, there is an edge of some copy of T_m which cannot be monitored by M . On the other hand, let

$M = \{(u_i, v_i), (u_i, v_{i+m}) \mid 1 \leq i \leq m\} \cup \{(u_m, v_j) \mid 2m+1 \leq j \leq n\}$. Obviously, any edge $e \in E(T_m \square C_n)$ can be monitored by M , so $\text{dem}(T_m \square C_n) \leq n$, and hence $\text{dem}(T_m \square C_n) = n$.

If $n \leq 2m$, we have $|M| \geq 2m$, otherwise, there is an edge of some copy of C_n cannot be monitored by M . Next we want to show $\text{dem}(T_m \square C_n) \leq 2m$. Let

$$M = \{(u_i, v_i), (u_i, v_{i+\lfloor n/2 \rfloor}) \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\} \cup \{(u_i, v_n), (u_i, v_{n-2}) \mid \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq m\}.$$

Obviously, any edge $e \in E(T_m \square C_n)$ can be monitored by M . Hence $\text{dem}(T_m \square C_n) = 2m$. \square

According to Theorem 3.11, the following result is obvious.

Corollary 3.12. *Let m and n be two integers with $m \geq 3$ and $n \geq 2$. Then*

$$\text{dem}(P_m \square C_n) = \begin{cases} n & \text{if } n \geq 2m+1, \\ 2m & \text{if } n < 2m+1. \end{cases}$$

Theorem 3.13. *Let m and n be two integers with $m \geq 3$ and $n \geq 2$. If $\text{dem}(C_m \square P_n) = 2n$, then the following conditions hold.*

(1) *For any vertex $x \in V(C_m)$, there exists a distance-edge-monitoring set M_{C_m} with $x \in M_{C_m}$.*

(2) *P_n has at least k distance-edge-monitoring sets, say $M_{P_n}^j$ ($j = 1, 2, \dots, k$), such that $M_{P_n}^a \cap M_{P_n}^b = \emptyset$ ($1 \leq a, b \leq k$, $a \neq b$) for any two sets $M_{P_n}^a$ and $M_{P_n}^b$, where $k = \min\{s \mid \bigcup_{i=1}^s M_{C_m}^i = V(C_m)\} \geq 2$.*

Proof. It is clear, that for any two non-adjacent vertices $u_i, u_j \in V(C_m)$, $\{u_i, u_j\}$ is a distance-edge-monitoring set of C_m . So (1) holds. Then there exist some distance-edge-monitoring sets, say $M_{C_m}^1, M_{C_m}^2, \dots, M_{C_m}^s$, such that $\bigcup_{i=1}^s M_{C_m}^i = V(C_m)$. If $i+2 > m$, then let $u_{i+2} = u_j$, where $i+2 \equiv j \pmod{m}$. Let $M_{C_m}^i = \{u_i, u_{i+2}\}$. Then

$$k = \min\{s \mid \bigcup_{i=1}^s M_{C_m}^i = V(C_m)\} = \left\lceil \frac{m}{2} \right\rceil.$$

It is clear that for any vertex $v_j \in P_n$, $\{v_j\}$ is a distance-edge-monitoring set of P_n . Therefore, P_n has n distance-edge-monitoring sets, and for any two distance-edge-monitoring sets $M_{P_n}^{j_1}$ and $M_{P_n}^{j_2}$, we have $M_{P_n}^{j_1} \cap M_{P_n}^{j_2} = \emptyset$. It follows from Corollary 3.12, that $m \leq 2n$. So $k = \lceil \frac{m}{2} \rceil \leq n$, and hence (2) holds. \square

Next we study the Cartesian product between the circles C_m and C_n .

