



HAL
open science

Nonsingular Poisson Suspensions

Alexandre Danilenko, Zemer Kosloff, Emmanuel Roy

► **To cite this version:**

Alexandre Danilenko, Zemer Kosloff, Emmanuel Roy. Nonsingular Poisson Suspensions. Journal d'analyse mathématique, 2022, 146 (2), pp.741-790. 10.1007/s11854-022-0213-8 . hal-03861103

HAL Id: hal-03861103

<https://hal.science/hal-03861103>

Submitted on 19 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NONSINGULAR POISSON SUSPENSIONS

ALEXANDRE I. DANILENKO, ZEMER KOSLOFF, AND EMMANUEL ROY

ABSTRACT. The classical Poisson functor associates to every infinite measure preserving dynamical system (X, μ, T) a probability preserving dynamical system (X^*, μ^*, T_*) called the Poisson suspension of T . In this paper we generalize this construction: a subgroup $\text{Aut}_2(X, \mu)$ of μ -nonsingular transformations T of X is specified as the largest subgroup for which T_* is μ^* -nonsingular. Topological structure of this subgroup is studied. We show that a generic element in $\text{Aut}_2(X, \mu)$ is ergodic and of Krieger type III₁. Let G be a locally compact Polish group and let $A : G \rightarrow \text{Aut}_2(X, \mu)$ be a G -action. We investigate dynamical properties of the Poisson suspension A_* of A in terms of an affine representation of G associated naturally with A . It is shown that G has property (T) if and only if each nonsingular Poisson G -action admits an absolutely continuous invariant probability. If G does not have property (T) then for each generating probability κ on G and $t > 0$, a nonsingular Poisson G -action is constructed whose Furstenberg κ -entropy is t .

1. INTRODUCTION

1.1. Poisson suspensions: measure preserving and nonsingular. In this paper we initiate a systematic study of *nonsingular* Poisson suspensions in the framework of ergodic theory. Poisson point processes have convenient mathematical properties and are often used as mathematical models for seemingly random phenomena. Say, in statistical physics they provide a model for ideal gas consisting of countably many randomly moving noninteracting points (particles) of a standard σ -finite infinite nonatomic measure space (X, \mathcal{A}, μ) . The space of states X^* of the gas consists of countable subsets (configurations) of X . It is endowed with the natural Borel σ -algebra \mathcal{A}^* and a probability measure μ^* such that for each subset $A \in \mathcal{A}$, the μ^* -expected value of the number of particles in A has the Poisson distribution with parameter $\mu(A)$. Given a transformation $T : X \rightarrow X$, the Poisson suspension $T_* : X^* \rightarrow X^*$ of T models the motion of the particles by T . In other words, $T_*\omega = \{Tx : x \in \omega\}$ for all $\omega \in X^*$. If $\mu^* \circ T_* \sim \mu^*$ then we call the system $(X^*, \mathcal{A}^*, \mu^*, T_*)$ the *nonsingular Poisson suspension* of (X, \mathcal{A}, μ, T) .

In ergodic theory, only the *measure preserving* case $\mu \circ T = \mu$ (and hence $\mu^* \circ T_* = \mu^*$) of Poisson suspensions has been considered so far. We refer the reader to the classical sources [11] studying the dynamical properties

The research of Z.K. was partially supported by ISF grant No. 1570/17.

of the ideal gas and to [28] providing a Poisson model for Bernoulli actions of locally compact groups¹. Over the last 15 years we observe a boost of interest to Poisson suspensions in ergodic theory. The work [29] describes dynamical properties of T_* such as ergodicity, weak mixing, mixing, rigidity, K -property, etc. in terms of T and μ . Spectral properties, entropic properties, similarity and asymmetry, properties of joinings of the measure preserving Poisson suspensions are studied extensively in [29], [12, §8], [25], [15], [19], [24], etc. Summarizing this progress we can say that the *Poisson functor* $T \rightarrow T_*$ from the category of infinite measure preserving actions to the probability preserving actions is similar (and of similar importance in ergodic theory) to the *Gaussian functor* from the category of orthogonal representations to the probability preserving actions.

Our global task is to find some extensions of the aforementioned results to the *nonsingular* Poisson suspensions as well as to investigate purely nonsingular properties of them such as dissipativeness, Krieger's type, associated flow, Furstenberg entropy. We note that nonsingular Poisson suspensions are widely used in the representation theory to construct unitary representations of *large groups* such as diffeomorphism groups of non-compact manifolds and current groups (see the surveys [32], [33] and references therein and Chapter 10 of the book [27]). The fundamental (in fact, the only dynamical) property of the nonsingular Poisson suspensions of these group actions exploited there is the ergodicity, because it implies irreducibility of the associated unitary representation. In contrast to that, in this work we investigate dynamical properties of nonsingular Poisson suspensions for individual transformations or, more generally, *locally compact* group actions. For them, the aforementioned implication does not hold.

1.2. Main results. We now list the main results of this paper. The following fact can be deduced easily from Takahashi's version of Kakutani dichotomy for Poisson point processes:

The set of μ -nonsingular transformations T of (X, \mathcal{A}, μ) such that T_* is μ^* -nonsingular is exactly the group

$$\text{Aut}_2(X, \mathcal{A}, \mu) := \left\{ T : \mu \circ T^{-1} \sim \mu, \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - 1 \in L^2(\mu) \right\}.$$

We note that $\text{Aut}_2(X, \mathcal{A}, \mu)$ contains (properly) a subgroup

$$\text{Aut}_1(X, \mathcal{A}, \mu) := \left\{ T : \mu \circ T^{-1} \sim \mu, \frac{d\mu \circ T^{-1}}{d\mu} - 1 \in L^1(\mu) \right\},$$

¹The seemingly simpler Bernoulli model as the shiftwise G -action on the product space A^G , for a probability space A , drops out from the category of standard measure spaces if the group G is not countable.

which is the largest subgroup of nonsingular transformations for which the Poisson suspensions were defined in the literature ([32], [33], [27]²) so far. Let $\mathcal{U}_{\mathbb{R}}(L^2(\mu))$ denote the group of unitary operators in $L^2(X, \mu)$ preserving the real functions and let $\text{Aff}_{\mathbb{R}}(L^2(\mu)) := L^2_{\mathbb{R}}(X, \mu) \rtimes \mathcal{U}_{\mathbb{R}}(L^2(\mu))$ denote the group of affine operators in $L^2(X, \mu)$ preserving the real functions. We consider two natural representations of $\text{Aut}_2(X, \mathcal{A}, \mu)$ in $L^2(X, \mu)$: the *unitary* (well known) Koopman representation $U : \text{Aut}_2(X, \mathcal{A}, \mu) \ni T \mapsto U_T \in \mathcal{U}_{\mathbb{R}}(L^2(\mu))$ and the *affine* representation $A^{(2)} : \text{Aut}_2(X, \mathcal{A}, \mu) \ni T \mapsto A_T^{(2)} \in \text{Aff}_{\mathbb{R}}(L^2(\mu))$, given by the formulas

$$(1.1) \quad U_T f := f \circ T^{-1} \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} \quad \text{and} \quad A_T^{(2)} f := U_T f + \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - 1.$$

It appears, surprisingly for us, that the nonsingular Poisson suspensions are related closely to geometrical (affine) properties of the underlying Hilbert space.

Theorem A. *Let $C_{-1} := \{f \in L^2(\mu) : f \geq -1\}$. Then $\{A \in \text{Aff}_{\mathbb{R}}(L^2(\mu)) : AC_{-1} = C_{-1}\} = \left\{ \left(\sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - 1, U_T \right) : T \in \text{Aut}_2(X, \mathcal{A}, \mu) \right\}$.*

A similar results holds also for $\text{Aut}_1(X, \mathcal{A}, \mu)$ and the corresponding natural isometric and affine representations of this group in $L^1(\mu)$.

Let $W : \text{Aff}_{\mathbb{R}}(L^2(\mu)) \ni A \mapsto W_A \in \mathcal{U}(F(L^2(\mu)))$ stand for the well known unitary representation of $\text{Aff}_{\mathbb{R}}(L^2(\mu))$ by the Weyl operators W_A in the Fock space $F(L^2(\mu))$ constructed over $L^2(\mu)$ [22]. The operators W_A are of fundamental importance in representation theory and quantum probability. They also appear naturally in description of the unitary Koopman operators associated with nonsingular Poisson suspensions.

Theorem B. *Under the natural identification of $L^2(\mu^*)$ with $F(L^2(\mu))$, for each transformation $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$, the Koopman operator U_{T^*} generated by T^* equals $W_{A_T^{(2)}}$.*

Theorem A enables us to define a Polish topology, denoted by d_2 , on $\text{Aut}_2(X, \mathcal{A}, \mu)$. In a similar way, utilizing the aforementioned analogue of Theorem A for $\text{Aut}_1(X, \mathcal{A}, \mu)$ we also introduce a Polish topology d_1 on $\text{Aut}_1(X, \mathcal{A}, \mu)$. We show that the homomorphism $\chi : \text{Aut}_1(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$, defined in [27] by the formula $\chi(T) := \int_X \left(\frac{d\mu \circ T^{-1}}{d\mu} - 1 \right) d\mu$, is d_1 -continuous. We then prove the following results.

Theorem C.

- *The weak topology is strictly weaker than d_2 and d_2 is strictly weaker than d_1 .*
- *$\text{Aut}_p(X, \mu)$ endowed with d_p is a Polish group for $p = 1, 2$.*
- *The set $\{T^* : T \in \text{Aut}_2(X, \mathcal{A}, \mu)\}$ of nonsingular Poisson suspensions is a weakly closed subgroup of nonsingular transformations of $(X^*, \mathcal{A}^*, \mu^*)$.*

²Neretin's notation for $\text{Aut}_1(X, \mathcal{A}, \mu)$ is Gms_{∞} .

- $\text{Aut}_1(X, \mathcal{A}, \mu)$ is isomorphic to a semidirect product $\text{Ker } \chi \rtimes \mathbb{R}$ in such a way that χ corresponds to the projection onto the second coordinate.
- Every conservative transformation in $\text{Aut}_1(X, \mathcal{A}, \mu)$ belongs to $\text{Ker } \chi$.

The latter result (as well as the following theorem) shows that the group $\text{Ker } \chi$ is a more natural object than $\text{Aut}_1(X, \mathcal{A}, \mu)$ from the ergodic theory point of view.

Theorem D.

- $\text{Aut}_2(X, \mathcal{A}, \mu)$ endowed with d_2 has the Rokhlin property³. The subset of ergodic transformations of Krieger's type III₁ is a dense G_δ in $\text{Aut}_2(X, \mu)$.
- $\text{Ker } \chi$ endowed with d_1 has the Rokhlin property ($\text{Aut}_1(X, \mathcal{A}, \mu)$ does not have it). The subset of ergodic transformations of Krieger's type III₁ is a dense G_δ in $\text{Ker } \chi$.

It is well known that the group of all μ -nonsingular transformations of (X, \mathcal{A}, μ) endowed with the weak topology also possesses similar properties (see [23], [9], [10], [16]). However the proof of Theorem D is more difficult for several reasons. The first is that d_2 and d_1 are stronger than the weak topology. The second is that we have no freedom to replace μ by an arbitrary equivalent measure any more. Indeed, if $\nu \sim \mu$ but $\sqrt{\frac{d\nu}{d\mu}} - 1 \notin L^2(\mu)$ then $\text{Aut}_2(X, \mathcal{A}, \nu) \neq \text{Aut}_2(X, \mathcal{A}, \mu)$.

In a subsequent work [14] we show that the subset of $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$ such that T_* is ergodic and type III₁ is a dense G_δ . Combined with Theorem D this implies the existence of a type III₁ ergodic transformation whose Poisson suspension is also ergodic and of type III₁.

Let G be a locally compact second countable group. The affine isometric representations of G is a fundamental tool in geometric group theory connecting Kazhdan property (T), the Haagerup property, operator algebras, harmonic analysis, etc. (see [7], [17], [18], [5]). The Poisson G -actions deliver natural non-trivial examples of such representations. Indeed, if $T : G \ni g \mapsto T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ is a measurable G -action then the mapping $G \ni g \mapsto A_{T_g}^{(2)}$ (see (1.1)) is a continuous affine representation of G in $L^2(\mu)$.

Theorem E. *Let $T : G \ni g \mapsto T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ be a measurable G -action.*

- The Poisson suspension $T_* := \{(T_g)_*\}_{g \in G}$ of T has an absolutely continuous invariant probability measure if and only if the $L^2(\mu)$ -cocycle $c_T : G \ni g \mapsto c_T(g) := \sqrt{\frac{d\mu \circ T_g^{-1}}{d\mu}} - 1$ is bounded.
- If $\int_G e^{-\frac{1}{2}\|c_T(g)\|_2^2} d\lambda(g) < \infty$ then T_* is totally dissipative.
- T_* is of zero type if and only if $\|c_T(g)\|_2 \rightarrow \infty$ as $g \rightarrow \infty$.

³I.e. there is a dense conjugacy class in this group.

The aforementioned *zero type* is a nonsingular analogue of the mixing in the probability preserving case (see [16]). We could not find a criterion for ergodicity of T_* because it depends in a subtle way not only on the ergodic properties of T but also on a “right” choice of a measure inside the equivalence class of μ : we construct an example of an ergodic T admitting a μ -equivalent invariant probability such that T_* is totally dissipative.

Let κ be a probability on G . A nonsingular G -action $S = (S_g)_{g \in G}$ on a probability space (Y, \mathcal{B}, ν) is called κ -stationary if $\int_G \kappa(g) \nu \circ S_g d\kappa(g) = \nu$ (see a survey [20] for properties and applications of the stationary actions).

Theorem F. *Let T be as in the previous theorem. If T_* is κ -stationary for a generating probability κ on G then T_* preserves μ^* and T preserves μ .*

There are several equivalent characterizations of property (T) for G [6]. We provide one more in terms of the nonsingular Poisson suspensions.

Theorem G. *G has property (T) if and only if each nonsingular Poisson G -action T_* admits an absolutely continuous invariant probability.*

One more characterization was obtained in [8]: if a countable discrete group G does not have property (T) then for each generating probability κ on G , the Furstenberg κ -entropy $h_\kappa(\cdot)$ has no gap on the set of purely infinite ergodic nonsingular G -actions. This was refined in [13]: $h_\kappa(\cdot)$ takes all possible positive values on the subset of ergodic G -actions of type III_1 . We extend this result to arbitrary locally compact groups and partly refine it by considering only the nonsingular Poisson suspensions.

Theorem H. *Let a locally compact G do not have property (T) and let κ be a probability on G . Then there is a nonsingular G -action T on an infinite measure space (X, μ) such that the Poisson suspension T_* of T is μ^* -nonsingular and $\{h_\kappa(T_*, \mu_t^*) : t \in (0, +\infty)\} = (0, +\infty)$, where $\mu_t = t\mu$.*

During the course of work on this paper we learnt about [2], where it was constructed a functor from the affine G -representations to the nonsingular Gaussian systems. Certain nonsingular Gaussian counterparts of Theorems G and H are proved there. We believe that there should be some interplay between the theory of nonsingular Gaussian actions and the nonsingular Poisson suspensions.

1.3. Sections overview. In Section 2 we introduce a model $(X^*, \mathcal{A}^*, \mu^*, T_*)$ for the Poisson suspension of a dynamical system (X, \mathcal{A}, μ, T) , consider $L^2(\mu^*)$ as a Fock space over $L^2(\mu)$ and extend the exponential map $\mathcal{E} : L^2(\mu) \rightarrow L^2(\mu^*)$ to some non-square integrable functions. In Section 3 we extend and refine Takahashi’s theorem on equivalence and orthogonality of the Poisson suspensions of equivalent measures [31]. In Section 4 we introduce and study the topological groups $\text{Aut}_2(X, \mathcal{A}, \mu)$, $\text{Aut}_1(X, \mathcal{A}, \mu)$, the unitary and affine representations U and $A^{(2)}$ of $\text{Aut}_2(X, \mathcal{A}, \mu)$ and their analogues for $\text{Aut}_1(X, \mathcal{A}, \mu)$, and prove Theorems A, B, C and related results. Section 5

is devoted to generic properties of $\text{Aut}_2(X, \mathcal{A}, \mu)$ and $\text{Aut}_1(X, \mathcal{A}, \mu)$. Theorem D is proved there. In Section 6, for arbitrary locally compact groups G , we characterize some basic dynamical properties of the nonsingular Poisson suspensions T_* of G -actions T in terms of the underlying system (X, μ, T) . Theorem E is proved there. In Section 7 we consider stationary G -actions, prove Theorem F and compute the Furstenberg κ -entropy of T_* in terms of the underlying system (X, μ, T) . Section 8 is devoted to property (T). We prove Theorems G and H and related results there. The paper ends with Appendix which is devoted to infinitely divisible variables and stochastic integration. This material is used in the course of the proofs of Theorems 3.4 and 3.6.

2. POISSON SUSPENSIONS

2.1. Space of point processes. Let (X, \mathcal{A}, μ) be a σ -finite Lebesgue space with a non-atomic measure, that is (X, \mathcal{A}, μ) is mod 0 isomorphic to the real line if $\mu(X) = \infty$ or to a bounded closed interval if $\mu(X) < \infty$, endowed with Lebesgue measure and Lebesgue measurable sets.

The space of point processes over X is defined as the set X^* of all measures ω of the form $\omega = \sum_{i \in I} \delta_{x_i}$ where I is at most countable. X^* is endowed with the smallest σ -algebra such that the $\mathbb{Z}_+ \cup \{+\infty\}$ -valued maps $N_A : \omega \mapsto \omega(A)$ are measurable, for all $A \in \mathcal{A}$. We denote this σ -algebra on X^* by \mathcal{A}^* .

2.2. Poisson measures. Let \mathcal{A}_f^μ denote the collection of sets $A \in \mathcal{A}$ of finite μ -measure. There exists a unique probability measure μ^* on (X^*, \mathcal{A}^*) such that:

- For all $k \geq 1$ and pairwise disjoint sets A_1, \dots, A_k in \mathcal{A}_f^μ , the random variables N_{A_i} , $1 \leq i \leq k$, are independent.
- For any $A \in \mathcal{A}_f^\mu$, N_A is Poisson distributed with parameter $\mu(A)$.

The probability space $(X^*, \mathcal{A}^*, \mu^*)$ is called the *Poisson space* over the base (X, \mathcal{A}, μ) . When completed with respect to μ^* , $(X^*, \mathcal{A}^*, \mu^*)$ is a Lebesgue space. The random measure $A \mapsto N_A$, $A \in \mathcal{A}$ distributed as μ^* is called a *Poisson point process of intensity μ* . In most cases this object is presented on \mathbb{R}^d (or on a subset of it) with Lebesgue measure as intensity and then called *homogeneous Poisson point process*. As Lebesgue spaces with a continuous measure are either isomorphic to \mathbb{R} or to a bounded closed interval with Lebesgue measure depending on whether the measure is finite or not, there is essentially no loss in generality in dealing with homogeneous Poisson point process. Observe the following three important features of a Poisson measure:

- μ^* is supported on *simple counting measures*, that is:

$$\mu^*(\{\omega \in X^* : \forall x \in X, \omega(\{x\}) = 0 \text{ or } 1\}) = 1.$$

- The *intensity* of the random measure $A \mapsto N_A$ is μ , that is $\mathbb{E}_{\mu^*}[N_A] = \mu(A)$.

