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F. Cornu[1], H.J. Hilhorst[2] and M. Bauer[3]

[1,2] Université Paris-Saclay, CNRS

[1] Laboratoire de Physique Théorique et Modèles Statistiques, 91405 Orsay, France

[2] Laboratoire Irène Joliot-Curie, 91405 Orsay, France

[3] Université Paris-Saclay, CNRS, CEA

Institut de Physique Théorique, 91191 Gif-sur-Yvette, France

and PSL Research University, CNRS

École normale supérieure, Département de mathématiques et applications, 75005 Paris, France

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Abstract

We construct a new duality for two-dimensional Discrete Gaussian (DG) models. It is based on a known one-dimensional duality and on a mapping, implied by the Chinese remainder theorem, between the sites of an $N \times M$ torus and those of a ring of NM sites. The duality holds for an arbitrary translation invariant interaction potential $v(\mathbf{r})$ between the height variables on the torus. It leads to pairs $(v, \widetilde{v})$ of mutually dual potentials and to a temperature inversion according to $\tilde{\beta} = \pi^2/\beta$. When the potential $v(\mathbf{r})$ is isotropic, $\widetilde{v}$ inherits an anisotropy from the mapping. This is the case for the potential $\widetilde{v}$ that is dual to an isotropic nearest-neighbor potential v. In the thermodynamic limit the latter dual potential is shown to decay with distance according to an inverse square law with a simple angular dependence. There is a single self-dual pair of potentials $(v^\star, v^\star)$. At the self-dual temperature $\beta^\star = \widetilde{\beta^\star} = \pi$ the height-height correlation can be calculated explicitly; it is anisotropic and diverges logarithmically with distance.

Key words : SoS Discrete Gaussian model, Chinese remainder theorem, two-dimensional duality.

1 Introduction

The Discrete Gaussian (DG) model is a particular lattice model belonging to the class of the so-called Solid-on-Solid (SoS) models which aim to describe the fluctuations of a crystal surface. The most usual versions of SoS models are two-dimensional. In such a model a surface is described as a collection of integer valued height variables $\{h_{\mathbf{i}}\}$ associated with the sites $\mathbf{i}$ of a two-dimensional (2D) lattice. The interaction between two height variables $h_{\mathbf{i}}$ and $h_{\mathbf{i}'}$ is some function of their difference $|h_{\mathbf{i}} - h_{\mathbf{i}'}|$, which in the case of the DG model is a simple quadratic form. In the case where $v(\mathbf{r})$ represents an isotropic nearest neighbor coupling, the DG model is dual to the XY model in its Villain version [1], and therefore it undergoes a phase transition in the Kosterliz-Thouless universality class. This phase transition has been the main motivation for the interest in this short-ranged two-dimensional DG model.

The DG Hamiltonians of interest to us in this work take the form

$$\mathcal{H} = \frac{1}{2} \sum_{\mathbf{i}} \sum_{\mathbf{i}'} v(\mathbf{i} - \mathbf{i}')(h_{\mathbf{i}} - h_{\mathbf{i}'})^2, \tag{1}$$

where the coupling constants $v(\mathbf{r})$ constitute a pair potential that depends only on the difference of coordinates between two sites. We may impose without loss of generality the symmetry $v(\mathbf{r}) = v(-\mathbf{r})$ under parity transformation. We consider a toroidal lattice of $N \times M$ sites, which for $M = 1$ includes also the one-dimensional (1D) case. The partition function associated with Hamiltonian (1) reads $\sum'_{\{h_{\mathbf{i}}\}} \exp[-\beta\mathcal{H}]$ in which the $h_{\mathbf{i}}$ are summed over all integer values except for the condition, indicated by the prime on the summation sign, that one height, say $h_{\mathbf{i}_0}$, should be kept fixed, say $h_{\mathbf{i}_0} = 0$. This "global gauge" condition eliminates a trivial infinite factor in the partition function, which is due to $\mathcal{H}$ being invariant under the global translation $h_{\mathbf{i}} \mapsto h_{\mathbf{i}} + h$ of all height variables. After this trivial factor has been removed, it still remains to impose some positivity condition on $v(\mathbf{r})$ ensuring that (1) defines a positive quadratic form.

The one-dimensional DG model with arbitrary interaction potential v at inverse temperature β was studied by Kjaer and Hilhorst (KH) [2], who found that it is dual to another such model but with a dual potential $\widetilde{v}$ and a dual inverse temperature $\widetilde{\beta} = \pi^2/\beta$. Whereas in general $v \neq \widetilde{v}$, there is a unique, and explicitly known, self-dual potential $v^\star$ for which $v^\star = \widetilde{v^\star}$. The temperature $\beta^\star$ such that $\beta^\star = \widetilde{\beta^\star}$ is therefore a candidate for a critical temperature of this model. In the thermodynamic limit $N \to \infty$ the self-dual potential $v^\star(\mathbf{r})$ behaves as $1/[\pi(r^2 - \frac{1}{4})]$.

In this work we combine the one-dimensional KH results with a mapping between one- and two-dimensional lattices that occurs in number theory in the context of the Chinese remainder theorem for two coprime integers N

and M. Geometrically one may represent the one-dimensional ring lattice as corresponding to a helix wound around the two-dimensional torus in such a way that the helix returns to its origin after having passed through all sites on the torus.

The result is a new duality relating the *two*-dimensional DG model with arbitrary potential v to another such model but with a different potential $\widetilde{v}$ and with inverse temperature $\tilde{\beta} = \pi^2/\beta$. More precisely the partition function on the torus is shown to obey the duality

$$Z_{N,M}\left[\beta v\right] = \frac{1}{\beta^{(MN-1)/2}} c_{N,M}[v]\; Z_{N,M}\left[\frac{\pi^2}{\beta}\widetilde{v}\right], \tag{2}$$

where the constant $c_{N,M}[v]$, which is a functional of v, and the relation between the Fourier transforms of the potentials $v(r,s)$ and $\widetilde{v}(r,s)$ are given in section 4.2. Again, there is a self-dual potential $v^\star$ and a candidate critical temperature.

This paper is organized as follows. In section 2 we establish our notation for the 1D and 2D Discrete Gaussian models, and we recall the results about the duality on the ring. In section 3, by using the Chinese remainder theorem we introduce and discuss the mapping between a one-dimensional and a two-dimensional lattice and the corresponding transformation of periodic functions. In section 4 we show how for the two-dimensional DG Hamiltonian this mapping leads to a duality relation. We discuss several more technical aspects of this mapping. In section 5 we consider the special case of the self-dual potential $v^\star$. In section 6 we consider the well-known 2D DG Hamiltonian with isotropic nearest-neighbor interaction. In section 7 we point out the main features of the new duality.

2 Discrete Gaussian models

In this section we establish some notations and review some results on the one-dimensional DG model that will be fundamental in the following sections. In view of the discussion in the next sections the length of the ring is denoted as $\mathcal{N}$, the relative distance as R, and the potential as $V(R)$.

2.1 DG model on a ring

For a ring of length $\mathcal{N}$ we will write $\mathcal{H}_1$ for the DG Hamiltonian (1). The lattice site $\mathbf{i}$ becomes a scalar i to which we assign the values $i = 0, 1, \ldots, \mathcal{N}-1$. In a slightly more formal notation we then have

$$\mathcal{H}_1 = \frac{1}{2}\sum_{i\in\mathbb{Z}_\mathcal{N}}\sum_{i'\in\mathbb{Z}_\mathcal{N}} V(i-i')(h_i - h_{i'})^2, \tag{3}$$

where $i \in \mathbb{Z}_{\mathcal{N}}$ is the equivalence class of all integers equal to i up to a multiple of $\mathcal{N}$. Symmetry of the interaction under parity is expressed as

$$V(R) = V(-R). \tag{4}$$

Since the labels i and $i + \mathcal{N}$ refer to the same site, the potential $V(R)$ must be $\mathcal{N}$-periodic,

$$V(R) = V(R + \mathcal{N}). \tag{5}$$

The two equations (4) and (5) together imply the reflection symmetry

$$V(R) = V(\mathcal{N} - R). \tag{6}$$

We define the partition function with the same gauge as that used by KH [2], namely

$$Z_{\mathcal{N}}\,[\beta V] = \sum_{\{h_i | i \neq 0\}}{}' \exp[-\beta \mathcal{H}_1], \tag{7}$$

where the prime indicates the constraint $h_0 = 0$. The latter restriction entails that the mean height at any site i vanishes at any temperature.

