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# GENERIC NONSINGULAR POISSON SUSPENSION IS OF TYPE $III_1$

ALEXANDRE I. DANILENKO, ZEMER KOSLOFF AND EMMANUEL ROY

ABSTRACT. It is shown that for a dense  $G_\delta$ -subset of the subgroup of nonsingular transformations (of a standard infinite  $\sigma$ -finite measure space) whose Poisson suspensions are nonsingular, the corresponding Poisson suspensions are ergodic and of Krieger's type  $III_1$ .

## 0. INTRODUCTION

In this paper we continue to study nonsingular Poisson suspensions for nonsingular transformations of infinite Lebesgue spaces  $(X, \mathfrak{B}, \mu)$  that we initiated in our previous work [DaKoRo]. Let  $X^*$  denote the space of  $\sigma$ -finite integer-valued non-negative measures on  $X$ . We endow  $X^*$  with the smallest  $\sigma$ -algebra  $\mathfrak{B}^*$  of subsets such that the map  $N_B : X^* \ni \omega \mapsto \omega(B) \in \mathbb{Z}_+$  is  $\mathfrak{B}^*$ -measurable for each  $B \in \mathfrak{B}$ . There exists a unique probability measure  $\mu^*$  on  $\mathfrak{B}^*$  such that

- for each finite family  $B_1, \dots, B_n$  of mutually disjoint subsets from  $\mathfrak{B}$  of finite  $\mu$ -measure, the random variables  $N_{B_1}, \dots, N_{B_n}$  are independent,
- for each  $B \in \mathfrak{B}$  with  $0 < \mu(B) < \infty$ , the measure  $\mu^* \circ N_B^{-1}$  is the Poisson distribution on  $\mathbb{Z}_+$  with parameter  $\mu(B)$ .

The triplet  $(X^*, \mathfrak{B}^*, \mu^*)$  is called *the Poisson suspension of  $(X, \mathfrak{B}, \mu)$* . Completing  $\mathfrak{B}^*$  with respect to  $\mu$ , we obtain that the Poisson suspension is a Lebesgue space. We showed in [DaKoRo, Corollary 4.1] that there is a maximal subset  $\text{Aut}_2(X, \mu)$  of the group  $\text{Aut}(X, \mu)$  of all nonsingular transformations  $T$  of  $(X, \mathfrak{B}, \mu)$  for which the Poisson suspension  $T_*$  is well defined as a nonsingular invertible transformation of  $(X^*, \mathfrak{B}^*, \mu^*)$ . As in the classical measure preserving case, by  $T_*\omega$  we mean the measure  $\omega \circ T^{-1} \in X^*$  for each  $\omega \in X^*$ . Moreover,  $\text{Aut}_2(X, \mu)$  is a Polish group in an appropriate topology  $d_2$  which is stronger than the usual weak topology [DaKoRo, Theorem D]. The suspension  $T_*$  admits an equivalent invariant probability measure if and only if  $T$  admits an equivalent invariant measure  $\nu$  such that  $\sqrt{\frac{d\nu}{d\mu}} - 1 \in L^2(X, \mu)$ . In this paper we consider the problem:

*Let  $T_*$  do not admit an equivalent probability measure. Can  $T_*$  be ergodic?  
If yes, what is the Krieger type of  $T_*$ ?*

We answer affirmatively the first question and contribute partly to the second one. Though we are unable so far to construct a concrete example of an ergodic conservative  $T$  of type  $III$  whose Poisson suspension is ergodic and of type  $III$ , we instead utilize the Baire category tools to prove a stronger “existence” result.

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**Theorem A (main result).** *The subset of all  $T \in \text{Aut}_2(X, \mu)$  such that  $T$  is ergodic and of type  $III_1$  and  $T_*$  is ergodic and of type  $III_1$  is a dense  $G_\delta$  in  $(\text{Aut}_2(X, \mu), d_2)$ .*

To prove this theorem we first construct a concrete example of a totally dissipative  $T$  with  $T_*$  being ergodic and of type  $III_1$ . The construction is motivated by a recent progress in the theory of nonsingular Bernoulli shifts achieved in [Ko1] and [DaLe], because if  $T$  is totally dissipative then  $T_*$  is always a nonsingular Bernoulli shift. However the aforementioned papers deal only with the shifts on  $\{0, 1\}^{\mathbb{Z}}$  while we encounter indeed with the product spaces  $A^{\mathbb{Z}}$  with  $A$  uncountable. This situation is considerably more difficult and we are not sure that the techniques developed in [Ko1] and [DaLe] extends to it in the full generality. However, we need only a very particular case which, in turn, can be reduced further to the shift on  $(\mathbb{Z}_+^{\mathbb{N}}, \bigotimes_{n \in \mathbb{Z}} \kappa_n)$ , where  $(\kappa_n)_{n \in \mathbb{Z}}$  is a specially selected sequence of Poisson distributions. Then we prove that  $T_*$  is of type  $III_1$  by showing that the Maharam extension of  $T_*$  is conservative and has nonsingular property  $K$ . For that we use essentially properties of Skellam distributions (see Appendix A) and Lévy's continuity theorem in addition to the theory of nonsingular endomorphisms and measurable orbit theory that were utilized in [Ko1] and [DaLe].

Secondly, we prove that the conjugacy class of  $T$  is dense in  $\text{Aut}_2(X, \mu)$ . Main Theorem follows from that and an additional fact that the subset of ergodic type  $III_1$  transformations in  $\text{Aut}_2(X, \mu)$  is a dense  $G_\delta$  (in  $d_2$ ) [DaKoRo, Theorem E].

The group  $\text{Aut}_1(X, \mu)$  and a homomorphism  $\chi : \text{Aut}_1(X, \mu) \rightarrow \mathbb{R}$  were introduced in [Ne, Chapter X, §4]. We showed in [DaKoRo, Theorem D] that  $\text{Aut}_1(X, \mu) \subset \text{Aut}_2(X, \mu)$ ,  $\text{Aut}_1(X, \mu)$  is a Polish group in a topology  $d_1$  which is stronger than  $d_2$ , and  $\chi$  is continuous. If  $\chi(T) \neq 0$  then  $T$  is dissipative [DaKoRo, Proposition 4.13]. Thus, every ergodic transformation from  $\text{Aut}_1(X, \mu)$  is contained in the proper closed subgroup  $\text{Ker } \chi$ . The following statement is proved in the same way as Theorem A.

**Theorem B.** *The subset of all  $T \in \text{Ker } \chi$  such that  $T$  is ergodic and of type  $III_1$  and  $T_*$  is ergodic and of type  $III_1$  is a dense  $G_\delta$  in  $(\text{Ker } \chi, d_1)$ .*

It is interesting to note that dynamical properties of  $T_*$  are determined not only by the dynamical properties of  $T$  but also by the choice of  $\mu$  inside its equivalence class. Indeed, if  $(X, \mu, T)$  is totally dissipative then we can find three different measures  $\mu_1 \sim \mu_2 \sim \mu_3$  in the equivalence class of  $\mu$  such that  $(X^*, \mu_1^*, T_*)$  is a probability preserving Bernoulli shift,  $(X^*, \mu_2^*, T_*)$  is an ergodic system of type  $III_1$  and  $(X^*, \mu_3^*, T_*)$  is a totally dissipative system. We “refine” further this phenomenon in a rather surprising way: for each  $T \in \text{Aut}_2(X, \mu)$  and  $t > 0$  we consider a dynamical system  $(X^*, \mu_t^*, T_*)$ , where  $\mu_t$  is the scaling of  $\mu$  by  $t$ , i.e.  $\mu_t(A) := t\mu(A)$  for each Borel subset  $A \subset X$ . Of course,  $\frac{d\mu_t \circ T}{d\mu_t} = \frac{d\mu \circ T}{d\mu}$  for each  $t > 0$ . We then encounter with the following *phase transition* phenomenon.

**Theorem C.** *Let  $T \in \text{Aut}_1(X, \mu)$ . If there is  $\alpha > 1$  such that  $\alpha^{-1} < \frac{d\mu \circ T^n}{d\mu}(x) < \alpha$  for each  $n > 0$  at a.e.  $x \in X$  then there is  $t_0 \in [0, +\infty]$  such that the Poisson suspension  $(X^*, \mu_t^*, T_*)$  is conservative for each  $t \in (0, t_0)$  and the Poisson suspension  $(X^*, \mu_t^*, T_*)$  is totally dissipative for each  $t \in (t_0, +\infty)$ .*

The most interesting case is when the *bifurcation point*  $t_0$  is *proper*, i.e.  $0 < t_0 < +\infty$ .

**Example D.** There is a totally dissipative  $T \in \text{Aut}_1(X, \mu)$  and  $t_0 \in (\frac{1}{6}, 4)$  such that the Poisson suspension  $(X^*, (t\mu)^*, T_*)$  is weakly mixing of stable type  $III_1$  if  $0 < t < t_0$  and the Poisson suspension  $(X^*, (t\mu)^*, T_*)$  is totally dissipative for  $t > t_0$ .

We recall that a nonsingular transformation  $T$  has conservative index  $n \in \mathbb{N}$  if the  $n$ -th Cartesian power of  $T$  is conservative and the  $(n+1)$ -th is dissipative.

**Proposition E.** Let the condition of Theorem C hold and let the bifurcation point  $t_0$  be proper. Then for each  $n \in \mathbb{N}$ , if  $\frac{t_0}{n+1} < t < \frac{t_0}{n}$  then  $(X^*, (t\mu)^*, T_*)$  has conservative index  $n$ .

As a byproduct, we apply the techniques developed in this paper to study conservativeness of the Poisson suspensions of general dissipative transformations. In particular, we show the following.

**Theorem F.** If  $T \in \text{Aut}_1(X, \mu)$  and  $\chi(T) \neq 0$  then  $T_*$  is totally dissipative.

The outline of the paper is as follows. The first two sections are of preliminary nature. We present there some concepts and facts from the theory of measured equivalence relations and their cocycles (§1) and nonsingular endomorphisms and their extensions (§2) to be used below in the paper. In §3 we study Poisson suspensions of transformations defined on purely atomic measure spaces. Since the purely atomic case was not considered in our previous paper on the Poisson suspensions, we first establish some basic results related to the nonsingularity and conservativeness of such suspensions independently of [DaKoRo] (see Propositions 3.1 and 3.2). Then we prove some necessary (Proposition 3.3) and sufficient (Proposition 3.4) conditions for conservativeness of the Poisson suspensions in terms of the underlying dynamical system. Theorem 3.5 plus Corollary 3.7 provide some conditions on a measure on  $\mathbb{Z}$  under which the Poisson suspension of the underlying unit translation on  $\mathbb{Z}$  possesses the nonsingular property  $K$ . This is technically the most involved result of the paper. Example 3.8 gives a concrete example of a measure on  $\mathbb{Z}$  satisfying those conditions. In §4 we show how to pass from the purely atomic case studied in §3 to the continuous case. In particular, we construct a totally dissipative transformation on a nonatomic Lebesgue space whose Poisson suspension is weakly mixing and of type  $III_1$  (see Theorem 4.2 and a remark just below it). In §5 we prove Theorems A and B (see Theorem 5.3). §6 is devoted to Poisson extensions of general dissipative transformations. We prove there Theorem F (see Theorem 6.1). Some extension of Theorem F to the more general case where  $T \in \text{Aut}_2(X, \mu)$  is also discussed in that section (see Remark 6.2). In §7 we study the phase transitions for the conservativeness of Poisson suspensions when scaling the underlying intensity. Theorem C and Proposition E are proved there. Example D is also provided in §7. The paper has Appendix A which is devoted completely to the Skellam distributions and their properties that we utilize in the proof of Theorem 3.5.

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## 1. MEASURED EQUIVALENCE RELATIONS AND THEIR COCYCLES

Let  $(X, \mathfrak{B}, \mu)$  be a standard  $\sigma$ -finite measure space. A Borel equivalence relation  $\mathcal{R} \subset X \times X$  is *countable* if for each  $x \in X$ , the  $\mathcal{R}$ -equivalence class  $\mathcal{R}(x)$  is

countable. If  $\mathcal{R}$  is countable and  $A \in \mathfrak{B}$  then the  $\mathcal{R}$ -saturation  $\mathcal{R}(A) := \bigcup_{x \in A} \mathcal{R}(x)$  of  $A$  belongs to  $\mathfrak{B}$ . An  $\mathcal{R}$ -saturated subset is also called  $\mathcal{R}$ -invariant. If for each Borel subset  $A$  of zero measure,  $\mathcal{R}(A)$  is also of zero measure then  $\mathcal{R}$  is called  $\mu$ -nonsingular. If the sub- $\sigma$ -algebra  $\{\mathcal{R}(A) \mid A \in \mathfrak{B}\}$  of  $\mathcal{R}$ -invariant Borel subsets is trivial (mod 0) then  $\mathcal{R}$  is called  $\mu$ -ergodic.

From now on  $\mathcal{R}$  is countable and  $\mu$ -nonsingular. The *full group*  $[\mathcal{R}]$  of  $\mathcal{R}$  consists of all nonsingular transformations  $R$  of  $(X, \mu)$  such that  $Rx \in \mathcal{R}(x)$  at a.e.  $x \in X$ . Given a locally compact Polish group  $G$ , a Borel map  $\alpha : \mathcal{R} \rightarrow G$  is called a *cocycle* of  $\mathcal{R}$  if there is a Borel subset  $N$  of zero measure such that

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \quad \text{for all } x, y \in X \setminus N \text{ such that } (x, y), (y, z) \in \mathcal{R}.$$

In the later sections of this paper we deal only with the cases where  $G$  is either  $\mathbb{R}$  or the multiplicative group  $\mathbb{R}_+^*$ . A cocycle  $\beta : \mathcal{R} \rightarrow G$  is *cohomologous* to  $\alpha$  if there is a Borel function  $\phi : X \rightarrow G$  and a Borel subset  $N$  of zero measure such that

$$\beta(x, y) = \phi(x)\alpha(x, y)\phi(y)^{-1} \quad \text{for all } x, y \in X \setminus N.$$

Fix a left Haar measure  $\lambda_G$  on  $G$ . The  $\alpha$ -skew product equivalence relation  $\mathcal{R}(\alpha)$  on the product space  $(X \times G, \mu \times \lambda_G)$  is defined by:

$$(x, g) \sim (y, h) \quad \text{if } (x, y) \in \mathcal{R} \quad \text{and} \quad h = \alpha(x, y)g.$$

This equivalence relation is countable and  $(\mu \times \lambda_G)$ -nonsingular. If  $\mathcal{R}(\alpha)$  is ergodic then  $\alpha$  is called *ergodic*. Of course, if  $\alpha$  is ergodic then  $\mathcal{R}$  is ergodic. If a cocycle  $\beta$  is cohomologous to  $\alpha$  and  $\alpha$  is ergodic then  $\beta$  is also ergodic.

We now isolate an important cocycle of  $\mathcal{R}$  with values in the multiplicative group  $\mathbb{R}_+^*$ . It is called the *Radon-Nikodym cocycle* of  $\mathcal{R}$  and denoted by  $\Delta_{\mathcal{R}}$ . To define it, we first fix a countable subgroup  $\Gamma$  of Borel bijections of  $X$  that generates  $\mathcal{R}$ . Such a group exists according to [FeMo, Theorem 1]. Now we set

$$\Delta_{\mathcal{R}}(x, \gamma x) := \frac{d\mu \circ \gamma}{d\mu}(x), \quad x \in X, \gamma \in \Gamma.$$

It follows from the chain rule for the Radon-Nikodym derivatives that  $\Delta_{\mathcal{R}}$  is a cocycle of  $\mathcal{R}$ . This cocycle is well defined, i.e. it does not depend on the choice of  $\Gamma$  generating  $\mathcal{R}$ <sup>1</sup>.

