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Stable constant mean curvature surfaces with free boundary in slabs

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Abstract We study stable constant mean curvature (CMC) hypersurfaces Σ with free boundary in slabs in a product space $M \times \mathbb{R}$, where M is an orientable Riemannian manifold. We obtain a characterization of stable cylinders and prove that if Σ is not a cylinder then it is locally a vertical graph. Moreover, in case M is $\mathbb{H}^n, \mathbb{R}^n$ or $\mathbb{S}^n_+$, if each component of $\partial\Sigma$ is embedded, then Σ is rotationally invariant. When M has dimension 2 and Gaussian curvature bounded from below by a positive constant κ, we prove there is no stable CMC with free boundary connecting the boundary components of a slab of width $l \geq 4\pi/\sqrt{3\kappa}$. We also show that a stable capillary surface of genus 0 in a warped product $[0, l] \times_f M$ where $M = \mathbb{R}^2, \mathbb{H}^2$ or $\mathbb{S}^2$, is rotationally invariant. Finally, we prove that a stable CMC immersion of a closed surface in $M \times \mathbb{S}^1(r)$, where M is a surface with Gaussian curvature bounded from below by a positive constant κ and $\mathbb{S}^1(r)$ the circle of radius r, lifts to $M \times \mathbb{R}$ provided $r \geq 4/\sqrt{3\kappa}$.

Keywords constant mean curvature, free boundary, capillary surfaces, stability.

Mathematics Subject Classification (2010) 53A10, 49Q10, 53C42, 76B45.

1. Introduction

Let W be an oriented smooth Riemannian manifold of dimension $n \geq 3$ and $\mathcal{B} \subset W$ a closed domain with smooth boundary. We consider constant mean curvature (CMC) hypersurfaces with free boundary in $\mathcal{B}$, that is, CMC hypersurfaces whose interior is contained in $\mathcal{B}$ and whose boundary lies on $\partial\mathcal{B}$ and which meet orthogonally $\partial\mathcal{B}$. They are stationary for the area functional for variations preserving the *enclosed volume*. It is interesting to study CMC hypersurfaces with free boundary which are stable, that is, those for which the second variation of the area is nonnegative for all volume-preserving variations. This problem arises naturally when studying the isoperimetric problem in $\mathcal{B}$. More generally, a capillary hypersurface in $\mathcal{B}$ is a CMC hypersurface in $\mathcal{B}$ with boundary on $\partial\mathcal{B}$ and meeting $\partial\mathcal{B}$ at a constant angle. Capillary hypersurfaces are also stationary for a functional which is a linear combination of the area of the hypersurface and the area of the *domain enclosed* by its boundary on $\partial\mathcal{B}$, for volume-preserving variations. The study of stability of capillary hypersurfaces in balls or with planar boundaries in space forms has attracted a lot of attention very recently, see for instance [1, 4, 7, 12, 13, 18].

We are mainly interested in this paper in stable CMC hypersurfaces with free boundary in slabs of a product space $M \times \mathbb{R}$, where M is an orientable Riemannian manifold of dimension ≥ 2. A slab in $M \times \mathbb{R}$ is the region between two slices $M \times \{t\}$. Without loss of generality, a slab of width $l > 0$ in $M \times \mathbb{R}$ will be taken to be the domain $M \times [0, l]$. The special case $M = \mathbb{R}^n$ was considered by Athanassenas [3] and Vogel [20], for $n = 2$, and by Pedrosa and Ritoré [15], and by Ainouz and Souam [1] for any $n \geq 2$.

The simplest examples of CMC hypersurfaces with free boundary in a slab $M \times [0, l]$ are obtained by taking cylinders above closed CMC hypersurfaces in M. Our first result (Theorem 3.3) gives a characterization of stable cylinders in terms of the first eigenvalue of the stability operator of the base and the width of the slab. When M is the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ or the unit sphere $\mathbb{S}^n$, this characterization of stable tubes is explicit in terms of their radius and the width (Corollary 3.4).

In the general case, we show (Theorem 3.5) that a stable free boundary CMC hypersurface in $M \times [0, l]$ that is not a cylinder has to be locally a vertical graph. It is moreover a global graph in case each of its boundary components is embedded and M is simply connected. When M is the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ or a hemisphere $\mathbb{S}^n_+$, the previous hypotheses imply that the hypersurface is rotationally invariant. This extends results obtained in [1] in the Euclidean case.

When M is two-dimensional with curvature bounded from below by a positive constant κ, we prove (Theorem 3.7) that no stable CMC surface with free boundary can connect the boundary components of a slab $M \times [0, l]$ whose width satisfies $l \geq 4\pi/\sqrt{3\kappa}$.

In section 4, we consider stable capillary surfaces in warped products of the type $[0, l] \times_f M$ where $M = \mathbb{R}^2, \mathbb{H}^2$ or $\mathbb{S}^2$. We prove they have to be rotationally invariant when they have genus zero, extending a result obtained in [1] in the Euclidean case. This applies, in particular, to the region bounded by two parallel horospheres in the hyperbolic space $\mathbb{H}^3$ (Corollary 4.2).

Finally, in Section 5, we obtain a result of the same type as Theorem 3.7 for closed stable CMC surfaces in the product $M \times \mathbb{S}^1(r)$ of an orientable surface M, with curvature bounded from below by a positive constant κ, by the circle $\mathbb{S}^1(r)$ of radius $r > 0$. More precisely, we show that if $r \geq 4/\sqrt{3\kappa}$, then the immersion lifts to a stable CMC immersion in $M \times \mathbb{R}$ (Theorem 5.1).

2. Preliminaries

Let W be an oriented smooth Riemannian manifold of dimension $n \geq 3$ and $\mathcal{B} \subset W$ a closed domain with smooth boundary. A capillary hypersurface in $\mathcal{B}$ is a compact and constant mean curvature (CMC) hypersurface, with non-empty boundary, which is contained in $\mathcal{B}$ and meets $\partial\mathcal{B}$ at a constant angle along its boundary. When the angle of contact is $\pi/2$, that is, when the hypersurface is orthogonal to $\partial\mathcal{B}$, the hypersurface is said to be a CMC hypersurface with free boundary in $\mathcal{B}$. Capillary hypersurfaces are critical points of an energy functional for volume-preserving variations. In the free boundary case the energy functional is the area functional. We refer to [1] for more details.