We shall also make use of the following important theorem:

Theorem 2.1. (*Rényi*) *Let m be a probability measure on (X^*, \mathcal{A}^*) supported on simple counting measures. If for all $A \in \mathcal{A}_f^\mu$,*

$$m(\{\omega : N_A(\omega) = 0\}) = \mu^*(\{\omega : N_A(\omega) = 0\}),$$

then $m = \mu^$.*

2.3. Poisson suspensions. At the core of this paper is an easy yet fundamental observation: If φ is a measurable map between two Lebesgue spaces (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) such that $\mu \circ \varphi^{-1} = \nu$ then φ_* also acts measurably between (X^*, \mathcal{A}^*) and (Y^*, \mathcal{B}^*) by $\varphi_*\omega = \omega \circ \varphi^{-1}$ and satisfies

$$\mu^* \circ \varphi_*^{-1} = \nu^*.$$

In particular, if T is an invertible transformation of (X, \mathcal{A}, μ) , we have the following picture:

$$\begin{aligned} (X, \mathcal{A}, \mu) &\xrightarrow{T} (X, \mathcal{A}, \mu \circ T^{-1}) \\ (X^*, \mathcal{A}^*, \mu^*) &\xrightarrow{T_*} (X^*, \mathcal{A}^*, \mu^* \circ T_*^{-1}). \end{aligned}$$

We will be interested in the situation where T is a non-singular automorphism, that is $\mu \sim \mu \circ T^{-1}$. It is not always true that $\mu^* \sim (\mu \circ T^{-1})^*$. We will recall necessary and sufficient conditions to get the equivalence of measures. When it is the case, the non-singular dynamical system $(X^*, \mathcal{A}^*, \mu^*, T_*)$ will be called the *Poisson suspension* over (X, \mathcal{A}, μ, T) .

2.4. Fock space structure of $L^2(\mu^*)$ and coherent vectors. We recall a very important structural feature of Poisson measures (see [3] or [27]): there is a canonical isometry between $L^2(\mu^*)$ and the *symmetric Fock space* $F(L^2(\mu))$ over $L^2(\mu)$. We recall that

$$F(L^2(\mu)) := \bigoplus_{n=0}^{\infty} L^2(\mu)^{\odot n},$$

where $L^2(\mu)^{\odot 0} := \mathbb{C}$ and each factor $L^2(\mu)^{\odot n}$ is equipped with the normalized scalar product $n! \langle \cdot, \cdot \rangle_{L^2(\mu)^{\odot n}}$. The Hilbert space $L^2(\mu)^{\odot n}$ considered as a subspace of $L^2(\mu^*)$ is called the *chaos of order n* . We now explain how to construct the canonical isometry. For that, we choose a distinguished family of vectors in $F(L^2(\mu))$, called the *coherent vectors*, defined, for $f \in L^2(\mu)$, by

$$\mathcal{E}(f) := \sum_{k=0}^{\infty} \frac{1}{k!} f^{\otimes k} \in F(L^2(\mu)).$$

They form a total family in $F(L^2(\mu))$ and satisfy the exponential relation

$$(2.1) \quad \langle \mathcal{E}(f), \mathcal{E}(g) \rangle_{F(L^2(\mu))} = e^{\langle f, g \rangle_{L^2(\mu)}}.$$

Denote by $\mathcal{B}_0(X)$ the subspace of $L^2(\mu)$ -functions with finite μ -measure support. Then the family $\{\mathcal{E}(f) : f \in \mathcal{B}_0(X)\}$ is also total in $F(L^2(\mu))$ (see e.g. [27], where it is shown that even a subspace of finitely valued functions from $\mathcal{B}_0(X)$ generates a total family in $F(L^2(\mu))$). On the other hand, for $f \in \mathcal{B}_0(X)$, define a bounded function $\exp(f)$ on X^* by setting:

$$(2.2) \quad \exp(f)(\omega) = e^{-\int_X f d\mu} \prod_{\{x \in X: \omega(\{x\})=1\}} (1 + f(x)), \quad \omega \in X^*.$$

In particular, for any set $A \in \mathcal{A}_f^\mu$,

$$(2.3) \quad \exp(-1_A) = e^{\mu(A)} 1_{\{\omega: N_A(\omega)=0\}}.$$

A standard calculation shows that

$$(2.4) \quad \langle \exp(f), \exp(g) \rangle_{L^2(\mu^*)} = e^{\langle f, g \rangle_{L^2(\mu)}}.$$

Due to the Rényi Theorem, the family $\{\exp(f) : f \in \mathcal{B}_0(X)\}$ is total in $L^2(\mu^*)$. Hence we deduce from (2.1) and (2.4) the map $\mathcal{E}(f) \mapsto \exp(f)$ extends to an isometry between $F(L^2(\mu))$ and $L^2(\mu^*)$. In the sequel, we will not distinguish between $\mathcal{E}(f)$ and $\overline{\exp(f)}$. We will use the following properties of coherent vectors: $\mathcal{E}(\overline{f}) = \overline{\mathcal{E}(f)}$, $\mathcal{E}(f) \in L^1(\mu^*)$ and $\mathbb{E}_{\mu^*}[\mathcal{E}(f)] = 1$ for all $f \in L^2(\mu)$.

2.5. Product formula and extended coherent vectors. For every two functions f and g in $L^2(\mu)$, we define a function $f \bullet g$ by setting

$$f \bullet g := (1 + f)(1 + g) - 1.$$

We now define a function space

$$\mathcal{L}(\mu) := \{\varphi : X \rightarrow \mathbb{R} : \exists f, g \in L^2(\mu), \varphi = f \bullet g\}.$$

Clearly, $L^2(\mu) \subset \mathcal{L}(\mu)$. If $f, g \in \mathcal{B}_0(X)$, $f \bullet g$ is integrable and has finite measure support. Moreover, one can deduce from (2.2) the following *product formula*:

$$(2.5) \quad \mathcal{E}(f)(\omega)\mathcal{E}(g)(\omega) = e^{\int_X fg d\mu} e^{-\int_X f \bullet g d\mu} \prod_{\{x \in X: \omega(\{x\})=1\}} (f \bullet g(x) + 1).$$

This formula enables us to extend the definition of coherent vectors to functions in $\mathcal{L}(\mu)$. Namely, we set

$$(2.6) \quad \mathcal{E}(f \bullet g) := e^{-\int_X fg d\mu} \mathcal{E}(f)\mathcal{E}(g)$$

for all $f, g \in L^2(\mu)$. We have to verify that this formula is well defined. To that end, we first define an auxiliary map $\Psi : L^2(\mu) \times L^2(\mu) \rightarrow L^1(\mu^*)$ by setting

$$(2.7) \quad \Psi(f, g) := e^{-\int fg d\mu} \mathcal{E}(f)\mathcal{E}(g).$$

Since the map $L^2(\mu) \ni f \mapsto \mathcal{E}(f) \in L^2(\mu^*)$ is continuous, it follows that Ψ is continuous. Now we consider $f, g, f', g' \in L^2(\mu)$ such that $f \bullet g = f' \bullet g'$. Select an increasing sequence $A_1 \subset A_2 \subset \dots$ of subsets of finite measure

in X such that $\bigcup_{n=1}^{\infty} A_n = X$. In we now set $f_n := f1_{A_n}$, $g_n := g1_{A_n}$, $f'_n := f'1_{A_n}$ and $g'_n := g'1_{A_n}$ then $f_n, g_n, f'_n, g'_n \in \mathcal{B}_0(X)$ for each $n \in \mathbb{N}$ and $f_n \rightarrow f$, $g_n \rightarrow g$, $f'_n \rightarrow f'$, $g'_n \rightarrow g'$ in $L^2(\mu)$ as $n \rightarrow \infty$. Moreover, for all $n \in \mathbb{N}$,

$$f_n \bullet g_n = (f + g + fg)1_{A_n} = (f' + g' + f'g')1_{A_n} = f'_n \bullet g'_n.$$

It now follows from (2.7) and (2.5) that $\Psi(f_n, g_n) = \Psi(f'_n, g'_n)$. Taking limits as $n \rightarrow \infty$ and using the continuity of Ψ , we obtain that

$$\Psi(f, g) = \Psi(f', g'),$$

as desired. Thus, utilizing (2.6), we can define $\mathcal{E}(\phi)$ for each $\phi \in \mathcal{L}(\mu)$. We call such $\mathcal{E}(\phi)$ *the extended coherent vectors*. If $\phi \in L^2(X)$ then $\phi = \phi \bullet 0$ and (2.6) implies that the extended coherent vector $\mathcal{E}(\phi)$ coincides with the ‘‘standard’’ coherent vector defined by ϕ . We also note that $\mathbb{E}_{\mu^*}(\mathcal{E}(\varphi)) = 1$ for each $\varphi \in \mathcal{L}(\mu)$.

2.6. More properties of coherent vectors.

Proposition 2.2. *An extended coherent vector $\mathcal{E}(\varphi)$, $\varphi \in \mathcal{L}(\mu)$, is in $L^2(\mu^*)$ if and only if $\varphi \in L^2(\mu)$.*

Proof. If $\varphi \in L^2(\mu)$ then the extended coherent vector $\mathcal{E}(\varphi)$ is the classical coherent vector and hence it belongs to $L^2(\mu^*)$.

Now we prove the converse. Let $f, g \in L^2(\mu)$ and $\mathcal{E}(f \bullet g) \in L^2(\mu^*)$. Denote by \mathcal{S} the subspace of finitely valued functions from $L^2(\mu)$. Then for each $h \in \mathcal{S}$, the function $g \bullet h$ is in $L^2(\mu)$. We now have:

$$\begin{aligned} \mathbb{E}_{\mu^*}[\mathcal{E}(f \bullet g)\mathcal{E}(h)] &= \mathbb{E}_{\mu^*}[e^{-\int_X fg d\mu} \mathcal{E}(f)\mathcal{E}(g)\mathcal{E}(h)] \\ &= e^{-\int_X fg d\mu + \int_X gh d\mu} \mathbb{E}_{\mu^*}[\mathcal{E}(f)\mathcal{E}(g \bullet h)] \\ &= e^{-\int_X fg d\mu + \int_X gh d\mu + \int_X f \cdot (g \bullet h) d\mu} \\ &= e^{\int_X (f \bullet g) \cdot h d\mu} \end{aligned}$$

Hence a linear functional $L : \mathcal{S} \ni h \mapsto \int_X (f \bullet g) \cdot h d\mu \in \mathbb{C}$ is well defined. It is continuous at 0. Indeed, if a sequence $(h_n)_{n=1}^{\infty}$ with $h_n \in \mathcal{S}$, $n \in \mathbb{N}$, goes to 0 in $L^2(\mu)$ as $n \rightarrow \infty$ then $\mathcal{E}(h_n) \rightarrow \mathcal{E}(0) = 1$ as $n \rightarrow \infty$ in $L^2(\mu^*)$. Hence

$$\mathbb{E}_{\mu^*}[\mathcal{E}(f \bullet g)\mathcal{E}(h_n)] \rightarrow \mathbb{E}_{\mu^*}[\mathcal{E}(f \bullet g)] = 1 \quad \text{as } n \rightarrow \infty.$$

Therefore $e^{\int_X (f \bullet g) \cdot h_n d\mu} \rightarrow 1$, i.e. $\int_X (f \bullet g) \cdot h_n d\mu \rightarrow 0$ as $n \rightarrow \infty$. Since L is linear, it follows that L is continuous on the entire \mathcal{S} . Since \mathcal{S} is dense in $L^2(\mu)$, we deduce that L extends uniquely to a continuous linear functional on $L^2(\mu)$. In view of the Riesz representation theorem, we conclude that $f \bullet g \in L^2(\mu)$. \square

Lemma 2.3. *Let a function $\varphi \in \mathcal{L}(\mu)$ take only real values. Then the function $\tilde{\varphi} := |1 + \varphi| - 1$ belong to $\mathcal{L}(\mu)$ and*

$$(2.8) \quad |\mathcal{E}(\varphi)| = e^{-2 \int_{\{x \in X: \varphi(x)+1 < 0\}} (\varphi+1) d\mu} \mathcal{E}(\tilde{\varphi}).$$

In particular, $\mathcal{E}(\varphi)$ is non-negative μ^* -a.s. if and only if $\varphi \geq -1$ μ -almost everywhere.

Proof. We consider separately three cases. Suppose first that $\varphi \in \mathcal{B}_0(X)$. Then $|1 + \varphi| - 1 \in \mathcal{B}_0(X)$ and, in view of (2.2),

$$\begin{aligned}
 (2.9) \quad |\mathcal{E}(\varphi)(\omega)| &= e^{-\int_X \varphi d\mu} \prod_{\{x \in X: \omega(\{x\})=1\}} |1 + \varphi(x)| \\
 &= e^{-\int_X \varphi d\mu} \prod_{\{x \in X: \omega(\{x\})=1\}} (1 + \tilde{\varphi}(x)) \\
 &= e^{-\int_X (\varphi - \tilde{\varphi}) d\mu} e^{-\int_X \tilde{\varphi} d\mu} \prod_{\{x \in X: \omega(\{x\})=1\}} (1 + \tilde{\varphi}(x)) \\
 &= e^{-\int_X (\varphi - \tilde{\varphi}) d\mu} \mathcal{E}(\tilde{\varphi})(\omega) \\
 &= e^{-2 \int_{\{x \in X: \varphi(x)+1 < 0\}} (\varphi+1) d\mu} \mathcal{E}(\tilde{\varphi})(\omega),
 \end{aligned}$$

as desired.

Suppose now that $\varphi \in L^2(\mu)$. We let $A_\varphi := \{x \in X : \varphi(x) < -1\}$. Then $\mu(A_\varphi) < \infty$. Select a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ of functions $\varphi_n \in \mathcal{B}_0(X)$ such that

- $\varphi_n \rightarrow \varphi$ in $L^2(\mu)$ as $n \rightarrow \infty$,
- $A_{\varphi_n} = A_\varphi$ for each $n \in \mathbb{N}$ and
- $\varphi_n(x) = \varphi(x)$ if $x \in A_\varphi$.

It is straightforward to verify that $|\tilde{\varphi}| \leq |\varphi|$ and $|\tilde{\varphi}_n - \tilde{\varphi}| \leq |\varphi_n - \varphi|$. This yields that $\tilde{\varphi} \in L^2(\mu)$ and $\tilde{\varphi}_n \rightarrow \tilde{\varphi}$ in $L^2(\mu)$ as $n \rightarrow \infty$. Therefore $\mathcal{E}(\tilde{\varphi}_n) \rightarrow \mathcal{E}(\tilde{\varphi})$ in $L^2(\mu^*)$ as $n \rightarrow \infty$. By the first case and the properties of φ_n ,

$$|\mathcal{E}(\varphi_n)| = e^{-2 \int_{A_{\varphi_n}} (\varphi_n+1) d\mu} \mathcal{E}(\tilde{\varphi}_n) = e^{-2 \int_{A_\varphi} (\varphi+1) d\mu} \mathcal{E}(\tilde{\varphi}_n).$$

Passing to the limit as $n \rightarrow \infty$, we obtain (2.8), as desired. Before we proceed to the general case, we rewrite (2.8) in the following equivalent form:

$$(2.10) \quad |\mathcal{E}(\varphi)| = e^{-\int_X (\varphi - \tilde{\varphi}) d\mu} \mathcal{E}(\tilde{\varphi}).$$

Now, in the general case, let $\varphi = f \bullet g$ for arbitrary vectors $f, g \in L^2(\mu)$. A straightforward verification shows that $\tilde{\varphi} = \tilde{f} \bullet \tilde{g}$. We deduce from (2.6) and (2.10) that

$$\begin{aligned}
 |\mathcal{E}(\varphi)| &= e^{-\int_X fg d\mu} |\mathcal{E}(f)| |\mathcal{E}(g)| \\
 &= e^{-\int_X fg d\mu} e^{\int_X (\tilde{f}-f) d\mu} \mathcal{E}(\tilde{f}) e^{\int_X (\tilde{g}-g) d\mu} \mathcal{E}(\tilde{g}) \\
 &= e^{-\int_X (fg - \tilde{f} + f - \tilde{g} + g - \tilde{f}\tilde{g}) d\mu} \mathcal{E}(\tilde{f} \bullet \tilde{g}) \\
 &= e^{-\int_X (\varphi - \tilde{\varphi}) d\mu} \mathcal{E}(\tilde{\varphi}),
 \end{aligned}$$

and (2.8) follows.

To prove the second claim of the lemma, we assume first that $\mathcal{E}(\varphi) \geq 0$ μ^* -a.s. for some $\varphi \in \mathcal{L}(\mu)$. Then

$$\begin{aligned} 1 &= \mathbb{E}[\mathcal{E}(\varphi)] \\ &= \mathbb{E}[|\mathcal{E}(\varphi)|] \\ &= \mathbb{E}\left[e^{-2\int_{\{x \in X: \varphi(x)+1 < 0\}}(\varphi+1)d\mu} \mathcal{E}(\tilde{\varphi})\right] \\ &= e^{-2\int_{\{x \in X: \varphi(x)+1 < 0\}}(\varphi+1)d\mu}. \end{aligned}$$

Therefore $\int_{\{x \in X: \varphi(x)+1 < 0\}}(\varphi+1)d\mu = 0$, which implies that $\varphi \geq -1$, μ -almost everywhere. Conversely, if $\varphi \geq -1$ μ -almost everywhere then $\int_{\{x \in X: \varphi(x)+1 < 0\}}(\varphi+1)d\mu = 0$ and $\tilde{\varphi} = \varphi$. Therefore $|\mathcal{E}(\varphi)| = \mathcal{E}(\tilde{\varphi}) = \mathcal{E}(\varphi)$. \square

3. ABSOLUTE CONTINUITY AND EQUIVALENCE OF POISSON MEASURES

Let (X, \mathcal{A}, μ) be a non-atomic standard σ -finite measure space. We single out four important sets of measures:

- $\mathcal{M}_{\mu,2}^+$ is the set of σ -finite measures ν on (X, \mathcal{A}) such that $\nu \ll \mu$ and $\sqrt{\frac{d\nu}{d\mu}} - 1 \in L^2(\mu)$,
- $\mathcal{M}_{\mu,2}^{\circ,+} := \{\nu \in \mathcal{M}_{\mu,2}^+ : \nu \sim \mu\}$,
- $\mathcal{M}_{\mu,1}^+$ is the set of σ -finite measures ν on (X, \mathcal{A}) such that $\nu \ll \mu$ and $\frac{d\nu}{d\mu} - 1 \in L^1(\mu)$ and
- $\mathcal{M}_{\mu,1}^{\circ,+} := \{\nu \in \mathcal{M}_{\mu,1}^+ : \nu \sim \mu\}$.

Since $(\sqrt{x} - 1)^2 \leq |x - 1|$ for each $x > 0$, it follows that $\mathcal{M}_{\mu,1}^+ \subset \mathcal{M}_{\mu,2}^+$ and hence $\mathcal{M}_{\mu,1}^{\circ,+} \subset \mathcal{M}_{\mu,2}^{\circ,+}$.

- Remark 3.1.* (1) Let μ be a finite measure. Then $\nu \in \mathcal{M}_{\mu,2}^+$ if and only if it is a finite measure and $\nu \ll \mu$. Hence $\mathcal{M}_{\mu,1}^+ = \mathcal{M}_{\mu,2}^+$ and $\mathcal{M}_{\mu,1}^{\circ,+} = \mathcal{M}_{\mu,2}^{\circ,+}$.
- (2) If $\nu \in \mathcal{M}_{\mu,2}^{\circ,+}$, then $\mathcal{M}_{\nu,2}^{\circ,+} = \mathcal{M}_{\mu,2}^{\circ,+}$. In a similar way, if $\nu \in \mathcal{M}_{\mu,1}^{\circ,+}$, then $\mathcal{M}_{\nu,1}^{\circ,+} = \mathcal{M}_{\mu,1}^{\circ,+}$.
- (3) If μ is infinite and $\nu \in \mathcal{M}_{\mu,2}^+$ then $\mu\left(\left\{x \in X : \frac{d\nu}{d\mu}(x) = 0\right\}\right) < \infty$.

Lemma 3.2. *Let $\nu \in \mathcal{M}_{\mu,2}^+$. Then for any set $A \in \mathcal{A}$, we have that $\mu(A) < \infty$ if and only if $\nu(A) < \infty$, that is $\mathcal{A}_f^\mu = \mathcal{A}_f^\nu$.*

Proof. We set $\phi := \frac{d\nu}{d\mu}$. If $\mu(A) < \infty$, then $\sqrt{\phi} - 1 \in L^2(\mu|_A) \subset L^1(\mu|_A)$. Since $L^1(\mu) \ni (\sqrt{\phi} - 1)^2 = (\phi - 1) - 2(\sqrt{\phi} - 1)$, we now obtain that $\phi - 1 \in L^1(\mu|_A)$, which implies that $\nu(A) < \infty$.