We observe that the Hamiltonian (3) is independent of the value of $V(0)$. In the Fourier transforms below we will consider that $V(0)$ has been assigned an arbitrary value, knowing that the results cannot depend on it.

The Fourier transforms of the variables are defined as

$$\widehat{h}_K = \frac{1}{\sqrt{\mathcal{N}}} \sum_{j \in \mathbb{Z}_{\mathcal{N}}} \mathrm{e}^{-\mathrm{i}Kj}\, h_j\,, \tag{8}$$

and the Fourier transform of the potential is

$$\widehat{V}(K) = \sum_{R \in \mathbb{Z}_{\mathcal{N}}} \mathrm{e}^{-\mathrm{i}KR} V(R), \tag{9}$$

where $K = 2\pi p/\mathcal{N}$ with $p \in \mathbb{Z}_{\mathcal{N}}$. Then the Hamiltonian (3) takes the form

$$\mathcal{H}_1[V] = \sum_{K \neq 0} W(K) \widehat{h}_K \widehat{h}_{-K}, \tag{10}$$

in which

$$W(K) \equiv \sum_{R=1}^{\mathcal{N}-1} [1 - \cos(KR)]\, V(R) = \widehat{V}(0) - \widehat{V}(K), \quad K \neq 0. \tag{11}$$

The last equality in (11) comes from the symmetry (6). Equation (10) also shows, incidentally, that the conditions for the partition function (7) to exist is that $W(K) > 0$ for all $K \neq 0$.

Equation (9) is the usual Fourier transform in the space of $\mathcal{N}$ points with inverse

$$V(R) = \frac{1}{\mathcal{N}} \sum_K \mathrm{e}^{\mathrm{i}KR} \widehat{V}(K), \qquad R = 0, 1, \ldots, \mathcal{N}-1. \tag{12}$$

Equation (11) transforms only the subset $\{V(1), V(2), \ldots, V(\mathcal{N}-1)\}$ which excludes the nonphysical variable $V(0)$, and its inverse is

$$V(R) = -\frac{1}{\mathcal{N}} \sum_{K \neq 0} \mathrm{e}^{\mathrm{i}KR} W(K), \qquad R = 1, 2, \ldots \mathcal{N}-1. \tag{13}$$

The symmetries (4)-(6) lead for $\widehat{V}(K)$ to the corresponding symmetries

$$\widehat{V}(-K) = \widehat{V}(K), \quad \widehat{V}(K+2\pi) = \widehat{V}(K), \quad \widehat{V}(2\pi - K) = \widehat{V}(K), \tag{14}$$

of which only two are independent. Clearly $W(K)$ satisfies the same symmetries as $\widehat{V}(K)$.

2.2 Duality in one dimension

It was shown in [2] that the one-dimensional DG model with arbitrary potential $V(R)$ obeying the symmetries (4)-(6) and for which the partition function exists is dual to a similar one-dimensional DG model with a potential $\widetilde{V}(R)$. In particular, the partition functions of the two models are related by

$$Z_{\mathcal{N}}[\beta V] = \frac{1}{\beta^{(\mathcal{N}-1)/2}} C_{\mathcal{N}}[V] \; Z_{\mathcal{N}}\left[\frac{\pi^2}{\beta}\widetilde{V}\right], \tag{15}$$

where

$$C_{\mathcal{N}}[V] = \sqrt{\mathcal{N}} \sqrt{\prod_{K \neq 0} \frac{\pi}{W(K)}}, \tag{16}$$

with $K = 2\pi p/\mathcal{N}$, $p \in \mathbb{Z}_{\mathcal{N}}$, and $W(K)$ is defined in (11).[1] The relation between $V(R)$ and $\widetilde{V}(R)$ takes its simplest form in terms of $W(K)$ and $tW(K)$, namely

$$\widetilde{W}(K) = \frac{4\sin^2 \frac{1}{2}K}{W(K)}, \qquad K \neq 0. \tag{17}$$

When expressed in terms of $\widehat{V}(K)$ and $\widehat{\widetilde{V}}(K)$ this relation becomes

$$\widehat{\widetilde{V}}(0) - \widehat{\widetilde{V}}(K) = \frac{2[1-\cos K]}{\widehat{V}(0) - \widehat{V}(K)}, \qquad K \neq 0, \tag{18}$$

[1] The partition function depends on $\tilde{\beta}\widetilde{V}$ and in [2] the normalizations of the potentiel $\widetilde{V}$ and the inverse temperature $\tilde{\beta}$ are such that $\tilde{\beta} = 1/\beta$.

where $\widehat{\widetilde{V}}(0)$ is arbitrary since $\widetilde{V}(0)$ is arbitrary according to the structure (3) of $\mathcal{H}_1$. The real-space expression $\widetilde{V}(R)$ of the dual potential may be obtained by inverse Fourier transformation of (17) according to (13) with the result

$$\widetilde{V}(R) = -\frac{1}{\mathcal{N}} \sum_{K \neq 0} \mathrm{e}^{\mathrm{i}KR} \frac{4 \sin^2 \frac{1}{2} K}{W(K)}, \qquad R = 1, 2, \ldots \mathcal{N} - 1. \tag{19}$$

In other words

$$\widetilde{V}(R) = -\frac{1}{\mathcal{N}} \sum_{K \neq 0} \mathrm{e}^{\mathrm{i}KR} \frac{2[1 - \cos K]}{\widehat{V}(0) - \widehat{V}(K)}, \qquad R = 1, \ldots, \mathcal{N} - 1. \tag{20}$$

We recall that $\widehat{\widetilde{V}}(0)$ is arbitrary. Furthermore, neither $W(K)$ nor $\widetilde{W}(K)$ appears in the transformations with argument $K = 0$.

We also notice that, according to (14) and (18), $\widehat{\widetilde{V}}(K)$ obeys the same reflection symmetry as $\widehat{V}(K)$, namely

$$\widehat{\widetilde{V}}(2\pi - K) = \widehat{\widetilde{V}}(K). \tag{21}$$

As a consequence $\widetilde{V}(R)$ obeys the same reflection symmetry as $V(R)$,

$$\widetilde{V}(R) = \widetilde{V}(\mathcal{N} - R). \tag{22}$$

For future discussion we introduce another potential $U(R)$, which appears in the derivation presented in [2] (but with another normalization), and such that $U(0) = 0$. In the following the potential U will be sometimes called a "charge potential" because it arises through the well-known correspondence between a DG model and a lattice model whose Hamiltonian reads $\frac{1}{2} \sum_{i \in \mathbb{Z}_\mathcal{N}} \sum_{i' \in \mathbb{Z}_\mathcal{N}} U(i - i') q_i q_{i'}$ with the constraint that the configurations of integer charges q_i are neutral, $\sum_i q_i = 0$. Without resorting to the latter derivation the potential $U(R)$ can be introduced as follows. In the expression (20), the factor $\mathrm{e}^{\mathrm{i}KR}$ can be replaced by $\cos KR$ by virtue of (21), and one rewrites $2[1 - \cos K] \cos KR = 2 \cos KR - \cos K(R+1) - \cos K(R-1)$. Then the dual potential $\widetilde{V}(R)$ reads

$$\widetilde{V}(R) = U(R + 1) + U(R - 1) - 2U(R), \tag{23}$$

where, if we choose to set $U(0) = 0$,

$$U(R) = \frac{1}{MN} \sum_{K \neq 0} [\cos(KR) - 1] \frac{1}{\widehat{V}(0) - \widehat{V}(K)} \tag{24}$$

with $\widehat{V}(0) - \widehat{V}(K) > 0$ for $K \neq 0$. According to the relation (23) the dual potential $\widetilde{V}(R)$ may be interpreted as a potential created by a (neutral)

quadrupolar charge configuration where charges 1 ,-2, and 1 sit on the sites $R=-1$, $R=0$, and $R=1$, respectively. Therefore $\widetilde{V}(R)$ will be sometimes referred to as a "quadrupolar U-charge interaction". In the case where $V(R)$ is a nearest neighbor interaction the "U-charge" interaction is the so-called Coulomb interaction.