Theorem 3.14. *For two integers $m, n \geq 3$, we have*

$$\text{dem}(C_m \square C_n) = \max\{2m, 2n\}.$$

Proof. Without loss of generality, suppose that $n \geq m$. It suffices to prove that $\text{dem}(C_m \square C_n) = 2n$. Suppose that M' is a distance-edge-monitoring set of $C_m \square C_n$ with $|M'| = 2n - 1$. Then there exists a subgraph $C_m(v_a)$ such that $|M' \cap V(C_m(v_a))| = 1$, and so there is at least an edge $e \in E(C_m(v_a))$, which cannot be monitored by M' , a contradiction. Hence $\text{dem}(C_m \square C_n) \geq 2n$.

Next, we need to show that $\text{dem}(C_m \square C_n) \leq 2n$. If $i + 2 > m$, then let $u_{i+2} = u_j$, where $i + 2 \equiv j \pmod{m}$. Let $M_i = \{(u_i, v_i), (u_{i+2}, v_i)\}$, then $\bigcup_{i=1}^m M_i$ can monitor all the edges of $\bigcup_{i=1}^m (E(C_m(v_i)) \cup E(C_n(u_i)))$. So let

$$M = \left(\bigcup_{i=1}^m M_i \right) \cup \left(\bigcup_{j=m+1}^n \{(u_1, v_j), (u_3, v_j)\} \right),$$

then for any edge $e \in E(C_m \square C_n)$ can be monitored by M , and $|M| = 2n$. So $\text{dem}(C_m \square C_n) \leq 2n$, and hence $\text{dem}(C_m \square C_n) = 2n$. The case $n \leq m$ can be also proved similarly. \square

Finally, we determine the Cartesian product between two complete graphs K_m and K_n .

Theorem 3.15. *Let m and n be two integers with $m, n \geq 3$. Then*

$$\text{dem}(K_m \square K_n) = mn - \min\{m, n\}.$$

Proof. Denote the vertex set of $K_m \square K_n$ by

$$V(K_m \square K_n) = \{(u_i, v_j) | u_i \in V(K_m), v_j \in V(K_n)\}.$$

Suppose that $m \geq n$. It suffices to show that $\text{dem}(K_m \square K_n) = mn - n$. Let M_i be the distance-edge-monitoring set of $K_m(u_i)$ ($i = 1, 2, \dots, n$), and $M_i = V(K_m(u_i)) - (u_i, v_i)$. It is clear that $M = \bigcup_{i=1}^n M_i$ is a distance-edge-monitoring set of $\text{dem}(K_m \square K_n)$. Moreover, $|M| = mn - n$, and hence $\text{dem}(K_m \square K_n) \leq mn - n$. Next we need to show that $\text{dem}(K_m \square K_n) \geq mn - n$. Since the vertices of each $K_m(u_i)$ cannot monitor the edges of $K_m(u_i)$ ($j = 1, 2, \dots, n$ and $j \neq i$), and each $K_m(u_i)$ needs at least $n - 1$ vertices to monitor all the edges of $K_m(u_i)$, it follows that n complete graphs $K_m(u_i)$ need at least $mn - n$ vertices to monitor all the edges. We have that $\text{dem}(K_m \square K_n) \geq mn - n$, and hence $\text{dem}(K_m \square K_n) = mn - n$. Similarly, if $m \leq n$, then $\text{dem}(K_m \square K_n) = mn - m$. The proof is complete. \square

4 Conclusion

In this paper, we have continued the study of *distance-edge-monitoring sets*, a new graph parameter recently introduced by Foucaud *et al.* [8], which is useful in the area of network monitoring. In particular, we have studied the distance-edge-monitoring numbers of join,

corona, cluster, and Cartesian products, and obtained the exact values of some specific networks.

For future work, it would be interesting to study distance-edge monitoring sets in further standard graph classes, including pyramids, Sierpiński-type graphs, circulant graphs, or line graphs. In addition, characterizing the graphs with $dem(G) = n - 2$ would be of interest, as well as clarifying further the relation of the parameter $dem(G)$ to other standard graph parameters, such as arboricity, vertex cover number and feedback edge set number.

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