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Stable constant mean curvature surfaces with free boundary in slabs

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Abstract We study stable constant mean curvature (CMC) hypersurfaces Σ with free boundary in slabs in a product space $M \times \mathbb{R}$, where M is an orientable Riemannian manifold. We obtain a characterization of stable cylinders and prove that if Σ is not a cylinder then it is locally a vertical graph. Moreover, in case M is \mathbb{H}^n , \mathbb{R}^n or \mathbb{S}_+^n , if each component of $\partial\Sigma$ is embedded, then Σ is rotationally invariant. When M has dimension 2 and Gaussian curvature bounded from below by a positive constant κ , we prove there is no stable CMC with free boundary connecting the boundary components of a slab of width $l \geq 4\pi/\sqrt{3\kappa}$. We also show that a stable capillary surface of genus 0 in a warped product $[0, l] \times_f M$ where $M = \mathbb{R}^2, \mathbb{H}^2$ or \mathbb{S}^2 , is rotationally invariant. Finally, we prove that a stable CMC immersion of a closed surface in $M \times \mathbb{S}^1(r)$, where M is a surface with Gaussian curvature bounded from below by a positive constant κ and $\mathbb{S}^1(r)$ the circle of radius r , lifts to $M \times \mathbb{R}$ provided $r \geq 4/\sqrt{3\kappa}$.

Keywords constant mean curvature, free boundary, capillary surfaces, stability.

Mathematics Subject Classification (2010) 53A10, 49Q10, 53C42, 76B45.

1. Introduction

Let W be an oriented smooth Riemannian manifold of dimension $n \geq 3$ and $\mathcal{B} \subset W$ a closed domain with smooth boundary. We consider constant mean curvature (CMC) hypersurfaces with free boundary in \mathcal{B} , that is, CMC hypersurfaces whose interior is contained in \mathcal{B} and whose boundary lies on $\partial\mathcal{B}$ and which meet orthogonally $\partial\mathcal{B}$. They are stationary for the area functional for variations preserving the *enclosed volume*. It is interesting to study CMC hypersurfaces with free boundary which are stable, that is, those for which the second variation of the area is nonnegative for all volume-preserving variations. This problem arises naturally when studying the isoperimetric problem in \mathcal{B} . More generally, a capillary hypersurface in \mathcal{B} is a CMC hypersurface in \mathcal{B} with boundary on $\partial\mathcal{B}$ and meeting $\partial\mathcal{B}$ at a constant angle. Capillary hypersurfaces are also stationary for a functional which is a linear combination of the area of the hypersurface and the area of the *domain enclosed* by its boundary on $\partial\mathcal{B}$, for volume-preserving variations. The study of stability of capillary hypersurfaces in balls or with planar boundaries in space forms has attracted a lot of attention very recently, see for instance [1, 4, 7, 12, 13, 18].

We are mainly interested in this paper in stable CMC hypersurfaces with free boundary in slabs of a product space $M \times \mathbb{R}$, where M is an orientable Riemannian manifold of dimension ≥ 2 . A slab in $M \times \mathbb{R}$ is the region between two slices $M \times \{t\}$. Without loss of generality, a slab of width $l > 0$ in $M \times \mathbb{R}$ will be taken to be the domain $M \times [0, l]$. The special case $M = \mathbb{R}^n$ was considered by Athanassenas [3] and Vogel [20], for $n = 2$, and by Pedrosa and Ritoré [15], and by Ainouz and Souam [1] for any $n \geq 2$.

The simplest examples of CMC hypersurfaces with free boundary in a slab $M \times [0, l]$ are obtained by taking cylinders above closed CMC hypersurfaces in M . Our first result (Theorem 3.3) gives a characterization of stable cylinders in terms of the first eigenvalue of the stability operator of the base and the width of the slab. When M is the Euclidean space \mathbb{R}^n , the hyperbolic space \mathbb{H}^n or the unit sphere \mathbb{S}^n , this characterization of stable tubes is explicit in terms of their radius and the width (Corollary 3.4).

In the general case, we show (Theorem 3.5) that a stable free boundary CMC hypersurface in $M \times [0, l]$ that is not a cylinder has to be locally a vertical graph. It is moreover a global graph in case each of its boundary components is embedded and M is simply connected. When M is the Euclidean space \mathbb{R}^n , the hyperbolic space \mathbb{H}^n or a hemisphere \mathbb{S}_+^n , the previous hypotheses imply that the hypersurface is rotationally invariant. This extends results obtained in [1] in the Euclidean case.

When M is two-dimensional with curvature bounded from below by a positive constant κ , we prove (Theorem 3.7) that no stable CMC surface with free boundary can connect the boundary components of a slab $M \times [0, l]$ whose width satisfies $l \geq 4\pi/\sqrt{3\kappa}$.

In section 4, we consider stable capillary surfaces in warped products of the type $[0, l] \times_f M$ where $M = \mathbb{R}^2, \mathbb{H}^2$ or \mathbb{S}^2 . We prove they have to be rotationally invariant when they have genus zero, extending a result obtained in [1] in the Euclidean case. This applies, in particular, to the region bounded by two parallel horospheres in the hyperbolic space \mathbb{H}^3 (Corollary 4.2).

Finally, in Section 5, we obtain a result of the same type as Theorem 3.7 for closed stable CMC surfaces in the product $M \times \mathbb{S}^1(r)$ of an orientable surface M , with curvature bounded from below by a positive constant κ , by the circle $\mathbb{S}^1(r)$ of radius $r > 0$. More precisely, we show that if $r \geq 4/\sqrt{3\kappa}$, then the immersion lifts to a stable CMC immersion in $M \times \mathbb{R}$ (Theorem 5.1).

2. Preliminaries

Let W be an oriented smooth Riemannian manifold of dimension $n \geq 3$ and $\mathcal{B} \subset W$ a closed domain with smooth boundary. A capillary hypersurface in \mathcal{B} is a compact and constant mean curvature (CMC) hypersurface, with non-empty boundary, which is contained in \mathcal{B} and meets $\partial\mathcal{B}$ at a constant angle along its boundary. When the angle of contact is $\pi/2$, that is, when the hypersurface is orthogonal to $\partial\mathcal{B}$, the hypersurface is said to be a CMC hypersurface with free boundary in \mathcal{B} . Capillary hypersurfaces are critical points of an energy functional for volume-preserving variations. In the free boundary case the energy functional is the area functional. We refer to [1] for more details.