2.3 DG model on a torus

In the special case of an $N \times M$ lattice with toroidal boundary conditions we will write the DG Hamiltonian (1) as $\mathcal{H}_2$. Sites will be labeled by $\mathbf{i} = (i, j)$, where $i = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots, M-1$. The Hamiltonian (1) reads

$$\mathcal{H}_2 = \frac{1}{2} \sum_{(i,j)\in\mathbb{Z}_N\times\mathbb{Z}_M} \sum_{(i',j')\in\mathbb{Z}_N\times\mathbb{Z}_M} v(i-i', j-j')(h_{i,j} - h_{i',j'})^2. \tag{25}$$

Parity symmetry is now expressed as

$$v(r, s) = v(-r, -s). \tag{26}$$

Since the labels (i, j), $(i + N, j)$, and $(i, j + M)$ refer to the same site, the potential v must have the periodicity properties

$$v(r + N, s) = v(r, s), \qquad v(r, s + M) = v(r, s). \tag{27}$$

As a consequence of (26) and (27) we have the reflection symmetry

$$v(N - r, M - s) = v(r, s). \tag{28}$$

For $M = 1$ this system reduces to the ring model described above.

Again, the Hamiltonian (25) is independent of the value of the interaction constant $v(0, 0)$. The partition function is defined as in (7)

$$Z_{N,M}[\beta v] = \sum_{\{h_{i,j}\,|(i,j)\neq(0,0)\}}{}' \exp[-\beta\mathcal{H}_2], \tag{29}$$

where v is now the two-dimensional potential $v(r, s)$ and the prime denotes the gauge condition $h_{0,0} = 0$,

We will consider in this work only functions on the ring and on the torus that have the symmetry properties (4)-(6) and (26)-(28), respectively.

3 Mapping between a torus and a ring

In this section we show how, under the condition that N and M are coprime, the Chinese remainder theorem allows us to introduce a mapping between the ring $\mathbb{Z}_{NM}$ and the torus $\mathbb{Z}_N \times \mathbb{Z}_M$ for both coordinates and periodic functions. For the Chinese remainder theorem at an elementary level see [5] and for more advanced topics see [6].

3.1 Mapping for spatial coordinates

3.1.1 Chinese remainder theorem

For any bijection of the sites (r, s) of the torus $\mathbb{Z}_N \times \mathbb{Z}_M$ onto the integers R of the ring $\mathbb{Z}_{NM}$ the Hamiltonian (25) becomes formally a one-dimensional Hamiltonian. We wish, however, to apply a bijection that transfers the properties of the group structure (translational invariance,...) from the torus to the ring. The Chinese remainder theorem provides such a bijection at the condition that N and M be coprime, that is, when their only positive common divisor is unity. We will henceforth take M and N such that this condition is met.

In the case of two integers $N, M > 1$ the Chinese remainder theorem can be stated as follows. For any given pair of integers (r, s) the set of equations with unknown R,

$$r = R \pmod{N}, \qquad s = R \pmod{M}, \tag{30}$$

where $r = R \pmod{N}$ means that r and R differ by a multiple of N, has a solution given by

$$R = aNs + bMr \pmod{NM}, \tag{31a}$$

in which the pair of integer *Bézout coefficients* (a, b) is, in turn, a solution of

$$aN + bM = 1. \tag{31b}$$

Bézout's theorem guarantees that there exists such a pair (a, b), which can be found by the so-called extended Euclidean algorithm. The linear combination in (31a) is readily shown to satisfy the set of equations (30) as follows. By construction $aNs + bMr = bMr \pmod{N}$; then according to the identity (31b), bMr can be rewritten as $r - aNr$ and $r - aNr = r \pmod{N}$. As a result $aNs+bMr = r \pmod{N}$. A similar argument leads to $aNs+bMr = s \pmod{M}$.

We notice that from the definition (31b) of the Bézout coefficients it immediately follows that any other pair of the form $(a + cM, b - cN)$, where c is any integer, is also a solution. We may make the solution of (31b) unique by imposing, for example, that $0 < a < M$ and $-N < b < 0$, or, alternatively, that $-M < \overline{a} < 0$ and $0 < \overline{b} < N$. With the constraint $0 < a < M$ and $-N < b < 0$, we give the solutions for various special cases of N and M. For $N > 1$ and $M = 2$, the solution is $a = 1$ and $b = (1 - N)/2$; for $N > 2$ and $M = N - 1$ one gets $a = 1$ and $b = -1$; and for $M > 1$ and $N = qM + 1$, where $q = 2, 3, \ldots$, one finds $a = 1$ and $b = -q$.

3.1.2 Geometrical interpretation

The Chinese remainder theorem can be interpreted as a helicoidal mapping of a one-dimensional path around the $N \times M$ torus as follows. We let R

take the successive values $R = 0, 1, 2, \ldots, NM$ and consider the path traced out on the torus by the pair (r, s) parametrized by R according to equation (30). At $R = 0$ the path starts in the origin $(r, s) = (0, 0)$, and as long as $R < \min(N, M)$ we have $(r, s) = (R, R)$, that is, the path follows the main diagonal, undergoing at each step an increment $(1, 1)$. For larger R the path continues to undergo increments $(1, 1)$, but the N, M-periodicity of the lattice has to be taken into account. This leads to a path that winds around the torus until it returns to the origin. The condition that N and M be coprime guarantees that this return will occur only after the path has visited all sites of the torus.

3.1.3 Another mapping and corresponding helicoidal wrapping

In next section (3.2) we will show that the functions $F(R) \equiv f(r, s)$ defined with the mapping (31a) are periodic with period MN. We notice that we could have chosen another helix which wraps around the torus while also preserving the periodicity of the lattice, so that the corresponding mapping $(r, s) \mapsto R'$ leads to the same periodicity NM for functions $F'(R') \equiv f(r, s)$.

For instance another proper helix is built by again mapping the origin $(r, s) = (0, 0)$ onto the integer $R = 0$ and then by incrementing the position on the $N \times M$ lattice by steps of $(-1, 1)$ while at the same time increasing R by one unit. After R' steps on the helix the corresponding coordinates on the torus are

$$\begin{cases} r = -R' \pmod N \\ s = R' \pmod M. \end{cases} \tag{32}$$

A simple argument similar to that presented for the derivation of the Chinese theorem (31) shows that the linear combination

$$R' = aNs - bMr \tag{33}$$

is a solution of (32), because a and b are the solutions of (31b). We will see in subsection (4.3) how the results of interest in the present paper depend on the choice of one among the two mappings $R = aNs + bMr$ or $R' = aNs - bMr$.

3.2 Mapping for periodic functions

3.2.1 Periods on the torus and on the ring

Let $f(r, s)$ be a biperiodic function on the $N \times M$ torus obeying the symmetry properties (26)-(28).

$$f(r + N, s) = f(r, s + M) = f(r, s) \tag{34}$$

and

$$f(N - r, M - s) = f(r, s). \tag{35}$$

We define a corresponding function $F(R)$ on the ring of length NM by

$$F(R) = f(r,s) \quad \text{with} \quad R = aNs + bMr \pmod{NM}. \tag{36}$$

We will show that the symmetries of f imply those of F and that the reciprocal is also true.

The pairs $(r+N, s)$ and $(r, s+M)$ are associated with $R + bNM$ and $R + aNM$, respectively. The periodicity properties (34) then lead to

$$F(R + bMN) = F(R + aMN) = F(R).$$

It follows that $F(R) = F(R + N(aNM) + M(bNM))$, whence, with the use of (31b), we find that

$$F(R + NM) = F(R). \tag{37}$$

Similarly, by virtue of (31b), the reflected point $(N-r, M-s)$ is associated with $aN(M-s) + bM(N-r) = NM - R$, and the symmetry property (35) leads to

$$F(NM - R) = F(R). \tag{38}$$

As a consequence we also have the third symmetry, $F(-R) = F(R)$; that is, F obeys the symmetries (26)-(28) for a ring of NM sites. We point out that similar arguments show that the periodicity (37) and the reflection symmetry (38) are also valid for the second mapping (33).

