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# WELL-POSEDNESS, GLOBAL EXISTENCE AND LARGE TIME BEHAVIOR FOR HARDY-HÉNON PARABOLIC EQUATIONS

BYRAME BEN SLIMENE<sup>1,2</sup>, SLIM TAYACHI<sup>1</sup>, AND FRED B. WEISSLER<sup>2</sup>

ABSTRACT. In this paper we study the nonlinear parabolic equation  $\partial_t u = \Delta u + a|x|^{-\gamma}|u|^\alpha u$ ,  $t > 0$ ,  $x \in \mathbb{R}^N \setminus \{0\}$ ,  $N \geq 1$ ,  $a \in \mathbb{R}$ ,  $\alpha > 0$ ,  $0 < \gamma < \min(2, N)$  and with initial value  $u(0) = \varphi$ . We establish local well-posedness in  $L^q(\mathbb{R}^N)$  and in  $C_0(\mathbb{R}^N)$ . In particular, the value  $q = N\alpha/(2 - \gamma)$  plays a critical role.

For  $\alpha > (2 - \gamma)/N$ , we show the existence of global self-similar solutions with initial values  $\varphi(x) = \omega(x)|x|^{-(2-\gamma)/\alpha}$ , where  $\omega \in L^\infty(\mathbb{R}^N)$  is homogeneous of degree 0 and  $\|\omega\|_\infty$  is sufficiently small. We then prove that if  $\varphi(x) \sim \omega(x)|x|^{-(2-\gamma)/\alpha}$  for  $|x|$  large, then the solution is global and is asymptotic in the  $L^\infty$ -norm to a self-similar solution of the nonlinear equation. While if  $\varphi(x) \sim \omega(x)|x|^{-\sigma}$  for  $|x|$  large with  $(2 - \gamma)/\alpha < \sigma < N$ , then the solution is global but is asymptotic in the  $L^\infty$ -norm to  $e^{t\Delta}(\omega(x)|x|^{-\sigma})$ .

The equation with more general potential,  $\partial_t u = \Delta u + V(x)|u|^\alpha u$ ,  $V(x)|x|^\gamma \in L^\infty(\mathbb{R}^N)$ , is also studied. In particular, for initial data  $\varphi(x) \sim \omega(x)|x|^{-(2-\gamma)/\alpha}$ ,  $|x|$  large, we show that the large time behavior is linear if  $V$  is compactly supported near the origin, while it is nonlinear if  $V$  is compactly supported near infinity.

## 1. INTRODUCTION

In this paper we consider the singular nonlinear parabolic equation

$$\partial_t u = \Delta u + a|\cdot|^{-\gamma}|u|^\alpha u, \tag{1.1}$$

$u = u(t, x) \in \mathbb{R}$ ,  $t > 0$ ,  $x \in \mathbb{R}^N \setminus \{0\}$ ,  $N \geq 1$ ,  $a \in \mathbb{R}$ ,  $\alpha > 0$ ,  $\gamma > 0$  and with initial value

$$u(0) = \varphi. \tag{1.2}$$

The case  $\gamma = 0$  corresponds to the standard nonlinear heat equation. For  $\gamma < 0$  it is known in the literature as a Hénon parabolic equation, while if  $\gamma > 0$  it is known as a Hardy parabolic equation. In this paper we are concerned with the case  $\gamma > 0$ . We are interested in the well-posedness of (1.1) with initial data  $\varphi \in L^q(\mathbb{R}^N)$ ,  $1 \leq q < \infty$ , and in  $C_0(\mathbb{R}^N)$ . We also study the existence of global solutions, including self-similar solutions and prove the existence of asymptotically self-similar solutions.

In what follows, we denote  $\|\cdot\|_{L^q(\mathbb{R}^N)}$  by  $\|\cdot\|_q$ ,  $1 \leq q \leq \infty$ . For all  $t > 0$ ,  $e^{t\Delta}$  denotes the heat semi-group

$$(e^{t\Delta} f)(x) = \int_{\mathbb{R}^N} G(t, x - y) f(y) dy, \tag{1.3}$$

where

$$G(t, x) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^N, \tag{1.4}$$

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and  $f \in L^q(\mathbb{R}^N)$ ,  $q \in [1, \infty)$  or  $f \in C_0(\mathbb{R}^N)$ . For  $f \in \mathcal{S}'(\mathbb{R}^N)$ ,  $e^{t\Delta} f$  is defined by duality. A mild solution of the problem (1.1)-(1.2) is a solution of the integral equation

$$u(t) = e^{t\Delta} \varphi + a \int_0^t e^{(t-s)\Delta} (|\cdot|^{-\gamma} |u(s)|^\alpha u(s)) ds, \quad (1.5)$$

and it is in this form that we consider problem (1.1)-(1.2).

We first consider local well-posedness for the integral equation (1.5). To our knowledge, there is only one previous result of this type, Wang [15], who works in the space  $C_B(\mathbb{R}^N)$  of continuous bounded functions. For  $N \geq 3$ ,  $a > 0$  and  $\gamma < 2$ , he proves local existence of solutions to (1.5) in  $C([0, T]; C_B(\mathbb{R}^N))$  for all  $\varphi \in C_B(\mathbb{R}^N)$ . See [15, Theorem 2.3, p. 563].

In this paper, we prove local well-posedness in  $C_0(\mathbb{R}^N)$ , the space of continuous functions vanishing at infinity, and in  $L^q(\mathbb{R}^N)$  for certain values of  $q$ . We also require the condition  $\gamma < 2$ , and in fact  $0 < \gamma < 2$ . Throughout the paper we put, for  $\alpha > 0$ ,  $0 < \gamma < 2$ ,

$$q_c = \frac{N\alpha}{2 - \gamma}. \quad (1.6)$$

The critical exponent  $q_c$  plays a crucial role in this theory. We will say that  $q$  is subcritical, critical or supercritical, according to whether  $1 \leq q < q_c$ ,  $q = q_c$  or  $q > q_c$ . We have obtained the following results.

**Theorem 1.1** (Local well-posedness). *Let  $N \geq 1$  be an integer,  $\alpha > 0$  and  $\gamma$  such that*

$$0 < \gamma < \min(2, N). \quad (1.7)$$

*Let  $q_c$  be given by (1.6). Then we have the following.*

- (i) *Equation (1.5) is locally well-posed in  $C_0(\mathbb{R}^N)$ . More precisely, given  $\varphi \in C_0(\mathbb{R}^N)$ , then there exist  $T > 0$  and a unique solution  $u \in C([0, T]; C_0(\mathbb{R}^N))$  of (1.5). Moreover,  $u$  can be extended to a maximal interval  $[0, T_{\max})$  such that either  $T_{\max} = \infty$  or  $T_{\max} < \infty$  and  $\lim_{t \rightarrow T_{\max}} \|u(t)\|_\infty = \infty$ .*
- (ii) *If  $q$  is such that*

$$q > \frac{N(\alpha + 1)}{N - \gamma}, \quad q > q_c \quad \text{and} \quad q < \infty,$$

*then equation (1.5) is locally well-posed in  $L^q(\mathbb{R}^N)$ . More precisely, given  $\varphi \in L^q(\mathbb{R}^N)$ , then there exist  $T > 0$  and a unique solution  $u \in C([0, T]; L^q(\mathbb{R}^N))$  of (1.5). Moreover,  $u$  can be extended to a maximal interval  $[0, T_{\max})$  such that either  $T_{\max} = \infty$  or  $T_{\max} < \infty$  and  $\lim_{t \rightarrow T_{\max}} \|u(t)\|_q = \infty$ .*

- (iii) *Assume that  $q \geq q_c$  with  $1 < q < \infty$ . It follows that equation (1.5) is locally well-posed in  $L^q(\mathbb{R}^N)$  as in part (ii) except that uniqueness is guaranteed only among continuous functions  $u : [0, T] \rightarrow L^q(\mathbb{R}^N)$  which also verify*
  - (a)  *$t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{r})} \|u(t)\|_r$  is bounded on  $(0, T]$ , where  $r$  satisfies (3.3) below,  $q > q_c$ ;*
  - (b)  *$\sup_{t \in (0, T]} t^{\frac{N}{2}(\frac{1}{q_c} - \frac{1}{r})} \|u(t)\|_r$  is sufficiently small, where  $r$  is given in (4.1) below,  $q = q_c$ .*

*Moreover,  $u$  can be extended to a maximal interval  $[0, T_{\max})$  such that, in the case  $q > q_c$ , either  $T_{\max} = \infty$  or  $T_{\max} < \infty$  and  $\lim_{t \rightarrow T_{\max}} \|u(t)\|_q = \infty$ .*

- (iv) *In all the above cases, except where  $q = q_c$ , the minimal existence time of the solution, denoted by  $T$ , depends only on  $\|\varphi\|_\infty$  or  $\|\varphi\|_q$  respectively.*

**Remark 1.1.** Wang shows the equivalence of  $C_B(\mathbb{R}^N)$  solutions of the integral equation (1.5) and weak continuous solutions of (1.1)-(1.2), which are also distribution solutions. See the remark in [15] just after Definition 2.1 of [15] on page 563. The  $C_0(\mathbb{R}^N)$  solutions above are included in this situation. For  $t > 0$  the  $L^q(\mathbb{R}^N)$  solutions are also  $C_0(\mathbb{R}^N)$  solutions, and therefore have the same regularity. See Part (i) of Proposition 3.2 below.

**Remark 1.2.** Well-posedness of (1.5) breaks down for  $q$  subcritical. The proof is based on the existence of a positive forward rapidly decaying self-similar solution [6, Theorem 1.1, p. 625]. It follows from this result, in analogy with [5], that if  $1 \leq q < q_c$ ,  $a > 0$ ,  $N \geq 3$  and  $\frac{2-\gamma}{N} < \alpha < 2\frac{2-\gamma}{N-2}$ , then there exists a positive solution  $u$  of (1.5) with initial data  $\varphi = 0$  in  $L^q(\mathbb{R}^N)$ . This shows a non-uniqueness result in  $L^q(\mathbb{R}^N)$  and thus the ill-posedness when  $q < q_c$ .

The proof of Theorem 1.1 is based on arguments and results in [16, 17]. To apply these methods, a key new ingredient is needed. See Proposition 2.1 below. As the proofs will show, these solutions depend continuously on the initial data in an appropriate sense. Moreover, if a given  $\varphi$  belongs to two different spaces as described in Theorem 1.1, then the resulting solutions in the different spaces coincide, and in particular, the maximal existence time does not depend on the space. In addition we have the following lower estimate for the blow-up rate.

**Theorem 1.2** (Lower blow-up rate). *Under the hypotheses of Theorem 1.1, let  $\varphi \in C_0(\mathbb{R}^N)$ , respectively  $\varphi \in L^q(\mathbb{R}^N)$ ,  $q > 1$  with  $q > q_c$ , and suppose that  $T_{\max} < \infty$ , where  $T_{\max}$  is the existence time of the resulting maximal solution of (1.5). It follows*

$$\|u(t)\|_q \geq C (T_{\max} - t)^{\frac{N}{2q} - \frac{2-\gamma}{2\alpha}}, \quad \forall t \in [0, T_{\max}), \quad (1.8)$$

where  $C$  is a positive constant.

We are unaware of any previous lower blow-up estimates. On the hand, the following upper blow-up estimate has been established by [1, Theorems 1.2 and 1.3] and [7, Theorem 1.6] in the case  $q = \infty$  (with various restriction on  $\alpha$ ):

$$\|u(t)\|_\infty \leq C (T_{\max} - t)^{-\frac{1}{\alpha}}, \quad \forall t \in [0, T_{\max}). \quad (1.9)$$

Note that there is a gap between the above lower and upper estimates (1.8) and (1.9), in particular for solutions blowing up at the origin. The blow-up rate (1.8) can in fact be realized as shown by the existence of a backward self-similar solution in [3, Theorem A, p. 470 and Proposition 3.1, p. 477]. We believe that the techniques of [19] can be adapted to show that, under certain conditions, (1.8) gives an upper bound for the blow up rate. To carry out these arguments in the present context seems nontrivial.

We now turn to the global existence of solutions. We have obtained the following result.

**Theorem 1.3** (Global existence). *Let  $N \geq 1$  be an integer,  $\alpha, \sigma, \gamma > 0$ . Suppose that (1.7) is satisfied and*

$$\alpha > \frac{2-\gamma}{N}, \quad \text{equivalently } q_c > 1,$$

where  $q_c$  is given by (1.6). Then we have the following.

- (i) If  $\varphi \in L^{q_c}(\mathbb{R}^N)$  and  $\|\varphi\|_{q_c}$  is sufficiently small, then  $T_{\max} = \infty$ .
- (ii) If  $\varphi \in C_0(\mathbb{R}^N)$  such that  $|\varphi(x)| \leq c(1 + |x|^2)^{-\frac{\sigma}{2}}$ , for all  $x \in \mathbb{R}^N$  with  $c$  sufficiently small and

$$\sigma > \frac{2-\gamma}{\alpha}, \quad (1.10)$$

then  $T_{\max} = \infty$ .

- (iii) Let  $\varphi \in L^1_{loc}(\mathbb{R}^N)$  be such that  $|\varphi(\cdot)| \leq c|\cdot|^{-\frac{2-\gamma}{\alpha}}$ , for  $c$  sufficiently small. Then there exists a global in time solution of (1.5),  $u \in C((0, \infty); L^q(\mathbb{R}^N))$  for all  $q > q_c$ . Moreover  $u(t) \rightarrow \varphi$  in  $\mathcal{S}'(\mathbb{R}^N)$  as  $t \rightarrow 0$ .

**Remark 1.3.** Using the backwards self-similar solution constructed in [3, Proposition 3.1, p. 477] and arguing as in [13, Corollary 1.4, p. 659], we can prove the following result. Suppose  $N \geq 3$ ,  $a > 0$  and  $\alpha$  such that  $\frac{2-\gamma}{N} < \alpha < 2\frac{2-\gamma}{N-2}$ . There exists  $C > 0$  such that if  $\varphi \in C_0(\mathbb{R}^N)$ ,  $\varphi > 0$  and

$$\liminf_{|x| \rightarrow \infty} |x|^{\frac{2-\gamma}{\alpha}} \varphi(x) \geq C,$$

then  $T_{\max} < \infty$ .