**Example 1.1.** Let  $A$  be a countable set and let  $\lambda_n$  be a non-degenerated distribution on  $A$ , i.e.  $\lambda_n(a) > 0$  if  $a \in A$  and  $\sum_{a \in A} \lambda_n(a) = 1$ , for each  $n \geq 1$ . We set  $(X, \lambda) := (A^{\mathbb{N}}, \bigotimes_{n=1}^{\infty} \lambda_n)$ . Denote by  $\mathcal{S}$  the *tail equivalence relation* on  $X$ , i.e. two points  $x = (x_n)_{n=1}^{\infty}$  and  $y = (y_n)_{n=1}^{\infty}$  from  $X$  are  $\mathcal{S}$ -equivalent if there is  $N > 0$  such that  $x_n = y_n$  for all  $n > N$ . Then  $\mathcal{S}$  is an ergodic  $\lambda$ -nonsingular countable equivalence relation on  $X$  and

$$\Delta_{\mathcal{S}}(x, y) = \prod_{n=1}^{\infty} \frac{\lambda_n(y_n)}{\lambda_n(x_n)} = \prod_{n=1}^N \frac{\lambda_n(y_n)}{\lambda_n(x_n)}$$

<sup>1</sup>In this paper we do not distinguish between objects (such as subsets, maps, cocycles, etc.) which agree almost everywhere.

for all  $(x, y) \in \mathcal{S}$ .

If we change  $\mu$  with an equivalent  $\sigma$ -finite measure then the Radon-Nikodym cocycle of  $\mathcal{R}$  related to the new measure is cohomologous to the original  $\Delta_{\mathcal{R}}$ . The  $\Delta_{\mathcal{R}}$ -skew product equivalence relation  $\mathcal{R}(\Delta_{\mathcal{R}})$  is called the *Maharam extension* of  $\mathcal{R}$ . If it is ergodic, i.e.  $\Delta_{\mathcal{R}}$  is ergodic, then  $\mathcal{R}$  is said to be of *Krieger's type III<sub>1</sub>*. It follows from the aforementioned cohomology property of  $\Delta_{\mathcal{R}}$  that the property of  $\mathcal{R}$  to be of type III<sub>1</sub> does not change if we replace  $\mu$  with an equivalent  $\sigma$ -finite measure.

Suppose now that  $\mathcal{R}$  is  $\mu$ -ergodic. Given a cocycle  $\alpha$  of  $\mathcal{R}$  with values in an Abelian locally compact Polish group  $G$ , we say that an element  $g \in G$  is an *essential value* of  $\alpha$  if for each neighborhood  $U$  of  $g$  in  $G$  and each Borel subset  $A \subset X$  of positive measure there are a Borel subset  $B \subset A$  of positive measure and a Borel one-to-one map  $\gamma : B \rightarrow A$  such that  $(x, \gamma x) \in \mathcal{R}$  and  $\alpha(x, \gamma x) \in U$  for each  $x \in B$ . The set  $r(\alpha)$  of all essential values of  $\alpha$  is a closed subgroup of  $G$  ([Sc, Lemma 3.3], [FeMo, Proposition 8.5]). The cocycle  $\alpha$  is ergodic if and only if  $r(\alpha) = G$  [Sc, Corollary 5.4] (see also [FeMo, Theorem 8]).

The following standard approximation lemma (see [ChHaPr, Lemma 2.1]) is useful for computation of  $r(\alpha)$ .

**Lemma 1.2.** *Let  $\mathfrak{A} \subset \mathfrak{B}$  be a semiring such the corresponding ring is dense in  $\mathfrak{B}$ . Let  $0 < \delta < 1$  and let  $g \in G$ . If for each  $A \in \mathfrak{A}$  of positive measure and a neighborhood  $U$  of  $g$  there are a subset  $B \subset A$  and a one-to-one Borel map  $\gamma : B \rightarrow A$  such that  $\mu(B) > \delta\mu(A)$ ,  $(x, \gamma x) \in \mathcal{R}$ ,  $\alpha(x, \gamma x) \in U$  and  $\delta < \Delta_{\mathcal{R}}(x, \gamma x) < \delta^{-1}$  for all  $x \in B$  then  $g$  is an essential value of  $\alpha$ .*

An ergodic invertible transformation  $T$  of  $(X, \mathfrak{B}, \mu)$  is called of *Krieger's type III<sub>1</sub>* if the  $T$ -orbit equivalence relation is of type III<sub>1</sub>. If, moreover,  $T \times S$  is of type III<sub>1</sub> for each ergodic probability preserving transformation  $S$  then  $T$  is called of *stable type III<sub>1</sub>*.

For more information on the measurable orbit theory and detailed proofs of the aforementioned facts we refer the reader to [FeMo], [Sc], [DaSi].

## 2. NONSINGULAR ENDOMORPHISMS, THEIR EXTENSIONS AND ASSOCIATED EQUIVALENCE RELATIONS

Let  $(X, \mathfrak{B}, \mu)$  be a standard  $\sigma$ -finite measure space. A Borel map  $T : X \rightarrow X$  is called a  $\mu$ -nonsingular endomorphism if  $\mu \circ T^{-1} \sim \mu$ . Consider the following decreasing sequence  $\mathfrak{B} \supset T^{-1}\mathfrak{B} \supset T^{-2}\mathfrak{B} \supset \dots$  of sub- $\sigma$ -algebras in  $\mathfrak{B}$ . If  $\bigwedge_{n=1}^{+\infty} T^{-n}\mathfrak{B} = \{\emptyset, X\} \pmod{0}$  then  $T$  is called *exact*. We will consider only *aperiodic* endomorphisms, i.e. we assume that

$$\mu \left( \bigcup_{n>0} \{x \in X \mid T^n x = x\} \right) = 0.$$

Let  $T$  be  $\mu$ -nonsingular endomorphism. We recall that a measurable function  $\omega : X \rightarrow \mathbb{R}_+^*$  is called *markovian* for  $(X, \mu, T)$  if

$$\int f \circ T \omega d\mu = \int f d\mu \quad \text{for each } f \in L^1(X, \mu) \text{ [SiTh1].}$$

Such a function may not be unique (see [SiTh1, Example 1.3(c)]). However if we assume that the measure  $\mu \circ T^{-1}$  is  $\sigma$ -finite then a standard verification shows that there exists a unique  $T^{-1}\mathfrak{B}$ -measurable markovian function for  $(X, \mu, T)$ . We call it the *Radon-Nikodym derivative of  $T$*  and denote by  $\rho_T$ . One can check that  $\rho_T = \frac{d\mu}{d\mu \circ T^{-1}} \circ T$ . Of course, if  $T$  is invertible then  $\rho_T = \frac{d\mu \circ T}{d\mu}$ .

Let  $\kappa$  denote a measure on  $\mathbb{R}_+^*$  equivalent to the Lebesgue measure and such that  $\kappa(aB) = a^{-1}\kappa(B)$ . Given  $T$  such that  $\mu \circ T^{-1}$  is  $\sigma$ -finite, we can define a  $\sigma$ -finite measure preserving endomorphism  $T_{\rho_T}$  of the product space  $(X \times \mathbb{R}_+^*, \mu \times \kappa)$  by setting  $T_{\rho_T}(x, t) := (Tx, \rho_T(x)t)$ . It is called *the Maharam extension of  $T$* . If  $(Y, \mathfrak{Y}, \nu)$  is a standard  $\sigma$ -finite measure space,  $S$  is a transformation from  $\text{Aut}(Y, \nu)$ ,  $\pi : Y \rightarrow X$  is a Borel map such that

$$\mu \circ \pi^{-1} = \nu, \pi S = T\pi, \rho_S = \rho_T \circ \pi \quad \text{and} \quad \mathfrak{Y} = \bigvee_{n>0} S^n \pi^{-1} \mathfrak{B}$$

then  $S$  is called *the natural extension of  $T$*  [SiTh2, Definition 4.1]. The existence and uniqueness (up to a natural isomorphism) of the natural extension was proved in [SiTh2, Theorem 4.2]. We will denote it by  $\tilde{T}$ . The natural extension of the Maharam extension of  $T$  is canonically isomorphic to the Maharam extension of the natural extension of  $T$ . If  $S \in \text{Aut}(Y, \nu)$  is the natural extension of an exact endomorphism then  $S$  is called *a nonsingular  $K$ -automorphism*. We recall that a nonsingular invertible transformation is called *weakly mixing* if the Cartesian product of it with every ergodic probability preserving transformation is ergodic. A nonsingular  $K$ -automorphism is weakly mixing whenever it is conservative [SiTh2, Proposition 4.8].

If for a.e.  $x \in X$ , the set  $T^{-1}\{x\}$  is at most countable then  $T$  is called *countable-to-one*. From now on we will consider only endomorphisms which are countable-to-one. Given such a  $T$ , we can associate an equivalence relation  $\mathcal{S}_T$  on  $X$  by setting:

$$(2-1) \quad (x, y) \in \mathcal{S}_T \quad \text{if there is } n \geq 0 \text{ such that } T^n x = T^n y.$$

Then  $\mathcal{S}_T$  is countable and  $\mu$ -nonsingular. It is ergodic if and only if  $T$  is exact [Ha]. If (2-1) holds, we set

$$\alpha_{\rho_T}(x, y) := \rho_T(x) \cdots \rho_T(T^{n-1}x) \rho_T(T^{n-1}y)^{-1} \cdots \rho_T(y)^{-1}.$$

Then  $\alpha_{\rho_T}$  is a well defined cocycle of  $\mathcal{S}_T$  with values in  $\mathbb{R}_+^*$ .<sup>2</sup>

*Remark 2.1.* We note that  $\alpha_{\rho_T}$  depends also on  $\mu$ , i.e., in fact,  $\alpha_{\rho_T} = \alpha_{\rho_T, \mu}$ . If we replace  $\mu$  with an equivalent measure  $\lambda$  and the Radon-Nikodym derivative  $\frac{d\lambda}{d\mu}$  is measurable with respect to the  $\sigma$ -algebra  $T^{-1}\mathfrak{B}$  then the cocycle  $\alpha_{\rho_T, \lambda}$  is cohomologous to  $\alpha_{\rho_T, \mu}$ .

Denote by  $\mathcal{S}_T(\alpha_{\rho_T})$  the  $\alpha_{\rho_T}$ -skew product extension of  $\mathcal{S}$ . Then  $\mathcal{S}_{T_{\rho_T}} = \mathcal{S}_T(\alpha_{\rho_T})$ . From the aforementioned facts we deduce the following proposition (see the beginning of the proof of [DaLe, Theorem 4.1, Claim B] for details).

<sup>2</sup>The reader should not confuse  $\alpha_{\rho_T}$  with  $\Delta_{\mathcal{S}_T}$ . The former is the cocycle generated by the Radon-Nikodym derivative of  $T$ . The later is the Radon-Nikodym cocycle of  $\mathcal{S}_T$ . If  $T$  is invertible then  $\alpha_{\rho_T} = \Delta_{\mathcal{S}_T}$ . However, in the general case, the two cocycles are different—see the proof of Theorem 3.5 below.

**Proposition 2.2.** *Let  $T$  be countable-to-one nonsingular endomorphism of a probability space. The Maharam extension of the natural extension of  $T$  is a  $K$ -automorphism if and only if the cocycle  $\alpha_{\rho_T}$  of  $\mathcal{S}_T$  is ergodic.*

We illustrate the aforementioned concepts with the following example.

**Example 2.3.** Let  $(X, \mu) := (A^{\mathbb{N}}, \bigotimes_{n=1}^{\infty} \mu_n)$ , where  $A$  and  $(\mu_n)_{n=1}^{\infty}$  stand for the same objects as in Example 1.1. Denote by  $T : X \rightarrow X$  the one-sided Bernoulli shift on  $X$ . Of course, it is countable-to-one. By the Kakutani criterion [Ka],  $T$  is  $\mu$ -nonsingular if and only if

$$\sum_{n=1}^{\infty} \sum_{a \in A} \left( \sqrt{\mu_n(a)} - \sqrt{\mu_{n+1}(a)} \right)^2 < \infty.$$

Moreover, if  $T$  is nonsingular then we can compute the Radon-Nikodym derivative of  $T$ :

$$(2-2) \quad \rho_T(x) = \prod_{n=1}^{\infty} \frac{\mu_n(x_{n+1})}{\mu_{n+1}(x_{n+1})} \quad \text{at } \mu\text{-a.e. } x = (x_n)_{n \in \mathbb{N}} \in X.$$

It is easy to see that  $\mathcal{S}_T$  is the tail equivalence relation on  $X$ . Of course,  $T$  is exact. The natural extension of  $T$  is the two-sided Bernoulli shift on the space  $(A^{\mathbb{Z}}, \bigotimes_{n \in \mathbb{Z}} \tilde{\mu}_n)$ , where  $\tilde{\mu}_n := \mu_n$  if  $n > 0$  and  $\tilde{\mu}_n := \mu_1$  if  $n \leq 0$ . The corresponding projection  $\pi : A^{\mathbb{Z}} \rightarrow A^{\mathbb{N}}$  is given by the formula  $\pi((x_n)_{n \in \mathbb{Z}}) := (x_n)_{n \in \mathbb{N}}$ .

Given a standard probability space  $(Y, \mathfrak{Y}, \nu)$  and a transformation  $S \in \text{Aut}(Y, \nu)$ , we denote by  $U_S$  the associated (with  $S$ ) Koopman unitary operator in  $L^2(Y, \nu)$ , i.e.  $U_S f := f \circ S^{-1} \cdot \sqrt{S'}$  for all  $f \in L^2(Y, \nu)$ <sup>3</sup>.

We recall that  $S$  is said to be

- *conservative* if for each subset  $B \in \mathfrak{Y}$  of positive measure, there is  $n > 0$  such that  $\nu(S^{-n}B \cap B) > 0$ ,
- *dissipative* if it is not conservative,
- *totally dissipative* if there is  $B \in \mathfrak{Y}$  such that  $\bigsqcup_{n \in \mathbb{Z}} T^n B = Y \pmod{0}$ .

The following lemma from [Ko1, Lemma 2.2] will be used in the subsequent sections.

**Lemma 2.4.** *If  $\sum_{n=1}^{\infty} \langle U_S^n 1, 1 \rangle < \infty$  then  $S$  is totally dissipative.*

We remind Hopf's criterion of conservativeness for nonsingular maps.

**Lemma 2.5** (see, e.g., [Aa, Proposition 1.3.1]). *A nonsingular transformation  $S$  of a standard probability space  $(Y, \mathfrak{Y}, \nu)$  is conservative if and only if  $\sum_{n=1}^{\infty} \frac{d\nu \circ S^n}{d\nu}(y) = +\infty$  at a.e.  $y \in Y$ .*

We will also need the following sufficient condition for conservativeness of nonsingular transformations.

**Proposition 2.6.** *Let  $S$  be a nonsingular transformation on a standard probability space  $(Y, \nu)$ . Assume that  $\frac{d\nu}{d\nu \circ S^{-n}} \in L^2(Y, \nu)$  for each  $n > 0$  and write  $b(n) := \left\| \frac{d\nu}{d\nu \circ S^{-n}} \right\|_2^2$ . If there is a sequence  $(a(n))_{n=1}^{\infty}$  of positive reals such that  $\sum_{n=1}^{\infty} a(n) = \infty$  but  $\sum_{n=1}^{\infty} a(n)^2 b(n) < \infty$  then  $S$  is conservative.*

<sup>3</sup>For the sake of simplicity, we use here and below the notation  $S'$  for the Radon-Nikodym derivative of  $S$ , i.e.  $S' := \frac{d\nu \circ S^{-1}}{d\nu}$ .



*Proof.* We set  $A_n := \{y \in Y \mid \frac{d\nu \circ S^{-n}}{d\nu}(y) < a(n)\}$ . By Markov's inequality,

$$\nu(A_n) = \nu\left(\left\{y \in Y \mid \left(\frac{d\nu}{d\nu \circ S^{-n}}(y)\right)^2 > \frac{1}{a(n)^2}\right\}\right) \leq a(n)^2 b(n).$$

Since  $\sum_{n=1}^{\infty} a(n)^2 b(n) < \infty$ , it follows from the Borel-Cantelli lemma that for  $\nu$ -a.e.  $y$ , there is  $N(y) > 0$  such that for each  $n > N(y)$ ,

$$\frac{d\nu \circ S^{-n}}{d\nu}(y) \geq a(n).$$

Now the condition  $\sum_{n=1}^{\infty} a(n) = \infty$  yields that  $\sum_{n=1}^{\infty} \frac{d\nu \circ S^{-n}}{d\nu}(y) = \infty$ . Hence  $S$  is conservative by Lemma 2.5.  $\square$

We also recall definition of an r.f.m.p. extension for nonsingular maps. A nonsingular endomorphism  $T$  of  $(X, \mathfrak{B}, \mu)$  is called a *relatively finite measure preserving* (r.f.m.p.) extension of a nonsingular endomorphism  $S$  of  $(Y, \mathfrak{Y}, \nu)$  if there is a Borel map  $\pi : X \rightarrow Y$  such that  $\nu = \mu \circ \pi^{-1}$ ,  $\pi T = S\pi$  and  $\rho_T = \rho_S \circ \pi$ . For instance, the natural extension of a nonsingular endomorphism is r.f.m.p.

