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The «materialization» of forces: Why confounding mathematical concept and physical entity makes the design of metamaterials arduous

Francesco dell'Isola | Maximilian Stilz

In this paper we show why the postulation scheme based on forces and couples is detrimental when designing novel metamaterials. Instead, the most effective postulation scheme for mechanics is that based on the Principle of Virtual Work formulated by, for example, d'Alembert, Lagrange, Piola, and Paul Germain. In fact, generalized continuum mechanics can be formulated in a coherent way only by basing its foundations on the Principle of Virtual Work while the problem of synthesis of novel metamaterials can be efficiently confronted only when introducing generalized continua as their model at macroscopic level. Once work functionals are postulated, the set of involved generalized forces are identified using the representation theorem for distributions by Laurent Schwartz, forces in particular being zeroth order distributions. Actually, only when limiting the class of considered internal work functionals to first order distributions is the Principle of Virtual Work equivalent to the balances of forces and couples. The predominant Role of the Principle of Virtual Work has been denied by those inductivist scholars who claim that balance laws can be induced by experimentation. The Principle of Virtual Work postulation more truly fits in a falsificationist approach to epistemology.

1 INTRODUCTION

The main aim of this paper is to discuss the methodological approach to be used when designing novel metamaterials.

We want to show that the conception of novel metamaterials must be based on a theory-driven design process in which we can distinguish three main steps: (i) the characterization of the metamaterial desired properties by choosing the mathematical model that is assumed to govern the metamaterial behavior, (ii) the extensive study of the chosen mathematical model aiming to predict the metamaterial behavior in all physical situations of interest, and (iii) the synthesis of the possibly multiscale microstructure (micro-architecture) that, once homogenized, will realize at macro-level the searched metamaterial.

Since the very nature of the concept of force can vary among different postulation schemes in that force can be seen as a primitive or derivate concept, and considered that there exists a wide spectrum of applicable loads (or generalized forces) applicable on continua [5], it is of utmost importance to always specify what one means with (generalized) forces, by using a precise mathematical formalism. In fact, depending on the considered specific model used, the applicable loads (that is generalized forces acting on continua) could be, for instance, couples or double-forces per unit line, area or volume, as well as quantities dual to nonmechanical kinematical variables, like temperature or concentration of a solute in a medium.

In the more general setting (see Refs. [6–8]) of n th gradient continua applicable loads on their boundary can be k -forces (that is the dual in work of k th order normal derivatives of virtual displacements).

The theory of mechanical metamaterials and architected materials seem to us particularly suitable to highlight the importance of the discussion about the choice of the most suitable postulation scheme for mechanics. In this discussion, while clarifying the role of the various concepts introduced, it will become clear why and how the mental process which lead to the «materialization» of forces was formed and emerged. Indeed, the microstructure of such materials allows for the combination of «simpler forces» to produce, at macro-level, more complex ones. For instance, the microscopically homogeneous materials described by Cauchy continua cannot support distributed couples or double-forces [9–12] which, instead, can be supported by strain-gradient continua describing, at macro-level, microscopically heterogeneous materials [13–19]. The description of the mental process, which produced the materialization of forces, is also relevant when one wants to firmly found (especially if he needs to avoid ambiguities and contradictions in their formulation) the mathematical models suitable to describe and predict nonstandard generalized and desired exotic mechanical properties [20–25]. It is also relevant when one wants to solve the fundamental problem of the theory of metamaterials (also known as the synthesis problem), namely the problem of finding a microstructure which, once homogenized, is exhibiting the behavior demanded by the designer by choosing a specific macro-continuum model [26].

Some examples of solution of some synthesis problem are provided by recent research in pantographic metamaterials [27–34] and duoskelion structures [35, 36]: in fact in Alibert et al. [37]. Linear elastic trusses leading to continua with exotic mechanical interactions. In Seppecher et al. [38], the problem of synthesis of a beam whose deformation energy depends on the second gradient of axial displacement and on the third gradient of transverse displacement was at first solved.

The partial results obtained up to now must motivate the efforts for getting more general results. In fact, a suitable algorithm has to be found that allows for the calculation of a specific microstructure which, at the macro-scale, behaves as predicted by the specifically chosen *desired-a-priori* generalized continuum model.

The problem of determining such a synthesis algorithm is not new in mathematical physics: In the design of analog computers (see e.g., the pioneering works by Refs. [39–42]), it is demanded, given the equations for which the analog computer must produce solutions, to find an electric circuit that is governed by exactly the same equations.

The problem of synthesis of circuits did produce interesting mathematical problems (see e.g. [43–47]) and it is likely that will suggest methods to solve the problem of synthesis of metamaterial micro-architectures and micro-structures (about this subject and for some interesting results see Refs. [48, 49]).

1.1 | Balances of forces and torques are only necessary and not sufficient conditions for equilibrium

As proven by Gabrio Piola, the Principle of Virtual Work implies as necessary condition for equilibrium the balances of forces and torques. Already Piola (see dell’Isola et al. [50]) was aware of the fact that this necessary condition is not, for a generic n th gradient continuum, sufficient for equilibrium. However Cauchy, via his celebrated tetrahedron augment (based on some ad hoc assumptions among which we recall so-called Cauchy postulate), did prove that in the case of so-called first gradient continua (that is continua whose internal virtual work functional is a first order distribution), the balances of force and torques are indeed also sufficient for equilibrium. The Continuum Mechanics for first gradient continua has been developed based on this specific constitutive assumption for considered continua: While its range of applicability still is surprisingly wide, in the theory of metamaterials, it proves all its limits, as it excludes from the synthesis process some possibilities that may have very important technological applications. In the framework of first gradient continuum theories, sustainable external work functionals reduce to zeroth-order distributions, in which the dual quantities to virtual displacements are covectors to which the tradition (see below the relevant considerations made already by d’Alembert) uses the name «forces». It has to be remarked in this context that, trying to intrude the literature with a greater awareness about the importance of the Principle of Virtual Work, Lagrange called «powers» the aforementioned covectors. Unfortunately, the nomenclature proposed by Lagrange has not been accepted in the literature. As a consequence of both the refuse of Lagrangian nomenclature and the Cauchy development of the theory of a particular class of continua, the existence of generalized forces has often been denied. Instead, the scholars who developed generalized continuum theories are aware of the need of introducing external work functionals in which covectors dual to gradients of virtual displacement field appear (see Refs. [51–53]). These external work functionals are distributions of order greater than zero and do not appear in the balances of forces and torques, as their resultant forces and torques are always vanishing, albeit

their value greatly influences the equilibrium configurations. It is, therefore, clear that for generalized continua, balances of forces and torques cannot be sufficient to characterize equilibrium configurations.

1.2 | Exegesis of d'Alembert explanation of the concept of force

The fundamental work by d'Alembert “*Traité de dynamique*” has been very often misunderstood (see e.g., Refs. [54, 55]) for what concerns its content, its epistemological viewpoint and in the used methodology.

We believe that in his treaty, d'Alembert revives the ancient concepts that were placed at the basis of mechanics in the more ancient treatise *Μηχανικά Προβλήματα* (*Mēchanica Problēmata*) which, albeit having been attributed to Aristotle by ancient sources, has been attributed to Archytas of Tarentum (see Winter [56]). In the *Mēchanica Problēmata*, a certain number of mechanical problems arising in Engineering are fully solved by using systematically the Principle of Virtual Work. This work, dating to the 4th century BC, is the most ancient known text in which mechanical problems are studied using a mathematical formalism: in it nothing, which resembles the concept of force, can be found. In the present subsection, we will try to give what seems to be the correct interpretation to the epistemological opinions expressed by d'Alembert.

In his *Discours Préliminaire de L'Encyclopédie* [57], d'Alembert writes that

Les principes [...] se trouvent d'ailleurs réduits au plus petit nombre possible, & par ce moyen ils ne peuvent manquer d'acquérir en même tems plus d'étendue, puisque l'objet d'une Science étant nécessairement déterminé, les principes en sont d'autant plus féconds, qu'ils sont en plus petit nombre.

As, to our knowledge, there is not any available translation of d'Alembert's treaty into English, the translation of all the excerpts from d'Alembert into English is ours:

The principles [...] are found, moreover, to be reduced to the smallest possible number and because of this means they cannot avoid to acquire at the same time a bigger extent, as being the object of a Science necessarily determined, the principles are for it the more fruitful the more they are in smaller number.

D'Alembert shows in this statement to be fully aware of the need of using the Occam razor while building a theory aimed to describe the nature. At the same time, he prepares the reader to the unification of the mathematical theory for physical phenomena induced by the Principle of Virtual Work.

Later, in the same *Discours*, we can read

A l'égard des démonstrations de ces Principes en eux-mêmes, le plan que j'ai suivi pour leur donner toute la clarté & la simplicité dont elles m'ont paru susceptibles, a été de les déduire toujours de la considération seule du Mouvement, envisagé de la manière la plus simple & la plus claire. Tout ce que nous voyons bien distinctement dans le Mouvement d'un Corps, c'est qu'il parcourt un certain espace, & qu'il employe un certain temps à le parcourir.

For what concerns the demonstrations of these Principles themselves, the plan, which I have followed for giving them all the clarity and simplicity of which these [demonstrations] seemed to me to be susceptible, has been that of deducing them always only by the consideration of the Movement of one body, i.e. that it travels a certain space and that it needs some time to travel this space.

This statement clearly explains that d'Alembert places at the basis of mechanics the description of the movement in the geometrical space. In other words that he considers the kinematics to be at the basis of mechanics, as only kinematics clearly relates to experimental evidence. All various principles (that had been separately and independently postulated before d'Alembert) are deduced by d'Alembert from a unique Principle, as he clearly declares in his preface, by reducing them the simplest and clearest form based on their kinematical study. D'Alembert here proudly declares that he can «demonstrate» many principles starting from the kinematics of considered systems and applying the Principle of Virtual Work.

D'Alembert then continues by stating that

C'est donc de cette seule idée qu'on doit tirer tous les Principes de la Méchanique, quand on veut les démontrer d'une manière nette & précise; ainsi on ne sera point surpris qu'en conséquence de cette réflexion, j'ai, pour ainsi dire, détourné la vûe de dessus les causes motrices, pour n'envisager uniquement que le Mouvement qu'elles produisent; que j'aie entièrement proscrit les forces inhérentes au Corps en Mouvement, êtres obscurs & Métaphysiques, qui ne sont capables que de répandre les ténèbres sur une Science claire par elle-même

The translation of this excerpt can be

It is therefore from this only idea that one must deduce all the Principles of Mechanics, when one wants to show them in a precise and clear way; similarly one will not be surprised that, as a consequence of this reflection, I have, in a manner of speaking, diverted the view from the mover causes in order to envisage uniquely the Movement that they produce; [and that] I had completely proscribed the forces inherent to the Bodies in Movement, entities obscure and Metaphysical, which are only capable to spread the darkness on a Science that is clear by itself.

In this statement, d'Alembert clearly refuses to try to give a «physical» meaning to the concept of force. The ontology of forces is simple: They are mathematical objects invented while developing a mathematical theory and they do not have any immediate physical interpretation. Instead for the concepts introduced in kinematics, one can give immediate physical interpretation: Therefore to a displacement, one can attach a mathematical and a corresponding physical meaning. Nobody has ever experienced directly anything which can be called «a force».

Finally close to the end of the *Discours* we read

Au reste, comme cette seconde Partie est destinée principalement à ceux, qui déjà instruits du calcul différentiel & intégral, se seront rendus familiers les principes établis dans la première, ou seront déjà exercés à la solution des Problèmes connus & ordinaires de la Méchanique ; je dois avertir que pour éviter les circonlocutions, je me suis souvent servi du terme obscur de force, & de quelques autres qu'on employe communément quand on traite du mouvement des Corps ; mais je n'ai jamais prétendu attacher à ces termes d'autres idées que celles qui résultent des principes que j'ai établis, soit dans ce Discours, soit dans la première Partie de ce Traité.

whose translation is

On the other hand, as this second part is addressed mainly to those who being already learned in differential and integral calculus managed to become familiar with the principles established in the first one, **I must warn [these readers] that for avoiding the circumlocutions I have often used the obscure term “force”,** and some other terms which one commonly employs when he treats the motion of bodies; but I never wanted to attribute to these terms any other ideas [different] from those which result from the principles which I have established, either in this *Discours*, or in the first Part of this Treatise.

The ancient Greek nominalistic attitude towards mathematical language now comes back in the mathematical physics treatises. A force is a mathematical object, it is precisely defined in terms of the mathematical entities introduced in his theory: Therefore, d'Alembert uses it not as done by many others, that is, trying to explain the concept in terms of «phenomenological evidences», so transforming them into «entities obscure and Metaphysical». Instead he defines in his mathematical theories some concepts, in terms of the primitive concepts introduced as basis of such theories, and calls them «forces». Using these definitions, he speaks about forces exclusively as mathematical concepts and knows that, in this way, he avoids all possible misunderstanding occurring when one tries to understand forces as objects corresponding to «physical» objects in real world or to invoke «metaphysics» for giving an «operational» definition of forces.

1.3 | The importance of the Principle of Virtual Work in homogenization process

Our thesis is that the theory-driven synthesis process of novel metamaterials can be efficiently performed only when both at the micro-level (i.e., at the level in which the synthesized microstructure is modelled in full detail) and at macro-level

(i.e., at the level in which the collective behavior of the complex considered microstructure gives the desired properties of the novel metamaterial), the introduced descriptive mechanical models are built starting from the Principle of Virtual Work.

In fact, the most efficient heuristic homogenization methods ever conceived are based on the micro–macro identification based on the equality of micro-virtual work with macro virtual work (see e.g., Refs. [27, 58, 59]) once a micro–macro kinematical identification is accepted (by stating what is sometimes called Piola’s Ansatz, see Refs. [60–63]). The mentioned homogenization methods introduce a small parameter ε controlling the size of the elementary cell characterizing the microstructure and identifies the expression for virtual work valid at macro-level starting from the expression of virtual work at micro-level and replacing in it the expression of micro-placement as given by the micro–macro identification Ansatz: The leading term in ε will produce the expression for the virtual work functional valid at macro-level.

It has to be remarked that in many multiscale complex architectures, a sufficiently detailed description of micro-kinematics cannot be attained by simply using a macro-placement function. In fact, macro-placement function is capable to identify the micro-position of only some among the relevant micro-material particles: Some extra kinematical macro-descriptors may be needed in order to take into account at macro-level all the work expended at micro-level. Therefore to describe in an appropriate way the macro-behavior of complex multiscale microstructures, the description at macro-level must be richer than standard Continuum Mechanics (see Paul [3]) so that so-called generalized Continuum Mechanics [64–66] has to be formulated as target model in the process of micro–macro homogenization, also when at micro-level, standard continua or even standard discrete Lagrangian models are introduced.

In fact, it is very difficult to formulate macro-models starting from micro-models and simply using averaging methods. Instead, usually, heuristic homogenization is based on the conjecture of (i) a macro-kinematics (which includes together with the macro-placement also some extra kinematical descriptors), (ii) a micro–macro kinematical identification, which identifies a specific micro-configuration in correspondence of every macro-configuration, (iii) a class of macro-models, which represent the potential target models in the homogenization procedure, and (iv) an expression for the virtual work functionals postulated at micro-level. The heuristic homogenization is finally completed by equating the macro-work functionals calculated on the macro-virtual displacement with the micro-work functional calculated on the micro-virtual displacement corresponding to the macro-one as specified in the identification (ii).

Clearly a complete knowledge of the introduced class of macro-models is necessary to proceed to get a satisfactory micro–macro identification.

Only after having obtained, via a formal asymptotic expansion, the micro-macro-models identification one can try to prove mathematically rigorous results (see e.g., Abdoul-Anziz et al. [67]).

1.4 | Balance law postulations are not suited for micro–macro identification

On the other hand when trying to use the approach based on postulating balance laws (as done in [68–71]) then the set of balance laws, which hold at macro-level, is different from those which are valid at micro-level and the whole identification process becomes very difficult. To our knowledge, only few simple homogenization processes were possible when starting from models based on the postulations of micro- and macro-balance laws. A lot of problems of internal coherence for the models in their identification processes arise and very often some energy terms are lost or added in the transition from micro- to macro-levels.

We claim that balance laws appeared historically in the literature (see Refs. [50, 58, 72]) at first as necessary conditions implied by the Principle of Virtual Work to be used when trying to solve specific problems. Then, because of the reformulation of Continuum Mechanics as imagined by Cauchy (see Refs. [73–75]), what was a mere sufficient condition was forced to become the basis of all Continuum Mechanics. One is not too much surprised that a mere necessary condition once chosen to be the most fundamental hypothesis for Continuum Mechanics did limit the set of possible models that one could formulate. In particular in the postulation scheme set by Cauchy (and strongly supported by Truesdell), second gradient continua cannot exist (see Gurtin [76]). On the other hand, second gradient continua could be introduced (see Dunn and Serrin[77]) in the framework of postulations based on balance laws only by introducing some ad hoc assumptions: Instead these ad hoc assumptions are natural consequences of the Principle of Virtual Work when it is assumed to be the basis of the postulation process for Continuum Mechanics (see dell’Isola and Seppecher [78]).

1.5 | Why it is so important to understand how and when new theories were first formulated?

The meta-theoretical problem of finding the best postulation scheme in mechanics has a great importance in the development of scientifically based technology. The debate which this meta-theoretical problem sparked has been sometimes harsh, as already reported in the works by Piola (see Refs. [50, 58]). The meta-theory, which allows us to choose among different postulation schemes, includes the criterion represented by the so-called Occam Razor and by some “phenomenological” evidences.

In fact, one must ask themselves: using which postulation scheme new theories were first formulated?

If a predominance of novel and effective models, which consists in models at first formulated based on the Principle of Virtual Work, is observed, then one can conclude that this is, most likely, the most effective Principle to be used for inventing new theories.

Therefore, we can conclude that one is obliged to investigate the genesis of mathematical and mechanical theories not exclusively for a mere not-better-specified philological purpose¹ if he wants to get a guidance in the process of inventing novel and original theories.

Some examples that clarify our thesis are given in dell’Isola et al. [79]. The most specific examples regarding the process of developing model of planetary motion and Continuum Mechanics are presented in Spagnuolo et al. [80].

In fact, the effort required to establish in which form and when a certain mathematical model was first formulated has a strong motivation: It paves our way towards the invention of novel models, by showing us the example of our predecessors.

It is not enough to present in a rigorous and precise way a certain theory or model, using one of its equivalent formulations. The range of applicability of a theory is always limited and therefore one must be ready to generalize it in presence of different demands from the applications. To learn how a theory was first formulated may facilitate its possible generalization, albeit its equivalent formulations have clearly the same predictivity. Such generalizations are to be found learning how the successful theories have been formulated at first, and how they were subsequently developed: Learning how well-established theories were conceived supplies a secure guidance in the invention of novel ones.

This is the reason for which we believe that a presentation of well-established theories to interested scholars must follow, as carefully as possible, their original invention path.

2 | THE PROCESS OF KNOWLEDGE TRANSMISSION AND KNOWLEDGE FRAGMENTATION: SOCIOLOGICAL PHENOMENA TO BE DESCRIBED BY USING THE SCIENTIFIC METHOD

One should never accept a deleterious organizational concept that is, too often, considered fundamental in too many research institutions: The hyper-specialization of knowledge, which is subdivided into too many subdisciplines that are, in addition, rigidly separated by sharp boundaries. This organization criterion leads to a hyper-fragmentation of knowledge that produces two deleterious consequences: (i) the incapacity of the produced hyper-specialists to understand the complexity of phenomena and (ii) the crystallization of postulation schemes and cultural paradigms. This second circumstance is particularly negative for the advancement of science which must proceed: (i) by comparing the available experimental knowledge with the predictions produced by the mathematical models used in the temporarily accepted scientific paradigm and (ii) by replacing the paradigm with a novel one when accurate and efficient predictions cannot be obtained in the framework of the old one without recurring to ad hoc assumptions.