Consider a capillary hypersurface defined by a smooth immersion $\psi : \Sigma \longrightarrow \mathcal{B}$ and let N be a global unit normal to Σ along ψ chosen so that its (constant) mean curvature satisfies $H \geq 0$. Denote by $\overline{N}$ the exterior unit normal to $\partial\mathcal{B}$. The angle of contact is the angle $\theta \in (0, \pi)$ between N and $\overline{N}$. We denote by ν the exterior unit normal to $\partial\Sigma$ in Σ

and by $\bar{\nu}$ the unit normal to $\partial\Sigma$ in $\partial\mathcal{B}$ so that $\{N, \nu\}$ and $\{\overline{N}, \bar{\nu}\}$ have the same orientation in $(T\partial\Sigma)^{\perp}$. The angle between ν and $\bar{\nu}$ is also equal to θ.

The index form $\mathcal{I}$ of ψ is the symmetric bilinear form defined on the first Sobolev space $H^1(\Sigma)$ of Σ by

$$\mathcal{I}(f,g) = \int_{\Sigma} \left(\langle \nabla f, \nabla g \rangle - (|\sigma|^2 + \mathrm{Ric}(N)) f g \right) d\Sigma - \int_{\partial\Sigma} q\, f g\, d(\partial\Sigma),$$

where ∇ stands for the gradient for the metric induced by ψ, σ is the second fundamental form of ψ, $\mathrm{Ric}(N)$ is the Ricci curvature of W in the direction N and

$$q = \frac{1}{\sin\theta} \mathrm{II}(\bar{\nu}, \bar{\nu}) + \cot\theta\, \sigma(\nu, \nu).$$

Here II denotes the second fundamental form of $\partial\mathcal{B}$ associated to the unit normal $-\bar{N}$.

Let $\mathcal{F} = \{f \in H^1(\Sigma), \int_{\Sigma} f\, d\Sigma = 0\}$. The immersion ψ is capillarily stable if $\mathcal{I}(f,f) \geq 0$ for all $f \in \mathcal{F}$.

The stability operator on Σ is the linear operator defined on $\mathcal{C}^2(\Sigma)$ by

$$L = \Delta + |\sigma|^2 + \mathrm{Ric}(N)$$

where Δ is the Laplacian on Σ induced by ψ.

A function $f \in \mathcal{F}$ is said to be a Jacobi function of ψ if it lies in the kernel of $\mathcal{I}$, that is, if $\mathcal{I}(f,g) = 0$ for all $g \in \mathcal{F}$. By standard arguments, this is equivalent to saying that $f \in \mathcal{C}^{\infty}(\Sigma)$ and satisfies

$$\begin{aligned} Lf &= \text{constant} && \text{on } \Sigma \\ \frac{\partial f}{\partial \nu} &= q\, f && \text{on } \partial\Sigma \end{aligned}$$

We will also need to consider compact immersed CMC hypersurfaces without boundary. Given such an immersion $\psi : \Sigma \longrightarrow W$ in an orientable Riemannian manifold W, it is said to be stable if $\mathcal{I}(f,f) \geq 0$ for all $f \in \mathcal{F} = \{f \in H^1(\Sigma), \int_{\Sigma} f\, d\Sigma = 0\}$ where $\mathcal{I}$ is the index form defined on $H^1(\Sigma)$ by

$$\mathcal{I}(f,g) = \int_{\Sigma} \left(\langle \nabla f, \nabla g \rangle - (|\sigma|^2 + \mathrm{Ric}(N)) f g \right) d\Sigma.$$

An immersion ψ of a closed or capillary CMC is said to be *strongly stable* if $\mathcal{I}(f,f) \geq 0$ for all $f \in H^1(\Sigma)$.

It is also interesting to consider, as a more general setting for capillarity, the case where the contact angle is constant along each component of $\partial\Sigma$, allowing it to vary from one component to the other. All the previous discussion on stability extends to this more general case (cf. [1]).

3. Stable CMC hypersurfaces with free boundary in a slab in a product space $M \times \mathbb{R}$

Let M be a connected oriented Riemannian manifold of dimension $n \geq 2$. We discuss in this section the case of CMC immersed hypersurfaces with free boundary in slabs

$M \times [0,l], l > 0$, of the product space $M \times \mathbb{R}$. The function q vanishes identically and the index form takes a simple form:

$$\mathcal{I}(f,g) = \int_\Sigma \left(\langle \nabla f, \nabla g \rangle - (|\sigma|^2 + \mathrm{Ric}(N)) f g \right) d\Sigma.$$

The simplest examples of free boundary CMC hypersurfaces in a slab $M \times [0,l]$ are cylinders over closed CMC hypersurfaces in M. We will discuss the stability of cylinders. For this, we will need a criterion for stability of closed CMC hypersurfaces. It was established by Koiso ([10], Theorem 1.3) for compact CMC surfaces with boundary in $\mathbb{R}^3$ but the result readily extends to the closed case in general ambient manifolds.

Theorem 3.1. *Let $\psi : \Sigma \longrightarrow M$ be a closed and two-sided immersed CMC hypersurface in a Riemannian manifold M of dimension $n \geq 2$. Denote by $\lambda_1(\Sigma)$ and $\lambda_2(\Sigma)$, respectively, the first and second eigenvalues of the stability operator L on Σ. The following hold*

(I) $\lambda_1(\Sigma) \geq 0$ if and only if ψ is strongly stable.

(II) If $\lambda_1(\Sigma) < 0$ and $\lambda_2(\Sigma) > 0$, then there exists a uniquely determined smooth function u on Σ such that $Lu = 1$ and ψ is stable if and only if $\int_\Sigma u \, d\Sigma \geq 0$.

(III) If $\lambda_1(\Sigma) < 0$ and $\lambda_2(\Sigma) = 0$ and there exists an eigenfunction associated to $\lambda_2(\Sigma)$ satisfying $\int_\Sigma g \, dA \neq 0$, then ψ is unstable.

(IV) If $\lambda_1(\Sigma) < 0$ and $\lambda_2(\Sigma) = 0$ and $\int_\Sigma g \, dA = 0$ for any eigenfunction g associated to λ_2. Denoting by E the eigenspace associated to $\lambda_2(\Sigma) = 0$, there exists a uniquely determined smooth function $u \in E^\perp$ (the orthogonal to E in the L^2 sense) which satisfies $Lu = 1$. Then ψ is stable if and only if $\int_\Sigma u \, dA \geq 0$.

(V) If $\lambda_2(\Sigma) < 0$, then ψ is unstable.

Remark 3.2. *If $\lambda_1(\Sigma) < 0$ and $\lambda_2(\Sigma) \geq 0$ and we know beforehand the existence of a smooth function u satisfying $Lu = 1$, then, it is easily checked that (II) and (IV) can be restated simply by saying that ψ is stable if and only if $\int_\Sigma u \, dA \geq 0$.*

We can now give the following characterization of stable cylinders.