Consider a capillary hypersurface defined by a smooth immersion $\psi : \Sigma \rightarrow \mathcal{B}$ and let N be a global unit normal to Σ along ψ chosen so that its (constant) mean curvature satisfies $H \geq 0$. Denote by \bar{N} the exterior unit normal to $\partial\mathcal{B}$. The angle of contact is the angle $\theta \in (0, \pi)$ between N and \bar{N} . We denote by ν the exterior unit normal to $\partial\Sigma$ in Σ

and by $\bar{\nu}$ the unit normal to $\partial\Sigma$ in $\partial\mathcal{B}$ so that $\{N, \nu\}$ and $\{\bar{N}, \bar{\nu}\}$ have the same orientation in $(T\partial\Sigma)^\perp$. The angle between ν and $\bar{\nu}$ is also equal to θ .

The index form \mathcal{I} of ψ is the symmetric bilinear form defined on the first Sobolev space $H^1(\Sigma)$ of Σ by

$$\mathcal{I}(f, g) = \int_{\Sigma} (\langle \nabla f, \nabla g \rangle - (|\sigma|^2 + \text{Ric}(N))fg) d\Sigma - \int_{\partial\Sigma} q fg d(\partial\Sigma),$$

where ∇ stands for the gradient for the metric induced by ψ , σ is the second fundamental form of ψ , $\text{Ric}(N)$ is the Ricci curvature of W in the direction N and

$$q = \frac{1}{\sin \theta} \text{II}(\bar{\nu}, \bar{\nu}) + \cot \theta \sigma(\nu, \nu).$$

Here II denotes the second fundamental form of $\partial\mathcal{B}$ associated to the unit normal $-\bar{N}$.

Let $\mathcal{F} = \{f \in H^1(\Sigma), \int_{\Sigma} f d\Sigma = 0\}$. The immersion ψ is capillaryly stable if $\mathcal{I}(f, f) \geq 0$ for all $f \in \mathcal{F}$.

The stability operator on Σ is the linear operator defined on $\mathcal{C}^2(\Sigma)$ by

$$L = \Delta + |\sigma|^2 + \text{Ric}(N)$$

where Δ is the Laplacian on Σ induced by ψ .

A function $f \in \mathcal{F}$ is said to be a Jacobi function of ψ if it lies in the kernel of \mathcal{I} , that is, if $\mathcal{I}(f, g) = 0$ for all $g \in \mathcal{F}$. By standard arguments, this is equivalent to saying that $f \in \mathcal{C}^\infty(\Sigma)$ and satisfies

$$\begin{aligned} Lf &= \text{constant} && \text{on } \Sigma \\ \frac{\partial f}{\partial \nu} &= qf && \text{on } \partial\Sigma \end{aligned}$$

We will also need to consider compact immersed CMC hypersurfaces without boundary. Given such an immersion $\psi : \Sigma \rightarrow W$ in an orientable Riemannian manifold W , it is said to be stable if $\mathcal{I}(f, f) \geq 0$ for all $f \in \mathcal{F} = \{f \in H^1(\Sigma), \int_{\Sigma} f d\Sigma = 0\}$ where \mathcal{I} is the index form defined on $H^1(\Sigma)$ by

$$\mathcal{I}(f, g) = \int_{\Sigma} (\langle \nabla f, \nabla g \rangle - (|\sigma|^2 + \text{Ric}(N))fg) d\Sigma.$$

An immersion ψ of a closed or capillary CMC is said to be *strongly stable* if $\mathcal{I}(f, f) \geq 0$ for all $f \in H^1(\Sigma)$.

It is also interesting to consider, as a more general setting for capillarity, the case where the contact angle is constant along each component of $\partial\Sigma$, allowing it to vary from one component to the other. All the previous discussion on stability extends to this more general case (cf. [1]).

3. Stable CMC hypersurfaces with free boundary in a slab in a product space

$$M \times \mathbb{R}$$

Let M be a connected oriented Riemannian manifold of dimension $n \geq 2$. We discuss in this section the case of CMC immersed hypersurfaces with free boundary in slabs

$M \times [0, l], l > 0$, of the product space $M \times \mathbb{R}$. The function q vanishes identically and the index form takes a simple form:

$$\mathcal{I}(f, g) = \int_{\Sigma} (\langle \nabla f, \nabla g \rangle - (|\sigma|^2 + \text{Ric}(N))fg) d\Sigma.$$

The simplest examples of free boundary CMC hypersurfaces in a slab $M \times [0, l]$ are cylinders over closed CMC hypersurfaces in M . We will discuss the stability of cylinders. For this, we will need a criterion for stability of closed CMC hypersurfaces. It was established by Koiso ([10], Theorem 1.3) for compact CMC surfaces with boundary in \mathbb{R}^3 but the result readily extends to the closed case in general ambient manifolds.

Theorem 3.1. *Let $\psi : \Sigma \rightarrow M$ be a closed and two-sided immersed CMC hypersurface in a Riemannian manifold M of dimension $n \geq 2$. Denote by $\lambda_1(\Sigma)$ and $\lambda_2(\Sigma)$, respectively, the first and second eigenvalues of the stability operator L on Σ . The following hold*

(I) $\lambda_1(\Sigma) \geq 0$ if and only if ψ is strongly stable.

(II) If $\lambda_1(\Sigma) < 0$ and $\lambda_2(\Sigma) > 0$, then there exists a uniquely determined smooth function u on Σ such that $Lu = 1$ and ψ is stable if and only if $\int_{\Sigma} u d\Sigma \geq 0$.

(III) If $\lambda_1(\Sigma) < 0$ and $\lambda_2(\Sigma) = 0$ and there exists an eigenfunction associated to $\lambda_2(\Sigma)$ satisfying $\int_{\Sigma} g dA \neq 0$, then ψ is unstable.

(IV) If $\lambda_1(\Sigma) < 0$ and $\lambda_2(\Sigma) = 0$ and $\int_{\Sigma} g dA = 0$ for any eigenfunction g associated to λ_2 . Denoting by E the eigenspace associated to $\lambda_2(\Sigma) = 0$, there exists a uniquely determined smooth function $u \in E^{\perp}$ (the orthogonal to E in the L^2 sense) which satisfies $Lu = 1$. Then ψ is stable if and only if $\int_{\Sigma} u dA \geq 0$.

(V) If $\lambda_2(\Sigma) < 0$, then ψ is unstable.

Remark 3.2. *If $\lambda_1(\Sigma) < 0$ and $\lambda_2(\Sigma) \geq 0$ and we know beforehand the existence of a smooth function u satisfying $Lu = 1$, then, it is easily checked that (II) and (IV) can be restated simply by saying that ψ is stable if and only if $\int_{\Sigma} u dA \geq 0$.*

We can now give the following characterization of stable cylinders.