2.1 | Fragmentation of culture brings the inability to deal with complexity

Until very recently also (Applied) Mathematicians together with Physicists, Mechanicians, and Engineering Scientists have been obliged, during their curricular studies, to master a variety of scientific disciplines, as the organization of studies stemming from the illuministic revolution did great value multidisciplinary.

Illuminism considered extremely dangerous the tendency to hyper-specialization, which unfortunately became fashionable in the modern academic and scientific milieu.

In fact, recently even the perception of the fundamental unity of human knowledge is being lost and it is surely not recognized, as it should be, as the founding and most fruitful feature of scholarly activity.

As visible consequence of this loss, quickly and systematically, Western Culture is losing its celebrated capacity of solving the problems arising when modeling complexity in natural phenomena, which requires the widest unitary vision of knowledge. In fact, it can be observed, in a period including few scholars' generations, a considerable change in the perception of what is scientific research: (i) science is not perceived anymore as a unique body of knowledge which has to be used as a whole for developing our understanding of reality; (ii) the fields of specialistic expertise appear to be separated by more and more high walls, which are more and more efficiently blocking interdisciplinary research.

Parceling out human knowledge leads to the incapacity to understand and model phenomena: Too often in the past, difficult scientific problems were solved by exploiting methods that had been used in apparently far domains of knowledge so that it is unlikely that scholars mastering only those methods, which gave good results in the past, are ready to confront novel problems in their specific discipline. In fact,

No amount of research on the candle would have given us the electric light bulb [81]

and one could complete the argument talking about the technology of light produced by LEDs.

2.2 | The difficulties in the process of knowledge transmission due to linguistic barriers

That the multidisciplinary approach is the most effective one is proven by many historical examples: Among the most prominent and most ancient one, we can cite that of Archimedes of Syracuse.

We refer to Spagnuolo et al. [80] for a discussion about the long and tormented process of transmission of the fundamental works by Archimedes to our modern science from Hellenistic one. Here we simply recall that, obviously, when a change of the used language in science occurs very often a large part of the knowledge stored in the books written in the old language is misunderstood and sometimes forgotten and therefore apparently lost.

Very often, this apparently lost knowledge re-appears in text written in the new dominant language² and is formulated several times in different more or less equivalent forms.

As the process of establishment of the new dominant culture and language is usually not so smooth and quick, the rediscoveries may occur in many different places, times, and languages.

The reader is invited to consider how Archimedes works arrived in Europe: via the Latin translation by William de Morbeke (see Refs. [60–62]), but also via the Arabic and Persian Translations. In fact, Latin needed some time for establishing itself as the new dominant language for science.

Many apparent anachronisms and “strange” or “unexplained” anticipations or incomplete or distorted theory formulations may be explained by considering that they are only apparently original, as they simply represent some “resurfacing” manifestations of a more ancient knowledge.

This cyclic process of loss, systematic incomplete and distorted rediscovery results into a series of regressions and/or slowing downs of knowledge and of technology.

We want to stress that examples of the rediscovery of a body of knowledge cannot be found in the Hellenistic science only. A more modern example can be described and is relevant to our subject: the relatively “small” and “low” linguistic barriers between Italian Mechanics of the first half of the 19th century and the American Mechanics of the second half of the 20th century.

It concerns the works by Gabrio Piola (see Refs. [50, 58]). In fact, Piola's fundamental works in Continuum Mechanics seemed to be lost for a long period albeit they did remain completely available in many world-spread libraries. Piola's works were written in Italian in a period when a competition between French and English had started for the predominance as preferred language for science. In fact, their content could be completely recovered rather simply, as a reasonably large number of mechanicians remained capable to speak both English and Italian, so that their faithful translation was possible. However, linguistic barriers are not the only cause of deformations and reformulations of scientific theories.

2.3 | Theories deformed in reformulation: The paradigmatic case of Tartaglia and the importance of the Maestro–Pupil lineage in science transmission

Some psychological and barely survival strategies used by scholars for getting notoriety also play a role in theories transmission process, whatever is believed by some holier-than-thou historians of science.

These strategies have a clear effect on the advancement of science and knowledge transmission processes: In our analysis, one cannot ignore the global social effects of the actions caused by both positive and negative states of mind like: jealousy, ambition, revenge, desire to help a friend, inflated self-esteem, insecurity, desire to contribute to science, ignorance, erudition, modesty, or arrogance. When these states of mind are associated to the necessity to earn money by occupying academic positions or by getting grants, to nepotism and to the aspiration to get power by exploiting scientific knowledge they can produce serious effect on the transmission and preservation of knowledge.

We believe that it is not needed to prove a very obvious statement: Very effective scientific theories or mathematical models and technologically relevant bodies of knowledge may be erased, distorted, partially or completely lost, or forgotten for a certain time period in a specific scientific milieu, because of the conjunction some states of mind like those which we have listed. These loss-distortion-delay processes occur to theories independently to the scientific importance and value of the theories to which they apply.

Aforementioned holier-than-thou scholars will keep maintaining, maybe, that listed sometimes ominous states of mind cannot influence scientific thought, which is objective and beyond human meanness. They will arrive to believe that describing the influence of the human emotions and mind attitudes on science development is even harmful for humankind. Unfortunately, exactly as nobody can avoid the effects of possible infections simply ignoring their existence, also in the descriptions of science transmission processes, all possible relevant caused of phenomena must be taken into account.

Tartaglia (1499–1557) (see Refs. [60–62, 80] for a detailed study of the sources describing the competences and ambitions shown by him) needed to persuade his contemporary academic milieu that he could master Greek and Latin languages so that he could translate and fully appreciate the scientific content of the works by Archimedes and Euclid.

Heiberg has proven, with incontrovertible evidence, that Tartaglia did not manage to read reasonably well Greek language and that his knowledge of Latin one was similar to that of a child at his first steps in his training. Moreover, it could be found a clear evidence that Tartaglia’s Latin translation of Archimedes’ work is exactly that to be attributed to William of Moerbeke (Tartaglia, in his published text, copied also the typos later found in the Moerbeke manuscript! A more detailed discussion about the work of this scholar can be found, e.g., in Refs.[50, 58, 60–62, 80]). Historians of science will agree that the analysis and chronology of the process of rediscovery of Hellenistic science cannot be mystified by the personal needs and problems, which afflicted the life of Tartaglia, so that Heiberg’s reconstruction of his state of mind (see also Feldmann [82]) cannot be regarded historically irrelevant.

To maintain the quality and efficacy of the process of science transmission, the survival chains of Maestro–Pupil along scholars generations is essential. Competent Maestri must be allowed to select and supervise competent Pupils. Albeit it may happen that this chain has some link, which may be constituted by incompetent scholars, if these weak links are not too much, then the efficacy of the chain is, usually, not compromised. Of course, the transmission of science is also based on the study of written texts: However, the role of competent scholars contributing to the texts and explaining them to their pupils, and controlling that the pupils really understood their content, is probably essential.

After the invention of the printing technology, another big obstacle to science transmission was indeed removed: The enormous costs of handwritten manuscripts did operate a not-so-effective selection by incompetent scholars of texts to be transmitted to future generations so that very often the best books were not copied to be preserved for future generation (see Spagnuolo et al. [80] for a description of what happened to the most important works by Archimedes when their content could not be anymore understood).

Unfortunately, incompetent scholars selecting “important” textbooks are active also in the era of printed books: Many books virtually are hidden to younger generations by the establishment of a dominant cultural paradigm. However, usually the traces of the point of view different from the dominant one remain well preserved in the libraries, to be eventually rediscovered. In fact, when the postulation scheme based on balance laws started showing all its limits those scholars who wanted to look for more effective conceptual tools did manage to find, in the libraries, the best books among those that presented mechanics as based on the Principle of Virtual Work.

2.4 | Modern obstacles to the transmission of knowledge: The distortive effect of “impact” metrics

The present fashion of using citation metrics for evaluating research quality has introduced another mechanism for condemning to oblivion scientific works (may them be papers or textbooks), their authors, and some theories. The strategy is simple: It is enough that the dominant group in a certain discipline «forgets» to cite the works to be erased. As in the culture of *publish or perish*, there is a huge amount of published works, after a while the «condemned» works are forgotten, and it is very difficult to find them among the enormous amount of papers that appear in the databases. Literature is practically overflowed by too many repetitive and not-well-thought works and serious contributions are even too easily lost among them. Moreover, it can happen that a popular compiler of a certain textbook, having many «followers» (in the sense of modern social networks) may impose his biased choices about relevant literature, arriving to «create» not-so-important research fields, research personalities, and results.

There are, also, examples of textbooks that deformed the true intent of their sources, imposing to too many younger scholars wrong points of view and definitively making for them every original research extremely difficult. The authors of these textbooks sometimes manage to reformulate, sometimes even in an equivalent and rigorous way, the content of some important sources they have found in the literature: However, they obliterate the original methods, postulation scheme, and heuristic procedure that lead the first discoverers to their novel results. Therefore, albeit the content of these textbooks are correct and one cannot find in them any wrong statement, they represent a dramatic loss of knowledge, as the original path to discovery results to be blurred.

2.5 | How the postulation scheme based on balance equations has been imposed in the second half of 20th century?

The paradigmatic case of a textbook, which imposed an obliteration of the original methods, used for developing a new theory can be found in continuum mechanics, in the textbook by Truesdell [74].

In the previous version of this textbook [83], that Truesdell co-authored with Toupin, variational methods were cited only for trying to prove that they are not suitable for founding Continuum Mechanics. Their short review conceived by Toupin [84] for the textbook [83], notwithstanding being accompanied by negative remarks, directed many scholar towards variational principles: A revival of Hellinger’s Encyclopedia was thus initiated. In Truesdell [74], also this review disappears: There even the same word variational never occurs.

Truesdell believes that Continuum Mechanics must be formulated as he believes was conceived by Cauchy: The reader should consult the illuminating work [73] where it is difficult to distinguish between Cauchy original ideas and their interpretation by Truesdell.

In Cauchy’s approach, the concepts of force and stress are regarded to be the most fundamental ones. The force is considered as a primitive concept and balance of forces is postulated for it. Of course, then one remarks that the concept of balance of torque is necessary, as also observed by Truesdell himself [85]. The proliferation of «fundamental» principles then starts and, unfortunately, does not and cannot stop there! In fact (see Refs. [68–71]), postulations based on balance principles need to postulate a balance equation for getting an evolution equation for each relevant and independent kinematical quantity.

Truesdell declares very often (see [74]) that his effort is to translate in a mathematical language, clarify explain and make more rigorous all the previous works found in the literature.

This attitude is shared with Tartaglia that included in the title of his *Opera Archimedis Syracusani philosophi et mathematici ingeniosissimi* (1543) the statement: “here I make clearer what it was not possible to understand in Archimedes works.” Both Tartaglia and Truesdell declare that they “clarified” some “obscure” theories when it is clear that they simply misunderstand their content.

Truesdell’s capacity of imposing his viewpoint in some groups of scholars is legendary and was based on the credit that he managed to acquire in many publishers companies: Actually his textbook with Noll became the pillar of a paradigm in 20th century Continuum Mechanics.

One important role in science transmission is played by the systematization of discipline produced by the compilers of Encyclopedias entries. If well-written, these entries describe the most important aspects of the part of knowledge that they resume and may manage to establish a formalism and vision for a whole group of scholars. Encyclopedias are a support for transmission of knowledge to younger scholars generations and may offer the key for a global vision on a discipline state of the art.

In the context of the evolution and diffusion of the postulation scheme in Continuum Mechanics, the 1913 Hellinger's Entry of German Encyclopedia of Mathematics [86] has a very important place. It intended to describe the 1913 current state of the art in Continuum Mechanics and highlight some promising research perspectives. This text was translated into English only recently (see Refs. [60–62]): While reading it, one can be easily persuaded that language barriers are active also in the present epoch: In fact, one can claim that the development of Continuum Mechanics has been somehow relented when English became the novel *lingua franca*: Too many scholars in Mechanics cannot read French, Italian or German, previously used for formulating important results in many original works.

Hellinger's entry is very well-balanced, precise, and comprehensive: However the reader is impressed by a specific among its peculiarity. Hellinger presents some research perspectives that, in 2022!, still seems to be visionary. Hellinger did forecast, many years before, the most important research streams that were actually developed.

Why Hellinger's entry did not represent the most cited source in Continuum Mechanics in the 20th century? In our opinion, the entry was written in German, the author was Jewish and was obliged to escape to the United States to avoiding being killed in a concentration camp, the mathematical concepts used were too much advanced for the more engineering-oriented readership, the propagandistic capacity of Truesdell obliterated the entry by citing it in a very negative way.

The lack of consciousness about the content of this entry, especially in German speaking Mechanics where (obviously!) the linguistic barrier has never been active, had a specific effect: Many scholars seem to be aware of many of the results presented in it, knew about the most important results presented but the information about the fact that they were obtained and presented based on variational principles was apparently lost. Therefore, the secondary sources based on Hellinger's entry (some of their authors being emigrated to the United States) presented the results from Hellinger's entry hiding the heuristic method used for deducing them. These sources managed to avoid any reference to Hellinger (a typical *damnatio memoriae*, which is inflicted to those who support variational principles) and pretended to present completely original results. As the structure of these results is rather intricate the authors manage to impress in the readers, the persuasion that scientific results are exoteric and can be obtained only by particularly genial scientists.

Unfortunately, the way in which second gradient generalized continua theory was developed shows that: (i) as already delineated by Hellinger, only by using variational principles, one can get a coherent mathematical theory (see Refs. [1–3, 87]), (ii) when trying to generalize standard Continuum Mechanics, as formulated by Truesdell and Noll (see Truesdell and Noll [88]), one most likely gets not-mathematically-coherent theories (see Gurtin [76] and iii) only by using variational principles, the incoherence of the proposed models can be cured (see dell'Isola and Seppecher [78]).

Unfortunately, following the paradigm of Truesdell and Noll [88], the content of some modern textbooks in Continuum Mechanics consists in a presentation, which hides the variational procedure used for finding them the first time. Some ad hoc balance arguments, plus some assumptions obtained «out of the blue» *induced* by experimental evidence (*sic!*) are thrown together apparently in a chaotic way and the miracle of getting well-posed models (in which, for instance boundary conditions are compatible with the obtained partial differential equations) happen only when somebody else obtained them based on variational principles.

A final comment is maybe appropriate here: The existence of Encyclopedia Entry make easier the existence of so called-polymath scholars. These scholars claim to have a universal knowledge: Instead they are scholars who had access to an Encyclopedic Entry or to a big body of knowledge (maybe temporarily) lost or forgotten.

We are sure that a careful exegesis of Hellinger's work will prove that it represents the source of many works in Continuum Mechanics, also authored by those scholars who claim that «Balance of Force» postulations are to be preferred to Variational Postulations.

In fact, if one knows in advance the formulation of well-posed models, he can manage to get them by a series of ad hoc postulates, invoking, if the objections are too critical, «experimental evidence» for inducing them.

We do not want here to delve into the fascinating epistemological debate, which opposed Inductivism and Falsificationism, in all their manifold aspects and expressions. There are many works about this subject: We found particularly clear the presentations in Refs. [89–91] or [92] and we refer also to the fundamental work by Popper [93].

In the period after the Renaissance between the 17th and 20th centuries, the great majority of scholars believed that inductivism was the only effective method capable to produce scientific theories. Unfortunately, also in many contemporary popularizations of science and scientific discovery, it is naively and erroneously presented as it were the only method used by scientists.

Moreover, also the great majority of contemporary scientists share this belief: It seems that in the 20th century, the epistemological debate was abandoned by scientists, as a further manifestation of the already lamented process of fragmentation of knowledge. It seems that epistemological considerations can be formulated only by professors of epistemology and that scientists are not allowed to discuss their own activity. On the contrary earlier scientists extensively debated and pondered their philosophies of method (or meta-physical methods, as called by Piola). About this regrettable circumstance Einstein remarked, «Science without epistemology is—in so far as it is thinkable at all—primitive and muddled» (see Refs. [89, 90]).

The lack of epistemological awareness became more and more diffused after the 1960s:³ following the trend towards hyper-specialization scientists started to ignore completely the historical and philosophical knowledge necessary to consciously develop their own scientific programs. This circumstance lead to a erasure of the knowledge necessary to formulate correctly mathematical models: also Continuum Mechanics suffered because of this circumstance. The debates about postulations to be chosen for founding Continuum Mechanics have shown how regrettable such an erasure is.

We underline that, in the second half of the 20th century, aware scientists and philosophers of science accepted that the best description possible for the process of scientific discovery and modeling is given by hypothetico-deductivism. There are various versions of this description: Karl Popper's falsificationism, which is also sometimes called deductivism, seems to us the most successful one.

3.1 | As every other theory, also Cauchy continua theory cannot be induced experimentally

However it is necessary here to clarify few epistemological concepts because in the textbooks following the Cauchy Postulation scheme, Continuum Mechanics is presented as a theory in which a sequence of postulates are accepted (randomly) one after the other during the development of the theory. Some of such postulates, as the so-called Cauchy Postulate for Stress, seems to be clearly an ad hoc assumption while others are claimed to be *induced* by experience.

The too-wide-spread acceptance of Cauchy Postulation scheme imposed by Truesdell textbooks diffusion represented the main reason for which Generalized Continuum Theories development was blocked for so long time, after their conception and first formulation by Piola in the period 1828–1850.

Cauchy first gradient continua (i.e., those continua in which elastic energy density depends on the first gradient of displacement only) were, indeed, to be recognized already by Piola to form a particular class of a larger one: Piola was aware of the modeling limitations intrinsic in the Cauchy approach and proposed to use the theory of n th gradient continua to describe a larger class of deformation phenomena. We can say that the undoubted successes of Cauchy continua in applications does not imply that they must be used in modeling all conceivable phenomenology: In fact, the exact span and limitations of their possibility of application were already clear to Piola [50, 58].

One has to recognize that there is no such thing like a «definitively true» theory, whose validity is not questionable once it has been established. Therefore, one has to accept that, as probably also Cauchy would agree, and as Piola explicitly stated, Cauchy continua are a class of models that forecast, in a first approximation, the modalities of the phenomena occurring in deformable bodies. When they lose their predictive capacity, then more sophisticated models, as for example those proposed by Piola, must be introduced.

Inductivists, and all the scholars believing in the universal applicability of Cauchy continua theory, believe that experimental evidence can determine uniquely the «correct» model to be used to describe a specific class of phenomena. They seem to ignore the epistemological debate (and its results) about the so-called underdetermination of the scientific theories [94]: A finite set of experimental evidence is not capable to specify the theory to be used for describing a specific phenomenology, as there are possibly infinite theories capable to predict it equally well.

The reader will, therefore, agree that it is necessary, when discussing the foundations of Continuum Mechanics, to spend some intellectual forces to try to solve the epistemological question concerning the relationship between experimental evidence and mathematical models used for predicting it.

3.2 | **Falsificationism or inductivism?** σώζειν τὰ φαινόμενα!