Theorem 3.3. *Let $\psi : \Gamma \to M$ be an immersion with constant mean curvature H of a closed oriented manifold Γ of dimension $n \geq 1$ in an $(n+1)$-dimensional oriented manifold M and let $l > 0$. The map $\overline{\psi}$*

$$\overline{\psi} := \psi \times \{id\} : \Gamma \times [0,l] \longrightarrow M \times [0,l],$$

is an immersion of constant mean curvature $\frac{(n-1)}{n} H$ with free boundary. Denote by $\lambda_1(\Gamma)$ the first eigenvalue of the stability operator L on Γ. The following hold

(i) If ψ is unstable, then $\overline{\psi}$ is unstable too.

(ii) If ψ is stable, then $\overline{\psi}$ is stable if and only if

$$\lambda_1(\Gamma) + \left(\frac{\pi}{l}\right)^2 \geq 0. \tag{3.1}$$

Proof. The first statement about $\overline{\psi}$ is straightforward.

Denote by $\overline{L}$ the stability operator of $\overline{\psi}$ and by $\overline{\mathcal{I}}$ the associated index form.

(i) Suppose ψ is unstable, then there exists $u \in H^1(\Gamma)$ such that $\int_\Gamma u = 0$ and $\mathcal{I}(u,u) < 0$. Define $\bar{u}$ on $\Gamma \times [0,l]$ by $\bar{u}(p,t) = u(p)$, for all $(p,t) \in \Gamma \times [0,l]$. Then clearly $\bar{u} \in H^1(\Gamma \times [0,l])$, $\int_{\Gamma\times[0,l]} \bar{u} = 0$ and $\overline{\mathcal{I}}(\bar{u},\bar{u}) = \mathcal{I}(u,u) < 0$, which shows that $\overline{\psi}$ is unstable.

(ii) Suppose ψ is stable.

If $\lambda_1(\Gamma) + \left(\frac{\pi}{l}\right)^2 < 0$, let u be an eigenfunction of L associated to $\lambda_1(\Gamma)$. Then the function $v := u\cos(\pi t/l)$ on $\Gamma \times [0,l]$ satisfies $\int_{\Gamma\times[0,l]} v = 0$ and $\overline{\mathcal{I}}(v,v) = (\lambda_1(\Gamma) + \left(\frac{\pi}{l}\right)^2) \int_{\Gamma\times[0,l]} v^2 < 0$. So $\overline{\psi}$ is unstable.

Suppose now that $\lambda_1(\Gamma) + \left(\frac{\pi}{l}\right)^2 \geq 0$. We will first show that the CMC immersion $\widehat{\psi} := \psi \times \{\mathrm{id}\} : \Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right) \longrightarrow M \times \mathbb{S}^1\left(\frac{l}{\pi}\right)$ is stable, where $\mathbb{S}^1\left(\frac{l}{\pi}\right)$ denotes the circle of radius $\frac{l}{\pi}$.

Let Ric and $\widehat{\mathrm{Ric}}$, be the Ricci curvature tensors of M and $M \times \mathbb{R}$, respectively, and denote by σ, and $\widehat{\sigma}$, the second fundamental forms of ψ and $\widehat{\psi}$, respectively. We note that $|\widehat{\sigma}| = |\sigma|$ and if N is a unit field normal to Γ along ψ, then it is also normal to $\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)$ along $\widehat{\psi}$ and $\widehat{\mathrm{Ric}}(N,N) = \mathrm{Ric}(N,N)$. We denote by $\widehat{L}$ be the stability operator of $\widehat{\psi}$ and let $\widehat{\mathcal{I}}$ be the associated index form.

The eigenvalues and eigenfunctions of the eigenvalue problem

$$f'' + \mu f = 0$$

on $\mathbb{S}^1\left(\frac{l}{\pi}\right)$ are given by $\mu_n = (n\pi/l)^2$, $n = 0,1,2,\ldots$ and $f_n(t) = \cos\sqrt{\mu_n}t$.

Let $\lambda_1(\Gamma) < \lambda_2(\Gamma) \leq .... \leq \lambda_k(\Gamma) \leq ...$ be the sequence of eigenvalues of the operator L on the closed manifold Γ. As the function $|\widehat{\sigma}|^2 + \widehat{\mathrm{Ric}}(N,N) = |\sigma|^2 + \mathrm{Ric}(N,N)$ depends only on the first factor of the product $\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)$, a standard argument shows that the eigenvalues of the operator $\widehat{L}$ on $\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)$ are given by $\lambda_m(\Gamma) + n^2\pi^2/l^2$, $m,n \in \mathbb{N}$.

In particular, the first and second eigenvalues of $\widehat{L}$ on $\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)$ verify:

$$\lambda_1\left(\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)\right) = \lambda_1(\Gamma) \tag{3.2}$$

$$\lambda_2\left(\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)\right) = \min\{\lambda_1(\Gamma) + \frac{\pi^2}{l^2}, \lambda_2(\Gamma)\} \tag{3.3}$$

If $\lambda_1(\Gamma) \geq 0$, then, by (3.2), $\lambda_1(\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)) \geq 0$, that is, $\widehat{\psi}$ is strongly stable. So we assume $\lambda_1(\Gamma) < 0$. In this case we know that $\lambda_2(\Gamma) \geq 0$ (Theorem 3.1 (V)). Therefore, by (3.3), $\lambda_2\left(\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)\right) \geq 0$.

Consider then the solution $u \in \mathcal{C}^\infty(\Gamma)$ to the equation $Lu = 1$ described in Theorem 3.1, (II) or (IV). Then, the function $\hat{u}$, defined on $\Gamma \times \mathbb{S}^1(\frac{l}{\pi})$ by $\hat{u}(p,t) = u(p)$, for $(p,t) \in \Gamma \times \mathbb{S}^1(\frac{l}{\pi})$, verifies $\widehat{L}\hat{u} = Lu = 1$ and $\int_{\Gamma\times\mathbb{S}^1(\frac{l}{\pi})} \hat{u} = 2l\int_\Gamma u \geq 0$. Theorem 3.1 (and Remark 3.2) shows that $\widehat{\psi}$ is stable.

We now show that stability of $\widehat{\psi}$ implies stability of $\overline{\psi}$.