Now if $\nu(A) < \infty$ then $\sqrt{\phi}$ is in $L^2(\mu|_A)$. As $\sqrt{\phi} - 1 \in L^2(\mu|_A)$, this implies that the constant function 1 is in $L^2(\mu|_A)$ too. Hence $\mu(A) < \infty$. \square

The following theorem is the ground for the rest of the paper. The necessary and sufficient condition for the absolute continuity for Poisson measures was found by Takahashi in [31]. However he did not write the explicit formula for the Radon-Nikodym derivative as a coherent vector. We only prove this formula and show that it generalizes the formula obtained by Neretin in [27] for the smaller class of measures $\mathcal{M}_{\mu,1}^+$.

Theorem 3.3. *Let ν be a σ -finite measure on (X, \mathcal{A}) . Then $\nu^* \ll \mu^*$ if and only if $\nu \in \mathcal{M}_{\mu,2}^+$. If $\nu \in \mathcal{M}_{\mu,2}^+$ then $\frac{d\nu^*}{d\mu^*} = \mathcal{E}(\frac{d\nu}{d\mu} - 1)$. If $\nu \notin \mathcal{M}_{\mu,2}^+$, then $\nu^* \perp \mu^*$.*

Proof. Assume $\nu \in \mathcal{M}_{\mu,2}^+$ and set $\phi := \frac{d\nu}{d\mu}$. We now observe that

$$(3.1) \quad \phi - 1 = (\sqrt{\phi} - 1) \bullet (\sqrt{\phi} - 1)$$

with $\sqrt{\phi} - 1 \in L^2(\mu)$. Hence $\mathcal{E}(\phi - 1)$ is well defined. Take a subset $A \in \mathcal{A}_f^\nu$. Then $A \in \mathcal{A}_f^\mu$ by Lemma 3.2. Applying the product formula (2.6) three times, we obtain that

$$\begin{aligned} \mathcal{E}(-1_A)\mathcal{E}(\phi - 1) &= \mathcal{E}(-1_A)e^{-\int_X(\sqrt{\phi}-1)^2 d\mu}\mathcal{E}(\sqrt{\phi} - 1)\mathcal{E}(\sqrt{\phi} - 1) \\ &= e^{-\int_X((\sqrt{\phi}-1)^2+(\sqrt{\phi}-1)1_A) d\mu}\mathcal{E}((-1_A) \bullet (\sqrt{\phi} - 1))\mathcal{E}(\sqrt{\phi} - 1) \\ &= e^{-\int_X(\phi-1)1_A d\mu}\mathcal{E}((-1_A) \bullet (\phi - 1)) \\ &= e^{\mu(A)-\nu(A)}\mathcal{E}(1_{A^c}\phi - 1). \end{aligned}$$

Taking the mathematical expectation and using (2.3) twice, we obtain that

$$\mathbb{E}_{\mu^*}[1_{\{\omega \in X^*: N_A(\omega)=0\}}\mathcal{E}(\phi - 1)] = e^{-\nu(A)} = \nu^*(\{\omega \in X^* : N_A(\omega) = 0\}).$$

Hence, by the Rényi's theorem, $\nu^* \ll \mu^*$ and $\frac{d\nu^*}{d\mu^*} = \mathcal{E}(\frac{d\nu}{d\mu} - 1)$. The second claim of the theorem was proved in [31]. \square

We note that Theorem 3.3 highlights the connection between extended coherent vectors and Radon-Nikodym derivatives of equivalent Poisson point process measures. The following result can be seen as an explicit description of $\frac{d\nu^*}{d\mu^*}$ for $\nu \in \mathcal{M}_{\mu,2}^{\circ,+}$ as a function from X^* to \mathbb{R} .

Theorem 3.4. *Let $\nu \in \mathcal{M}_{\mu,2}^{\circ,+}$ and set $\phi := \frac{d\nu}{d\mu}$. Then*

(1) *We can represent $\log \frac{d\nu^*}{d\mu^*}$ as the following limit in probability:*

$$(3.2) \quad \log \frac{d\nu^*}{d\mu^*}(\omega) = \lim_{\epsilon \rightarrow 0} \left(\int_{\{x \in X: |\log \phi(x)| > \epsilon\}} \log \phi \, d\omega - \int_{\{x \in X: |\log \phi(x)| > \epsilon\}} (\phi - 1) \, d\mu \right).$$

(2) *Moreover, $\log \frac{d\nu^*}{d\mu^*}$ is an infinitely divisible random variable whose Lévy measure is the image of μ by $\log \phi$, restricted to $\mathbb{R} \setminus \{0\}$.*

(3) We have that

$$\mathbb{E}_{\mu^*} \left[\log \frac{d\nu^*}{d\mu^*} \right] = - \int_X (\phi - 1 - \log \phi) d\mu \in [-\infty, 0].$$

It is finite if and only if $\int_{\{x \in X: |\log \phi(x)| > 1\}} |\log \phi| d\mu < \infty$.

Proof. Given $\epsilon > 0$, we let $X_\epsilon := \{x \in X : |\log \phi(x)| > \epsilon\}$. As usual, X_ϵ^c denotes the compliment to X_ϵ . We first prove three auxiliary claims.

Claim A: $\mu(X_\epsilon) < \infty$ for each $\epsilon > 0$.

Indeed,

$$\begin{aligned} \mu(X_\epsilon) &= \mu(\{x \in X : \phi(x) > e^\epsilon\} \cup \{x \in X : \phi(x) < e^{-\epsilon}\}) \\ &\leq \mu(\{x \in X : |\sqrt{\phi(x)} - 1| > \alpha\}) \\ &\leq \frac{1}{\alpha^2} \int_X (\sqrt{\phi} - 1)^2 d\mu < +\infty, \end{aligned}$$

where $\alpha = \min(e^\epsilon - 1, 1 - e^{-\epsilon})$.

Claim B: $(\log \phi)^2 \wedge 1 \leq \kappa (\sqrt{\phi} - 1)^2$ for some constant $\kappa > 0$.

This claim follows from the standard inequality $\log t \leq t - 1$ for $t > 0$.

Claim C: $\phi - 1 - \log \phi \cdot 1_{X_\epsilon^c} \in L^1(\mu)$.

To prove this claim we first write the function $\phi - 1 - \log \phi \cdot 1_{X_\epsilon^c}$ as the following sum:

$$(\sqrt{\phi} - 1)^2 + 2(\sqrt{\phi} - 1 - \log \sqrt{\phi})1_{X_\epsilon^c} + 2(\sqrt{\phi} - 1)1_{X_1}.$$

The first term in this sum is in $L^1(\mu)$ because $\sqrt{\phi} - 1 \in L^2(\mu)$. Since

$$0 \leq \sqrt{\phi} - 1 - \log \sqrt{\phi} \leq (\sqrt{\phi} - 1)^2,$$

it follows that the second term is in $L^1(\mu)$ too. The Cauchy-Schwarz inequality and Claim A yield that the third term is also integrable. Claim C follows.

It follows from Claim B that the stochastic integral $I_\mu(\log \phi) : X^* \rightarrow \mathbb{R}$ is well defined (see Appendix). Claim C implies that the real

$$\beta := - \int_X (\phi - 1 - \log \phi \cdot 1_{X_\epsilon^c}) d\mu$$

is well defined. It follows that

$$(3.3) \quad I_\mu(\log \phi)(\omega) + \beta = \lim_{\epsilon \rightarrow 0} \left(\int_{X_\epsilon} \log \phi d\omega - \int_{X_\epsilon} (\phi - 1) d\mu \right),$$

where the limit means the convergence in probability (see Appendix). Our purpose now is to identify the lefthand side of this formula as a Radon-Nikodym derivative. It is straightforward to verify that for each subset $B \in \mathcal{A}$, we have that $(\phi - 1)1_B = ((\sqrt{\phi} - 1)1_B) \bullet ((\sqrt{\phi} - 1)1_B)$. Since

$\mathcal{E}((\sqrt{\phi} - 1)1_{X_\epsilon}) \rightarrow \mathcal{E}(\sqrt{\phi} - 1)$ in $L^2(\mu^*)$, we can apply (2.6) to obtain that

$$\begin{aligned} \mathcal{E}((\phi - 1)1_{X_\epsilon}) &= e^{-\int_{X_\epsilon} (\sqrt{\phi}-1)^2 d\mu} \mathcal{E}((\sqrt{\phi} - 1)1_{X_\epsilon})^2 \\ (3.4) \quad &\rightarrow e^{-\int_X (\sqrt{\phi}-1)^2 d\mu} \mathcal{E}((\sqrt{\phi} - 1))^2 \\ &= \mathcal{E}(\phi - 1) \end{aligned}$$

in $L^1(\mu^*)$ as $\epsilon \rightarrow 0$. Since $(\sqrt{\phi} - 1)1_{X_\epsilon} \in \mathcal{B}_0(X)$, it follows from (2.2) that for a.e. $\omega \in X^*$,

$$\begin{aligned} \mathcal{E}((\phi - 1)1_{X_\epsilon})(\omega) &= e^{-\int_{X_\epsilon} (\sqrt{\phi}-1)^2 d\mu - 2 \int_{X_\epsilon} (\sqrt{\phi}-1) d\mu} \prod_{\{x \in X_\epsilon: \omega(\{x\})=1\}} \phi(x) \\ &= e^{\int_{X_\epsilon} \log \phi d\omega - \int_{X_\epsilon} (\phi-1) d\mu}. \end{aligned}$$

From this and (3.4) we deduce that

$$\lim_{n \rightarrow \infty} \left(\int_{X_\epsilon} \log \phi d\omega - \int_{X_\epsilon} (\phi - 1) d\mu \right) = \log \mathcal{E}(\phi - 1)$$

where the limit is in μ^* -probability. This formula, (3.3) and Theorem 3.3 yield that

$$(3.5) \quad I_\mu(\log \phi) + \beta = \log \frac{d\nu^*}{d\mu^*}.$$

Thus, (1) is proved. Moreover, $I_\mu(f)$ is infinitely divisible (see Appendix) so (3.5) implies (2).

Since $\phi - 1 - \log \phi \geq 0$, the integral $\int_X (\phi - 1 - \log \phi) d\mu$ is always well defined. Combining this observation with Claim C, we obtain that

$$(3.6) \quad \int_X (\phi - 1 - \log \phi) d\mu < \infty \iff \int_{X_1} |\log \phi| d\mu < \infty.$$

By Proposition 9.2, the latter inequality is equivalent to the fact that $I_\mu(\log \phi) \in L^1(\mu^*)$. The latter, in turn, is equivalent to $\log \frac{d\nu^*}{d\mu^*} \in L^1(\mu^*)$ in view of (3.5).

Firstly we consider the case where $\log \frac{d\nu^*}{d\mu^*} \in L^1(\mu^*)$. Then it follows from (3.5), Proposition 9.2 and the definition of β that

$$\begin{aligned} \mathbb{E}_{\mu^*} \left[\log \frac{d\nu^*}{d\mu^*} \right] &= \mathbb{E}_{\mu^*}(I_\mu(\log \phi)) + \beta \\ &= \int_{X_1} \log \phi d\mu + \beta \\ &= \int_X (\log \phi - \phi - 1) d\mu \end{aligned}$$

Consider now the second case, where $\int_{X_1} |\log \phi| d\mu = +\infty$. Then from (3.6) we deduce that $\int_X (\phi - 1 - \log \phi) d\mu = +\infty$. On the other hand, by the Jensen inequality, $\mathbb{E}_{\mu^*}[-\log \frac{d\nu^*}{d\mu^*}] \geq 0$. Hence the fact $\log \frac{d\nu^*}{d\mu^*} \notin L^1(\mu^*)$ implies $\mathbb{E}_{\mu^*}[-\log \frac{d\nu^*}{d\mu^*}] = +\infty$. The proof of (3) is now complete. \square

Remark 3.5. With additional efforts, using the martingale convergence theorem, it is possible to prove the almost sure convergence in (3.2) instead of the convergence in probability.

The special case where $\phi - 1 \in L^1(\mu)$ has been considered in [31]. In this case we have the following results.

Theorem 3.6. *Let $\nu \in \mathcal{M}_{\mu,1}^{\circ,+}$ and set $\phi := \frac{d\nu}{d\mu}$. Then*

- (1) $\log \phi \in L^1(\omega)$ for μ^* -almost every $\omega \in X^*$ and

$$\frac{d\nu^*}{d\mu^*}(\omega) = e^{-\int_X (\phi-1)d\mu} \prod_{\{x \in X: \omega(\{x\})=1\}} \phi(x),$$

where the infinite product converges absolutely.

- (2) The integral $\int_X \log \phi d\mu$ is well defined and takes values in the extended interval $[-\infty, \int_X (\phi - 1)d\mu]$. Moreover,

$$\mathbb{E}_{\mu^*} \left[\log \frac{d\nu^*}{d\mu^*} \right] = - \int_X (\phi - 1) d\mu + \int_X \log \phi d\mu.$$

Proof. In the course of proof we will use the notation X_ϵ and X_1^c and refer to Claims B and C from the proof of Theorem 3.4.

If $\nu \in \mathcal{M}_{\mu,1}^{\circ,+}$ then $\phi - 1 \in L^1(\mu)$ and hence by Claim C, $(\log \phi)1_{X_1^c} \in L^1(\mu)$. By Claim B, $|\log \phi|^2 1_{X_1^c} \in L^1(\mu)$, thus the stochastic integral $I_\mu(|\log \phi|)$ is well defined as

$$(3.7) \quad I_\mu(|\log \phi|) = \lim_{\epsilon \rightarrow 0} \left(\int_{X_\epsilon} |\log \phi| d\omega - \int_{X_\epsilon} |\log \phi| 1_{X_1^c} d\mu \right),$$

where the limit is in probability. In particular, $I_\mu(|\log \phi|)$ is finite μ^* -almost surely. By the monotone convergence theorem and the integrability of $|\log \phi| 1_{X_1^c}$,

$$\lim_{\epsilon \rightarrow 0} \int_{X_\epsilon} |\log \phi| 1_{X_1^c} d\mu = \int_X |\log \phi| 1_{X_1^c} d\mu < \infty.$$

It follows from this and (3.7) that there exists $\lim_{\epsilon \rightarrow 0} \int_{X_\epsilon} |\log \phi| d\omega < \infty$ for a.e. $\omega \in X^*$. By the monotone convergence theorem,

$$\lim_{\epsilon \rightarrow 0} \int_{X_\epsilon} |\log \phi| d\omega = \int_X |\log \phi| d\omega.$$

Thus, $\log \phi \in L^1(\omega)$ for μ^* -a.e. This fact combined with the integrability of $\phi - 1$ imply the almost everywhere convergence in (3.2). Passing to this limit, we obtain now that

$$\log \frac{d\nu^*}{d\mu^*}(\omega) = \int_X \log \phi d\omega - \int_X (\phi - 1) d\mu$$

for a.e. ω . This proves (1). The second claim follows from Theorem 3.4(3). \square

Remark 3.7. • Firstly, we note that the formula for the Radon-Nikodym derivative in Theorem 3.6 is well known and follows immediately from Theorem 3.4(1) and the fact that $\phi - 1 \in L^1(\mu)$. However the absolute convergence of the infinite product (or the fact that $\log \phi \in L^1(\omega)$ for a.e. ω) requires an additional reasoning.

• Secondly, it is worth mentioning that the combination of $\nu \in \mathcal{M}_{\mu,2}^{\circ,+}$ and $\log \phi \in L^1(\mu)$ implies that ν is in $\mathcal{M}_{\mu,1}^{\circ,+}$ and $\log \frac{d\nu^*}{d\mu^*} \in L^1(\mu^*)$. Indeed, from $\nu \in \mathcal{M}_{\mu,2}^{\circ,+}$ we get that $\phi - 1 - (\log \phi)1_{X_1^c} \in L^1(\mu)$ (see Claim C). On the other hand, the fact $\log \phi \in L^1(\mu)$ implies that $(\log \phi)1_{X_1^c} \in L^1(\mu)$. Therefore $\phi - 1 \in L^1(\mu)$, i.e. $\nu \in \mathcal{M}_{\mu,1}^{\circ,+}$. The integrability of $\log \frac{d\nu^*}{d\mu^*}$ follows now from Theorem 3.6(2).

4. POISSON SUSPENSIONS OF NONSINGULAR TRANSFORMATIONS AND RELATED KOOPMAN REPRESENTATIONS

4.1. The unitary Koopman representation of the group of nonsingular transformations. Let (Y, \mathcal{B}, ρ) be a σ -finite Lebesgue space. Denote by $\mathcal{U}(L^2(\rho))$ the group of unitary operators in $L^2(\mu)$ and by $\mathcal{U}_{\mathbb{R}}(L^2(\rho))$ the subgroup of unitaries that preserve invariant the \mathbb{R} -subspace $L^2_{\mathbb{R}}(\rho)$ of real valued functions in $L^2(\mu)$. Let $\text{Aut}(Y, \mathcal{B}, \rho)$ stand for the group of all nonsingular transformations of (Y, \mathcal{B}, ρ) . For each $S \in \text{Aut}(Y, \mathcal{B}, \rho)$, we set $S' := \frac{d\rho \circ S^{-1}}{d\rho}$ and define a unitary operator $U_S \in \mathcal{U}_{\mathbb{R}}(L^2(\rho))$ by setting $U_S f := \sqrt{S'} f \circ S^{-1}$. Then the mapping

$$U : \text{Aut}(Y, \mathcal{B}, \rho) \ni S \mapsto U_S \in \mathcal{U}_{\mathbb{R}}(L^2(\rho))$$

is a unitary one-to-one representation of $\text{Aut}(Y, \mathcal{B}, \rho)$ in $L^2(Y, \rho)$. It is called *the unitary Koopman representation* of $\text{Aut}(Y, \mathcal{B}, \rho)$.

Let $C_0 := \{f \in L^2(\rho) : f \geq 0\}$. Then C_0 is a closed cone in $L^2(\rho)$. It is well known that

$$(4.1) \quad \{V \in \mathcal{U}(L^2(\rho)) : VC_0 = C_0\} = \{U_S : S \in \text{Aut}(Y, \mathcal{B}, \rho)\}.$$

Endow $\mathcal{U}_{\mathbb{R}}(L^2(\rho))$ with the weak (equivalently, strong) operator topology. We recall that *the weak topology* on $\text{Aut}(Y, \mathcal{B}, \rho)$ is the weakest topology in which U is continuous. It follows from (4.1) that $\text{Aut}(Y, \mathcal{B}, \rho)$ furnished with the weak topology is a Polish group.

4.2. Nonsingular Poisson suspensions and related transformation groups. Let the measure space (X, \mathcal{A}, μ) be as in the previous section and let T be a nonsingular (invertible) transformation of this space. Theorem 3.3 provides us with an “if and only if” criteria for when T_* is non-singular and Theorems 3.4 and Theorem 3.6 give an explicit pointwise description of the Radon-Nikodym derivative of T_* as follows.

Corollary 4.1. *T_* is a nonsingular automorphism of $(X^*, \mathcal{A}^*, \mu^*)$ if and only if $\sqrt{T'} - 1 \in L^2(\mu)$. In this case $(T_*)' = \mathcal{E}(T' - 1)$. Moreover,*

(1) We can represent $\log(T_*)'$ as the following limit in probability:

$$\log(T_*)'(\omega) = \lim_{\epsilon \rightarrow 0} \left(\int_{\{x \in X: |\log T'(x)| > \epsilon\}} \log T' d\omega - \int_{\{x \in X: |\log T'(x)| > \epsilon\}} (T' - 1) d\mu \right).$$

(2) The function $X^* \ni \omega \mapsto \log(T_*)'(\omega) \in \mathbb{R}$ is an infinitely divisible random variable whose Lévy measure is the (restriction to $\mathbb{R} \setminus \{0\}$ of) the image of μ by $\log T'$.

(3) If $T' - 1 \in L^1(\mu)$, then $\log T' \in L^1(\omega)$ for μ^* -almost every $\omega \in X^*$ and

$$(T_*)'(\omega) = e^{-\int_X (T' - 1) d\mu} \prod_{\{x \in X: \omega(\{x\}) = 1\}} T'(x),$$

where the infinite product converges absolutely.