Theorem 3.3. *Let $\psi : \Gamma \rightarrow M$ be an immersion with constant mean curvature H of a closed oriented manifold Γ of dimension $n \geq 1$ in an $(n + 1)$ -dimensional oriented manifold M and let $l > 0$. The map $\bar{\psi}$*

$$\bar{\psi} := \psi \times \{\text{id}\} : \Gamma \times [0, l] \rightarrow M \times [0, l],$$

is an immersion of constant mean curvature $\frac{(n-1)}{n}H$ with free boundary. Denote by $\lambda_1(\Gamma)$ the first eigenvalue of the stability operator L on Γ . The following hold

(i) *If ψ is unstable, then $\bar{\psi}$ is unstable too.*

(ii) *If ψ is stable, then $\bar{\psi}$ is stable if and only if*

$$(3.1) \quad \lambda_1(\Gamma) + \left(\frac{\pi}{l}\right)^2 \geq 0.$$

Proof. The first statement about $\bar{\psi}$ is straightforward.

Denote by \bar{L} the stability operator of $\bar{\psi}$ and by $\bar{\mathcal{I}}$ the associated index form.

(i) Suppose ψ is unstable, then there exists $u \in H^1(\Gamma)$ such that $\int_{\Gamma} u = 0$ and $\mathcal{I}(u, u) < 0$. Define \bar{u} on $\Gamma \times [0, l]$ by $\bar{u}(p, t) = u(p)$, for all $(p, t) \in \Gamma \times [0, l]$. Then clearly $\bar{u} \in H^1(\Gamma \times [0, l])$, $\int_{\Gamma \times [0, l]} \bar{u} = 0$ and $\overline{\mathcal{I}}(\bar{u}, \bar{u}) = \mathcal{I}(u, u) < 0$, which shows that $\overline{\psi}$ is unstable.

(ii) Suppose ψ is stable.

If $\lambda_1(\Gamma) + (\frac{\pi}{l})^2 < 0$, let u be an eigenfunction of L associated to $\lambda_1(\Gamma)$. Then the function $v := u \cos(\pi t/l)$ on $\Gamma \times [0, l]$ satisfies $\int_{\Gamma \times [0, l]} v = 0$ and $\overline{\mathcal{I}}(v, v) = (\lambda_1(\Gamma) + (\frac{\pi}{l})^2) \int_{\Gamma \times [0, l]} v^2 < 0$. So $\overline{\psi}$ is unstable.

Suppose now that $\lambda_1(\Gamma) + (\frac{\pi}{l})^2 \geq 0$. We will first show that the CMC immersion $\widehat{\psi} := \psi \times \{\text{id}\} : \Gamma \times \mathbb{S}^1(\frac{l}{\pi}) \rightarrow M \times \mathbb{S}^1(\frac{l}{\pi})$ is stable, where $\mathbb{S}^1(\frac{l}{\pi})$ denotes the circle of radius $\frac{l}{\pi}$.

Let Ric and $\widehat{\text{Ric}}$, be the Ricci curvature tensors of M and $M \times \mathbb{R}$, respectively, and denote by σ , and $\widehat{\sigma}$, the second fundamental forms of ψ and $\widehat{\psi}$, respectively. We note that $|\widehat{\sigma}| = |\sigma|$ and if N is a unit field normal to Γ along ψ , then it is also normal to $\Gamma \times \mathbb{S}^1(\frac{l}{\pi})$ along $\widehat{\psi}$ and $\widehat{\text{Ric}}(N, N) = \text{Ric}(N, N)$. We denote by \widehat{L} be the stability operator of $\widehat{\psi}$ and let $\widehat{\mathcal{I}}$ be the associated index form.

The eigenvalues and eigenfunctions of the eigenvalue problem

$$f'' + \mu f = 0$$

on $\mathbb{S}^1(\frac{l}{\pi})$ are given by $\mu_n = (n\pi/l)^2$, $n = 0, 1, 2, \dots$ and $f_n(t) = \cos \sqrt{\mu_n} t$.

Let $\lambda_1(\Gamma) < \lambda_2(\Gamma) \leq \dots \leq \lambda_k(\Gamma) \leq \dots$ be the sequence of eigenvalues of the operator L on the closed manifold Γ . As the function $|\widehat{\sigma}|^2 + \widehat{\text{Ric}}(N, N) = |\sigma|^2 + \text{Ric}(N, N)$ depends only on the first factor of the product $\Gamma \times \mathbb{S}^1(\frac{l}{\pi})$, a standard argument shows that the eigenvalues of the operator \widehat{L} on $\Gamma \times \mathbb{S}^1(\frac{l}{\pi})$ are given by $\lambda_m(\Gamma) + n^2\pi^2/l^2$, $m, n \in \mathbb{N}$.

In particular, the first and second eigenvalues of \widehat{L} on $\Gamma \times \mathbb{S}^1(\frac{l}{\pi})$ verify:

$$(3.2) \quad \lambda_1\left(\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)\right) = \lambda_1(\Gamma)$$

$$(3.3) \quad \lambda_2\left(\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)\right) = \min\left\{\lambda_1(\Gamma) + \frac{\pi^2}{l^2}, \lambda_2(\Gamma)\right\}$$

If $\lambda_1(\Gamma) \geq 0$, then, by (3.2), $\lambda_1(\Gamma \times \mathbb{S}^1(\frac{l}{\pi})) \geq 0$, that is, $\widehat{\psi}$ is strongly stable. So we assume $\lambda_1(\Gamma) < 0$. In this case we know that $\lambda_2(\Gamma) \geq 0$ (Theorem 3.1 (V)). Therefore, by (3.3), $\lambda_2(\Gamma \times \mathbb{S}^1(\frac{l}{\pi})) \geq 0$.

Consider then the solution $u \in C^\infty(\Gamma)$ to the equation $Lu = 1$ described in Theorem 3.1, (II) or (IV). Then, the function \hat{u} , defined on $\Gamma \times \mathbb{S}^1(\frac{l}{\pi})$ by $\hat{u}(p, t) = u(p)$, for $(p, t) \in \Gamma \times \mathbb{S}^1(\frac{l}{\pi})$, verifies $\widehat{L}\hat{u} = Lu = 1$ and $\int_{\Gamma \times \mathbb{S}^1(\frac{l}{\pi})} \hat{u} = 2l \int_{\Gamma} u \geq 0$. Theorem 3.1 (and Remark 3.2) shows that $\widehat{\psi}$ is stable.

We now show that stability of $\widehat{\psi}$ implies stability of $\overline{\psi}$.