Let $u \in H^1(\Gamma \times [0,l])$ satisfying $\int_{\Gamma\times[0,l]} u = 0$. We extend u to a function in $H^1(\Gamma \times [0,2l])$ by reflection across $M \times \{l\}$, that is, we set $u(p,t) = u(p,2l-t)$ for

all $p \in \Gamma$ and $t \in [l, 2l]$. Clearly this function gives rise to a function $\hat{u} \in H^1\left(\Gamma \times \mathbb{S}^1\left(\frac{l}{\pi}\right)\right)$ satisfying $\int_{\Gamma\times\mathbb{S}^1(\frac{l}{\pi})} \hat{u} = 0$ and $\mathcal{I}(u,u) = \frac{1}{2}\widehat{\mathcal{I}}(\hat{u},\hat{u}) \geq 0$. Therefore $\overline{\psi}$ is stable. This completes the proof. □

For instance, consider the case of a tube of radius ρ in $\mathbb{M}^n(\kappa) \times [0,l]$, $\kappa = -1, 0, 1$, $n \geq 2$, where $\mathbb{M}^n(-1) = \mathbb{H}^n, \mathbb{M}^n(0) = \mathbb{R}^n$ and $\mathbb{M}^n(1) = \mathbb{S}^n$. The stability operator of the tube writes

$$L = \Delta + (n-1)(c_\kappa(\rho)^{-1} + \kappa)$$

with

$$\begin{aligned} c_{-1}(\rho) &= \tanh^2 \rho \\ c_0(\rho) &= \rho^2 \\ c_1(\rho) &= \tan^2 \rho \end{aligned}$$

As round spheres in space forms are stable, we get the following characterization of stable tubes in $\mathbb{M}^n(\kappa) \times \mathbb{R}$, the case of tubes in $\mathbb{R}^3$ was treated by Athanassenas [3] and Vogel [20].

Corollary 3.4. *A tube of radius ρ and height l in $\mathbb{H}^n \times [0,l]$ is stable if and only if $\pi \sinh \rho \geq \sqrt{n-1}\, l$.*

A tube of radius ρ and height l in $\mathbb{R}^n \times [0,l]$ is stable if and only if $\pi\rho \geq \sqrt{n-1}\, l$.

A tube of radius ρ and height l in $\mathbb{S}^n \times [0,l]$ is stable if and only if $\pi \sin \rho \geq \sqrt{n-1}\, l$.

We now prove a general fact about the geometry of stable free boundary CMC hypersurfaces in a slab in $M \times \mathbb{R}$. It extends the result proved in [1] for the case $M = \mathbb{R}^n$.

Theorem 3.5. *Let M be an $n-$dimensional connected and oriented Riemannian manifold and $\psi : \Sigma \to M \times [0,l]$ a stable free boundary immersion connecting the two boundary components $M \times \{0\}$ and $M \times \{l\}$, $l > 0$, of a slab in the Riemannian product $M \times \mathbb{R}$.*

Then, either $\psi(\Sigma)$ is a vertical cylinder over a closed immersed stable CMC in M or $\psi(\Sigma)$ is locally a vertical graph and in this case the lowest eigenvalue of the stability operator on Σ with Dirichlet boundary condition is equal to 0.

Furthermore, suppose $\psi(\Sigma)$ is not a vertical cylinder. If M is simply connected and the restriction of ψ to each component of $\partial\Sigma$ is an embedding, then $\psi(\Sigma)$ is a global vertical graph over a domain in M. In particular, if M is either the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ or a hemisphere $\mathbb{S}^n_+$, then, under the same hypotheses, $\psi(\Sigma)$ is rotationally invariant around an axis $\{p\} \times \mathbb{R}$, for some $p \in M$.

Proof. We start with the following general fact. Let $\psi : \Sigma \to M$ be a CMC immersion of an orientable manifold Σ of dimension n in an orientable Riemannian manifold M of dimension $n+1$ and N a global unit normal to Σ. If X a Killing vector field on M then the function $\langle X \circ \psi, N\rangle$ lies in the kernel of the Jacobi operator L of Σ. We include a proof of this fact for the reader's convenience. The Jacobi operator is also known to be the linearized operator of the mean curvature functional. More precisely, let ψ_t be a smooth deformation, through immersions, of $\psi_0 = \psi$ and let H_t be the mean curvature function of ψ_t, then:

$$2\frac{\partial H_t}{\partial t}|_{t=0} = L(\langle \frac{\partial \psi_t}{\partial t}|_{t=0}, N\rangle). \tag{3.4}$$

Now, let ϕ_t be the local flow of isometries of M generated by the Killing field X and consider the deformation $\psi_t = \phi_t \circ \psi$ of ψ. Clearly, $H_t = H$ for all t and so the left-hand side of (3.4) vanishes. Furthermore, $\frac{\partial \psi_t}{\partial t}|_{t=0} = X \circ \psi$. This shows that the function $\langle X \circ \psi, N\rangle$ lies in the kernel of L.

In our case, as the vertical unit vector field $\frac{\partial}{\partial t}$ is a Killing field of the product $M \times \mathbb{R}$, the function $v = \langle \frac{\partial}{\partial t}, N\rangle$ verifies $Lv = 0$.

If v is identically zero, then $\psi(\Sigma)$ is a vertical cylinder whose base is an immersed CMC hypersurface in M and which is moreover stable by Theorem 3.3.

Assume v is not identically zero. We will show it has a sign in the interior of Σ. Suppose by contradiction that v changes sign and consider the functions $v_+ = \max\{v, 0\}$ and $v_- = \min\{v, 0\}$ which lie in the Sobolev space $H^1(\Sigma)$. Using the fact that $v = 0$ on $\partial\Sigma$, one has

$$\begin{aligned}
\mathcal{I}(v_+, v_+) &= \int_\Sigma \{\langle \nabla v_+, \nabla v_+\rangle - (|\sigma|^2 + \mathrm{Ric}(N))(v_+)^2\}\, d\Sigma \\
&= \int_\Sigma \{\langle \nabla v, \nabla v_+\rangle - (|\sigma|^2 + \mathrm{Ric}(N)) v v_+\}\, d\Sigma \\
&= -\int_\Sigma \{(\Delta v) v_+ + (|\sigma|^2 + \mathrm{Ric}(N)) v v_+\}\, d\Sigma \\
&= 0
\end{aligned}$$

and similarily $\mathcal{I}(v_-, v_-) = 0$. As we supposed that v changes sign, there exists $a \in \mathbb{R}$ such that $\int_\Sigma (v_+ + a v_-)\, d\Sigma = 0$. So we can use $\widetilde{v} := v_+ + a v_-$ as a test function for stability. We have

$$\mathcal{I}(\widetilde{v}, \widetilde{v}) = \mathcal{I}(v_+, v_+) + 2a\mathcal{I}(v_+, v_-) + a^2 \mathcal{I}(v_-, v_-) = 0.$$

Since ψ is stable, we conclude that $\widetilde{v}$ is a Jacobi function and so is, in particular, smooth. We distinguish two cases:

(i) If $a \neq 1$, then we conclude that v_+ and v_- are smooth and so we can write $Lv = Lv_+ + Lv_- = 0$ and $L\widetilde{v} = Lv_+ + aLv_- = 0$. It follows that $Lv_+ = 0$ and so, by the unique continuation principle [2], v_+ has to be identically 0, which is a contradiction.