By Fujita type blow-up results, the condition  $\alpha > (2-\gamma)/N$  ( $q_c > 1$ ) is optimal in Theorem 1.3. In fact, if  $a > 0$  and  $\alpha \leq (2-\gamma)/N$  then the solutions of (1.1) with positive initial data blow-up in finite time. See [9, Theorem 1.6, p. 126]. In [9, Theorem 1.2, p. 125], examples of global solutions to (1.1) with positive initial data for  $q_c > 1$  and  $a > 0$  are constructed. Still for  $a > 0$ , other global existence results for positive initial data are proved in [15] and [6] but only for  $N \geq 3$  and under supplementary conditions on  $\alpha$ . See [6, Theorem 1.3, p. 626]. Our global existence results are established without any restriction on  $a$  or on the initial data (other than a smallness condition). In [15], Wang observed that, by modifying the arguments of Lee and Ni [14], if (1.1) has a global solution, then necessarily  $\liminf_{|x| \rightarrow \infty} |x|^{\frac{2-\gamma}{\alpha}} \varphi(x) < \infty$ , so that our results in Theorem 1.3 (ii) and (iii) are consistent with this condition.

We now turn to the existence of forward self-similar solutions. In [6, 15] the existence of radially symmetric self-similar solutions for  $\frac{2-\gamma}{N} < \alpha < 2\frac{2-\gamma}{N-2}$  and  $N \geq 3$  is established. Here we do not impose radial symmetry of the self-similar solutions. We have obtained the following result.

**Theorem 1.4** (Self-similar solutions). *Let  $N \geq 1$  be an integer,  $\alpha, \gamma > 0$ . Suppose that  $0 < \gamma < \min(2, N)$  and*

$$\alpha > \frac{2-\gamma}{N}. \quad (1.11)$$

*Let  $\varphi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$ , where  $\omega \in L^\infty(\mathbb{R}^N)$  is homogeneous of degree 0 and  $\|\omega\|_\infty$  is sufficiently small. Then there exists a global mild self-similar solution  $u_S$  of (1.1)-(1.2). This solution verifies  $u_S(t) \rightarrow \varphi$  in  $\mathcal{S}'(\mathbb{R}^N)$  as  $t \rightarrow 0$ .*

Concerning the asymptotic behavior of global solutions, we have the following result.

**Theorem 1.5** (Asymptotic Behavior). *Let  $N \geq 1$  be an integer,  $\alpha, \sigma, \gamma > 0$ . Suppose that  $0 < \gamma < \min(2, N)$  and*

$$\frac{2-\gamma}{\alpha} \leq \sigma < N. \quad (1.12)$$

*Let  $\varphi \in C_0(\mathbb{R}^N)$  be such that*

$$|\varphi(x)| \leq \frac{c}{(1+|x|^2)^{\sigma/2}}, \quad \forall x \in \mathbb{R}^N,$$

*for  $c > 0$  sufficiently small, and*

$$\varphi(x) = \omega(x)|x|^{-\sigma}, \quad |x| \geq A,$$

*for some constant  $A > 0$  and some  $\omega \in L^\infty(\mathbb{R}^N)$ , homogeneous of degree 0, with  $\|\omega\|_\infty$  sufficiently small.*

Let  $u$  be the unique global solution of (1.5) with initial data  $\varphi$  given by Theorem 1.3. Let  $u_S$  be the global self-similar solution of (1.5) with initial data  $\omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$ , given by Theorem 1.4. Then we have the following.

(i) *Nonlinear behavior:* If  $\sigma = \frac{2-\gamma}{\alpha}$ , then there exists  $\delta > 0$  such that

$$\|u(t) - u_S(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}-\delta}, \quad \forall t > 0.$$

where  $C$  is a positive constant. In particular, there exists  $C_1, C_2$  two positive constants such that for  $t$  large

$$C_1 t^{-\frac{2-\gamma}{2\alpha}} \leq \|u(t)\|_\infty \leq C_2 t^{-\frac{2-\gamma}{2\alpha}}.$$

(ii) *Linear behavior:* If  $\sigma > \frac{2-\gamma}{\alpha}$ , then there exists  $\delta > 0$  such that

$$\|u(t) - e^{t\Delta}(\omega(\cdot)|\cdot|^{-\sigma})\|_\infty \leq Ct^{-\frac{\sigma}{2}-\delta}, \quad \forall t > 0,$$

where  $C$  is a positive constant. In particular, there exists  $C_1, C_2$  two positive constants such that for  $t$  large

$$C_1 t^{-\frac{\sigma}{2}} \leq \|u(t)\|_\infty \leq C_2 t^{-\frac{\sigma}{2}}.$$

For  $\gamma = 0$ , analogous results were obtained in [2] and [12]. For  $\gamma > 0$  the only known result to our knowledge, is a decay rate of the sup norm. See [15]. To prove the previous theorems, we use some arguments from [2, 12] (see also reference therein), combined with the estimates of Proposition 2.1 below.

The rest of the paper is organized as follows. In Section 2, we establish the estimate for the heat semi-group needed to treat the singular potential in (1.1). See Proposition 2.1 below. In Section 3 we prove Theorem 1.1, except the case  $q = q_c$ , and we also prove Theorem 1.2. In Section 4, we prove Theorem 1.1 for  $q = q_c$ , Theorem 1.3 and Theorem 1.4. In Section 5 we prove the nonlinear asymptotic behavior in Theorem 1.5. Section 6 is devoted to the proof of the linear asymptotic behavior in Theorem 1.5. Finally in Section 7, we consider a more general equation where  $|x|^{-\gamma}$  is replaced by a function  $V(x)$ . In all the paper  $C$  will be a positive constant which may have different values at different places.

## 2. A KEY ESTIMATE

In this section we prove the new estimate which is needed for the proofs of essentially all the results in this paper. Let  $e^{t\Delta}$  be the linear heat semi-group defined by:  $e^{t\Delta}\varphi = G(t, \cdot) * \varphi$ ,  $t > 0$ , where  $G$  is the heat kernel defined by (1.4). We recall the well-known smoothing effect of the heat semi-group on Lebesgue spaces,

$$\|e^{t\Delta}u\|_{s_2} \leq (4\pi t)^{-\frac{N}{2}(\frac{1}{s_1} - \frac{1}{s_2})} \|u\|_{s_1}, \quad (2.1)$$

for  $1 \leq s_1 \leq s_2 \leq \infty$ ,  $t > 0$  and  $u \in L^{s_1}(\mathbb{R}^N)$ .

To treat the nonlinear term in the equation (1.1), which includes the factor  $|\cdot|^{-\gamma}$ , we establish the following estimate, analogous to (2.1).

**Proposition 2.1.** *Let  $N \geq 1$  be an integer. Let  $\gamma$  such that  $0 < \gamma < N$ . Let  $q_1 \in (1, \infty]$  and  $q_2 \in (1, \infty]$  satisfy*

$$0 \leq \frac{1}{q_2} < \frac{\gamma}{N} + \frac{1}{q_1} < 1.$$

Then, for all  $t > 0$ , the following are bounded maps

- (i)  $e^{t\Delta}|\cdot|^{-\gamma} : L^{q_1}(\mathbb{R}^N) \rightarrow L^{q_2}(\mathbb{R}^N)$ ,  $q_2 < \infty$ ;
- (ii)  $e^{t\Delta}|\cdot|^{-\gamma} : L^{q_1}(\mathbb{R}^N) \rightarrow C_0(\mathbb{R}^N)$ ,  $q_2 = \infty$ .

Furthermore, there exists a constant  $C > 0$  depending on  $N, \gamma, q_1$  and  $q_2$  such that

$$\|e^{t\Delta}(|\cdot|^{-\gamma}u)\|_{q_2} \leq Ct^{-\frac{N}{2}(\frac{1}{q_1}-\frac{1}{q_2})-\frac{\gamma}{2}}\|u\|_{q_1}, \quad \forall t > 0, \quad \forall u \in L^{q_1}(\mathbb{R}^N). \quad (2.2)$$

*Proof.* We set  $m = \frac{N}{\gamma}$ . Let  $\epsilon, \delta > 0$  satisfy

$$\epsilon < m, \quad \frac{1}{q_2} \leq \frac{1}{m+\delta} + \frac{1}{q_1} \leq \frac{1}{m-\epsilon} + \frac{1}{q_1} \leq 1.$$

Let us consider the following decomposition

$$|\cdot|^{-\gamma} = \psi_1 + \psi_2; \quad \psi_1 \in L^{m-\epsilon}(\mathbb{R}^N), \quad \psi_2 \in L^{m+\delta}(\mathbb{R}^N).$$

Using the Hölder inequality

$$\|\psi_1 u\|_{r_1} \leq \|\psi_1\|_{m-\epsilon} \|u\|_{q_1},$$

where

$$\frac{1}{r_1} = \frac{1}{m-\epsilon} + \frac{1}{q_1}.$$

Similarly

$$\|\psi_2 u\|_{r_2} \leq \|\psi_2\|_{m+\delta} \|u\|_{q_1},$$

where

$$\frac{1}{r_2} = \frac{1}{m+\delta} + \frac{1}{q_1}.$$

So that by the smoothing effect of the heat equation (2.1) we have

$$\begin{aligned} \|e^{\Delta}(|\cdot|^{-\gamma}u)\|_{q_2} &\leq \|e^{\Delta}(\psi_1 u)\|_{q_2} + \|e^{\Delta}(\psi_2 u)\|_{q_2} \\ &\leq C\|\psi_1 u\|_{r_1} + C\|\psi_2 u\|_{r_2} \\ &\leq C\left(\|\psi_1\|_{m-\epsilon} + \|\psi_2\|_{m+\delta}\right)\|u\|_{q_1}, \end{aligned}$$

where we used  $1 \leq r_1 \leq r_2 \leq q_2$ . Therefore we obtain

$$\|e^{\Delta}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(N, \gamma, q_1, q_2)\|u\|_{q_1}. \quad (2.3)$$

Since  $r_1 < \infty$  and  $r_2 < \infty$ ,  $e^{\Delta}(\psi_1 u)$  and  $e^{\Delta}(\psi_2 u)$  are in  $C_0(\mathbb{R}^N)$ , and so therefore is  $e^{\Delta}(|\cdot|^{-\gamma}u)$ .

We now prove (2.2) by a scaling argument. Given  $\lambda > 0$ , we define the dilation operator  $D_\lambda$  by  $D_\lambda \varphi(x) = \varphi(\lambda x)$  for all  $\varphi \in \mathcal{S}(\mathbb{R}^N)$ . This operator is extended by duality to  $\mathcal{S}'(\mathbb{R}^N)$ . Clearly we have

- (i)  $D_\lambda(e^{\lambda^2 t \Delta} \varphi) = e^{t \Delta}(D_\lambda \varphi)$  for all  $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ ,
- (ii)  $D_\lambda(D_{\frac{1}{\lambda}} \varphi) = \varphi$  for all  $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ ,
- (iii)  $\|D_\lambda \varphi\|_r = \lambda^{-\frac{N}{r}} \|\varphi\|_r$  for all  $\varphi \in L^r(\mathbb{R}^N)$ ,  $r \geq 1$ ,
- (iv)  $D_\lambda(\varphi \psi) = D_\lambda \varphi D_\lambda \psi$ , for all  $\varphi, \psi, \varphi \psi \in L^1_{loc}(\mathbb{R}^N)$ ,
- (v)  $D_\lambda(|\cdot|^{-\gamma}) = \lambda^{-\gamma} |\cdot|^{-\gamma}$ , for all  $\gamma > 0$ .

It follows that  $e^{\lambda^2 t \Delta} \varphi = D_{\frac{1}{\lambda}} e^{t \Delta} D_\lambda \varphi$ , and so  $e^{\Delta} \varphi = D_{\sqrt{t}} e^{t \Delta} D_{\frac{1}{\sqrt{t}}} \varphi$  for all  $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ .

Then, from (2.3), we have that

$$\|D_{\sqrt{t}} e^{t \Delta} D_{\frac{1}{\sqrt{t}}}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(N, \gamma, q_1, q_2)\|u\|_{q_1},$$

and so

$$t^{-\frac{N}{2q_2}} \|e^{t \Delta} D_{\frac{1}{\sqrt{t}}}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(N, \gamma, q_1, q_2)\|u\|_{q_1}.$$

Therefore

$$t^{-\frac{N}{2q_2}} t^{\frac{\gamma}{2}} \|e^{t \Delta}(|\cdot|^{-\gamma} D_{\frac{1}{\sqrt{t}}} u)\|_{q_2} \leq C(N, \gamma, q_1, q_2)\|u\|_{q_1}.$$

Replacing  $u$  by  $D_{\sqrt{t}}u$ , we obtain

$$t^{-\frac{N}{2q_2}} t^{\frac{\gamma}{2}} \|e^{t\Delta}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(N, \gamma, q_1, q_2) \|D_{\sqrt{t}}u\|_{q_1},$$

which gives

$$t^{-\frac{N}{2q_2}} t^{\frac{\gamma}{2}} \|e^{t\Delta}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(N, \gamma, q_1, q_2) t^{-\frac{N}{2q_1}} \|u\|_{q_1}.$$

Hence

$$\|e^{t\Delta}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(N, \gamma, q_1, q_2) t^{-\frac{N}{2}(\frac{1}{q_1} - \frac{1}{q_2}) - \frac{\gamma}{2}} \|u\|_{q_1}.$$

This shows (2.2) and the boundness of the maps.  $\square$

**Remark 2.2.** The conditions on  $q_1$  and  $q_2$  in the previous proposition can be expressed as

$$q_1 > \frac{N}{N - \gamma}, \quad q_2 > \frac{Nq_1}{N + \gamma q_1}.$$

Note also that for  $\gamma > 0$ , we may take  $q_2 < q_1$ , unlike the case  $\gamma = 0$ , where we must have  $q_2 \geq q_1$ .