Let $u \in H^1(\Gamma \times [0, l])$ satisfying $\int_{\Gamma \times [0, l]} u = 0$. We extend u to a function in $H^1(\Gamma \times [0, 2l])$ by reflection across $M \times \{l\}$, that is, we set $u(p, t) = u(p, 2l - t)$ for

all $p \in \Gamma$ and $t \in [l, 2l]$. Clearly this function gives rise to a function $\hat{u} \in H^1(\Gamma \times \mathbb{S}^1(\frac{l}{\pi}))$ satisfying $\int_{\Gamma \times \mathbb{S}^1(\frac{l}{\pi})} \hat{u} = 0$ and $\mathcal{I}(u, u) = \frac{1}{2} \widehat{\mathcal{I}}(\hat{u}, \hat{u}) \geq 0$. Therefore $\bar{\psi}$ is stable. This completes the proof. \square

For instance, consider the case of a tube of radius ρ in $\mathbb{M}^n(\kappa) \times [0, l]$, $\kappa = -1, 0, 1$, $n \geq 2$, where $\mathbb{M}^n(-1) = \mathbb{H}^n$, $\mathbb{M}^n(0) = \mathbb{R}^n$ and $\mathbb{M}^n(1) = \mathbb{S}^n$. The stability operator of the tube writes

$$L = \Delta + (n-1)(c_\kappa(\rho)^{-1} + \kappa)$$

with

$$\begin{aligned} c_{-1}(\rho) &= \tanh^2 \rho \\ c_0(\rho) &= \rho^2 \\ c_1(\rho) &= \tan^2 \rho \end{aligned}$$

As round spheres in space forms are stable, we get the following characterization of stable tubes in $\mathbb{M}^n(\kappa) \times \mathbb{R}$, the case of tubes in \mathbb{R}^3 was treated by Athanassenas [3] and Vogel [20].

Corollary 3.4. *A tube of radius ρ and height l in $\mathbb{H}^n \times [0, l]$ is stable if and only if $\pi \sinh \rho \geq \sqrt{n-1} l$.*

A tube of radius ρ and height l in $\mathbb{R}^n \times [0, l]$ is stable if and only if $\pi \rho \geq \sqrt{n-1} l$.

A tube of radius ρ and height l in $\mathbb{S}^n \times [0, l]$ is stable if and only if $\pi \sin \rho \geq \sqrt{n-1} l$.

We now prove a general fact about the geometry of stable free boundary CMC hypersurfaces in a slab in $M \times \mathbb{R}$. It extends the result proved in [1] for the case $M = \mathbb{R}^n$.

Theorem 3.5. *Let M be an n -dimensional connected and oriented Riemannian manifold and $\psi : \Sigma \rightarrow M \times [0, l]$ a stable free boundary immersion connecting the two boundary components $M \times \{0\}$ and $M \times \{l\}$, $l > 0$, of a slab in the Riemannian product $M \times \mathbb{R}$.*

Then, either $\psi(\Sigma)$ is a vertical cylinder over a closed immersed stable CMC in M or $\psi(\Sigma)$ is locally a vertical graph and in this case the lowest eigenvalue of the stability operator on Σ with Dirichlet boundary condition is equal to 0.

Furthermore, suppose $\psi(\Sigma)$ is not a vertical cylinder. If M is simply connected and the restriction of ψ to each component of $\partial\Sigma$ is an embedding, then $\psi(\Sigma)$ is a global vertical graph over a domain in M . In particular, if M is either the Euclidean space \mathbb{R}^n , the hyperbolic space \mathbb{H}^n or a hemisphere \mathbb{S}_+^n , then, under the same hypotheses, $\psi(\Sigma)$ is rotationally invariant around an axis $\{p\} \times \mathbb{R}$, for some $p \in M$.

Proof. We start with the following general fact. Let $\psi : \Sigma \rightarrow M$ be a CMC immersion of an orientable manifold Σ of dimension n in an orientable Riemannian manifold M of dimension $n+1$ and N a global unit normal to Σ . If X a Killing vector field on M then the function $\langle X \circ \psi, N \rangle$ lies in the kernel of the Jacobi operator L of Σ . We include a proof of this fact for the reader's convenience. The Jacobi operator is also known to be the linearized operator of the mean curvature functional. More precisely, let ψ_t be a smooth deformation, through immersions, of $\psi_0 = \psi$ and let H_t be the mean curvature function of ψ_t , then:

$$(3.4) \quad 2 \frac{\partial H_t}{\partial t} \Big|_{t=0} = L \left(\left\langle \frac{\partial \psi_t}{\partial t} \Big|_{t=0}, N \right\rangle \right).$$

Now, let ϕ_t be the local flow of isometries of M generated by the Killing field X and consider the deformation $\psi_t = \phi_t \circ \psi$ of ψ . Clearly, $H_t = H$ for all t and so the left-hand side of (3.4) vanishes. Furthermore, $\frac{\partial \psi_t}{\partial t}|_{t=0} = X \circ \psi$. This shows that the function $\langle X \circ \psi, N \rangle$ lies in the kernel of L .

In our case, as the vertical unit vector field $\frac{\partial}{\partial t}$ is a Killing field of the product $M \times \mathbb{R}$, the function $v = \langle \frac{\partial}{\partial t}, N \rangle$ verifies $Lv = 0$.

If v is identically zero, then $\psi(\Sigma)$ is a vertical cylinder whose base is an immersed CMC hypersurface in M and which is moreover stable by Theorem 3.3.

Assume v is not identically zero. We will show it has a sign in the interior of Σ . Suppose by contradiction that v changes sign and consider the functions $v_+ = \max\{v, 0\}$ and $v_- = \min\{v, 0\}$ which lie in the Sobolev space $H^1(\Sigma)$. Using the fact that $v = 0$ on $\partial\Sigma$, one has

$$\begin{aligned} \mathcal{I}(v_+, v_+) &= \int_{\Sigma} \{ \langle \nabla v_+, \nabla v_+ \rangle - (|\sigma|^2 + \text{Ric}(N))(v_+)^2 \} d\Sigma \\ &= \int_{\Sigma} \{ \langle \nabla v, \nabla v_+ \rangle - (|\sigma|^2 + \text{Ric}(N))vv_+ \} d\Sigma \\ &= - \int_{\Sigma} \{ (\Delta v)v_+ + (|\sigma|^2 + \text{Ric}(N))vv_+ \} d\Sigma \\ &= 0 \end{aligned}$$

and similarly $\mathcal{I}(v_-, v_-) = 0$. As we supposed that v changes sign, there exists $a \in \mathbb{R}$ such that $\int_{\Sigma} (v_+ + av_-) d\Sigma = 0$. So we can use $\tilde{v} := v_+ + av_-$ as a test function for stability. We have

$$\mathcal{I}(\tilde{v}, \tilde{v}) = \mathcal{I}(v_+, v_+) + 2a\mathcal{I}(v_+, v_-) + a^2\mathcal{I}(v_-, v_-) = 0.$$

Since ψ is stable, we conclude that \tilde{v} is a Jacobi function and so is, in particular, smooth. We distinguish two cases:

(i) If $a \neq 1$, then we conclude that v_+ and v_- are smooth and so we can write $Lv = Lv_+ + Lv_- = 0$ and $L\tilde{v} = Lv_+ + aLv_- = 0$. It follows that $Lv_+ = 0$ and so, by the unique continuation principle [2], v_+ has to be identically 0, which is a contradiction.