Our purpose in this paper is to study nonsingular Poisson suspensions. Therefore in view of Corollary 4.1 we introduce some special subgroups of nonsingular transformations that are related naturally to these suspensions.

Definition 4.2. We set

$$\begin{aligned} \text{Aut}_2(X, \mathcal{A}, \mu) &:= \left\{ T \in \text{Aut}(X, \mathcal{A}, \mu), \sqrt{T'} - 1 \in L^2(\mu) \right\}, \\ \text{Aut}_1(X, \mathcal{A}, \mu) &:= \left\{ T \in \text{Aut}(X, \mathcal{A}, \mu), T' - 1 \in L^1(\mu) \right\} \quad \text{and} \\ \text{Aut}_{\mathcal{P}}(X^*, \mathcal{A}^*, \mu^*) &:= \left\{ T_* \in \text{Aut}(X^*, \mathcal{A}^*, \mu^*), T \in \text{Aut}_2(X, \mathcal{A}, \mu) \right\}. \end{aligned}$$

Of course, $\text{Aut}_1(X, \mathcal{A}, \mu) \subset \text{Aut}_2(X, \mathcal{A}, \mu)$. By Remark 3.1(2), the two objects are subgroups of $\text{Aut}(X, \mathcal{A}, \mu)$. Since the map $T \mapsto T_*$ is a homomorphism from $\text{Aut}_2(X, \mathcal{A}, \mu)$ to $\text{Aut}(X^*, \mathcal{A}^*, \mu^*)$, the set $\text{Aut}_{\mathcal{P}}(X^*, \mathcal{A}^*, \mu^*)$ is a subgroup of $\text{Aut}(X^*, \mathcal{A}^*, \mu^*)$.

Definition 4.3. The map $\text{Aut}_2(X, \mathcal{A}, \mu) \ni T \mapsto T^* \in \text{Aut}(X^*, \mathcal{A}^*, \mu^*)$ will be called *the Poisson homomorphism*.

In the next three subsections we study $\text{Aut}_2(X, \mathcal{A}, \mu)$, $\text{Aut}_1(X, \mathcal{A}, \mu)$ and $\text{Aut}_{\mathcal{P}}(X^*, \mathcal{A}^*, \mu^*)$ respectively in more detail.

4.3. Polish group $\text{Aut}_2(X, \mathcal{A}, \mu)$ and the associated affine Koopman representation. We denote by $\text{Aff}_{\mathbb{R}}(L^2(\mu))$ the subgroup of invertible affine operators in $L^2(\mu)$ that preserve invariant the \mathbb{R} -subspace $L^2_{\mathbb{R}}(\mu)$. Then $\text{Aff}_{\mathbb{R}}(L^2(\mu)) := L^2_{\mathbb{R}}(\mu) \rtimes \mathcal{U}_{\mathbb{R}}(L^2(\mu))$. We recall that an operator $A = (f, V) \in \text{Aff}_{\mathbb{R}}(L^2(\mu))$ acts on $L^2(\mu)$ by the formula $Ah := f + Vh$. One can verify that the multiplication law in $\text{Aff}_{\mathbb{R}}(L^2(\mu))$ is given by:

$$(f, V)(f', V') := (f + Vf', VV').$$

$\text{Aff}_{\mathbb{R}}(L^2(\mu))$ is a Polish group when endowed with the product of the norm topology on $L^2(\mu)$ and the weak operator topology on $\mathcal{U}_{\mathbb{R}}(L^2(\mu))$. We now

let

$$C_{-1} := \{f \in L^2(\mu) : f \geq -1\}.$$

Then C_{-1} is a closed semispace in $L^2_{\mathbb{R}}(\mu)$. We now establish an ‘‘affine’’ analogue of (4.1).

Theorem 4.4.

$$\{A \in \text{Aff}_{\mathbb{R}}(L^2(\mu)) : AC_{-1} = C_{-1}\} = \{(\sqrt{S'} - 1, U_S) : S \in \text{Aut}_2(X, \mathcal{A}, \mu)\}.$$

Proof. If $A := (\sqrt{S'} - 1, U_S)$ for some transformation $S \in \text{Aut}_2(Y, \mathcal{B}, \rho)$ then $Ah = (h \circ S^{-1} + 1)\sqrt{S'} - 1 \geq -1$ for each $h \in C_{-1}$. Hence $AC_{-1} \subset C_{-1}$. The same is true if we take S^{-1} in place of S . Therefore we obtain that $A^{-1}C_{-1} \subset C_{-1}$. Hence $AC_{-1} = C_{-1}$, as desired.

Conversely, let $A = (f, V) \in \text{Aff}_{\mathbb{R}}(L^2(\mu))$ and $AC_{-1} = C_{-1}$. The following properties are verified straightforwardly:

- $C_0 + C_{-1} = C_{-1}$.
- If $a + C_{-1} \subset C_{-1}$ for some $a \in L^2(\mu)$ then $a \in C_0$.

Then $A(C_0 + C_{-1}) = C_{-1}$. On the other hand,

$$A(C_0 + C_{-1}) = AC_0 + AC_{-1} - A0 = VC_0 + C_{-1}.$$

Therefore $VC_0 + C_{-1} = C_{-1}$. Hence $VC_0 \subset C_0$. Since $A^{-1}C_{-1} = C_{-1}$, a similar reasoning yields that $V^{-1}C_0 \subset C_0$. Therefore $VC_0 = C_0$. In view of (4.1), there is $S \in \text{Aut}(X, \mathcal{A}, \mu)$ such that $V = U_S$. Hence

$$(4.2) \quad VC_{-1} = \{h \circ S\sqrt{S'} + f \mid h \in C_{-1}\} = C_{-1}.$$

Let $L(\mu)$ stand for the space of all measurable real valued functions on X . Endowed with the natural order, $L(\mu)$ is an ordered vector space. Considering the semispace C_{-1} as a subset of $L(\mu)$, we deduce from (4.2) that

$$L(\mu) \ni -1 = \inf C_{-1} = \inf VC_{-1} = -\sqrt{S'} + f \in L(\mu).$$

Thus, $\sqrt{S'} - 1 = f \in L^2_{\mathbb{R}}(\mu)$. □

We note that Theorem 4.4 provides an alternative characterization of $\text{Aut}_2(X, \mathcal{A}, \mu)$. This characterization is not related straightforwardly to Poisson suspensions. Moreover, Theorem 4.4 determines a one-to-one representation of $\text{Aut}_2(X, \mathcal{A}, \mu)$ in $\text{Aff}_{\mathbb{R}}(L^2(\mu))$.

Definition 4.5. We call the homomorphism

$$A^{(2)} : \text{Aut}_2(X, \mathcal{A}, \mu) \ni S \mapsto A_S^{(2)} := (\sqrt{S'} - 1, U_S) \in \text{Aff}_{\mathbb{R}}(L^2(\mu))$$

the *affine Koopman representation* of $\text{Aut}_2(X, \mathcal{A}, \mu)$. We call the weakest topology on $\text{Aut}_2(X, \mathcal{A}, \mu)$ in which the affine Koopman representation is continuous the *d_2 -topology*.

Thus, a sequence $(T_n)_{n=1}^{\infty}$ of transformations $T_n \in \text{Aut}_2(X, \mathcal{A}, \mu)$ converges in d_2 to a transformation $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$ as $n \rightarrow \infty$ if and only if $T_n \rightarrow T$ weakly and $\|\sqrt{T'_n} - \sqrt{T'}\|_2 \rightarrow 0$ as $n \rightarrow \infty$. It follows from Theorem 4.4 that the image of $\text{Aut}_2(X, \mathcal{A}, \mu)$ under $A^{(2)}$ is closed in

$\text{Aff}(L^2(\mu))$. Hence $\text{Aut}_2(X, \mathcal{A}, \mu)$ endowed with the d_2 -topology is a Polish group. We state the next proposition without proof. It follows easily from Remark 3.1(2).

Proposition 4.6. *Let $\nu \in \mathcal{M}_{\mu,2}^{\circ,+}$. Then $\text{Aut}_2(X, \mathcal{A}, \nu) = \text{Aut}_2(X, \mathcal{A}, \mu)$ as topological groups furnished with the corresponding d_2 -topologies.*

4.4. Polish group $\text{Aut}_1(X, \mathcal{A}, \mu)$, the associated affine Koopman representation and the structure of semidirect product. Let $\mathcal{U}(L^1(\mu))$ stand for the group of isometries in $L^1(\mu)$ and let $\mathcal{U}_{\mathbb{R}}(L^1(\mu))$ stand for the subgroup of isometries that preserve invariant the \mathbb{R} -subspace $L_{\mathbb{R}}^1(\mu)$ of real valued functions in $L^1(\mu)$. We denote by $\text{Aff}_{\mathbb{R}}(L^1(\mu)) := L_{\mathbb{R}}^1(\mu) \rtimes \mathcal{U}_{\mathbb{R}}(L^1(\mu))$ the group of invertible affine operators in $L^1(\rho)$ that preserve invariant $L_{\mathbb{R}}^1(\mu)$. The multiplication law in $\text{Aff}_{\mathbb{R}}(L^1(\mu))$ is given by the same formula as the multiplication law in $\text{Aff}_{\mathbb{R}}(L^2(\mu))$. We also note that $\text{Aff}_{\mathbb{R}}(L^1(\mu))$ is a Polish group when endowed with the product of the norm topology on $L_{\mathbb{R}}^1(\mu)$ and the strong (not the weak!) operator topology on $\mathcal{U}_{\mathbb{R}}(L^1(\mu))$. We now let

$$C_{-1}^{(1)} := \{f \in L^1(\mu) : f \geq -1\}.$$

Then $C_{-1}^{(1)}$ is a closed semispace in $L_{\mathbb{R}}^1(\mu)$. Given $S \in \text{Aut}_1(X, \mathcal{A}, \mu)$, we define an isometric invertible operator $U_S^{(1)}$ on $L^1(\mu)$ by setting $U_S^{(1)}f := f \circ S^{-1} \cdot S'$. Then we call the one-to-one homomorphism

$$U^{(1)} : \text{Aut}_1(X, \mathcal{A}, \mu) \ni S \mapsto U_S^{(1)} \in \mathcal{U}_{\mathbb{R}}(L^1(\mu))$$

the *isometric Koopman representation* of $\text{Aut}_1(X, \mathcal{A}, \mu)$. The following theorem is an analogue of Theorem 4.4.

Theorem 4.7.

$$\{A \in \text{Aff}_{\mathbb{R}}(L^1(\mu)) : AC_{-1}^{(1)} = C_{-1}^{(1)}\} = \{(S' - 1, U_S^{(1)}) : S \in \text{Aut}_1(X, \mathcal{A}, \mu)\}.$$

We do not provide a proof of this theorem because it is very similar to the proof of Theorem 4.4.

Definition 4.8. We call the one-to-one homomorphism

$$A^{(1)} : \text{Aut}_1(X, \mathcal{A}, \mu) \ni S \mapsto A_S^{(1)} := (S' - 1, U_S^{(1)}) \in \text{Aff}_{\mathbb{R}}(L^1(\mu))$$

the *affine Koopman representation* of $\text{Aut}_1(X, \mathcal{A}, \mu)$. We call the weakest topology on $\text{Aut}_1(X, \mathcal{A}, \mu)$ in which the affine Koopman representation is continuous the *d_1 -topology*.

Thus, a sequence $(T_n)_{n=1}^{\infty}$ of transformations $T_n \in \text{Aut}_1(X, \mathcal{A}, \mu)$ converges in d_1 to a transformation $T \in \text{Aut}_1(X, \mathcal{A}, \mu)$ as $n \rightarrow \infty$ if and only if $T_n \rightarrow T$ weakly and $\|T_n' - T'\|_1 \rightarrow 0$ as $n \rightarrow \infty$. It follows from Theorem 4.7 that the image of $\text{Aut}_1(X, \mathcal{A}, \mu)$ under $A^{(1)}$ is closed in $\text{Aff}_{\mathbb{R}}(L^1(\mu))$. Hence $\text{Aut}_1(X, \mathcal{A}, \mu)$ endowed with the d_1 -topology is a Polish group. We state the next proposition without proof. It follows easily from Remark 3.1(2).

Proposition 4.9. *Let $\nu \in \mathcal{M}_{\mu,1}^{\circ,+}$. Then $\text{Aut}_1(X, \mathcal{A}, \nu) = \text{Aut}_1(X, \mathcal{A}, \mu)$ as topological groups furnished with the corresponding d_1 -topologies.*

The following important group homomorphism was introduced in [27]:

$$\chi : \text{Aut}_1(X, \mathcal{A}, \mu) \ni T \mapsto \chi(T) := \int_X (T' - 1) d\mu \in \mathbb{R}.$$

Of course, χ depends on μ . However, we now show that χ does not depend on the choice of measure within the class $\mathcal{M}_{\mu,1}^{\circ,+}$.

Proposition 4.10. *Let $\nu \in \mathcal{M}_{\mu,1}^{\circ,+}$. Then $\chi(T) = \int_X \left(\frac{d\nu \circ T^{-1}}{d\nu} - 1 \right) d\nu$ for each $T \in \text{Aut}_1(X, \mathcal{A}, \mu)$.*

Proof. By Remark 3.1, μ and ν share the same family of subsets of finite measure. Let $\{A_n\}_{n \in \mathbb{N}}$ be an increasing sequence of sets of finite measure such that $\bigcup_{n=1}^{\infty} A_n = X$. Then

$$b := \int_X \left(\frac{d\nu}{d\mu} - 1 \right) d\mu = \lim_{n \rightarrow +\infty} \int_{A_n} \left(\frac{d\nu}{d\mu} - 1 \right) d\mu = \lim_{n \rightarrow +\infty} (\nu(A_n) - \mu(A_n)).$$

In a similar way, $b = \lim_{n \rightarrow +\infty} (\nu(T^{-1}A_n) - \mu(T^{-1}A_n))$. Hence

$$\begin{aligned} 0 &= \lim_{n \rightarrow +\infty} (\nu(A_n) - \mu(A_n) - \nu(T^{-1}A_n) + \mu(T^{-1}A_n)) \\ &= \lim_{n \rightarrow +\infty} (\mu(T^{-1}A_n) - \mu(A_n)) - \lim_{n \rightarrow +\infty} (\nu(T^{-1}A_n) - \nu(A_n)) \\ &= \chi(T) - \int_X \left(\frac{d\nu \circ T^{-1}}{d\nu} - 1 \right) d\nu. \end{aligned}$$

□

Theorem 4.11. *The homomorphism χ is d_1 -continuous and the quotient group $\text{Aut}_1(X, \mathcal{A}, \mu)/\text{Ker } \chi$ is isomorphic to \mathbb{R} . In other words, the following short sequence of Polish groups is exact*

$$\{1\} \rightarrow \text{Ker } \chi \rightarrow \text{Aut}_1(X, \mathcal{A}, \mu) \rightarrow \mathbb{R} \rightarrow \{0\}$$

Moreover, this sequence splits, i.e. there is a continuous one-to-one homomorphism $\sigma : \mathbb{R} \rightarrow \text{Aut}_1(X, \mathcal{A}, \mu)$ such that $\chi \circ \sigma = \text{id}_{\mathbb{R}}$.

Proof. Of course, χ is continuous. Next, there is no loss in generality if we take $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}, m)$ where m is defined by $\frac{dm}{dx} = 1_{\mathbb{R}^*} + 2 \times 1_{\mathbb{R}^+}$. For $t \in \mathbb{R}$, denote by $T_t : \mathbb{R} \rightarrow \mathbb{R}$ the translation by t . Then it is easy to verify that $T_t \in \text{Aut}_1(X, \mathcal{A}, \mu)$ and $\chi(T_{-t}) = t$. Of course, the homomorphism $\sigma : \mathbb{R} \ni t \mapsto T_{-t} \in \text{Aut}_1(X, \mathcal{A}, \mu)$ is continuous. □

It follows from the second claim of the theorem that there is a topological isomorphism $\theta : \text{Aut}_1(X, \mathcal{A}, \mu) \rightarrow \text{Ker } \chi \rtimes_{\sigma} \mathbb{R}$ such that the following diagram

commutes:

$$\begin{array}{ccccccc}
 \{1\} & \longrightarrow & \text{Ker } \chi & \xrightarrow{i} & \text{Aut}_1(X, \mathcal{A}, \mu) & \xrightarrow{\chi} & \mathbb{R} \longrightarrow \{0\} \\
 & & \text{id} \downarrow & & \theta \downarrow & & \text{id} \downarrow \\
 \{1\} & \longrightarrow & \text{Ker } \chi & \xrightarrow{i} & \text{Ker } \chi \rtimes_{\sigma} \mathbb{R} & \xrightarrow{\chi} & \mathbb{R} \longrightarrow \{0\}
 \end{array}$$

Thus, we have showed that $\text{Aut}_1(X, \mathcal{A}, \mu)$ has a natural structure of semidirect product of $\text{Ker } \chi$ and \mathbb{R} .

In order to state one more property of χ we need to recall a definition of conservativeness.

Definition 4.12. A transformation $T \in \text{Aut}(X, \mathcal{A}, \mu)$ is called *conservative* if for each subset $A \in \mathcal{A}$ of positive measure, there is $n > 0$ such that $\mu(T^{-1}A \cap A) > 0$.

We recall that a nonsingular transformation T is conservative if and only if $\sum_{k=0}^{\infty} U_{T^k}^{(1)} f = +\infty$ a.e. for each measurable function $f > 0$ [1].

Proposition 4.13. *Let $T \in \text{Aut}_1(X, \mathcal{A}, \mu)$. If T is conservative then $\chi(T) = 0$.*

Proof. Let $\phi := T' - 1$. Then $\phi \in L^1(\mu)$ and $\int_X \phi d\mu = \chi(T)$. Take $g \in L^1(\mu)$ such that $g > 0$ and $\int_X g d\mu = 1$. By the Hurewicz ratio ergodic theorem, there exists $\psi \in L^1(\mu)$ such that $\psi \circ T = \psi$, $\int_X \psi d\mu = \int_X \phi d\mu = \chi(T)$ and

$$\frac{\sum_{k=0}^n U_{T^k}^{(1)} \phi}{\sum_{k=0}^n U_{T^k}^{(1)} g} \rightarrow \psi \quad \text{almost everywhere as } n \rightarrow \infty.$$

However $\sum_{k=0}^n U_{T^k}^{(1)} \phi = (T^{n+1})' - 1 \geq -1$ while $\sum_{k=0}^n U_{T^k}^{(1)} g \rightarrow +\infty$ as $n \rightarrow \infty$. Consequently, $\psi \geq 0$ a.e. and hence $\chi(T) \geq 0$. However the transformation T^{-1} is also conservative and the above reasoning yields that $\chi(T^{-1}) \geq 0$. As $\chi(T^{-1}) = -\chi(T)$, we obtain that $\chi(T) = 0$. \square

We conclude this subsection with a discussion about relationship among d_1 , d_2 and the weak topology. Since $\|\sqrt{T'} - 1\|_2^2 \leq \|T' - 1\|_1$, it follows that d_1 is stronger than d_2 .

Proposition 4.14.

- d_1 is strictly stronger than d_2 restricted to $\text{Aut}_1(X, \mathcal{A}, \mu)$ and d_2 is strictly stronger than the weak topology restricted to $\text{Aut}_2(X, \mathcal{A}, \mu)$.
- $\text{Aut}_1(X, \mathcal{A}, \mu)$ is dense and meager in $\text{Aut}_2(X, \mathcal{A}, \mu)$ endowed with d_2 and $\text{Aut}_2(X, \mathcal{A}, \mu)$ is dense and meager in $\text{Aut}(X, \mathcal{A}, \mu)$ endowed with the weak topology.
- χ is not d_2 -continuous. Hence χ does not extend by continuity to $\text{Aut}_2(X, \mathcal{A}, \mu)$.
- If \mathcal{B}_1 , \mathcal{B}_2 and \mathcal{B} stand for the Borel σ -algebras generated by τ_1 , τ_2 and the weak topology respectively then $\mathcal{B}_2 \upharpoonright \text{Aut}_1(X, \mathcal{A}, \mu) = \mathcal{B}_1$ and $\mathcal{B} \upharpoonright \text{Aut}_2(X, \mathcal{A}, \mu) = \mathcal{B}_2$.