Let us now try to specify more precisely the needed epistemological concepts. In philosophy of science, two alternative descriptions of the process of invention and use of mathematical models for describing physical reality have been formulated. Most likely, the roots arrive at the Hellenistic start of science (see e.g., dell'Isola et al. [95]): We will here simply summarize the most important point of the epistemological analysis, which we need for our purposes. In the literature, the names given to these two descriptions are inductivism and falsificationism. Albeit more sophisticated conceptual frames may be needed to describe precisely human scientific efforts, the discussion of only these two seems enough for the present purposes.

Inductivism claims that scientific theories can be obtained by carefully observing a well-defined phenomenology, measuring all relevant physical quantities, infer generally valid «physical laws» from examined phenomena (by using so called «inductive reasoning») and by finally objectively «discover»⁴ the only possible theory describing the observed phenomenology.

Inductivists declare that it is possible to infer from «observations of similar effects to similar causes», via a unrestricted generalization inference process, universally valid laws.

In other words, the idea underlying inductivism is that «theories can be derived from facts»: an abused and misleading statement that is really spreading darkness upon the factual essence of scientific research.

As most likely already had happened during Hellenistic science parabola, inductivism has been replaced by hypothetico-deductivism (deductivism in short) as a more realistic description of scientific discovery.

Hypothetico-deductivism describes the process of scientific discovery as follows: The scientist invents (invention: the key word, which replaces for falsificationists the word discovery) some principles at the basis of his mathematical theory using imagination, intuition, or even dreams. Then he deduces all logical consequences from his principles, by correctly using the rules of mathematical logic. Finally, he specifies which ones, among his mathematical entities and proven mathematical properties have a correspondence in physical reality so being able to check experimentally his theory. Experimental verification can never prove directly the fundamental principles, which have been postulated (because of an inventive act of our mind) at the beginning of the process of formulation of the theory. Experimental verification can involve only some specific consequences deduced from the fundamental principles: Therefore, experiments can only falsify a theory, by checking that some of the statements, which are logically proven starting from its principles, are indeed false.

In the version by Popper of hypothetico-deductivism, that is, falsificationism, the selection of most suitable theories is obtained by trying to refute (falsify!) them empirically. The more refutation efforts fail the more predictive and effective the theory is. The more risky predictions are not falsified the more one can have confidence in the theory. When the theory's unexpected predictions are verified experimentally, then one can say that the theory is strongly corroborated. However, one can never say that it is «true»: If one really wants to use the mental category of truth, then he can at most say that for a theory, which was not yet falsified, notwithstanding serious and systematic efforts, it is granted a certain «verisimilitude», or, if one prefers, that the appearance of truth and maybe, a likeness to truth, has been «saved».

It is remarkable that in Hellenistic science, Archimedes claims that Aristarchus is capable to «save the phenomena», (σώζειν τὰ φαινόμενα) by formulating his heliocentric theory.

While for inductivists, the physical objects modeled, once a true theory is established, can be «identified» with their mathematical models, it is clear to falsificationists that a distinction must be maintained between a mathematical model and the modeled object, as different models most likely need to be introduced for describing different phenomena occurring to the same physical object.

Inductivists believe that Cauchy first gradient continua are models «induced» by experimental evidence with absolute certainty and therefore they fully identify them with the physical objects (i.e., deformable bodies) that they are aimed to describe. It is clear, therefore, that they do not manage to accept that one could introduce a different model for deformable bodies: For them to change a «true» model is equivalent to assume that there exist physical objects with «contradictory» behavior. For instance (see the subsequent sections), they arrive to believe that contact forces in deformable solids can be only forces distributed per unit areas (as modeled in Cauchy continua): Therefore, they believe

that, as reality is ontologically unique, one cannot have contact forces distributed per unit line (as modeled in second gradient continua).

3.3 | The ontological interpretation of mathematical objects used in the modeling process

Both inductivists and falsificationists need mathematical objects for modeling physical reality. When inductivistic vision is adopted, one believes to obtain a full identification of the mathematical objects used as models with the physical objects, which one wants to model. In this way, inductivists manage to «materialize» mathematical entities. For instance, they did manage to «materialize» forces, which, by themselves, are specific mathematical quantities for which correspondent physical entities do not exist.

The inductivistic viewpoint is not accepted by falsificationists: The mathematical model can describe only some of the properties of the real objects and phenomenology in consideration. The mathematical model is constituted of mathematical entities among which some relationships are established based on the accepted postulates and by using logical deduction procedures. Some mathematical objects directly correspond to physical quantities or physical objects, while there are other mathematical objects that are precisely defined to facilitate the logical deduction process necessary to develop the theory but do not correspond to any physical entity or quantity.

In Continuum Mechanics placement, elongation and shear deformation, for instance, are well-defined mathematical quantities, which correspond to some measurable physical quantities. The predictions obtained in the framework of the developed theory can be verified experimentally. Instead energy, work, force are mathematical quantities which, while are well-defined, do not correspond to any measurable physical quantity. When one claims that he has measured a force or an energy, he is in fact measuring positions or deformations and then using theoretical relationships linking these mathematical entities corresponding to them with forces or energies.

In fact, in all models, one can find mathematical entities that do not have any physical correspondent: These entities are, however, essential in the logical development of the postulated mathematical model.

The inductivist confusion between the mathematical model with the modeled physical reality, when it is verified that a certain model is falsified, may lead to believe that an ontological perception of the world is impossible, that the real world is not self-consistent and natural phenomena are intrinsically paradoxical. On the contrary, the falsificationist careful distinction between model and reality, in the same situation, leads to conclude simply that the model is not suitable to describe observed phenomena.

3.4 | Forces and their materialization

In history of science, it can be observed rather frequently the occurrence of the confusion between mathematical objects and mathematical models with the physical objects and physical phenomena.

The list of these occurrences could be very long and we limit ourselves (for more details see, e.g., Spagnuolo et al. [80]) to cite here two among the most paradigmatic ones: (i) when confusing the concept of rational number (a mathematical entity from modeling quantities) with the length of a segment, some Pythagoreans are said to have believed that reality is paradoxical, as the hypotenuse of an isosceles rectangular triangle, with catheti whose length is 1, does not exist, as its length would be $\sqrt{2}$, which is not rational; (ii) when misunderstanding the nature of mathematical model of the “moving spheres” of the planets, many scholars materialized them arriving to believe that planets were physically connected to gigantic metallic “armillary” spheres in the sky.

Albeit nearly nobody nowadays still believes anymore that armillary spheres are real objects, unfortunately too many contemporary scholars did persuade themselves about the existence of real objects that we can call «forces». They discuss experimental evidence concerning forces and treat them as if they were objects observed in the real world and in everyday life: In their mind to forces, one can attribute the same ontological reality as we do for walls, boats, wind, pulleys, and cords.

3.5 | Confutation of inductivism: Russel anecdote and Einstein criticism

Once one has distinguished between mathematical models and real objects, it is easy to confute the so-called inductivist vision of Philosophy of Science. Inductivism claims that the true scientific method requires the work of «diligent»

scientists, which can induce from careful observations the underlying physical laws, that, immutable, can be established once forever. Unfortunately, the existence of an efficient process of induction of physical laws is illusory. The prescriptions of inductivism include a very ambiguous clause: scientists experiments require a «due diligence». This ad hoc clause can be used if an induced physical law is falsified: The inductivists will claim that the scientist inducing it was not «diligent enough».

Inductivists were ridiculed by Bertrand Russel using a famous anecdote about the inductivist chicken. In Chalmers [92], it is formulated as follows:

[We present] a gruesome example attributed to Bertrand Russell. It concerns a turkey who noted on his first morning at the turkey farm that he was fed at 9 am. After this experience had been repeated daily for several weeks the turkey felt safe, in drawing the conclusion “I am always fed at 9 am”. Alas, this conclusion was shown to be false in no uncertain manner when, on Christmas Eve, instead of being fed, the turkey’s throat was cut.

Einstein also felt the need of criticizing inductivism:

Any amount of experiments may prove that I am right; a single experiment can prove that I am wrong [letter to Max Born on the December 4th of 1926].

The same idea that theories can be induced from experimental facts seems to us completely wrong.

4 | THE DEBATE ON THE CHOICE OF THE BEST POSTULATION FOR THE FOUNDATIONS OF MECHANICS

Our short description of the debate on the formulated postulations for founding mechanics starts from the works (see Refs. [50, 58, 96–99]) of Gabrio Piola (1794–1850), who based his postulation on the Principle of Virtual Work. Piola himself refers to the 1815 edition of the *Mécanique Analytique* as the starting point of his studies, attributing to Lagrange the introduction of the concept of Stress in Continuum Mechanics. Cauchy, instead, preferred [73, 75, 100] to postulate, as a basis of Continuum Mechanics, the balance of forces and momentum of forces. We report some continuations of this debate in contemporary times: Differently to what happens for the Hellenistic theories and models (see Refs. [80, 95]), all sources are fully available, albeit they are written in many different languages (mainly French, German, Italian, English).

4.1 | The object of Continuum Mechanics 1: Deformation phenomena

In this section, we draft an informal description of the object of Continuum Mechanics. We have recognized that in the works by Piola [50, 58, 98], one can find a description in substance equivalent to the one which we present here.

It is usually stated that Continuum Mechanics, or Mechanics of deformable bodies, aims to mathematically forecast how the shapes of bodies are influenced by the bodies’ interactions with the external world. In fact, more recently (see dell’Isola and Steigmann [7]), it has been recognized that, since its origins, Continuum Mechanics had also as object the study of the design of possible materials, which may exist in nature. In fact, when Lamé and Green (see Refs. [53, 101]) determined the structure of the most general linear elastic isotropic material by introducing two elastic parameters (which may be equivalently either Young and Poisson moduli, or both Lamé moduli), finding the famous restrictions on their possible values did solve the problem of characterizing the wider class of these materials, which may exist. Their results opened, later, the path to the design, for instance, of auxetic materials.

The most fundamental primitive mathematical concepts introduced in Continuum Mechanics as formulated *à la d’Alembert* are that of: (i) material particles, which all together constitute the reference configuration; (ii) placement function, and (iii) volume density of deformation energy.

The reference configuration is a subset (suitably regular) of the three-dimensional Euclidean space. The current shape of a body is defined (since the fundamental work by Lagrange (1736–1813)) by a placement function mapping every particle in the reference configuration to its current position. Each material particle is the mathematical model for the “small part” of the real body, which is localized, in the reference configuration, in its vicinity. From the placement function, we can

define the deformation measures (see later sections or Refs. [7, 102]), which model how the shape of the body is changed in the vicinity of every material particle.

The mechanical behavior of every “small part” of the body is mathematically modeled by the postulated a priori deformation energy volume density dependence on the deformation measure. In other words, deformation energy volume density is specified by a “constitutive equation.”

Coherently with the falsificationist approach that we adopt, no experimental justification for these choices must even be tried: Only a posteriori checks of the predictions obtained using our model are meaningful. These experimentally verifiable predictions will regard simply the physical quantities correspondent to placement and to the deformation measures calculated from the placement. A further model specification is required: One has to specify in which range of deformation measures the local behavior of the body is satisfactorily described by the postulated deformation energy density. One may need to introduce further constitutive functions and kinematical descriptors when modeling damage, plastic phenomena, and so forth.

Finally, the Principle of Virtual Work must be postulated, which allows for the determination of the evolution equations governing the motion of the continuum.

Virtual Work functionals are linear and continuous functionals (see Refs. [1–3, 6]) mapping any pair constituted by the instantaneous value for the field of placement and its first and second time derivative and a virtual displacement into the set of real numbers. This functional must be continuous and linear with respect to the virtual displacement. As underlined by Germain, this part of d’Alembert–Lagrange postulation scheme renders Continuum Mechanics dependent on functional analysis (see again Germain [6]).

Piola assumed that internal work functionals describing the internal deformation phenomena of the considered bodies may depend on virtual work displacement via the local values of its derivatives up to the n th order, thus introducing n th gradient continua (see dell’Isola et al. [8]). Piola assumes that internal virtual work functionals can be given as the sum of two addends: The first being the functional derivative of the total deformation energy functional and the second being the first variation of a Hamilton–Rayleigh Dissipation functional with respect to the velocity calculated on the displacement.

Piola conjectured that it is possible to account for phenomena which, while occurring at lower length microscales, have a relevant influence at macro-level by judiciously postulating for the virtual work functionals suitable constitutive forms. At this point, our reformulation of what has been conceived by Piola ends.

Instead Cauchy wanted to limit his attention to a less general class of continua, which, therefore, are capable to predict the behavior of deformable bodies in a smaller phenomenological range. It has been proven recently, using modern rigor standard (see Refs. [8, 78, 103, 104]) that the ad hoc assumptions accepted by Cauchy in his theory formulation is equivalent to assume that Internal Work Functional for Cauchy continua virtual depend only on the first gradient of displacement: For this reason, Cauchy Continua are often called also First Gradient Continua.

From an engineering point of view, Cauchy Continua are suitable to model materials that have a simple microstructure. The nearly exclusive use of Cauchy continua to design engineering artifacts clearly justify the reason for which only materials with a simple microstructure are used in engineering applications.

4.2 | The object of Continuum Mechanics 2: Metamaterial design

Both Piola and Cauchy analyses were assuming that the deformable body to be modeled already exist, so that their aim was to find the best possible model for it. Piola did want to include in his theory a wider class of models and therefore he could argue that the predictivity capacity that he expected from them had to be expected to be wider.

Of course, we can try to attack a more general problem: Is it possible to design a novel material whose behavior can be described by given constitutive functions? Indeed, as we have seen in the previous subsection, in the framework of Continuum Mechanics, specific materials are specified by constitutive assumptions: those specifying virtual work functionals. In this context, it is very important to establish some general «physical admissibility» criteria for characterizing the class of all conceivable constitutive functions in a specified modeling framework.

We will call metamaterials those materials, which have been scientifically designed and did not exist before being designed. Sometimes one reads that metamaterials are «exotic» materials. An obvious question may arise: What does one mean with the expression «exotic behavior»? We terminate any possible discussion with a sharp answer: Simply, an exotic mechanical behavior is a behavior which, in a certain epoch and in a certain group of scholars, is not known. In fact, what is exotic in a certain contingent situation may be considered standard in another one. The most classical example is given by the already cited debate about the linear elasticity isotropic constants. Lamé (1795–1870), Navier (1785–1836),

Cauchy (1789–1857), Poisson (1781–1840) surely regarded zero or negative Poisson's ratio to be very exotic: Many scholars did believe that such a material (see Refs. [53, 101]) could not exist. However, so-called auxetic metamaterials have been built and are very important for their engineering applications.

The Principle of Virtual Work supplies an important guidance also in the design of novel metamaterials (see for more details dell'Isola and Steigmann [7]): An important research program, which is being accomplished, consists in finding variational techniques for synthesizing (that is finding the appropriate micro-architecture) novel metamaterials, as specified by the constitutive equations characterizing their behavior. Some results obtained in this direction are listed in dell'Isola et al. [105].

However, there are, in the literature, some works in which the authors try to discover exotic metamaterials simply using a data-driven approach. Manipulating a lot of data, without the guidance of a theoretical approach, sometimes with the support of large computers, they more or less randomly divine and test microstructures trying to find some which may be of interest in applications. This modern form of inductivism does not seem to have been very fruitful.

4.3 | Does there exist an all-describing model for deformable bodies?

Many authors believe that the theory of first gradient continua is an all-describing model for the deformation phenomena and, therefore, deny that there is any interest in generalized continua theories.

Believing that variational principles are able to model only conservative phenomena (for a confutation of such statement see e.g., dell'Isola and Placidi [106]), Truesdell claims that the most fundamental concept in Continuum Mechanics is that of Stress. Trying to prove that what is left to study is only the choice of the stress strain constitutive equation, he introduces the concept of hyperelastic materials (Green elastic) and distinguishes them from elastic materials (Cauchy elastic). The set of constitutive equations giving stress as a function of strain characterize, in his definition, elastic materials: Hyperelastic materials are those materials for which a deformation energy exists from which stress constitutive equation can be deduced. However in Carroll [107], it is simply proven that elastic materials, which are not also hyperelastic, cannot exist. The presented proof is very simple: Every stress–strain relationship, which does not derive from a deformation energy, allows for closed cycles, which are producing (or, by changing the cycle direction, losing) energy. Clearly Cauchy elastic materials would be an example of a mechanical system for which the Principle of conservation of energy does not hold: Most likely, therefore, Cauchy elastic materials, which are not Green Elastic, are not physically admissible.

Truesdell, in all his huge scientific production, tried to prove that Continuum Mechanics can be founded on a postulation based on two fundamental Principles: balance of forces and balance of couples. Following the ideas expressed by Noll (see [88, 108]), the concept of force and couples are primitive concepts whose meaning has to be specified via a system of axioms. Simply adding some constitutive equations, in this conceptual framework, Truesdell and Noll believed to have formulated an all-describing model for all deformation phenomena.

This belief and the efforts for proving it true are in vain. In fact (as recognized by Truesdell him-self, see [83]), already Piola has proven that in every generalized theory of continua, once formulating an objectivity principle (that is that the expression of the Internal Virtual Work Functionals do not depend on the observer), balance of forces and moments of forces are necessary conditions for the validity of the Principle of Virtual Work: This necessary condition becomes sufficient for first gradient continua only.

Piola himself was already aware of the fact that there are generalized continua (for instance, second gradient continua see e.g., Auffray et al. [109]) for which the balance of forces and couples do not represent sufficient conditions to the validity of the Principle of Virtual Work.

Seen the results available in the literature (see Refs. [103, 110–116] for specific consideration to Continuum Mechanics and [86, 117–120] more generally for examples from different parts of physics), it can be stated that the Principle of Virtual Work is the most fundamental one ever formulated in Mechanics.

4.4 | Some (mainly inductivistic) objections to generalized continuum models

In the debate on the foundation of Continuum Mechanics, the statement: «second gradient (or microstructured or higher gradient) materials do not exist because used materials in engineering do not exhibit their properties and standard theoretical framework does not forecast them» has been heard too often.

The objection is even not well-formulated: In fact, second gradient (or microstructured or higher gradient) continua surely exist as mathematical models. The corresponding theories are well-posed, if the theory of real numbers is well-posed.

The objection should be formulated more precisely as follows: «second gradient (or microstructured or higher gradient) models are not descriptive of the behavior of materials used in engineering so that standard theoretical framework is sufficient».

The first statement proves the diffusion of the following misconceptions:

- a. the inductivistic confusion of first gradient continuum model (a mathematical model) with all present, past, and future observable and usable deformable bodies (physical objects existing in nature);
- b. the inductivistic persuasion that the well-established Cauchy model, having been proven experimentally, allows for the description of every conceivable phenomena;
- c. the naive belief that the set of materials used in engineering is immutable: No new materials can be invented and used.

As a consequence, the supporters of these misconceptions arrive to deny the existence of materials that they can have in their own hands, as such the properties of such materials cannot be described by the theory that they believe universally true.

A simple explanation can be given to the observation that: «used materials in engineering do not exhibit the properties of second gradient materials». In Engineering Sciences, it happens, as it is advisable, that once an effective theory is developed, it is used systematically in its (limited) range of applicability. As a consequence in Engineering practice, only those materials, whose behavior can be described with available mathematical models, can be systematically used in designing Engineering artifacts. In fact, one must carefully avoid, in engineering applications, to use those materials and those physical situations for which a mathematical description is not available.