(ii) If $a = 1$. Then $v = \tilde{v}$ is a Jacobi function and so satisfies $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Sigma$. Let $p \in \partial\Sigma$ with $\psi(p) \in M \times \{0\}$ and Ω a neighborhood of p in Σ , diffeomorphic to the halfspace \mathbb{R}_+^n , such that $\psi : \Omega \rightarrow M \times [0, l]$ is an embedding. Identify $S = \psi(\Omega)$ with Ω and call τ the reflection in $M \times \mathbb{R}$ through $M \times \{0\}$. Then $\bar{S} := S \cup \tau(S)$ is a C^2 hypersurface in $M \times \mathbb{R}$ with constant mean curvature; \bar{S} is therefore smooth. We now extend v to a function \bar{v} on \bar{S} by setting $\bar{v} = 0$ on $\tau(S)$. As $\frac{\partial v}{\partial \nu} = 0$ on ∂S , we have $\bar{v} \in C^1(\bar{S})$. Furthermore, because $v = \frac{\partial v}{\partial \nu} = 0$ on ∂S , one checks that \bar{v} satisfies the equation $\bar{L}\bar{v} = 0$ on \bar{S} in the sense of distributions; \bar{L} denoting the Jacobi operator on \bar{S} . By regularity theory for elliptic PDEs, \bar{v} is smooth. As \bar{v} vanishes on a non empty subset, the unique continuation principle [2] implies that $\bar{v} \equiv 0$ on S . Again, by the unique continuation principle, one has $v \equiv 0$ on Σ , which is a contradiction.

We have thus shown that the function v does not change sign in Σ . We will assume $v \geq 0$, the case $v \leq 0$ being similar. The function v satisfies

$$\begin{cases} v & \geq 0, \\ Lv & = 0. \end{cases}$$

As we are assuming v is not identically zero, by the strong maximum principle (Theorem 2.9 in [9]) we know that $v > 0$ on the interior of Σ . So the interior of $\psi(\Sigma)$ is a local vertical graph. Moreover, since v does not vanish in the interior of Σ , it is a first eigenfunction of L and so 0 is the lowest eigenvalue of L for the Dirichlet problem on Σ . This proves the first statement.

Let us now prove the second statement. Assuming M is simply connected and $\psi(\Sigma)$ is not a cylinder, we will show it is globally a vertical graph by analyzing its behaviour near its boundary.

For the sake of simplicity, we set $M_0 = M \times \{0\}$ and $M_1 = M \times \{l\}$. Let $\Gamma_1, \dots, \Gamma_k$ denote the connected components of $\partial\Sigma$. By hypothesis, ψ restricted to Γ_i is an embedding and so $\psi(\Gamma_i)$ separates the slice containing it, among M_0 and M_1 , into two connected components ([8]).

Denote by $P : M \times \mathbb{R} \rightarrow M$ the projection on the second factor and set $F = P \circ \psi$. Fix $i = 1, \dots, k$ and let $\varphi : \mathcal{U} \rightarrow \mathbb{R}$ be a smooth function defined in a neighborhood $\mathcal{U} \subset M$ of $\psi(\Gamma_i)$ such that $\psi(\Gamma_i) = \{\varphi = 0\}$ and for each $p \in \Gamma_i$, $\nabla\varphi(\psi(p)) = N(p)$. For instance we may take φ to be the signed distance function to $\psi(\Gamma_i)$.

Let us take a point $p \in \Gamma_i$, and a curve $\gamma : (-\epsilon, 0] \rightarrow \Sigma$ parametrized by arc-length with $\gamma(0) = p$ and $\dot{\gamma}(0) = \nu(p)$. Set $\bar{\gamma}(s) := \psi(\gamma(s)) = (\alpha(s), t(s))$ where $\alpha = F \circ \gamma : (-\epsilon, 0] \rightarrow M$ and define the real function h on $(-\epsilon, 0]$ by $h(s) = \varphi(\alpha(s))$.

Denoting by D the connection on M , and by \bar{D} the one on $M \times \mathbb{R}$, we have

$$\begin{aligned} \dot{h}(0) &= h(0) = 0, \\ \ddot{h}(0) &= \nabla^2\varphi(\dot{\alpha}(0), \dot{\alpha}(0)) + \langle \nabla\varphi(\psi(p)), \frac{D\dot{\alpha}}{ds}(0) \rangle \\ &= \langle N(p), \frac{\bar{D}\dot{\bar{\gamma}}}{ds}(0) \rangle \\ &= \sigma(\nu(p), \nu(p)). \end{aligned}$$

Note that

$$\frac{\partial v}{\partial \nu} = -\sigma(\nu, \nu) \langle \nu, e_{n+1} \rangle = \begin{cases} +\sigma(\nu, \nu) & \text{if } \psi(\Gamma_i) \subset M_0, \\ -\sigma(\nu, \nu) & \text{if } \psi(\Gamma_i) \subset M_1. \end{cases}$$

By the boundary point maximum principle (see, for instance, Theorem 2.8 in [9]), $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Sigma$. So for t small, $F(\gamma(t))$ lies in the component of $M \setminus F(\Gamma_i)$ which has $N(p)$ as outwards (resp. inwards) pointing normal at $F(p)$ if $\psi(\Gamma_i) \subset M_0$ (resp. if $\psi(\Gamma_i) \subset M_1$). It follows that there is a thin strip in the interior of Σ around Γ_i which projects on this component. We call D_i the component of $M \setminus F(\Gamma_i)$ which does not intersect this projection.

We let $\tilde{\Sigma}$ denote the (topological) manifold obtained by attaching, for each $i = 1, \dots, k$, D_i to Σ using the diffeomorphism $\psi|_{\Gamma_i} : \Gamma_i \rightarrow \partial D_i$ (see, for instance, Theorem 9.29 in [11] for the details of such a construction).