Finally, we mention that Proposition 2.1 can be proved, except in the cases  $q_1 = \infty$  or  $q_2 = \infty$ , using Hölder's inequality in weak spaces [4, p. 15] and the generalized Young's inequality [4, p. 63]. The authors thank N. Chikami for pointing this out.

### 3. LOCAL WELL-POSEDNESS

In this section we establish the well-posedness results for the equation (1.1)-(1.2) in Lebesgue spaces  $L^q(\mathbb{R}^N)$  and in  $C_0(\mathbb{R}^N)$ . We do this study via the nonlinear integral equation

$$u(t) = e^{t\Delta}\varphi + a \int_0^t e^{(t-s)\Delta} (|\cdot|^{-\gamma}|u(s)|^\alpha u(s)) ds, \quad (3.1)$$

$t > 0$ ,  $x \in \mathbb{R}^N$ ,  $\alpha > 0$ ,  $a \in \mathbb{R}$ ,  $\gamma > 0$ . Our aim is to prove Theorem 1.1 for  $q > q_c$  and Theorem 1.2. The proof of Theorem 1.1 for  $q = q_c$  is given in Section 4.

We first show that Theorem 1.1 Parts (i) and (ii) are immediate consequence of Theorem 1, page 279 in [16].

*Proof of Theorem 1.1, (i)-(ii).* Let us define the maps

$$K_t(u) = e^{t\Delta} (|\cdot|^{-\gamma}|u|^\alpha u), \quad t > 0.$$

(i) Let  $u \in C_0(\mathbb{R}^N)$ . By Proposition 2.1,  $K_t(u) \in C_0(\mathbb{R}^N)$ . Moreover, by Proposition 2.1, for each  $t > 0$ ,  $K_t : C_0(\mathbb{R}^N) \rightarrow C_0(\mathbb{R}^N)$  is locally Lipschitz with

$$\begin{aligned} \|K_t(u) - K_t(v)\|_\infty &\leq Ct^{-\frac{\gamma}{2}} \| |u|^\alpha u - |v|^\alpha v \|_\infty \\ &\leq Ct^{-\frac{\gamma}{2}} (\|u\|_\infty^\alpha + \|v\|_\infty^\alpha) \|u - v\|_\infty \\ &\leq 2CM^\alpha t^{-\frac{\gamma}{2}} \|u - v\|_\infty, \end{aligned}$$

for  $\|u\|_\infty \leq M$  and  $\|v\|_\infty \leq M$ . We have also that  $t^{-\frac{\gamma}{2}} \in L^1_{loc}(0, \infty)$ , since  $\gamma < 2$ . Obviously  $t \mapsto \|K_t(0)\|_\infty = 0 \in L^1_{loc}(0, \infty)$ , also  $e^{s\Delta}K_t = K_{t+s}$  for  $s, t > 0$ . Then the results of (i) follows by [16, Theorem 1, p. 279].



(ii) Similarly, by Proposition 2.1, for each  $t > 0$  and if  $q > \frac{N(\alpha+1)}{N-\gamma}$ ,  $K_t : L^q(\mathbb{R}^N) \rightarrow L^q(\mathbb{R}^N)$  is locally Lipschitz with

$$\begin{aligned} \|K_t(u) - K_t(v)\|_q &\leq Ct^{-\frac{N}{2}(\frac{\alpha+1}{q}-\frac{1}{q})-\frac{\gamma}{2}} \| |u|^\alpha u - |v|^\alpha v \|_{\frac{q}{\alpha+1}} \\ &\leq Ct^{-\frac{N\alpha}{2q}-\frac{\gamma}{2}} (\|u\|_q^\alpha + \|v\|_q^\alpha) \|u - v\|_q \\ &\leq 2CM^\alpha t^{-\frac{N\alpha}{2q}-\frac{\gamma}{2}} \|u - v\|_q, \end{aligned}$$

for  $\|u\|_q \leq M$  and  $\|v\|_q \leq M$ . We have also, that  $t^{-\frac{N\alpha}{2q}-\frac{\gamma}{2}} \in L^1_{loc}(0, \infty)$ , since  $q > q_c = \frac{N\alpha}{2-\gamma}$ . Obviously  $t \mapsto \|K_t(0)\|_\infty = 0 \in L^1_{loc}(0, \infty)$ , also  $e^{s\Delta}K_t = K_{t+s}$  for  $s, t > 0$ . Then the proof of (ii) follows by [16, Theorem 1, p. 279].  $\square$

We now consider the case where initial data are in  $L^q(\mathbb{R}^N)$ ,  $q > q_c$ , where  $q_c$  is given by (1.6). We use the method introduced in [17]. However, the abstract Theorem in [17] does not directly apply if  $\gamma > 0$ . Thus we need to give the details of the proofs.

*Proof of Theorem 1.1 (iii)-(iv) when  $q > q_c$ .* Let  $q$  be such that

$$q > q_c \quad \text{and} \quad 1 < q < \infty. \quad (3.2)$$

We begin with the observation that, since  $q > 1$ , there exists  $r > q$  satisfying

$$\frac{1}{q(\alpha+1)} - \frac{\gamma}{N(\alpha+1)} < \frac{1}{r} < \frac{N-\gamma}{N(\alpha+1)}. \quad (3.3)$$

The two inequalities in (3.3) imply that we may apply Proposition 2.1 with  $q_1 = \frac{r}{\alpha+1}$  and  $q_2 = q$ . Also we may apply Proposition 2.1 with  $q_1 = \frac{r}{\alpha+1}$  and  $q_2 = r$ .

We then observe that, since  $q > q_c$ , we have

$$\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{1}{q(\alpha+1)} - \frac{\gamma}{N(\alpha+1)}.$$

Hence any  $r > q$  satisfying (3.3) verifies

$$\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{1}{r}.$$

This last inequality implies that

$$\beta(\alpha+1) < 1,$$

where

$$\beta = \frac{N}{2q} - \frac{N}{2r}. \quad (3.4)$$

This estimate is crucial to the local existence argument below.

In what follows we fix a value of  $r > q$  satisfying (3.3), and let  $\beta$  be given by (3.4). Let  $M > 0$ ,  $\rho > 0$ ,  $T > 0$  and  $\varphi \in L^q(\mathbb{R}^N)$  be such that

$$\|\varphi\|_q \leq \rho, \quad (3.5)$$

$$\rho + KM^{\alpha+1}T^{1-\frac{N\alpha}{2q}-\frac{\gamma}{2}} \leq M \quad \text{and} \quad KM^\alpha T^{1-\frac{N\alpha}{2q}-\frac{\gamma}{2}} < 1, \quad (3.6)$$

where  $K$  is a positive constant. We will show that there exists a unique solution  $u$  of (3.1) such that  $u \in C([0, T]; L^q(\mathbb{R}^N)) \cap C((0, T]; L^r(\mathbb{R}^N))$  with

$$\sup_{t \in [0, T]} \|u(t)\|_q \leq M \quad \text{and} \quad \sup_{t \in (0, T]} t^\beta \|u(t)\|_r \leq M.$$

The proof is based on a contraction mapping argument in the set

$$Y_M = \{u \in C([0, T]; L^q(\mathbb{R}^N)) \cap C((0, T]; L^r(\mathbb{R}^N)); \max[\sup_{t \in [0, T]} \|u(t)\|_q, \sup_{t \in (0, T]} t^\beta \|u(t)\|_r] \leq M\}.$$

Endowed with the metric

$$d(u, v) = \max[\sup_{t \in [0, T]} \|u(t) - v(t)\|_q, \sup_{t \in (0, T]} t^\beta \|u(t) - v(t)\|_r],$$

$Y_M$  is a nonempty complete metric space. Given  $u \in Y_M$ , we set

$$\mathcal{F}_\varphi(u)(t) = e^{t\Delta}\varphi + a \int_0^t e^{(t-s)\Delta} (|\cdot|^{-\gamma} |u(s)|^\alpha u(s)) ds, \quad (3.7)$$

where  $\varphi \in L^q(\mathbb{R}^N)$ . We will show that  $\mathcal{F}_\varphi$  is a strict contraction on  $Y_M$ .

Let  $\varphi, \psi \in L^q(\mathbb{R}^N)$  and  $u, v \in Y_M$ . Using Proposition 2.1 with  $q_1 = r/(\alpha + 1)$ ,  $q_2 = q$ , it follows that

$$\begin{aligned} & \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_q \leq \\ & \|e^{t\Delta}(\varphi - \psi)\|_q + |a| \int_0^t \|e^{(t-s)\Delta} [|\cdot|^{-\gamma} (|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s))]\|_q ds \\ & \leq \|\varphi - \psi\|_q + |a| C \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})-\frac{\gamma}{2}} \| |u(s)|^\alpha u(s) - |v(s)|^\alpha v(s) \|_{\frac{r}{\alpha+1}} ds \\ & \leq \|\varphi - \psi\|_q + \left( 2|a|(\alpha+1)CM^\alpha \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})-\frac{\gamma}{2}} s^{-\beta(\alpha+1)} ds \right) d(u, v). \end{aligned}$$

Using the fact that  $\beta = \frac{N}{2q} - \frac{N}{2r}$ , we get

$$\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_q \leq \|\varphi - \psi\|_q + \left( 2|a|(\alpha+1)CM^\alpha t^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)} d\sigma \right) d(u, v).$$

Since  $r > q > \frac{N\alpha}{2-\gamma} := q_c$ , it follows that

$$1 - \frac{\gamma}{2} - \frac{N\alpha}{2q} > 0, \quad \frac{N}{2} \left( \frac{\alpha+1}{r} - \frac{1}{q} \right) + \frac{\gamma}{2} < \frac{N}{2} \left( \frac{\alpha+1}{r} - \frac{1}{r} \right) + \frac{\gamma}{2} = \frac{N\alpha}{2r} + \frac{\gamma}{2} < 1.$$

Using also the fact that  $\beta(\alpha+1) < 1$ , we get

$$\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_q \leq \|\varphi - \psi\|_q + C_1 M^\alpha T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} d(u, v), \quad (3.8)$$

where  $C_1 = 2|a|(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)} d\sigma$ , is a finite positive constant.

Similarly, using the smoothing effect of the heat semi-group (2.1) with  $s_1 = q < s_2 = r$  and Proposition 2.1 with  $q_1 = r/(\alpha + 1)$ ,  $q_2 = r$ , we have

$$\begin{aligned}
& \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r \leq \\
& \|e^{t\Delta}(\varphi - \psi)\|_r + |a| \int_0^t \|e^{(t-s)\Delta}[|\cdot|^{-\gamma}(|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s))]\|_r ds \\
& \leq t^{-\beta} \|\varphi - \psi\|_q + |a|C \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{r})-\frac{\gamma}{2}} \| |u(s)|^\alpha u(s) - |v(s)|^\alpha v(s) \|_{\frac{r}{\alpha+1}} ds \\
& \leq t^{-\beta} \|\varphi - \psi\|_q + \left( 2|a|(\alpha+1)CM^\alpha \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{r})-\frac{\gamma}{2}} s^{-\beta(\alpha+1)} ds \right) d(u, v) \\
& \leq t^{-\beta} \|\varphi - \psi\|_q + \left( 2|a|(\alpha+1)CM^\alpha \int_0^t (t-s)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} s^{-\beta(\alpha+1)} ds \right) d(u, v).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
& t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r \leq \\
& \|\varphi - \psi\|_q + \left( 2|a|(\alpha+1)CM^\alpha t^\beta \int_0^t (t-s)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} s^{-\beta(\alpha+1)} ds \right) d(u, v) \\
& \leq \|\varphi - \psi\|_q + \left( 2|a|(\alpha+1)CM^\alpha t^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)} d\sigma \right) d(u, v),
\end{aligned}$$

and so, by the conditions on  $\beta$ ,  $q$  and  $r$  we have

$$t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r \leq \|\varphi - \psi\|_q + C_2 M^\alpha T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} d(u, v), \quad (3.9)$$

where  $C_2 = 2|a|(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)} d\sigma$ , is a finite positive constant. From (3.8) and (3.9) it follows that

$$d(\mathcal{F}_\varphi(u), \mathcal{F}_\psi(v)) \leq \|\varphi - \psi\|_q + KM^\alpha T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} d(u, v), \quad (3.10)$$

where  $K = \max(C_1, C_2)$ .

It is clear that if  $u \in Y_M$  and since  $1 - \frac{\gamma}{2} - \frac{N\alpha}{2q} > 0$  (i.e.  $q > q_c$ ), then  $\mathcal{F}_\varphi(u) \in C([0, T]; L^q(\mathbb{R}^N)) \cap C((0, T]; L^r(\mathbb{R}^N))$ . Setting  $\psi = 0$  and  $v = 0$  in (3.10) and using (3.5) and (3.6), we obtain

$$d(\mathcal{F}_\varphi(u), 0) \leq \rho + KM^{\alpha+1} T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} \leq M.$$

And so  $\mathcal{F}_\varphi$  maps  $Y_M$  into itself. Letting  $\varphi = \psi$  in (3.10), we get

$$d(\mathcal{F}_\varphi(u), \mathcal{F}_\varphi(v)) \leq KM^\alpha T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}} d(u, v).$$

Hence, using (3.6), it follows that  $\mathcal{F}_\varphi$  is a strict contraction mapping from  $Y_M$  into itself. So  $\mathcal{F}_\varphi$  has a unique fixed point in  $Y_M$  which is solution of (3.1). The proof of uniqueness for arbitrary  $M$  follows by taking  $T$  sufficiently small in (3.6). This solution can be extended to a maximal solution by well known argument. The proof of part (iv) follows by the previous calculations. We note also that by the previous calculations, precisely (3.10) we have the following continuous dependence property: Let  $\varphi, \psi \in L^q(\mathbb{R}^N)$  and let  $u$  and  $v$  be the solutions of (1.5) with initial values  $\varphi$  and respectively  $\psi$ , with  $\sup_{t \in [0, T]} \|u(t)\|_q \leq M$  and  $\sup_{t \in [0, T]} \|v(t)\|_q \leq M$  for some  $M > 0$ . Then

$$\sup_{t \in [0, T]} \|u(t) - v(t)\|_q \leq (1 - KM^\alpha T^{1-\frac{\gamma}{2}-\frac{N\alpha}{2q}})^{-1} \|\varphi - \psi\|_q,$$

for  $t \in [0, T]$  and for some positive constant  $K$ .  $\square$

**Remark 3.1.** Using a fixed point argument on

$$Y'_M = \left\{ u \in C((0, T]; L^r(\mathbb{R}^N)); \sup_{t \in (0, T]} t^\beta \|u(t)\|_r \leq M \right\},$$

endowed with the metric  $d'(u, v) = \sup_{t \in (0, T]} t^\beta \|u(t) - v(t)\|_r$ , we can prove the local existence in  $Y'_M$  with initial data in  $L^q(\mathbb{R}^N)$ ,

$$q > q_c, \quad 1 \leq q < \infty. \quad (3.11)$$

That is we may include the case  $q = 1$  if  $q_c < 1$ . In fact, we can choose  $r > q$  satisfying:

$$\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{1}{r} < \frac{N-\gamma}{N(\alpha+1)}. \quad (3.12)$$

The inequalities (3.12) imply that we may apply Proposition 2.1 with  $q_1 = \frac{r}{\alpha+1}$  and  $q_2 = r$ , and implies also that  $\beta(\alpha+1) < 1$ . Hence, we may perform a fixed point argument on  $Y'_M$ . But it is not clear that if  $q = 1$ , the solution is still in  $L^1(\mathbb{R}^N)$ , for  $t > 0$ .