These obvious considerations can be of help in answering to an objection raised too often in the second half of the 20th century: In fact, many engineers did maintain that nonlinear phenomena should not be of interest in Engineering, as in its «practice», they are not relevant. This objection represents a typical manifestation of the process of removal and erasure of the complexity. To it, one can answer with a simple reasoning: If the available mathematical models allow, only, for the computation of the deformed shape of a body in the small deformation regime (that is only when linearized equations are predictive), then it will be wise to design only artifacts to which corresponding small external loads are applied. As a consequence, it may appear that in engineering, nonlinear phenomena are not relevant. Instead they are simply avoided because of the incapacity of controlling them with effective theories. In fact, it is true that the limits of the mathematical capacities, in a certain epoch and a certain social group, limit its predictive capacity and consequently its design capacity. History of technology proves that what can be designed by means of a mathematical model coincides with what becomes exactly engineering practice.

4.5 | Data-driven metamaterials design: A modern form of inductivism

As very often happens in many parts of human knowledge, old misconceptions in epistemology are periodically resurfacing, assuming novel, more modern, and apparently different forms.

Data-driven mechanics seem to represent a simple restyling of inductivistic mechanics. As inductivists believe that by means of «diligent» and systematic experimentation, mathematical theories can be induced, data-driven mechanics believe that, by using enormous amount of data and powerful computers and algorithms, one can get predictions about the behavior of mechanical systems.

The ambiguity consists in what we mean with an «algorithm» used to treat data. If they incorporate the prescriptions of a model, then one is simply developing a kind of theory-driven mechanics with the support of big computing facilities. Otherwise, the same objections raised by falsificationists against inductivism apply, and data-driven mechanics is bound to fail to reach its objectives. It is well-known that every data-driven series of trials, without the guidance of a mathematical model, never led to any effective engineering solution.

The importance of computing tools in science and technology cannot be underestimated, but the a priori formulation of a model is the essence of scientific and technological development. Of course, when Artificial Intelligence will be able to «conjecture» falsifiable models, then computers will be able to formulate theory-driven mechanics.

4.6 | The importance of available computing tools in the selection of postulations and models to be used

In general, in engineering practice, one has to choose, among the conceivable mechanical systems, those for which a mathematical model exist: Moreover, he has to limit his designing efforts by considering only the models using which some mathematical predictions are possible, given the available computing tools. Clearly, the epistemological evaluation of the merits of a theory cannot prescind from the availability of the available computational tools to be employed for getting its predictions. A theory, which may appear extremely powerful, by potentially including the detailed description of many phenomena can reveal itself completely useless if it cannot produce quantitative predictions. In fact, if it was tried to take into account, by introducing the chosen mathematical model, too many phenomena, neglecting or underestimating the eventual presence of insurmountable computing difficulties faced in calculating the predictions using it, then the Engineering practice will not get any advantage by the formulated model. A too detailed model «being in principle capable to predict everything leads to the incapacity to predict anything».

A famous example in Continuum Mechanics is the model formulated by Navier [101] for deformable bodies, which was based on the theory of interacting molecules. Navier's discrete model was so detailed that it appeared immediately clear that no prediction was possible based on it. Hence, in order to be able to predict, for instance, the deformation phenomena occurring in uniformly loaded beams, Navier did try to homogenize his discrete equations for getting an effective continuous model, capable to describe the global macroscopic behavior of a deformable body.

In the framework of the Navier continuum model, it was possible to deduce some partial differential equations predicting the equilibrium shapes of considered body, once suitable boundary conditions were added. By solving these equations with analytical methods, it was possible to calculate the predicted shapes in some elongation processes finding that Poisson's ratio for isotropic materials could only assume the value 0.3. Such a statement was soon falsified by experimental evidence, and Navier model was rejected, while the debate about the needed elastic constants in isotropic materials started. Later, it was discovered that Navier averaging hypotheses for producing homogenized models deduced from his discrete micro-model contained an error.

It is interesting here to remark that, at the epoch of Navier, homogenized continuum models were introduced in order to find a way for calculating the solutions of problems regarding discrete systems having many degrees of freedom. Instead modern computing algorithms introduce discretizations for solving the equilibrium problem for continuous systems.

4.7 | Piola versus Cauchy vision of Continuum Mechanics

Coming back to the debate that led to the first formulation of Continuum Mechanics, we must observe that Piola introduced already in 1848 a continuum model for deformable bodies more general than the model introduced by Cauchy in Cauchy [100]: Piola claims in Ref. [96] that already Lagrange, in the last edition of his masterpiece [72], had based Continuum Mechanics on variational principles.

Piola's generalized continuum model is based on the assumption that the elastic part of the Internal Virtual Work Functional is the first variation of a deformation energy functional whose volume density depends on the n th gradients of displacement. We attribute to Piola the invention of the corresponding class of continuum models and we call them Piola, higher gradient or n th gradient continua. While discussing his definition Piola shows clearly that he is fully aware of the conjectural nature of the mathematical models that he proposes [58].

On the contrary, many supporters of Cauchy continuum models showed to believe that he was formulating the «ultimate» continuum model, deduced from the «true» properties of matter. The champions of Cauchy postulation show to believe that its principles are experimentally induced truths: They often claim that Cauchy Continuum Mechanics can be simply improved by «adding some epicycles», that is, by introducing some specific constitutive equations, by fixing some details and by correcting some small mistakes.

Such attitude blocked the development of generalized models in Continuum Mechanics. Albeit Eringen [68–70] seemed persuaded that such generalizations were necessary for describing novel materials and eventually relevant phenomenology, he was «prisoner» of what we can call the Cauchy «straitjacket»: In order to acquiesce to Truesdellian balance law prescriptions, he was lead to somehow postulate the existence of a balance law for each extra-kinematical descriptor that he had introduced in his «micropolar» media. Notwithstanding Eringen adhesion to Truesdell inductivistic approach, and albeit he had accepted to formulate, as basic principles, a long list of balance laws, micropolar media were opposed

by some scholars of Truesdellian school. This is not surprising: Truesdell did believe to have induced «the true» model for deformable bodies, so it could not accept any generalization.

The postulation choice by Eringen (see also Capriz [71]) caused a «proliferation» of basic principles (a tendency which had been so bitterly contrasted by d'Alembert, Lagrange, and Piola) that clearly contradicts the Occam Razor epistemological principle.

To prove the inductivist attitude to be attributed by the great majority of presentations of Cauchy Continuum Mechanics, we recall that they develop it with a series of ad hoc assumptions, which appear during the logical development of the theory, when they need to be used: Systematically, these assumptions are justified on the basis of «experimental evidence».

As an example of one among these ad hoc assumptions, we cite the so-called Cauchy postulate. It assumes that (i) «contact» forces, inside continua, can be forces per unit area only and that (ii) they depend on the shape of Cauchy cuts via only the cuts normals. Some believe that the second of these assumptions is not so relevant, as by Hamel–Noll Theorem (see e.g., Truesdell [74]) one can prove that (ii) is implied by the postulate of balance of forces. However, in the theorem, a hidden assumption appears evident after a careful analysis: The assumption (i) is systematically accepted in the demonstration. One can conclude, therefore, that the hypothesis (i) is the essence, in Cauchy postulations scheme of the so-called Cauchy Postulate.

Already in Ref. [96], Piola concluded that so-called Cauchy Postulate (i) has a limited phenomenological applicability, as it may be considered predictive of the behavior of a limited class of materials and therefore (ii) be regarded as a condition characterizing a class of constitutive equations.

Unfortunately, many scholars did accept that the Cauchy postulate has an ultimate, experimentally proven truth that cannot be doubted anymore. This viewpoint has, however, been abandoned by some among its strongest supporters: Morton Gurtin, albeit having been a strictly orthodox Cauchy–Truesdellian follower (see Gurtin [55]), in his late papers (see e.g., Fried and Gurtin [121]), changed his fundamental postulation approach. However, by ignoring, for instance d'Alembert, Lagrange, Piola, and Landau, he did attribute, showing what can be called a «modernist» attitude, to an «explicit» Lagrangian self-defining epigone (i.e., Toupin, who himself declared, [84], to have been inspired mainly by the reading of Landau work [119]) the formulation of what is the most appropriate postulation of Continuum Mechanics.

Piola's works explicitly resurfaced to the scholars' attention at the end of the 20th century only: In fact, 50 years before, they had been re-invented and studied with attention in view of their potential technological application (see dell'Isola et al. [122]) by scholars whose works had been obscured by Truesdell's ones.

4.8 | Why was Piola's pioneering work discovered 150 years later only? Why Cauchy Continuum Mechanics school ignored Piola's results for over a century?

We can manage to find a concomitance of causes, which give an answer to the two asked questions.

In order to make our reasoning smoother, let us start by recalling that, only recently, it could be proven (see Refs. [8, 78, 103]) that Cauchy Continua are a subclass of Piola continua: They can be characterized by the condition that deformation energy volume density must depend objectively on the first gradient of the displacement field only. This simple characterization must be compared by the longer list of assumptions used in the standard formulation of Cauchy model, which also must prove the existence of Cauchy Stress locally in every point of the current configuration using the famous tetrahedron argument. Instead, Piola, on the basis of arguments of logical coherence and the Principle of Virtual Work, was lead to assume the dependence of deformation energy on higher displacement gradients and managed to characterize the set of microscopic interactions corresponding to the corresponding class of generalized continua.

A first answer to the asked question is that differential geometry was not, at the time when Piola was active, developed enough to allow him to determine the dependence of internal contact forces, in the case of higher gradient continua, on the shape of Cauchy cuts (see dell'Isola et al. [8]). The reader, therefore, will not be surprised in discovering that Levi-Civita (see dell'Isola et al. [123]) is a direct descendent of Piola in the genealogy of mathematicians: Levi-Civita direct absolute tensor calculus is the tool, which lacked Piola for completing his analysis. On the contrary, basing his demonstrations on a simple application of the Gauss divergence theorem, Piola did manage to find that, in the case of first gradient continua, the surface density of contact force depends linearly on the normal of Cauchy cuts.

After about a century Paul Germain presented, in an elegant way (see Refs. [1, 3]) and using Levi-Civita absolute tensor calculus, the proof of the representation theorem for contact forces in terms of the shape of Cauchy cuts for second gradient and microstructured continua.

An obvious consequence of Germain's results is that the so-called Cauchy postulate is not valid for second gradient continua, as contact forces are seen to depend not only on the normal but also on the curvature of Cauchy cuts. Moreover in dell'Isola and Seppecher [8], it was proven that for $n + 1$ th gradient contact forces depend on the n th (and lower) surface gradients of the normal of Cauchy cuts.

Germain, also, duly stressed that a novel kind of contact interaction arises in second gradient continua, which he called *contact double force*. These interactions, whose interpretation and definition are not possible in the framework of Cauchy postulation (and for this reason were considered non-existing by many scholars), can be easily defined in the framework of a postulation based on the Principle of Virtual Work: They are those interactions with the external world, which expend work on the normal derivative (at Cauchy cuts) of virtual displacement. For a discussion of those new external contact interactions to be introduced in higher gradient continua, we refer to dell'Isola and Seppecher [8].

The difficulties in handling these generalized contact interactions when one is limited by the Cauchy postulation scheme is another reason of the rejection of generalized continua in many groups of scholars. It has to be recalled that in the debate that started between 1960 and 1970, some American and Soviet scientists tried to champion the d'Alembert–Lagrange approach (see e.g., Refs.[51, 122]): unfortunately without a great success. Later, it could be demonstrated, by using rigorous homogenization arguments, that (see Refs. [67, 104, 124]) second gradient models are necessary to model, at macroscopic level, deformable bodies whose microstructure, or micro-architecture, at a lower scale show high contrasts of elastic properties. We claim that Piola himself had guessed this result [50, 58].

A final cause of the rediscovery of Piola's models at the beginning of the 21st century concerns the simultaneous development of powerful computational tools and the invention of 3D printing technology. Indeed, if one wants to forecast via numerical simulations and reveal experimentally the mechanical behavior typical of second gradient continua, it is absolutely necessary to have a computer driven technology capable to easily producing, with the required precision, microstructured materials exhibiting high contrast in their microscopic elastic properties.

The role played by the invention of 3D printing in the debate about the foundations of mechanics gives a further example of how a limited technological capacity can indeed block the advancement of scientific knowledge: Until when one can calculate predictions in the framework of new theories and when he can test experimentally, with the available technology, the obtained predictions these new theories are bound to be blocked, ignored, or considered useless.

Unfortunately, also nowadays, in presence of the technological ability to produce materials whose behavior is described by Piola continua (and not by Cauchy ones, see dell'Isola et al. [105]), there are still scholars who refuse to accept their importance.

5 | THE FUNDAMENTAL PROBLEM IN THE THEORY OF METAMATERIALS

As already discussed in the previous sections, the systematization proposed by Truesdellians of Continuum Mechanics, both because of its inductivistic inspiration and because of its pretended and self-proclaimed completeness, seemed to have exhausted any further need of original theoretical research in Continuum Mechanics. As claimed many times and in many circumstances by Truesdell himself, novel research seemed limited to the search of particular constitutive equations that had to be «induced» from experimental evidence for any specific material.

5.1 | What we learn from history must guide our research strategy

As usually happened in history of science (see Kuhn [125]) every time when such kind of pretensions were put forward, after a stasis in the advancement of the specific discipline, the cultural paradigm dominating in a certain group of scholars is simply overwhelmed by a paradigm revolution. Inductivism hope is vain, as history is teaching us:

Historia vero testis temporum, lux veritatis, vita memoriae, magistra vitae, nuntia vetustatis

History is, in reality, witness of the times, light of truth, life of memory, teacher of life, messenger of antiquity

(Cicerone, De Oratore, II, 9, 36).

In fact, using the true words by Kuhn:

History, if viewed as a repository for more than anecdote or chronology, could produce a decisive transformation in the image of science by which we are now possessed. That image has previously been drawn, even by scientists themselves, mainly from the study of finished scientific achievements as these are recorded in the classics and, more recently, in the textbooks from which each new scientific generation learns to practice its trade. Inevitably, however, the aim of such books is persuasive and pedagogic; a concept of science drawn from them is no more likely to fit the enterprise that produced them than an image of a national culture drawn from a tourist brochure or a language text. This essay attempts to show that we have been misled by them in fundamental ways. Its aim is a sketch of the quite different concept of science that can emerge from the historical record of the research activity itself.

What happened is that, immediately after the main champions of the postulations based on balance of forces left the scientific scene, an abrupt change of paradigm occurred in Continuum Mechanics. The works by Germain, Mindlin, Toupin, and Sedov, just for listing some names, after having been ignored for some decades attracted finally the attention of the majority of researchers, novel problems were formulated and the corresponding challenges were faced.

One among the most interesting contemporary mathematical challenges in Continuum Mechanics arising in the paradigm, which is being newly established, consists in the problem of designing those metamaterials, which can be modeled with the generalized continuum theory. Indeed, generalized continua mechanical behavior seems to open the way towards «exotic» phenomenology whose technological applications may have an enormous impact.

Piola's theoretical result, which were at the origin of Continuum Mechanics, became suddenly also the vanguard of 21th century's research [99].

5.2 | How can one precisely specify the desired behavior of a metamaterial?

It is clear that one cannot specify the properties of novel metamaterials with statements formulated using plain English (or any other language). To give a statement of a problem having some chances to be solved one has to use a formal, mathematical language.

The postulation scheme based on the Principle of Virtual Work, because of its «concise» and «meaningful» formulation, seems the most suitable for such aim. The problem to be faced is then: «Given an Internal Work, External Work and Inertia functionals (as defined in Refs. [2, 3, 6]) one has to find a material whose mechanical behavior is governed by the Principle of Virtual work, postulated by using them». A particular case of this general problem is obtained when the internal and external works are calculated starting from an appropriate action functional and a dissipation Rayleigh functional (see dell'Isola and Placidi [106]).

It appears immediately clear that the search of the novel materials so characterized can become possible if one chooses to search a microstructure (or a micro-architecture) in which he organizes a «simpler» material such that, at a macro-level, its global behavior is the desired one. Of course such behavior can be manifested only at the chosen macroscopic length-scale, that is at the length-scale, in which the microstructure shows specific and «macro» global properties (see dell'Isola and Steigmann [7]). Therefore, a possible strategy for solving the fundamental problem in modern theory of metamaterials requires the solution of a «synthesis» problem:

«Given an Internal Work, External Work and Inertia functionals (as defined in Refs. [2, 3, 6]) one has to synthesize a microstructure that, once homogenized, at a macro-level is governed by the Principle of Virtual work, postulated by using them».

The formulated problem can be, therefore, reformulated as follows: «Given an Internal Work, External Work, and Inertia functionals (as defined in Refs. [2, 3, 6]), one has to find an algorithm to design a microstructure that, once homogenized, at a macro-level is governed by the Principle of Virtual work, postulated by using them». In other words, we are facing the difficult mathematical problem consisting into establishing a two-way link between microstructures and macro-behaviors.

In this context, the construction of more and more powerful computers can be of help. It is clear, in fact, that the synthesis problem we are facing must be itself solved by using the techniques of the calculus of variations and of optimization. As the materials, which we are looking for must, in general, be designed to function in nonlinear regimes, it is not possible to dream to calculate their equilibrium configurations and their motions finding analytical closed form solutions. In

our search of innovative microstructures, the use of effective computational algorithms and powerful computers seems unavoidable, once a solid theory will be established for driving our microstructure search.

5.3 | Is the problem of synthesis so new?

Roughly speaking the fundamental problem of synthesis of metamaterials consists in finding at least a micro-architecture, which makes «true» a given continuum macro-model: that is a micro-architecture whose behavior can be predicted by given continuum model.

The methodologies needed for synthesizing metamaterials must be borrowed, or at least start to be developed from, the theory of design of so-called «analog circuits». Analog circuits were conceived as analog computers designed to solve a specific Differential Equations when initial and, when necessary, boundary conditions are assigned [40, 46]. Kron [40, 126–128] and MacNeal [41, 129–131] greatly developed the theory of synthesis of electric circuits that was, for several decades, the most powerful predictive computing tool in engineering and scientific design. The most interesting (from our point of view) result obtained in the theory of synthesis of analog circuits (see also [132]) is that

every passive linear n-port circuitual element is algorithmically synthesized using inductors, capacitors, resistors and transformers.

To be more precise, it is proven that, for any given desired passive impedance, it is possible to find (i) a graph and (ii) for each branch of the graph, a suitable circuitual element such that the resulting discrete circuit has the given impedance.

In fact, the procedure conceived by Kron and McNeal, for any given quadratic discrete Lagrangian Action and Rayleigh Dissipation potential, allows for the algorithmic calculation simultaneously for (i) the graph structure, (ii) the circuitual elements for any of its branches of the discrete circuit, which synthesizes a linear n-port element governed by the starting Lagrange and Rayleigh potentials.

We can, therefore, conclude that, simply paralleling the methods used for the synthesis of linear and passive circuits, it should be easy to synthesize the generic higher gradient linear viscoelastic material: One has to simply introduce a “discrete” element (paralleling a finite element) for such a material.

Remark that, starting from finite elements for plates, shells, beams of 3D linear elastic continua, it has been already been possible to synthesize the electric elements governed by the same Lagrange and Rayleigh potential as its mechanical analog element (see Refs. [41, 131, 133]): The mathematical correspondence between mechanical and electric systems is carefully investigated in Crandall et al. [132].

The challenge represented by the metamaterial synthesis problem can, most likely, be faced by conjecturing that Kron and McNeal’s ideas can be generalized to produce a method allowing us to synthesize nonlinear mechanical (and also multiphysics) metamaterials. Some partial initial results obtained for metamaterial synthesis problems, concerning metamaterials capable to undergo large deformations in elastic regimes, are presented in Refs. [134–138] starting from the results obtained in dell’Isola et al. [139].