(ii) If $a = 1$. Then $v = \widetilde{v}$ is a Jacobi function and so satisfies $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Sigma$. Let $p \in \partial\Sigma$ with $\psi(p) \in M \times \{0\}$ and Ω a neighborhood of p in Σ, diffeomorphic to the halfspace $\mathbb{R}^n_+$, such that $\psi : \Omega \to M \times [0, l]$ is an embedding. Identify $S = \psi(\Omega)$ with Ω and call τ the reflection in $M \times \mathbb{R}$ through $M \times \{0\}$. Then $\overline{S} := S \cup \tau(S)$ is a $\mathcal{C}^2$ hypersurface in $M \times \mathbb{R}$ with constant mean curvature; $\overline{S}$ is therefore smooth. We now extend v to a function $\bar{v}$ on $\overline{S}$ by setting $\bar{v} = 0$ on $\tau(S)$. As $\frac{\partial v}{\partial \nu} = 0$ on ∂S, we have $\bar{v} \in \mathcal{C}^1(\overline{S})$. Furthermore, because $v = \frac{\partial v}{\partial \nu} = 0$ on ∂S, one checks that $\bar{v}$ satisfies the equation $\overline{L}\bar{v} = 0$ on $\overline{S}$ in the sense of distributions; $\overline{L}$ denoting the Jacobi operator on $\overline{S}$. By regularity theory for elliptic PDEs, $\bar{v}$ is smooth. As $\bar{v}$ vanishes on a non empty subset, the unique continuation principle [2] implies that $\bar{v} \equiv 0$ on S. Again, by the unique continuation principle, one has $v \equiv 0$ on Σ, which is a contradiction.

We have thus shown that the function v does not change sign in Σ. We will assume $v \geq 0$, the case $v \leq 0$ being similar. The function v satisfies

$$\begin{cases} v & \geq 0, \\ Lv & = 0. \end{cases}$$

As we are assuming v is not identically zero, by the strong maximum principle (Theorem 2.9 in [9]) we know that $v > 0$ on the interior of Σ. So the interior of $\psi(\Sigma)$ is a local vertical graph. Moreover, since v does not vanish in the interior of Σ, it is a first eigenfunction of L and so 0 is the lowest eigenvalue of L for the Dirichlet problem on Σ. This proves the first statement.

Let us now prove the second statement. Assuming M is simply connected and $\psi(\Sigma)$ is not a cylinder, we will show it is globally a vertical graph by analyzing its behaviour near its boundary.

For the sake of simplicity, we set $M_0 = M \times \{0\}$ and $M_1 = M \times \{l\}$. Let $\Gamma_1, \dots, \Gamma_k$ denote the connected components of $\partial\Sigma$. By hypothesis, ψ restricted to Γ_i is an embedding and so $\psi(\Gamma_i)$ separates the slice containing it, among M_0 and M_1, into two connected components ([8]).

Denote by $P : M \times \mathbb{R} \to M$ the projection on the second factor and set $F = P \circ \psi$. Fix $i = 1, \dots, k$ and let $\varphi : \mathcal{U} \longrightarrow \mathbb{R}$ be a smooth function defined in a neighborhood $\mathcal{U} \subset M$ of $\psi(\Gamma_i)$ such that $\psi(\Gamma_i) = \{\varphi = 0\}$ and for each $p \in \Gamma_i$, $\nabla\varphi(\psi(p)) = N(p)$. For instance we may take φ to be the signed distance function to $\psi(\Gamma_i)$.

Let us take a point $p \in \Gamma_i$, and a curve $\gamma : (-\epsilon, 0] \to \Sigma$ parametrized by arc-length with $\gamma(0) = p$ and $\dot\gamma(0) = \nu(p)$. Set $\overline{\gamma}(s) := \psi(\gamma(s)) = (\alpha(s), t(s))$ where $\alpha = F \circ \gamma : (-\epsilon, 0] \to M$ and define the real function h on $(-\epsilon, 0]$ by $h(s) = \varphi(\alpha(s))$.

Denoting by D the connection on M, and by $\overline{D}$ the one on $M \times \mathbb{R}$, we have

$$\begin{aligned}
\dot h(0) &= h(0) = 0,\\
\ddot h(0) &= \nabla^2\varphi(\dot\alpha(0), \dot\alpha(0)) + \langle \nabla\varphi(\psi(p)), \frac{D\dot\alpha}{ds}(0)\rangle\\
&= \langle N(p), \frac{\overline{D}\dot{\overline{\gamma}}}{ds}(0)\rangle\\
&= \sigma(\nu(p), \nu(p)).
\end{aligned}$$

Note that

$$\frac{\partial v}{\partial \nu} = -\sigma(\nu,\nu)\langle \nu, e_{n+1}\rangle = \begin{cases} +\sigma(\nu,\nu) & \text{if} \quad \psi(\Gamma_i) \subset M_0,\\ -\sigma(\nu,\nu) & \text{if} \quad \psi(\Gamma_i) \subset M_1.\end{cases}$$

By the boundary point maximum principle (see, for instance, Theorem 2.8 in [9]), $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Sigma$. So for t small, $F(\gamma(t))$ lies in the component of $M \setminus F(\Gamma_i)$ which has $N(p)$ as outwards (resp. inwards) pointing normal at $F(p)$ if $\psi(\Gamma_i) \subset M_0$ (resp. if $\psi(\Gamma_i) \subset M_1$). It follows that there is a thin strip in the interior of Σ around Γ_i which projects on this component. We call D_i the component of $M \setminus F(\Gamma_i)$ which does not intersect this projection.

We let $\widetilde\Sigma$ denote the (topological) manifold obtained by attaching, for each $i = 1, \dots, k$, D_i to Σ using the diffeomorphism $\psi|_{\Gamma_i} : \Gamma_i \to \partial D_i$ (see, for instance, Theorem 9.29 in [11] for the details of such a construction).