For more information on nonsingular endomorphisms we refer to [Aa], [Ha], [SiTh2] and [DaLe].

### 3. POISSON SUSPENSIONS OF TYPE $III_1$ OVER A DISCRETE BASE

Let  $(X, \mathfrak{B}, \mu)$  be a  $\sigma$ -finite measure space and  $\mu(X) = \infty$ . We first recall some definitions, notation and facts from [DaKoRo].  $\text{Aut}(X, \mu)$  stands for the group of all  $\mu$ -nonsingular invertible transformations. The weak topology on  $\text{Aut}(X, \mu)$  is induced from the weak (equivalently, strong) operator topology on the unitary group  $\mathcal{U}$  of the Hilbert space  $L^2(X, \mu)$  via the embedding  $\text{Aut}(X, \mu) \ni S \mapsto U_S \in \mathcal{U}$ . We also let

$$\begin{aligned} \text{Aut}_2(X, \mu) &:= \{T \in \text{Aut}(X, \mu) \mid \sqrt{T'} - 1 \in L^2(X, \mu)\} \quad \text{and} \\ \text{Aut}_1(X, \mu) &:= \{T \in \text{Aut}(X, \mu) \mid T' - 1 \in L^1(X, \mu)\}. \end{aligned}$$

The two sets are Borel subgroups of  $\text{Aut}(X, \mu)$  endowed with the weak topology and  $\text{Aut}_1(X, \mu) \subset \text{Aut}_2(X, \mu)$  [DaKoRo, Proposition 4.14]. For  $j = 1, 2$ , we define a topology  $d_j$  on  $\text{Aut}_j(X, \mu)$  by saying that a sequence  $(T_n)_{n=1}^{\infty}$  of transformations from  $\text{Aut}_j(X, \mu)$  converges to a transformation  $T$  from  $\text{Aut}_j(X, \mu)$  if  $T_n \rightarrow T$  weakly and  $\|(T_n')^{1/j} - (T')^{1/j}\|_j \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\text{Aut}_j(X, \mu)$  endowed with  $d_j$  is a Polish group [DaKoRo, Theorem D]. There exists a continuous homomorphism  $\chi : \text{Aut}_1(X, \mu) \rightarrow \mathbb{R}$  defined by the formula

$$\chi(T) := \int_X (T' - 1) d\mu$$

(see [Ne, Chapter X, §4] and [DaKoRo, Theorem 4.11]).

From now on and till the end of the section let  $X = \mathbb{Z}$  and let  $T$  denote the unit translation, i.e.  $Tn := n + 1$  for all  $n \in \mathbb{Z}$ . Given a measure  $\mu$  on  $X$ , we set  $a_n := \mu(n)$  for each  $n \in \mathbb{Z}$ . Then  $T$  is  $\mu$ -nonsingular if and only if  $\mu$  is non-degenerated, i.e.  $a_n > 0$  for each  $n \in \mathbb{Z}$ . Moreover,

$$T'(n) = \frac{a_{n-1}}{a_n} \quad \text{for all } n \in \mathbb{Z}.$$

Of course,  $\text{Aut}(X, \mu)$  is the group of all permutations of  $\mathbb{Z}$ . We see that  $T \in \text{Aut}_2(X, \mu)$  if and only if

$$(3-1) \quad \sum_{n \in \mathbb{Z}} \left( \sqrt{\frac{a_{n-1}}{a_n}} - 1 \right)^2 a_n = \sum_{n \in \mathbb{Z}} (\sqrt{a_{n-1}} - \sqrt{a_n})^2 < \infty.$$

In a similar way,  $T \in \text{Aut}_1(X, \mu)$  if and only if  $\sum_{n \in \mathbb{Z}} |a_{n-1} - a_n| < \infty$ . If the latter inequality holds then there exist the two limits  $a_{+\infty} := \lim_{n \rightarrow +\infty} a_n$  and  $a_{-\infty} := \lim_{n \rightarrow -\infty} a_n$ . Moreover, it is easy to verify that

$$(3-2) \quad \chi(T) = a_{+\infty} - a_{-\infty}.$$

Let  $(X^*, \mu^*, T_*)$  denote the Poisson suspension of the dynamical system  $(X, \mu, T)$ . The mapping  $\omega \mapsto (\omega(n))_{n \in \mathbb{Z}}$  is an isomorphism of  $(X^*, \mu^*)$  onto the infinite product space  $(\mathbb{Z}_+^{\mathbb{Z}}, \bigotimes_{n \in \mathbb{Z}} \kappa_n)$ , where  $\kappa_n$  is the Poisson distribution with parameter  $a_n$  for each  $n \in \mathbb{Z}$ . Moreover, this mapping conjugates  $T_*$  with the shift on  $\mathbb{Z}_+^{\mathbb{Z}}$ . For this reason, from now on we view  $T_*$  as the shift  $\mathbb{Z}_+^{\mathbb{Z}}$  defined by

$$(T_* y)_n := y_{n+1} \quad \text{for all } y = (y_n)_{n \in \mathbb{Z}} \in \mathbb{Z}_+^{\mathbb{Z}}.$$

It was shown in [DaKoRo, Theorem A] that in the case where  $\mu$  is non-atomic,  $T_*$  is  $\mu^*$ -nonsingular if and only if  $T \in \text{Aut}_2(X, \mu)$ . We now verify that the same holds also in our (purely atomic) case.

**Proposition 3.1.**  *$T_*$  is  $\mu^*$ -nonsingular if and only if (3-1) is satisfied. In this case for a.e.  $y = (y_n)_{n \in \mathbb{Z}} \in X^*$ ,*

$$(T_*)'(y) = \prod_{n \in \mathbb{Z}} \frac{\kappa_{n-1}(y_n)}{\kappa_n(y_n)}.$$

*Proof.* We first recall definition of the Hellinger distance  $H(\lambda, \xi)$  between two probability measures  $\lambda, \xi$  on a countable set  $C$ :

$$H^2(\lambda, \xi) := 1 - \int_C \sqrt{\frac{d\lambda}{d\xi}} d\xi = \frac{1}{2} \sum_{c \in C} \left( \sqrt{\lambda(c)} - \sqrt{\xi(c)} \right)^2.$$

Given two Poisson distributions  $\nu_a$  and  $\nu_b$  on  $\mathbb{Z}_+$  with parameters  $a$  and  $b$  respectively, then the Hellinger distance  $H(\nu_a, \nu_b)$  between  $\nu_a$  and  $\nu_b$  satisfies

$$H^2(\nu_a, \nu_b) = 1 - e^{-\frac{1}{2}(\sqrt{a} - \sqrt{b})^2}.$$

Therefore it follows from the Kakutani criterion [Ka] that  $T_*$  is  $\mu^*$ -nonsingular if and only if

$$(3-3) \quad \infty > 2 \sum_{n \in \mathbb{Z}} H^2(\kappa_n, \kappa_{n+1}) = 2 \sum_{n \in \mathbb{Z}} (1 - e^{-\frac{1}{2}(\sqrt{a_n} - \sqrt{a_{n+1}})^2}).$$

The series in the righthand side of (3-3) converges if and only if (3-1) holds.  $\square$

In a similar way, the Kakutani criterion [Ka] can be applied to prove the following claim.

**Proposition 3.2.** *Let  $\lambda$  be another non-degenerated measure on  $\mathbb{Z}$ . Then  $\lambda^*$  is equivalent to  $\mu^*$  if and only if  $\sum_{n \in \mathbb{Z}} \left( \sqrt{\lambda(n)} - \sqrt{\mu(n)} \right)^2 < \infty$ .*

Our next purpose is to investigate when  $T_*$  is conservative or dissipative in terms of the original system  $(X, \mu, T)$ . Let  $\mathcal{L}_-(\mu)$  denote the set of limit points of the sequence  $(a_n)_{n < 0}$  and let  $\mathcal{L}_+(\mu)$  denote the set of limit points of the sequence  $(a_n)_{n > 0}$ . If  $T \in \text{Aut}_2(X, \mu)$  then (3-1) implies that there exist reals  $\beta_+ \geq \alpha_+ \geq 0$  and  $\beta_- \geq \alpha_- \geq 0$  such that  $\mathcal{L}_-(\mu) = [\alpha_-, \beta_-]$  and  $\mathcal{L}_+(\mu) = [\alpha_+, \beta_+]$ .

**Proposition 3.3.**

- (i) *If  $T \in \text{Aut}_2(X, \mu)$  and  $\mathcal{L}_-(\mu) \cap \mathcal{L}_+(\mu) = \emptyset$  then  $T_*$  is totally dissipative.*
- (ii) *If  $T \in \text{Aut}_1(X, \mu)$  and  $\chi(T) \neq 0$  then  $T_*$  is totally dissipative.*

*Proof.* Since (ii) follows from (i) and (3-2), it suffices to prove (i). Let  $H(\cdot, \cdot)$  denote the Hellinger distance on the set of probability measures on  $\mathbb{Z}_+$ . It follows from the condition of (i) that there is  $\delta > 0$  and  $N > 0$  such that  $H(\kappa_i, \kappa_j) > \delta$  whenever  $i < -N$  and  $j > N$ . This inequality and Proposition 3.1 yield that if  $n > 3N$  then

$$\begin{aligned} \langle U_{T_*}^n 1, 1 \rangle &= \prod_{n \in \mathbb{Z}} \int_{\mathbb{Z}_+} \sqrt{\frac{d\kappa_{k-n}}{d\kappa_k}}(y_k) d\kappa_k(y_k) \\ &= \prod_{n \in \mathbb{Z}} (1 - H^2(\kappa_{k-n}, \kappa_k)) \\ &\leq \prod_{\frac{n}{3} < k < \frac{2n}{3}} (1 - H^2(\kappa_{k-n}, \kappa_k)) \\ &< (1 - \delta^2)^{n/3}. \end{aligned}$$

Hence  $\sum_{n=1}^{\infty} \langle U_{T_*}^n 1, 1 \rangle < \sum_{n=1}^{\infty} (1 - \delta^2)^{n/3} < \infty$ . It remains to use Lemma 2.4.  $\square$

We will also need a sufficient condition for conservativeness of  $T^*$ .

**Proposition 3.4.** *Let  $T \in \text{Aut}_1(X, \mu)$ ,  $\chi(T) = 0$  and  $\left( \frac{d\mu}{d\mu \circ T^{-n}} \right)^2 - 1 \in L^1(X, \mu)$  for each  $n \in \mathbb{N}$ . If there is a sequence  $(b_n)_{n=1}^{\infty}$  of positive reals such that  $\sum_{n=1}^{\infty} b_n = \infty$  but*

$$\sum_{n=1}^{\infty} b_n^2 e^{\int_X \left( \left( \frac{d\mu}{d\mu \circ T^{-n}} \right)^2 - 1 \right) d\mu} < \infty$$

*then  $T_*$  is conservative.*

*Proof.* Since  $\chi(T) = 0$ , it follows that  $\sum_{k \in \mathbb{Z}} (a_k - a_{k-n}) = 0$  for each  $n > 0$ . Therefore

$$\begin{aligned} \log \left( \frac{d\mu^*}{d\mu^* \circ T_*^{-n}}(y) \right)^2 &= - \sum_{k \in \mathbb{Z}} 2 \log \frac{\kappa_{k-n}(y_k)}{\kappa_k(y_k)} \\ &= -2 \sum_{k \in \mathbb{Z}} \left( y_k \log \frac{a_{k-n}}{a_k} + a_k - a_{k-n} \right) \\ &= -2 \sum_{k \in \mathbb{Z}} y_k \log \frac{a_{k-n}}{a_k} = - \left\langle y, 2 \log \frac{d\mu \circ T^{-n}}{d\mu} \right\rangle \end{aligned}$$

at a.e.  $y \in X^*$ . Here we consider  $y$  as a measure on  $X$  and use the notation  $\langle y, f \rangle$  for the integral of a function  $f \in L^1(X, y)$  with respect to  $y$ . Utilizing the above formula and the Laplace transform we obtain that

$$\begin{aligned} \left\| \frac{d\mu^*}{d\mu^* \circ T_*^{-n}} \right\|_2^2 &= \int_{X^*} e^{-\langle y, 2 \log \frac{d\mu \circ T^{-n}}{d\mu} \rangle} d\mu^*(y) \\ &= e^{\int_X \left( e^{-2 \log \frac{d\mu \circ T^{-n}}{d\mu}} - 1 \right) d\mu} \\ &= e^{\int_X \left( \left( \frac{d\mu}{d\mu \circ T^{-n}} \right)^2 - 1 \right) d\mu}. \end{aligned}$$

It remains to apply Proposition 2.6.  $\square$

From now on we will assume that  $a_n := ae^{\epsilon_n}$  for some  $a > 0$  and a sequence  $(\epsilon_n)_{n \in \mathbb{Z}}$  of reals such that

$$(3-4) \quad \epsilon_n = 0 \text{ if } n \leq 1, \quad \lim_{n \rightarrow +\infty} \epsilon_n = 0, \quad \sum_{n=1}^{\infty} \epsilon_n^2 = \infty \text{ but } \sum_{n=1}^{\infty} \epsilon_n^4 < +\infty.$$

Let  $R$  denote the one-sided Bernoulli shift on the space  $((\mathbb{Z}_+)^{\mathbb{N}}, \bigotimes_{n=1}^{\infty} \kappa_n)$ , and let  $\pi$  denote the projection

$$(\mathbb{Z}_+)^{\mathbb{Z}} \ni (x_n)_{n \in \mathbb{Z}} \mapsto (x_n)_{n > 0} \in (\mathbb{Z}_+)^{\mathbb{N}}.$$

Then  $R\pi = \pi T_*$ . In view of Example 2.3,  $T_*$  is the natural extension of  $R$ .