We do not provide a proof of this proposition because it will not be used below in the paper. We only note that it can be deduced from the general theorems of the descriptive topology combined with several facts that are proved in the next section: Propositions 5.2, 5.4, Theorem 5.8. The interested reader can also prove it independently of the next section by constructing appropriate concrete counterexamples.

4.5. Unitary Koopman representations of $\text{Aut}_{\mathcal{P}}(X^*, \mathcal{A}^*, \mu^*)$. Our objective in this subsection is to clarify relationship between the Koopman operators associated to T and T_* respectively.

Given an affine operator $A = (f, V) \in \text{Aff}_{\mathbb{R}}(L^2(\mu))$, we define an operator W_A in $L^2(\mu^*)$ by setting

$$(4.3) \quad W_A \mathcal{E}(h) := e^{-\frac{\|f\|_2^2}{2} - \langle f, Vh \rangle_{L^2(\mu)}} \mathcal{E}(Ah)$$

for all $h \in L^2(\mu)$ and then extending W_A by linearity and continuity to the entire $L^2(\mu^*)$. It is shown in [22, § 2.2] that W_A is well defined and $W_A \in \mathcal{U}(L^2(\mu^*))$. It is called a *Weyl operator*. We observe that $W_A \in \mathcal{U}_{\mathbb{R}}(L^2(\mu^*))$ and $W_A W_B = W_{AB}$ for all $A, B \in \text{Aff}_{\mathbb{R}}(L^2(\mu))$. It is possible to define W_A for arbitrary $V \in \mathcal{U}(L^2(\mu))$ and $f \in L^2(\mu)$ by the same formula (4.3). Then $W_A \in \mathcal{U}_{\mathbb{R}}(L^2(\mu^*))$ if and only if $A \in \text{Aff}_{\mathbb{R}}(L^2(\mu))$. We leave the proof of this fact as an exercise for the reader. We also need one more auxiliary result that follows easily from [22, Theorem 2.1].

Theorem. *The map $W : \text{Aff}_{\mathbb{R}}(L^2(\mu)) \ni A \mapsto W_A \in \mathcal{U}_{\mathbb{R}}(L^2(\mu^*))$ is a continuous one-to-one group homomorphism. Its image $\mathcal{W} := \{W_A : A \in \text{Aff}_{\mathbb{R}}(L^2(\mu))\}$ is a Polish subgroup in the induced topology.*

We will call W the *Weyl homomorphism*. It follows from the above auxiliary theorem and [4, Proposition 1.2.1] that \mathcal{W} is closed in $\mathcal{U}_{\mathbb{R}}(L^2(\mu^*))$ endowed with the weak operator topology.

It is well known that that if a transformation $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$ preserves μ then the associated unitary Koopman operator U_{T_*} can be written as $U_{T_*} = W_{(0, U_T)} = W_{A_T^{(2)}}$. We now extend this result to arbitrary (nonsingular) elements of $\text{Aut}_2(X, \mathcal{A}, \mu)$.

Theorem 4.15. *Let $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$. Under the natural identification of $F(L^2(\mu))$ with $L^2(\mu^*)$ described in § 2.4, we obtain that $U_{T_*} = W_{A_T^{(2)}}$. In other words, the composition of the Poisson homomorphism of $\text{Aut}_2(X, \mathcal{A}, \mu)$ with the unitary Koopman representation of $\text{Aut}_{\mathcal{P}}(X^*, \mathcal{A}^*, \mu^*)$ equals the composition of the affine Koopman representation of $\text{Aut}_2(X, \mathcal{A}, \mu)$ with the Weyl homomorphism.*

Proof. It is sufficient to verify that $U_{T_*} \mathcal{E}(f) = W_{A_T^{(2)}} \mathcal{E}(f)$ for every simple (i.e. finite valued) function f from $\mathcal{B}_0(X)$. We note that $f \circ T^{-1}$ is also a

simple function from $\mathcal{B}_0(X)$ and

$$\begin{aligned}\mathcal{E}(f \circ T^{-1})(\omega) &= e^{-\int_X f \circ T^{-1} d\mu} \prod_{\{x \in X: \omega(\{x\})=1\}} (1 + f(T^{-1}x)) \\ &= e^{\int_X (f - f \circ T^{-1}) d\mu} \mathcal{E}(f)(T_*^{-1}\omega)\end{aligned}$$

at a.e. $\omega \in X^*$. Using this and Corollary 4.1 we obtain that

$$(4.4) \quad \begin{aligned}U_{T_*} \mathcal{E}(f) &= \sqrt{(T_*)'} \mathcal{E}(f) \circ T_*^{-1} \\ &= \sqrt{\mathcal{E}(T' - 1)} e^{\int_X (f \circ T^{-1} - f) d\mu} \mathcal{E}(f \circ T^{-1}).\end{aligned}$$

By the product formula (2.6) (see also (3.1)),

$$\mathcal{E}(T' - 1) = e^{-\|\sqrt{T'} - 1\|_2^2} \mathcal{E}(\sqrt{T'} - 1)^2.$$

Due to Lemma 2.3, $\mathcal{E}(\sqrt{T'} - 1) \geq 0$ and hence

$$(4.5) \quad \sqrt{\mathcal{E}(T' - 1)} = e^{-\frac{1}{2}\|\sqrt{T'} - 1\|_2^2} \mathcal{E}(\sqrt{T'} - 1).$$

By a straightforward computation, $(\sqrt{T'} - 1) \bullet (f \circ T^{-1}) = A_T^{(2)} f$. Hence, in view of (2.6),

$$(4.6) \quad \mathcal{E}(\sqrt{T'} - 1) \mathcal{E}(f \circ T^{-1}) = e^{\int_X (\sqrt{T'} - 1) f \circ T^{-1} d\mu} \mathcal{E}(A_T^{(2)} f).$$

Substituting first (4.5) and then (4.6) into (4.4), we obtain that

$$\begin{aligned}U_{T_*} \mathcal{E}(f) &= e^{-\frac{1}{2}\|\sqrt{T'} - 1\|_2^2 + \int_X (-f + U_T f) d\mu} \mathcal{E}(A_T^{(2)} f) \\ &= e^{-\frac{1}{2}\|\sqrt{T'} - 1\|_2^2 - \langle \sqrt{T'} - 1, U_T f \rangle_{L^2(\mu)}} \mathcal{E}(A_T^{(2)} f) \\ &= W_{A_T^{(2)}} \mathcal{E}(f).\end{aligned}$$

□

Thus, the group \mathcal{W} contains the unitary Koopman operator generated by every transformation from $\text{Aut}_{\mathcal{P}}(X^*, \mathcal{A}^*, \mu^*)$. We now show that it does not contain any Koopman operator generated by transformations from the set theoretical difference $\text{Aut}(X^*, \mathcal{A}^*, \mu^*) \setminus \text{Aut}_{\mathcal{P}}(X^*, \mathcal{A}^*, \mu^*)$.

Proposition 4.16. $\mathcal{W} \cap \{U_S : S \in \text{Aut}(X^*, \mathcal{A}^*, \mu^*)\} = \{W_{A_T^{(2)}} : T \in \text{Aut}_2(X, \mathcal{A}, \mu)\}$.

Proof. Suppose that for some operator $A \in \text{Aff}_{\mathbb{R}}(L^2(\mu))$, the unitary W_A is the Koopman operator generated by a nonsingular transformation of $(X^*, \mathcal{A}^*, \mu^*)$. Then, according to (4.1), W_A preserves invariant the cone $L_+^2(\mu^*)$ of non-negative functions in $L^2(\mu^*)$. It follows from Lemma 2.3 that

$$L_+^2(\mu^*) \cap \{\mathcal{E}(h) : h \in L_{\mathbb{R}}^2(X, \mu)\} = \{\mathcal{E}(h) : h \in C_{-1}\}.$$

Hence $W_A(\{\mathcal{E}(h) : h \in C_{-1}\}) \subset L_+^2(\mu^*)$. In view of (4.3) and Lemma 2.3 this is equivalent to $AC_{-1} \subset C_{-1}$. Since A^{-1} is also a Koopman operator, a similar reasoning yields that $A^{-1}C_{-1} \subset C_{-1}$. Hence $AC_{-1} = C_{-1}$. By Theorem 4.4, $A = A_T^{(2)}$ for some $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$. Thus, we showed that

$\mathcal{W} \cap \{U_S : S \in \text{Aut}(X^*, \mathcal{A}^*, \mu^*)\} \subset \{W_{A_T^{(2)}} : T \in \text{Aut}_2(X, \mathcal{A}, \mu)\}$. The converse inclusion was established in Theorem 4.15. \square

Since \mathcal{W} is a closed subgroup of $\mathcal{U}(L^2(\mu^*))$, we obtain the following corollary from the above proposition.

Corollary 4.17. *$\text{Aut}_{\mathcal{P}}(X^*, \mathcal{A}^*, \mu^*)$ is weakly closed in $\text{Aut}(X^*, \mathcal{A}^*, \mu^*)$.*

5. GENERIC PROPERTIES IN $\text{AUT}_2(X, \mathcal{A}, \mu)$ AND $\text{AUT}_1(X, \mathcal{A}, \mu)$

As the groups $\text{Aut}_2(X, \mathcal{A}, \mu)$ and $\text{Aut}_1(X, \mathcal{A}, \mu)$ are Polish, it is natural to ask: which dynamical properties (or, more rigorously, the subsets of elements possessing these properties) are generic in these groups in the Baire category sense? Recall that a set in a Polish space is generic if it contains a subset which is a dense G_δ in this space.

We first list the well known generic properties for $\text{Aut}(X, \mathcal{A}, \mu)$, the definitions of these (and other) properties will be given just below the Theorem.

Theorem ([9], [10]). *The following subsets of nonsingular transformations:*

- $\text{Cons}(X, \mathcal{A}, \mu) := \{T \in \text{Aut}(X, \mathcal{A}, \mu) : T \text{ is conservative}\},$
- $\text{Erg}(X, \mathcal{A}, \mu) := \{T \in \text{Aut}(X, \mathcal{A}, \mu) : T \text{ is ergodic}\},$
- $\text{Erg}^{\text{III}_1}(X, \mathcal{A}, \mu) := \{T \in \text{Aut}(X, \mathcal{A}, \mu) : T \text{ is of type III}_1\}$

are dense G_δ in $\text{Aut}(X, \mathcal{A}, \mu)$ endowed with the weak topology.

Recall that a transformation $T \in \text{Aut}(X, \mathcal{A}, \mu)$ is *conservative* if every wandering set $W \in \mathcal{A}$ for T , that is a set such that $\{T^n W\}_{n \in \mathbb{Z}}$ are pairwise disjoint, is a null set. The transformation T is *ergodic* if every T -invariant subset is either \emptyset or X modulo null sets. The transformation T is *aperiodic* if there is a conull subset $X' \subset X$ such that for all $x \in X'$ and $n > 0$, $T^n x \neq x$. There are several ways to define the type III_1 property, in this paper we will use the definition involving the Maharam extension. The *Maharam extension* of $T \in \text{Aut}(X, \mathcal{A}, \mu)$, is the transformation \tilde{T} on $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}})$ defined by

$$\tilde{T}(x, y) := \left(Tx, y - \log \frac{d\mu \circ T}{d\mu}(x) \right).$$

For every $T \in \text{Aut}(X, \mathcal{A}, \mu)$, its Maharam extension preserves the measure $\tilde{\mu}$ defined by the formula $\tilde{\mu}(A \times I) := \mu(A) \int_I e^t dt$ for every $A \in \mathcal{A}$ and each interval $I \subset \mathbb{R}$. The transformation T is *of type III₁* if its Maharam extension is ergodic.

The proof of genericity results usually consists of two steps: the first one is to show that the set under consideration is G_δ which may involve making use of ergodic theorems and countably many conditions defining the set. The second step is to prove that this set is dense. For the second step the following conjugacy lemma is often a key ingredient.

Lemma. *For every aperiodic $T \in \text{Aut}(X, \mathcal{A}, \mu)$, the conjugacy class*

$$\{STS^{-1} : S \in \text{Aut}(X, \mathcal{A}, \mu)\}$$

of S is weakly dense in $\text{Aut}(X, \mathcal{A}, \mu)$ [9].

We also recall that a Polish group G has the *Rokhlin property* if it has a dense conjugacy class. For instance, by the above lemma, $\text{Aut}(X, \mathcal{A}, \mu)$ has the Rokhlin property.

We note that $\text{Aut}_1(X, \mathcal{A}, \mu)$ appeared in [27] as a generalization of earlier work on representation theory of groups of diffeomorphisms on non-compact manifolds which are the identity outside a compact set (see [33], [21]). In this connection, it seems natural to introduce the following definition: a transformation $T \in \text{Aut}(X, \mathcal{A}, \mu)$ is *local* if there is a set $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that $Tx = x$ for all $x \notin A$. Denote by $\text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu) \subset \text{Aut}_1(X, \mathcal{A}, \mu)$ the set of all local transformations. Of course, $\text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$ is a group. Since each transformation in $\text{Aut}_2(X, \mathcal{A}, \mu)$ preserves the class of subsets of finite measure, $\text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$ is a normal subgroup of $\text{Aut}_1(X, \mathcal{A}, \mu)$ and $\text{Aut}_2(X, \mathcal{A}, \mu)$. (However it is not normal in $\text{Aut}(X, \mathcal{A}, \mu)$.) The following consequence of Theorem 4.11 (specifically the topological isomorphism to a semidirect product) shows that $\text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$ is “too small” in $\text{Aut}_1(X, \mathcal{A}, \mu)$.

Corollary 5.1. *The group $\text{Aut}_1(X, \mathcal{A}, \mu)$ does not have the Rokhlin property. The subgroup of local transformations, being a subset of $\ker \chi$, is d_1 -nowhere dense in $\text{Aut}_1(X, \mathcal{A}, \mu)$.*

The situation with $\text{Aut}_2(X, \mathcal{A}, \mu)$ is different. We preface the statement of the corresponding result with the following notation that will be of wide use in this section. Given $A, B \subset X$ which are both of finite measure, let $\tau_{A,B}$ denote a μ -nonsingular bijection from A to B such that for all $x \in A$, $\frac{d\mu \circ \tau_{A,B}}{d\mu}(x) = \frac{\mu(B)}{\mu(A)}$.

Proposition 5.2. *$\text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$ is d_2 -dense in $\text{Aut}_2(X, \mathcal{A}, \mu)$.*

Proof. Let $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$. There exists an increasing sequence $A_n \in \mathcal{A}$ of finite measure subsets satisfying $\bigcup_{n=1}^{\infty} A_n = X$ and

$$\int_{X \setminus A_n} \left(1 - \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} \right)^2 d\mu =: \epsilon_n \xrightarrow{n \rightarrow \infty} 0.$$

Select subsets $B_n \subset X$ of finite measure such that $A_n \cup TA_n \subset B_n$ and

$$\sqrt{\mu(B_n \setminus A_n)} + \sqrt{\mu(B_n \setminus T^{-1}A_n)} \geq n(\mu(A_n) + \mu(T^{-1}A_n)).$$

We define $T_n \in \text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$ via

$$T_n^{-1}x := \begin{cases} T^{-1}x, & x \in A_n \\ \tau_{B_n \setminus A_n, B_n \setminus T^{-1}A_n}x, & x \in B_n \setminus A_n \\ x, & x \notin B_n. \end{cases}$$

We will now show that T_n converges in d_2 to T . Since $T_n^{-1}x = T^{-1}x$ for $x \in A_n$ we see that T_n converges weakly to T as $n \rightarrow \infty$. As

$$(5.1) \quad \frac{d\mu \circ T_n^{-1}}{d\mu}(x) := \begin{cases} \frac{d\mu \circ T^{-1}}{d\mu}(x), & x \in A_n \\ \frac{\mu(B_n \setminus T^{-1}A_n)}{\mu(B_n \setminus A_n)}, & x \in B_n \setminus A_n \\ 1, & x \notin B_n, \end{cases}$$

we deduce that

$$\begin{aligned} \left\| \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - \sqrt{\frac{d\mu \circ T_n^{-1}}{d\mu}} \right\|_2 &\leq \left\| \left(\sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - 1 \right) 1_{X \setminus A_n} \right\|_2 + \left\| \left(\sqrt{\frac{d\mu \circ T_n^{-1}}{d\mu}} - 1 \right) 1_{X \setminus A_n} \right\|_2 \\ &\leq \sqrt{\epsilon_n} + \left(\int_{B_n \setminus A_n} \left(\sqrt{\frac{\mu(B_n \setminus T^{-1}A_n)}{\mu(B_n \setminus A_n)}} - 1 \right)^2 d\mu \right)^{1/2} \\ &= \sqrt{\epsilon_n} + \left| \frac{\mu(B_n \setminus A_n) - \mu(B_n \setminus T^{-1}A_n)}{\sqrt{\mu(B_n \setminus A_n)} + \sqrt{\mu(B_n \setminus T^{-1}A_n)}} \right| \\ &\leq \sqrt{\epsilon_n} + \frac{\mu(A_n) + \mu(T^{-1}A_n)}{\sqrt{\mu(B_n \setminus A_n)} + \sqrt{\mu(B_n \setminus T^{-1}A_n)}} \\ &\leq \sqrt{\epsilon_n} + \frac{1}{n}. \end{aligned}$$

It follows that $\lim_{n \rightarrow \infty} \left\| \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} - \sqrt{\frac{d\mu \circ T_n^{-1}}{d\mu}} \right\|_2 = 0$. Thus we have shown that T_n converges to T in d_2 as $n \rightarrow \infty$. \square

For $A \in \mathcal{A}$, we can consider the group $\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$ as the subset of $S \in \text{Aut}_2(X, \mathcal{A}, \mu)$ such that $S^{-1}A = A$ and $S|_{X \setminus A} = \text{id}|_{X \setminus A}$. We note that if $\mu(A) < \infty$ then the map $S \mapsto S|_A$ is a topological isomorphism of $\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$ with the d_2 -topology onto $\text{Aut}(A, \mathcal{A} \cap A, \mu|_A)$ with the weak topology.

Corollary 5.3.

- (1) *The subset of periodic transformations from $\text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$ is d_2 -dense in $\text{Aut}_2(X, \mathcal{A}, \mu)$.*
- (2) *The subset of conservative transformations is a dense G_δ -subset of $\text{Aut}_2(X, \mathcal{A}, \mu)$.*

Proof. (1) By [23] and the metric isomorphism mentioned above, for every subset $A \in \mathcal{A}$ of finite measure, the subset of periodic transformations in $\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$ is d_2 -dense in $\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$. This implies that the subset of periodic transformations is d_2 -dense in $\text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$. The claim (1) follows from this and Proposition 5.2.

(2) Since d_2 is stronger than the weak topology and the subset of conservative transformations is a G_δ in $\text{Aut}(X, \mathcal{A}, \mu)$, it follows that the subset

of conservative transformations is a G_δ in $\text{Aut}_2(X, \mathcal{A}, \mu)$. As every periodic transformation is conservative, we deduce from (1) that subset of conservative transformations is d_2 -dense in $\text{Aut}_2(X, \mathcal{A}, \mu)$. \square

We say that a transformation $T \in \text{Aut}_0(X, \mathcal{A}, \mu)$ is *locally aperiodic* if there exists $A \in \mathcal{A}$ of positive finite measure such that $T \in \text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$ and $T|_A$ is aperiodic.

Proposition 5.4. *Let $T \in \text{Aut}_0(X, \mathcal{A}, \mu)$ be locally aperiodic. Then the conjugacy class of T is d_2 -dense in $\text{Aut}_2(X, \mathcal{A}, \mu)$. In particular, $\text{Aut}_2(X, \mathcal{A}, \mu)$ has the Rokhlin property.*

Proof. In view of Proposition 5.2, it is enough to show that each local transformation is in the d_2 -closure of the conjugacy class of T . Select a subset $A \in \mathcal{B}$ of positive finite measure such that $T|_A$ is aperiodic. It follows from [9, Theorem 2] that $\{S^{-1}TS : S \in \text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)\}$ is d_2 -dense in $\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A)$. Since for every subset $B \subset X$ of positive finite measure, the map

$$\text{Aut}_2(A, \mathcal{A} \cap A, \mu|_A) \ni T \mapsto (\tau_{B,A})^{-1} T \tau_{B,A} \in \text{Aut}_2(B, \mathcal{A} \cap B, \mu|_B)$$

is a topological group isomorphism and $\tau_{B,A} \in \text{Aut}_2(X, \mathcal{A}, \mu)$, every local transformation is in the d_2 -closure of the conjugacy class of T . \square

We now state one of the main result of this section.