We define $\widetilde F : \widetilde\Sigma \to M$ by

$$\widetilde F = \begin{cases} F & \text{on} \quad \Sigma\\ \text{identity} & \text{on} \quad D_i, i = 1, \dots, k.\end{cases}$$

Note that $\widetilde{F}$ is well defined since, by construction of $\widetilde{\Sigma}$, each $x \in \Gamma_i$ is identified with $\psi(x) \in \partial D_i$. As the interior of $\psi(\Sigma)$ is locally a vertical graph and thanks to the above analysis of the behaviour of ψ near $\partial\Sigma$, we see that $\widetilde{F}$ is a local homeomorphism. It is also clear that $\widetilde{F}$ is a proper map, thus it is a covering map. Therefore $\widetilde{F}$ is a global homeomorphism onto M. Consequently, $\psi(\Sigma)$ is a graph over a domain in M. In particular, when $M = \mathbb{R}^n, \mathbb{H}^n$ or $\mathbb{S}^n_+$, Alexandrov's reflection technique shows that $\psi(\Sigma)$ is a hypersurface of revolution around a vertical axis, see [19]. □

Remark 3.6. *It is clear from the proof that a similar result is true for stable CMC hypersurfaces with free boundary in a half-space* $M \times [0, +\infty)$.

It is not difficult to deduce from Theorem 3.3 that if the Ricci curvature of M is bounded from below by a positive constant κ, then there is no stable cylinder in $M \times [0, l]$ for $l > \pi/\sqrt{\kappa}$ (see the argument below). When M has dimension 2, we have the following stronger statement.

Theorem 3.7. *Let M be an orientable Riemannian surface with Gaussian curvature $K \geq \kappa > 0$. If*

$$l \geq \frac{4\pi}{\sqrt{3\kappa}}$$

then there is no immersed stable free boundary CMC surface in $M \times \mathbb{R}$ connecting $M \times \{0\}$ to $M \times \{l\}$.

Proof. Let $\psi : \Sigma \to M \times [0, l]$ be a stable CMC immersion with free boundary. Suppose $\psi(\Sigma)$ is not a cylinder. By Theorem 3.5, we know that 0 is the lowest eigenvalue of the stability operator L on Σ with Dirichlet boundary condition. Denote by H the mean curvature of Σ and by S the scalar curvature function of $M \times \mathbb{R}$. We have

$$3H^2 + S(p, t) = 3H^2 + K(p) \geq \kappa, \quad \text{for } (p, t) \in M \times \mathbb{R}.$$

We can therefore apply Theorem 1 in [17] which gives an upper bound for the intrinsic distance in Σ to $\partial\Sigma$. More precisely, for each $x \in \Sigma$,

$$d_\Sigma(x, \partial\Sigma) \leq \frac{2\pi}{\sqrt{3\kappa}}.$$

If $\psi : \Sigma \to M \times [0, l]$ connects $M \times \{0\}$ to $M \times \{l\}$, take $x \in \Sigma$ such that $\psi(x) \in M \times \{\frac{l}{2}\}$. Then

$$\frac{l}{2} \leq d_\Sigma(x, \partial\Sigma) \leq \frac{2\pi}{\sqrt{3\kappa}}.$$

Moreover, if we have the equality $l/2 = 2\pi/\sqrt{3\kappa}$, then $d_\Sigma(x, \partial\Sigma) = l/2$ and so at least one of the geodesic segments $\{x\} \times [0, l/2]$ or $\{x\} \times [l/2, l]$ is contained in Σ. In this case, by Theorem 3.5, $\psi(\Sigma)$ has to be a cylinder.

To conclude we need to check that cylinders over closed constant geodesic curves are unstable for $l \geq 4\pi/\sqrt{3\kappa}$. In fact a stronger statement is true. Consider a curve γ of constant geodesic curvature h in M. The lowest eigenvalue $\lambda_1(\gamma)$ of the stability operator on γ satisfies

$$\lambda_1(\gamma) \leq \frac{\mathcal{I}(1, 1)}{\int_\gamma ds} = -\frac{\int_\gamma (h^2 + K) ds}{\int_\gamma ds} \leq -\kappa.$$

For $l > \pi/\sqrt{\kappa}$, we get

$$\lambda_1(\gamma) + \frac{\pi^2}{l^2} \leq -\kappa + \frac{\pi^2}{l^2} < 0$$

and so, according to Theorem 3.3, the cylinder is unstable. This completes the proof. □

4. Stable capillary surfaces of genus zero in certain warped products

In this section we are interested in stable capillary surfaces in warped product spaces of the type $[0,l] \times_f M$ where $M = \mathbb{R}^2, \mathbb{S}^2$ or $\mathbb{H}^2$ and $f : [0,l] \to \mathbb{R}$ is a smooth positive function. More precisely, $[0,l] \times_f M$ is the product space $[0,l] \times M$ endowed with the metric (with some abuse of notation)

$$g = dt^2 + f(t)^2 g_0,$$

g_0 being the metric on M.

An important feature of these spaces we will use is that if $\varphi : M \longrightarrow M$ is an isometry, then its trivial extension $\{\mathrm{id}\} \times \varphi : [0,l] \times_f M \longrightarrow [0,l] \times_f M$ is an isometry too.

We have the following extension to these spaces of a theorem proved in the Euclidean case in [1].

Theorem 4.1. *Let $\psi : \Sigma \to [0,l] \times_f M$ be an immersed capillary surface of genus zero connecting the two boundary components of $[0,l] \times_f M$ where $M = \mathbb{R}^2, \mathbb{H}^2$, or $\mathbb{S}^2$ and having constant contact angles θ_0 and θ_1 with $\{0\} \times M$ and $\{l\} \times M$, respectively.*

If ψ is stable then $\psi(\Sigma)$ is a surface of revolution around an axis $\mathbb{R} \times \{x\}$, $x \in M$.

Proof. Let γ be a connected component of $\partial\Sigma$ such that $\psi(\gamma)$ lies on $\{0\} \times M$. We claim there is a circle $\mathcal{C}$ in M bounding a disk containing $\psi(\gamma)$ and touching $\psi(\gamma)$ at least at 2 points. For $M = \mathbb{R}^2$, the circumscribed circle about $\psi(\gamma)$ has this property (cf. [14]). When $M = \mathbb{H}^2$, we can just take the (Euclidean) circumscribed circle about $\psi(\gamma)$ in the half-space model since Euclidean circles in this model are also metric circles for the hyperbolic metric. When $M = \mathbb{S}^2$, we pick a point $p \in \mathbb{S}^2 \setminus \psi(\gamma)$ and perform stereographic projection σ from the point p onto the Euclidean plane $\mathbb{R}^2$. Let $\mathcal{C}^*$ be the circumscribed circle in $\mathbb{R}^2$ about $\sigma(\psi(\gamma))$. Its inverse image $\mathcal{C} := \sigma^{-1}(\mathcal{C}^*)$ is a circle in $\mathbb{S}^2$ which has the required property.

We will show that $\psi(\Sigma)$ is a surface of revolution around the vertical axis passing through the center x of $\mathcal{C}$. In the case of $\mathbb{S}^2$, we take x to be the center of the disk bounded by $\mathcal{C}$ that contains $\psi(\gamma)$.