To see how one can choose  $r > q$  satisfying (3.12), note first that the inequality  $\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{N-\gamma}{N(\alpha+1)}$  is equivalent to  $q > \frac{N(\alpha+1)}{N+2-\gamma} \equiv q_r$ . If  $q_c = \frac{N\alpha}{2-\gamma} < 1$ , then  $q_r < 1$  and so  $q \geq 1 > q_r$ . If  $q_c = \frac{N\alpha}{2-\gamma} \geq 1$ , then  $q_c \geq q_r$  and so  $q > q_c \geq q_r$ . The conditions (3.11) therefore imply  $\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{N-\gamma}{N(\alpha+1)}$ , which means that there exists  $r > q$  which verifies (3.12).

We have also the following result.

**Proposition 3.2.** *Let  $\alpha > 0$  and let  $\gamma$  be such that  $0 < \gamma < \min(2, N)$ . Assume the hypotheses of Theorem 1.1. Let  $T_{\max}(\varphi, q)$  denotes the maximal existence time of the solution of (1.5) with initial data  $\varphi \in L^q(\mathbb{R}^N)$ . Then we have the following.*

- (i) *If  $\varphi \in L^q(\mathbb{R}^N)$ , then for  $t \in (0, T_{\max}(\varphi, q))$ ,  $u(t) \in C_0(\mathbb{R}^N)$ .*
- (ii) *If  $\varphi \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ ,  $1 < q < p \leq \infty$  and  $q > q_c$ . Then  $T_{\max}(\varphi, p) = T_{\max}(\varphi, q)$ .*

*Proof.* (i) Let  $\varphi \in L^q(\mathbb{R}^N)$ ,  $q > q_c$  and  $q > 1$ . Let  $r$  and  $\beta$  be as in (3.3) and (3.4). Let  $p$  be such that  $r < p \leq \infty$ . Hence  $p > q$ ,

$$0 \leq \frac{1}{p} < \frac{\gamma}{N} + \frac{\alpha+1}{r} < 1$$

and for  $0 < T < T_{\max}(\varphi, q)$ , we have

$$\begin{aligned} \|u(t)\|_p &\leq \|e^{t\Delta}\varphi\|_p + |a|C \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\frac{\gamma}{2}} \|u(\sigma)\|_r^{\alpha+1} d\sigma \\ &\leq (4\pi t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \|\varphi\|_q + |a|C t^{1-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\frac{\gamma}{2}-\beta(\alpha+1)} \sup_{s \in (0, T]} \left( s^{\beta(\alpha+1)} \|u(s)\|_r^{\alpha+1} \right) \times \\ &\quad \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)} d\sigma \\ &\leq (4\pi t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \|\varphi\|_q + |a|M^{\alpha+1} C t^{1-\frac{\gamma}{2}-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)} d\sigma. \end{aligned}$$

Since  $r > q > q_c$ , it follows that if

$$\frac{\alpha+1}{r} - \frac{2-\gamma}{N} < \frac{1}{p} < \frac{1}{r},$$

then  $u(t)$  is in  $L^p(\mathbb{R}^N)$  for all  $t \in (0, T_{\max}(\varphi, q))$ . The result for general  $p > q$  follows by iteration. Hence  $u(t)$  is in  $L^\infty(\mathbb{R}^N)$ , for  $t \in (0, T_{\max}(\varphi, q))$ . The fact that  $u(t) \in C_0(\mathbb{R}^N)$ , for  $t \in (0, T_{\max}(\varphi, q))$  follows by Proposition 2.1.

(ii) To emphasize the dependence on  $q$  let us denote the metric space  $(Y_M, d)$  by  $(Y_M(q), d_q)$ . Let  $\varphi \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ . Let  $u_q$  be the solution with initial data  $\varphi \in L^q(\mathbb{R}^N)$ . Let  $u_p$  be the solution with initial data  $\varphi \in L^p(\mathbb{R}^N)$ . We can show, by similar calculations that the mapping defined by (3.7) is a contraction on  $Y_M(q) \cap Y_M(p)$  endowed with the metric  $d = \max(d_q, d_p)$ . This gives the existence of a unique solution in  $Y_M(q) \cap Y_M(p)$ . Hence the solutions  $u_q = u_p$  for  $t \in [0, T]$ ,  $T$  small. We deduce also, by well known arguments, that the maximal time of existence is independent of  $q$ . This finishes the proof of the proposition.  $\square$

*Proof of Theorem 1.2.* Let  $\varphi \in L^q(\mathbb{R}^N)$  be such that  $T_{\max} < \infty$  and  $u \in C\left([0, T_{\max}), L^q(\mathbb{R}^N)\right)$  be the maximal solution of (1.5). Fix  $s \in [0, T_{\max})$  and let

$$w(t) = u(t + s), \quad t \in [0, T_{\max} - s),$$

with  $w(0) = u(s)$ . Then we have, as in the proof of Theorem 1.1 Part (iii),

$$\|u(s)\|_q + KM^{\alpha+1}(T_{\max} - s)^{1 - \frac{N\alpha}{2q} - \frac{\gamma}{2}} > M, \quad \forall M > 0. \quad (3.13)$$

In fact, if not, there exists  $M > 0$  such that

$$\|u(s)\|_q + KM^{\alpha+1}(T_{\max} - s)^{1 - \frac{N\alpha}{2q} - \frac{\gamma}{2}} \leq M,$$

$w$  will be defined on  $[0, T_{\max} - s]$  in particular  $u(T_{\max})$  is well defined, a contradiction. Hence (3.13) is verified, for any  $t \in [0, T_{\max})$  fixed and for all  $M > 0$ . Let

$$M = 2\|u(t)\|_q.$$

From (3.13) we have

$$\|u(t)\|_q + K2^{\alpha+1}\|u(t)\|_q^{\alpha+1}(T_{\max} - t)^{1 - \frac{N\alpha}{2q} - \frac{\gamma}{2}} > 2\|u(t)\|_q.$$

Then

$$K2^{\alpha+1}\|u(t)\|_q^\alpha(T_{\max} - t)^{1 - \frac{N\alpha}{2q} - \frac{\gamma}{2}} > 1, \quad \forall t \in [0, T_{\max}).$$

Hence we derive a lower bound of the blow-up rate

$$\|u(t)\|_q \geq C(T_{\max} - t)^{\frac{N}{2q} - \frac{2-\gamma}{2\alpha}}, \quad \forall t \in [0, T_{\max}).$$

The proof in the case where  $\varphi \in C_0(\mathbb{R}^N)$  is similar. This finishes the proof of Theorem 1.2.  $\square$

**Remark 3.3.** By the same methods used to prove Theorem 1.1 (i), (ii) and Proposition 3.2, one can show that (1.5) is well posed in

- $C_0^1(\mathbb{R}^N) := \{u \in C_0(\mathbb{R}^N), \nabla u \in C_0(\mathbb{R}^N)\}$ , if  $0 < \gamma < 1$ ,
- $W^{1,q}(\mathbb{R}^N)$ , if  $0 < \gamma < 1$ ,  $q > \frac{N\alpha}{1-\gamma}$  and  $q > \frac{N(\alpha+1)}{N-\gamma}$ .

Moreover under some conditions on  $\gamma$ ,  $\alpha$ , and  $q$ , the solutions constructed in Theorem 1.1, are in  $C_0^1(\mathbb{R}^N)$  and  $W^{1,q}(\mathbb{R}^N)$  for  $t > 0$ .

## 4. GLOBAL EXISTENCE

In this section we prove Theorem 1.1 Part (iii) for  $q = q_c > 1$ , Theorem 1.3 and Theorem 1.4. We consider the solutions of the integral equation

$$u(t) = e^{t\Delta}\varphi + a \int_0^t e^{(t-s)\Delta} (|\cdot|^{-\gamma}|u(s)|^\alpha u(s)) ds,$$

where  $t > 0$ ,  $x \in \mathbb{R}^N$ ,  $a \in \mathbb{R}$ ,  $0 < \gamma < \min(2, N)$  and  $\alpha > (2 - \gamma)/N$ , i.e.  $q_c = \frac{N\alpha}{2-\gamma} > 1$ . Given such an  $\alpha$ , one can choose  $r > q_c$  such that

$$\frac{1}{q_c} - \frac{2}{N(\alpha + 1)} < \frac{1}{r} < \frac{N - \gamma}{N(\alpha + 1)}. \quad (4.1)$$

This relationship is analogous to (3.3) with  $q = q_c$ . In fact

$$\frac{1}{q_c} - \frac{2}{N(\alpha + 1)} = \frac{1}{q_c(\alpha + 1)} - \frac{\gamma}{N(\alpha + 1)}.$$

The existence of such an  $r > q_c$  follows from the fact that  $q_c > 1$ .

While  $r$  is not uniquely determined, we consider  $r$  fixed and set

$$\beta = \frac{N}{2q_c} - \frac{N}{2r} = \frac{2 - \gamma}{2\alpha} - \frac{N}{2r}. \quad (4.2)$$

One verifies that

$$\beta(\alpha + 1) < 1, \quad \frac{N\alpha}{2r} + \frac{\gamma}{2} < 1, \quad \beta + 1 - \left( \frac{N\alpha}{2r} + \frac{\gamma}{2} \right) - \beta(\alpha + 1) = 0. \quad (4.3)$$

We have obtained the following global existence result.

**Theorem 4.1** (Global existence). *Let  $0 < \gamma < \min(2, N)$  and  $\alpha > (2 - \gamma)/N$ . Let  $r$  and  $\beta$  verify (4.1) and (4.2). Suppose that  $\rho > 0$  and  $M > 0$  satisfy the inequality*

$$\rho + KM^{\alpha+1} \leq M, \quad (4.4)$$

where  $K = K(\alpha, N, \gamma, r) > 0$  is a constant and can explicitly be computed. Let  $\varphi$  be a tempered distribution such that

$$\sup_{t>0} t^\beta \|e^{t\Delta}\varphi\|_r \leq \rho. \quad (4.5)$$

It follows that there exists a unique global solution  $u$  of (3.1) such that

$$\sup_{t>0} t^\beta \|u(t)\|_r \leq M. \quad (4.6)$$

Furthermore,

- (i)  $u(t) - e^{t\Delta}\varphi \in C([0, \infty); L^s(\mathbb{R}^N))$ ,  $\frac{1}{q_c} < \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha+1}{r}$ .
- (ii)  $u(t) - e^{t\Delta}\varphi \in L^\infty((0, \infty); L^s(\mathbb{R}^N))$ ,  $\frac{1}{q_c} \leq \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha+1}{r}$ .
- (iii)  $\lim_{t \rightarrow 0} u(t) = \varphi$  in the sense of distributions.
- (iv)  $\sup_{t>0} t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2q}} \|u(t)\|_q < \infty$ ,  $\forall q \in [r, \infty]$ .

Moreover, let  $\varphi$  and  $\psi$  satisfy (4.5) and let  $u$  and  $v$  be respectively the solutions of (3.1) with initial values  $\varphi$  and  $\psi$ . Then

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2q}} \|u(t) - v(t)\|_q \leq C \sup_{t>0} t^\beta \|e^{t\Delta}(\varphi - \psi)\|_r, \quad \forall q \in [r, \infty]. \quad (4.7)$$

If in addition,  $e^{t\Delta}(\varphi - \psi)$  has the stronger decay property

$$\sup_{t>0} t^{\beta+\delta} \|e^{t\Delta}(\varphi - \psi)\|_r < \infty, \quad (4.8)$$

for some  $\delta > 0$  such that  $\beta(\alpha + 1) + \delta < 1$ , and with  $M$  perhaps smaller, then

$$\sup_{t>0} t^{\beta+\delta} \|u(t) - v(t)\|_r \leq C \sup_{t>0} t^{\beta+\delta} \|e^{t\Delta}(\varphi - \psi)\|_r, \quad (4.9)$$

where  $C > 0$  is a constant.

**Remarks 4.1.** (a) If we suppose that  $\varphi - \psi \in L^s(\mathbb{R}^N)$ ,  $\frac{1}{q_c} < \frac{1}{s} < \frac{\alpha+1}{r} + \frac{\gamma}{N}$ , then (4.8) is verified with  $\delta = \frac{N}{2s} - \frac{2-\gamma}{2\alpha} > 0$ . By the conditions on  $s$  we have  $\delta < \delta_0$  where

$$\delta_0 = \frac{N(\alpha + 1)}{2r} + \frac{\gamma}{2} - \frac{2 - \gamma}{2\alpha}. \quad (4.10)$$

Since  $\beta(\alpha + 1) + \delta_0 = 1$ , it follows that (4.9) holds for all  $\delta \in (0, \delta_0)$ .

(b) If the hypotheses of Theorem 4.1 are verified on some finite interval  $t \in (0, T)$ , instead of  $\forall t > 0$ , the conclusion still holds, but only on the interval  $(0, T)$ .