Theorem 5.5. *The subset*

$$\text{Erg}_2^{\text{III}_1}(X, \mathcal{A}, \mu) := \{T \in \text{Aut}_2(X, \mathcal{A}, \mu) : T \text{ is ergodic of type III}_1\}$$

is a dense G_δ in $\text{Aut}_2(X, \mathcal{A}, \mu)$.

The proof of Theorem 5.5 relies on the method of inducing which we now describe. Given a transformation $T \in \text{Aut}(X, \mathcal{A}, \mu)$, a subset $A \in \mathcal{A}$ is called *T -sweeping out* if $\mu(X \setminus \bigcup_{n=1}^\infty T^{-n}A) = 0$. If T is ergodic then each subset $A \in \mathcal{A}$ of positive measure is T -sweeping out. Given a T -sweeping out subset A , we can define the *induced map*, also known as the first return map, to A as a nonsingular transformation T_A of the space $(A, \mathcal{A} \cap A, \mu|_A)$ defined on a full measure subset of A by

$$T_A x := T^{\varphi_A(x)} x,$$

where $\varphi_A(x) := \inf \{n \in \mathbb{N} : T^n x \in A\}$ is the *first return time function* to A . We note T_A is well defined because A is T -sweeping out. The following facts will be used in the proof of Theorem 5.5:

- If A is T -sweeping out then T is ergodic if and only if T_A is ergodic [1, Proposition 1.5.2].
- If T is ergodic and A is of positive measure then T is of type III_1 if and only if T_A is of type III_1 .⁴

⁴This can be proved via inducing in the Maharam extension to the subset $A \times \mathbb{R}$ and noticing that A is T -sweeping out if and only if $A \times \mathbb{R}$ is sweeping out for the Maharam extension of T .

Proof of Theorem 5.5. Since $\text{Erg}^{\text{III}_1}(X, \mathcal{A}, \mu)$ is a G_δ in $\text{Aut}(X, \mathcal{A}, \mu)$ and d_2 is stronger than the weak topology, it follows that $\text{Erg}_2^{\text{III}_1}(X, \mathcal{A}, \mu)$ is a G_δ in $\text{Aut}_2(X, \mathcal{A}, \mu)$. It remains to show that $\text{Erg}_2^{\text{III}_1}(X, \mathcal{A}, \mu)$ is dense in $\text{Aut}_2(X, \mathcal{A}, \mu)$. For that, take a local transformation $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$. Our purpose is to find a sequence of transformations from $\text{Erg}_2^{\text{III}_1}(X, \mathcal{A}, \mu)$ that converges to T in d_2 and apply Proposition 5.2. Let A be a subset of finite measure such that $T|_{X \setminus A} = \text{id}_{X \setminus A}$. Take a sequence $\{B_n\}_{n=1}^\infty$ of subsets of finite measure and a sequence $\{\mathcal{P}_n\}_{n=1}^\infty$ of countable partitions of $X \setminus A$ into subsets of finite measure such that the following are satisfied:

- $A \subset B_n \subset B_{n-1}$, $\mu(B_n \setminus A) > 0$ for each n and $\lim_{n \rightarrow \infty} \mu(B_n) = \mu(A)$,
- $\mathcal{P}_n = \{p_{l,j}^{(n)} : l \in \mathbb{N}, 1 \leq j \leq J_l^{(n)}\}$ for some $J_l^{(n)} > n$ for all $n, l > 0$,
- $\mu(p_{l,1}^{(n)}) = \mu(p_{l,2}^{(n)}) = \cdots = \mu(p_{l,J_l^{(n)}}^{(n)})$ for all $n, l > 0$,
- $B_n \setminus A = \bigsqcup_{l=1}^\infty p_{l,1}^{(n)}$,
- if $\mathcal{Q}_n := \left\{ \bigsqcup_{j=1}^{J_l^{(n)}} p_{l,j}^{(n)} : l \in \mathbb{N} \right\}$ then $\mathcal{Q}_1 \prec \mathcal{Q}_2 \prec \cdots$ and $\mathcal{Q}_n \rightarrow \mathcal{A}|_{X \setminus A}$ as $n \rightarrow \infty$.

Fix also a sequence $\{S_n\}_{n=1}^\infty$ of transformations $S_n \in \text{Erg}^{\text{III}_1}(B_n, \mathcal{A} \cap B_n, \mu|_{B_n})$ that weakly converges to $T|_A$. By this we mean that $\|U_{S_n} f - U_T f\|_2 \rightarrow 0$ as $n \rightarrow \infty$ for each $f \in L^2(A)$. We now can construct, for each $n > 0$, a nonsingular transformation T_n of X satisfying the following conditions:

- $T_n x = S_n x$ for all $x \in A$,
- $T_n p_{l,j}^{(n)} = \begin{cases} p_{l,j+1}^{(n)}, & \text{if } j \neq J_l^{(n)} \\ S_n p_{l,1}^{(n)}, & \text{if } j = J_l^{(n)} \end{cases}$
- $T_n'(x) = 1$ for each $x \notin \bigsqcup_{l=1}^\infty S_n p_{l,1}^{(n)}$.

It follows straightforwardly from the definition of T_n that B_n is T_n -sweeping out and S_n is induced by T_n . Hence T_n is ergodic of type III_1 for each $n \in \mathbb{N}$. Of course, $T_n \in \text{Aut}_2(X, \mathcal{A}, \mu)$. We claim that $T_n \rightarrow T$ in d_2 as $n \rightarrow \infty$.

Take an atom q of \mathcal{Q}_n . Then $q = \bigsqcup_{j=1}^{J_l^{(n)}} p_{l,j}^{(n)}$ for some $l > 0$ and

$$\|U_T 1_q - U_{T_n} 1_q\|_2 = \left\| 1_{p_{l,1}^{(n)}} - U_{T_n} 1_{p_{l,J_l^{(n)}}^{(n)}} \right\|_2 \leq 2 \left\| 1_{p_{l,1}^{(n)}} \right\|_2 = \frac{2 \|1_q\|_2}{\sqrt{J_l^{(n)}}}.$$

Hence for each function $f \in L^2(X \setminus A)$ which is \mathcal{Q}_m -measurable for some $m > 0$, we have that

$$\|U_T f - U_{T_n} f\|_2 \leq \frac{2 \|f\|_2}{\sqrt{n}}$$

whenever $n \geq m$. Hence $U_{T_n} f \rightarrow U_T f$ as $n \rightarrow \infty$. Since $\mathcal{Q}_n \rightarrow \mathcal{A}|_{X \setminus A}$ as $n \rightarrow \infty$, it follows that $U_{T_n} f \rightarrow U_T f$ as $n \rightarrow \infty$ for each function $f \in L^2(X \setminus A)$. On the other hand, $U_{T_n} g = U_{S_n} g$ for each $g \in L^2(A)$ and $U_{S_n} g \rightarrow U_T g$ weakly as $n \rightarrow \infty$. We deduce that $T_n \rightarrow T$ weakly as $n \rightarrow \infty$.

Since $T'_n(x) \neq 1$ only if $x \in S_n A \sqcup \bigsqcup_{l=1}^{\infty} S_n p_{l,1}^{(n)} = S_n B_n = B_n$, we obtain that

$$\begin{aligned} \|\sqrt{T'} - \sqrt{T'_n}\|_2 &= \|(\sqrt{T'} - \sqrt{T'_n})1_{B_n}\|_2 \\ &= \|U_T 1_{B_n} - U_{T_n} 1_{B_n} + \sqrt{T'_n}(1_{T_n B_n} - 1_{B_n})\|_2 \\ &\leq \|(U_T - U_{T_n})1_{B_n}\|_2 + \sqrt{\mu(B_n \Delta T_n^{-1} B_n)} \end{aligned}$$

and $\mu(B_n \Delta T_n^{-1} B_n) = \mu\left(\left(\bigsqcup_{l=1}^{\infty} p_{l,1}^{(n)}\right) \Delta \left(\bigsqcup_{l=1}^{\infty} p_{l,J_l^{(n)}}^{(n)}\right)\right) = 2\mu(B_n \setminus A) \rightarrow 0$ as $n \rightarrow \infty$. It follows that $\|\sqrt{T'} - \sqrt{T'_n}\|_2 \rightarrow 0$, as desired. \square

Remark 5.6. Let B be a subset of positive finite measure in the standard σ -finite non-atomic measure space (X, \mathcal{A}, μ) . Arguing as in the proof of the above theorem, we see that given an ergodic nonsingular transformation S of $(B, \mathcal{A} \cap B, \mu|_B)$, we can construct an ergodic transformation $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$ such that $S = T_B$. It is well known that given an ergodic nonsingular flow W (i.e. an \mathbb{R} -action $(W(t))_{t \in \mathbb{R}}$), there is an ergodic nonsingular transformation whose associated flow is isomorphic to W . On the other hand, the associated flow of T is isomorphic to the associated flow of each transformation induced by T . It follows from these facts that given an ergodic nonsingular flow W , there is $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$ such that the associated flow of T is W . In particular, for each $\lambda \in [0, 1]$, the group $\text{Aut}_2(X, \mathcal{A}, \mu)$ contains an ergodic transformation of Krieger's type III_λ . For the definition of the associated flow, Krieger's type and other concepts of orbit theory we refer to [16].

The following assertion is an analogue of Proposition 5.2 for $\text{Aut}_1(X, \mathcal{A}, \mu)$ furnished with d_1 . It can not hold for the entire group $\text{Aut}_1(X, \mathcal{A}, \mu)$ because $\text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$ is a subgroup of the proper closed subgroup $\text{Ker } \chi$ of $\text{Aut}_1(X, \mathcal{A}, \mu)$. However, it holds for $\text{Ker } \chi$.

Proposition 5.7. $\text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$ is d_1 -dense in $\text{Ker } \chi$.

Proof. Let $T \in \text{Ker } \chi$. Then there exists an increasing sequence $A_n \in \mathcal{A}$ of finite measure subsets satisfying $\bigcup_{n=1}^{\infty} A_n = X$,

$$\int_{X \setminus A_n} \left| \frac{d\mu \circ T^{-1}}{d\mu} - 1 \right| d\mu := \epsilon_n \xrightarrow{n \rightarrow \infty} 0.$$

and

$$|\mu(T^{-1}A_n) - \mu(A_n)| < \epsilon_n.$$

Select subsets $B_n \subset X$ of finite measure such that $A_n \cup T^{-1}A_n \subset B_n$. As in the proof of Proposition 5.2, we define a transformation $T_n \in \text{Aut}_{\hat{0}}(X, \mathcal{A}, \mu)$ by setting

$$T_n^{-1}x := \begin{cases} T^{-1}x, & x \in A_n \\ \tau_{B_n \setminus A_n, B_n \setminus T^{-1}A_n} x, & x \in B_n \setminus A_n \\ x, & x \notin B_n. \end{cases}$$

We will now show that T_n converges in d_1 to T as $n \rightarrow \infty$. Since $T_n^{-1}x = T^{-1}x$ for $x \in A_n$, it follows that T_n converges weakly to T . Next, as in the proof of Proposition 5.2, we see that (5.1) holds and hence

$$\begin{aligned} \left\| \frac{dm \circ T^{-1}}{d\mu} - \frac{d\mu \circ T_n^{-1}}{d\mu} \right\|_1 &\leq \int_{X \setminus A_n} \left| \frac{d\mu \circ T^{-1}}{d\mu} - 1 \right| d\mu \\ &\quad + \int \left| \frac{\mu(B_n \setminus T^{-1}A_n)}{\mu(B_n \setminus A_n)} - 1 \right| d\mu \\ &\leq \epsilon_n + |\mu(T^{-1}A_n) - \mu(A_n)| \leq 2\epsilon_n. \end{aligned}$$

□

Proceeding from here in a similar way as we have done for $\text{Aut}_2(X, \mathcal{A}, \mu)$ but utilizing Proposition 5.7 instead of Proposition 5.2, we arrive at the following analogues of Corollary 5.3, Proposition 5.4 and Theorem 5.5 for $\text{Ker } \chi$.

Theorem 5.8.

- (1) *The subset of periodic transformations from $\text{Aut}_{\bar{0}}(X, \mathcal{A}, \mu)$ is a d_1 -dense subset of $\text{Ker } \chi$.*
- (2) *For each locally aperiodic transformation T , the conjugacy class of T (in $\text{Aut}_1(X, \mathcal{A}, \mu)$) is a d_1 -dense subset of $\text{Ker } \chi$. In particular, $\text{Ker } \chi$ has the Rokhlin property.*
- (3) *The following three sets:*

$$\begin{aligned} &\{T \in \text{Aut}_1(X, \mathcal{A}, \mu) : T \text{ is conservative}\}, \\ &\{T \in \text{Aut}_1(X, \mathcal{A}, \mu) : T \text{ is ergodic}\} \text{ and} \\ &\{T \in \text{Aut}_1(X, \mathcal{A}, \mu) : T \text{ is of type III}_1\} \end{aligned}$$

are dense G_δ -subsets of $\text{Ker } \chi$ endowed with d_1 .

6. BASIC DYNAMICAL PROPERTIES OF POISSON SUSPENSIONS FOR LOCALLY COMPACT GROUP ACTIONS

6.1. Unitary and affine Koopman representations. Let G be a locally compact non-compact second countable group. A nonsingular G -action on a standard σ -finite measure space (X, \mathcal{A}, μ) is a Borel map

$$G \times X \ni (g, x) \mapsto T_g x \in X$$

such that the mapping $T_g : X \ni x \mapsto T_g x \in X$ is a μ -nonsingular bijection of X for each $g \in G$. Equivalently, a nonsingular G -action can be defined as a continuous group homomorphism $T : G \ni g \mapsto T_g \in \text{Aut}(X, \mathcal{A}, \mu)$, where the later group is furnished with the weak topology. The corresponding unitary Koopman representation $G \ni g \mapsto U_{T_g} \in \mathcal{U}_{\mathbb{R}}(L^2(\mu))$ of G is continuous in the weak (and strong) operator topology. Since the real Hilbert space $L^2_{\mathbb{R}}(\mu)$ is invariant under the unitary Koopman representation, we can consider $L^2_{\mathbb{R}}(\mu)$ as a G -module. Denote by $Z^1(G, L^2_{\mathbb{R}}(\mu))$ the vector space of all continuous

1-cocycles of G in $L^2_{\mathbb{R}}(\mu)$, i.e. the mappings $c : G \ni g \mapsto c(g) \in L^2_{\mathbb{R}}(\mu)$ such that $c(gh) = c(g) + U_{T_g}c(h)$ for all $g, h \in G$. By $B^1(G, L^2_{\mathbb{R}}(\mu))$ we denote the subspace of 1-coboundaries, i.e. those 1-cocycles $c \in Z^1(G, L^2_{\mathbb{R}}(\mu))$ for which there is $f \in L^2_{\mathbb{R}}(\mu)$ such that $c(g) = U_{T_g}f - f$. We also recall that a 1-cocycle $c \in Z^1(G, L^2_{\mathbb{R}}(\mu))$ is called *proper* if $\|c(g)\|_2 \rightarrow \infty$ as $g \rightarrow \infty$.

Suppose now that we are given a nonsingular G -action T such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ for each $g \in G$. Since the d_2 -topology is stronger than the weak topology, it follows that the restriction (to $\text{Aut}_2(X, \mathcal{A}, \mu)$) of the Borel structure generated by the weak topology coincides with the Borel structure generated by d_2 . Hence the map T considered as a homomorphism from G to $\text{Aut}_2(X, \mathcal{A}, \mu)$ is d_2 -Borel. Since each Borel homomorphism from a Polish group to another Polish group is continuous, we obtain that T is continuous as a map from G to $\text{Aut}_2(X, \mathcal{A}, \mu)$ furnished with d_2 . We recall that $\text{Aut}_2(X, \mathcal{A}, \mu)$ embeds continuously into $\text{Aff}_{\mathbb{R}}(L^2(\mu))$ via the affine Koopman representation $A^{(2)}$ (see Definition 4.5). Thus, we obtain the following proposition.

Proposition 6.1. *If T is a nonsingular G -action such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ for each $g \in G$ then the 1-cocycle*

$$c_T : G \ni g \mapsto c_T(g) := \sqrt{T'_g} - 1 \in L^2_{\mathbb{R}}(\mu)$$

of G in $L^2_{\mathbb{R}}(\mu)$ is continuous.

Thus, $c_T \in Z^1(G, L^2_{\mathbb{R}}(\mu))$. It follows that under the condition of the above proposition, a (weakly) continuous affine representation $A_T : G \ni g \mapsto A_{T_g} \in \text{Aff}_{\mathbb{R}}(L^2(\mu))$ of G in $L^2(\mu)$ is well defined by the restriction of $A^{(2)}$ to G , i.e.

$$A_{T_g}h := U_{T_g}h + c_T(g).$$

Definition 6.2. We call A_T the *affine Koopman* representation of G generated by T .

In the next proposition we compare the property to have a fixed vector for the unitary and affine Koopman representations.

Proposition 6.3.

- (1) *The unitary Koopman representation has a non-trivial fixed vector if and only if T admits a non-trivial absolutely continuous invariant probability measure.*
- (2) *The affine Koopman representation has a fixed vector if and only if T admits an absolutely continuous invariant measure belonging to $\mathcal{M}^+_{\mu,2}$.*

Proof. (1) is standard. We leave its proof to the reader.

- (2) If there is $h \in L^2(\mu)$ such that $A_g h = h$ for all $g \in G$ then

$$\sqrt{T'_g}(h \circ T_g^{-1} + 1) = h + 1.$$

We now define a measure ν which is absolutely continuous with respect to μ and such that $\frac{d\nu}{d\mu} := (h+1)^2$. Then ν is invariant under T . Since $\sqrt{\frac{d\nu}{d\mu}} - 1 = |h+1| - 1 \in L^2_{\mathbb{R}}(\mu)$, it follows that $\nu \in \mathcal{M}_{\mu,2}^+$, as desired. The converse assertion is proved in a similar way by “reversing” the argument. \square

6.2. Existence of an equivalent probability measure. We now examine when the Poisson suspension $T_* := \{(T_g)_*\}_{g \in G}$ of T admits a μ^* -absolutely continuous invariant probability measure.

Proposition 6.4. *Let $T := \{T_g\}_{g \in G}$ be a nonsingular G -action such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ for every $g \in G$. Then the following assertions are equivalent:*

- (1) *The 1-cocycle c_T is bounded, i.e. there is $d > 0$ such that $\|c_T(g)\|_2 \leq d$ for each $g \in G$.*
- (2) *$c_T \in B^1(G, L^2_{\mathbb{R}}(\mu))$.*
- (3) *There exists a T -invariant measure $\nu \in \mathcal{M}_{\mu,2}^+$ (hence the probability measure ν^* is T_* -invariant and $\nu^* \ll \mu^*$).*
- (4) *There exists a T_* -invariant probability measure $\rho \ll \mu^*$.*

Proof. (1) \iff (2) is classical, see [6, Proposition 2.2.9], for a proof.

(2) \implies (3) because if $c_T(g) = U_{T_g}h - h$ for some $h \in L^2_{\mathbb{R}}(\mu)$ and all $g \in G$ then h is a fixed vector for the affine Koopman representation of G generated by T . It remains to apply Proposition 6.3(2).

(3) \implies (4) is obvious if we set $\rho := \nu^*$.