Let us consider the Jacobi function u on Σ induced by the rotations around the vertical axis passing through x. The function u verifies

$$\begin{cases} Lu = 0 & \text{on} \quad \Sigma \\ \frac{\partial u}{\partial \nu} = q\, u & \text{on} \quad \partial\Sigma \end{cases} \tag{4.1}$$

We will prove that $u \equiv 0$ on Σ.

Suppose, by contradiction, u is not identically zero. Then its nodal set $u^{-1}(0)$ in the interior of Σ has the structure of a graph (cf. [6]). We will show that u has at least 3 nodal domains by analyzing the set $u^{-1}(0) \cap \gamma$; a nodal domain of u being a connected component of $\Sigma \setminus u^{-1}(0)$.

We first note that, because of the boundary condition satisfied by u, if $p \in u^{-1}(0) \cap \partial\Sigma$ then $\frac{\partial u}{\partial \nu}(p) = 0$. It follows from the boundary point maximum principle (see Theorem 2.8 in [9]) that u changes sign in any neighborhood of $p \in u^{-1}(0) \cap \partial\Sigma$ unless u is identically zero in a neighborhood of p. In the latter case, by the unique continuation principle [2], u would vanish everywhere on Σ, contradicting our assumption. Consequently each such point lies on the boundary of at least 2 components of the set $\{u \neq 0\}$.

If $p \in \partial\Sigma$ is a critical point of the distance function to x restricted to γ, then one can check that $u(p) = 0$. Now, we observe that by the choice of $\mathcal{C}$, there are at least 3 points in $u^{-1}(0) \cap \gamma$. Indeed, we already know that there are two points in $u^{-1}(0) \cap \gamma$. A third one is a point of γ whose image by ψ is a closest one to the center x of $\mathcal{C}$.

Since by hypothesis Σ is topologically a planar domain, using the above information and the Jordan curve theorem, it is easy to see this implies that u has at least 3 nodal domains.

Denote by Σ_1 and Σ_2 two of these components and consider the following function in the Sobolev space $H^1(\Sigma)$:

$$\widetilde{u} = \begin{cases} u & \text{on} \quad \Sigma_1 \\ \alpha\, u & \text{on} \quad \Sigma_2 \\ 0 & \text{on} \quad \Sigma \setminus (\Sigma_1 \cup \Sigma_2) \end{cases}$$

where $\alpha \in \mathbb{R}$ is chosen so that $\int_\Sigma \widetilde{u}\, dA = 0$. Using (4.1) we compute

$$\begin{aligned} \int_{\Sigma_1} \{\langle \nabla \widetilde{u}, \nabla \widetilde{u}\rangle - (|\sigma|^2 + \mathrm{Ric}(N))\widetilde{u}^2\} dA &= \int_{\Sigma_1} \{\langle \nabla u, \nabla \widetilde{u}\rangle - (|\sigma|^2 + \mathrm{Ric}(N))\, u\widetilde{u}\} dA \\ = -\int_{\Sigma_1} (\Delta u + (|\sigma|^2 + \mathrm{Ric}(N))u)\widetilde{u}\, dA &+ \int_{\partial\Sigma_1} \widetilde{u}\frac{\partial u}{\partial \nu}\, ds \\ &= \int_{\partial\Sigma_1 \cap \partial\Sigma} q\widetilde{u}^2\, ds \end{aligned}$$

Using a similar computation on Σ_2, we deduce that $\mathcal{I}(\widetilde{u}, \widetilde{u}) = 0$. As Σ is stable, we conclude that $\widetilde{u}$ is a Jacobi function. Indeed, the quadratic form on $\mathcal{F}$ associated to $\mathcal{I}$ has a minimum at $\widetilde{u}$ and so $\widetilde{u}$ lies in the kernel of $\mathcal{I}$. However, $\widetilde{u}$ vanishes on a non empty open set. By the unique continuation principle [2], $\widetilde{u}$ has to vanish everywhere, which is a contradiction.

Therefore $u \equiv 0$. This means that $\psi(\Sigma)$ is a surface of revolution around the axis through x. □

The above result applies to the region bounded by two spheres centered at the origin for metrics on $\mathbb{R}^3$ that are invariant under the group $SO(3)$. This is, for instance, the case for the region bounded by two concentric spheres in $\mathbb{R}^3, \mathbb{H}^3$ or $\mathbb{S}^3$. Another interesting case to which the result applies is the region bounded by two parallel horospheres in $\mathbb{H}^3$. Indeed, consider the half-space model of $\mathbb{H}^3$, that is,

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3, x_3 > 0\}$$

equipped with the metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}.$$

Making the change of variables $t = \log x_3$, we see that the metric writes

$$ds^2 = e^{-2t}(dx_1^2 + dx_2^2) + dt^2.$$

Otherwise said, $\mathbb{H}^3$ can be viewed as the warped product $\mathbb{R} \times_{e^{-t}} \mathbb{R}^2$ and we may assume, up to a rigid motion, that the two parallel horospheres are slices $\{x_3 = \text{cst}\}$. We can thus state the following

Corollary 4.2. *Let $\psi : \Sigma \to \mathbb{H}^3$ be an immersed capillary surface of genus zero connecting two parallel horospheres. If ψ is stable then $\psi(\Sigma)$ is a surface of revolution around an axis orthogonal to the horospheres.*

The isoperimetric problem in regions bounded by two parallel horospheres in $\mathbb{H}^3$ was studied in [5] by Chaves, da Silva and Pedrosa.

5. Stable closed CMC surfaces in a product $M \times \mathbb{S}^1(r)$

In this section, we consider stable CMC surfaces in a product $M \times \mathbb{S}^1(r)$, where M is a Riemannian surface with curvature positively bounded from below, $K \geq \kappa$, with $\kappa > 0$ a constant and $\mathbb{S}^1(r)$ is the circle of radius $r > 0$. We have the following result which is in the same vein as Theorem 3.7.

Theorem 5.1. *Let $\psi : \Sigma \to M \times \mathbb{S}^1(r)$ be a stable CMC immersion of a closed connected and oriented surface Σ in $M \times \mathbb{S}^1(r)$ where M is an orientable surface with Gaussian curvature satisfying $K \geq \kappa > 0$, for some constant κ, and $\mathbb{S}^1(r)$ is the circle of radius $r > 0$. Denote by $p : M \times \mathbb{S}^1(r) \to \mathbb{S}^1(r)$ the canonical projection. Suppose that*

$$r \geq \frac{4}{\sqrt{3\kappa}},$$

then the induced homomorphism

$$(p \circ \psi)_* : \pi_1(\Sigma) \to \pi_1(\mathbb{S}^1(r))$$

is trivial and therefore ψ lifts to a stable CMC immersion $\widetilde{\psi} : \Sigma \to M \times \mathbb{R}$.