We notice that conversely, if $F(R + NM) = F(R)$ then $f(r+N, s) = f(r, s+M) = f(r,s)$. Indeed if $F(R+NM) = F(R)$ then $F(R + aNM) = F(R)$, and $R + aNM$ corresponds to $(r, s+N)$.

3.2.2 Mapping for Fourier transforms from the torus to the ring

The Fourier transform of an NM-periodic function $F(R)$ on the ring of length NM is

$$\widehat{F}(K) = \sum_{R \in \mathbb{Z}_{NM}} \mathrm{e}^{-\mathrm{i}KR} F(R), \tag{39}$$

with the wavenumbers

$$K = \frac{2\pi}{NM} p \tag{40}$$

where $p \in \mathbb{Z}_{NM}$. Similarly, the Fourier transform of an N, M-biperiodic function $f(r,s)$ on the torus is

$$\widehat{f}(k_1, k_2) = \sum_{r \in \mathbb{Z}_N} \sum_{s \in \mathbb{Z}_M} \mathrm{e}^{-\mathrm{i}(k_1 r + k_2 s)} f(r,s), \tag{41}$$

with the wavenumbers

$$k_1 = \frac{2\pi}{N} n, \qquad k_2 = \frac{2\pi}{M} m \tag{42}$$

where $n \in \mathbb{Z}_N$ and $m \in \mathbb{Z}_M$. We now investigate the relation between these two Fourier transforms that results in the case $F(R) = f(r, s)$.

The Chinese remainder theorem allows us to make a bijection between the index p of the wavenumber on the ring and the index pair (n, m) on the torus,

$$p = aNm + bMn \pmod{NM}, \tag{43}$$

which is analogous to $R = aNs + bMr \pmod{NM}$. Hence the wavenumber on the torus may be expressed in terms of those on the ring as

$$K = bk_1 + ak_2. \tag{44}$$

After some rewriting and use of identity (31b) we find

$$\mathrm{e}^{-\mathrm{i}KR} = \mathrm{e}^{-\mathrm{i}(bk_1 r + ak_2 s)}. \tag{45}$$

When we substitute (44) in (39) and identify $F(R) = f(r, s)$, we obtain

$$\widehat{F}(K) = \sum_{r \in \mathbb{Z}_N} \sum_{s \in \mathbb{Z}_M} \mathrm{e}^{-\mathrm{i}(bk_1 r + ak_2 s)} f(r, s) = \widehat{f}(bk_1, ak_2) \tag{46}$$

The Fourier transform $\widehat{F}(K)$ on the ring proves to coincide with a scaled Fourier transform on the torus. The relation (46) allows to determine the Fourier transform $\widehat{F}(K)$ on the ring when the Fourier transform $\widehat{f}(k_1, k_2)$ on the torus is given.

3.2.3 Mapping for Fourier transforms from the ring to the torus

We will now see how to determine the Fourier transform $\widehat{f}(k_1, k_2)$ on the torus when the Fourier transform $\widehat{F}(K)$ on the ring is given. If $\widehat{F}(K)$ is given we write $f(r, s) = F(R) = \frac{1}{MN} \sum_K \mathrm{e}^{\mathrm{i}KR} \widehat{F}(K)$, and from (45) and (46) we get

$$f(r, s) = \frac{1}{MN} \sum_{(k_1, k_2)} \mathrm{e}^{\mathrm{i}(bk_1 r + ak_2 s)} \widehat{F}(bk_1 + ak_2), \tag{47}$$

where (k_1, k_2) is related to a pair of integers (n, m) through (42).

We notice that, according to (31b), the coefficient b is coprime with N (because if b and N had a common divisor different from 1 or -1 then $aN + bM$ could not be equal to 1). By virtue of Gauss lemma, the fact that there exists no common divisor of b and N entails that if $b(n - n')$ is a multiple of N then $n - n'$ is also a multiple of N. Equivalently $n \neq n' \pmod{N}$ implies $bn \neq bn' \pmod{N}$ and $n \mapsto bn$ is a one-to-one correspondence from $\mathbb{Z}_N$ to $\mathbb{Z}_N$. Similarly, according to (31b), the coefficient a is coprime with M and $m \mapsto am$ is a one-to-one correspondence from $\mathbb{Z}_M$ to $\mathbb{Z}_M$. As a result if the

function $A(n,m)$ is N,M-periodic, nameley $A(n+N,M) = A(n,m+M) = A(n,m)$, then

$$\sum_{(n,m)\in\mathbb{Z}_N\times\mathbb{Z}_M} A(bn,am) = \sum_{(n,m)\in\mathbb{Z}_N\times\mathbb{Z}_M} A(n,m). \tag{48}$$

Hence the sum in (47) can be rewritten without the coefficients a and b, and eventually

$$f(r,s) = \frac{1}{MN}\sum_{(k_2,k_1)} \mathrm{e}^{\mathrm{i}(k_1 r + k_2 s)}\widehat{F}(k_1+k_2). \tag{49}$$

Upon Fourier transforming both members of this equation we find

$$\widehat{f}(k_1,k_2) = \widehat{F}(k_1+k_2), \tag{50}$$

which is the desired relation that yields $\widehat{f}(k_1,k_2)$ when $\widehat{F}(K)$ is given.

4 New duality for two-dimensional DG models

4.1 Mapping between torus and ring Hamiltonians

In the previous section we have defined a mapping between the sites of the torus and those of the ring, and an identification of functions defined on the torus with functions defined on the ring. Now we consider how a Hamiltonian given on the torus transforms into one defined on the ring.

Let the Hamiltonian $\mathcal{H}_2$ of equation (25) be given. A mapping between this model, defined on the torus $\mathbb{Z}_N \times \mathbb{Z}_M$, onto a model on the ring $\mathbb{Z}_{NM}$ is constructed as follows. We relabel the sites of the height variables $h_{r,s}$ according to (31)

$$h_R = h_{r,s}\,, \tag{51}$$

and the potential function $V(R)$ is defined through the same mapping,

$$V(R) = v(r,s). \tag{52}$$

According to section (3.2) the periodicity properties (27) and the reflection symmetry (28) of $v(r,s)$ entail that

$$V(R) = V(R+MN) \tag{53}$$

and

$$V(R) = V(MN-R). \tag{54}$$

When we express the two-dimensional DG Hamiltonian $\mathcal{H}_2$ defined in (25) with potential $v(r,s)$ and variables $h_{i,j}$ in terms of the new quantities $V(R)$

and h_R, we find that $\mathcal{H}_2$ becomes a one-dimensional DG Hamiltonian of type (3),

$$\mathcal{H}_1[V] = \frac{1}{2} \sum_{i \in \mathbb{Z}_{MN}} \sum_{R \in \mathbb{Z}_{MN}} V(R)(h_i - h_{i+R})^2, \tag{55}$$

and the partition functions of the two models are identical,

$$Z_{N,M}\left[\beta v\right] = Z_{NM}\left[\beta V\right]. \tag{56}$$

The mapping allows to identify the partition function on the torus with the partition function on the ring.

4.2 Duality on the torus

In the same way as the potential on the ring has been defined from the potential on the torus by $V(R) = v(r,s)$, a dual potential can be defined on the torus as

$$\widetilde{v}(r,s) = \widetilde{V}(R), \tag{57}$$

where $\widetilde{V}(R)$ is given in terms of the Fourier transform $\widehat{\widetilde{V}}(K)$ by (20). Because of (22) the dual potential also has the reflection property on the torus

$$\widetilde{v}(r,s) = \widetilde{v}(N-r, M-s). \tag{58}$$

The corresponding DG partition function is given by the identity (56),

$$Z_{N,M}\left[\beta\, \widetilde{v}\right] = Z_{NM}\left[\beta\, \widetilde{V}\right]. \tag{59}$$

The duality relation (15) between the partition functions on the ring can be rewritten as a two-dimensional duality

$$Z_{N,M}\left[\beta\, v\right] = \frac{1}{\beta^{(MN-1)/2}} C_{NM}[V]\ Z_{N,M}\left[\frac{\pi^2}{\beta}\, \widetilde{v}\right], \tag{60}$$

where the expression of $C_{NM}[V]$ is given in (16).