**Theorem 3.5.** *If (3-4) holds then the cocycle  $\log \alpha_{\rho_R}$  of the equivalence relation  $\mathcal{S}_R$  associated with  $R$  is ergodic.*

*Proof.* According to Example 2.3,  $\mathcal{S}_R$  is the tail equivalence relation on  $(\mathbb{Z}_+)^{\mathbb{N}}$ . By (2-2),  $\rho_R(v) = \prod_{j=2}^{\infty} \frac{\kappa_{j-1}(v_j)}{\kappa_j(v_j)}$  for each  $v = (v_j)_{j=1}^{\infty} \in (\mathbb{Z}_+)^{\mathbb{N}}$ . Hence for each  $n > 0$ ,

$$\rho_R(v) \cdots \rho_R(R^{n-1}v) = \prod_{j=2}^n \frac{\kappa_1(v_j)}{\kappa_j(v_j)} \prod_{j>n} \frac{\kappa_{j-n}(v_j)}{\kappa_j(v_j)}.$$

Suppose now that  $R^n v = R^n w$ . Then  $v_j = w_j$  for all  $j > n$ . Therefore

$$\alpha_{\rho_R}(v, w) := \frac{\rho_R(v) \cdots \rho_R(R^{n-1}v)}{\rho_R(w) \cdots \rho_R(R^{n-1}w)} = \prod_{j=2}^n \frac{\kappa_j(w_j)}{\kappa_j(v_j)} \prod_{j=2}^n \frac{\kappa_1(v_j)}{\kappa_1(w_j)} = \prod_{j=2}^n \left( \frac{a_j}{a_1} \right)^{w_j - v_j}.$$

Thus we obtain that for each  $(v, w) \in \mathcal{S}_R$ ,

$$\log \alpha_{\rho_R}(v, w) = \sum_{j>1} (w_j - v_j) \epsilon_j.$$

Let  $\Delta_{\mathcal{S}_R}$  denote the Radon-Nikodym cocycle of  $\mathcal{S}_R$ . Utilizing Example 1.1 and the definition of  $\kappa_j$  we obtain that

$$\log \Delta_{\mathcal{S}_R}(v, w) = \log \left( \prod_{j=1}^n \frac{\kappa_j(w_j)}{\kappa_j(v_j)} \right) = \sum_{j>0} \left( (w_j - v_j) \log a_j - \log \frac{w_j!}{v_j!} \right).$$

Now we fix an infinite subset  $J \subset \{2, 3, 4, \dots\}$  with  $\sum_{j \in J} \epsilon_j^2 < \infty$ . Then it follows from (3-4) that  $\sum_{j \notin J} \epsilon_j^2 = \infty$ . Hence we can change  $(\epsilon_n)_{n \in \mathbb{Z}}$  with another sequence coinciding with the original one if  $n \notin J$  and such that  $\epsilon_n = 0$  for each  $n \in J$ . Of course, the new sequence satisfies (3-4) and determines a new measure  $\widehat{\mu}$  on  $X$ . Proposition 3.2 yields that  $\widehat{\mu}^* \sim \mu^*$ . Moreover, the Radon-Nikodym derivative  $v \mapsto \frac{d\widehat{\mu}^*}{d\mu^*}(v)$  does not depend on the first coordinate of  $v = (v_j)_{j=1}^\infty$  because  $1 \notin J$ . Therefore according to Remark 2.1, the cocycle  $\alpha_{\rho_R, \widehat{\mu}^*}$  is cohomologous to  $\alpha_{\rho_R, \mu^*}$ . Hence  $\alpha_{\rho_R, \widehat{\mu}^*}$  is ergodic if and only if  $\alpha_{\rho_R, \mu^*}$  is ergodic. In the sequel, by  $\alpha_{\rho_R}$  we mean  $\alpha_{\rho_R, \widehat{\mu}^*}$ . Thus, we may assume without loss of generality that the triplet  $((\mathbb{Z}_+)^{\mathbb{N}}, \bigotimes_{n=1}^\infty \kappa_n, \mathcal{S}_R)$  is isomorphic to  $(Z, \eta, \mathcal{T})$ , where  $Z = (\mathbb{Z}_+ \times \mathbb{Z}_+)^{\mathbb{N}}$ ,  $\eta = \bigotimes_{j=1}^\infty (\kappa_j \otimes \kappa_0)$  and  $\mathcal{T}$  is the tail equivalence relation on  $Z$ . The corresponding isomorphism is given by the map  $\mathbb{Z}_+^{\mathbb{N}} \ni v \mapsto ((v_j)_{j \notin J}, (v_j)_{j \in J}) \in (\mathbb{Z}_+ \times \mathbb{Z}_+)^{\mathbb{N}}$ . Indeed, we use here the fact that the sets  $J$  and  $\mathbb{N} \setminus J$  are both infinite and hence admit a bijection onto  $\mathbb{N}$  and, furthermore,  $\kappa_j = \kappa_0$  if  $j \in J$ .

Let  $z = (v_j, w_j)_{j>0}$  and  $z' = (v'_j, w'_j)_{j>0}$  are two  $\mathcal{T}$ -equivalent points with  $v_j, w_j, v'_j, w'_j \in \mathbb{Z}_+$ . Computing  $\log \alpha_{\rho_R}$  and  $\log \Delta_{\mathcal{S}_R}$  in the “new coordinates” we obtain that

$$\begin{aligned} \log \alpha_{\rho_R}(z, z') &= \sum_{j>0} (v'_j - v_j) \epsilon_j \quad \text{and} \\ \log \Delta_{\mathcal{S}_R}(z, z') &= \sum_{j>0} \left( (v'_j - v_j) \log a_j + (w'_j - w_j) \log a_0 - \log \frac{v'_j!}{v_j!} - \log \frac{w'_j!}{w_j!} \right). \end{aligned}$$

Denote by  $\tau$  the flip on  $\mathbb{Z}_+ \times \mathbb{Z}_+$ , i.e.  $\tau(v, w) = (w, v)$ . Define a transformation  $\tau_n$  of  $Z$  by setting  $\tau_n := \underbrace{\tau \times \dots \times \tau}_{n \text{ times}} \times \text{Id} \times \text{Id} \times \dots$ . Of course,  $\tau_n \in [\mathcal{T}]$  and

$$(3-5) \quad \log \alpha_{\rho_R}(z, \tau_n z) = \sum_{j=1}^n (w_j - v_j) \epsilon_j = \log \Delta_{\mathcal{S}_R}(z, \tau_n z)$$

at a.e.  $z \in Z$  for each  $n > 0$ .

*Claim 1.* For each  $p > 0$ ,

$$\lim_{n \rightarrow \infty} \eta(\{z \in Z \mid \log \alpha_{\rho_R}(z, \tau_n z) > -p\}) = 0.$$

To prove this claim we first define mappings  $X_j : Z \rightarrow \mathbb{R}$  by setting

$$X_j(z) := (w_j - v_j) \epsilon_j$$

for each  $z = (v_m, w_m)_{m>0}$  and  $j \in \mathbb{N}$ . Then the following are satisfied:

- $X_1, X_2, \dots$  is a sequence of independent random variables,
- for each  $j > 1$ , the measure  $\eta \circ (\frac{1}{\epsilon_j} X_j)^{-1}$  is the Skellam distribution  $\chi_{a_0, a_j}$  with parameters  $(a_0, a_j)$  (see Appendix A) and
- for each  $n > 0$  and a.e.  $z \in Z$ ,

$$(3-6) \quad \log \alpha_{\rho_R}(z, \tau_n z) = \sum_{j=1}^n X_j(z).$$

We now have (see Appendix A):

$$\begin{aligned} E(X_j) &= \epsilon_j(a_0 - a_j) = a\epsilon_j(1 - e^{\epsilon_j}), \\ \sigma^2(X_j) &= \epsilon_j^2(a_0 + a_j) = a\epsilon_j^2(1 + e^{\epsilon_j}) \end{aligned}$$

and for each  $t \in \mathbb{R}$ ,

$$\begin{aligned} \phi_{X_j - E(X_j)}(t) &:= E(e^{it(X_j - E(X_j))}) \\ &= e^{-(a + ae^{\epsilon_j}) + ae^{it\epsilon_j} + ae^{\epsilon_j} e^{-it\epsilon_j} - itE(X_j)} \\ (3-7) \quad &= e^{a((e^{it\epsilon_j} - 1) + e^{-it\epsilon_j} - 1) + (e^{\epsilon_j} - 1)(e^{-it\epsilon_j} - 1 + it\epsilon_j)} \\ &= e^{-4a \sin^2 \frac{t\epsilon_j}{2} + a(e^{\epsilon_j} - 1)(e^{-it\epsilon_j} - 1 + it\epsilon_j)}. \end{aligned}$$

Let  $\beta_n := (\sum_{j=1}^n \epsilon_j^2)^{-\frac{1}{2}}$ . Utilizing (3-7), we now compute the characteristic function of the random variable  $Y_n := \beta_n \sum_{j=1}^n (X_j - E(X_j))$  at a point  $t \in \mathbb{R}$ :

$$\begin{aligned} \phi_{Y_n}(t) &= \prod_{j=1}^n \phi_{X_j - E(X_j)}(\beta_n t) \\ (3-8) \quad &= e^{-4a \sum_{j=1}^n \sin^2 \frac{t\epsilon_j \beta_n}{2} + a \sum_{j=1}^n (e^{\epsilon_j} - 1)(e^{-it\beta_n \epsilon_j} - 1 + it\epsilon_j \beta_n)}. \end{aligned}$$

Since the sequence  $(\epsilon_n)_{n=1}^\infty$  is bounded and  $\sum_{j=1}^\infty \epsilon_j^2 = \infty$ , it follows that

$$\beta_n \max_{1 \leq j \leq n} |\epsilon_j| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore there is  $C = C(t) > 0$  such that

$$(3-9) \quad \left| \sum_{j=1}^n \sin^2 \frac{t\epsilon_j \beta_n}{2} - \sum_{j=1}^n \left( \frac{t\epsilon_j \beta_n}{2} \right)^2 \right| \leq C\beta_n^4 \sum_{j=1}^n \epsilon_j^4 \quad \text{and}$$

$$(3-10) \quad \left| \sum_{j=1}^n (e^{\epsilon_j} - 1)(e^{-it\beta_n \epsilon_j} - 1 + it\epsilon_j \beta_n) \right| \leq C\beta_n^2 \sum_{j=1}^n |\epsilon_j|^3.$$

For each  $\epsilon > 0$ , there is  $N > 0$  such that  $|\epsilon_n| < \epsilon$  whenever  $n > N$ . Therefore

$$\begin{aligned} \beta_n^4 \sum_{j=1}^n \epsilon_j^4 &\leq \beta_n^4 \sum_{j=1}^N \epsilon_j^4 + \frac{\epsilon^2 \sum_{j=N+1}^n \epsilon_j^2}{(\sum_{j=1}^n \epsilon_j^2)^2} < 2\epsilon \quad \text{and} \\ \beta_n^2 \sum_{j=1}^n |\epsilon_j|^3 &\leq \beta_n^2 \sum_{j=1}^N |\epsilon_j|^3 + \frac{\epsilon \sum_{j=N+1}^n \epsilon_j^2}{\sum_{j=1}^n \epsilon_j^2} < 2\epsilon \end{aligned}$$

if  $n$  is large enough. It follows that the lefthand sides in (3-9) and (3-10) go to 0 as  $n \rightarrow \infty$ . Hence (3-9) yields that  $\lim_{n \rightarrow \infty} \sum_{j=1}^n \sin^2 \frac{t\epsilon_j \beta_n}{2} = \frac{t^2}{4}$  for each  $t \in \mathbb{R}$ . From this, (3-8) and (3-10), we deduce that

$$\lim_{n \rightarrow \infty} \phi_{\beta_n \sum_{j=1}^n (X_j - E(X_j))}(t) = e^{-at^2} \quad \text{for each } t \in \mathbb{R}.$$

However, the map  $\mathbb{R} \ni t \mapsto e^{-at^2}$  is the characteristic function of a normal distribution. Hence by Lévy's continuity theorem,  $(\beta_n \sum_{j=1}^n (X_j - E(X_j)))_{n=1}^\infty$  converges in distribution to a Gaussian random variable. Since  $\sum_{j=1}^n \epsilon_j^2 \rightarrow \infty$ , it follows that

$$\lim_{n \rightarrow \infty} \beta_n \sum_{j=1}^n E(X_j) = \lim_{n \rightarrow \infty} \frac{a \sum_{j=1}^n \epsilon_j (1 - e^{\epsilon_j})}{\sqrt{\sum_{j=1}^n \epsilon_j^2}} = -\infty.$$

Therefore  $\beta_n \sum_{j=1}^n X_j \rightarrow -\infty$  in distribution. This, in turn, yields that  $\sum_{j=1}^n X_j \rightarrow -\infty$  in distribution. Thus Claim 1 is proved.

*Claim 2.* For  $\eta$ -a.e.  $z \in Z$  and each  $\epsilon > 0$ , there is  $N$  such that  $|X_n(z)| < \epsilon$  for each  $n > N$ . To prove this claim we let

$$B_n := \{z = (v_j, w_j)_{j \in \mathbb{N}} \in Z \mid |X_n(z)| > \sqrt{|\epsilon_n|}\} = \{z \in Z \mid |w_n - v_n| > |\epsilon_n|^{-1/2}\}.$$

Then  $\eta(B_n) = \chi_{a_0, a_n}(\{k \in \mathbb{Z} \mid |k| > |\epsilon_n|^{-1/2}\})$ . According to (A-1), there is  $L > 0$  such that if  $|\epsilon_n|^{-1/2} > L$  then

$$\sum_{|k| > |\epsilon_n|^{-1/2}} \chi_{a_0, a_n}(k) < \epsilon_n^4.$$

Hence  $\eta(B_n) < \epsilon_n^4$ . Therefore  $\sum_{n=1}^\infty \eta(B_n) < \sum_{n=1}^\infty \epsilon_n^4 < \infty$  according to (3-4). Claim 2 follows from this via the Borel-Cantelli lemma.

*Claim 3.* Each  $r < -1$  is an essential value of  $\log \alpha_{\rho_R}$ .

We first recall that given  $d > 0$  and elements  $c_1, \dots, c_d \in \mathbb{Z}_+ \times \mathbb{Z}_+$ , the *cylinder*  $[c_1, \dots, c_d]_1^d$  is the set  $\{z = (z_n)_{n \in \mathbb{Z}} \in Z \mid z_j = c_j, 1 \leq j \leq d\}$ . To prove the claim, take  $\epsilon > 0$ , a positive integer  $k$  and a cylinder  $C = [(a_1, b_1), \dots, (a_k, b_k)]_1^k \subset Z$ , where  $a_1, b_1, \dots, a_k, b_k \in \mathbb{Z}_+$ . It follows from Claim 1 and Claim 2 that there are  $N > M > k$  and a subset  $I \subset (\mathbb{Z}_+ \times \mathbb{Z}_+)^{N-k}$  such that the subset

$$A := \bigsqcup_{(z_{k+1}, \dots, z_N) \in I} [(a_1, b_1), \dots, (a_k, b_k), z_{k+1}, \dots, z_N]_1^N \subset C$$

satisfies the following three conditions:

- $\eta(A) > 0.5\eta(C)$ ,
- $\max_{z \in A} |X_j(z)| < \epsilon$  whenever  $N \geq j > M$  and
- $\max_{z \in A} \sum_{j=M+1}^N X_j(z) < -r$ .

For  $z \in A$ , let  $l(z)$  be the smallest number  $l > M$  such that  $\sum_{j=M+1}^l X_j(z) < -r$ . Then  $l(z) \leq N$  and

$$(3-11) \quad \left| \sum_{j=M+1}^{l(z)} X_j(z) - r \right| \leq |X_{l(z)}(z)| < \epsilon.$$

We now set

$$\psi(z) := ((a_1, b_1), \dots, (a_k, b_k), z_{k+1}, \dots, z_M, \tau(z_{M+1}), \dots, \tau(z_{l(z)}), z_{l(z)+1}, \dots)$$

for each  $z \in A$ . Then  $\psi(z) \in C$ ,  $(z, \psi(z)) \in \mathcal{T}$  and

$$(3-12) \quad |\log \alpha_{\rho_R}(z, \psi(z)) - r| < \epsilon$$

in view of (3-11) and (3-6). We now show that the map  $\psi : A \ni z \mapsto \psi(z) \in C$  is one-to-one. Suppose that  $\psi(z) = \psi(z')$  for some  $z = (z_j)_{j \in \mathbb{Z}} \in A$  and  $z' = (z'_j)_{j \in \mathbb{Z}} \in A$ . If  $l(z) = l(z')$  then obviously  $z = z'$ . Therefore suppose that  $l(z) > l(z')$ . Then the equality  $\psi(z) = \psi(z')$  implies that  $z_j = z'_j$  if  $1 \leq j \leq M$  and  $\tau(z_j) = \tau(z'_j)$  if  $M < j \leq l(z')$ . Hence  $z_j = z'_j$  if  $1 \leq j \leq l(z')$ . Therefore  $-r > \sum_{j=M+1}^{l(z')} X_j(z') = \sum_{j=M+1}^{l(z')} X_j(z)$ . This yields that  $l(z) \leq l(z')$ , a contradiction. Thus,  $\psi$  is one-to-one. It follows from (3-5) and (3-12) that  $-r - \epsilon < \log \Delta_{\mathcal{S}_R}(z, \tau_n z) < -r + \epsilon$  for all  $z \in A$ . Lemma 1.2 implies now that  $r$  is an essential value of  $\log \alpha_{\rho_R}$ . Thus, Claim 3 is proved.

Since the essential range of  $\log \alpha_{\rho_R}$  is a closed subgroup of  $\mathbb{R}$  containing an arbitrary real less than  $-1$  (in view of Claim 3), it follows that this subgroup is  $\mathbb{R}$ . Hence  $\log \alpha_{\rho_R}$  is ergodic.  $\square$

*Remark 3.6.* While proving Theorem 3.5 we also showed as a byproduct that the cocycle  $\Delta_{\mathcal{S}_R}$  is also ergodic, i.e.  $\mathcal{S}_R$  is of type  $III_1$ . In fact, we proved a stronger result. Let  $\mathcal{E}$  denote the orbit equivalence relation on  $(\mathbb{Z}_+)^{\mathbb{N}}$  generated by the group of finite permutations of the coordinates. Then  $\mathcal{E}$  is a proper subrelation of  $\mathcal{S}_R$ . It follows from the proof of Theorem 3.5 that  $\mathcal{E}$  is of type  $III_1$  and the restriction of the cocycle  $\log \alpha_{\rho_R}$  to  $\mathcal{E}$  is ergodic.

The next corollary is the main result of this section.