The following corollary gives the proof of Theorem 1.1 Part (iii) for  $q = q_c > 1$ . The proof of parts (i)-(iii) below is similar to that in [11, Corollary 2.6, p. 1296], so we omit it. The proof of part (iv) is similar to that in [10, Theorem 20.19(iii)], so we likewise omit it.

**Corollary 4.2.** *Suppose the hypotheses of Theorem 4.1 are satisfied.*

- (i) *If  $\varphi \in L^{q_c}(\mathbb{R}^N)$  and  $\|\varphi\|_{q_c}$  is sufficiently small, then  $\varphi$  satisfies (4.5).*
- (ii) *If  $\varphi \in L^{q_c}(\mathbb{R}^N)$  (without any assumption of smallness), then there exists  $T > 0$ , such that  $\varphi$  satisfies (4.5), but only on  $(0, T)$ .*
- (iii) *In the above two cases, if  $u$  is the resulting solution of (1.5), then  $u \in C([0, \infty); L^{q_c}(\mathbb{R}^N))$ , respectively,  $u \in C([0, T]; L^{q_c}(\mathbb{R}^N))$ .*
- (iv) *If  $\varphi \in L^{q_c}(\mathbb{R}^N)$  and  $\|\varphi\|_{q_c}$  is sufficiently small and if  $u \in C([0, \infty); L^{q_c}(\mathbb{R}^N))$  is the resulting solution, then  $\|u(t)\|_{q_c} \rightarrow 0$  as  $t \rightarrow \infty$ .*

*Proof of Theorem 1.1 Part (iii) for  $q = q_c > 1$ .* The proof follows by Theorem 4.1 and Corollary 4.2. The fact that  $u : [0, T] \rightarrow L^{q_c}(\mathbb{R}^N)$  is continuous and the condition (iii) (b), imply (4.6) on  $(0, T)$  and then is sufficient to guarantee uniqueness.  $\square$

Using now the previous results we give the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Part (i) follows by Corollary 4.2 and Theorem 4.1.

(ii) By the condition on  $\sigma$ ,  $\varphi$  verifies the hypothesis of (i).

(iii) Since  $\varphi \in L^1_{loc}(\mathbb{R}^N)$  and  $|\varphi(\cdot)| \leq c|\cdot|^{-\frac{2-\gamma}{\alpha}}$ , then  $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ . By writing  $|\cdot|^{-\frac{2-\gamma}{\alpha}} = \varphi_1 + \varphi_2$ , with  $\varphi_1 \in L^s(\mathbb{R}^N)$ ,  $1 \leq s < N\alpha/(2 - \gamma)$ ,  $\varphi_2 \in L^\tau(\mathbb{R}^N)$ ,  $\tau > q_c$  it follows by the smoothing properties of the heat semigroup that  $e^{t\Delta}|\cdot|^{-\frac{2-\gamma}{\alpha}} \in L^r(\mathbb{R}^N)$ ,  $\forall t > 0$  and by homogeneity, we have that  $\sup_{t>0} t^\beta \|e^{t\Delta}|\cdot|^{-\frac{2-\gamma}{\alpha}}\|_r < \infty$ . Since  $|\varphi(\cdot)| \leq c|\cdot|^{-\frac{2-\gamma}{\alpha}}$ ,  $\varphi$  verifies (4.5)

for  $c$  sufficiently small. Then the first statement of Part (iii) follows by Theorem 4.1. The fact that  $u \in C((0, \infty); L^q(\mathbb{R}^N))$  for all  $q > q_c$  follows by iteration as in the proof of Proposition 3.2 and by Theorem 4.1 (i). The last statement of Part (iii) follows by Theorem 4.1 (iii). This completes the proof of Theorem 1.3.  $\square$

*Proof of Theorem 4.1.* The proof is based on a contraction mapping argument. Let  $X$  be the set of Bochner measurable functions  $u : (0, \infty) \rightarrow L^r(\mathbb{R}^N)$  such that  $\sup_{t>0} t^\beta \|u(t)\|_r$  is finite. We denote by  $X_M$  the set of  $u \in X$  such that  $\sup_{t>0} t^\beta \|u(t)\|_r \leq M$ . Endowed with the metric,  $d(u, v) = \sup_{t>0} t^\beta \|u(t) - v(t)\|_r$ ,  $X_M$  is a nonempty complete metric space. Consider the mapping defined by

$$\mathcal{F}_\varphi(u)(t) = e^{t\Delta}\varphi + a \int_0^t e^{(t-s)\Delta} (|\cdot|^{-\gamma} |u(s)|^\alpha u(s)) ds, \quad (4.11)$$

where  $\varphi$  is a tempered distribution satisfying (4.5). We will show that  $\mathcal{F}_\varphi$  is a strict contraction on  $X_M$ .

Let  $\varphi$  and  $\psi$  satisfy (4.5) and  $u, v \in X_M$ . It follows that

$$\begin{aligned} t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r &\leq t^\beta \|e^{t\Delta}(\varphi - \psi)\|_r + \\ &|a| t^\beta \int_0^t \|e^{(t-s)\Delta} [|\cdot|^{-\gamma} (|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s))]\|_r ds. \end{aligned}$$

Using Proposition 2.1 with  $(q_1, q_2) = (\frac{r}{\alpha+1}, r)$ , we obtain

$$\begin{aligned} \|e^{(t-s)\Delta} [|\cdot|^{-\gamma} (|u(s)|^\alpha u(s) - |v(s)|^\alpha v(s))]\|_r &\leq C(t-s)^{-\frac{N\alpha}{2r} - \frac{\gamma}{2}} \| |u(s)|^\alpha u(s) - |v(s)|^\alpha v(s) \|_{\frac{r}{\alpha+1}} \\ &\leq C(t-s)^{-\frac{N\alpha}{2r} - \frac{\gamma}{2}} (\alpha+1) (\|u(s)\|_r^\alpha + \|v(s)\|_r^\alpha) \|u(s) - v(s)\|_r \\ &\leq 2(\alpha+1) C(t-s)^{-\frac{N\alpha}{2r} - \frac{\gamma}{2}} s^{-\beta(\alpha+1)} M^\alpha d(u, v). \end{aligned}$$

By (4.3), we obtain

$$t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r \leq t^\beta \|e^{t\Delta}(\varphi - \psi)\|_r + KM^\alpha d(u, v), \quad (4.12)$$

where  $K = 2|a|(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r} - \frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)} d\sigma$  is a finite positive constant.

Setting  $\psi = 0$  and  $v = 0$  in (4.12), we see that

$$\sup_{t>0} t^\beta \|\mathcal{F}_\varphi(u)(t)\|_r \leq \rho + KM^{\alpha+1} \leq M.$$

That is,  $\mathcal{F}_\varphi$  maps  $X_M$  into itself. Letting  $\varphi = \psi$  in (4.12), we observe that

$$d(\mathcal{F}_\varphi(u) - \mathcal{F}_\varphi(v)) \leq KM^\alpha d(u, v).$$

Since  $KM^\alpha < 1$ , we see that  $\mathcal{F}_\varphi$  is a strict contraction on  $X_M$ , and so  $\mathcal{F}_\varphi$  has a unique fixed point  $u$  in  $X_M$  solution of (3.1).

We now prove that  $u(t) - e^{t\Delta}\varphi \in C([0, \infty); L^s(\mathbb{R}^N))$  for  $s$  satisfying

$$\frac{2-\gamma}{N\alpha} < \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha+1}{r}. \quad (4.13)$$

Since continuity for  $t > 0$  can be handled by well known arguments, we only give the proof at  $t=0$ . Write

$$u(t) - e^{t\Delta}\varphi = a \int_0^t e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma} |u(\sigma)|^\alpha u(\sigma)) d\sigma.$$



Then for  $s$  satisfying (4.13) and by Proposition 2.1 with  $(q_1, q_2) = (\frac{r}{\alpha+1}, s)$ , we obtain

$$\begin{aligned} \|u(t) - e^{t\Delta}\varphi\|_s &\leq |a| \int_0^t \|e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)]\|_s d\sigma \\ &\leq |a|C \int_0^t (t-\sigma)^{\frac{N}{2s} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2r}} \|u(\sigma)\|_r^{\alpha+1} d\sigma \\ &\leq |a|CM^{\alpha+1} \int_0^t (t-\sigma)^{\frac{N}{2s} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \\ &= |a|CM^{\alpha+1} t^{\frac{N}{2s} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2r} + 1 - \beta(\alpha+1)} \int_0^1 (1-\sigma)^{\frac{N}{2s} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma. \end{aligned}$$

Therefore we obtain

$$\|u(t) - e^{t\Delta}\varphi\|_s \leq |a|CM^{\alpha+1} t^{\frac{N}{2s} - \frac{2-\gamma}{2\alpha}} \int_0^1 (1-\sigma)^{\frac{N}{2s} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma. \quad (4.14)$$

By (4.13),  $\int_0^1 (1-\sigma)^{\frac{N}{2s} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma$  is finite and that  $t^{\frac{N}{2s} - \frac{2-\gamma}{2\alpha}}$  converges to zero as  $t \searrow 0$ . This proves the statement (i) and (iii) of Theorem 4.1. Statement (ii) with  $s = N\alpha/(2-\gamma)$  follows from (4.14) which still holds if  $s = N\alpha/(2-\gamma) := q_c$ .

To prove the stronger decay estimate (4.9), we observe that, by the previous calculations we have,

$$\|u(t) - v(t)\|_r \leq \|e^{t\Delta}(\varphi - \psi)\|_r + 2|a|M^\alpha C(\alpha+1) \int_0^t (t-s)^{-\frac{N\alpha}{2r} - \frac{\gamma}{2}} s^{-\beta\alpha} \|u(s) - v(s)\|_r ds.$$

Let  $\delta > 0$  be such that  $\beta(\alpha+1) + \delta < 1$ . For arbitrary  $T > 0$ , we have

$$\begin{aligned} t^{\beta+\delta} \|u(t) - v(t)\|_r &\leq t^{\beta+\delta} \|e^{t\Delta}(\varphi - \psi)\|_r + 2|a|M^\alpha C(\alpha+1)t^{\beta+\delta} \times \\ &\quad \left( \sup_{0 < t \leq T} t^{\beta+\delta} \|u(t) - v(t)\|_r \right) \int_0^t (t-s)^{-\frac{N\alpha}{2r} - \frac{\gamma}{2}} s^{-\beta(\alpha+1) - \delta} ds \\ &\leq t^{\beta+\delta} \|e^{t\Delta}(\varphi - \psi)\|_r + KM^\alpha t^{\beta+\delta - \frac{N\alpha}{2r} - \frac{\gamma}{2} + 1 - \beta(\alpha+1) - \delta} \times \\ &\quad \left( \sup_{0 < t \leq T} t^{\beta+\delta} \|u(t) - v(t)\|_r \right) \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r} - \frac{\gamma}{2}} \sigma^{-\beta(\alpha+1) - \delta} d\sigma \\ &\leq t^{\beta+\delta} \|e^{t\Delta}(\varphi - \psi)\|_r + KM^\alpha \left( \sup_{0 < t \leq T} t^{\beta+\delta} \|u(t) - v(t)\|_r \right) \times \\ &\quad \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r} - \frac{\gamma}{2}} \sigma^{-\beta(\alpha+1) - \delta} d\sigma, \quad \forall 0 < t \leq T. \end{aligned}$$

Since the constants does not depend on  $T$  and finite by the hypothesis on  $\delta$ , the result follows.

We now prove (iv) of Theorem 4.1 for  $q = \infty$ . The result for the other values of  $q$  will then follows by using the Hölder inequality and (4.6). We need the following Lemma.

**Lemma 4.3.** *Let  $N$  be a positive integer and  $s, q$  be two real numbers, and suppose that*

$$\alpha+1 < s < q \leq \infty, \quad 0 \leq \frac{1}{q} < \frac{\gamma}{N} + \frac{\alpha+1}{s} < 1, \quad \frac{N}{2} \left( \frac{\alpha+1}{s} - \frac{1}{q} \right) < 1 - \frac{\gamma}{2}.$$

*Let  $u$  be the solution of (3.1) with initial data  $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ . Assume that*

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2s}} \|u(t)\|_s < \infty.$$

It follows that

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2q}} \|u(t)\|_q < \infty.$$

Assuming this Lemma for the moment, we continue the proof of the Theorem. Let us consider the solution of (3.1) constructed by the first part of Theorem 4.1. We have

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2r}} \|u(t)\|_r < \infty,$$

for  $r$  satisfying the conditions specified in (4.1). We use the Lemma 4.3 with an iterative argument as in [12]:  $s_i$  will play the role of  $s$ , and  $s_{i+1}$  will play the role of  $q$ , for  $i = 0, 1, \dots$ . Let  $s_0 = r$ , and choose  $s_1$  satisfy the hypothesis of Lemma 4.3. Then by Lemma 4.3 we get that

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2s_1}} \|u(t)\|_{s_1} < \infty.$$

We iterate this procedure. For the next step it is clear we can choose  $s_2$  so that

$$\alpha + 1 < s_1 < s_2 \leq \infty, \quad 0 \leq \frac{1}{s_2} < \frac{\gamma}{N} + \frac{\alpha + 1}{s_1} < 1, \quad \frac{N}{2} \left( \frac{\alpha + 1}{s_1} - \frac{1}{s_2} \right) < 1 - \frac{\gamma}{2}.$$

Then we conclude that

$$\sup_{t>0} t^{\frac{2-\gamma}{\alpha} - \frac{N}{2s_2}} \|u(t)\|_{s_2} < \infty.$$

One can check easily that by this iterative procedure, we can reach  $s_{i+1} = \infty$  for some finite  $i$ . This proves (iv) for  $q = \infty$ . The other cases follows by interpolation. The continuous dependence relation (4.7) with  $q = r$ , of the solution on the initial data can be easily deduced by setting in (4.12)  $\mathcal{F}_\varphi(u) = u$  and  $\mathcal{F}_\varphi(v) = v$ . Formula (4.7) for all  $q \in [r, \infty]$ , can be proved by using an iterative procedure similar to the proof of (iv). In particular, one can prove a version of Lemma 4.3 with  $u$  replaced by  $u - v$ . This completes the proof of Theorem 4.1.  $\square$