In science, it happens very often that a method, a theory, some results, once very popular and widely spread can be partially or nearly completely lost after only few years and forgotten by the great majority of scholars. The destiny of synthesis methods used in the theory of analog circuits give a further example of such a removal social phenomenon: Indeed it is very difficult, even in our epoch, when books are all stored also in electronic form and when the biggest, in whole human history, number of scholars is active, to manage to find in databases and in the library, some texts describing precisely and exhaustively what is, indeed, the most effective among synthesis methods.

The reasons for its erasure are easily explained: When it became clear, in the 1960s, that digital computing methods were more effective, the analog computing methods were considered useless and the corresponding theoretical knowledge simply removed. This circumstance gives a further example of the dangers intrinsic in the belief that the «immediate applications» are the only reason for which human knowledge and scientific methods deserve to be preserved and transmitted.

The observation of the just-described sociological phenomenon does not falsify three conjectures that seem very well grounded:

- i. the sociological process of loss and removal of important parts of knowledge continues to occur very often in every kind of scholars social groups;

- ii. notwithstanding what may appear at first sight, in science, it is not possible to classify a mathematical methodology as exclusively specific for a particular field of applications;
- iii. one can never classify a part of knowledge as being obsolete, especially if it has deep mathematical roots.

6 | THE PRINCIPLE OF VIRTUAL WORK AND ITS IMPORTANCE IN METAMATERIAL THEORY

We will present here our argument for supporting the following thesis: Only by accepting the Principle of Virtual Work, it is possible to develop Metamaterial Theory. An historical perspective may help us in our argumentation.

6.1 | The principle of virtual work in the original sources of Continuum Mechanics

We examine now, shortly, how the original sources in Continuum Mechanics were sometimes distorted and how consequently the basic principles of Continuum Mechanics were modified: One should not be too surprised to discover that sometimes the most ancient sources are those, which based on a wider conceptual basis, therefore, encompassing the possibility to describe more phenomena.

This philological analysis is not so difficult. Since d'Alembert's *Traité de dynamique* (1743), all sources are easily available, albeit they are written in different languages: In fact, English, French, German, and Italian are the languages of all the texts that we have consulted. A lot of information of relevance in our discussion should become available in the Euler *Opera Omnia*, when completed. It had the intent to make available in modern languages the original Euler Latin text, but it seems to proceed slowly: In Euler [140], not much of Euler original works have been translated (see Spagnuolo et al. [80] for more comments). Moreover, it has to be remarked that also the original Euler Latin text is not completely accessible to the larger audience of scholars.

Lagrange's masterpiece *Mécanique Analytique*, in its last edition, contains the formulation of the Principle of Virtual Work adapted to Continuum Mechanics and suitable to encompass not only the description of fluids but also of solids mechanical behavior. It is starting from this source, as he himself explicitly states, that Gabrio Piola, in his fundamental contributions to Continuum Mechanics (see Refs. [50, 58, 98]), develops the first formulation of Generalized Continuum Mechanics and lays the foundations of a method, that has been exploited systematically in subsequent literature, for the identification of macro-constitutive parameters of homogenized continua in terms of the properties of their microstructure.

Piola fully develops the techniques needed to deduce, from the Principle of Virtual Work, the so-called strong form of equilibrium conditions. In the case of first gradient continua, he gets together with the so-called indefinite equilibrium conditions (that is the PDE, which equates to zero the sum of the divergence of Piola Stress plus what Piola defines to be the «Lagrangian volume density of external force») also the corresponding natural boundary conditions, which state the equality between Piola stress applied to the Normal to the boundary of the continuum reference configuration and the Lagrangian external contact forces per unit surfaces. Piola deduces, simply via an integration by parts, in an extremely simpler way and in its version valid for the reference configuration, the same result that Cauchy obtains via his tetrahedron argument. Piola, using his celebrated Lagrangian–Eulerian transformations (see, e.g., Refs. [141–144]) proves, also, that his own results are equivalent to those obtained by Cauchy, in the particular cases of first gradient continua. In the case of n th gradient continua, Piola finds the corresponding indefinite equilibrium conditions, and starts the calculations needed to determine the corresponding natural boundary conditions [50, 58]. Unfortunately, in his posthumous work (see Piola [145], published by his pupil Francesco Brioschi), one cannot find their completion. To our knowledge, these natural boundary conditions have been found, in the case of second and third gradient continua only in Refs. [1, 51, 146], while for the most general case, it was needed to wait for the work [8].

This long delay can be explained easily: In fact, starting from Cauchy, Navier, and Poisson, the Principle of Virtual Work has been replaced with the independent postulation of the balance of forces and of the moments of forces. Cauchy changes the logical role of the concept of force: He considers it to be a primitive one, that must verify some postulates, while Piola (following d'Alembert and Lagrange) considers it to be a derived concept, defined in terms of others (that is work functionals, see Refs. [1, 2, 6]) that are, instead, considered as primitive. It is not simple to understand (see [75, 147]) if Cauchy did believe that the «truth» of the balance of force could be «induced» experimentally: For sure, this is what many of his epigones stated to justify its validity.

It is now become clear that, when the physical system to be modeled has a complex microstructure, the Principle of Virtual Work: (i) allows for a more effective formulation of evolution equations, (ii) it is easier to apply, and (iii) it produces well-posed mathematical problems. On the contrary, the balances of forces and moments of forces are not sufficient to characterize uniquely equilibrium and motion (see Refs. [80, 109]).

Here some considerations from history of science may be of help in clarifying our arguments: In fact, it is clear that the mechanical systems, which first attracted the attention of Engineering Sciences were, surely, studied by using the Principle of Virtual Work. This is proven by what appears to be the most ancient text of mathematical mechanics, which has been transmitted to us. It is the pseudo-Aristotelian «Mechanica Problemata», which most likely has to be attributed to Archytas of Tarentum (see Refs. [56, 80]).

The *Mechanica Problemata* appears to be a set of «exercises» of interest in the control and design of some among those machines, which contributed to the development of technology, at first, during Hellenistic period and then during the Industrial Revolution.

From the available text, it is not possible to understand in which formalism Archytas had postulated the Principle of Virtual Work. However in his text, this principle is used systematically to predict the functioning of machines and levers. Two equally likely conjectures can be formulated, at the present state of our knowledge, about the origins of mathematical mechanics. Either the Principle of Virtual Work was formulated by «abstraction» from the long sequence of exercises, which were solved all together by using it, without having the awareness of its essence. Or it was consciously «invented» by somebody (Archytas?) after having tried to «unify» the solution of different problems on the basis of a unique principle and then systematically applied to get the description of all machines conceived at the epoch.

What we can assess, without any doubt, is that all subsequent Hellenistic texts, which reached us show the needed awareness that leads modern mathematical physicists to conjecture more and more abstract principles in order to describe, on the basis of the minimum possible set of fundamental principles, the widest possible phenomenology.

Reading the masterpieces by d'Alembert, Lagrange, and Piola, we are assured that at least in the first half of the 18th century, this epistemological awareness had been fully recovered.

Unfortunately, the idea that leads inductivists to believe that principles can be induced from specific examples and experiments remains at the basis of too many modern textbooks whose authors believe that the students can «arrive» to see the «truth» by inducing it after having seen how to solve «many exercises». In these textbooks mathematical formulas, abstract concepts, physical phenomena, and experimental evidence are all «identified» misunderstanding and confusing their role in the development of mathematical models for physical phenomena.

6.2 | D'Alembert's and Lagrange's formulation of the principle of virtual work

The Principle of Virtual Work has the ambition to have a wide scope of applicability: It is intended to be applicable to find the evolution equations for any physical system (see Landau and Lifshitz [119]). Therefore, it must be formulated in a very abstract way: This ambition to abstraction has been pursued in every epoch always using the most powerful mathematical structures then available. Presently, the analysis by Paul Germain indicates that modern functional analysis and theory of distribution gives the appropriate formulation framework, at least for all known field theories. It has to be recalled that Lagrange could use, for his formulation, the relatively more primitive mathematical tools (i.e., essentially Euclidean geometry), which he had available: It is remarkable that he did manage to give all the same the needed impetus to grandiose physico-mathematical studies.

6.2.1 | The principle of virtual velocities, in its modern version at least, originates from the principle of minimum of total energy

A principle that can be effectively used for calculating the stable equilibrium configurations for a large class of mathematical models is the Principle of Minimum Total Energy. This Principle has been systematically applied by Archimedes (see Spagnuolo et al. [80]) and is a powerful tool often used in applied mathematics. This principle is not capable, however, to produce predictions about the motion of physical systems.

The Principle of Minimum Total Energy must be, therefore, replaced by a more general one. To this aim, we start observing that, with not so complicated mathematical reasoning, it can be demonstrated that a necessary condition to min-

imality for total energy, that is, the stationarity condition for total energy, can be generalized for predicting the evolution of considered physical systems. We describe, in the following, the basic ideas of how this can be done.

The stationarity condition consists in stating that the first variation of total energy, calculated from an equilibrium configuration, must vanish for every «small» admissible variation of configuration. It is exactly the structure of this stationarity condition that has been retained by d'Alembert and Lagrange for formulating the most general Principle of Virtual Work.

To proceed further, we must introduce, however, the most debated and controversial concept in the formulation of the Principle of Virtual Work: that of «virtual displacement». The reader who found the interest in consulting the *Mechanica Problemata* will discover that this concept was already formulated clearly, and systematically used, in the 4th century B.C. It is remarkable that also after so many centuries of fruitful applications it can be, still, considered controversial.

To this aim, let us come back to the minimality characterization postulated for determining stable equilibrium: The total energy must have a local minimum in the vicinity of a stable configuration. Therefore, when varying a stable configuration and considering an arbitrary variation of configuration, the total energy must increase. Assume that the total energy's first variation can be calculated (this condition is equivalent to assume a kind of differentiability of the total energy as a function of the configuration). This first variation consists in the Taylor expansion for energy truncated at the first order, which is valid for small variations. Because of its linearity, as a function of the small variation, the first variation of total energy must vanish in the vicinity of a minimum: This condition is equivalent, in the case of real functions of many real independent variables, to the condition requiring that the function gradient is vanishing.

In conclusion, if one adds to a stable equilibrium configuration every admissible small variation of configuration, the corresponding first variation of the total energy must vanish. The concept of «virtual» displacement, or virtual variation of configuration, when used for testing the variation of the value of total energy appears, therefore, very natural.

The generalization of the stationarity necessary condition obtained from the Principle of Minimum of Energy gives the conceptual basis, and the correct inspiration, for the formulation of the Principle of Virtual Work by d'Alembert.

It is unfortunate that we do not have other ancient texts in addition to the *Mechanica Problemata*, which discusses the Principle of Virtual Velocities (or how was called later: Principle of Virtual Work). It could be interesting to see if also Archytas (or whoever was the scholar who formulated this principle at the dawn of Hellenistic science) had conjectured his formulation of the Principle of Virtual Work starting from a minimum necessary condition. We suspect that this may be true. In fact, in the Hellenistic tradition, one can find many references to minimization principles:

For this would be agreed by all: that Nature does nothing in vain nor labours in vain. [148]

(by Olympiodorus, Commentary on Aristotle's *Meteora*). Unfortunately, the available sources does not give us much further support to our conjecture.

6.2.2 | D'Alembertian formulation of the principle of virtual work

A «virtual displacement» is, therefore, in its original definition, simply defined as a «small variation» (or, using a standard nomenclature of classical mathematical analysis: an infinitesimal variation) of the configuration of our continuum, which is «admissible»: that is, which respects all imposed kinematical constraints. This small variation has to be added to the candidate minimum energy configuration in order to verify that this candidate configuration verifies the necessary stationarity condition for minimality.

Using a modern mathematical language, in order to formulate the more general Principle of Virtual Work for any mechanical system, d'Alembert assumes that two work functionals can be introduced which linearly map any admissible virtual displacement into, respectively:

- i. the interaction work functional allowing for the calculation of the work expended by any interaction between the pair of particles constituting the system and any pair of particles the first one from the system the other one belonging to the system's external world,
- ii. the inertial work functional giving the work expended by inertial forces.

Remark that work functionals depend on two variables: the current value of motion and the virtual displacement.

D'Alembert postulates that the motion is uniquely determined by the following condition: In every instant, the total work expended by the sum of virtual work functionals on any admissible virtual displacement is zero. It is trivial to

see that, when the interaction work functional is the first variation of total energy and the motion coincides with the permanence in the stable equilibrium configuration, this formulation of the Principle of Virtual Work reduces to the stationarity condition for the total energy.

6.2.3 | The mathematical difficulties met in the formulation of the principle of virtual work and their consequences

D'Alembertian formulation has the undoubted advantage of being synthetic, unifying and suitable to formulate new models. Using the words written in 1913 by Hellinger (see Eugster and dell'Isola [60]): In order to found Continuum Mechanics,

As unifying mathematical form, in which all individual [methodological] fundamentals are included in the easiest, the variational principle is applied.

Hellinger points out the two main reasons for which this choice is preferable: The first reason is

The advantage, which the application of such a variational principle as foundation allows versus other possible formulations or also the direct consideration of the fundamental equations, consists especially therein that the variational principle is capable to determine the behavior of the considered media in all points and to every instant of time in a single formula, in particular to contain besides the equations for the interior also the boundary conditions and the initial conditions.

The second is expressed as follows:

Furthermore, from a certain point of view, it is in its concise brevity clearer than the equations and consequently it is [more suitable] for the treatment of new fields, for the formulation of further generalizations and it is, therefore, of essential heuristic relevance.

Such a powerful heuristic tool, however, needs a solid mathematical background for being handled. Variational formulation of mathematical models requires the use and invention of sophisticated nontrivial mathematical theories: Therefore, when one wants to find Mechanics on variational principles, he needs to master them.

We conjecture that this is the «fundamental» reason for which many scholars decided to «erase» the variational approach either in their textbooks or in their research strategies or in both. They believe to have, in this way, obtained the «simplification»⁵ needed to spread the main concepts of Continuum Mechanics in Engineering Sciences.

The formulation of Continuum Mechanics based on the postulation of balance of forces and moments of forces is the most popular approach alternative to that based on variational principles: Unfortunately, it is, in a way made already clear by Piola (see Auffray et al. [109] and the following), incomplete and, therefore, unfit to facilitate the process of discovery of novel models.

In order to have an idea of the mathematical abstraction needed when using variational principles, it is useful to recall that some of the most difficult mathematical theories, the Calculus of Variations and the Theory of Distributions, have been developed to respond to the postulation demands arising in Continuum Mechanics. The Calculus of Variations has its roots in Hellenistic Science: As proven by the fact that the simplest isoperimetric problem is called the «Dido's problems» and that Pappus (in his *Collectio*) states that «the hexagon is the regular polygon with the least perimeter for a given area». The sources report that Zenodorus (2nd century BC), in a lost treatise, solved many isoperimetric plane problems: Unfortunately, we do not have any idea about the methods that he used.

6.2.4 | Lagrangian formulation of the principle of virtual work as a basis for the definition of Piola continua

As we have discovered by reading the secondary source represented by the works by Piola, in the last edition of Lagrange's *Mécanique Analytique*, one can find the general formulation of the Principle of Virtual Work for a continuum, which Piola himself developed arriving to introduce n th gradient continua.

The reader may be puzzled in reading Lagrange: He calls «power» (*puissance*) and «moment» what we call now «force» and «power», respectively.

However, if one is patient enough to read all the introductory comments by Lagrange himself, they will discover that, with this choice, Lagrange wanted to follow Galileo Galilei. Unfortunately, this nomenclature choice has not been accepted in the mechanical literature: probably again because of the «erasure» of the variational postulation scheme occurred in too many texts.

The ideas of d'Alembert were elaborated and generalized by Lagrange, who formulated his Principle of Virtual Velocities as follows (our translation of the French text):

If a system constituted by bodies or points, each of which is pushed by any power, is in equilibrium and if a small movement is given to this system, by virtue of which each point will cover an infinitesimally small distance that will express its virtual velocity, then the sum of the powers multiplied by the distance covered by the points where it is applied along the line of application of this same power will be equal to zero, if we consider as positive the small distances covered in the same direction of the power and as negative the distances covered in the opposite direction.

A comparison of this statement with those presented in Germain (see Refs. [1–3, 6]) may lead the reader believe that it is «too simple». In fact, Germain understands that the most suitable framework for formulating the Principle of Virtual Work in Continuum Mechanics must include the tensor algebra and mathematical analysis for introducing the preliminary fundamental kinematical concepts and functional analysis and the theory of distributions for the statement of the principle itself. However, reading carefully the formulation by Lagrange, one can agree that, albeit being somehow generic, it does include all the versions that have been formulated so far, assuming as known Euclidean geometry only. This observation may give an idea of how the principle may be formulated in Hellenistic mathematics.

Piola established as his duty to complete the scientific project that the death of Lagrange had interrupted and added to the Lagrangian original objectives also the establishment of a micro–macro identification procedure linking the properties of bodies microstructure to their macro-properties.

In Piola's 1848 and 1856 fundamental works (see Refs. [96, 145]), one finds

- i) conjectured the kinematics and the work functional structure to be used for introducing a macroscopic model suitable to describe, at macro-level, a system constituted by a large number of interacting «elementary» particles,
- ii) defined n th gradient continua and established the conditions for which this generalized continuum model can be used for describing microscopically the considered set of particles,
- iii) used the micro–macro virtual work identification for obtaining the macroscopic constitutive equations when microscopic geometric and mechanical properties of the system are known;
- iv) determined the structure of contact forces for first gradient continua, obtaining the same representation as Cauchy,
- v) the promise of finding an expression for contact interactions for n th gradient continua in a future paper, which could not be completed because of his death.

Unfortunately, it was necessary to wait until 2012 [8] for seeing this promise kept. This long time delay is related to the relevance, in this context, of too many concepts from differential geometry: The most relevant ones were invented by Gauss and Riemann and by the Italian school of Piola's scientific lineage, that is, Francesco Brioschi (1824–1897), Eugenio Beltrami (1835–1900), Gregorio Ricci Curbastro (1853–1925), and Tullio Levi-Civita (1873–1941).

6.3 | Metamaterial theory: Principle of virtual work versus balance of forces approach

The fundamental problem in the theory of metamaterials, already formulated in the previous sections, consists in the design and construction of materials having prescribed physical behavior.

As already clear in the analysis developed by Piola, and as accepted also by Cauchy and Navier (see Refs. [53, 101]), the physical behavior of materials depend on their microstructure. Albeit they did not have an idea of the real dimensions of molecules, Piola, Cauchy, and Navier tried to deduce the fundamental equation of Continuum Mechanics, valid at macro-level, from the equations valid at the micro-level. In particular, Navier tried to get the equilibrium equations for a continuum starting from the law of interaction between «molecules» (see Refs. [101, 150]). He needed somehow to «average» the balance laws at micro-level to get the corresponding laws at macro-level. Unfortunately, Navier averaging

process had some (not easily removable, as we will discuss in the following) flaws and the equilibrium equations for the displacement named after Navier are applicable only in the case of a very specific value of Poisson constant.

It is, therefore, not controversial (being accepted also by the supporters of Cauchy postulation approach) that one must deduce macro-equations from those postulated at micro-level.

6.3.1 | The problem of unavoidable difference in the kinematical description at micro- and macro-level

What makes the difference between the d'Alembert–Lagrange approaches is in the possibility of identifying macro-models properties in terms of micro-models properties also when the set of macro-kinematical descriptors is different from the set chosen at micro-level.