We define $\tilde{F} : \tilde{\Sigma} \rightarrow M$ by

$$\tilde{F} = \begin{cases} F & \text{on } \Sigma \\ \text{identity} & \text{on } D_i, i = 1, \dots, k. \end{cases}$$

Note that \tilde{F} is well defined since, by construction of $\tilde{\Sigma}$, each $x \in \Gamma_i$ is identified with $\psi(x) \in \partial D_i$. As the interior of $\psi(\Sigma)$ is locally a vertical graph and thanks to the above analysis of the behaviour of ψ near $\partial\Sigma$, we see that \tilde{F} is a local homeomorphism. It is also clear that \tilde{F} is a proper map, thus it is a covering map. Therefore \tilde{F} is a global homeomorphism onto M . Consequently, $\psi(\Sigma)$ is a graph over a domain in M . In particular, when $M = \mathbb{R}^n, \mathbb{H}^n$ or \mathbb{S}_+^n , Alexandrov's reflection technique shows that $\psi(\Sigma)$ is a hypersurface of revolution around a vertical axis, see [19]. \square

Remark 3.6. *It is clear from the proof that a similar result is true for stable CMC hypersurfaces with free boundary in a half-space $M \times [0, +\infty)$.*

It is not difficult to deduce from Theorem 3.3 that if the Ricci curvature of M is bounded from below by a positive constant κ , then there is no stable cylinder in $M \times [0, l]$ for $l > \pi/\sqrt{\kappa}$ (see the argument below). When M has dimension 2, we have the following stronger statement.

Theorem 3.7. *Let M be an orientable Riemannian surface with Gaussian curvature $K \geq \kappa > 0$. If*

$$l \geq \frac{4\pi}{\sqrt{3\kappa}}$$

then there is no immersed stable free boundary CMC surface in $M \times \mathbb{R}$ connecting $M \times \{0\}$ to $M \times \{l\}$.

Proof. Let $\psi : \Sigma \rightarrow M \times [0, l]$ be a stable CMC immersion with free boundary. Suppose $\psi(\Sigma)$ is not a cylinder. By Theorem 3.5, we know that 0 is the lowest eigenvalue of the stability operator L on Σ with Dirichlet boundary condition. Denote by H the mean curvature of Σ and by S the scalar curvature function of $M \times \mathbb{R}$. We have

$$3H^2 + S(p, t) = 3H^2 + K(p) \geq \kappa, \quad \text{for } (p, t) \in M \times \mathbb{R}.$$

We can therefore apply Theorem 1 in [17] which gives an upper bound for the intrinsic distance in Σ to $\partial\Sigma$. More precisely, for each $x \in \Sigma$,

$$d_\Sigma(x, \partial\Sigma) \leq \frac{2\pi}{\sqrt{3\kappa}}.$$

If $\psi : \Sigma \rightarrow M \times [0, l]$ connects $M \times \{0\}$ to $M \times \{l\}$, take $x \in \Sigma$ such that $\psi(x) \in M \times \{\frac{l}{2}\}$. Then

$$\frac{l}{2} \leq d_\Sigma(x, \partial\Sigma) \leq \frac{2\pi}{\sqrt{3\kappa}}.$$

Moreover, if we have the equality $l/2 = 2\pi/\sqrt{3\kappa}$, then $d_\Sigma(x, \partial\Sigma) = l/2$ and so at least one of the geodesic segments $\{x\} \times [0, l/2]$ or $\{x\} \times [l/2, l]$ is contained in Σ . In this case, by Theorem 3.5, $\psi(\Sigma)$ has to be a cylinder.

To conclude we need to check that cylinders over closed constant geodesic curves are unstable for $l \geq 4\pi/\sqrt{3\kappa}$. In fact a stronger statement is true. Consider a curve γ of constant geodesic curvature h in M . The lowest eigenvalue $\lambda_1(\gamma)$ of the stability operator on γ satisfies

$$\lambda_1(\gamma) \leq \frac{\mathcal{I}(1, 1)}{\int_\gamma ds} = -\frac{\int_\gamma (h^2 + K) ds}{\int_\gamma ds} \leq -\kappa.$$

For $l > \pi/\sqrt{\kappa}$, we get

$$\lambda_1(\gamma) + \frac{\pi^2}{l^2} \leq -\kappa + \frac{\pi^2}{l^2} < 0$$

and so, according to Theorem 3.3, the cylinder is unstable. This completes the proof. \square

4. Stable capillary surfaces of genus zero in certain warped products

In this section we are interested in stable capillary surfaces in warped product spaces of the type $[0, l] \times_f M$ where $M = \mathbb{R}^2, \mathbb{S}^2$ or \mathbb{H}^2 and $f : [0, l] \rightarrow \mathbb{R}$ is a smooth positive function. More precisely, $[0, l] \times_f M$ is the product space $[0, l] \times M$ endowed with the metric (with some abuse of notation)

$$g = dt^2 + f(t)^2 g_0,$$

g_0 being the metric on M .

An important feature of these spaces we will use is that if $\varphi : M \rightarrow M$ is an isometry, then its trivial extension $\{\text{id}\} \times \varphi : [0, l] \times_f M \rightarrow [0, l] \times_f M$ is an isometry too.

We have the following extension to these spaces of a theorem proved in the Euclidean case in [1].

Theorem 4.1. *Let $\psi : \Sigma \rightarrow [0, l] \times_f M$ be an immersed capillary surface of genus zero connecting the two boundary components of $[0, l] \times_f M$ where $M = \mathbb{R}^2, \mathbb{H}^2$, or \mathbb{S}^2 and having constant contact angles θ_0 and θ_1 with $\{0\} \times M$ and $\{l\} \times M$, respectively.*

If ψ is stable then $\psi(\Sigma)$ is a surface of revolution around an axis $\mathbb{R} \times \{x\}$, $x \in M$.

Proof. Let γ be a connected component of $\partial\Sigma$ such that $\psi(\gamma)$ lies on $\{0\} \times M$. We claim there is a circle \mathcal{C} in M bounding a disk containing $\psi(\gamma)$ and touching $\psi(\gamma)$ at least at 2 points. For $M = \mathbb{R}^2$, the circumscribed circle about $\psi(\gamma)$ has this property (cf. [14]). When $M = \mathbb{H}^2$, we can just take the (Euclidean) circumscribed circle about $\psi(\gamma)$ in the half-space model since Euclidean circles in this model are also metric circles for the hyperbolic metric. When $M = \mathbb{S}^2$, we pick a point $p \in \mathbb{S}^2 \setminus \psi(\gamma)$ and perform stereographic projection σ from the point p onto the Euclidean plane \mathbb{R}^2 . Let \mathcal{C}^* be the circumscribed circle in \mathbb{R}^2 about $\sigma(\psi(\gamma))$. Its inverse image $\mathcal{C} := \sigma^{-1}(\mathcal{C}^*)$ is a circle in \mathbb{S}^2 which has the required property.