(4) \implies (1) It follows from (4.5) that for each $g \in G$,

$$\sqrt{\mathcal{E}(T'_g - 1)} = e^{-\frac{1}{2}\|c_T(g)\|_2^2} \mathcal{E}(\sqrt{T'_g} - 1).$$

Integrating this equality and using Corollary 4.1 we obtain that

$$(6.1) \quad \langle U_{(T_g)_*}1, 1 \rangle_{L^2(\mu^*)} = e^{-\frac{1}{2}\|c_T(g)\|_2^2}.$$

Therefore if c_T were unbounded then there would exist a sequence $\{g_n\}_{n=1}^{\infty}$ of elements of G such that $\langle U_{(T_{g_n})_*}1, 1 \rangle_{L^2(\mu^*)} \rightarrow 0$ as $n \rightarrow \infty$. Hence for all subsets A, B of finite measure in (X^*, μ^*) , we have that

$$\langle U_{(T_{g_n})_*}1_A, 1_B \rangle_{L^2(\mu^*)} \leq \langle U_{(T_{g_n})_*}1, 1 \rangle_{L^2(\mu^*)} \rightarrow 0 \quad (\text{as } n \rightarrow \infty).$$

This implies, in turn, that $U_{(T_{g_n})_*}h \rightarrow 0$ for each $h \in L^2(X^*, \mu^*)$ in the weak topology. However, taking $h := \sqrt{\frac{d\rho}{d\mu^*}}$, we obtain that $U_{(T_{g_n})_*}h = h$, a contradiction. \square

Corollary 6.5. *Assume that μ is infinite and T is ergodic. There exists a T_* -invariant probability measure $\rho \sim \mu^*$ if and only if there exists a T -invariant measure $\nu \in \mathcal{M}_{\mu,2}^{\circ,+}$. In this case $\rho = \nu^*$ and T_* is ergodic (and weakly mixing).*

Proof. The ergodicity of T_* when ν^* is invariant follows from the lack of T -invariant set of non-zero and finite ν -measure which is the classical ergodicity criteria in the measure preserving case. \square

6.3. Conservativeness and zero type for Poisson G -actions. We first recall some standard definitions for locally compact group actions.

Definition 6.6. Let $T = \{T_g\}_{g \in G}$ be a nonsingular G -action on (X, \mathcal{A}, μ) .

- T is called *conservative* if for each subset A of positive measure and a compact subgroup $K \subset G$, there is $g \in G \setminus K$ such that $\mu(A \cap T_g A) > 0$.
- T is called *dissipative* if it is not conservative.
- T is called *totally dissipative* if the restriction of T to every T -invariant subset of X is dissipative.
- T is called *of zero type* if $U_{T_g} \rightarrow 0$ as $g \rightarrow \infty$ in the weak operator topology.

In case $G = \mathbb{Z}$, these definitions for conservativeness and dissipativeness are equivalent to those given above. We will need the following lemma from [2, Proposition A.34].

Lemma 6.7. *Let λ be a left Haar measure on G . If there is $s \in \mathbb{R}$ such that $\int_G (T'_g)^s d\lambda(g) < \infty$ then T is totally dissipative.*

The next corollary follows from Lemma 6.7 and (6.1).

Corollary 6.8. *If $T = \{T_g\}_{g \in G}$ is a nonsingular G -action such that $T_g \in \text{Aut}_2(X, \mu)$ for each $g \in G$ and*

$$\int_G e^{-\frac{1}{2} \|c_T(g)\|_2^2} d\lambda(g) < \infty$$

then T_ is totally dissipative.*

Let $T_g \in \text{Aut}_2(X, \mu)$ for all $g \in G$ and let $A_T = \{A_{T_g}\}_{g \in G}$ stand for the affine Koopman representation of G generated by T .

Proposition 6.9. *T_* is of zero type if and only if c_T is proper.*

Proof. We first note that T_* is of zero type if and only if $\langle U_{(T_g)_*} 1, 1 \rangle \rightarrow 0$ as $g \rightarrow \infty$. This fact was shown (for arbitrary nonsingular G -actions) in the proof of Proposition 6.4. It remains to apply (6.1). \square

A class of groups *having property* (BP₀) was introduced in [18]. This class includes groups with property (T), solvable groups (in particular, Abelian groups), connected Lie groups, linear algebraic groups over a local field of characteristic zero, etc. We need only the following fact: if G has property (BP₀) and T is of zero type then c_T is either proper or bounded [18]. This fact and Propositions 6.9 and 6.4 imply the next corollary.

Corollary 6.10. *Let G have property (BP₀). Let $T = \{T_g\}_{g \in G}$ be a non-singular G -action such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ for every $g \in G$. Suppose that there is no any T -invariant measure in $\mathcal{M}_{\mu,2}^+$. If T is of zero type then T_* is of zero type.*

7. FURSTENBERG ENTROPY AND STATIONARITY

7.1. Furstenberg entropy. Let G be a locally compact group second countable group and let κ be a generating probability measure on G (i.e. the support of κ generates a dense subgroup of G). The Furstenberg κ -entropy of a non-singular action $S = (S_g)_{g \in G}$ on a probability space (Y, \mathcal{B}, ν) is the quantity

$$h_\kappa(S, \nu) := - \int_G \left(\int_Y \log S'_g d\nu \right) d\kappa(g) \in [0, +\infty].$$

We note that $h_\kappa(S, \nu) = 0$ if and only if S preserves ν . We will need the following corollary from Theorems 3.4 and 3.6.

Corollary 7.1. *Let T be in $\text{Aut}_2(X, \mathcal{A}, \mu)$. We have that*

$$\mathbb{E}_{\mu^*}[\log(T_*)'] = - \int_X (T' - 1 - \log T') d\mu,$$

it is finite if and only if $\int_{\{x \in X: |\log T'(x)| > 1\}} |\log T'| d\mu < \infty$. In particular, if $T \in \text{Aut}_1(X, \mathcal{A}, \mu)$, then $\int_X \log T' d\mu$ is well defined and takes its value in the extended interval $[-\infty, \chi(T)]$. We get in this case:

$$\mathbb{E}_{\mu^*}[\log(T_*)'] = -\chi(T) + \int_X \log T' d\mu.$$

Given $c > 0$, we denote by μ_c the c -scaling of μ , i.e. $\mu_c(B) = c\mu(B)$ for each $B \in \mathcal{A}$.

Corollary 7.2. *Let T be in $\text{Aut}_2(X, \mathcal{A}, \mu)$. Then for each $c > 0$, the following formula holds:*

$$\mathbb{E}_{\mu_c^*} \left[\log \frac{d\mu_c^* \circ T_*^{-1}}{d\mu_c^*} \right] = c \mathbb{E}_{\mu^*} \left[\log \frac{d\mu^* \circ T_*^{-1}}{d\mu^*} \right].$$

Proof. Since $\frac{d\mu_c \circ T^{-1}}{d\mu_c} = \frac{d\mu \circ T^{-1}}{d\mu}$, we deduce from Corollary 7.1 that

$$\begin{aligned} \mathbb{E}_{\mu_c^*} \left[\log \frac{d\mu_c^* \circ T_*^{-1}}{d\mu_c^*} \right] &= -c \int_X \left(\frac{d\mu \circ T^{-1}}{d\mu} - 1 - \log \frac{d\mu \circ T^{-1}}{d\mu} \right) d\mu \\ &= c \mathbb{E}_{\mu^*} \left[\log \frac{d\mu^* \circ T_*^{-1}}{d\mu^*} \right]. \end{aligned}$$

□

In view of the above results, we obtain the following.

Proposition 7.3. *Let $T = (T_g)_{g \in G}$ be a non-singular G -action on (X, \mathcal{A}, μ) such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ for each $g \in G$. Then:*

$$h_\kappa(T_*, \mu^*) = - \int_G \left(\int_X (\log T'_g - T'_g + 1) d\mu \right) d\kappa(g).$$

Moreover $h_\kappa(T_*, \mu_c^*) = ch_\kappa(T_*, \mu^*)$, for any $c > 0$. If $T_g \in \text{Aut}_1(X, \mathcal{A}, \mu)$ for each $g \in G$ then

$$h_\kappa(T_*, \mu^*) = \int_G \left(\chi(T_g) - \int_X \log T'_g d\mu \right) d\kappa(g).$$

Corollary 7.4. *Let $T = (T_g)_{g \in G}$ be a non-singular G -action on (X, \mathcal{A}, μ) such that $T_g \in \text{Aut}_1(X, \mathcal{A}, \mu)$ for each $g \in G$. If one of the following conditions is satisfied:*

- (1) T_g is conservative for each $g \in G$,
- (2) κ is symmetric,

then we have

$$h_\kappa(T_*, \mu^*) = - \int_G \left(\int_X \log T'_g d\mu \right) d\kappa(g).$$

Proof. The first point follows from Proposition 4.13. For the second, take an increasing sequence of compact symmetric sets $\{A_n\}_{n \in \mathbb{N}}$ such that $G = \bigcup_{n \in \mathbb{N}} A_n$. Then

$$\int_G \left(\chi(T_g) - \int_X \log T'_g d\mu \right) d\kappa(g) = \lim_{n \rightarrow \infty} \int_{A_n} \left(\chi(T_g) - \int_X \log T'_g d\mu \right) d\kappa(g).$$

Since $G \ni g \mapsto \chi(T_g) \in \mathbb{R}$ is a continuous group homomorphism, it is integrable on A_n . Moreover, $\chi(T_g^{-1}) = -\chi(T_g)$ and hence, by the symmetry of κ and A_n , we obtain that $\int_{A_n} \chi(T_g) d\kappa(g) = 0$. The result follows. \square

7.2. Stationarity for Poisson suspensions. Let κ be a generating probability measure on G . We recall that a non-singular action $(S_g)_{g \in G}$ on a probability space (Y, \mathcal{B}, ν) is called κ -stationary if $\int_G \nu \circ S_g d\kappa(g) = \nu$.

Proposition 7.5. *Let $T = (T_g)_{g \in G}$ be a non-singular G -action on (X, \mathcal{A}, μ) such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ for each $g \in G$. The Poisson suspension $T_* = ((T_g)_*)_{g \in G}$ of T acting on the space $(X^*, \mathcal{A}^*, \mu^*)$ is stationary if and only if T_* preserves μ^* .*

Proof. Assume that T_* is κ -stationary. Then

$$\mu^* = \int_G \mu^* \circ (T_g)_* d\kappa(g) = \int_G (\mu \circ T_g)^* d\kappa(g).$$

Computing the intensity of the two sides of this equation we obtain that

$$(7.1) \quad \mu = \int_G \mu \circ T_g d\kappa(g).$$

Take a set $A \in \mathcal{A}$ with $0 < \mu(A) < +\infty$ and compute the void probabilities of A using μ^* and $\int_G (\mu \circ T_g)^* d\kappa(g)$. We obtain that

$$e^{-\mu(A)} = \int_G e^{-\mu \circ T_g(A)} d\kappa(g).$$

From this and the Jensen inequality we deduce that

$$\mu(A) = -\log \int_G e^{-\mu \circ T_g(A)} d\kappa(g) \leq \int_G \mu \circ T_g(A) d\kappa(g)$$

In view of (7.1) we see that the equality case takes place in the Jensen inequality. This happens if and only if $\mu(A) = \mu(T_g A)$ for κ -a.e. $g \in G$. Thus $\mu = \mu \circ T_g$ for κ -a.e. $g \in G$. Since κ is generating, μ is invariant under T_g for g belonging to a dense subgroup in G . Therefore $\mu = \mu \circ T_g$ for every $g \in G$. \square

8. PROPERTY (T), POISSON SUSPENSIONS AND FURSTENBERG ENTROPY

Let G be a locally compact second countable group. The group has Kazhdan property (T) if every unitary representation of G which has almost invariant vectors admits a non zero invariant vector. There are numerous equivalent characterizations of property (T), among them is that for every unitary representation of G every cocycle is a coboundary [6]. Theorem 8.2 is a new characterization of property (T) in terms of nonsingular Poisson suspensions.

Definition 8.1. A nonsingular action $R = \{R_g\}_{g \in G}$ of G is called *Poisson* if there is a σ -finite standard measure space (X, \mathcal{A}, μ) and a nonsingular action $T = \{T_g\}_{g \in G}$ on X such that $T_g \in \text{Aut}_2(X, \mathcal{A}, \mu)$ and the Poisson suspension $T_* = \{(T_g)_*\}_{g \in G}$ of T is isomorphic to R .

We establish the following characterization of property (T) in terms of nonsingular Poisson suspensions.

Theorem 8.2. G has property (T) if and only if each nonsingular Poisson G -action admits an absolutely continuous invariant probability measure.

Proof. (\implies) Let G has property (T) and let $R = \{R_g\}_{g \in G}$ be a nonsingular Poisson G -action. Then there is a σ -finite standard measure space (X, \mathcal{A}, μ) with $\mu(X) = \infty$ and a G -action $T : G \ni g \mapsto T_g \in \text{Aut}_2(X, \mu)$ such that the Poisson suspension $T_* = \{(T_g)_*\}_{g \in G}$ of T is isomorphic to R . By the Delorme-Guichardet theorem (see [6]), c_T is a coboundary. Hence there is $f \in L^2(X, \mu)$ such that

$$(8.1) \quad c_T(g) = f \circ T_g^{-1} \cdot \sqrt{T'_g} - f.$$

We define a measure ν on X be setting $\frac{d\nu}{d\mu} = (1 - f)^2$. Of course, $\nu(X) = \infty$ and $\nu \in \mathcal{M}_{\mu, 2}^+$. The equation (8.1) yields that ν is invariant under T_g . $\nu \in \mathcal{M}_{\mu, 2}^+$. Hence ν^* is a T_* -invariant probability absolutely continuous with respect to μ^* .

(\Leftarrow) Suppose now that G does not have property (T). Then there is a weakly mixing measure preserving action $S = \{S_g\}_{g \in G}$ on a standard probability space (X, \mathcal{B}, m) which is not strongly ergodic. The latter implies that there is a sequence $\{A_n\}_{n=1}^\infty$ of measurable subsets in X such that $m(A_n) = \frac{1}{2}$ for every $n \in \mathbb{N}$ and for each compact subset $K \subset G$,

$$\lim_{n \rightarrow \infty} \sup_{g \in K} m(A_n \Delta S_g A_n) = 0.$$

Fix an increasing sequence $\{K_n\}_{n=1}^\infty$ of symmetric compact subsets of G with $1 \in K_1$ and $G = \bigcup_{n=1}^\infty K_n$. By passing to a subsequence in $\{A_n\}_{n=1}^\infty$, we can assume without loss of generality that for all $n \in \mathbb{N}$ and $g \in K_n$,

$$(8.2) \quad m(A_n \Delta S_g A_n) \leq 2^{-n}.$$

Let $(\Omega, \mathcal{C}, \mathbb{P}) := (X^\mathbb{N}, \mathcal{B}^\mathbb{N}, m^\mathbb{N})$. We will consider the diagonal (measure preserving) action

$$T : G \ni g \mapsto T_g := S_g \times S_g \times \cdots \in \text{Aut}(\Omega, \mathcal{C}, \mathbb{P})$$

of G on Ω . Since S is weakly mixing, so is T . For each $n \in \mathbb{N}$, we let

$$B_n := \left\{ x = (x_j)_{j=1}^\infty \in \Omega : \forall j \in \left[\frac{n(n-1)}{2}, \frac{n(n+1)}{2} \right), x_j \in A_n \right\}.$$

Of course, $B_n \in \mathcal{C}$ and $\mathbb{P}(B_n) = 2^{-n}$. Now we define a function $F : \Omega \rightarrow [1, +\infty)$ by setting

$$F(x) = \sqrt{1 + \sum_{n=1}^\infty 2^n 1_{B_n}(x)}.$$

Since $\sum_{n=1}^\infty \mathbb{P}(B_n) < \infty$, it follows from the Borel-Cantelli Lemma that $F(x) < \infty$ at a.e. $x \in \Omega$. We note that $F \notin L^2(\mathbb{P})$.

Claim. We claim that $\|F^2 - F^2 \circ T_g\|_1 < \infty$ for each $g \in G$. To prove this inequality, we choose $N \in \mathbb{N}$ such that $g^{-1} \in K_n$ for all $n \geq N$. Then we have in view of (8.2):

$$(8.3) \quad \begin{aligned} \|1_{B_n} - 1_{B_n} \circ T_g\|_1 &= \mathbb{P}(B_n \Delta T_g^{-1} B_n) \\ &= \prod_{k=\frac{n(n-1)}{2}}^{\frac{n(n+1)}{2}-1} m(A_n \Delta S_g A_n) \leq 2^{-n^2} \end{aligned}$$

for all $n \geq N$. Hence

$$\begin{aligned} \|F^2 - F^2 \circ T_g\|_1 &\leq \sum_{n=1}^\infty 2^n \|1_{B_n} - 1_{B_n} \circ T_g\|_1 \\ &\leq \sum_{n=1}^{N-1} 2^n + \sum_{n=N}^\infty 2^n 2^{-n^2} < \infty, \end{aligned}$$

as claimed.

Now we define a new measure μ on (Ω, \mathcal{C}) by setting: $\mu \sim \mathbb{P}$ and $\frac{d\mu}{d\mathbb{P}} := F^2$. Then μ is σ -finite and $\mu(X) = \infty$. For each $g \in G$,

$$(8.4) \quad \int_X \left| \frac{d\mu \circ T_g}{d\mu} - 1 \right| d\mu = \int_X |F^2 \circ T_g - F^2| d\mathbb{P} < \infty$$

in view of Claim. Thus, $T_g \in \text{Aut}_1(\Omega, \mathcal{C}, \mu)$. Hence the nonsingular Poisson G -action $T_* = \{(T_g)_*\}_{g \in G}$ is well defined on $(\Omega^*, \mathcal{C}^*, \mu^*)$. If T_* had an invariant μ^* -absolutely continuous probability measure then by Proposition 6.4 there would exist an infinite T -invariant measure $\nu \in \mathcal{M}_{\mu, 2}^+$ action. This contradicts to the fact that T is ergodic and has an equivalent invariant probability measure. \square

We are interested now in the computation of the Furstenberg entropy of Poisson G -actions. For that, we first prove the following proposition.

Proposition 8.3. *Let (Ω, μ, T) be as in the proof of Theorem 8.2. Then $\log T'_g \in L^1(\Omega, \mathbb{P})$ for each $g \in G$.*

Proof. We first let $B_0 := \Omega$ and define a sequence $\{E_n\}_{n=0}^\infty$ of subsets of Ω by setting $E_n := B_n \setminus (\bigcup_{k=n+1}^\infty B_k)$. It follows from the definition of F^2 that

$$E_n = \{x \in \Omega : 2^n < F(x)^2 \leq 2^{n+1}\}.$$

Hence E_0, E_1, \dots form a countable partition of Ω . Fix $g \in G$ and note that

$$\int_\Omega \left| \log \frac{d\mu \circ T_g}{d\mu} \right| d\mu = \sum_{n=0}^\infty \int_{E_n} F^2 \left| \log \left(\frac{F^2 \circ T_g}{F^2} \right) \right| d\mathbb{P}.$$

We will show that the right hand side of this equality is finite. If for some $n \in \mathbb{N}$ and $x \in \Omega$, we have that $x \in D_{n,g} := E_n \cap T_g^{-1}(\bigcup_{k=n}^\infty E_k)$ then $F(T_g x)^2 > 2^n \geq F(x)^2/2$. Since $|\log y| \leq 2|y - 1|$ for each $y > 1/2$, we see that

$$\begin{aligned} \sum_{n=0}^\infty \int_{D_{n,g}} F^2 \left| \log \left(\frac{F^2 \circ T_g}{F^2} \right) \right| d\mathbb{P} &\leq \sum_{n=0}^\infty 2 \int_{D_{n,g}} F^2 \left| \frac{F^2 \circ T_g}{F^2} - 1 \right| d\mathbb{P} \\ &\leq 2 \sum_{n=0}^\infty \int_{E_n} |F^2 \circ T_g - F^2| d\mathbb{P} \\ &= 2 \|F^2 - F^2 \circ T_g\|_1 < \infty \end{aligned}$$

in view of (8.4). Now if $x \in E_n \setminus T_g^{-1}(\bigcup_{k=n}^\infty E_k)$ then

$$F^2(x) \left| \log \left(\frac{F(T_g x)^2}{F(x)^2} \right) \right| \leq 2^{n+1} \left| \log \left(2^{-(n+1)} \right) \right|.$$

As $E_n \subset B_n$ and $\bigcup_{k=n}^\infty E_k \supset B_n$, we obtain that $E_n \setminus T_g^{-1}(\bigcup_{k=n}^\infty E_k) \subset B_n \triangle T_g^{-1} B_n$. Let N be the smallest natural number such that $g^{-1} \in K_n$ for

all $n \geq N$ then,

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{E_n \setminus D_{n,g}} F^2 \left| \log \left(\frac{F^2 \circ T_g}{F^2} \right) \right| d\mathbb{P} &\leq \sum_{n=0}^{\infty} (n+1)2^{n+1} \int_{B_n \triangle T_g^{-1} B_n} d\mathbb{P} \\ &\leq \sum_{n=1}^{N-1} (n+1)2^{n+1} + \sum_{n=N}^{\infty} (n+1)2^{n+1}2^{-n^2}, \end{aligned}$$

where the latter bound follows from (8.3). We conclude that

$$\int_{\Omega} \left| \log \frac{d\mu \circ T_g}{d\mu} \right| d\mu < \infty,$$

as desired. \square

We can now prove the following result.