In particular, one must often face the following situation: At macro-level, it is necessary to introduce more kinematical descriptors than it can be done at micro-level.

This circumstance has its conceptual basis also on a clear phenomenological consideration: One is ready to model a material with a much refined description (that is at a micro-level) only if he can use a simplified interaction law among particles at this level. The epistemological attitude aiming to reduce macroscopic «richness» of phenomenology to the presence of a large number of «simple» single entities is becoming more and more frequent in recent research, in the field which is called «theory of complexity» (see Thurner et al. [151] for an interesting description of the state of the art in this field).

It is easy to understand that even when using the improved version proposed by Eringen (see Refs. [70, 71]) of postulation of mechanics based on «many» balance laws, a nearly unsurmountable difficulty arises if one refuses the postulation based on the Principle of Virtual Work.

In fact, if one has k balance laws at micro-level and H balance laws at macro-level, it is not clear from which balance laws, at micro-level, we get the contributions at balances at macro-level. No general criterion can be given, in the framework of a Postulation *à la Cauchy*, for establishing which micro-balances correspond to how many and which macro-balances.

A simple example, just chosen for illustrative purposes, of such a situation is given in the study of pantographic microstructures (see Refs. [27, 105]): In these microstructures, two families of beams are interconnected by elastic pivots. It is one of the first examples of microstructures whose macro-behavior is modeled by second gradient continua. In these microstructures, the torsional equilibrium of every pivot contributes, at macro-level, at the shear equilibrium of every pantographic cell. For them, it is rather difficult to perform the micro–macro identification based on the consideration of balance laws at both micro- and macro-level also when the correct results have been obtained using variational arguments [152, 153].

6.3.2 | Optimization of microstructure synthesis: Data-driven or theory- driven synthesis of metamaterials?

The scientific program that has to be pursued, in our opinion, in the theory of metamaterial consists in the search of algorithms that allow for the construction of microstructures once the macroscopic models that describe their behavior are fixed. It is more than obvious that given a macroscopic behavior there are eventually many, if not infinite, microstructures that will show the desired behavior. Therefore, to the main synthesis problem just recalled, one will need to associate an optimization problem having two different aspects: (i) specified a given macrobehavior to find among the macroscopically equivalent microstructures the «optimal» one and (ii) given a family of macro-equations and a corresponding family of microstructures to find those pairs macro-equations and microstructures that are «optimal» for a certain specific purpose.

An obvious criterion of optimality, for what concerns the point (i), may involve the request of «easiness» in physical realization: A convenient macro-equation may be synthesized by a microstructure that is difficult to build given the available technology. However other criteria are conceivable, in the different application contexts.

We claim that the hope to obtain interesting metamaterials with a data-driven procedure is surely bound to meet important difficulties and will probably result in vain, at least until when really enormous computing capacities will be effectively available. Moreover, a purely data-driven research is clearly incapable to get meaningful synthesis results. We do not claim that a huge data manipulation could not lead to design exotic metamaterials: What we claim is that without a sufficiently

robust theory, it will not be possible to teach the algorithm producing systematically microstructures how to recognize that a certain microstructure has the specifically desired exotic behavior.

A theory guiding the algorithmic homogenization of the randomly produced micro-architectures seems necessary: Although some partial results in this aspect have been obtained (see e.g., Refs. [67, 104]), we believe that a lot has to be done in this direction.

Moreover, an algorithmic criterion for verifying if a certain microstructure, after homogenization, shows a specific kind of exoticity must be invented: In this direction, we do not see any relevant result yet. In fact, one should not believe that an exotic behavior can be easily recognized or that exotic microstructures can be easily conceived.

The attention paid to pantographic microstructures is a clear indication of the difficulties to be confronted in the optimal synthesis problem: In Refs. [27, 105], it is reported that, for instance, the change of a very small percentage of volume of the total specimen concentrated in the pivots may determine a passage from second gradient to first gradient continuum behavior at macro-level (see De Angelo et al. [154]). An algorithm blindly searching for such a passage may miss it simply because of its «discrete» computing capacities.

6.3.3 | The «power» of the Principle of Virtual Work: micro–macro identification of models having different sets of kinematical descriptors

We believe that theory-driven synthesis of novel metamaterials is made possible by the systematic use of the Principle of Virtual Work, as conceived by Gabrio Piola.

As it is based on a multiple scale microstructure, the design of novel metamaterials requires the simultaneous description of the involved mechanical (or, if the case, multiphysics) phenomena at all relevant different scales. The modeling process, therefore, involves the description at one, or more, micro-levels: Going down and down in the length scale, the microstructure is modeled in more and more detail. The macro-level is defined by the length-scale where the designed microstructure collective behavior arises, which produces the desired properties of the novel metamaterial.

At each level, a suitable kinematical description needs to be introduced: At different levels, depending on the complexity of the involved geometries or of the relevant phenomena, the kinematics can be extremely different. If, for instance, the macro-phenomena are obtained (as in the case of pantographic structures) with complex geometries of the reference configuration, then the kinematical descriptors needed reduce to the placement field, while at meso and macro-level extra rotations kinematical descriptors may be necessary (see e.g., Refs. [155, 156]). Therefore, in this case, the complexity of geometrical microstructure will be described at macro-level with new kinematical fields: While at micro-level, one kinematical descriptor is sufficient, at macro-level, extra kinematical descriptors are needed. On the contrary if one conceives a metamaterial where at micro-level, some electromagnetic phenomena are important, which have a macro-effect on placement field only, then it may happen that the micro-description is richer than the macro-description.

A modeling approach based on balance laws will require the formulation of a balance law for each kinematical descriptor at each relevant length-scale and «clever», to be adapted case by case, physical reasonings for understanding which balance equations, at a certain length-scale, are relevant for a certain balance equation at higher length-scales. One of the authors, in his PhD thesis (see Refs. [157, 158]), was obliged to operate such a difficult identification process and it seemed to him absolutely nonefficient.

Instead, by using the identification of Virtual Work at different scales, as suggested by Piola, we get a «regular and uniform» analytical procedure allowing for the determination of the macro-constitutive equations at macro-level in terms of all relevant micro-properties of the microstructures.

In this endeavor, Piola continues the program set down by Lagrange in his famous preface to his *Mécanique Analytique*:

Les méthodes que j'y expose ne demandent ni constructions, ni raisonnemens géométriques ou mécaniques, mais seulement des opérations algébriques, assujetties à une marche régulière et uniforme.

6.3.4 | Piola's Ansatz

Piola establishes what seems to be the most efficient heuristic homogenization methods: It relies upon the equality of virtual work at the lower length-scale with the virtual work at the immediately higher length-scale (see e.g., Refs. [50, 58, 99]). In order to get a relationship between the different scales virtual work expressions, it is necessary to conjecture a

micro-macro kinematical identification: This conjecture is called a Piola's Ansatz (see Refs. [60–63]). In other words, the modeler must conjecture which is the micro-configuration which «best fits» each macro-configuration, from the energetic viewpoint. Of course, the efficacy of the macro-model completely relies on this only Ansatz.

In the expression of micro-kinematical descriptors, all relevant macro-kinematical descriptors must appear together with some parameters characteristic of the geometry and the mechanical properties at lower level. Most important, in this expression, is the small parameter ε , which must be introduced: It controls the length-scale characterizing the lower level of the microstructure and all relevant properties of the lower-level-structure, including elastic coefficients and any other necessary physical parameter.

The ε parameter appears in the expression of lower-level-placement in terms of the higher-level-placement and eventually in all other relevant physical properties of the material constituting the microstructure. Once the lower-level expression for virtual work functional is expressed in terms of higher-level kinematical descriptors, by using the conjectured expression for lower-level-placement, the expression for virtual work valid at higher level is simply identified by equating the values of the two levels work functionals when neglecting the higher-order terms in ε .

It may happen that in the expression for higher-level virtual work functional, some diverging terms in ε appear: This means that, with the chosen kinematical identification and constitutive parameters' dependence on ε , some constraints appear in the kinematics at the higher level: These constraints are a mathematical consequence of the request that the coefficient of diverging terms in ε appearing in the expression of macroscopic work functional must vanish (see e.g., Barchiesi et al. [134]).

We stress the fact that in presence of multiscale architectures, it is not reasonable to expect that a macro-placement function allows for a detailed enough description of the micro-placement field, as obviously macro-placement function allows for the identification of the micro-position of some and not all among the micro-material particles, which are relevant in the calculation of macro-virtual work. The complexity of the micro-architecture imposes the introduction of some extra kinematical macro-descriptors so that one can, also at macro-level, calculate all the work expended at micro-level.

As a consequence, one should expect that, in general, the macro-behavior of multiscale microstructures requires richer kinematics than the placement, contrarily to what is done in standard Continuum Mechanics (see Germain [3]). The so-called generalized Continuum Mechanics [64–66] is the most suitable «target model» in micro–macro homogenization. Even when the geometrical and mechanical «complexity» at micro-level may allow for the use, at some lower-scale level of Cauchy continua or even discrete Lagrangian models, it is unavoidable at macro-level, but also at some intermediate level, the use of generalized continuum models.

6.3.5 | The need of studying a priori mathematically consistent generalized continuum models

The efforts spent by Navier, but also those in modern statistical mechanics, prove that it is not yet possible to “deduce” from micro-models the most suitable macro-model simply using averaging techniques. The partial success by Navier did produce some equilibrium equations in terms of displacement that are valid in linear elasticity only and that do not account for Poisson effect carefully. In statistical mechanics, only fluid macro-equations were deduced, using Boltzmann equations, from micro-atomistic theories (see e.g. Saint-Raymond [159]).

Albeit this efforts need to be pursued further, the present state of the art advises to continue to start from a conjectured «target» macro-theory in the micro–macro identification process, being aware that rigorous results will be later necessary to mathematically found the proposed conjecture. In fact Gamma-convergence techniques, albeit giving rigorous results proving the energy convergence of micro-models towards macro-model, must be based on a heuristic asymptotic analysis, which leads to a judicious conjecture about the theorem to be proven.

Following the ideas by Piola, as improved and modified in more recent literature, in the heuristic homogenization technique, which seems the most effective one, we can recognize the following subsequent choice-steps:

- a. the definition of a micro-kinematics, which is judged sufficient to describe the class of micro-architectures to which one wants to limit the search in his metamaterial synthesis process: In this way, the space of micro-configurations is specified;
- b. the definition of a «sufficiently rich» macro-kinematics, that will add to the macro-placement also those extra kinematical descriptors, which are considered necessary to account for the deformative phenomena, in the microstructure, that have a macroscopic relevance: In this way, the space of macro-configurations is specified;

- c. a class of generalized continuum models based on the introduced macro-kinematics, characterized with the structure of virtual work functionals to be used in the Principle of Virtual Work applied at macro-level;
- d. a class of generalized continuum or Lagrangian discrete models, based on the micro-kinematics and on the structure of micro-virtual work functional;
- e. a micro–macro kinematical identification: One must select a specific micro-configuration, which represents «at best» the class of micro-configurations corresponding to each of the macro-configurations; this representative ϵ -micro-configuration is given in terms of the corresponding macro-configuration, of the structural cell dimension parameter ϵ , and of all relevant geometrical and mechanical properties of the micro-architecture, which all may be chosen to be rescaled with ϵ ;
- f. the identification of macro-virtual work functionals, and eventual macro-constraints, via an asymptotic expansion in the variable ϵ , obtained calculating the micro-work functionals using the ϵ -micro-configuration: The obtained macro-work functional will depend, therefore, not only on the macro-kinematical kinematical fields but also explicitly on all relevant properties of the microstructure.

It is clear that in order to proceed effectively a preliminary complete knowledge of the properties of generalized continuum models is needed.

Once the just-described procedure has calculated a micro–macro identification, it is then possible to prove mathematically rigorous convergence results (see e.g., Abdoul-Anziz et al. [67]).

6.3.6 | Balance law postulations are less suited for an effective micro–macro identification and for the formulation of generalized continuum models

It is clear, from what we have discussed up to now, that it is not possible to refrain from using both as micro-models, but mainly also for macro-models, generalized continua.

The deduction of a coherent set of evolution and equilibrium PDE equations, initial conditions and, mainly, boundary conditions is easily assured by using the Principle of Virtual Work: The unriveted and concise argumentation supporting this statement is given by already cited excerpts from the masterpiece by Hellinger (see Refs. [60–62, 86]).

On the other hand, one may insist to want to use the postulation based on balance laws (as done in Refs. [68–71]). In this case, he needs to accept a sequence of postulation choices, one logically independent of the others. Instead of postulating the Principle of Virtual Work, that is, a single principle, as one should prefer based on the Occam razor epistemological criterion, he is obliged to resort to «physical intuition»⁶. The balance laws must then be completed by assuming, again on the basis of physical intuition and a series of vague and doubtful (see Rivlin [52]) general criteria (as, for instance, the principles of Equipresence or, more generally, of unification).

The potential incoherences among so many postulates chosen one independently from the others is a sure cause of logical incoherence: The most frequent (but surely not the unique one) cause of «ill-posedness» of the resulting theories is the resulting lack of validity of the conservation of total energy shown by many proposed theories. As already mentioned, Truesdell arrived to imagine a difference between hyper-elasticity and elasticity, which clearly (see Carroll [107]) cannot exist.

Coleman and Noll (see e.g., Coleman and Noll [160]) and their epigones did try to cure the unavoidable inconsistencies by imposing a selection criterion for constitutive equations based on the requirement of some thermodynamic compatibility. Their efforts seem to direct, in a very complicated way, them towards the rediscovery of Hamilton and Rayleigh modeling indicating that a part of described thermo-mechanical phenomena are conservative (so that they can be modeled in the Hamiltonian formalism) while other phenomena are dissipative (and, therefore, need a Rayleigh potential for being described): see, for example, dell’Isola and Placidi [106].

Again it seems that the approach based on balance laws seems to strive painfully to recover, based on physical intuition, the results more easily and elegantly obtained using variational methods.

In fact, it is tempting to identify, when starting from a formulation of the Principle of Virtual Work, the corresponding Euler–Lagrange equations as the volume version of a balance equation and to try to recognize the corresponding boundary conditions as the “boundary version” of the same balance equation. This identification, when possible, gives an impressive source of inspiration to supporters of the balance laws postulations: It is easier to get, from Euler–Lagrange conditions, the right inspiration, and then induce it from «physical evidence» or to conjecture it based on «physical intuition». Remark, however, that in second gradient continua, two boundary conditions are associated to only one PDE,

representing «volume» equilibrium condition: In fact (see Refs. [1, 2, 109]), one has to assign, when dealing with the natural boundary conditions, the dual in work, on the boundary surface, not only of the virtual displacement (what is usually called surface density of contact force⁷) but also the dual in work, on the boundary surface, of the normal derivative of virtual displacement. This second dual vector quantity can be split into its tangent and normal part: The tangent part (see Refs. [1, 2, 109]) can be recognized to be a surface density of couples.

The existence of this kind of contact interaction allowed Richard Toupin [84] to publish his paper [146] in Archives of Rational Mechanics and Analysis. In this paper, the content does not correspond to the title: In the title, only couple stresses are mentioned. Instead, carefully reading it, one discovers that the full second gradient theory is treated, and not only its particular case in which double stress expends work only on the skew-symmetric part of the gradient of virtual displacement.

Toupin's second gradient theory reached a large part of scholars in mechanics simply thanks to a stratagem very similar in spirit to the one which, most likely, allowed us to read the *Mechanica Problemata*. While the clever copyist (we conjecture against the opinion of his head librarian) added a text by Archytas (most likely) to the collection of works by Aristotle (which the Catholic orthodoxy had chosen as the “true” precursor of the Christian dogma), Toupin [84] published the most persuasive «counterexample» to Truesdellian Postulation of mechanics, pretending, instead, that it fully entered the orthodoxy. Toupin called the particular case of second second gradient continua inside which only contact surface couples can be exerted «couple-stress» continua.

In the most general second gradient continua, the «dual in work of the normal component of the normal derivative of virtual displacement» has been called by Paul Germain a «surface contact double force». This kind of interaction of a deformable body spends work on the bodies elongations: The resultant and the resultant moment of a double force distribution are clearly vanishing, albeit the double force participates to the change of deformation energy, for instance. Ignoring the existence of double forces causes serious problems in the formulation of many promising models, of relevance not only in metamaterial theory, but also, for instance, in the theory of capillary fluids (for a detailed discussion of this last point see e.g., Refs. [77, 78]).

The existence of double forces definitively proves that the balance of forces and momentum of forces cannot, alone, be considered the suitable conceptual basis for a comprehensive postulation of mechanics. It has to be remarked that it has been proven that (see Bersani et al. [161]) second gradient continua can be regarded as generalized continua of the kind studied by Eringen [70] when a suitable constraint is introduced, linking the gradient of placement and a tensor of microstructure to which a balance law has been associated. The mathematical difficulties to be overcome for getting this identification are not easy and, once more, the postulation scheme based on the Principle of Virtual Work seems a more elegant and secure guidance.

In conclusion, we believe that it is clear that novel mathematical models, including every kind of generalized continuum, are formulated more confidently when basing them on a particular version of the Principle of Virtual Work (for further arguments supporting this statement see Refs. [7, 102, 106]).

6.3.7 | The principle of virtual work seems the most fundamental conceptual tool in the theory of metamaterials

More specifically, in the context of the theory of metamaterials and metamaterial synthesis, some further final considerations are necessary:

- i. in the problem of synthesis of metamaterials, it is necessary to analyze, homogenize, and optimize a large class of multiscale microstructures in a unified way: In fact, the desired macro-behavior must be obtained by means of the design of microstructures exploiting the micro-geometry in which «simple» materials are organized or by means of the judicious exploitation of multi-physics couplings (an example is given by piezo-electromechanical systems, see e.g., Refs. [48, 162–164], where more specifically, the multiphysics coupling is assured by piezoelectric transducers [165, 166], aiming to get mechanical vibrations control [167, 168]).
- ii. the set of evolution equations postulated to describe the behavior of metamaterials microstructures both at macro- and micro-level must be easily checked to be well-posed: In fact, only when one is assured about existence and uniqueness of the initial boundary values problems conceived to get predictions the search of the optimized tailored metamaterials can proceed expeditiously;

- iii. efficient numerical methods must be developed and used to get predictions about all conceived microstructures, at every scale, where their behavior may be of macroscopic relevance: In fact, it is not conceivable, especially in the case of systems governed by nonlinear equations, to use closed form analytical methods to this aim;
- iv. an efficient theory-driven algorithm must be conceived for selecting, among the candidate microstructures, those which respond to the demanded design requests: Indeed, a data-driven search cannot be used to find extremely «unlikely» configurations⁸, as those that show the most interesting behaviors.

It seems to us that variational methods, and their most abstract version, that is, the Principle of Virtual Work, supply the unifying postulation framework for modeling, at the same time, the behavior at every scale and every involved coupled physical phenomenon in multiscale and multiphysics microstructures. It is not conceivable that postulation schemes based on the choice of a set of balance laws may give such a unifying vision of the complexity involved in the theory of metamaterials.

It may be extremely difficult (without knowing the results as given by a suitable variational method) to identify which balance laws, at lower length-scales, are relevant for a certain balance law at macro-scale: The micro–macro identification process, essential in the theory of metamaterials, may become nearly impossible. In fact, only few specific systems could be homogenized when using models based on the postulations of micro- and macro-balance laws and formidable problems of internal logical coherence arise when using them. Moreover, some energy terms are systematically lost in the transition from micro- to macro-levels. The hard lesson taught to us by the innovative and genial failure of Navier homogenization should be always present in our mind.