Proof. We first derive an upper bound for the diameter of Σ. Denote by d_Σ the intrinsic distance on Σ and by $\text{diam}(\Sigma)$ its diameter and let $x, y \in \Sigma$ be such that $d := d_\Sigma(x, y) = \text{diam}(\Sigma)$. Then the disks $B_{d/2}(x)$ and $B_{d/2}(y)$ of radius $d/2$ centered at x and y, respectively, have disjoint interiors. Denote by $\lambda_1(B_{d/2}(x))$ (resp. $\lambda_1(B_{d/2}(y))$) the first eigenvalue of the operator L with Dirichlet boundary condition on $B_{d/2}(x)$ (resp. on $B_{d/2}(y)$). Since ψ is stable, the second eigenvalue of L on Σ verifies $\lambda_2(\Sigma) \geq 0$ (cf. Theorem 3.1). By the min-max characterization of the eigenvalues of L, we have

$$\lambda_2(\Sigma) = \inf_{E \subset H^1(\Sigma), \dim E = 2} \sup_{u \in E \setminus \{0\}} \frac{-\int_\Sigma uLu \, d\Sigma}{\int_\Sigma u^2 \, d\Sigma}.$$

Since $B_{d/2}(x)$ and $B_{d/2}(y)$ have disjoint interiors, it follows that

$$\lambda_2(\Sigma) \leq \max\{\lambda_1(B_{d/2}(x)), \lambda_1(B_{d/2}(y))\}.$$

Therefore at least one of the numbers $\lambda_1(B_{d/2}(x))$ and $\lambda_1(B_{d/2}(y))$ is nonnegative. Denoting by H the mean curvature of Σ and by S the scalar curvature function of $M \times \mathbb{S}^1(r)$, we have

$$3H^2 + S(p, \theta) = 3H^2 + K(p) \geq \kappa, \quad \text{for } (p, \theta) \in M \times \mathbb{S}^1(r).$$

We can thus apply Theorem 1 in [17] which gives the upper bound $d/2 \leq 2\pi/\sqrt{3\kappa}$, that is,

$$\text{diam}(\Sigma) \leq \frac{4\pi}{\sqrt{3\kappa}}. \tag{5.1}$$

Suppose now that the induced homomorphism $\pi_1(\Sigma) \to \pi_1(\mathbb{S}^1(r))$ is not trivial and consider a loop $\gamma : \mathbb{S}^1 \to \Sigma$ such that the loop $p \circ \psi \circ \gamma : \mathbb{S}^1 \to \mathbb{S}^1(r)$ is nontrivial. We may assume that γ is a piecewise $\mathcal{C}^1$-immersion. Denote by d_0 the distance on $M \times \mathbb{S}^1(r)$, by d_Σ the one on Σ and by d_1 the one on $\mathbb{S}^1(r)$. Let $x \in \gamma(\mathbb{S}^1)$, then $p \circ \psi \circ \gamma(\mathbb{S}^1)$ is not entirely contained in an interval of radius $< \pi r$ centered at $p(\psi(x))$ in $\mathbb{S}^1(r)$, because otherwise the loop $p \circ \psi \circ \gamma$ would be trivial. Therefore, there exists a point $y \in \gamma(\mathbb{S}^1)$ such that

$$\pi r = d_1(p(\psi(x)), p(\psi(y)) \leq d_0(\psi(x), \psi(y) \leq d_\Sigma(x, y) \leq \operatorname{diam}(\Sigma). \tag{5.2}$$

It follows from (5.1) and (5.2) that $r \leq 4/\sqrt{3\kappa}$.

Moreover, if $r = 4/\sqrt{3\kappa}$, then all the inequalities in (5.2) are equalities. In particular $\pi r = d_1(p(\psi(x)), p(\psi(y))) = d_0(\psi(x), \psi(y))$. It follows that if $\psi(x) = (x_0, \theta) \in M \times \mathbb{S}^1(r)$ then, we necessarily have $\psi(y) = (x_0, \theta^*)$, where $\theta^* \in \mathbb{S}^1(r)$ is the antipodal point of θ. Now, we can deform slightly the loop γ to a homotopic one $\bar{\gamma}$ satisfying $x \in \bar{\gamma}(\mathbb{S}^1)$ and $\psi(y) \notin \psi(\bar{\gamma}(\mathbb{S}^1))$. By the same argument as before, there must exist a point $\bar{y} \in \bar{\gamma}(\mathbb{S}^1)$ satisfying $\pi r = d_1(p(\psi(x)), p(\psi(\bar{y}))) = d_0(\psi(x), \psi(\bar{y}))$. Again this implies that $\psi(\bar{y}) = (x_0, \theta^*) = \psi(y)$, which is a contradiction. Therefore $r < 4/\sqrt{3\kappa}$ if the induced homomorphism $\pi_1(\Sigma) \to \pi_1(\mathbb{S}^1(r))$ is not trivial. This completes the proof. $\square$

As an application, consider the sphere $\mathbb{S}^2$ endowed with a Riemannian metric g of Gaussian curvature $K \geq \kappa > 0$, for some positive constant κ and let $r \geq 4/\sqrt{3\kappa}$. Under these hypotheses, it follows from Theorem 5.1 and Theorem 3 in [16] that an isoperimetric region in $\mathbb{S}^2(g) \times \mathbb{S}^1(r)$ is either a slab or a domain bounded by a surface of genus ≤ 2. Indeed, the boundary Σ of an isoperimetric domain Ω in $(\mathbb{S}^2, g) \times \mathbb{S}^1(r)$ lifts to an embedded stable closed CMC surface in $(\mathbb{S}^2, g) \times \mathbb{R}$ by our result. By Theorem 3 in [16], Σ has genus at most 2 if it is connected and is a union of a finite number of horizontal slices if it is disconnected. In the latter case, Ω has to be a finite union of disjoint slabs $\mathbb{S}^2 \times [\theta_i, \theta_i + \alpha_i], i = 1, \ldots, n$, in $\mathbb{S}^2 \times \mathbb{S}^1(r)$. Note that a slab of width $\sum_{i=1}^n \alpha_i$ has the same volume as the union $\mathcal{U} = \cup_{i=1}^n \mathbb{S}^2 \times [\theta_i, \theta_i + \alpha_i]$ and its boundary has area less than the area of $\partial\mathcal{U}$. As Ω is an isoperimetric domain, it has to be a slab in this case.

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