*Proof of Lemma 4.3.* We use similar argument as in [12]. Set

$$A = \sup_{t>0} t^{\beta(s)} \|u(t)\|_s, \quad \beta(s) = \frac{2-\gamma}{2\alpha} - \frac{N}{2s}.$$

We use the integral equation (3.1) from  $\frac{t}{2}$  to  $t$ :

$$u(t) = e^{\frac{t}{2}\Delta} u(t/2) + a \int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma} |u(\sigma)|^\alpha u(\sigma)) d\sigma. \quad (4.15)$$

It follows from the smoothing properties of the heat semigroup, with  $s_2 = q$ ,  $s_1 = s$  and (2.2) with  $q_2 = q$ ,  $q_1 = s/(\alpha + 1)$ , that

$$\begin{aligned} \|u(t)\|_q &\leq \|e^{\frac{t}{2}\Delta} u(t/2)\|_q + |a| \int_{\frac{t}{2}}^t \|e^{(t-s)\Delta} (|\cdot|^{-\gamma} |u(\sigma)|^\alpha u(\sigma))\|_q d\sigma \\ &\leq C t^{-\frac{N}{2}(\frac{1}{s} - \frac{1}{q})} \|u(t/2)\|_s + C |a| \int_{\frac{t}{2}}^t (t-\sigma)^{\frac{N}{2q} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2s}} \|u(\sigma)\|_s^{\alpha+1} d\sigma \\ &\leq C t^{-\frac{N}{2}(\frac{1}{s} - \frac{1}{q})} \|u(t/2)\|_s + C |a| A^{\alpha+1} t^{\frac{N}{2q} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2s} + 1 - \beta(s)(\alpha+1)} \times \\ &\quad \int_{\frac{1}{2}}^1 (1-\sigma)^{\frac{N}{2q} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2s}} \sigma^{-\beta(s)(\alpha+1)} d\sigma \\ &\leq C' t^{\frac{N}{2q} - \frac{2-\gamma}{2\alpha}} A + C |a| A^{\alpha+1} t^{-\frac{1}{\alpha} + \frac{\gamma}{2\alpha} + \frac{N}{2q}} \int_{\frac{1}{2}}^1 (1-\sigma)^{\frac{N}{2q} - \frac{\gamma}{2} - \frac{N(\alpha+1)}{2s}} \sigma^{-\beta(s)(\alpha+1)} d\sigma. \end{aligned}$$

Therefore we obtain

$$t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2q}} \|u(t)\|_q \leq C'A + C|a|A^{\alpha+1}C_1,$$

where

$$C_1 = \int_{\frac{1}{2}}^1 (1-\sigma)^{-\frac{N}{2}\left(\frac{\alpha+1}{s} - \frac{1}{q}\right) - \frac{\gamma}{2}} \sigma^{-\beta(s)(\alpha+1)} d\sigma.$$

By the conditions imposed on the parameters in the Lemma,  $C_1$  is positive and finite. Thus, we get

$$\sup_{t>0} t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2q}} \|u(t)\|_q \leq C(A) < \infty,$$

where  $C(A)$  is a positive constant. Remark that  $C(A) \rightarrow 0$  as  $A \rightarrow 0$ . This completes the proof of the Lemma.  $\square$

We now give the proof of Theorem 1.4.

*Proof of Theorem 1.4.* Due to the homogeneity properties of (1.1)-(1.2), and hence (3.1), it is clear that the set of solutions of (3.1) is invariant under the transformation  $u \rightarrow u_\lambda$  for all  $\lambda > 0$ , where

$$u_\lambda(t, x) = \lambda^{\frac{2-\gamma}{\alpha}} u(\lambda^2 t, \lambda x), \quad t > 0, \quad x \in \mathbb{R}^N. \quad (4.16)$$

A self-similar solution is a solution such that  $u_\lambda = u$  for all  $\lambda > 0$ . We claim that solutions given in Theorem 4.1 with homogeneous initial data of degree  $-(2-\gamma)/\alpha$  are self-similar.

In fact, let  $\varphi$  be a tempered distribution satisfying (4.5). Let  $u$  be the solution of (1.5) with initial data  $\varphi$  given by Theorem 4.1. Now define the scaling function  $\varphi_\lambda$  by

$$\varphi_\lambda(x) = \lambda^{\frac{2-\gamma}{\alpha}} \varphi(\lambda x) \quad \forall \lambda > 0, \quad x \in \mathbb{R}^N.$$

This makes sense for distributions by duality. Since  $\sup_{t>0} t^\beta \|e^{t\Delta} \varphi_\lambda\|_r = \sup_{t>0} t^\beta \|e^{t\Delta} \varphi\|_r$  for all  $\lambda > 0$ , it follows that  $\varphi_\lambda$  satisfies also (4.5) for all  $\lambda > 0$ . We can easily compute that the function  $u_\lambda$  given by (4.16) is the solution of (3.1) with initial data  $\varphi_\lambda$ . Finally, if  $\varphi_\lambda = \varphi$ , that is  $\varphi$  homogeneous of degree  $-(2-\gamma)/\alpha$ , and since  $\sup_{t>0} t^\beta \|u_\lambda(t)\|_r = \sup_{t>0} t^\beta \|u(t)\|_r$ ,  $\forall \lambda > 0$ , then  $u_\lambda = u$  and thus  $u$  is self-similar. Let us denote it by  $u_S$ . The fact that  $u_S(t) \rightarrow \varphi$  in  $\mathcal{S}'(\mathbb{R}^N)$  as  $t \rightarrow 0$  follows by Theorem 4.1 part (iii).  $\square$

## 5. ASYMPTOTIC BEHAVIOR: NONLINEAR CASE

In this section we give the proof of Theorem 1.5 Part (i). In fact, we will prove the following more general version. In particular the asymptotic behavior is given in  $L^q(\mathbb{R}^N)$  for all  $q \geq r$ .

**Theorem 5.1** (Nonlinear behavior). *Let  $0 < \gamma < \min(2, N)$ ,  $\alpha > (2-\gamma)/N$ . Let  $r$  and  $\beta$  verify (4.1) and (4.2). Define  $\beta(q)$  by*

$$\beta(q) = \frac{2-\gamma}{2\alpha} - \frac{N}{2q}, \quad q > 1.$$

Let  $\Phi$  be given by

$$\Phi(x) = \omega(x) |x|^{-\frac{2-\gamma}{\alpha}}$$

with  $\omega$  homogeneous of degree 0,  $\omega \in L^\infty(S^{N-1})$  and  $\|\omega\|_\infty$  is sufficiently small. Let

$$u_S(t, x) = t^{-\frac{2-\gamma}{2\alpha}} u_S\left(1, \frac{x}{\sqrt{t}}\right)$$

be the self-similar solution of (3.1) with initial data  $\Phi$  given by Theorem 1.4.

Let  $\varphi \in C_0(\mathbb{R}^N)$  be such that

$$|\varphi(x)| \leq \frac{c}{(1 + |x|^2)^{\frac{2-\gamma}{2\alpha}}}, \quad \forall x \in \mathbb{R}^N, \quad \varphi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}, \quad |x| \geq A,$$

for some constant  $A > 0$ , where  $c$  is a small positive constant. (We take  $\|\omega\|_\infty$  and  $c$  sufficiently small so that (4.5) is satisfied by  $\Phi$  and  $\varphi$ ).

Let  $u$  be the global solution of (1.5) with initial data  $\varphi$  constructed by Theorem 1.3. Then there exists  $\delta > 0$  sufficiently small such that

$$\|u(t) - u_{\mathcal{S}}(t)\|_q \leq C_\delta t^{-\beta(q)-\delta}, \quad \forall t > 0, \quad (5.1)$$

for all  $q \in [r, \infty]$ . Also, we have

$$\|t^{\frac{2-\gamma}{2\alpha}} u(t, \cdot \sqrt{t}) - u_{\mathcal{S}}(1, \cdot)\|_q \leq C_\delta t^{-\delta}, \quad \forall t > 0, \quad (5.2)$$

for all  $q \in [r, \infty]$ .

**Remark 5.1.** In the previous theorem, if  $\delta$  is sufficiently small, then all the quantities on the right hand side of inequalities (5.1)-(5.2) converge to zero as  $t \rightarrow \infty$ . Also, the difference  $u - u_{\mathcal{S}}$  goes to zero as  $t$  goes to infinity more rapidly than each of them do separately.

*Proof.* We have that  $|\Phi(x) - \varphi(x)| = 0$  for  $|x| \geq A$  and  $|\Phi(x) - \varphi(x)| \leq (\|\omega\|_\infty + c)|x|^{-\frac{2-\gamma}{\alpha}}$  for  $|x| \leq A$ . Then

$$|\Phi - \varphi| \leq (\|\omega\|_\infty + c)\varphi_1,$$

with  $\varphi_1 = |\cdot|^{-\frac{2-\gamma}{\alpha}} 1_{\{|x| \leq A\}} \in L^s(\mathbb{R}^N)$ ,  $1 \leq s < N\alpha/(2-\gamma)$ . By the smoothing properties of the heat semigroup (2.1), we have that  $e^{t\Delta}\varphi_1 \in L^r(\mathbb{R}^N)$  and

$$\sup_{t>0} t^{\beta+\delta} \|e^{t\Delta}\varphi_1\|_r < \infty, \quad \text{for } 0 < \delta < \frac{N}{2} - \frac{2-\gamma}{2\alpha}.$$

From the latter part of Theorem 4.1, and, in particular, formula (4.9), we have that

$$\sup_{t>0} t^{\beta+\delta} \|u(t) - u_{\mathcal{S}}(t)\|_r \leq C \sup_{t>0} t^{\beta+\delta} \|e^{t\Delta}(\Phi - \varphi)\|_r = C \sup_{t>0} t^{\beta+\delta} \|e^{t\Delta}\varphi_1\|_r.$$

That is

$$\sup_{t>0} t^{\beta+\delta} \|u(t) - u_{\mathcal{S}}(t)\|_r \leq C, \quad (5.3)$$

for  $\delta > 0$  sufficiently small and  $C$  a finite positive constant. This gives (5.1) directly, and (5.2) by a simple dilation argument for  $q = r$ .

We now turn to the asymptotic result in the  $L^\infty$ -norm. Write

$$u(t) - u_{\mathcal{S}}(t) = e^{\frac{t}{2}\Delta}(u(t/2) - u_{\mathcal{S}}(t/2)) + a \int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma} (|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))] d\sigma.$$

Let  $T > 0$  be an arbitrary real number. By using the smoothing properties of the heat semi-group and Proposition 2.1 with  $(q_1, q_2) = (\infty, \infty)$ , we have that

$$\begin{aligned} t^{\frac{2-\gamma}{2\alpha}+\delta} \|u(t) - u_{\mathcal{S}}(t)\|_\infty &\leq t^{\frac{2-\gamma}{2\alpha}+\delta} \left\| e^{\frac{t}{2}\Delta}(u(t/2) - u_{\mathcal{S}}(t/2)) \right\|_\infty + |a| t^{\frac{2-\gamma}{2\alpha}+\delta} \times \\ &\quad \int_{\frac{t}{2}}^t \left\| e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma} (|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))] \right\|_\infty d\sigma \\ &\leq C t^{\beta+\delta} \|u(t/2) - u_{\mathcal{S}}(t/2)\|_r + \\ &\quad |a| C t^{\frac{2-\gamma}{2\alpha}+\delta} \int_{\frac{t}{2}}^t (t-\sigma)^{-\frac{\gamma}{2}} (\|u(\sigma)\|_\infty^\alpha + \|u_{\mathcal{S}}(\sigma)\|_\infty^\alpha) \|u(\sigma) - u_{\mathcal{S}}(\sigma)\|_\infty d\sigma. \end{aligned}$$

Using (5.3) to estimate the first term and the fact that  $\|u_S(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}}$ ,  $\|u(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}}$  to estimate the last term, we get

$$t^{\frac{2-\gamma}{2\alpha}+\delta}\|u(t) - u_S(t)\|_\infty \leq C(\delta) + 2C^\alpha|a|C \times \left[ \int_{\frac{1}{2}}^1 (1-\sigma)^{-\frac{\gamma}{2}} \sigma^{-(\alpha+1)\frac{2-\gamma}{2\alpha}-\delta} d\sigma \right] \sup_{t \in (0, T]} \left( t^{\frac{2-\gamma}{2\alpha}+\delta} \|u(t) - u_S(t)\|_\infty \right).$$

This gives that,  $\sup_{t \in (0, T]} t^{\frac{2-\gamma}{2\alpha}+\delta} \|u(t) - u_S(t)\|_\infty \leq C'(\delta)$ . Since the constant  $C'(\delta)$  does not depend on  $T > 0$ , one can take the supremum over  $(0, \infty)$ . It follows that for all  $\delta > 0$  sufficiently small there exists a constant  $C'(\delta)$  such that

$$\|u(t) - u_S(t)\|_\infty \leq C'(\delta)t^{-\frac{2-\gamma}{2\alpha}-\delta},$$

for all  $t > 0$ . This prove (5.1) for  $r = \infty$ . The general result (5.1) follows now by the interpolation inequality. The estimate (5.2) follows by a simple dilation argument.  $\square$

## 6. ASYMPTOTIC BEHAVIOR: LINEAR CASE

In this section we give the proof of Theorem 1.5 Part (ii). To prove the asymptotic linear behavior, we need to establish an adequate global existence result.

**6.1. More global existence results.** We have the following lemma used to establish the needed global existence result. We denote, for  $a \in \mathbb{R}$ ,  $a_+$  by  $a_+ := \max(a, 0)$  and  $\frac{1}{a_+} = \frac{1}{a}$  if  $a > 0$ , and  $\infty$  if  $a \leq 0$ .