7 | FORCE: A DERIVED MATHEMATICAL CONCEPT WHICH IS TWISTED FOR BECOMING A PRIMITIVE CONCEPT

We dare to claim, however, without any evidence in available sources, that balance laws appeared in the literature, at first, as necessary conditions implied by the Principle of Virtual Work to be used when trying to solve specific problems. The only certain fact is that Archimedes uses some balance laws in his hydrostatic studies some (possibly many) years after the pseudo-Aristotelian text, which we like to attribute to Archytas (the following excerpts are from Loeb Classical Library edition of Greek Mathematics Vol.II Translated into English by Ivor Thomas [148]): On pages 242–243, one reads the Postulates from Archimedes On Floating Bodies

Let the nature of a fluid be assumed to be such that, of its parts which lie evenly and are continuous, that which is under the lesser pressure is driven along by that under the greater pressure, and each of its parts is under pressure from the fluid which is perpendicularly above it, except when the fluid is enclosed in something and is under pressure from something else.

On page 247, one sees the first example, as formulated by Archimedes, of a Cauchy cut in a current equilibrium configuration:

Similarly it may be shown that, in whatever other manner the surface be cut by a plane through the center of the earth, the section is an arc of a circle and its center will also be the centre of the earth

Much later (see Refs. [50, 58, 72]), during and immediately after the *l'âge des lumières*, d'Alembert, Lagrange, and Piola applied the Principle of Virtual Work to the mechanics of deformable bodies while Cauchy developed, based on the balance of forces and on his celebrated tetrahedron argument, the concept of contact force and stress. Detailed philological studies are necessary to establish the temporal sequence of the development of the various versions of the concept of stress, as all sources are available albeit they need, still, to be carefully examined (see Refs. [50, 58, 75, 97, 98, 145, 147]). In fact, in the framework of a postulation of Continuum Mechanics based on the Principle of Virtual Work, the stress in a material point is, simply, the volume density dual in the internal work functional of the virtual displacement gradient. Therefore, it is a mathematical object that is primitive: Its existence does not need to be demonstrated, as it is an essential part of a formula accepted as a postulate.

On the other hand, in a postulation based on the law of force balance, the chosen primitive concept is that of «resultant force» applied to a continuum and to its parts: In this framework, the concept of stress needs to be defined after that its existence has been proven. This is what Cauchy did in his celebrated memoir [100].

In the postulation scheme chosen by Cauchy, the particular instance of the Principle of Virtual Work, which is valid for first gradient continua, is proven as a theorem. This circumstance lead many scholars to state a frequent oxymoron, which we have heard even too often: In fact, this last demonstration is often called «the theorem of the Principle of Virtual Work». In this theorem, the Cauchy stress is proven to play the role of the volume density dual in internal work functional of the virtual displacement gradient.

For this reason, many state that the two postulations are proven to be mathematically equivalent: They confuse the particular formulation of the Principle of Virtual Work valid for first gradient continua with its more general possible formulation based on functional analysis concepts (see Refs. [6–8]). As clearly explained by Germain (see Refs. [1, 2]), and in the previous section, already in the case of second gradient continua, this equivalence does not hold anymore.

7.1 | Piola's theorem: A necessary condition for equilibrium involving the mathematical concept, which we call «force»

In fact, when starting from the postulation based on the Principle of Virtual Work, it is easy to prove a theorem that, in agreement to what also stated by Truesdell, we attribute to Piola⁹.

Piola could prove, for a generic generalized continuum in whose kinematics the placement function is included, the balance of forces and momentum of forces theorem, starting from the Principle of Virtual Velocities and from what has been called later the Principle of Objectivity of Internal Work (see Germain [1]):

If a deformable continuum is in an equilibrium configuration, then the resultant and resultant momentum of externally applied forces, calculated in the current configuration of the continuum, must vanish.

Piola, following the nomenclature prescriptions established by d'Alembert cited in the introduction, defines any distribution of externally applied force as the dual in the internal work of a virtual displacement having the same support as the applied force distribution.

The resultant forces and the resultant momentum of forces for a distribution of forces are the vectors, which allow for the representation of the work of this distribution expended in a virtual translation or in a virtual rotation, respectively, as the Internal Work is zero for these virtual displacements, because of the Principle of Objectivity of the Internal Work.

Clearly, resultant forces and resultant momenta of forces are mathematical abstract constructions, which have no immediate correspondence with quantities that can be determined via some measurements. For what we manage to understand, Truesdell in his preface to the first edition of the work [74] clearly criticizes Lagrange on the basis of an inductivist epistemological position: He believes that one can recognize forces as physical objects and can «induce» for them some universally valid laws. A strong hint that this was really his viewpoint is given by Truesdell's epigraph to the textbook [74] citing the paragraph 475 of C.S. Pierce Scientific Metaphysics. Resultant forces and resultant moment of forces have been systematically used, before the advent of digital computers, for calculating equilibrium configurations for deformable structures. For this reason, many mechanicians became so accustomed to them that many started to see them in the physical world.

It is also understandable that in many engineering schools, mechanics has been based on a postulation based on balance laws: In this way, one could pass directly from the postulation to the «practical» exercises. Unfortunately, this very convenient situation became extremely inconvenient when the use of digital computers rendered the formulation of the Principle of Virtual Work unavoidable for getting predictions via numerical simulations based, for instance, on finite element method. With a strange change of roles, the Principle of Virtual Work, having been conceived as the epistemological tool of choice for formulating theories in the most general way, has been rediscovered, then, as a tool necessary for performing calculations and, ironically, demonstrated (*sic!*) starting from balance laws.

Once more, it is true that for first gradient continua, Piola's necessary condition is also sufficient for characterizing equilibrium configurations. However for second gradient continua, which are not couple stress continua and for all n th gradient continua with $n \geq 3$, the balance of forces and momentum of forces are not capable to characterize equilibrium configurations. This fact can be easily proven by remarking that in higher-order continua, contact interactions are not

limited to forces and couples (see dell’Isola and Seppacher [8]) as *kth order contact forces and kth order contact couples are present*.

Unfortunately, the formulation of Continuum Mechanics as imagined by Cauchy (see Refs. [73–75]), has lead many scholars to believe that a mere sufficient condition could, indeed, give a firm foundation of all Continuum Mechanics. The choice of a mere necessary condition for equilibrium, if chosen to be the most fundamental hypothesis for Continuum Mechanics, is bound to limit the set of models, which can be formulated. It has been a lucky circumstance that first gradient continua can describe a relatively rich phenomenology, so that they could be the conceptual tool for developing many engineering design techniques and technologies.

The apparent generality of postulation schemes based on balance laws has been supported by some very formal theorems that gave the illusion of extreme generality: In Truesdell [74], the postulation of balance of forces and momentum of forces is replaced (see page 156) by an Axiom on forces in Continuum Mechanics. It states, using, without true need, a complex formalism taken from set and measure theories, that the interaction force between two continua is bounded by their volume and the area of their contact surface. Therefore, this axiom assumes that contact forces concentrated on edges are vanishing. This axiom hides in the fogs of the abstract language of measure theory¹⁰, a statement which is very simple: The scholar accepting this axiom do not want to include in Continuum Mechanics edge contact forces.

No surprise, then, that in the postulation scheme strongly supported by Truesdell second gradient continua cannot exist (see Gurtin [76]): In these continua, one can have contact forces concentrated on edges. Moreover, as proven in Refs. [78, 103], the existence of edge forces implies the existence of contact double forces. Finally, some experimental evidence about pantographic sheets[27, 105] indicates that the macro-modeling of this class of metamaterials is greatly simplified, both theoretically and numerically, if one allows for the introduction of the abstract concept of surface contact double forces and edge contact forces.

7.2 | Various kinds of forces and stresses: Auxiliary derived concepts defined in the context of a mathematical theory without any direct physical counterpart

Forces, and, in particular, contact forces, are a mathematical concept defined in order to apply Piola theorem to solve some specific exercises. The reader is invited to compare the solution of the problem of the equilibrium of a lever as obtained by applying the Principle of Virtual Work and the Momentum of Forces Balance law (see e.g., Benvenuto [101]): It is clear that the algorithm given by the balance of momentum of forces is more expedite in this specific application, albeit it is clear that the problem was solved at first by using the Principle of Virtual Work.

The use of the concept of resultant forces and resultant momentum of forces outside the specific context given by the applicability range of Piola’s theorem can be, and effectively was, misleading. In order to base Continuum Mechanics on balance laws, it is essential to introduce the concept of contact force inside a deformable continuum. In fact, contact forces are a mathematical invention, already developed most likely in the *On the floating bodies* by Archimedes, for studying the mechanical behavior of fluids. Cauchy generalized the concept to include also the case of solids: This had been also an important part of the scientific program left incomplete by Lagrange and successfully continued by Piola.

This program wanted to formulate Continuum Mechanics as a self-contained theory, not based on any micro–macro identification (as attempted by Navier), but formulated as founded on its own basic principles. While Lagrange and Piola wanted to use, to this aim, the Principle of Virtual Work, Cauchy, instead, wanted to use as fundamental principles the balances of forces and momentum of forces.

It is difficult to understand which were the interactions between these two approaches in the process of founding Continuum Mechanics: (i) what can appear to be a formulation of the Principle of Virtual Work for a general continuum can be found already in Lagrange (Note II of [72]), (ii) Piola started his investigation in 1822 (see Refs. [50, 58]) and Cauchy presented a note (not published) to the French Academy of Sciences in 1822 (see Maugin [147]). The preliminary study of the particular case of fluid continua has been the object of many ancient textbooks and was fully developed by d’Alembert and Lagrange, both by applying the Principle of Virtual Work.

It has to be remarked that Truesdell attributes to Euler the choice of basing mechanics on balance laws (see Truesdell [169]):

Lagrange’s best ideas in mechanics derive from his earliest period, when he was studying Euler’s papers and had not yet fallen under the personal influence of d’Alembert [...].

It seems that Truesdell considers Lagrange as a «weak» personality easily influenced by the stronger personalities around him. This conjecture is clearly falsified when reading any Lagrange's biography. Moreover, it is well-known that Euler himself (see Benvenuto [101]) has deduced the equations of the *Elastica* from the Principle of Total Potential Energy Minimum.

The relationship between the work functional and any kind of distributed forces, k -forces, couples, and k -couples can be comprehensively described by using the concepts of functional analysis (see Refs. [6, 8]). In fact, given the space of virtual displacements, to be chosen suitably depending on the particular mechanical model introduced, the set of work functionals coincides with the set of the linear and continuous functionals defined in the space of virtual displacements and all kind of forces are derived concepts defined in terms of work functionals.

Therefore, as already remarked by Germain, the theory of distributions by Laurent Schwartz seems the most suitable conceptual tool for framing Continuum Mechanics: In fact, the set of distributions is the widest conceived set of linear and continuous functionals available, up to now, in the mathematical literature.

To be more mathematically precise: If the set of virtual displacements includes the set of infinitely differentiable functions having a compact support included in the continuum reference configuration (i.e., the set of test functions) and if this set is endowed with the Fréchet topology, then any work functional must be a distribution.

In the particular case, in which the set of virtual displacements is the L^2 Hilbert space defined on a volume, a surface or a line, then by Riesz representation theorem, any linear and continuous functional defined in the set of virtual displacements can be represented via the inner product with a uniquely determined element of L^2 . If the virtual work functional is postulated to be defined in L^2 then the specific element found by Riesz theorem, being the dual in work of a virtual displacement, is called a volume, surface or line distributed «force».

When the virtual displacement space includes only rigid virtual displacements, that is, infinitesimal translations and rotations, then the Hilbert space reduces to the pairs of translation vectors and rotation vectors: The pair of vectors, which allow for the Riesz representation by means of the Euclidean inner product, is called, respectively, Resultant Force and Resultant Momentum of Forces.

Due to a theorem of representation by Laurent Schwartz [4], the most general Lebesgue absolutely continuous distribution D having a volume compact support Ω (see dell'Isola et al. [8]) can be represented by the formula

$$D(\phi) = \sum_{i=0}^N \int_{\Omega} T^i | \nabla^i \phi \quad (1)$$

where ϕ is the generic test compact function concentrated in Ω , ∇^i is the i th gradient operator, the symbol $|$ denotes the saturation between tensors and the i th order tensorial measure T^i , whose existence is assured by Laurent Schwartz theorem, is called i th order stress tensor. When the internal work functional for a continuum has the form given by 1, we say that the continuum is a n th order continuum. A particular class of these continua is those for which a deformation energy exists, which depends on all the gradients of placement up to the n th order. A more particular class is that of first gradient continua or Cauchy continua, where $N = 1$.

When the distribution D is concentrated on a regular submanifold M without boundary embedded in the Euclidean three-dimensional space, then the Schwartz representation theorem reads

$$D(\phi) = \sum_{i=0}^N \int_M f^i | \nabla_{\perp}^i \phi \quad (2)$$

where ϕ is the generic test compact function concentrated in M , ∇_{\perp}^i is the i th normal gradient operator defined on M and the i th order orthogonal tensorial measure f^i whose existence is assured by Schwartz theorem can be split into what is called i th force or i th couples. When the external work functional concentrated on surfaces or edges or points of the boundary of a continuum has the form given by 2, we say that the applied external i th force or i th couples are concentrated on surfaces, lines, or points.

The statement of the representation theorem by Riesz and Schwartz clearly shows the concept of i th force, i th couples, or i th order stress tensors are obtained by means of definitions based on existence and uniqueness demonstrations. Such definitions are made possible by the development of a mathematical deduction, which is used for obtaining logical consequences from the basic principle of Mechanics: the Principle of Virtual Work.

Albeit these concepts of forces, couples and stresses are of great importance, the logical structure of Mechanics implies that they do not correspond to any quantity, which can be directly measured: They represent pure mathematical artifices useful in mathematical reasoning, exactly as stated by d'Alembert.

These artifices may be very useful in calculating equilibrium configurations, especially when one cannot or does not want to use numerical methods. Of course, in the practice of structural engineers, before the advent of digital computers, the so-called method of forces (due to Maxwell (see Refs. [170, 171]) and demonstrated by using the Principle of Virtual Work) for calculating the deformed shapes of statically indetermined structures by using repeatedly the forces and momentum of forces balance laws was the only viable «practical» technique for getting predictions from the theory of elasticity: As a result, it became more expeditious to start its presentation, in structural mechanics, founding it on balance laws. Albeit some resistance to this tendency was exerted by those who kept calculating the reactive forces using Castigliano Theorem, the method of forces became paradigmatic and widely diffused. It is clear that such a deeply rooted tradition of calculations, surely played a role in accustoming scholars in Engineering Sciences and Physics to the concept of force and momentum of forces at the point that they started seeing forces in the real world, giving to the concept a surreptitious ontological status, recognizing them as objects belonging to the physical worlds.

The free body diagrams are the computational tool, inherited from the Hellenistic algorithmic tradition based on drawings, which has been used systematically in engineering practice until very recently, and is still imposed to engineering students, as if the revolution of digital computers had not occurred yet. In presence of constraints, by introducing the corresponding Lagrange multipliers, one can complete the free body diagrams by including, together with active forces also the reactive forces. The definition of reactive forces, introduced as Lagrange multipliers again using variational methods, and assuming that they act together with the active forces leads to enlarge the validity of fundamental balance equations for Mechanics also in the case of structures constituted by deformable constrained bodies. With the ad hoc assumption concerning the existence of reactive forces, once more!, one replaces a consequence proven by using a variational principles with a «law» obtained «by induction» from experience.

It can be concluded that balance laws represented a powerful computational tool, whose efficacy had been deduced from more fundamental variational principles and that, because of a natural drift of the literature towards «simplifications» they became fundamental principles themselves.

On the other hand, after the digital computing paradigm change, it was clear that it was necessary to resort again either to the Total Potential Energy Minimum Principle or to the Principle of Virtual Work. These principles, in the dominating paradigm, were recovered (in a strange «coming back» logical process) as theorems starting from the believed most fundamental induced physical laws, and the Principle of Virtual Work was disqualified as a «technical» and less important concept useful only in some applications: The Truesdellian despise and deep contempt for computers (see Truesdell [172]) made the rest for leading his school to undervalue it.

The materialization of the abstract mathematical concepts of force and stress is the ultimate consequence of an epistemological attitude, which considers «induction» of physical laws the main objective of mathematical physics. In the presentation of Truesdell's Continuum Mechanics, one finds statements that attribute to the concept of forces a physical reality and that considers the laws of balance of forces and moments of forces as sure consequences of «physical evidence».

Such viewpoint risks to lead to a more than Platonistic vision of mathematics: In fact, Platonistic epistemology considers physical objects to have a reality, which is different from the reality of mathematical objects. Plato considers the «world of ideas» as having a distinct existence from the «world of real objects» and believes that mathematicians are exploring the world of ideas with their mathematical investigations: Mathematicians, for Plato, are not inventing but simply discovering mathematical results! Instead inductivism leads to the confusion between mathematical objects and the physical objects that they are modeling and, worse, to the materialization of every mathematical object, which is used in the development of a mathematical theory. In a sense, Truesdellian are materializing a functional defined in a Sobolev space and try to induce physical laws about them based on experiments or, worse, on «physical intuitions».

On the contrary, the concepts of force, stress, and couple have a very precise role in the development of that theory called Continuum Mechanics in which they were defined in a rigorous and precise way. As claimed by d'Alembert, they can be used only in that precise context and with the precise meaning given to them by their mathematical definition. In particular, they can be of use in studying some specific equilibrium configurations or some specific motions.

7.4 | The concept of moment of forces and torques: A derived concept, which also «must» become primitive. The start of addition of ad hoc fundamental laws

The balance of force was assumed to be an experimentally «induced» physical law or, in the best case, a postulate on which one conjecturally bases mechanics. This single law appeared immediately to be not sufficient to characterize equilibrium configurations since the works by Euler.

Therefore, in the Truesdellian approach (see Truesdell [85]), it appeared necessary to complement the law of balance of forces with, at least, a further one: the balance of torques. However, the «true» fundamental concept is believed to be that of «force». Therefore, even when postulating an independent balance law, this new law must be related somehow to the concept of forces, by relating the concept of torque to the concept of force. *Dans la pensée* de Truesdell, one can see, an evolution: He manages to get rid of the word force, when postulating (or inducing?) the further needed balance law: In his Truesdell [74], he introduces—as an independent postulate—the balance of torques, albeit somewhere else he claims that «torques have been defined in terms of forces and couples». To explain his viewpoint, one has to conclude that either he has difficulties also with Toupin’s couple stress continua, or he imagines a “micro-level” corresponding to them in which only forces are «dominating the scene». Eringen and Capriz, on their turn, arrive to the only tenable position in the field of balance laws supporters: They state that one has to postulate many balance laws, each for every kinematical field introduced.

Probably Truesdell was aware of the need to obey to the Occam razor criterion in formulating Continuum Mechanics and for this reason did oppose to Eringen’s views: On the other hand, Eringen was aware of the need of considering generalized continua with richer kinematics. Neither of them did seem to have realized that the principle of Virtual Work does manage to give the correct conceptual framework tempering both their well-founded epistemological doubts.

This incapacity of recognizing, which is the correct conceptual framework, is often related to the scientific epistemological consciousness decay. When a school of scholars fails to understand the difference between models and reality, a total conceptual confusion arises in their theories and it systematically happens that objects of lesser importance, such as forces, torques, and stress as compared to energy or work in Continuum Mechanics, appear to have a more central role.