We will show that $\psi(\Sigma)$ is a surface of revolution around the vertical axis passing through the center x of \mathcal{C} . In the case of \mathbb{S}^2 , we take x to be the center of the disk bounded by \mathcal{C} that contains $\psi(\gamma)$.

Let us consider the Jacobi function u on Σ induced by the rotations around the vertical axis passing through x . The function u verifies

$$(4.1) \quad \begin{cases} Lu = 0 & \text{on } \Sigma \\ \frac{\partial u}{\partial \nu} = qu & \text{on } \partial\Sigma \end{cases}$$

We will prove that $u \equiv 0$ on Σ .

Suppose, by contradiction, u is not identically zero. Then its nodal set $u^{-1}(0)$ in the interior of Σ has the structure of a graph (cf. [6]). We will show that u has at least 3 nodal domains by analyzing the set $u^{-1}(0) \cap \gamma$; a nodal domain of u being a connected component of $\Sigma \setminus u^{-1}(0)$.

We first note that, because of the boundary condition satisfied by u , if $p \in u^{-1}(0) \cap \partial\Sigma$ then $\frac{\partial u}{\partial \nu}(p) = 0$. It follows from the boundary point maximum principle (see Theorem 2.8 in [9]) that u changes sign in any neighborhood of $p \in u^{-1}(0) \cap \partial\Sigma$ unless u is identically zero in a neighborhood of p . In the latter case, by the unique continuation principle [2], u would vanish everywhere on Σ , contradicting our assumption. Consequently each such point lies on the boundary of at least 2 components of the set $\{u \neq 0\}$.

If $p \in \partial\Sigma$ is a critical point of the distance function to x restricted to γ , then one can check that $u(p) = 0$. Now, we observe that by the choice of \mathcal{C} , there are at least 3 points in $u^{-1}(0) \cap \gamma$. Indeed, we already know that there are two points in $u^{-1}(0) \cap \gamma$. A third one is a point of γ whose image by ψ is a closest one to the center x of \mathcal{C} .

Since by hypothesis Σ is topologically a planar domain, using the above information and the Jordan curve theorem, it is easy to see this implies that u has at least 3 nodal domains.

Denote by Σ_1 and Σ_2 two of these components and consider the following function in the Sobolev space $H^1(\Sigma)$:

$$\tilde{u} = \begin{cases} u & \text{on } \Sigma_1 \\ \alpha u & \text{on } \Sigma_2 \\ 0 & \text{on } \Sigma \setminus (\Sigma_1 \cup \Sigma_2) \end{cases}$$

where $\alpha \in \mathbb{R}$ is chosen so that $\int_{\Sigma} \tilde{u} dA = 0$. Using (4.1) we compute

$$\begin{aligned} \int_{\Sigma_1} \{ \langle \nabla \tilde{u}, \nabla \tilde{u} \rangle - (|\sigma|^2 + \text{Ric}(N)) \tilde{u}^2 \} dA &= \int_{\Sigma_1} \{ \langle \nabla u, \nabla \tilde{u} \rangle - (|\sigma|^2 + \text{Ric}(N)) u \tilde{u} \} dA \\ &= - \int_{\Sigma_1} (\Delta u + (|\sigma|^2 + \text{Ric}(N)) u) \tilde{u} dA + \int_{\partial\Sigma_1} \tilde{u} \frac{\partial u}{\partial \nu} ds \\ &= \int_{\partial\Sigma_1 \cap \partial\Sigma} q \tilde{u}^2 ds \end{aligned}$$

Using a similar computation on Σ_2 , we deduce that $\mathcal{I}(\tilde{u}, \tilde{u}) = 0$. As Σ is stable, we conclude that \tilde{u} is a Jacobi function. Indeed, the quadratic form on \mathcal{F} associated to \mathcal{I} has a minimum at \tilde{u} and so \tilde{u} lies in the kernel of \mathcal{I} . However, \tilde{u} vanishes on a non empty open set. By the unique continuation principle [2], \tilde{u} has to vanish everywhere, which is a contradiction.

Therefore $u \equiv 0$. This means that $\psi(\Sigma)$ is a surface of revolution around the axis through x . \square

The above result applies to the region bounded by two spheres centered at the origin for metrics on \mathbb{R}^3 that are invariant under the group $SO(3)$. This is, for instance, the case for the region bounded by two concentric spheres in \mathbb{R}^3 , \mathbb{H}^3 or \mathbb{S}^3 . Another interesting case to which the result applies is the region bounded by two parallel horospheres in \mathbb{H}^3 . Indeed, consider the half-space model of \mathbb{H}^3 , that is,

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3, x_3 > 0\}$$

equipped with the metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}.$$

Making the change of variables $t = \log x_3$, we see that the metric writes

$$ds^2 = e^{-2t}(dx_1^2 + dx_2^2) + dt^2.$$

Otherwise said, \mathbb{H}^3 can be viewed as the warped product $\mathbb{R} \times_{e^{-t}} \mathbb{R}^2$ and we may assume, up to a rigid motion, that the two parallel horospheres are slices $\{x_3 = \text{cst}\}$. We can thus state the following

Corollary 4.2. *Let $\psi : \Sigma \rightarrow \mathbb{H}^3$ be an immersed capillary surface of genus zero connecting two parallel horospheres. If ψ is stable then $\psi(\Sigma)$ is a surface of revolution around an axis orthogonal to the horospheres.*

The isoperimetric problem in regions bounded by two parallel horospheres in \mathbb{H}^3 was studied in [5] by Chaves, da Silva and Pedrosa.

5. Stable closed CMC surfaces in a product $M \times \mathbb{S}^1(r)$

In this section, we consider stable CMC surfaces in a product $M \times \mathbb{S}^1(r)$, where M is a Riemannian surface with curvature positively bounded from below, $K \geq \kappa$, with $\kappa > 0$ a constant and $\mathbb{S}^1(r)$ is the circle of radius $r > 0$. We have the following result which is in the same vein as Theorem 3.7.

Theorem 5.1. *Let $\psi : \Sigma \rightarrow M \times \mathbb{S}^1(r)$ be a stable CMC immersion of a closed connected and oriented surface Σ in $M \times \mathbb{S}^1(r)$ where M is an orientable surface with Gaussian curvature satisfying $K \geq \kappa > 0$, for some constant κ , and $\mathbb{S}^1(r)$ is the circle of radius $r > 0$. Denote by $p : M \times \mathbb{S}^1(r) \rightarrow \mathbb{S}^1(r)$ the canonical projection. Suppose that*

$$r \geq \frac{4}{\sqrt{3\kappa}},$$

then the induced homomorphism

$$(p \circ \psi)_* : \pi_1(\Sigma) \rightarrow \pi_1(\mathbb{S}^1(r))$$

is trivial and therefore ψ lifts to a stable CMC immersion $\tilde{\psi} : \Sigma \rightarrow M \times \mathbb{R}$.