**Lemma 6.1.** *Assume that  $0 < \gamma < \min(2, N)$  and  $\alpha > (2 - \gamma)/N$ . Let  $\alpha_1$  be a real number such that*

$$\alpha > \alpha_1 > \frac{2 - \gamma}{N}.$$

Let  $r_1$  be a real number satisfying

$$\max\left(\frac{N(\alpha_1 + 1)}{N - \gamma}, \frac{N\alpha_1}{2 - \gamma}\right) < r_1 < \frac{N\alpha_1(\alpha_1 + 1)}{(2 - \gamma(\alpha_1 + 1))_+}, \quad (6.1)$$

Let

$$r_2 = \frac{\alpha}{\alpha_1} r_1, \quad (6.2)$$

$$\beta_1 = \frac{2 - \gamma}{2\alpha_1} - \frac{N}{2r_1}, \quad (6.3)$$

$$\beta_2 = \frac{2 - \gamma}{2\alpha} - \frac{N}{2r_2}. \quad (6.4)$$

Define  $r_{12}$  and  $\beta_{12}$  by

$$r_{12} = \frac{\alpha + 1}{\alpha_1 + 1} r_1, \quad \beta_{12} = \frac{\alpha_1 + 1}{\alpha + 1} \beta_1. \quad (6.5)$$

Then we have the following

- (i)  $\beta_1 > 0, \beta_2 > 0, \beta_{12} > 0,$
- (ii)  $\frac{1}{r_1} < \frac{\gamma}{N} + \frac{\alpha+1}{r_{12}} < 1, \frac{1}{r_2} < \frac{\gamma}{N} + \frac{\alpha+1}{r_2} < 1,$
- (iii)  $\frac{N}{2} \left( \frac{\alpha+1}{r_{12}} - \frac{1}{r_1} \right) + \frac{\gamma}{2} = \frac{N\alpha}{2r_2} + \frac{\gamma}{2} < 1,$
- (iv)  $\beta_2(\alpha+1) < 1, \beta_{12}(\alpha+1) < 1,$
- (v)  $\beta_2 - \frac{N\alpha}{2r_2} - \frac{\gamma}{2} - \beta_2(\alpha+1) + 1 = 0,$

$$(vi) \quad \beta_1 - \frac{N}{2} \left( \frac{\alpha + 1}{r_{12}} - \frac{1}{r_1} \right) - \frac{\gamma}{2} - \beta_{12}(\alpha + 1) + 1 = 0.$$

The proof of Lemma 6.1 is given in the appendix. We now give the following global existence result.

**Theorem 6.1.** *Let  $0 < \gamma < \min(2, N)$  and  $\alpha > (2 - \gamma)/N$ . Let  $\alpha_1$  be a real number such that*

$$\alpha > \alpha_1 > \frac{2 - \gamma}{N}.$$

*Let  $r_1, r_2, r_{12}, \beta_1$  and  $\beta_2$  be real numbers as in Lemma 6.1. Suppose further that  $M > 0$  satisfies the inequality*

$$KM^\alpha < 1, \quad (6.6)$$

*where  $K$  is a positive constant. Choose  $R > 0$  such that*

$$R + KM^{\alpha+1} \leq M. \quad (6.7)$$

*Let  $\varphi$  be a tempered distribution such that*

$$\max \left[ \sup_{t>0} t^{\beta_1} \|e^{t\Delta} \varphi\|_{r_1}, \sup_{t>0} t^{\beta_2} \|e^{t\Delta} \varphi\|_{r_2} \right] \leq R. \quad (6.8)$$

*It follows that there exists a unique global solution  $u$  of (3.1) such that*

$$\max \left[ \sup_{t>0} t^{\beta_1} \|u(t)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|u(t)\|_{r_2} \right] \leq M. \quad (6.9)$$

*Furthermore,*

- (i)  $u(t) - e^{t\Delta} \varphi \in C([0, \infty); L^s(\mathbb{R}^N))$ , for  $s$  satisfying  $\frac{2-\gamma}{N\alpha_1} < \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha+1}{r_{12}}$ .
- (ii)  $u(t) - e^{t\Delta} \varphi \in L^\infty((0, \infty); L^s(\mathbb{R}^N))$ , for  $s$  satisfying  $\frac{2-\gamma}{N\alpha_1} \leq \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha+1}{r_{12}}$ .
- (iii)  $\lim_{t \rightarrow 0} u(t) = \varphi$  in the sense of distributions.
- (iv)  $\sup_{t>0} t^{\frac{2-\gamma}{2\alpha_1} - \frac{N}{2q}} \|u(t)\|_q < \infty, \forall q \in [r_1, \infty]$ .
- (v)  $\sup_{t>0} t^{\frac{2-\gamma}{2\alpha} - \frac{N}{2q}} \|u(t)\|_q < \infty, \forall q \in [r_2, \infty]$ .

*Moreover, let  $\varphi$  and  $\psi$  satisfying (6.8) and let  $u$  and  $v$  be respectively the solutions of (3.1) with initial values  $\varphi$  and  $\psi$  respectively. Then*

$$\begin{aligned} \max \left[ \sup_{t>0} t^{\beta_1} \|u(t) - v(t)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|u(t) - v(t)\|_{r_2} \right] &\leq (1 - KM^\alpha)^{-1} \times \\ &\max \left[ \sup_{t>0} t^{\beta_1} \|e^{t\Delta}(\varphi - \psi)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|e^{t\Delta}(\varphi - \psi)\|_{r_2} \right] \end{aligned} \quad (6.10)$$

*and*

$$\begin{aligned} \max \left[ \sup_{t>0} t^{\frac{2-\gamma}{2\alpha_1}} \|u(t) - v(t)\|_\infty, \sup_{t>0} t^{\frac{2-\gamma}{2\alpha}} \|u(t) - v(t)\|_\infty \right] &\leq \\ &C \max \left[ \sup_{t>0} t^{\beta_1} \|e^{t\Delta}(\varphi - \psi)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|e^{t\Delta}(\varphi - \psi)\|_{r_2} \right]. \end{aligned} \quad (6.11)$$

*Proof.* The proof is based on a contraction mapping argument and uses some idea of [12]. Let  $X$  be the Bochner of measurable functions  $u : (0, \infty) \rightarrow L^{r_1}(\mathbb{R}^N) \cap L^{r_2}(\mathbb{R}^N)$  such that

$$\max \left[ \sup_{t>0} t^{\beta_1} \|u(t)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|u(t)\|_{r_2} \right] < \infty.$$

We denote by  $X_M$  the set of  $u \in X$  such that  $\max [\sup_{t>0} t^{\beta_1} \|u(t)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|u(t)\|_{r_2}] \leq M$ . Endowed with the metric:  $d(u, v) = \max \left[ \sup_{t>0} t^{\beta_1} \|u(t) - v(t)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|u(t) - v(t)\|_{r_2} \right]$ ,  $X_M$  is a nonempty complete metric space. Let  $M, R$  be two real numbers satisfying (6.6)-(6.7). Consider the mapping defined by

$$\mathcal{F}_\varphi(u)(t) = e^{t\Delta} \varphi + a \int_0^t e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma} |u(\sigma)|^\alpha u(\sigma)) d\sigma, \quad (6.12)$$

where  $\varphi$  is a tempered distribution satisfying (6.8). We will show that  $\mathcal{F}_\varphi$  is a strict contraction on  $X_M$ .

Let  $\varphi$  and  $\psi$  satisfy (6.8) and  $u, v \in X_M$ . It follows that

$$\begin{aligned} t^{\beta_1} \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_1} &\leq t^{\beta_1} \|e^{t\Delta}(\varphi - \psi)\|_{r_1} + \\ &|a| t^{\beta_1} \int_0^t \|e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma} (|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma))]\|_{r_1} d\sigma. \end{aligned}$$

It follows, by Proposition 2.1 with  $(q_1, q_2) = (\frac{r_{12}}{\alpha+1}, r_1)$  due to Lemma 6.1 Part (ii) and Hölder inequality that

$$\begin{aligned} \|e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma} (|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma))]\|_{r_1} &\leq C(t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) - \frac{\gamma}{2}} \times \\ &\| |u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma) \|_{\frac{r_{12}}{\alpha+1}} \\ &\leq C(t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) - \frac{\gamma}{2}} (\alpha+1) \left( \|u(\sigma)\|_{r_{12}}^\alpha + \|v(\sigma)\|_{r_{12}}^\alpha \right) \|u(\sigma) - v(\sigma)\|_{r_{12}}. \end{aligned}$$

Using the interpolation inequality

$$\|u(\sigma)\|_s \leq \|u(\sigma)\|_{r_1}^\theta \|u(\sigma)\|_{r_2}^{1-\theta}, \quad \frac{1}{s} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}, \quad (6.13)$$

where  $\theta = \frac{1}{\alpha+1}$  and  $s = r_{12}$  along with the fact that  $u, v$  are in  $X_M$  we see that

$$\begin{aligned} \|e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma} (|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma))]\|_{r_1} &\leq 2(\alpha+1)C \times \\ &(t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) - \frac{\gamma}{2}} \sigma^{-\beta_{12}(\alpha+1)} M^\alpha d(u, v). \end{aligned}$$

Hence,

$$\begin{aligned} \left\| \int_0^t e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma} (|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma))]\ d\sigma \right\|_{r_1} &\leq 2(\alpha+1)CM^\alpha \times \\ &\left[ \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) - \frac{\gamma}{2}} \sigma^{-\beta_{12}(\alpha+1)} d\sigma \right] d(u, v) \\ &\leq 2(\alpha+1)CM^\alpha d(u, v) t^{-\beta_{12}(\alpha+1) - \frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) - \frac{\gamma}{2} + 1} \times \\ &\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) - \frac{\gamma}{2}} \sigma^{-\beta_{12}(\alpha+1)} d\sigma. \end{aligned}$$

Then, using Part (vi) of Lemma 6.1, we obtain

$$\begin{aligned} t^{\beta_1} \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_1} &\leq t^{\beta_1} \|e^{t\Delta}(\varphi - \psi)\|_{r_1} + 2|a|(\alpha+1)CM^\alpha \times \\ &\left[ \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) - \frac{\gamma}{2}} \sigma^{-\beta_{12}(\alpha+1)} d\sigma \right] d(u, v) \\ &\leq t^{\beta_1} \|e^{t\Delta}(\varphi - \psi)\|_{r_1} + C_1 M^\alpha d(u, v), \end{aligned} \quad (6.14)$$

where

$$C_1 = 2|a|(\alpha + 1)C \int_0^1 (1 - \sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}} - \frac{1}{r_1}) - \frac{\gamma}{2}} \sigma^{-\beta_{12}(\alpha+1)} d\sigma.$$

On the other hand, we have

$$\begin{aligned} t^{\beta_2} \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_2} &\leq t^{\beta_2} \|e^{t\Delta}(\varphi - \psi)\|_{r_2} + |a|t^{\beta_2} \times \\ &\int_0^t \|e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma} (|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma))]\|_{r_2} d\sigma. \end{aligned}$$

It follows by Proposition 2.1 with  $(q_1, q_2) = (r_2/(\alpha + 1), r_2)$  and with the fact that  $u, v$  are in  $X_M$  that

$$\begin{aligned} \|e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma} (|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma))]\|_{r_2} &\leq C(t - \sigma)^{-\frac{N\alpha}{2r_2} - \frac{\gamma}{2}} \times \\ &\| |u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma) \|_{\frac{r_2}{\alpha+1}} \\ &\leq C(t - \sigma)^{-\frac{N\alpha}{2r_2} - \frac{\gamma}{2}} (\alpha + 1) \left( \|u(\sigma)\|_{r_2}^\alpha + \|v(\sigma)\|_{r_2}^\alpha \right) \|u(\sigma) - v(\sigma)\|_{r_2} \\ &\leq 2(\alpha + 1)C(t - \sigma)^{-\frac{N\alpha}{2r_2} - \frac{\gamma}{2}} \sigma^{-\beta_2(\alpha+1)} M^\alpha d(u, v). \end{aligned}$$

This together with Part (v) of Lemma 6.1 gives

$$t^{\beta_2} \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_2} \leq t^{\beta_2} \|e^{t\Delta}(\varphi - \psi)\|_{r_2} + C_2 M^\alpha d(u, v), \quad (6.15)$$

where  $C_2 = 2|a|(\alpha + 1)C \int_0^1 (1 - \sigma)^{-\frac{N\alpha}{2r_2} - \frac{\gamma}{2}} \sigma^{-\beta_2(\alpha+1)} d\sigma$ .

Due to Parts (iii) and (iv) of Lemma 6.1,  $C_1$  and  $C_2$  are finite positive constants. Now we get by (6.14) and (6.15) that,

$$\begin{aligned} \max \left[ \sup_{t>0} t^{\beta_1} \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{r_2} \right] &\leq \\ \max \left[ \sup_{t>0} t^{\beta_1} \|e^{t\Delta}(\varphi - \psi)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|e^{t\Delta}(\varphi - \psi)\|_{r_2} \right] &+ KM^\alpha d(u, v), \end{aligned} \quad (6.16)$$

where  $K = \max(C_1, C_2)$ . Setting  $\psi = 0$  and  $v = 0$ , and using (6.8), (6.7) we obtain

$$\max \left[ \sup_{t>0} t^{\beta_1} \|\mathcal{F}_\varphi(u)(t)\|_{r_1}, \sup_{t>0} t^{\beta_2} \|\mathcal{F}_\varphi(u)(t)\|_{r_2} \right] \leq R + KM^{\alpha+1} \leq M.$$

Then  $\mathcal{F}_\varphi$  maps  $X_M$  into itself. Letting  $\varphi = \psi$ , we get

$$d(\mathcal{F}_\varphi(u), \mathcal{F}_\varphi(v)) \leq KM^\alpha d(u, v).$$

Hence inequality (6.6) gives that  $\mathcal{F}_\varphi$  is a strict contraction mapping from  $X_M$  into itself. Then  $\mathcal{F}_\varphi$  has a unique fixed point  $u$  in  $X_M$  which is solution of (3.1).