**Corollary 3.7.** *If (3-4) holds and  $T_*$  is conservative then the Maharam extension of  $T_*$  is a weakly mixing  $K$ -automorphism. In particular,  $T_*$  is weakly mixing and of stable type  $III_1$ .*

*Proof.* It follows from Theorem 3.5 that  $\alpha_{\rho_R}$  is ergodic. Since  $T_*$  is the natural extension of  $R$ , the Maharam extension  $(T_*)_{\rho_{T_*}}$  of  $T_*$  is a nonsingular  $K$ -automorphism by Proposition 2.2. Since  $T_*$  is conservative,  $(T_*)_{\rho_{T_*}}$  is also conservative according to the Maharam theorem (see [Aa, Theorem 3.4.1], [Sc, Theorem 5.5]). It follows that  $(T_*)_{\rho_{T_*}}$  is weakly mixing. This implies that  $T_*$  is weakly mixing and of type  $III_1$ .  $\square$

We now provide concrete examples of conservative  $T_*$  such that (3-4) holds.

**Example 3.8.** Let  $\epsilon_n := 0$  if  $n \leq 1$  and  $\epsilon_n := -n^{-1/2}$  if  $n > 1$ . Then (3-4) holds. We claim that if  $0 < a < \frac{1}{6}$  then  $T_*$  is conservative. For that we will utilize Proposition 3.4. We first note that

$$\begin{aligned} \int_X \left( \left( \frac{d\mu}{d\mu \circ T^{-n}} \right)^2 - 1 \right) d\mu &= a \sum_{k \in \mathbb{Z}} \left( e^{2(\epsilon_k - \epsilon_{k-n})} - 1 \right) e^{\epsilon_k} \\ &= a \sum_{k \in \mathbb{Z}} \left( e^{3\epsilon_k - 2\epsilon_{k-n}} - e^{\epsilon_k} \right) \end{aligned}$$

and  $|3\epsilon_k - 2\epsilon_{k-n}| \leq 3|\epsilon_k|$ . Therefore it follows from the Taylor expansion of the exponential function that

$$e^{3\epsilon_k - 2\epsilon_{k-n}} - e^{\epsilon_k} = 2\epsilon_k - 2\epsilon_{k-n} + \frac{1}{2}(3\epsilon_k - 2\epsilon_{k-n})^2 - \frac{1}{2}\epsilon_k^2 + O(\epsilon_k^3)$$



as  $k \rightarrow \infty$ . Since  $\sum_{k \in \mathbb{Z}} |\epsilon_k|^3 < \infty$  and  $\sum_{k \in \mathbb{Z}} (\epsilon_k - \epsilon_{k-n}) = 0$ ,

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} (e^{3\epsilon_k - 2\epsilon_{k-n}} - e^{\epsilon_k}) &= \frac{1}{2} \sum_{k \in \mathbb{Z}} \left( (3\epsilon_k - 2\epsilon_{k-n})^2 - \epsilon_k^2 \right) + O(1) \\
&= 2 \sum_{k \in \mathbb{Z}} \left( 2\epsilon_k^2 + \epsilon_{k-n}^2 - 3\epsilon_k \epsilon_{k-n} \right) + O(1) \\
&= 4 \sum_{k=2}^{n+1} \frac{1}{k} + 2 \sum_{k > n+1} \left( \frac{1}{k-n} - \frac{1}{k} + \frac{3}{k} - \frac{3}{\sqrt{k(k-n)}} \right) + O(1) \\
&= 6 \sum_{k=2}^{n+1} \frac{1}{k} + 6 \sum_{k > n+1} \left( \frac{1}{k} - \frac{1}{\sqrt{k(k-n)}} \right) + O(1) \\
&= 6 \log n + 6 \int_{n+1}^{+\infty} \left( \frac{1}{t} - \frac{1}{\sqrt{t(t-n)}} \right) dt + O(1) \\
&= 6 \log n + 6 \left( \log t - \log \left( t - \frac{n}{2} + \sqrt{t(t-n)} \right) \right) \Big|_{n+1}^{+\infty} + O(1) \\
&= 6 \log n + O(1)
\end{aligned}$$

as  $n \rightarrow +\infty$ . Thus, we have shown that

$$\int_X \left( \left( \frac{d\mu}{d\mu \circ T^{-n}} \right)^2 - 1 \right) d\mu = 6a \log n + O(1).$$

It follows that there is a constant  $C > 0$  such that for each  $n > 0$ ,

$$(3-13) \quad \int_X \left( \left( \frac{d\mu}{d\mu \circ T^{-n}} \right)^2 - 1 \right) d\mu < 6a \log n + C.$$

Choose a real  $\beta$  such that  $1 \geq \beta > \frac{1}{2} + 3a$ . It exists because  $a < \frac{1}{6}$ . Now let  $b_n := n^{-\beta}$ . Then  $\sum_{n=1}^{\infty} b_n = \infty$  but in view of (3-13),

$$\sum_{n=1}^{\infty} b_n^2 e^{\int_X \left( \left( \frac{d\mu}{d\mu \circ T^{-n}} \right)^2 - 1 \right) d\mu} \leq e^C \sum_{n=1}^{\infty} \frac{1}{n^{2\beta - 6a}} < \infty.$$

It follows from Proposition 3.4 that  $T_*$  is conservative. It is weakly mixing and of type  $III_1$  by Corollary 3.7.

We now show that the restriction  $a < \frac{1}{6}$  in Example 3.8 can not be dropped. For that we'll need a criterion for dissipativity from [DaKoRo, Corollary 6.8].

**Lemma 3.9.** *If  $T \in \text{Aut}_2(X, \mu)$  and  $\sum_{n \geq 0} e^{-\frac{1}{2} \left\| \sqrt{\frac{d\mu \circ T^n}{d\mu}} - 1 \right\|_2^2} < \infty$  then  $T_*$  is totally dissipative.*

**Proposition 3.10.** *Let  $(X, \mu, T)$  be as in Example 3.8 but  $a > 4$ . Then  $T_*$  is totally dissipative.*

*Proof.* For each  $n > 0$ ,

$$\begin{aligned}
\left\| \sqrt{\frac{d\mu \circ T^n}{d\mu}} - 1 \right\|_2^2 &= a \sum_{k > -n} \left( e^{\frac{\epsilon_{n+k} - \epsilon_k}{2}} - 1 \right)^2 e^{\epsilon_k} \\
&= a \sum_{k=-n+1}^1 \left( e^{\frac{\epsilon_{n+k}}{2}} - 1 \right)^2 + a \sum_{k=2}^{\infty} \left( e^{\frac{\epsilon_{n+k} - \epsilon_k}{2}} - 1 \right)^2 e^{\epsilon_k} \\
&= a \sum_{k=1}^{n+1} \left( e^{\frac{\epsilon_k}{2}} - 1 \right)^2 + a \sum_{k=2}^{\infty} \left( e^{\frac{\epsilon_{n+k}}{2}} - e^{\frac{\epsilon_k}{2}} \right)^2 \\
&= \frac{a}{4} \sum_{k=1}^{n+1} \left( \frac{1}{\sqrt{k}} + O\left(\frac{1}{k}\right) \right)^2 + \frac{a}{4} \sum_{k=2}^{+\infty} \left( \frac{1}{\sqrt{n+k}} - \frac{1}{\sqrt{k}} + O\left(\frac{1}{k}\right) \right)^2 \\
&= \frac{a}{4} \sum_{k=1}^{n+1} \frac{1}{k} + \frac{a}{4} \sum_{k=2}^{+\infty} \left( \frac{1}{n+k} + \frac{1}{k} - \frac{2}{\sqrt{k(n+k)}} \right) + O(1)
\end{aligned}$$

as  $n \rightarrow \infty$ . Since  $\sum_{k=1}^{n+1} \frac{1}{k} = \log n + O(1)$  and

$$\begin{aligned}
\sum_{k=2}^{+\infty} \left( \frac{1}{n+k} + \frac{1}{k} - \frac{2}{\sqrt{k(n+k)}} \right) &= \int_2^{+\infty} \left( \frac{1}{n+t} + \frac{1}{t} - \frac{2}{\sqrt{t(n+t)}} \right) dt + o(1) \\
&= \left( \log(t(n+t)) - 2 \log \left( t + \frac{n}{2} + \sqrt{t(n+t)} \right) \right) \Big|_2^{+\infty} + o(1) \\
&= \log n + O(1),
\end{aligned}$$

we obtain that  $\left\| \sqrt{\frac{d\mu \circ T^n}{d\mu}} - 1 \right\|_2^2 = \frac{a}{2} \log n + O(1)$  as  $n \rightarrow \infty$ . It follows that there is a real  $D$  such that for each  $n > 0$ ,

$$(3-14) \quad \left\| \sqrt{\frac{d\mu \circ T^n}{d\mu}} - 1 \right\|_2^2 \geq D + \frac{a \log n}{2}.$$

Since  $a > 4$ , we deduce from (3-14) that

$$\sum_{n \geq 0} e^{-\frac{1}{2} \left\| \sqrt{\frac{d\mu \circ T^n}{d\mu}} - 1 \right\|_2^2} \leq \sum_{n \geq 0} e^{-\frac{1}{2}(D + \frac{a \log n}{2})} = e^{-\frac{D}{2}} \sum_{n \geq 0} n^{-\frac{a}{4}} < \infty.$$

It follows now from Lemma 3.9 that  $T_*$  is totally dissipative.  $\square$

#### 4. POISSON SUSPENSIONS OF TYPE $III_1$ OVER A NON-ATOMIC BASE

In this subsection we construct concrete examples of weakly mixing Poisson suspensions of type  $III_1$  which are Poisson suspensions of nonsingular transformations defined on non-atomic  $\sigma$ -finite spaces.

Let  $(X, \mu, T)$  be as in §3, i.e.  $X = \mathbb{Z}$ ,  $\mu$  a non-degenerated measure on  $X$  and  $T$  the unit translation on  $X$ . Let  $K = [0, 1]$ . Denote by  $\lambda$  the Lebesgue measure on  $K$ . Let  $Y := K \times \mathbb{Z}$  and let  $Q : Y \rightarrow Y$  be the direct product of the identity on  $K$

and  $T$  on  $X$ , i.e.  $Q(k, n) := (k, n + 1)$  for all  $k \in K$  and  $n \in \mathbb{Z}$ . Then the product measure  $\nu := \lambda \otimes \mu$  is a non-atomic measure on  $Y$ . Of course,  $Q$  is  $\nu$ -nonsingular. Moreover,

$$Q'(k, n) = T'(n) = \frac{a_{n-1}}{a_n},$$

where  $a_n := \mu(\{n\})$ . The dynamical system  $(X, \mu, T)$  is a factor of  $(Y, \nu, Q)$ . Moreover, the corresponding projection

$$(4-1) \quad \vartheta : Y \ni (k, n) \mapsto n \in X$$

is relatively finite measure preserving (r.f.m.p.). It is straightforward to verify that  $Q \in \text{Aut}_2(Y, \nu)$  if and only if  $T \in \text{Aut}_2(X, \mu)$ , i.e. if and only if (3-1) is satisfied. In a similar way,  $Q \in \text{Aut}_1(Y, \nu)$  if and only if  $T \in \text{Aut}_1(X, \mu)$ . Hence if  $Q \in \text{Aut}_1(Y, \nu)$  then the limits  $\lim_{n \rightarrow +\infty} a_n$  and  $\lim_{n \rightarrow -\infty} a_n$  exist and  $\chi(Q) = \chi(T) = \lim_{n \rightarrow +\infty} a_n - \lim_{n \rightarrow -\infty} a_n$ .

To describe the Poisson suspension of  $(Y, \nu, Q)$  we first write  $Y$  as the “disjoint” union  $Y = \bigsqcup_{n \in \mathbb{Z}} K \times \{n\}$ . Under the natural identification, we may assume that

$$Y^* = \prod_{n \in \mathbb{Z}} (K \times \{n\})^* \quad \text{and} \quad \nu^* = \bigotimes_{n \in \mathbb{Z}} \nu_n^*,$$

where  $\nu_n := (\nu \upharpoonright (K \times \{n\}))$ . If we identify naturally  $K \times \{n\}$  with  $K$  then  $\nu_n$  corresponds to  $a_n \cdot \lambda$  for each  $n \in \mathbb{Z}$ . Hence the Poisson suspension  $Q_*$  of  $Q$  corresponds to the 2-sided Bernoulli shift on the infinite product space

$$(Y^*, \nu^*) = \left( (K^*)^{\mathbb{Z}}, \bigotimes_{n \in \mathbb{Z}} (a_n \cdot \lambda)^* \right).$$

We also recall that we view  $(X^*, \mu^*)$  as the infinite product  $((\mathbb{Z}_+)^{\mathbb{Z}}, \bigotimes_{n \in \mathbb{Z}} \kappa_n)$ , where  $\kappa_n := (\mu \upharpoonright \{n\})^*$  (see §3). The factor map  $\vartheta$  from (4-1) generates a factor map  $\vartheta^* : Y^* \rightarrow X^*$ . Thus we have that  $\vartheta^* Q_* = T_* \vartheta^*$  and  $\mu^* = \nu^* \circ (\vartheta^*)^{-1}$ . Since  $\vartheta$  is r.f.m.p.,  $\vartheta^*$  is r.f.m.p. whenever  $Q_*$  is  $\nu^*$ -nonsingular. It is important to note that  $\vartheta^*$  has a “coordinate-wise” structure, i.e.  $\vartheta^* y = (\vartheta_n^* y_n)_{n \in \mathbb{Z}}$  for each  $y = (y_n)_{n \in \mathbb{Z}} \in Y^*$ , where  $\vartheta_n^*$  is a measure preserving mapping from the probability space  $(K^*, \nu_n^*)$  onto the probability space  $(\mathbb{Z}_+, \kappa_n)$  given by  $K^* \ni y_n \mapsto y_n(K) \in \mathbb{Z}_+$ .

**Corollary 4.1.** *The Bernoulli shift  $Q_*$  is  $\nu^*$ -nonsingular if and only if  $T_*$  is  $\mu^*$ -nonsingular, i.e. if and only if (3-1) is satisfied. In this case, the mapping*

$$\vartheta^* : Y^* \ni y = (y_n)_{n \in \mathbb{Z}} \mapsto (\vartheta_n^* y_n)_{n \in \mathbb{Z}} := (y_n(K))_{n \in \mathbb{Z}} \in X^*$$

*intertwines  $Q_*$  with  $T_*$ , maps  $\nu^*$  onto  $\mu^*$  and*

$$\frac{d\nu^* \circ Q_*}{d\nu^*}(y) = \frac{d\mu^* \circ T_*}{d\mu^*}(\vartheta^* y) = \prod_{n \in \mathbb{Z}} \frac{\kappa_{n-1}(\vartheta_n^* y_n)}{\kappa_n(\vartheta_n^* y_n)}$$

*for a.e.  $y = (y_n)_{n \in \mathbb{Z}} \in Y^*$ . Hence  $(Y^*, \nu^*)$  is isomorphic to the product space  $(K^{\mathbb{Z}} \times X^*, \lambda^{\mathbb{Z}} \times \mu^*)$  in such a way that*

- (i)  $\vartheta^*$  corresponds to the projection to the second coordinate and
- (ii)  $Q_*$  corresponds to the direct product  $B \times T_*$ , where  $B : K^{\mathbb{Z}} \rightarrow K^{\mathbb{Z}}$  is the Bernoulli shift preserving  $\lambda^{\mathbb{Z}}$ .