Proof. We first derive an upper bound for the diameter of Σ . Denote by d_Σ the intrinsic distance on Σ and by $\text{diam}(\Sigma)$ its diameter and let $x, y \in \Sigma$ be such that $d := d_\Sigma(x, y) = \text{diam}(\Sigma)$. Then the disks $B_{d/2}(x)$ and $B_{d/2}(y)$ of radius $d/2$ centered at x and y , respectively, have disjoint interiors. Denote by $\lambda_1(B_{d/2}(x))$ (resp. $\lambda_1(B_{d/2}(y))$) the first eigenvalue of the operator L with Dirichlet boundary condition on $B_{d/2}(x)$ (resp. on $B_{d/2}(y)$). Since ψ is stable, the second eigenvalue of L on Σ verifies $\lambda_2(\Sigma) \geq 0$ (cf. Theorem 3.1). By the min-max characterization of the eigenvalues of L , we have

$$\lambda_2(\Sigma) = \inf_{E \subset H^1(\Sigma), \dim E = 2} \sup_{u \in E \setminus \{0\}} \frac{-\int_\Sigma u L u \, d\Sigma}{\int_\Sigma u^2 \, d\Sigma}.$$

Since $B_{d/2}(x)$ and $B_{d/2}(y)$ have disjoint interiors, it follows that

$$\lambda_2(\Sigma) \leq \max\{\lambda_1(B_{d/2}(x)), \lambda_1(B_{d/2}(y))\}.$$

Therefore at least one of the numbers $\lambda_1(B_{d/2}(x))$ and $\lambda_1(B_{d/2}(y))$ is nonnegative. Denoting by H the mean curvature of Σ and by S the scalar curvature function of $M \times \mathbb{S}^1(r)$, we have

$$3H^2 + S(p, \theta) = 3H^2 + K(p) \geq \kappa, \quad \text{for } (p, \theta) \in M \times \mathbb{S}^1(r).$$

We can thus apply Theorem 1 in [17] which gives the upper bound $d/2 \leq 2\pi/\sqrt{3\kappa}$, that is,

$$(5.1) \quad \text{diam}(\Sigma) \leq \frac{4\pi}{\sqrt{3\kappa}}.$$

Suppose now that the induced homomorphism $\pi_1(\Sigma) \rightarrow \pi_1(\mathbb{S}^1(r))$ is not trivial and consider a loop $\gamma : \mathbb{S}^1 \rightarrow \Sigma$ such that the loop $p \circ \psi \circ \gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^1(r)$ is nontrivial. We may assume that γ is a piecewise \mathcal{C}^1 -immersion. Denote by d_0 the distance on $M \times \mathbb{S}^1(r)$, by d_Σ the one on Σ and by d_1 the one on $\mathbb{S}^1(r)$. Let $x \in \gamma(\mathbb{S}^1)$, then $p \circ \psi \circ \gamma(\mathbb{S}^1)$ is not entirely contained in an interval of radius $< \pi r$ centered at $p(\psi(x))$ in $\mathbb{S}^1(r)$, because otherwise the loop $p \circ \psi \circ \gamma$ would be trivial. Therefore, there exists a point $y \in \gamma(\mathbb{S}^1)$ such that

$$(5.2) \quad \pi r = d_1(p(\psi(x)), p(\psi(y))) \leq d_0(\psi(x), \psi(y)) \leq d_\Sigma(x, y) \leq \text{diam}(\Sigma).$$

It follows from (5.1) and (5.2) that $r \leq 4/\sqrt{3\kappa}$.

Moreover, if $r = 4/\sqrt{3\kappa}$, then all the inequalities in (5.2) are equalities. In particular $\pi r = d_1(p(\psi(x)), p(\psi(y))) = d_0(\psi(x), \psi(y))$. It follows that if $\psi(x) = (x_0, \theta) \in M \times \mathbb{S}^1(r)$ then, we necessarily have $\psi(y) = (x_0, \theta^*)$, where $\theta^* \in \mathbb{S}^1(r)$ is the antipodal point of θ . Now, we can deform slightly the loop γ to a homotopic one $\bar{\gamma}$ satisfying $x \in \bar{\gamma}(\mathbb{S}^1)$ and $\psi(y) \notin \psi(\bar{\gamma}(\mathbb{S}^1))$. By the same argument as before, there must exist a point $\bar{y} \in \bar{\gamma}(\mathbb{S}^1)$ satisfying $\pi r = d_1(p(\psi(x)), p(\psi(\bar{y}))) = d_0(\psi(x), \psi(\bar{y}))$. Again this implies that $\psi(\bar{y}) = (x_0, \theta^*) = \psi(y)$, which is a contradiction. Therefore $r < 4/\sqrt{3\kappa}$ if the induced homomorphism $\pi_1(\Sigma) \rightarrow \pi_1(\mathbb{S}^1(r))$ is not trivial. This completes the proof. \square

As an application, consider the sphere \mathbb{S}^2 endowed with a Riemannian metric g of Gaussian curvature $K \geq \kappa > 0$, for some positive constant κ and let $r \geq 4/\sqrt{3\kappa}$. Under these hypotheses, it follows from Theorem 5.1 and Theorem 3 in [16] that an isoperimetric region in $\mathbb{S}^2(g) \times \mathbb{S}^1(r)$ is either a slab or a domain bounded by a surface of genus ≤ 2 . Indeed, the boundary Σ of an isoperimetric domain Ω in $(\mathbb{S}^2, g) \times \mathbb{S}^1(r)$ lifts to an embedded stable closed CMC surface in $(\mathbb{S}^2, g) \times \mathbb{R}$ by our result. By Theorem 3 in [16], Σ has genus at most 2 if it is connected and is a union of a finite number of horizontal slices if it is disconnected. In the latter case, Ω has to be a finite union of disjoint slabs $\mathbb{S}^2 \times [\theta_i, \theta_i + \alpha_i]$, $i = 1, \dots, n$, in $\mathbb{S}^2 \times \mathbb{S}^1(r)$. Note that a slab of width $\sum_{i=1}^n \alpha_i$ has the same volume as the union $\mathcal{U} = \cup_{i=1}^n \mathbb{S}^2 \times [\theta_i, \theta_i + \alpha_i]$ and its boundary has area less than the area of $\partial\mathcal{U}$. As Ω is an isoperimetric domain, it has to be a slab in this case.

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