Theorem 8.4. *Let G do not have property (T) and let κ be a probability measure on G . Then there is a nonsingular G -action T on an infinite measure space (Ω, μ) such that the Poisson suspension T_* of T is μ^* -nonsingular and $\{h_{\kappa}(T_*, \mu_t^*) : t \in (0, +\infty)\} = (0, +\infty)$.*

Proof. For each $N > 0$, we let

$$C_N := \sum_{n=1}^{N-1} (n+3)2^{n+1} + \sum_{n=N}^{\infty} (n+2)2^{n+1}2^{-n^2}.$$

Then we choose an increasing sequence $(K_n)_{n=1}^{\infty}$ of compacts in G such that $\bigcup_{n=1}^{\infty} K_n = G$ and $\kappa(K_{n-1}^{-1}) > 1 - \frac{1}{2^n C_n}$ for each n . By K_n^{-1} we mean the subset $\{k^{-1} : k \in K_n\}$. Using this sequence, we construct the dynamical system (Ω, μ, T) exactly as in the proof of Theorem 8.2. It follows from the proof of Proposition 8.3 that

$$\sup_{g^{-1} \in K_N} \left| \int_{\Omega} \log \frac{d\mu \circ T_g}{d\mu} d\mu \right| < C_N$$

for each $N > 0$. Since T_g preserves a probability measure equivalent to μ , it follows that T_g is conservative for each $g \in G$. Therefore, by Proposition 7.3,

$$\begin{aligned} h_{\kappa}(T_*, \mu^*) &= \sum_{N=1}^{\infty} \int_{K_N^{-1} \setminus K_{N-1}^{-1}} \left(- \int_{\Omega} \log \frac{d\mu \circ T_g}{d\mu} d\mu \right) d\kappa(g) \\ &\leq \sum_{N=1}^{\infty} C_N \kappa(K_N^{-1} \setminus K_{N-1}^{-1}) \\ &\leq \sum_{N=1}^{\infty} \frac{1}{2^N} = 1. \end{aligned}$$

By Corollary 7.4,

$$\{h_{\kappa}(T_*, \mu_t^*) : t \in (0, +\infty)\} = \{th_{\kappa}(T_*, \mu^*) : 0 < t < +\infty\} = (0, +\infty).$$

\square

8.1. A remark on ergodicity of T_* . It is well known that if a transformation T preserves μ then T_* is ergodic if and only if there are no T -invariant sets of strictly positive finite measure. This is no longer true for general $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$. Moreover, using the techniques developed in this section, we show in the following example that even when T is ergodic, T_* can be non-ergodic.

Proposition 8.5. *There exists an ergodic transformation $T \in \text{Aut}_2(X, \mathcal{A}, \mu)$ such that T_* is totally dissipative.*

Proof. Fix a standard probability space (X, \mathcal{B}, m) and select a mixing m -preserving transformation S of X . Let $(\Omega, \mathcal{C}, \mathbb{P}) := (X^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}}, m^{\mathbb{N}})$ and let $T := S \times S \times \cdots$. Then T is an ergodic \mathbb{P} -preserving transformation of Ω . Suppose that we have a function $F : \Omega \rightarrow [0, +\infty)$ such that

- $F \notin L^2(\mathbb{P})$,
- $F - F \circ T^n \in L^2(\mathbb{P})$ for all $n \in \mathbb{Z}$ and
- $\sum_{n \in \mathbb{Z}} e^{-\frac{1}{2}\|F - F \circ T^n\|_2^2} < \infty$.

Then, as in the proof of Theorem 8.2, we define a new measure μ on (Ω, \mathcal{C}) by setting $\mu \sim \mathbb{P}$ and $\frac{d\mu}{d\mathbb{P}} := F^2$ to obtain that $T \in \text{Aut}_2(X^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}}, \mu)$. Moreover,

$$\sum_{n \in \mathbb{Z}} e^{-\frac{1}{2}\|\sqrt{(T^{-n})'} - 1\|_2^2} = \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2}\|F - F \circ T^n\|_2^2} < \infty.$$

Hence T_* is dissipative by Corollary 6.8 and the proposition is proved. Thus, it remains to build such an F . For that, we first apply the Rokhlin lemma to find a sequence $(A_n)_{n \in \mathbb{N}}$ of measurable subsets of X such that $m(A_n) = \frac{1}{2}$ and

$$(8.5) \quad m(A_n \Delta S^j A_n) < 2^{-n} \text{ whenever } |j| \leq n.$$

Then we construct inductively an increasing sequence $(l_n)_{n=1}^{\infty}$ of positive integers. Let $l_1 := 1$. Suppose that we have already defined l_n for some n . Since S is mixing, there exists $l_{n+1} > l_n$ such that if $|j| > l_{n+1}$ then

$$(8.6) \quad \begin{aligned} \|1_{A_{l_n}} - 1_{A_{l_n} \circ S^j}\|_2^2 &= 2(m(A_{l_n}) - m(A_{l_n} \cap S^j A_{l_n})) \\ &\geq 2 \left(m(A_{l_n}) - m(A_{l_n})^2 - \frac{1}{8} \right) = \frac{1}{4}. \end{aligned}$$

In this way we define $(l_n)_{n=1}^{\infty}$ entirely. For every $k > 1$, we now let

$$B_k := \left\{ w = (x_j)_{j=1}^{\infty} \in \Omega : x_j \in A_{l_{k-1}}, \text{ if } \sum_{i=1}^k l_i < j \leq \sum_{i=1}^{k+1} l_i \right\}$$

Note that $\mathbb{P}(B_k) = m(A_{l_{k-1}})^{l_{k+1}} = 2^{-l_{k+1}}$. If $l_{k-1} > |j|$ then in view of (8.5),

$$(8.7) \quad \mathbb{P}(B_k \Delta T^j B_k) = m(A_{l_{k-1}} \Delta S^j A_{l_{k-1}})^{l_{k+1}} \leq 2^{-l_{k-1} l_{k+1}}.$$

Since $\sum_{k=1}^{\infty} \mathbb{P}(B_k) < \infty$, it follows from the Borel-Cantelli lemma that the function $F := 1 + \sum_{k=1}^{\infty} 2^{l_{k+1}} l_{k+1} 1_{B_k}$ takes values in $[1, +\infty)$ almost everywhere. It is straightforward to verify that $\int_{\Omega} F d\mathbb{P} = +\infty$. Hence

$F \notin L^2(\mathbb{P})$. For each $n \in \mathbb{Z}$, we have that $F - F \circ T^n = \sum_{k=1}^{\infty} \Upsilon_{n,k}$, where $\Upsilon_{n,k} := 2^{l_{k+1}} l_{k+1} (1_{B_k} - 1_{B_k} \circ T^n)$. Since $\{\Upsilon_{n,k}\}_{k=1}^{\infty}$ is an independent sequence of centered (mean 0) random variables,

$$\|F - F \circ T^n\|_2^2 = \sum_{k=1}^{\infty} \|\Upsilon_{n,k}\|_2^2.$$

If $l_{k-1} > |n|$ then in view of (8.7),

$$\|\Upsilon_{n,k}\|_2^2 = 2^{2l_{k+1}} l_{k+1}^2 \mathbb{P}(B_k \triangle T^{-n} B_k) \leq 2^{2l_{k+1}} l_{k+1}^2 2^{-l_{k-1} l_{k+1}}.$$

As $l_k \geq 4$ for all $k > 3$ then $2^{2l_{k+1}} l_{k+1}^2 2^{-l_{k-1} l_{k+1}} = o(2^{-l_{k+1}})$. This implies that $\sum_{l_{k-1} > |n|} \|\Upsilon_{n,k}\|_2^2 < \infty$ and hence $F - F \circ T^n \in L^2(\mathbb{P})$.

On the other hand, if $l_k < |n| \leq l_{k+1}$ then

$$\begin{aligned} \|F - F \circ T^n\|_2^2 &\geq \|\Upsilon_{n,k}\|_2^2 \\ &= 4^{l_{k+1}} l_{k+1}^2 \left\| 1_{A_{l_{k-1}}} - 1_{A_{l_{k-1}}} \circ S^n \right\|_{L^2(m)}^{2l_{k+1}} \\ &\geq l_{k+1}^2 \end{aligned}$$

in view of (8.6). Hence

$$\begin{aligned} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{2} \|F - F \circ T^n\|_2^2} &\leq 3 + \sum_{k=1}^{\infty} \sum_{l_k < |n| \leq l_{k+1}} e^{-\frac{1}{2} \|F - F \circ T^n\|_2^2} \\ &\leq 3 + \sum_{k=1}^{\infty} 2(l_{k+1} - l_k) e^{-l_{k+1}^2/2} < \infty, \end{aligned}$$

as desired. \square

9. APPENDIX

9.1. Infinitely divisible random variables. We recall that a real valued random variable X defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *infinitely divisible* if the distribution p of X satisfies that, for each $k \geq 1$, there exists a probability distribution p_k on \mathbb{R} such that p is the k -th convolution power of p_k . Equivalently, there exists $\delta, \kappa \geq 0$ and a σ -finite Borel measure σ on \mathbb{R} satisfying the following conditions: $\sigma(\{0\}) = 0$, $\int_{\mathbb{R}} x^2 \wedge 1 d\sigma(x) < \infty$ and such that

$$\mathbb{E}[e^{iaX}] = \exp\left(-\frac{a^2 \kappa^2}{2} + ia\delta + \int_{\mathbb{R}} (e^{iax} - 1 - iax 1_{\{y \in \mathbb{R}: |y| \leq 1\}}(x)) d\sigma(x)\right)$$

The measure σ is called the *Lévy measure* of X . Conversely, each σ -finite measure σ such that $\sigma(\{0\}) = 0$ and $\int_{\mathbb{R}} x^2 \wedge 1 d\sigma < \infty(x)$ is the Lévy measure of some infinitely divisible random variable. For the proof of the following proposition we refer to [30, page 39, below formula (8.8)].

Proposition 9.1. *Let X be infinitely divisible. Then $X \in L^1(\mathbb{P})$ if and only if $\int_{\mathbb{R}} |x| 1_{\{y \in \mathbb{R}: |y| > 1\}}(x) d\sigma(x) < \infty$. In this case the characteristic function of X can be written as*

$$\mathbb{E}[e^{iaX}] = \exp\left(-\frac{a^2\kappa^2}{2} + ia\gamma + \int_{\mathbb{R}} (e^{iax} - 1 - iax) d\sigma(x)\right), \quad a \in \mathbb{R},$$

and $\mathbb{E}[X] = \gamma$.

9.2. Stochastic integrals against a Poisson measure. Let $(X^*, \mathcal{A}^*, \mu^*)$ be the Poisson space over a base (X, \mathcal{A}, μ) . If $f : X \rightarrow \mathbb{R}$ is a measurable function satisfying $\int_X f^2 \wedge 1 d\mu < \infty$ then a so-called *stochastic integral* $I_\mu(f) : X^* \rightarrow \mathbb{R}$ is well defined (see [26], up to a slight change) by the following formula

$$I_\mu(f)(\omega) = \lim_{\epsilon \rightarrow 0} \left(\int_{\{x \in X: |f(x)| > \epsilon\}} f d\omega - \int_{\{x \in X: |f(x)| > \epsilon\}} f 1_{\{x \in X: |f(x)| \leq 1\}} d\mu \right),$$

where the limit means the convergence in probability. It appears that the random variable $I_\mu(f)$ is infinitely divisible. The Lévy measure of $I_\mu(f)$ is $(\mu \circ f^{-1}) \upharpoonright \mathbb{R} \setminus \{0\}$. The characteristic function of $I_\mu(f)$ is

$$(9.1) \quad \mathbb{E}_{\mu^*}[e^{iaI_\mu(f)}] = \exp\left(\int_X (e^{iaf} - 1 - ia f 1_{\{x \in X: |f(x)| \leq 1\}}) d\mu\right).$$

for each $a \in \mathbb{R}$.

Proposition 9.2. *Let a function $f : X \rightarrow \mathbb{R}$ satisfy $\int_X f^2 \wedge 1 d\mu < \infty$. Then $I_\mu(f) \in L^1(\mu^*)$ if and only if $f 1_{\{x \in X: |f(x)| > 1\}} \in L^1(\mu)$. In this case $\mathbb{E}_{\mu^*}(I_\mu(f)) = \int_{\{x \in X: |f(x)| > 1\}} f d\mu$.*

Proof. The integrability criteria for $I_\mu(f)$ follows from Proposition 9.1 since $(\mu \circ f^{-1}) \upharpoonright \mathbb{R} \setminus \{0\}$ is the Lévy measure of $I_\mu(f)$. If the latter happens, we can rewrite (9.1) as

$$\mathbb{E}_{\mu^*}[e^{iaI_\mu(f)}] = \exp\left(ia \int_{\{x \in X: |f(x)| > 1\}} f d\mu + \int_X (e^{iaf} - 1 - ia f) d\mu\right).$$

By Proposition 9.1, $\mathbb{E}_{\mu^*}(I_\mu(f)) = \int_{\{x \in X: |f(x)| > 1\}} f d\mu$. □

REFERENCES

- [1] J. Aaronson. *An introduction to infinite ergodic theory*. American Mathematical Society, 1997.
- [2] Y. Arano, Y. Isono, and A. Marrakchi. Ergodic theory of affine isometric actions on Hilbert spaces. *arXiv:1911.04272*, November 2019.
- [3] S. Attal. Lectures in Quantum Noise Theory. <http://math.univ-lyon1.fr/~attal/chapters.html>.
- [4] H. Becker and Kechris A. S. *The descriptive set theory of Polish group actions*, volume 232 of *London Mathematical Society lecture note series*. 1996.
- [5] B. Bekka. Harmonic cocycles, von Neumann algebras, and irreducible affine isometric actions. *New York J. Math.*, 23:1205–1218, 2017.

- [6] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's property (T)*. Cambridge, 2008.
- [7] B. Bekka, T. Pillon, and A. Valette. Irreducible affine isometric actions on Hilbert spaces. *Münster J. Math.*, 9(1):1–34, 2016.
- [8] L. Bowen, Y. Hartman, and O. Tamuz. Property (T) and the Furstenberg entropy of nonsingular actions. *Proc. Amer. Math. Soc.*, 144(1):31–39, 2016.
- [9] J. R. Choksi and Shizuo Kakutani. Residuality of ergodic measurable transformations and of ergodic transformations which preserve an infinite measure. *Indiana Univ. Math. J.*, 28(3):453–469, 1979.
- [10] J.M Choksi, J.H. Hawkins, and V.S. Prasad. Abelian cocycles for nonsingular ergodic transformations and the genericity of type III₁ transformations. *Monatsh. Math.*, 103(3):187–205, 1987.
- [11] I.P. Cornfeld, S.V. Fomin, and Y. G. Sinaï. *Ergodic Theory*. Springer-Verlag, 1982.
- [12] A. I. Danilenko. A survey on spectral multiplicities of ergodic actions. *Ergodic Theory Dynam. Systems*, 33(1):81–117, 2013.
- [13] A. I. Danilenko. Furstenberg entropy values for nonsingular actions of groups without property (T). *Proc. Amer. Math. Soc.*, 145(3):1153–1161, 2017.
- [14] A. I. Danilenko, Z. Kosloff, and E. Roy. Generic nonsingular Poisson suspension is of type III₁. *arXiv:2002.05094*, February 2020.
- [15] A. I. Danilenko and V. Ryzhikov. On self-similarities of ergodic flows. *Proc. Lond. Math. Soc. (3)*, 104(3):431–454, 2012.
- [16] A. I. Danilenko and C. E. Silva. Ergodic Theory: Nonsingular Transformations. *arXiv:0803.2424*.
- [17] Y. de Cornulier, R. Tessera, and A. Valette. Isometric group actions on hilbert spaces: growth of cocycles. *Geom. Funct. Anal.*, 17(3):770–792, 2007.
- [18] Y. De Cornulier, R. Tessera, and A. Valette. Isometric group actions on banach spaces and representations vanishing at infinity. *Transform. Groups*, 13(1):125–147, 2008.
- [19] Y. Derriennic, K. Frączek, M. Lemańczyk, and F. Parreau. Ergodic automorphisms whose weak closure of off-diagonal measures consists of ergodic self-joinings. *Colloq. Math.*, 110:81–115, 2008.
- [20] H. Furstenberg and E. Glasner. Stationary dynamical systems. In *Dynamical numbers—interplay between dynamical systems and number theory*, volume 532 of *Contemp. Math.*, pages 1–28. Amer. Math. Soc., Providence, RI, 2010.
- [21] G. A. Goldin, J. Grodnik, R. T. Powers, and D. H. Sharp. Nonrelativistic current algebra in the N/V limit. *J. Mathematical Phys.*, 15:88–100, 1974.
- [22] A. Guichardet. *Symmetric Hilbert Spaces and Related Topics*, volume 261 of *Lecture Notes in Mathematics*. 1972.
- [23] A. Ionescu Tulcea. On the category of certain classes of transformations in ergodic theory. *Trans. Amer. Math. Soc.*, 114:261–279, 1965.
- [24] E. Janvresse, T. de la Rue, and E. Roy. Poisson Suspensions and SuShis. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(6):1301–1334, 2017.
- [25] E. Janvresse, T. Meyerovitch, T. de la Rue, and E. Roy. Poisson suspensions and entropy for infinite transformations. *Trans. Amer. Math. Soc.*, 362(6):3069–3094, 2010.
- [26] G. Maruyama. Infinitely divisible processes. *Theory Probab. Appl.*, 15(1):1–22, 1970.
- [27] Yu. A. Neretin. *Categories of symmetries and infinite-dimensional groups*. Oxford University press, 1996.
- [28] D. S. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.*, 48:1–147, 1987.
- [29] E. Roy. Poisson suspensions and infinite ergodic theory. *Ergodic Theory Dynam. Systems*, 29(2):667–683, 2009.
- [30] K.-I. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge University Press, 1999.
- [31] Y. Takahashi. Absolute continuity of Poisson random fields. *Publ. Res. Inst. Math. Sci.*, 26:629–647, 1990.

- [32] A. M. Vershik and M. I. Graev. A Poisson model of the Fock space and representations of current groups. *Algebra i Analiz*, 23(3):63–136, 2011.
- [33] A. M. Veršik, I. M. Gelfand, and M. I. Graev. Representations of the group of diffeomorphisms. *Uspehi Mat. Nauk*, 30(6(186)):1–50, 1975.

B. VERKIN INSTITUTE FOR LOW TEMPERATURE PHYSICS AND ENGINEERING OF NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 47 NAUKY AVE., KHARKIV, 61103, UKRAINE
Email address: alexandre.danilenko@gmail.com

EINSTEIN INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, EDMOND J. SAFRA CAMPUS, JERUSALEM 91904, ISRAEL
Email address: zemer.kosloff@mail.huji.ac.il

LABORATOIRE ANALYSE, GÉOMÉTRIE ET APPLICATIONS, CNRS UMR 7539, UNIVERSITÉ PARIS 13, INSTITUT GALILÉE, 99 AVENUE JEAN-BAPTISTE CLÉMENT F93430 VILLETANEUSE, FRANCE.
Email address: roy@math.univ-paris13.fr