*Proof.* For each  $n \in \mathbb{Z}$ , we disintegrate  $\nu_n^*$  relative to  $\kappa_n$  (via  $\vartheta_n^*$ ):

$$(4-2) \quad \nu_n^* = \sum_{k \in \mathbb{Z}_+} \kappa_n(k) \xi_{n,k},$$

where  $(\xi_{n,k})_{k \in \mathbb{Z}_+}$  is the corresponding sequence of conditional probabilities on  $K^*$ . Using (4-2), we obtain the disintegration of  $\nu^*$  relative to  $\mu^*$  (via  $\vartheta^*$ ):

$$\nu^* = \int_{X^*} \bigotimes_{n \in \mathbb{Z}} \xi_{n,k_n} d\mu^*((k_n)_{n \in \mathbb{Z}}).$$

Since  $\vartheta^*$  intertwines the Bernoulli shifts on  $Y^*$  and  $X^*$  respectively, the r.f.m.p. property of  $\vartheta^*$  means that  $\bigotimes_{n \in \mathbb{Z}} \xi_{n,k_{n+1}} = \bigotimes_{n \in \mathbb{Z}} \xi_{n+1,k_{n+1}}$  and hence  $\xi_{n,k_{n+1}} = \xi_{n+1,k_{n+1}}$  for all  $n \in \mathbb{Z}$  and  $\mu^*$ -a.e.  $(k_n)_{n \in \mathbb{Z}} \in X^*$ . Hence there is a sequence  $(\xi_k)_{k \in \mathbb{Z}_+}$  of probability measures on  $K^*$  such that  $\xi_{n,k} = \xi_k$  and therefore

$$\nu^* = \int_{X^*} \bigotimes_{n \in \mathbb{Z}} \xi_{k_n} d\mu^*((k_n)_{n \in \mathbb{Z}}).$$

The measure  $\xi_k$  is non-atomic for each  $k > 0$  and the measure  $\xi_0$  is a delta-measure. For a.e.  $\mathbf{k} = (k_n)_{n \in \mathbb{Z}} \in X^*$ , there exist infinitely many positive  $n$  with  $k_n > 0$  and infinitely many negative  $n$  with  $k_n > 0$ . Hence for such a  $\mathbf{k}$  and every  $n \in \mathbb{Z}$ , there are (uniquely defined) two integers  $l_n(\mathbf{k})$  and  $r_n(\mathbf{k})$  such that the following are satisfied:

- $l_n(\mathbf{k}) \leq n \leq r_n(\mathbf{k})$ ,
- $k_{l_n(\mathbf{k})} \neq 0$  and  $k_{r_n(\mathbf{k})+1} \neq 0$  and
- if  $l_n(\mathbf{k}) < n \leq r_n(\mathbf{k})$  then  $k_n = 0$ .

Of course, the mappings  $X^* \ni \mathbf{k} \mapsto l_n(\mathbf{k}) \in \mathbb{Z}$  and  $X^* \ni \mathbf{k} \mapsto r_n(\mathbf{k}) \in \mathbb{Z}$  are measurable for every  $n \in \mathbb{Z}$ . For each  $n \in \mathbb{Z} \setminus \{0\}$  and  $l \in \mathbb{N}$ , there is a measure theoretical isomorphism

$$\tau_{k,l} : ((K^*)^l, \xi_k \otimes \xi_0 \otimes \cdots \otimes \xi_0) \rightarrow (K^l, \lambda^l).$$

Next, for each point  $\mathbf{k} = (k_n)_{n \in \mathbb{Z}} \in X^*$  and  $\mathbf{y} = (y_n)_{n \in \mathbb{Z}} \in (K^*)^{\mathbb{Z}}$ , we define a point  $\langle \mathbf{k}, \mathbf{y} \rangle = (z_n)_{n \in \mathbb{Z}}$  of  $K^{\mathbb{Z}}$  in the following way:  $z_n$  is the  $(n - l_n(\mathbf{k}) + 1)$ -th symbol in the block

$$\tau_{l_n(\mathbf{k}), r_n(\mathbf{k}) - l_n(\mathbf{k}) + 1} [y_{l_n} \cdots y_{r_n}] \in K^{r_n(\mathbf{k}) - l_n(\mathbf{k}) + 1}.$$

We now define an isomorphism  $\tau$  of  $(Y^*, \nu^*)$  onto  $(K^{\mathbb{Z}} \times X^*, \lambda^{\mathbb{Z}} \otimes \mu^*)$  by setting

$$\tau(\mathbf{y}) := (\langle \vartheta^* \mathbf{y}, \mathbf{y} \rangle, \vartheta^* \mathbf{y}).$$

It is straightforward to verify that (i) and (ii) hold.  $\square$

We now state the main result of this section.

**Theorem 4.2.** *If (3-1) and (3-4) are satisfied then the Maharam extension of  $(Y^*, \nu^*, Q_*)$  is a  $K$ -automorphism. If, moreover,  $T_*$  is conservative then  $Q_*$  is weakly mixing and of stable Krieger's type III<sub>1</sub>.*

*Proof.* Consider  $Q_*$  as the direct product  $B \times T_*$  described in Corollary 4.1. Then the Maharam extension of  $Q_*$  is isomorphic to the product of  $B$  with the Maharam extension of  $T_*$ . It remains to apply Theorem 3.5 and a simple fact that the direct product of a (conservative)  $K$ -automorphism with a probability preserving Bernoulli shift is a (conservative)  $K$ -automorphism.  $\square$

*Remark 4.3.* Utilizing Example 3.8 we obtain concrete examples of ergodic conservative suspensions  $Q_*$  of type  $III_1$  for  $Q$  defined on a nonatomic infinite measure space. Moreover, it follows from Example 3.8 and Proposition 3.10 that  $Q_*$  is totally dissipative if  $a > 4$  and conservative (and hence weakly mixing and of stable type  $III_1$ ) if  $0 < a < \frac{1}{6}$ . This result will be improved in §7 below.

## 5. GENERIC POISSON SUSPENSION IS OF TYPE $III_1$

Throughout this section we assume that  $\mu$  is an infinite  $\sigma$ -finite non-atomic measure on standard Borel space  $(X, \mathfrak{B})$ . Given two subsets  $A, B \subset X$  of finite measure, let  $\tau_{A,B}$  denote a  $\mu$ -nonsingular bijection from  $A$  onto  $B$  such that  $\frac{d\mu \circ \tau_{A,B}}{d\mu}(x) = \frac{\mu(B)}{\mu(A)}$  for all  $x \in A$ . We now recall a definition and some facts from [DaKoRo].

**Definition 5.1.** A nonsingular transformation  $T$  of a  $\sigma$ -finite standard measure space  $(X, \mu)$  is *locally aperiodic* if there is a subset  $A \subset X$  of positive finite measure such that  $Tx = x$  if  $x \notin A$  and  $T^n x \neq x$  for each  $x \in A$  and  $n > 0$ .

Of course, each locally aperiodic transformation  $T$  belongs to  $\text{Aut}_1(X, \mu)$ . It is easy to verify that  $\chi(T) = 0$ . It was also shown in [DaKoRo, Proposition 5.4 and Theorem 5.8(2)] that:

- (F1) the conjugacy class of  $T$  in  $\text{Aut}_2(X, \mu)$  is  $d_2$ -dense in  $\text{Aut}_2(X, \mu)$  and
- (F2) the conjugacy class of  $T$  in  $\text{Aut}_1(X, \mu)$  is  $d_1$ -dense in  $\text{Ker } \chi$ .

**Proposition 5.2.** Let  $T \in \text{Aut}(X, \mu)$  and there are a partition  $X = \bigsqcup_{n \in \mathbb{Z}} W_n$  of  $X$  into subsets  $W_n$  of finite measure and a sequence  $(a_n)_{n \in \mathbb{Z}}$  of reals such that  $TW_n = W_{n+1}$  and  $T'(x) = a_n$  at a.e.  $x \in W_n$  and for each  $n \in \mathbb{Z}$ . Suppose that there exists a limit  $\lim_{|n| \rightarrow \infty} \mu(W_n) \in (0, +\infty)$ .

- (i) If  $T \in \text{Aut}_2(X, \mu)$  then the conjugacy class of  $T$  is dense in  $\text{Aut}_2(X, \mu)$ .
- (ii) If  $T \in \text{Aut}_1(X, \mu)$  then the conjugacy class of  $T$  is dense in  $\text{Ker } \chi$ .

*Proof.* (i) Since  $T \in \text{Aut}_2(X, \mu)$ ,

$$\infty > \|\sqrt{T'} - 1\|_2^2 = \sum_{n \in \mathbb{Z}} \int_{W_n} (\sqrt{T'(x)} - 1)^2 d\mu(x) = \sum_{n \in \mathbb{Z}} (\sqrt{a_n} - 1)^2 \mu(W_n).$$

Since  $a_n = \mu(W_{n+1})/\mu(W_n)$ , we obtain that

$$(5-1) \quad \sum_{n \in \mathbb{Z}} (\sqrt{\mu(W_{n+1})} - \sqrt{\mu(W_n)})^2 < \infty.$$

If we show that the  $d_2$ -closure of the conjugacy class of  $T$  contains a locally aperiodic transformation then (F1) yields that the conjugacy class of  $T$  is dense in  $\text{Aut}_2(X, \mu)$ . We now construct such a transformation. For each  $n > 0$ , we set  $h_n := 2^n$ . Let  $B_0$  be a subset of  $X$  with  $\mu(B_0) = 1$  and let  $S$  be a  $\mu$ -preserving transformation

of  $X$  such that  $Sx = x$  if  $x \notin B_0$  and the restriction of  $S$  to  $B_0$  is isomorphic to the 2-adic odometer. Then  $S$  is locally aperiodic and there exists a decreasing infinite sequence of subsets  $B_0 \supset B_1 \supset \dots$  in  $X$  such that  $S^i B_n \cap S^j B_n = \emptyset$  if  $0 \leq i < j < h_n$  and  $\bigsqcup_{i=0}^{h_n-1} S^i B_n = B_0$ . We see that  $S$  has a “cyclic structure of period  $h_n$  on  $B_n$ ”. The idea of the following argument is to define for each  $n > 0$ , a transformation  $S_n$  extending the cyclic structure of  $S$  from  $B_n$  to a larger subset in such a way  $S_n \rightarrow S$  and  $S_n$  is conjugate to a cyclic permutation of the finite sequence  $W_{-h_{n-1}}, \dots, W_{h_{n-1}-1}$  that approaches to  $T$  as  $n \rightarrow \infty$ : Since the convergence is considered in  $d_2$ , we should choose carefully the transformations in these constructions to control their Radon-Nikodym derivatives.

More precisely, we now construct a sequence of  $\mu$ -preserving transformations  $(S_n)_{n=1}^\infty$  and a sequence  $(B'_n)_{n=1}^\infty$  of subsets in  $X$  such that

- $S_n \upharpoonright B_0 = S$  for each  $n$  and  $S_n \rightarrow S$  weakly (and hence in  $d_2$ ) as  $n \rightarrow \infty$ ,
- $B'_n \supset B_n$ ,  $\mu(B'_n) = \mu(W_{-h_{n-1}})$ ,  $S_n^i B'_n \cap S_n^j B'_n = \emptyset$  for all  $0 \leq i < j < h_n$  and each  $n$ .
- If we set  $Y_n := \bigsqcup_{i=0}^{h_n-1} S_n^i B'_n$  and  $Y_n^\circ := Y_n \setminus B'_n$  then for each subset  $A \subset X$  of finite measure,  $\lim_{n \rightarrow \infty} \mu(A \cap Y_n^\circ) = \mu(A)$ .

To see that such a construction is possible, it is convenient to think that  $X = [0, +\infty)$ ,  $\mu$  is a Lebesgue measure on  $X$ ,  $B_0 = [0, 1)$ ,  $B_n := [0, h_n^{-1})$ . To define  $B'_n$  and  $S_n$ , we first select a real  $\delta_n > 0$  and an integer  $D_n > 0$  such that

- (a)  $\delta_n h_n < n^{-1}$  and
- (b)  $\mu(B_n) + \delta_n D_n = \mu(W_{-h_{n-1}})$ .

Now we set  $B'_n := B_n \sqcup \bigsqcup_{j=0}^{D_n-1} [1 + jh_n\delta_n, 1 + (j+1)h_n\delta_n)$  and

$$S_n x := \begin{cases} Sx, & \text{if } x \in B_0 \\ x + \delta_n, & \text{if } 1 \leq x < 1 + (D_n h_n - 1)\delta_n \\ x - (D_n h_n - 1)\delta_n, & \text{if } 1 + (D_n h_n - 1)\delta_n \leq x < 1 + D_n h_n \delta_n \\ x, & \text{if } x \geq 1 + D_n h_n \delta_n. \end{cases}$$

It follows from (a) that  $S_n \rightarrow S$  weakly as  $n \rightarrow \infty$ . (b) implies that  $\mu(B'_n) = \mu(W_{-h_{n-1}})$ . The sets  $S_n^i B'_n$ ,  $i = 0, \dots, h_n - 1$ , are mutually disjoint and their union  $Y_n$  equals  $[0, 1 + h_n D_n \delta_n)$ . In particular,  $Y_n \supset [0, 1 + h_n b/2)$  eventually in  $n$ , where  $b := \lim_{|m| \rightarrow \infty} \mu(W_m)$ . Hence  $\bigcup_{n \rightarrow \infty} Y_n = X$ . Since the set  $B'_n \setminus B_n$  is “uniformly distributed” along  $Y_n \setminus B_0$ , we have that  $\mu(Y_n^\circ \cap [0, E)) \rightarrow \mu([0, E))$  for each  $E > 1$ . Thus, the sequences  $(S_n)_{n=1}^\infty$  and  $(B'_n)_{n=1}^\infty$  are as desired.

We now set  $X_n := \bigsqcup_{i=0}^{h_n-1} T^i W_{-h_{n-1}}$  and  $X_n^\circ := X_n \setminus W_{-h_{n-1}}$ . Then  $X_1^\circ \subset X_2^\circ \subset \dots$  and  $\bigcup_{n=0}^\infty X_n^\circ = X$ . For each  $n > 0$ , we select a measure preserving Borel bijection  $\tau_n$  of  $X \setminus X_n$  onto  $X \setminus Y_n$  and define a transformation  $R_n$  of  $X$  by setting

$$R_n x := \begin{cases} S_n^i \tau_{W_{-h_{n-1}}, B'_n} T^{-i} & \text{if } x \in W_{i-h_{n-1}}, 0 \leq i < h_n. \\ \tau_n x & \text{if } x \notin X_n. \end{cases}$$

Then  $R_n \in \text{Aut}(X, \mu)$ ,  $R_n X_n = Y_n$  and hence  $R_n \in \text{Aut}_1(X, \mu)$ . A straightforward verification shows that  $R_n T^{-1} R_n^{-1} = S_n^{-1}$  on  $Y_n^\circ$ . Therefore  $S_n R_n T^{-1} R_n^{-1} \rightarrow \text{Id}$  weakly as  $n \rightarrow \infty$ . Next, we note that the Radon-Nikodym derivatives of  $T^{-1}$  and  $R_n$  are constant on each level  $W_j$ ,  $j \in \mathbb{Z}$ . Since  $S_n$  preserves  $\mu$  and  $T^{-1} W_j = W_{j-1}$  for each  $j \in \mathbb{Z}$ , it follows that the Radon-Nikodym derivative of the transformation

$S_n R_n T^{-1} R_n^{-1}$  is constant on the subset  $R_n W_j$  for each  $j \in \mathbb{Z}$ . Hence we compute easily that  $(S_n R_n T^{-1} R_n^{-1})'(x)$  equals

$$\begin{cases} 1, & \text{if } x \in Y_n^\circ \\ \frac{\mu(S_n R_n W_{i-h_{n-1}-1})}{\mu(R_n W_{i-h_{n-1}})}, & \text{if } x \in R_n W_{i-h_{n-1}} \text{ and } i \notin \{1, \dots, h_n - 1\}. \end{cases}$$

If  $i \notin \{1, \dots, h_n\}$  then  $\mu(S_n R_n W_{i-h_{n-1}-1}) = \mu(W_{i-h_{n-1}-1})$ . If  $i = h_n$  then  $\mu(S_n R_n W_{i-h_{n-1}-1}) = \mu(S_n S_n^{h_n-1} \tau_{W_{-h_{n-1}}, B'_n} W_{-h_{n-1}}) = \mu(W_{-h_{n-1}})$ . Therefore

$$(S_n R_n T^{-1} R_n^{-1})'(x) = \begin{cases} 1, & \text{if } x \in Y_n^\circ \\ \frac{\mu(W_{i-h_{n-1}-1})}{\mu(W_{i-h_{n-1}})}, & \text{if } x \in R_n W_{i-h_{n-1}} \text{ and } i \notin \{1, \dots, h_n\} \\ \frac{\mu(W_{-h_{n-1}})}{\mu(W_{h_n-h_{n-1}})}, & \text{if } x \in R_n W_{h_n-h_{n-1}}. \end{cases}$$

Therefore

$$\begin{aligned} \left\| \sqrt{(S_n R_n T^{-1} R_n^{-1})'} - 1 \right\|_2^2 &= \sum_{i \notin \{1, \dots, h_n\}} \left( \sqrt{\mu(W_{i-h_{n-1}-1})} - \sqrt{\mu(W_{i-h_{n-1}})} \right)^2 \\ &\quad + \left( \sqrt{\mu(W_{-h_{n-1}})} - \sqrt{\mu(W_{h_n-h_{n-1}})} \right)^2. \end{aligned}$$

Since  $b < \infty$ , it follows that  $\sqrt{\mu(W_{-h_{n-1}})} - \sqrt{\mu(W_{h_n-h_{n-1}})} \rightarrow 0$  as  $n \rightarrow \infty$ . Utilizing this fact and (5-1) we obtain that

$$\lim_{n \rightarrow \infty} \left\| \sqrt{(S_n R_n T^{-1} R_n^{-1})'} - 1 \right\|_2 = 0.$$

Therefore  $S_n R_n T^{-1} R_n^{-1} \rightarrow \text{Id}$  in  $d_2$  as  $n \rightarrow \infty$ . Since  $S_n \rightarrow S$  in  $d_2$ , we obtain that  $S R_n T^{-1} R_n^{-1} \rightarrow \text{Id}$  in  $d_2$  as  $n \rightarrow \infty$ . Thus  $S$  belongs to the  $d_2$ -closure of the conjugacy class of  $T$  in  $\text{Aut}_2(X, \mu)$ , as desired.