The constant $C_{NM}[V]$, which is a functional of the one-dimensional Fourier transform of V, can be re-expressed in terms of a constant $c_{N,M}[v]$ which is a functional of the two-dimensional Fourier transform of v. Indeed, the expression (16) for $C_{NM}[V]$ contains $W(K) = \widehat{V}(0) - \widehat{V}(K)$ and, according to (46), a Fourier transform on the ring is equal to a scaled Fourier transform on the torus. Therefore $C_{NM}[V]$ reads

$$C_{NM}[V] = \sqrt{NM} \left[\prod_{(n,m) \in \mathbb{Z}_N \times \mathbb{Z}_M \setminus (0,0)} \frac{\pi}{\widehat{v}(0,0) - \widehat{v}(2\pi bn/N, 2\pi am/M)} \right]^{1/2}. \tag{61}$$

With the aid of (48), the product can be rewritten without a or b, and

$$c_{N,M}[v] \equiv C_{NM}[V] = \sqrt{NM} \left[\prod_{(k_1,k_2)\setminus(0,0)} \frac{\pi}{\widehat{v}(0,0-\widehat{v}(k_1,k_2))} \right]^{1/2}, \qquad (62)$$

with the notation $k_1 = 2\pi n/N$ and $k_2 = 2\pi m/M$.

As in the one-dimensional case the expression of the dual potential $\widetilde{v}(r,s)$ can also be re-expressed in terms of the Fourier transform of $v(r,s)$ as follows. According to (50) the Fourier transform on the torus for $\widetilde{v}(r,s)$ is given in terms of the Fourier transform on the ring for $\widetilde{V}(R)$ by $\widehat{\widetilde{v}}(k_1,k_2) = \widehat{\widetilde{V}}(k_1+k_2)$, while the expression for $\widetilde{V}(K)$ in terms of $V(K)$ is given by (18). As a result $\widehat{\widetilde{v}}(k_1,k_2) - \widehat{\widetilde{v}}(0,0) = 2[1-\cos(k_1+k_2)]/[\widehat{V}(k_1+k_2) - \widehat{V}(0)]$. By using again the relation (50) to go back from the ring to the torus, namely $\widehat{V}(k_1+k_2) = \widehat{v}(k_1,k_2)$, we find that the Fourier transform of the dual potential on the torus takes the simple form

$$\widehat{\widetilde{v}}(k_1,k_2) - \widehat{\widetilde{v}}(0,0) = -\frac{2[1-\cos(k_1+k_2)]}{\widehat{v}(0,0) - \widehat{v}(k_1,k_2)}. \qquad (63)$$

Subsequently the expression of the dual potential $\widetilde{v}(r,s)$ in terms of $v(r,s)$ is given by the inverse Fourier transform on the torus, for $(r,s) \neq (0,0)$

$$\widetilde{v}(r,s) = -\frac{1}{MN} \sum_{(k_2,k_1)\in\mathbb{Z}_N\times\mathbb{Z}_M\setminus(0,0)} \mathrm{e}^{\mathrm{i}[k_1 r + k_2 s]} \frac{2[1-\cos(k_1+k_2)]}{\widehat{v}(0,0) - \widehat{v}(k_1,k_2)} \qquad (64)$$

with $\widehat{v}(0,0) - \widehat{v}(k_1,k_2) > 0$ for $(k_1,k_2) \neq (0,0)$.

Eventually the duality relation (60) between partition functions on the torus reads in terms of functions defined on the torus

$$Z_{N,M}[\beta v] = \frac{1}{\beta^{(NM-1)/2}} c_{N,M}[v]\ Z_{N,M}\left[\frac{\pi^2}{\beta}\widetilde{v}\right]. \qquad (65)$$

The constant $c_{N,M}[v]$ is a functional of v given in (62), while $\widetilde{v}(r,s)$ and its Fourier transform $\widehat{\widetilde{v}}(k_2,k_1)$ are given in terms of the Fourier transform of $v(r,s)$ by (64) and (63), respectively.

As in the one-dimensional case the square of the duality transformation is the identity. Indeed, the iteration of the duality relation (65) leads to $Z_{N,M}[\beta v] = (1/\pi)^{NM-1}\, c_{N,M}[v] c_{N,M}[\widetilde{v}]\, Z_{N,M}\left[\beta\widetilde{\widetilde{v}}\right]$, where $\widetilde{\widetilde{v}} = v$ according to (63), while the identity $\prod_{K\neq 0} 2|\sin(K/2)| = NM$ implies that $c_{N,M}[v] c_{N,M}[\widetilde{v}] = c_{NM}[V] c_{N,M}[\widetilde{V}] = \pi^{NM-1}$.

We notice that for a given mapping the expressions for the constant $c_{N,M}$ and the dual potential are independent from the Bézout coefficients (a,b)

according to (62) and (64). As a result we could have choosen the pair of Bézout coefficients $(\overline{a}, \overline{b})$ such that $-M < \overline{a} < 0$ and $0 < \overline{b} < N$ with the mapping $\overline{R} = \overline{a}Ns + \overline{b}Mr$ without changing the duality relation between the partition functions nor the expression of the dual potential $\widetilde{v}$ in terms of the potential v.

This achieves the purpose of establishing a duality relation for Hamiltonians on the torus.

4.3 Dependence of the dual potential upon the choice of the mapping

As noticed previously the constant $c_{N,M}$ as well as the relation between the Fourier transforms of the dual potentials on the torus prove to be independent of a and b for a given mapping. However the dependence upon the choice of the mapping can be exemplified by the comparison of the two mappings presented in section (3.1.2) and (3.1.3).

With the mapping $R = aNs + bMr$, the coordinates R and $R \pm 1$ correspond to (r, s) and $(r \pm 1, s \pm 1)$, respectively. Then the relation (23) between $\widetilde{V}(R)$ and $U(R)$ implies that $\widetilde{v}(r, s) = V(aNs + bMr)$ can be rewritten as

$$\widetilde{v}(r, s) = u(r + 1, s + 1) + u(r - 1, s - 1) - 2u(r, s), \tag{66}$$

where, by using (24),

$$u(r, s) = \frac{1}{MN} \sum_{(k_1,k_2)\neq(0,0)} [\cos(k_1 r + k_2 s)) - 1] \frac{1}{\widehat{v}(0, 0) - \widehat{v}(k_1, k_2)}. \tag{67}$$

(The latter relation can also be directly derived from the inverse Fourier transform representation (64) for $\widetilde{v}(r, s)$, as it was done to derive (23)-(24) from (20).) With the other mapping the coordinates $R' = aNs - bMr$ and $R' \pm 1$ correspond to (r, s) and $(r \mp 1, s \pm 1)$, respectively. Then the relation (23) on the ring implies that $\widetilde{v}(r, s) = V(aNs - bMr)$ can be rewritten as

$$\widetilde{v}'(r, s) = u(r + 1, s - 1) + u(r - 1, s + 1) - 2u(r, s), \tag{68}$$

with the same potential $u(r, s)$ as in the relation (66) for the first mapping.

With the terminology introduced after (24), in the case of the first mapping $\widetilde{v}(r, s)$ appears as a quadrupolar u-charge interaction, with charges $(1, -2, 1)$ aligned at points $(-1, -1)$, $(0, 0)$, and $(1, 1)$, respectively, along the direction of the first mapping helix. However, for the second mapping $\widetilde{v}'(r, s)$ still appears as a quadrupolar u-charge interaction with the same charge triplet, but the charges are located at different points, namely $(-1, 1)$, $(0, 0)$ and $(1, -1)$, respectively, along the direction of the second mapping helix at a given point.