To make clear the viewpoint expressed by Truesdell, we quote here the words with which he concludes his work [85] (the bold is ours):

It seems to me that the approach to mechanics outlined by the following steps is a practical one today:

1. The axioms satisfied by forces and torques are motivated, and to some extent verified by experiments, in statics. These axioms are taken over unchanged in kinetics.
2. The principles

$$F = 0, L = 0 \quad (8) \quad (3)$$

as **necessary and sufficient for equilibrium** are motivated, and to some extent verified by experiment, in statics.

7.5 | Confusing a necessary condition with the fundamental principle: The materialization of forces and stresses

Cauchy decided to found Continuum Mechanics by postulating Euler necessary conditions for equilibrium as the most fundamental principle in the work [100]: that is balance of forces and torques.

He needed

- i. to devise the imaginary «Cauchy’s cuts» dividing any continuum into disjoint parts,
- ii. to postulate that on each part, obtained the balance of forces and torques holds,
- iii. to state that on these cuts, some contact interactions between the separated parts arise because of the separation,
- iv. to assume the existence of the stress vector, in order to calculate resultant forces and moments of forces on the so-isolated part of the continuum,

v. to accept the so-called Cauchy «postulate», which assumes that the stress vector depends only on the normal of Cauchy cuts¹¹.

He explains the onset of contact interaction as a «logical» consequence of the removal of a part of the continuum obtained via the Cauchy cuts: They are necessary to keep the considered part of the body in equilibrium.

While the list of ad hoc assumptions made by Cauchy contradicts the Occam Razor demands, per se his idea is suggestive and undoubtedly elegant. It may be useful to interpret the process of integration by parts performed by Piola (as already suggested by Piola himself, see Refs. [50, 58] and for more details Refs. [1, 109]) for deducing the equilibrium PDEs and boundary conditions from the Principle of Virtual Work. It has to be explicitly remarked here that assuming the hypothesis (v) Cauchy excludes the possibility to introduce second gradient continua, as in them (see Germain [1]), the stress vector depends on the curvature of Cauchy cuts. Also for this reason, the approach *à la d'Alembert* seems preferable in Continuum Mechanics.

The Cauchy treatment introduces a number of mathematical abstract concepts in the model of first gradient continuum. They have the sole purpose of logically connecting different concepts mathematically defined in the framework of the model: Only few of these concept have a correspondence in the physical world while others have no observational relevance. We claim that it is impossible to imagine an experiment for directly measuring, in any material point of the continuum, the local value of stress, that is, that linear map that linearly associates to every normal to Cauchy cuts the Cauchy vector.

The clever use of the assumption (i)–(v), and by means of the celebrated tetrahedron argument, leads Cauchy to prove the existence of the stress tensor. The Principle of Virtual Work, in the form valid for first gradient continua, can be then proven by using the PDEs and the boundary conditions deduced from the balance of force and momentum of force. The elegance of Cauchy proof lead many scholar to believe that Continuum Mechanics can be found only by means of a tetrahedron argument: For this reason, this argument was applied in dell'Isola et al. [115] (see also [59]), relaxing the assumptions (i)–(v) to prove the existence of the n th order stresses from a postulate of quasi-balance of work¹².

To be able to accept the assumption (i)–(v) very often one persuades himself that he can «observe» the stress state of a deformed body (remark that already with this statement we confuse the physical object with some mathematical entity). In this context, the question arises immediately: What is a force and how one can measure it? In fact a stress is a force per unit area, and one can arrive to believe that he can measure such a force through a Cauchy cut.

In many physics textbooks, one reads that «a force is what is measured using a dynamometer». However, a dynamometer measures displacements only. One needs a theory, in this case, Hooke's law, to deduce from this measurement the measure of a force. Without the intermediation of a theory, one cannot measure directly a force. The concept of force does not have any direct observational meaning.

Notwithstanding, this obvious consideration one hears very often talking about stress as it was a more fundamental concept than elongation. The need of accepting «laws of physics» involving forces requires their materialization: This step is necessary to make such laws «intuitive» and «physically understandable».

The materialization of mathematical concepts is a very dangerous social phenomenon when it is accepted in a too large group of scholars. It leads to a dangerous detachment of the intellectual work from the world reality and technology. On the other hand, such a detachment usually produces a reaction: some skeptics¹³, while criticizing this materialization by underlining the obvious fact that the models cannot coincide with the real objects, violently discredit science and scientific technology. Skeptics' arguments are the reason for the diffusion of the belief that theoreticians are not capable to respond to the demands arising from «practice».

Of course, these skeptical beliefs completely ignore that, just to name few, Archytas of Tarentum, Archimedes, d'Alembert, Lagrange, Maxwell, and more recently Leonid Ivanovich Sedov, Richard Toupin, and Paul Germain did prove that theoretical knowledge and technological capacities are always simultaneously present in advanced societies and in eminent scientists.

The inductivist attitude leading to the materialization of stress and force was accepted by too many scholars active in Continuum Mechanics: This circumstance is alarming. The presence of this epistemological misunderstanding, in an epoch in which technology continues to develop may lead to a dangerous imbalance between scientific culture, which must consciously control and dominate the technological applications, and the «power» gained by the improvement of technological capacities.

Why one should bother with the epistemological misconception leading to the materialization of forces? The question is that skeptical criticism is always lurking: Starting to accept clearly false statements about real objects, because one

confuses them with the mathematical objects used in a specific model to describe them, systematically leads to the refuse of the whole scientific vision of nature.

8 | SOME CONCLUSIONS

The art of mathematical modeling requires a clear epistemological awareness. It is surprising how authoritative and otherwise very skilled scholars may direct generations of scientists towards the wrong directions simply for a lack of understanding of the basic methodological tools of scientific discovery [93].

8.1 | What happens when one confuses the objects of a mathematical theory with the objects in the real world? Skepticism!

Having discussed with too many students, who had developed it independently by themselves, this subject, we explain how inductivism leads to the skeptic philosophy. It was presented by Sextus Empiricus already in mid-late 2nd century AD in his *Against the Physicists* and *Against Those in the Disciplines* (see Refs. [173, 174]). Those who believe that history of philosophy is a waste of time for young scientists hopefully will change their opinion after having read the following excerpts. In fact, it is difficult to understand that the scientific method proceeds via a complex sequence of steps: (i) the formulation of falsifiable conjectures leading to the formulation of postulates, (ii) the development of mathematical theories for deducing from the postulates some theorems, (iii) the characterization of those mathematical objects that model some specific measurable quantities, and (iv) the corresponding measurements, aiming to check that the theorems formulated with the theory lead to predictions, which are not falsified.

Mathematical theories and postulates cannot be proven, and the mathematical abstraction is required to get powerfully predictive models. All this process requires a lot of study. In fact, etymologically mathematics means exactly: *what needs to be studied longly for being understood!*¹⁴

Of course, those who do not want (or are not capable) to study then decide to try to find simplifications. The simplest epistemological simplification leads to accept the postulates are not conjectures: they are «true» and can be «induced» as «physical laws». The next step is to confuse the models with the objects and finally many, like Sextus Empiricus, decide that all these theories are nothing else than a fraud conceived by those who call themselves theoreticians, who want to live without really working. These skeptics are really diffused even nowadays, and their persuasion is more and more deeply rooted by the existence of some theories, which are so formal to result to be actually empty.

Sextus Empiricus has to be placed in a special position in the history of science because he is the first known representative of such everywhere, in space-time and cultures, diffused «denouncers» of mentioned «fraud».

Postponing to further studies the many cues of discussion, which arise when reading Sextus, we limit ourselves to describe what happened when inductivism deformed the lost models for bodies and Euclidean geometry elaborated by Hellenistic science. Once the confusion between models and reality was complete, Sextus started denying even the existence of bodies. Moreover, he «easily» confuted the existence of points, lines, and surfaces.

In Bett [173] on page 69 and 70, we read

b. Mathematicians' conception of body, and some initial impasses to which they lead (367–75)

[367] *But the issue should now be organized in terms of the mathematicians' conceptions. They say that body is what has three dimensions, length, depth and breadth, [...]*

[368] *Now from this conception a rather large number of impasses seems to follow. For either body, in terms of its conception, is separate from these three dimensions, so that body is one thing, and body's length, depth and breadth are another, or body is a collection of these dimensions.*

[369] *But separate from these dimensions it is not possible to conceive body; for where there is neither length nor breadth nor depth, it is not possible to conceive body either. But if the collection of these constitutes body, then since each of them is incorporeal, and what is composed of incorporeals is surely incorporeal, the combination of them in common will also have to be not a body but an incorporeal.*

[370] *For just as the coming together of lines, which are incorporeal, and the collection of points, are never such as to make a solid body with resistance, so too the combination of length, depth and breadth, being a combination of incorporeals, will not produce a body. And if body is neither something apart from these, nor are these body, nothing is a body.*

The author has heard exactly this kind of criticisms in discussions about the so-called axioms about the universes of bodies (see page 7 and ff. in Truesdell [74]). The naive inductivistic attitude presented there did produce a skeptical reaction. Sextus conclusion is

[375] *If, then, the body being conceived is not there, either before their coming together or after their coming together, it is not possible to conceive body.*

Skeptics arrive to a drastic ontology: nothing can exist.

Also very interesting is how Sextus «demonstrates» that the basic concepts of Euclidean geometry are «non-existing»: In Bett [173] on pages 70–72, we read

c. Arguments for the nonexistence of length (or breadth or depth), in terms of which body is conceived (375–429)

i. Introduction

[375] *In addition to this, if length or breadth or depth are nothing, nor will there arise the body that is conceived in terms of participation in these; but length and breadth and depth are nothing, as we will establish; therefore there is not body.*

ii. Argument from the nonexistence of a “partless sign,” that is, point (376–379)

[376] *For there is not length (since this is the most important dimension of body), which is called line by the mathematicians; and the line is a point that has flowed, and the point is a partless sign without dimensions. Hence if there is no partless sign without dimensions, nor will line come into being; and if there is not line, nor will there be length; and if there is not length, nor will body subsist; for it is together with length that body is conceived.*

It happened to the author to hear this reasoning from a student in engineering: He did not manage to understand geometry. As a consequence, he developed again skeptic philosophy.

Sextus continues by stating

iii. Argument from the impossibility of generating a line out of one or more partless signs (380–389)

[380] *Besides, even if it is allowed that there is a sign, there will not be length. For length is a line, and the line is the flowing of a sign. Either, then, the line is one sign stretched out, or it is conceived as many signs lying in a row.*

[381] *But if it is one sign stretched out, it cannot be a line. For this sign either stays in the same place, or it changes from place to place. And if this sign stays in the same place, it will not be a line but a point;*

Here Sextus refuses to use the Euclidean method of constructing curves by allowing a point to move. Also, he criticizes the concept of motion, which is at the basis of kinematics.

Sextus refuses to distinguish the concept of defined mathematical object from the concept of existing real-world objects.

So the line is not one sign.

[386] *Then again, nor is it many signs lying in a row. For these signs either touch each other (as a feature of their conception), or they do not touch each other, but are marked off by some places that come in between them. And*

if places come in between them, they no longer make one line. But if they touch one another, either they touch as wholes on wholes, or they touch parts with parts.

[387] *But if they touch parts with parts, they will no longer be partless;*

The confutation of the concept of point and line leads to the confutation of the concept of body!

[389] *If, then, in order for body to be conceived, length has to be conceived, and in order for length to be conceived, line, and for it, sign, then since the line has been shown to be neither a sign nor a composite of signs, the line is nothing. But if there is no line, nor is there length; from which it follows that no body exists either.*

Finally, we conclude citing some ancient objections to the concept of surface that we have heard, some millennia later, for claiming that the theory of plates is «meaningless». One reads on page 78 of Bett [173] the

Argument against body based on the concept of a surface as the limit of a body (430–436)

[430] *Besides, even if we allow that the line is a breadthless length, the argument about body will be found no less intractable for the geometers. For just as the sign produces a line when it has flowed, so too the line, when it has flowed, produces a surface, which is the limit of a body and has two dimensions, length and breadth.*

[431] *Since, then, the surface is the limit of a body, the body is definitely limited. But if so, when a body is placed alongside a body, either the limits touch the limits, or the things limited touch the things limited, or both the things limited touch the things limited and the limits touch the limits.*

To make clearer the confused previous argument Sextus, being a practical man, makes an example!

For instance (for what is being said will be clear with the aid of an example), if we conceived the outside pottery of the wine-jar as the limit, and the wine in the wine-jar as the thing limited, then when two wine-jars are placed alongside one another, either the pottery will touch the pottery, or the wine the wine, or both the pottery will touch the pottery and the wine the wine

The following conclusion is unavoidable: It is impossible to decide if surfaces exist or do not exist:

[434] *Then again, if the surface is a limit, and the body is a thing limited, the surface is either a body or incorporeal. And if it is a body, it is false that the surface is depthless; for every body has a share of depth. Also, the limit will not even touch anything, but every body will become infinite in magnitude;*

[435] *for if the surface is a body, then since every body has a limit, that limit too, again being a body, will have a limit, and that one a third, and the third a fourth, and so on to infinity. But if the surface is incorporeal, since the incorporeal cannot touch anything or be touched by anything, the limits will not touch one another, and if these do not touch, nor will the things limited touch.*

[436] *So that even if we forget about the line, the argument about surface is intractable and puts us into suspension of judgement.*

8.2 | Was Cauchy an inductivist? D'Alembert, Lagrange, and Piola were not!

The «Cauchy straitjacket» consists in an epistemological belief that one can «induce» some laws of physics and that Continuum Mechanics can be based on balance of forces and balance of torques and that Cauchy tetrahedron argument leads to the proof of the existence of stress inside deformable bodies.

The Cauchy straitjacket has blocked for several decades the development of generalized continua theory and had, as a consequence, the delay of the development of metamaterials theory. It should be investigated if in the *corpus* of the works

by Cauchy if his epistemological viewpoint was really inductivist or if he had recovered somehow the Archimedean *ante litteram* falsificationist attitude (for a discussion about Archimedes epistemological awareness see, for example, Refs. [62, 80, 175]). In fact, in the literature, it is not a rare situation that «Aristotelians» are much more biased and erroneous than the Schoolmaster in their scientific positions and beliefs.

What is sure is that d'Alembert, Lagrange, and Piola (see Refs. [50, 58]) made clear the difference between mathematical model and physical systems that they describe, and, therefore, they could conceive to improve a model in order to get a more predictive one in the description of a specific phenomenology. They did never try to prove via induction the Principle of Virtual Work: Instead they carefully tried to get predictions from its formulation in specific modeling instances and, when they found that their predictions were falsified or even simply imagined that such a falsification may occur, they formulated a more general version of the principle to extend the predictivity of their theories.

This very effective attitude needs to be used in the difficult challenge represented by the problem of synthesis of meta-materials: exotic materials must be characterized by novel internal work functionals, microstructures must be designed that have a macro-behavior described by these functionals and 3D printing techniques must be conceived to construct so-designed materials.

8.3 | The sociological process of knowledge transmission needs careful investigation

In order to understand the reasons of (and the motivations for) the epistemological debate about the most suitable postulation to be used in mechanics the understanding of the sociological phenomena occurring in transmission and re-elaboration of scientific theories is necessary. While the interested reader is referred to Spagnuolo et al. [80], here we limit ourselves to observe that it is very important to study the phenomena involved in the process of transmission of an original theory and in the modalities of the theory's transformation during the transmission.

To get some solid foundation in this kind of studies, and in particular for what concerns the origins and the transmission of the Principle of Virtual Work, the most important problem to be faced concerns the determination of the «first» or «more original» sources of a certain theory.

For this reason, we are happy to have assessed a nearly unanimous agreement about the fact that the pseudo Aristotelian *Mechanica Problemata* dates to the 4th or 3rd century BC and that in it, without any doubt, the Principle of Virtual Work is applied to solve engineering problems.

The personalistic research of the first genius who did invent a theory is not so important: Instead, it is extremely important to understand which was the discovery process that, at first, lead to formulate a novel mathematized mechanical theory: We claim that such a first theory is that of the equilibrium of a lever or that of the functioning of machines¹⁵. It is clear that learning from history how novel theory were invented may help us to invent newer theories.

Unfortunately, it is very frustrating to observe in the History of Science a systematic distortion of scientific models from their original form: Mechanics being founded on the Principle of Virtual Work has been distorted by a vision, which considers a necessary condition, that is balances of forces and torques, to be the most fundamental principles, on the basis of a naive inductivistic position.

Albeit balance laws were formulated later, they are not the most adapted conceptual basis for mechanics. This circumstance falsifies an often assumed postulate in epistemology: When time is passing, then the accumulation of knowledge produces a permanent increase of scientific understanding and technological capacity. Unfortunately, historical evidence proves that human technological capacity and scientific knowledge did experience some dramatic regressions.

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ENDNOTES

¹Such a purpose is always meritorious, as the study of the sources of every discipline soon or later reveals precious for the advancement of human knowledge. The reader is invited to consider that Italian and then European Renaissance was initiated by the philological studies of Latin and Greek texts.

- ²For instance, some of the most famous results by Archimedes were rediscovered, thanks to Tartaglia, in a somehow modified form (see again Spagnuolo et al. [80]).
- ³One should remark that, at least up to the collapse of Soviet Union, every student in scientific faculty was obliged to take at least one course in epistemology.
- ⁴Inductivists «discover» laws, which are ontologically existing outside their mind, exactly how Colombo discovered the Americas.
- ⁵Already Gramsci did write that: «it will be necessary to resist the tendency to render easy that which cannot become easy without being distorted. [149]»
- ⁶What is physical intuition is not clear: Hopefully, this expression is not used as a kind of replacement for the concept of «induction», used to «prove» physical laws.
- ⁷The reader should recall that, as d’Alembert: *je dois avertir que pour éviter les circonlocutions, je me suis souvent servi du terme obscur de force*, see the Introduction.
- ⁸Albeit we cannot give, for the moment, any rigorous statement substantiating such a bold statement, we dare to say that in the set of conceivable candidate multiscale microstructures, the set of «exotic» ones to be used in metamaterial design have a vanishing measure. Of course for giving a more precise meaning to this vague statement, one should introduce both the set of candidate microstructures and a suitable measure in it.
- ⁹We suspect that a careful investigation in the works by Euler may lead to the discovery of such a statement also there.
- ¹⁰Condescending to the most extreme degenerate Bourbakistic attitude, which surely goes much beyond the intention of the original Bourbaki group.
- ¹¹We reiterate that the logical status of this assumption is not that of a postulate, as is, instead, the Principle of Virtual Work. As clearly proven by Germain it is, simply, a constitutive assumption.
- ¹²Clearly, after having postulated the Principle of Virtual Work, it is possible to deduce the corresponding Euler–Lagrange conditions, including a set of PDEs, each corresponding to an independent kinematical field and its virtual variation, and the associated boundary conditions. Knowing them it is easy to reformulate the same mechanical model postulating the needed balance laws, each one for each independent Euler–Lagrange condition already obtained from the original Principle of Virtual Work.
- ¹³In Western culture, the prototype of such a skeptical scholar is believed to be Sextus Empiricus (Σέξτος > Εμπειρικός): The interested reader may consult Refs. [60–62, 79] for a discussion about the role of skeptical philosophy on the decadence of scientific culture.
- ¹⁴See Thomas [148], Vol.I, page 2.
- ¹⁵In this we are also helped by the etymology of the word mechanics, which means the «science pertaining to machines».

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