(ii) is proved in a similar way. First, we note that the conditions  $T \in \text{Aut}_1(X, \mu)$  and  $\lim_{|n| \rightarrow \infty} \mu(W_n) < \infty$  imply that  $T \in \text{Ker } \chi$ . Indeed,

$$\chi(T) = \int_X (T' - 1) d\mu = \sum_{n \in \mathbb{Z}} \int_{W_n} (T' - 1) d\mu = \sum_{n \in \mathbb{Z}} (\mu(W_{n-1}) - \mu(W_n)) = 0.$$

Secondly, we will use (F2) in place of (F1). Instead of (5-1) we now have that  $\sum_{n \in \mathbb{Z}} |\mu(W_{n+1}) - \mu(W_n)| < \infty$ . Define  $R_n$  in the same way as in the proof of (i). As was noted there,  $R_n \in \text{Aut}_1(X, \mu)$ . Moreover, it is easy to see that  $R_n \in \text{Ker } \chi$ . Slightly modifying the above argument and considering the  $L^1$ -norm instead of the square of the  $L^2$ -norm, we obtain that  $\|(S_n R_n T^{-1} R_n^{-1})' - 1\|_1 \rightarrow 0$ . The latter yields that  $S R_n T^{-1} R_n^{-1} \rightarrow \text{Id}$  in  $d_1$  as  $n \rightarrow \infty$ .  $\square$

We recall two more facts from [DaKoRo]. Let  $\mathcal{E}$  denote the subset of all ergodic transformations of type  $III_1$  in  $\text{Aut}(X, \mu)$ .

(F3)  $\mathcal{E} \cap \text{Aut}_2(X, \mu)$  is a dense  $G_\delta$  in  $(\text{Aut}_2(X, \mu), d_2)$  [DaKoRo, Theorem 5.5].

(F4)  $\mathcal{E} \cap \text{Aut}_1(X, \mu) = \mathcal{E} \cap \text{Ker } \chi$  is a dense  $G_\delta$  in  $(\text{Ker } \chi, d_1)$  [DaKoRo, Theorem 5.8(3)].

We now state and prove the main result of the section.

**Theorem 5.3.**

- (i) The subset  $\mathcal{E}_2^* := \{T \in \mathcal{E} \cap \text{Aut}_2(X, \mu) \mid T_* \text{ is ergodic and of type III}_1\}$  is a dense  $G_\delta$  in  $(\text{Aut}_2(X, \mu), d_2)$ .
- (ii) The subset  $\mathcal{E}_1^* := \{T \in \mathcal{E} \cap \text{Ker } \chi \mid T_* \text{ is ergodic and of type III}_1\}$  is a dense  $G_\delta$  in  $(\text{Ker } \chi, d_1)$ .

*Proof.* (i) Since the set of ergodic transformations of type III<sub>1</sub> is  $G_\delta$  in  $\text{Aut}(X^*, \mu^*)$  endowed with the weak topology [ChHaPr] and the map

$$\text{Aut}_2(X, \mu) \ni T \mapsto T_* \in \text{Aut}(X^*, \mu^*)$$

is continuous in view of [DaKoRo, Theorem 4.15 and Definition 4.5], it follows that the subset

$$\mathcal{E}_2 := \{T \in \text{Aut}_2(X, \mu) \mid T_* \text{ is ergodic of type III}_1\}$$

is a  $G_\delta$  in  $(\text{Aut}_2(X, \mu), d_2)$ . By Theorem 4.2 and Example 3.8,  $\mathcal{E}_2 \neq \emptyset$ . Moreover, there exists a transformation  $T \in \mathcal{E}_2$  satisfying the conditions of Proposition 5.2. Hence, by Proposition 5.2, the conjugacy class of  $T$  is dense in  $(\text{Aut}_2(X, \mu), d_2)$ . Since  $\mathcal{E}_2$  is conjugacy invariant, it follows that  $\mathcal{E}_2$  is a dense  $G_\delta$ . Of course,  $\mathcal{E}_2^* = \mathcal{E}_2 \cap \mathcal{E} \cap \text{Aut}_2(X, \mu)$ . We finally deduce (i) from these facts and (F3).

(ii) is proved in a similar way with usage of (F4) instead of (F3).  $\square$

## 6. POISSON SUSPENSIONS OVER DISSIPATIVE BASES

Let  $(X, \mathfrak{B}, \mu)$  be a  $\sigma$ -finite infinite standard nonatomic measure space and let  $T \in \text{Aut}_2(X, \mu)$ . Let  $T$  be *totally dissipative*. This means that there is a subset  $B \in \mathfrak{B}$  such that  $X = \bigsqcup_{n \in \mathbb{Z}} T^n B$ . Suppose that  $\mu(B) < \infty$ . Since  $T \in \text{Aut}_2(X, \mu)$ , it follows from [DaKoRo, Lemma 3.2] that  $\mu(T^n B) < \infty$  for each  $n \in \mathbb{Z}$ . Then without loss of generality we can assume that  $X = [0, 1] \times \mathbb{Z}$  and  $T(y, n) = (y, n+1)$  for all  $y \in [0, 1]$  and  $n \in \mathbb{Z}$ . Let  $\lambda$  denote the Lebesgue measure on  $[0, 1]$ . We can also assume that there exist a sequence  $(a_n)_{n \in \mathbb{Z}}$  of functions  $a_n \in L^1([0, 1], \lambda)$  such that  $a_n > 0$  a.e. and for each  $F \in L^1([0, 1] \times \mathbb{Z}, \mu)$ ,

$$(6-1) \quad \int_X F d\mu = \sum_{n \in \mathbb{Z}} \int_{[0,1]} F(y, n) a_n(y) d\lambda(y).$$

It is straightforward to verify that  $T \in \text{Aut}_2(X, \mu)$  if and only if

$$\sum_{n \in \mathbb{Z}} \int_{[0,1]} \left( \sqrt{\frac{a_{n+1}(y)}{a_n(y)}} - 1 \right)^2 a_n(y) d\lambda(y) = \sum_{n \in \mathbb{Z}} \|\sqrt{a_{n+1}} - \sqrt{a_n}\|_2^2 < \infty.$$

In a similar way,  $T \in \text{Aut}_1(X, \mu)$  if and only if  $\sum_{n \in \mathbb{Z}} \|a_{n+1} - a_n\|_1 < \infty$ . The latter inequality implies that there exist two nonnegative functions  $a, b \in L^1([0, 1], \lambda)$  such that

$$(6-2) \quad \lim_{n \rightarrow \infty} \|a - a_n\|_1 = 0 \quad \text{and} \quad \lim_{n \rightarrow -\infty} \|b - a_n\|_1 = 0.$$

Since  $\chi(T) = \lim_{m \rightarrow \infty} (\mu(T([0, 1] \times \{-m, \dots, m\})) - \mu([0, 1] \times \{-m, \dots, m\}))$ , it follows that

$$(6-3) \quad \chi(T) = \lim_{m \rightarrow \infty} \left( \sum_{j=-m+1}^{m+1} \int a_j d\lambda - \sum_{j=-m}^m \int a_j d\lambda \right) = \|a\|_1 - \|b\|_1.$$

The following theorem is the main result of this section.



**Theorem 6.1.** *If  $T \in \text{Aut}_1(X, \mu)$  and  $\chi(T) \neq 0$  then  $T_*$  is totally dissipative.*

*Proof.* Consider the Hopf decomposition of  $X$ : there are two invariant subsets  $X_c$  and  $X_d$  of  $X$  such that the dynamical system  $(X_c, \mu \upharpoonright X_c, T \upharpoonright X_c)$  is conservative and the dynamical system  $(X_d, \mu \upharpoonright X_d, T \upharpoonright X_d)$  is totally dissipative [Aa, §1.1]. Since  $\chi(T) \neq 0$ , it follows that  $T$  is dissipative [DaKoRo, Proposition 4.13] and hence  $\mu(X_d) > 0$ . Of course,  $T \upharpoonright X_d \in \text{Aut}_1(X_d, \mu_d)$ ,  $T \upharpoonright X_c \in \text{Aut}_1(X_c, \mu_c)$  and

$$\chi(T) = \chi(T \upharpoonright X_d) + \chi(T \upharpoonright X_c) = \chi(T \upharpoonright X_d).$$

Hence  $\chi(T \upharpoonright X_d) \neq 0$ . Moreover,  $T_*$  is isomorphic to the Cartesian product of  $(T \upharpoonright X_d)_*$  and  $(T \upharpoonright X_c)_*$ . Therefore if  $(T \upharpoonright X_d)_*$  is totally dissipative then so is  $T_*$ . In view of this, we may assume without loss of generality that  $T$  is totally dissipative itself. Hence there is a subset  $B \subset X$  such that  $X = \bigsqcup_{n \in \mathbb{Z}} T^n B$ . We may also assume that  $\mu(B) < \infty$ . Indeed, if  $\mu(B) = \infty$  then we partition  $B$  into countably many subsets  $B_m$ ,  $m \in \mathbb{N}$  of finite measure. For each  $m > 0$ , let  $B'_m := \bigsqcup_{n \in \mathbb{Z}} T^n B_m$ . Then  $B'_m$  is invariant under  $T$  and  $\bigsqcup_{m=1}^{\infty} B'_m = X$ . Of course,  $T \upharpoonright B'_m \in \text{Aut}_1(B'_m, \mu \upharpoonright B'_m)$  for each  $m > 0$  and  $\chi(T) = \sum_{m=1}^{\infty} \chi(T \upharpoonright B'_m)$ . Therefore there exists  $m_0 > 0$  such that  $\chi(T \upharpoonright B'_{m_0}) \neq 0$ . Since  $T_*$  is isomorphic to the Cartesian product of  $(T \upharpoonright B'_{m_0})_*$  and  $(T \upharpoonright (X \setminus B'_{m_0}))_*$ , it follows that if  $(T \upharpoonright B'_{m_0})_*$  is totally dissipative then so is  $T_*$ . Thus, it suffices to consider only the case where  $\mu(B) < \infty$ .

In this case we can consider  $X$  as  $[0, 1] \times \mathbb{Z}$  and  $T$  as the unit rotation along the second coordinate. We will also use the notation  $(a_n)_{n \in \mathbb{Z}}$  and  $\lambda$  introduced in the beginning of this section to describe  $\mu$  via (6-1). Then  $T_*$  is the Bernoulli shift on the infinite product space  $X^* := \bigotimes_{n \in \mathbb{Z}} ([0, 1]^*, \kappa_n^*)$ , where  $\kappa_n$  is a measure equivalent to  $\lambda$  on  $[0, 1]$  with  $\frac{d\kappa_n}{d\lambda} = a_n$ . We are going to apply Lemma 3.9 to prove that  $T_*$  is totally dissipative. We will use the following equality:

$$(6-4) \quad \left\| \sqrt{\frac{d\mu \circ T^n}{d\mu}} - 1 \right\|_2^2 = \sum_{k \in \mathbb{Z}} \int_{[0,1]} \left( \sqrt{\frac{a_{k+n}}{a_k}} - 1 \right)^2 a_k d\lambda \\ = \sum_{k \in \mathbb{Z}} \left\| \sqrt{a_{k+n}} - \sqrt{a_k} \right\|_2^2.$$

Since for two nonnegative functions  $e, f \in L^1([0, 1], \lambda)$ ,

$$\|e - f\|_1 = \langle \sqrt{e} - \sqrt{f}, \sqrt{e} + \sqrt{f} \rangle \leq \|\sqrt{e} - \sqrt{f}\|_2 \cdot \|\sqrt{e} + \sqrt{f}\|_2,$$

and  $\|\sqrt{e} + \sqrt{f}\|_2 \leq \|\sqrt{e}\|_2 + \|\sqrt{f}\|_2 = \sqrt{\|e\|_1} + \sqrt{\|f\|_1}$ , we obtain that

$$(6-5) \quad \left\| \sqrt{a_{k+n}} - \sqrt{a_k} \right\|_2 \geq \frac{1}{2D} \|a_{k+n} - a_k\|_1$$

for all  $k$  and  $n$ , where  $D := \sup_{k \in \mathbb{Z}} \|a_k\|_1 < \infty$ . It follows from (6-2) and (6-3) that there is  $N > 0$  such that

$$(6-6) \quad \|a_i - a_j\|_1 > \frac{1}{3} |\chi(T)| \quad \text{whenever } i > N \text{ and } j < -N.$$

Taking  $n > 3N$  and utilizing (6-4), (6-5), (6-6) we obtain that

$$\begin{aligned} \left\| \sqrt{\frac{d\mu \circ T^n}{d\mu}} - 1 \right\|_2^2 &\geq \sum_{k \in \mathbb{Z}} \frac{\|a_{k+n} - a_k\|_1^2}{4D^2} \\ &\geq \sum_{-\frac{n}{3} > k > -\frac{2n}{3}} \frac{\|a_{k+n} - a_k\|_1^2}{4D^2} \\ &\geq n \frac{|\chi(T)|^2}{108D^2}. \end{aligned}$$

Therefore

$$\sum_{n>0} e^{-\frac{1}{2} \left\| \sqrt{\frac{d\mu \circ T^n}{d\mu}} - 1 \right\|_2^2} < \sum_{n>0} e^{-\frac{|\chi(T)|^2}{216D^2} n} < \infty.$$

Hence  $T_*$  is totally dissipative by Lemma 3.9.  $\square$

*Remark 6.2.* This theorem can be extended to the case where  $T \in \text{Aut}_2(X, \mu)$  in the following way. Suppose that  $T$  is dissipative. Select a subset  $B \subset X$  of finite positive measure such that  $T^n B \cap T^m B = \emptyset$  if  $n, m \in \mathbb{Z}$  and  $n \neq m$ . Then we represent the restriction of  $T$  to  $\bigsqcup_{n \in \mathbb{Z}} T^n B$  in the same way as in the beginning of this section such that (6-1) holds. If there exist  $\delta > 0$  and  $N > 0$  such that for all  $n, m > N$ , we have that  $\|\sqrt{a_n} - \sqrt{a_{-m}}\|_2 > \delta$  then  $T_*$  is totally dissipative. This fact is proved in the very same way as Theorem 6.1. We leave details to the reader.

## SECTION 7. PHASE TRANSITION FOR CONSERVATIVENESS WHEN SCALING THE INTENSITY OF POISSON SUSPENSIONS

Let  $(X, \mathfrak{B}, \mu)$  be a  $\sigma$ -finite standard measure space with  $\mu$  nonatomic and infinite. For  $t > 0$ , let  $\mu_t$  denote the measure on  $(X, \mathfrak{B})$  given by  $\mu_t(B) := t\mu(B)$ .

**Definition 7.1.** Let  $T \in \text{Aut}_2(X, \mu)$ . We say that  $T_*$  is *conservatively concrete for intensity scaling (CCIS)* if for each  $t > 0$ , the Poisson suspension  $(X^*, \mathfrak{B}^*, \mu_t^*, T_*)$  is either conservative or totally dissipative.