The interaction u is definitely independent of the mapping by virtue of (67). However the above discussion shows that the dual potential $\widetilde{v}$ depends on the mapping since it is a quadrupolar u-charge interaction and the locations of the charges in the quadrupole depend on the mapping.

5 Self-duality

5.1 Self-dual potential

As shown in [2] the relation between the potential $V(R)$ and the potential $\widetilde{V}(R)$ on the ring, which is given by the relation (18) between their Fourier transforms, allows the existence of a self-dual potential such that for any $R \neq 0$

$$\widetilde{V^\star}(R) = V^\star(R). \tag{69}$$

Indeed, according to (18), if for $K \neq 0$

$$\widehat{V^\star}(K) - \widehat{V^\star}(0) = -2|\sin(K/2)|, \tag{70}$$

then $\widehat{\widetilde{V^\star}}(K) - \widehat{\widetilde{V^\star}}(0) = \widehat{V^\star}(K) - \widehat{V^\star}(0)$, namely $\widetilde{W^\star}(K) = W^\star(K) = 2|\sin(K/2)|$. The expression of $\widetilde{V}(R)$ for $R \neq 0$ is obtained by inserting the latter expression in (13). The potential $\widetilde{V}(R)$ is a periodic function of R with period MN which can be written in various forms. For the following discussion we write

$$V^\star(R) = \frac{\frac{1}{MN}\sin\frac{\pi}{MN}}{\sin^2(\frac{\pi}{MN}R) - \sin^2(\frac{\pi}{2MN})}. \tag{71}$$

For the corresponding self-dual potential on the torus, $v^\star(r,s) = V^\star(R)$ with $R = aNs + bMr$. Moreover $c_{N,M}[v^\star] = C_{NM}[V^\star] = \pi^{(NM-1)/2}$ according to the definition (16) and the identity $\prod_{K\neq 0} 2|\sin(K/2)| = NM$. Therefore the duality relation (65) for partition functions becomes

$$Z_{N,M}\left[\beta v^\star\right] = \left(\frac{\beta^\star}{\beta}\right)^{(NM-1)/2} Z_{N,M}\left[\frac{\beta^{\star 2}}{\beta} v^\star\right] \tag{72}$$

with $\beta^\star = \pi$.

5.2 Self-dual potential for large N

We now investigate some of the properties of this two-dimensional self-dual potential. We wish to consider its limit for a strip of infinite length and finite width, $N \to \infty$ with M fixed, and for an infinite lattice, $N \to \infty$ and $M \to \infty$. By virtue of (71) the explicit expression of $\widetilde{v}(r,s) = \widetilde{V}(R)$ is in

fact a function of $R/(NM)$. In order to study the large-N limit of $\widetilde{v}(r,s)$ it is convenient to make the change of variables $(r,s) \mapsto (r,t)$ with

$$t = s - r, \tag{73}$$

which, with the use of the identities (31), leads to the rewriting of R as $R = aNt + r$. Then $R/(NM)$ becomes

$$\frac{R}{MN} = \frac{at}{M} + \frac{r}{MN}, \tag{74}$$

and, according to (71), the self-dual potential $\widetilde{v}(r,s)$ becomes the function

$$v^\star(r, r+t) = \frac{\frac{1}{MN}\sin\frac{\pi}{MN}}{\sin^2\pi\left(\frac{at}{M} + \frac{r}{MN}\right) - \sin^2\left(\frac{\pi}{2MN}\right)}. \tag{75}$$

For relative coordinates (r,r) we have that $t = 0$ and

$$v^\star(r,r) = \frac{\frac{1}{MN}\frac{\pi}{MN}}{\sin^2\pi\left(\frac{r}{MN}\right) - \sin^2\left(\frac{\pi}{2MN}\right)}, \tag{76}$$

which depends only on NM. Therefore when N goes to infinity, and whether or not M remains finite, the latter expression gives

$$\lim_{N\to\infty} v^\star(r,r) = \frac{1}{\pi\left[r^2 - \frac{1}{4}\right]}. \tag{77}$$

For relative coordinates (r,s) with $s \neq r$ we have to distinguish between M remaining finite or tending to infinity, and we must know the Bézout coefficient a as a function of N and M. We will choose to take

$$N = qM + 1, \tag{78}$$

which ensures that N and M are coprime. In this case $a = 1$ and $b = -q$. Then, by virtue of (75), $v^\star(r, r+t)$ becomes a function of $\frac{t}{M} + \frac{r}{MN}$.

In order to study finite strips of width M we consider the scaling (78) with M fixed and $q \to \infty$. For $M = 2$ the torus is the ladder lattice, with each interchain bond counting twice, and for $M = 4$ a beam with a rectangular section. Then for r and $t \neq 0$ fixed, t/M remains finite while $r/(MN)$ vanishes. Upon inserting this limit behavior in equation (75) and restoring the original coordinates r and s we obtain

$$v^\star(r,s) \simeq \frac{\pi}{M^2N^2\sin^2\pi\left(\frac{r-s}{M}\right)}, \qquad N \to \infty,\ M \text{ fixed}. \tag{79}$$

The case of an infinite lattice is obtained when both M and N go to infinity with q fixed. For r and t fixed, r/MN and $1/(2MN)$ vanish faster than t/M, and the expression (75) tends to the limit

$$v^\star(r,s) \simeq \frac{1}{N^2\pi(r-s)^2}. \tag{80}$$

In all cases considered above

$$\lim_{N\to\infty} v^\star(r,s) = \frac{\delta_{r,s}}{\pi\left[r^2-\frac{1}{4}\right]}\,. \tag{81}$$

This says that in the limit $N \to \infty$ each height interacts only with the heights on the sites on the diagonal passing through it in the direction $(1,1)$, and we recover the large distance behavior of the potential (71) on the one-dimensional chain of length MN in the limit $NM \to \infty$.

5.3 Self-dual height-height correlation at $\beta^\star$

First we consider the DG model in either dimension 1 or 2. Let $h_{\mathbf{r}_i+\mathbf{r}} - h_{\mathbf{r}_i}$ be the difference between two height variables at sites $\mathbf{r}_i$ and $\mathbf{r}_i + \mathbf{r}$. The parity symmetry of the potential $v(\mathbf{r})$ in the Hamiltonian (1) entails that the mean height is the same at all sites ; moreover the height is fixed to the zero value at site $\mathbf{r} = 0$ by the gauge condition in the partition function in any dimension (see (7) or (29)). Therefore the average $\langle h_{\mathbf{r}_i}\rangle_\beta$ vanishes at any site $\mathbf{r}_i$ and the average height difference at any relative position $\mathbf{r}$ vanishes as well, $\langle h_{\mathbf{r}_i+\mathbf{r}} - h_{\mathbf{r}_i}\rangle_\beta = 0$. However the correlation

$$g(\mathbf{r};\beta) \equiv \langle (h_{\mathbf{r}_i+\mathbf{r}} - h_{\mathbf{r}_i})^2\rangle_\beta \tag{82}$$

is a non-vanishing function of $\mathbf{r}$ at inverse temperature β.

For the DG model on a ring it was shown in [2] that, though the correlation $G(R;\beta) \equiv \langle (h_{i+r} - h_i)^2\rangle_\beta$ is not known for a generic potential $V(R)$ at any inverse temperature β, the duality relation (15) for the partition functions entails that the correlation can be explicitly determined fn the case of the self-dual potential $V^\star(R)$ at the specific inverse temperature $\beta^\star = \pi$ defined after (72). It reads

$$G^\star(R;\beta^\star) = -\frac{1}{2\pi} U^\star(R), \tag{83}$$

where the superscript $\star$ of the correlation G signals that the statistical average is made for the Hamiltonian with the potential $V^\star(R)$, while $U^\star(R)$ is the potential associated with $V^\star(R)$ by (23) and which vanishes at $R = 0$. The relation (23) can be seen as a finite difference equation to be solved for R in the set $\{0, 1, \ldots, MN\}$ with the boundary condition $U^\star(0) = U^\star(NM) = 0$. By rewriting the expression (71) for $V^\star(R)$ as a difference of cotangents with arguments proportional to $R + 1$ and R, respectively, one gets that for $R = 0, 1, \ldots, NM$

$$U^\star(R) = -\frac{1}{NM}\sum_{R'=1}^{R} \cot\frac{\pi}{NM}\left(R' - \frac{1}{2}\right). \tag{84}$$

The expression for $U^\star(R)$ for $R = -NM, -NM+1, \ldots, -1, 0$ is obtained by using the periodicity property $U^\star(-|R|) = U^\star(NM - |R|)$ derived from (24) and rewriting the sum for $U^\star(NM - |R|)$ by taking into account the value $U^\star(NM) = 0$. The result is that for $R = -NM, -NM+1, , \ldots, NM$

$$U^\star(R) = -\frac{1}{NM} \sum_{R'=1}^{|R|} \cot \frac{\pi}{NM} \left(R' - \frac{1}{2} \right). \tag{85}$$

For the DG model on a torus an argument similar to that presented in [2] shows that, for the potential $v^\star(r,s) = V^\star(R)$ at the specific inverse temperature $\beta^\star$, the correlation $g^\star(r,s;\beta^\star) = \langle (h_{i+r,j+s} - h_{i,j})^2 \rangle^\star_{\beta^\star}$ can be determined as

$$g^\star(r,s;\beta^\star) = -\frac{1}{2\pi} u^\star(r,s). \tag{86}$$

Since for the model on the ring $U^\star(R)$ is known, the identity

$$u^\star(r,s) = U^\star(R) \tag{87}$$

allows to determine the explicit expression for $g^\star(r,s;\beta^\star)$ on the torus.

5.4 Corresponding correlation in the thermodynamic limit for $N = M + 1$

In the present section we consider the *thermodynamic limit* where $N = M + 1$ and N goes to infinity. Then M and N are coprime, $a = 1$, and $R = r + N(s - r)$. Before taking the limit we consider the variables r and s in intervals centered at $(0, 0)$. If, for instance, M is even, the intervals read

$$-\frac{M}{2} \leq r \leq \frac{M}{2} \qquad \text{and} \qquad -\frac{M}{2} < s \leq \frac{M}{2}. \tag{88}$$

5.4.1 Fixed relative coordinates

In the case $r = s$ fixed, $u^\star(r,r) = U^\star(r)$ where $U^\star(r)$ is the sum up to $|r|$ given in (85). In the thermodynamic limit the argument of every cotangent in the sum is at least of order $\mathcal{O}(1/N^2)$ so that we can replace $\cot x$ by $1/x$ and $u^\star(r,r)$ becomes

$$u^\star(r,r) = -\frac{1}{\pi} \sum_{r'=1}^{|r|} \frac{1}{r' - \frac{1}{2}} + \mathcal{O}\left(\frac{1}{N^4} \right). \tag{89}$$

Therefore the correlation given by (86) is a non-vanishing function in the thermodynamic limit for $r = s$. It is denoted as $g^\star_{\mathrm{Th}}(r,r;\beta^\star)$ and reads

$$g^\star_{\mathrm{Th}}(r,r;\beta^\star) = \frac{1}{2\pi^2} \sum_{r'=1}^{|r|} \frac{1}{r' - \frac{1}{2}}. \tag{90}$$

For large r it behaves as

$$g^{\star}_{\mathrm{Th}}(r,r;\beta^{\star}) = \frac{1}{2\pi^2}\left[\ln|r| + A_0 + \mathcal{O}\left(\frac{1}{r^2}\right)\right] \tag{91}$$

with $A_0 = C + 2\ln 2$ where C denotes the Euler constant.

In the case $r \neq s$ it is more convenient to make the change of variables $(r,s) \mapsto (r,t)$ with $s = r + t$ and to consider the function

$$g^{\star}(r, r+t;\beta^{\star}) = -\frac{1}{2\pi}U^{\star}(Nt + r). \tag{92}$$

The expression for $U^{\star}(Nt+r)$ is the sum given in (85) with $M = N-1$ and $|t| < N$ according to (88). When r and s are kept fixed, while N and $M = N-1$ become very large, t is fixed and $|r + Nt| \simeq N|t|$ with $N \leq N|t| \ll N^2$. Therefore the argument of every cotangent in the sum is at least of order $\mathcal{O}(1/N)$ and one can again replace $\cot x$ by $1/x$, while the upper bound of the sum is of order N. As a result in the thermodynamic limit the leading contribution in the correlation $g^{\star}(r, r+t;\beta^{\star})$ is the large distance behavior (91) of the expression (90) where the argument r is to be replaced by $Nt = N(s-r)$,

$$g^{\star}(r,s;\beta^{\star}) \underset{r\neq s}{=} \frac{1}{2\pi^2}\left[\ln(N|s-r|) + A_0 + o(1)\right], \tag{93}$$

where $o(1)$ denotes a contribution that vanishes in the limit $N \to \infty$. Equation (93) expresses that in the case $N = M+1$, according to (80), two height variables on parallel diagonals have an interaction whose coupling constant decreases with N so that the variance of their difference increases with N.

5.4.2 Relative coordinates scaled with the lattice size

Whereas in the preceding subsection we investigated the height-height correlation $g^{\star}(r,s;\beta^{\star}) = -(1/2\pi)U^{\star}(R)$ in the regime of fixed r and s with $N = M+1$ and $N \to \infty$, it is also interesting to study the nature of this correlation at the scale of the system, that is, for fixed values of

$$\xi = \frac{r}{N}, \qquad \eta = \frac{s}{N-1} \tag{94}$$

where, according to (88), $-\frac{1}{2} < \xi, \eta < \frac{1}{2}$ and $N \to \infty$. Then

$$R = (\eta - \xi)N(N-1) \pmod{N(N-1)} \tag{95}$$

with $-1 < \eta - \xi < 1$. According to (85)

$$\begin{aligned} U^{\star}\left((\eta-\xi)N(N-1)\right) &= -\frac{1}{N(N-1)}\sum_{R'=1}^{|\eta-\xi|N(N-1)} \cot\frac{\pi(R'-\frac{1}{2})}{N(N-1)}, \\ &\equiv \overline{U^{\star}}(\eta-\xi). \end{aligned} \tag{96}$$

Since only the argument $|\eta-\xi|$ is involved in the upper bound of the sum in (96), it suffices to calculate $\overline{U^\star}(\eta-\xi)$ with $0<\eta-\xi<1$. Moreover, according to the expression (24) for $U^\star(R)$ as an inverse Fourier transform, and as can be checked on its explicit R-dependence given in (85), $U^\star(R)$ has the symmetry $U^\star(N(N-1)-R)=U^\star(R)$. Therefore $\overline{U^\star}(\eta-\xi)$ takes the same value for $\eta-\xi$ and $1-(\eta-\xi)$ and we may further restrict ourselves to $0<\eta-\xi<1/2$, which we will do now.

With the present scaling, when $0<\eta-\xi<1/2$, the argument $\pi(R'-\frac{1}{2})/(N(N-1))$ of the cotangent increment in the sum runs up to values of order $\pi/2$ and for every R' all terms in the large-$N(N-1)$ expansion of the cotangent contribute. Therefore we will write $U^\star=I_0+I_1$, where I_0 and I_1 are the sums of the contributions of the first term and of all remaining terms, respectively, in the full expansion. This gives

$$\begin{aligned} I_0 &= -\frac{1}{\pi}\sum_{R'=1}^{(\eta-\xi)N(N-1)}\frac{1}{R'-\frac{1}{2}} \\ &= -\frac{1}{\pi}\Big[\ln\Big((\eta-\xi)N(N-1)\Big)+A_0\Big]+\mathcal{O}\big((N(N-1))^{-2}\big), \end{aligned} \tag{97}$$

where we have used (91), and

$$\begin{aligned} I_1 &= -\frac{1}{N(N-1)}\sum_{R'=1}^{(\eta-\xi)N(N-1)}\Big[\cot\frac{\pi(R'-\frac{1}{2})}{N(N-1)}-\frac{N(N-1)}{\pi(R'-\frac{1}{2})}\Big] \\ &= -\frac{1}{\pi}\int_0^{(\eta-\xi)\pi}\mathrm{d}u\,\Big[\cot u-\frac{1}{u}\Big]+o(1) \\ &= -\frac{1}{\pi}\ln\frac{\sin\big((\eta-\xi)\pi\big)}{(\eta-\xi)\pi}+o(1). \end{aligned} \tag{98}$$