If  $T \in \text{Aut}_1(X, \mu)$  and  $\chi(T) > 0$  then  $T$  is CCIS. On the other hand, if  $T \in \text{Aut}_2(X, \mu)$  and there is a  $T$ -invariant subset  $A \subset X$  of finite positive measure such that  $T \upharpoonright A$  is totally dissipative and  $T \upharpoonright (X \setminus A)$  preserves  $\mu \upharpoonright (X \setminus A)$  then  $T_*$  is not CCIS. Indeed,  $(X^*, \mu^*, T_*)$  is isomorphic to the Cartesian product

$$(A^*, (\mu \upharpoonright A)^*, (T \upharpoonright A)_*) \times ((X \setminus A)^*, (\mu \upharpoonright (X \setminus A))^*, (T \upharpoonright (X \setminus A))_*).$$

We now set  $[A]_j := \{\omega \in X^* \mid \omega(A) = j\}$  for  $j = 0, 1$ . Then  $[A]_j$  is invariant under  $T_*$  and  $\mu^*([A]_j) > 0$  for  $j = 0, 1$ . Since  $[A]_0$  is a singleton (modulo  $(\mu \upharpoonright A)^*$ ), it follows that  $T_*$  is conservative when restricted to this subset. One can check that  $T_* \upharpoonright [A]_1$  is totally dissipative. Since  $(T \upharpoonright (X \setminus A))_*$  is conservative (because it is probability preserving), it follows that  $(T \upharpoonright A)_* \times (T \upharpoonright (X \setminus A))_*$  is conservative on  $[A]_0 \times (X \setminus A)^*$  and dissipative on  $[A]_1 \times (X \setminus A)^*$ . Hence  $T_*$  is not CCIS.

*Problem.* What are necessary and sufficient conditions on  $(X, \mu, T)$  under which  $T_*$  is CCIS?

In this paper we prove CCIS for a certain family of Poisson suspensions. In the proof of the following theorem we use an idea similar to what was used in [Ko2, Theorem 3] (see also [Da, Theorem 3.5]).

**Theorem 7.2.** *Let  $T \in \text{Aut}_1(X, \mu)$ . Suppose also that there is  $\alpha > 1$  such that  $\alpha^{-1} < (T^n)'(x) < \alpha$  for each  $n > 0$  at a.e.  $x \in X$ . Then  $T_*$  is either conservative or totally dissipative. Hence  $T_*$  is CCIS.*

*Proof.* In view of the aforementioned remark, it suffices to consider only the case where  $\chi(T) = 0$ .

Let  $\mathcal{S}$  be the subgroup of all  $\mu$ -preserving transformations on  $(X, \mu)$  such that for each  $S \in \mathcal{S}$ , there is a subset  $A_S \subset X$  of finite measure with  $Sx = x$  whenever  $x \notin A_S$ . Let  $D := \{\omega \in X^* \mid \sum_{n=1}^{\infty} (T^n)'(\omega) < \infty\}$ .

We first show that  $D$  is invariant under  $S_*$  for each  $S \in \mathcal{S}$ . Indeed, for each  $n > 0$ ,  $\omega \in D$  and  $S \in \mathcal{S}$ , we have (utilizing [DaKoRo, Corollary 4.1(3)]) that

$$\begin{aligned} (T_*^n)'(S_*\omega) &= \prod_{\omega(\{x\}) > 0} (T^n)'(Sx) \\ &= \prod_{\substack{x \notin A_S \\ \omega(\{x\}) > 0}} (T^n)'(x) \prod_{\substack{x \in A_S \\ \omega(\{x\}) > 0}} (T^n)'(Sx) \\ &= (T_*^n)'(\omega) \frac{\prod_{\omega(\{x\}) > 0}^{x \in A_S} (T^n)'(Sx)}{\prod_{\omega(\{x\}) > 0}^{x \in A_S} (T^n)'(x)} \\ &< (T_*^n)'(\omega) \cdot \alpha^{2\omega(A_S)}. \end{aligned}$$

Hence  $\sum_{n=1}^{\infty} (T_*^n)'(\omega) \leq \alpha^{2\omega(A_S)} \sum_{n=1}^{\infty} (T^n)'(\omega) < +\infty$ , i.e.  $S_*\omega \in D$ , as desired.

We now prove that the group  $\mathcal{S}_* := \{S_* \mid S \in \mathcal{S}\}$  of  $\mu^*$ -preserving transformations of  $X^*$  is ergodic<sup>4</sup>. For that, we select a  $\mu$ -preserving totally dissipative transformation  $Q$  of  $X$  and a sequence  $(S_n)_{n=1}^{\infty}$  of transformations from  $\mathcal{S}$  that converges weakly to  $Q$ . Then the sequence  $((S_n)_*)_{n=1}^{\infty}$  weakly converges to  $Q_*$  in  $\text{Aut}(X^*, \mu^*)$ . If  $A$  is an  $\mathcal{S}_*$ -invariant subset of  $X^*$  then  $A$  is invariant  $Q_*$  because  $\mu^*(Q_*A \Delta A) = \lim_{n \rightarrow \infty} \mu^*((S_n)_*A \Delta A) = 0$ . Since  $Q_*$  is isomorphic to a probability preserving Bernoulli shift, it is ergodic and hence  $\mu^*(A)(1 - \mu^*(A)) = 0$ . Hence  $\mathcal{S}_*$  is ergodic.

Since  $D$  is  $\mathcal{S}_*$ -invariant,  $\mu^*(D)(1 - \mu^*(D)) = 0$  and the assertion of the theorem follows.  $\square$

We also prove a general result related to CCIS.

**Proposition 7.3.** *Let  $T \in \text{Aut}_1(X, \mu)$ .*

- (i) *If  $(X^*, \mu^*, T_*)$  is totally dissipative then  $(X^*, \mu_t^*, T_*)$  is totally dissipative for each  $t > 1$ .*
- (ii) *If  $(X^*, \mu^*, T_*)$  is conservative then  $(X^*, \mu_t^*, T_*)$  is conservative for each  $t \in (0, 1)$ .*

*Proof.* (i) If  $\chi(T) \neq 0$  then the assertion of the proposition is trivial. Hence from now on we assume that  $\chi(T) = 0$ . It follows from the condition of the proposition that the Cartesian product  $(X^* \times X^*, \mu^* \otimes \mu_{t-1}^*, T_* \times T_*)$  is totally dissipative. Therefore, by Lemma 2.5,

$$(7-1) \quad \sum_{n=1}^{+\infty} \frac{d\mu^* \circ T_*^n}{d\mu^*}(\omega) \frac{d\mu_{t-1}^* \circ T_*^n}{d\mu_{t-1}^*}(\tau) < +\infty$$

<sup>4</sup>This follows also from [Sh, Theorem 2.3]. However we present here an alternative very short proof.

at  $(\mu^* \otimes \mu_{t-1}^*)$ -a.e.  $(\omega, \tau) \in X^* \times X^*$ . Since  $T^n \in \text{Aut}_1(X, \mu)$ , it follows from [DaKoRo, Corollary 4.1(3)] that for each  $n > 0$ ,

$$\begin{aligned} \frac{d\mu^* \circ T_*^n}{d\mu^*}(\omega) \frac{d\mu_{t-1}^* \circ T_*^n}{d\mu_{t-1}^*}(\tau) &= e^{\int_X \log \frac{d\mu \circ T^n}{d\mu} d\omega + \int_X \log \frac{d\mu \circ T^n}{d\mu} d\tau} \\ &= e^{\int_X \log \frac{d\mu_t \circ T^n}{d\mu_t} d(\omega + \tau)} \\ &= \frac{d\mu_t^* \circ T_*^n}{d\mu_t^*}(\omega + \tau). \end{aligned}$$

Since  $\mu^* * \mu_{t-1}^* = \mu_t^*$ , we deduce from (7-1) that  $\sum_{n=1}^{+\infty} \frac{d\mu_t^* \circ T_*^n}{d\mu_t^*}(\omega) < +\infty$  at  $\mu_t^*$ -a.e.  $\omega$ . It remains to apply Lemma 2.5.

(ii) Let  $D_t := \{\omega \in X^* \mid \sum_{n=1}^{+\infty} \frac{d\mu_t^* \circ T_*^n}{d\mu_t^*}(\omega) < \infty\}$  stand for the dissipative part of  $(X^*, \mu_t^*, T_*)$ . Assume in the contrapositive that  $\mu_t^*(D_t) > 0$ . Then the set  $D_t \times X^*$  is contained in the dissipative part of the product  $(X^* \times X^*, \mu_t^* \otimes \mu_{1-t}^*, T_* \times T_*)$ . Hence for  $(\mu_t^* \otimes \mu_{1-t}^*)$ -a.e.  $(\omega, \tau) \in D_t \times X^*$ ,

$$\infty > \sum_{n=1}^{+\infty} \frac{d\mu_t^* \circ T_*^n}{d\mu_t^*}(\omega) \frac{d\mu_{1-t}^* \circ T_*^n}{d\mu_{1-t}^*}(\tau) = \sum_{n=1}^{+\infty} \frac{d\mu^* \circ T_*^n}{d\mu^*}(\omega + \tau),$$

i.e.  $\omega + \tau$  is contained in the dissipative part of  $(X^*, \mu^*, T_*)$ . Since  $\mu_t^* * \mu_{1-t}^* = \mu^*$ , the dissipative part of  $(X^*, \mu^*, T_*)$  is of positive measure, a contradiction  $\square$

From Proposition 7.3 and Theorem 7.2 we deduce the main result of this section on a phase transition for conservativeness of Poisson suspensions while scaling the underlying intensity.

**Corollary 7.4.** *Let  $T \in \text{Aut}_1(X, \mu)$ . Suppose that there is  $\alpha > 1$  such that  $\alpha^{-1} < (T^n)'(x) < \alpha$  for each  $n > 0$  at a.e.  $x \in X$ . Then there is  $t_0 \in [0, +\infty]$  such that the Poisson suspension  $(X^*, \mu_t^*, T_*)$  is conservative for each  $t \in (0, t_0)$  and the Poisson suspension  $(X^*, \mu_t^*, T_*)$  is totally dissipative for each  $t \in (t_0, +\infty)$ .*

We call  $t_0$  the *bifurcation point*. Of course, it is interesting when the bifurcation point is *proper*, i.e.  $0 < t_0 < +\infty$ . A concrete example of such a Poisson suspension was constructed in §4 (see Remark 4.3). Combining it with Corollary 7.4 we obtain the following theorem.

**Theorem 7.5.** *Let  $X = \mathbb{R}$ ,  $Tx = x + 1$  for all  $x \in X$  and  $\mu$  be an absolutely continuous measure on  $X$  such that  $d\mu(x) = f(x)dx$  with*

$$f(x) = \begin{cases} 1, & \text{if } x < 2 \\ e^{-n^{-1/2}}, & \text{if } n \leq x < n + 1 \text{ for an integer } n > 1. \end{cases}$$

*Then there exists  $t_0 \in [\frac{1}{6}, 4]$  such that  $(X^*, \mu_t^*, T_*)$  is weakly mixing of stable type  $III_1$  for each  $t \in (0, t_0)$  and  $(X^*, \mu_t^*, T_*)$  is totally dissipative for each  $t \in (t_0, +\infty)$ .*

We note that Corollary 7.4 and Theorem 7.5 are the “nonsingular Poisson” analogues of a phase transition phenomenon discovered recently for the nonsingular Gaussian actions introduced by Y. Arano, Y. Isono and A. Marrakchi in [ArIsMa] (see Proposition 5.1 there).

We conclude this section with computing the conservativeness index of the Poisson suspension  $(X^*, \mu_t^*, T_*)$  in the case where the bifurcation point is proper.

**Proposition 7.6.** *Let the condition of Corollary 7.4 hold. Suppose that the bifurcation point  $t_0$  is proper. Then for each  $n \in \mathbb{N}$ , if  $\frac{t_0}{n+1} < t < \frac{t_0}{n}$  then  $(X^*, \mu_t^*, T_*)$  has conservative index  $n$ .*

*Proof.* We have to prove that the  $n$ -th Cartesian power of  $(X^*, \mu_t^*, T_*)$  is conservative and the  $(n+1)$ -th Cartesian power of  $(X^*, \mu_t^*, T_*)$  is dissipative. We first note that the  $n$ -th convolution of  $\mu_t$  is  $\mu_{nt}^*$ . Arguing as in the proof of Proposition 7.3(i), we obtain that for each  $m > 0$ ,

$$\frac{d\mu_t^* \circ T_*^m}{d\mu_t^*}(\omega_1) \cdots \frac{d\mu_t^* \circ T_*^m}{d\mu_t^*}(\omega_n) = \frac{d\mu_{nt}^* \circ T_*^m}{d\mu_{nt}^*}(\omega_1 + \cdots + \omega_n)$$

for  $(\mu_t^* \otimes \cdots \otimes \mu_t^*)$ -a.e.  $(\omega_1, \dots, \omega_n) \in X^* \times \cdots \times X^*$ . Since  $nt < t_0$ , the system  $(X^*, \mu_{nt}^*, T_*)$  is conservative and hence the above equality and Lemma 2.5 yield that the  $n$ -th Cartesian power of  $(X^*, \mu_t^*, T_*)$  is conservative. On the other hand, the  $(n+1)$ -th convolution power of  $\mu_t^*$  equals  $\mu_{(n+1)t}^*$  with  $(n+1)t > t_0$ . Now, for each  $m > 0$ , we have that

$$\frac{d\mu_t^* \circ T_*^m}{d\mu_t^*}(\omega_1) \cdots \frac{d\mu_t^* \circ T_*^m}{d\mu_t^*}(\omega_{n+1}) = \frac{d\mu_{(n+1)t}^* \circ T_*^m}{d\mu_{(n+1)t}^*}(\omega_1 + \cdots + \omega_{n+1})$$

for  $(\mu_t^* \otimes \cdots \otimes \mu_t^*)$ -a.e.  $(\omega_1, \dots, \omega_{n+1}) \in X^* \times \cdots \times X^*$ . Since  $(X^*, \mu_{(n+1)t}^*, T_*)$  is dissipative, the above equality and Lemma 2.5 yield that the direct product of the  $(n+1)$ -th Cartesian power of  $(X^*, \mu_t^*, T_*)$  is dissipative, as desired.  $\square$

#### APPENDIX A. SKELLAM DISTRIBUTIONS

The Skellam distribution with parameters  $(a, b)$  is the distribution  $\chi_{a,b}$  of the difference  $X - Y$  of two independent random variables  $X$  and  $Y$ , each Poisson-distributed with respective parameter (expected value)  $a \geq 0$  and  $b \geq 0$ . It is known that  $E(X - Y) = a - b$  and  $\sigma^2(X - Y) = a + b$ . The characteristic function  $\phi_{X-Y}(t) := E(e^{it(X-Y)})$  of  $X - Y$  is  $e^{-(a+b) + ae^{it} + be^{-it}}$  at each  $t \in \mathbb{R}$ . Of course,  $\chi_{a,b}(\mathbb{Z}) = 1$ . For each  $k \in \mathbb{Z}$ ,

$$\chi_{a,b}(k) = e^{a-b} \left(\frac{a}{b}\right)^{k/2} I_k(2\sqrt{ab}),$$

where  $I_k$  is the modified Bessel function of the first kind, i.e.

$$I_k(z) = I_{|k|}(z) := \left(\frac{z}{2}\right)^{|k|} \sum_{j=0}^{\infty} \frac{(z^2/4)^j}{j!(j+|k|)!}.$$

From the two above formulas we deduce that for each  $L > 0$ ,

$$\begin{aligned} \sum_{|k| \geq L} \chi_{a,b}(k) &\leq e^{a-b} \sum_{|k| \geq L} \left(\frac{a}{b}\right)^{k/2} (ab)^{|k|/2} \frac{1}{|k|!} \sum_{j=0}^{\infty} \frac{(ab)^j}{j!} \\ &= e^{a-b+ab} \left( \sum_{k \geq L} \frac{a^k}{k!} + \sum_{k \leq -L} \frac{1}{b^k (-k)!} \right) \\ &= e^{a-b+ab} \left( \sum_{k \geq L} \frac{a^k}{k!} + \sum_{k \geq L} \frac{b^k}{k!} \right) \\ &= e^{a-b+ab} \frac{a^L e^a + b^L e^b}{L!}. \end{aligned}$$

This yields the following estimation: for each  $A > 0$ , there is an integer  $L > 0$  such that for each  $l > L$

$$(A-1) \quad \sup_{0 < a, b < A} \sum_{|k| \geq l} \chi_{a,b}(k) \leq l^{-8}.$$

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