We obtain $U^\star$ by adding (97) to (98). When doing so, a factor $\eta-\xi$ in the argument of the logarithm cancels against its inverse, so that the only dependence on $\eta-\xi$ occurs in through $\sin\big((\eta-\xi)\pi\big)$. We have assumed $0<\eta-\xi<\frac{1}{2}$, but as already noticed $\overline{U^\star}(\eta-\xi)=\overline{U^\star}(1-(\eta-\xi))=\overline{U^\star}(|\eta-\xi|)$. By using $\sin\pi(1-\alpha)=\sin\alpha\pi$ we arrive at the result

$$g^\star(\xi N,\eta(N-1);\beta^\star)=\frac{1}{2\pi^2}\Big[\ln\left(N(N-1)\frac{\sin\big(|\eta-\xi|\pi\big)|}{\pi}\right)+A_0+o(1)\Big], \tag{99}$$

valid for all $-1<\eta-\xi<1$, that is, for all $(\xi,\eta)\in[-\frac{1}{2},\frac{1}{2}]^2$. It so happens that if in (99) we put again $\xi=r/N$ and $\eta=s/(N-1)$, and expand the resulting expression in powers of N, now at r and s fixed, we obtain equation (93).

6 Two-dimensional DG model with nearest-neighbor interaction

In this section we consider the standard DG model with homogeneous isotropic nearest-neighbor interaction $v^{\rm nn}(r,s)$ on the $N \times M$ torus, that is,

$$v^{\rm nn}(r,s) = J\left[\left(\delta_{r,-1}+\delta_{r,1}\right)\delta_{s,0} + \delta_{r,0}\left(\delta_{s,-1}+\delta_{s,1}\right)\right]. \tag{100}$$

Its Fourier transform reads

$$\widehat{v^{\rm nn}}(k_1,k_2) = 2J\left[\cos k_1 + \cos k_2\right]. \tag{101}$$

6.1 Dual potential on the $N \times M$ torus

The Fourier transform of the corresponding dual potential is given by (63),

$$\widehat{\widetilde{v^{\rm nn}}}(k_1,k_2) = \frac{1}{J} \times \frac{\cos(k_1+k_2)-1}{2-\cos k_1 - \cos k_2}. \tag{102}$$

where we have set $\widehat{\widetilde{v^{\rm nn}}}(0,0) = 0$, since this value is arbitrary, as well as the value $\widetilde{v^{\rm nn}}(0,0)$ which does not appear in the DG Hamiltonian (25).

We notice that on a two-dimensional lattice the Laplacian of a function $f(r,s)$ is defined as

$$\Delta_2 f(r,s) = f(r+1,s)+f(r-1,s)+f(r,s+1)+f(r,s-1)-4f(r), \tag{103}$$

and its Fourier transform reads

$$\widehat{\Delta_2 f}(k_1,k_2) = -2\left[2-\cos k_1 - \cos k_2\right]\widehat{f}(k_1,k_2). \tag{104}$$

As a result the 2D lattice Coulomb potential created by a neutral charge distribution $\rho(r,s)$, namely the solution $U^{\rm C}_{[\rho]}(r,s)$ of the Poisson equation

$$\Delta_2 U^{\rm C}_{[\rho]}(r,s) = -\rho(r,s), \tag{105}$$

has the following Fourier transform

$$\widehat{U^{\rm C}_{[\rho]}}(k_1,k_2) = \frac{\widehat{\rho}(k_1,k_2)}{2[2-\cos k_1 - \cos k_2]}. \tag{106}$$

By comparing the latter expression with (102) and by identifying $2[\cos(k_1+k_2)-1]$ as the Fourier transform of the quadrupolar charge distribution

$$\rho_{\rm quad}(r,s) = \delta_{r,1}\delta_{s,1} + \delta_{r,-1}\delta_{s,-1} - 2\,\delta_{r,0}\delta_{s,0}\,, \tag{107}$$

we interpret $J \times \widetilde{v^{\rm nn}}(r,s)$ as the two-dimensionnal lattice Coulomb potential created by the latter quadrupolar charge distribution $(1,-2,1)$ located at sites $(-1,-1)$, $(0,0)$ and $(1,1)$ respectively. In other words

$$\widetilde{v^{\rm nn}}(k_1,k_2) = \frac{1}{J}\left[u^{\rm C}(r+1,s+1) + u^{\rm C}(r-1,s-1) - 2u^{\rm C}(r,s)\right], \qquad (108)$$

where $u^{\rm C}(r,s)$ denotes the periodic 2D Coulomb potential created by the neutral distribution of a single unit charge at the origin and a negative uniform background with charge $-1/(MN)$ at each site (which ensures the periodicity). A priori the solution of the lattice Poisson equation (105) is defined up to an additive constant. The potential $u^{\rm C}(r,s)$ is chosen to vanish at the origin and reads

$$u^{\rm C}(r,s) = -\frac{1}{NM}\sum_{(k_2,k_1)\neq(0,0)} \frac{1-\cos(k_1 r)\cos(k_2 s)}{2[2-\cos k_1 - \cos k_2]}. \qquad (109)$$

6.2 Dual potential in the thermodynamic limit for $N = M+1$

We are now interested in the large-distance behavior of the quadrupolar potential (108). In the thermodynamic limit, where $N = M+1$ and N goes to infinity while r and s are kept fixed, the Coulomb potential $u^{\rm C}(r,s)$ of equation (109) tends to a function $u^{\rm C}_{\rm Th}(r,s)$ which is equal to $-(1/2\pi)\ln\sqrt{r^2+s^2}$ plus a constant term. This constant cancels out on the RHS of (108), with the result

$$\widetilde{v^{\rm nn}_{\rm Th}}(r,s) = -\frac{1}{4\pi J}\ln\left[1 - \frac{8rs}{(r^2+s^2)^2} + \frac{4}{(r^2+s^2)^2}\right]. \qquad (110)$$

In the large $\sqrt{r^2+s^2}$ limit the behavior of the latter potential reads

$$\widetilde{v^{\rm nn}_{\rm Th}}(r,s) \underset{\sqrt{r^2+s^2}\gg 1}{\simeq} \frac{2}{\pi J}\frac{rs}{(r^2+s^2)^2}, \qquad (111)$$

which decays as $1/(r^2+s^2)$ with an angular dependence. In the case of a square lattice we set $r = X\cos\phi$ and $s = X\sin\phi$; then the expression (110) for $\widetilde{v^{\rm nn}_{\rm Th}}(r,s)$ becomes

$$\widetilde{v^{\rm nn}_{\rm Th}}(X\cos\phi, X\sin\phi) = -\frac{1}{4\pi J}\ln\left[1 - \frac{4\sin(2\phi)}{X^2} + \frac{4}{X^4}\right]. \qquad (112)$$

In the limit $X\to\infty$ the latter potential behaves as $(1/\pi J)$ times $\sin(2\phi)/X^2$.

7 Conclusion

We have constructed a new duality for the Discrete Gaussian model on a torus with arbitrary translation invariant interactions, which inverts the temperature. The interactions are, however, in general anisotropic. There is a self-dual interaction potential which we have studied in particular at its self-dual temperature. We have also considered the well-known DG model with isotropic nearest-neighbor interactions. Our work is exact for an $N \times M$ torus with finite N and M, which however should be coprime. This condition has its origin in the Chinese Remainder Theorem, which we invoke to transpose known one-dimensional results to the two-dimensional torus. One simple way to satisfy this condition is by taking $M = N - 1$, where N is an arbitrary integer. At several points in our discussion we have taken the thermodynamic limit $N \to \infty$. Another similar duality can be derived for a neutral charge system corresponding to the Discrete Gaussian model and will be discussed elsewhere. The present paper contributes to the large body of exact results, and in particular of duality relations, for lattice models.

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