We now prove that  $u(t) - e^{t\Delta}\varphi \in C([0, \infty), L^s(\mathbb{R}^N))$  for  $s$  satisfying

$$\frac{2 - \gamma}{N\alpha_1} < \frac{1}{s} < \frac{\gamma}{N} + \frac{\alpha + 1}{r_{12}}. \quad (6.17)$$

First, the existence of such  $s$  is insured by Part (iv) of Lemma 6.1. Write

$$u(t) - e^{t\Delta}\varphi = a \int_0^t e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma} |u(\sigma)|^\alpha u(\sigma)) d\sigma.$$



Then for  $s$  satisfying (6.17), we obtain by Proposition 2.1, with  $(q_1, q_2) = (\frac{r_{12}}{\alpha+1}, s)$ ,

$$\begin{aligned}
\|u(t) - e^{t\Delta}\varphi\|_s &\leq |a| \int_0^t \|e^{(t-\sigma)\Delta}(|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma))\|_s d\sigma \\
&\leq |a|C \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}} \|u(\sigma)\|_{r_{12}}^{\alpha+1} d\sigma \\
&\leq |a|CM^{\alpha+1} \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}} \sigma^{-\beta_{12}(\alpha+1)} d\sigma \\
&= |a|CM^{\alpha+1} t^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}-\beta_{12}(\alpha+1)+1} \times \\
&\quad \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}} \sigma^{-\beta_{12}(\alpha+1)} d\sigma.
\end{aligned}$$

Therefore we obtain

$$\|u(t) - e^{t\Delta}\varphi\|_s \leq |a|CM^{\alpha+1} t^{\frac{N}{2s}-\frac{2-\gamma}{2\alpha_1}} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}} \sigma^{-\beta_{12}(\alpha+1)} d\sigma. \quad (6.18)$$

Owing to (6.17) we can see that  $\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r_{12}}-\frac{1}{s})-\frac{\gamma}{2}} \sigma^{-\beta_{12}(\alpha+1)} d\sigma$  is finite and that  $t^{\frac{N}{2s}-\frac{2-\gamma}{2\alpha_1}}$  converges to zero as  $t \searrow 0$ . This prove the statements (i) and (iii) of Theorem 6.1. Statement (ii) with  $s = \frac{N\alpha_1}{2-\gamma}$  follows from (6.18) which still holds if  $s = \frac{N\alpha_1}{2-\gamma}$ .

The proof of Parts (iv)-(v) for  $r = \infty$  follows by iterative argument as in the proof of Theorem 4.1 Part (v) so we omit it. The result for the other values of  $r$  will then follows by using the Hölder inequality and (6.9).

The continuous dependence relation (6.10) of the solution on the initial data can be easily deduced by (6.16) with  $\mathcal{F}_\varphi(u) = u$ ,  $\mathcal{F}_\psi(v) = v$ . Formula (6.11) can be proved starting with (6.10) and using an iterative procedure. This completes the proof of Theorem 6.1.  $\square$

We now turn to establish the linear behavior.

**6.2. Linear behavior.** The following technical Lemma will be needed in the proof of the linear asymptotic behavior.

**Lemma 6.2.** *Let  $0 < \gamma < \min(2, N)$ . Let the real numbers  $\alpha_1$  and  $\alpha$  be such that*

$$\alpha > \alpha_1 > \frac{2-\gamma}{N}.$$

*Let  $r_1$  and  $r_2$  be two real numbers as in Lemma 6.1. Let  $\beta_1$  and  $\beta_2$  be given by (6.3) and (6.4). Then there exists a real number  $\delta_0 > 0$  such that, for all  $0 < \delta < \delta_0$ , there exists a real number*

$$0 < \theta_\delta < 1, \quad (6.19)$$

*with the properties that, the two real numbers  $r'$  and  $\beta'$  given by*

$$\frac{1}{r'} = \frac{\theta_\delta}{r_1} + \frac{1-\theta_\delta}{r_2}, \quad \beta' = \theta_\delta\beta_1 + (1-\theta_\delta)\beta_2 = \frac{(2-\gamma)[\theta_\delta(\alpha-\alpha_1) + \alpha_1]}{2\alpha_1\alpha} - \frac{N}{2r'}, \quad (6.20)$$

*satisfy the following conditions*

- (a)  $\frac{1}{r_1} < \frac{\gamma}{N} + \frac{\alpha+1}{r'} < 1$ ,
- (b)  $\beta_1 + \delta - \frac{N}{2}(\frac{\alpha+1}{r'} - \frac{1}{r_1}) - \frac{\gamma}{2} - \beta'(\alpha+1) + 1 = 0$ ,
- (c)  $\frac{N}{2}(\frac{\alpha+1}{r'} - \frac{1}{r_1}) + \frac{\gamma}{2} < 1$ ,  $\beta'(\alpha+1) < 1$ .

Moreover, the real number  $\theta_\delta$  satisfies

$$\theta_\delta = \frac{1}{\alpha + 1} + \frac{2\alpha_1\alpha}{(2 - \gamma)(\alpha - \alpha_1)(\alpha + 1)}\delta. \quad (6.21)$$

The proof of the previous lemma is given in the appendix. We now give the asymptotic behavior result. We have the following more general version. In particular the asymptotic behavior is given in  $L^q(\mathbb{R}^N)$  for all  $q \geq r_1$ .

**Theorem 6.2** (Linear behavior). *Let  $0 < \gamma < \min(2, N)$ . Suppose that*

$$\alpha > \alpha_1 > \frac{2 - \gamma}{N}.$$

*Let  $r_1, r_2$  be two real numbers as in Lemma 6.1. Let  $\beta_1, \beta_2$  be given by (6.3), (6.4) and define  $\beta_1(q)$  by*

$$\beta_1(q) = \frac{2 - \gamma}{2\alpha_1} - \frac{N}{2q}, \quad q > 1. \quad (6.22)$$

*Let  $\Psi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha_1}}$ , where  $\omega \in L^\infty(S^{N-1})$  is homogeneous of degree 0. Let  $\varphi \in C_0(\mathbb{R}^N)$  be such that*

$$|\varphi(x)| \leq \frac{c}{(1 + |x|^2)^{\frac{2-\gamma}{2\alpha_1}}}, \quad \forall x \in \mathbb{R}^N, \quad \varphi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha_1}}, \quad |x| \geq A,$$

*for some constant  $A > 0$ , where  $c$  is a small positive constant and  $\|\omega\|_\infty$  is sufficiently small.*

*Let  $u$  be the solution of (3.1) with initial data  $\varphi$ , constructed by Theorem 6.1 and let  $w$  be the self-similar solution of (3.1) constructed by Theorem 6.1 with  $a = 0$ , and with initial data  $\Psi$ . Then there exists  $\delta_1 > 0$  such that for all  $\delta, 0 < \delta < \delta_1$ , and with  $M$  perhaps smaller, there exists  $C_\delta > 0$  such that*

$$\|u(t) - w(t)\|_q \leq C_\delta t^{-\beta_1(q) - \delta}, \quad \forall t > 0, \quad (6.23)$$

$$\|t^{\frac{2-\gamma}{2\alpha_1}} v(t, \sqrt{t}) - w(1, \cdot)\|_q \leq C_\delta t^{-\delta}, \quad \forall t > 0, \quad (6.24)$$

*for all  $q \in [r_1, \infty]$ . In particular, there exists  $d_1 > 0, d_2 > 0$  two constants, such that*

$$d_1 t^{-\beta_1(q)} \leq \|u(t)\|_q \leq d_2 t^{-\beta_1(q)},$$

*for large time and for all  $r_1 \leq q \leq \infty$ .*

*Proof.* By writing  $|\cdot|^{-\frac{2-\gamma}{\alpha_1}} = f_1 + f_2$ , with  $f_1 \in L^s(\mathbb{R}^N)$ ,  $1 \leq s < N\alpha_1/(2 - \gamma)$ ,  $f_2 \in L^{r_1}(\mathbb{R}^N)$ , it follows by the smoothing properties of the heat semigroup that  $e^{t\Delta}\Psi \in L^{r_1}(\mathbb{R}^N)$  and by homogeneity, we have that  $\sup_{t>0} t^{\beta_1} \|e^{t\Delta}\Psi\|_{r_1} < \infty$ . Since  $|\varphi(x)| \leq (c + \|\omega\|_\infty)|x|^{-\frac{2-\gamma}{\alpha_1}}$ , and because  $\alpha_1 < \alpha$ , and by conditions on  $\varphi$ , we have also  $|\varphi(x)| \leq (c + \|\omega\|_\infty)|x|^{-\frac{2-\gamma}{\alpha}}$ , then  $\sup_{t>0} t^{\beta_1} \|e^{t\Delta}\varphi\|_{r_1} < \infty$  and  $\sup_{t>0} t^{\beta_2} \|e^{t\Delta}\varphi\|_{r_2} < \infty$ . Hence  $\varphi$  verifies (6.8) for  $c$  and  $\|\omega\|_\infty$  sufficiently small. We have also that  $|\Psi(x) - \varphi(x)| = 0$  for  $|x| \geq A$  and  $|\Psi(x) - \varphi(x)| \leq (\|\omega\|_\infty + c)|x|^{-\frac{2-\gamma}{\alpha_1}}$  for  $|x| \leq A$ . Then

$$|\Psi - \varphi| \leq (\|\omega\|_\infty + c)\varphi_1,$$

with  $\varphi_1 = |\cdot|^{-\frac{2-\gamma}{\alpha_1}} \mathbf{1}_{\{|x| \leq A\}} \in L^s(\mathbb{R}^N)$ ,  $1 \leq s < N\alpha_1/(2 - \gamma)$ . By the smoothing properties of the heat semigroup (2.1), we have that  $e^{t\Delta}\varphi_1 \in L^{r_1}(\mathbb{R}^N)$  and

$$\sup_{t>0} t^{\beta_1 + \delta} \|e^{t\Delta}\varphi_1\|_{r_1} < \infty, \quad \text{for } 0 < \delta < \frac{N}{2} - \frac{2 - \gamma}{2\alpha_1}.$$

Let  $v$  be the solution of (3.1) with  $a = 0$  and with initial data  $\varphi$ . We have

$$u(t) - v(t) = a \int_0^t e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma} |u(\sigma)|^\alpha u(\sigma)) d\sigma,$$

and so

$$\|u(t) - v(t)\|_{r_1} \leq |a| \int_0^t \|e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma} |u(\sigma)|^\alpha u(\sigma))\|_{r_1} d\sigma. \quad (6.25)$$

Let  $0 < \delta < \delta_0$ , where  $\delta_0$  is as in Lemma 6.2 and consider the two real numbers  $r'$  and  $\beta'$  given by (6.20). Then thanks to (a) of Lemma 6.2, we obtain

$$\|u(t) - v(t)\|_{r_1} \leq |a| C \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r'} - \frac{1}{r_1}) - \frac{\gamma}{2}} \|u(\sigma)\|_{r'}^{\alpha+1} d\sigma. \quad (6.26)$$

Using the fact that  $u$  and  $v$  belong to  $X_M$  and the interpolation inequality (6.13) with  $s = r'$  and  $\theta_\delta$ , given by (6.21), we deduce from (6.26) that

$$\begin{aligned} t^{\beta_1+\delta} \|u(t) - v(t)\|_{r_1} &\leq |a| C M^{\alpha+1} t^{\beta_1+\delta - \frac{N}{2}(\frac{\alpha+1}{r'} - \frac{1}{r_1}) - \frac{\gamma}{2} - \beta'(\alpha+1)+1} \\ &\quad \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r'} - \frac{1}{r_1}) - \frac{\gamma}{2}} \sigma^{-\beta'(\alpha+1)} d\sigma. \end{aligned}$$

By Lemma 6.2  $\int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r'} - \frac{1}{r_1}) - \frac{\gamma}{2}} \sigma^{-\beta'(\alpha+1)} d\sigma$  is finite. Now by (b) of Lemma 6.2 we deduce that

$$t^{\beta_1+\delta} \|u(t) - v(t)\|_{r_1} \leq C_\delta.$$

And so

$$\|u(t) - v(t)\|_{r_1} \leq C_\delta t^{-\beta_1-\delta}, \quad \forall t > 0. \quad (6.27)$$

Then, we obtain

$$\|u(t) - w(t)\|_{r_1} \leq \|u(t) - v(t)\|_{r_1} + \|e^{t\Delta}(\varphi - \Psi)\|_{r_1} \leq C_\delta t^{-\beta_1-\delta} + C'_\delta t^{-\beta_1-\delta}, \quad (6.28)$$

where  $\delta > 0$  sufficiently small. Hence (6.28) gives (6.23) for  $q = r_1$ .

We now turn to the asymptotic result in the  $L^\infty$ -norm. Write

$$u(t) - w(t) = e^{\frac{t}{2}\Delta}(u(t/2) - w(t/2)) + a \int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma} |u(\sigma)|^\alpha u(\sigma)) d\sigma.$$

Let  $\delta > 0$  sufficiently small. Let  $\theta_\delta$  be given by (6.21), where  $\delta > 0$  is chosen such that  $\theta_\delta < 1$ . By using the smoothing properties of the heat semi-group and Proposition 2.1 with  $(q_1, q_2) = (\infty, \infty)$ , we have that

$$\begin{aligned} t^{\frac{2-\gamma}{2\alpha_1}+\delta} \|u(t) - w(t)\|_\infty &\leq t^{\frac{2-\gamma}{2\alpha_1}+\delta} \|e^{\frac{t}{2}\Delta}(u(t/2) - w(t/2))\|_\infty + |a| t^{\frac{2-\gamma}{2\alpha_1}+\delta} \times \\ &\quad \int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma} |u(\sigma)|^\alpha u(\sigma))\|_\infty d\sigma \\ &\leq C t^{\beta_1+\delta} \|u(t/2) - w(t/2)\|_{r_1} + \\ &\quad |a| C t^{\frac{2-\gamma}{2\alpha_1}+\delta} \int_{\frac{t}{2}}^t (t-\sigma)^{-\frac{\gamma}{2}} \|u(\sigma)\|_\infty^{\alpha+1} d\sigma. \end{aligned}$$

Using the fact that,  $\|u(\sigma)\|_\infty \leq C\sigma^{-[\theta_\delta \frac{2-\gamma}{2\alpha_1} + (1-\theta_\delta) \frac{2-\gamma}{2\alpha}]}$ , to estimate this last term, that is  $\|u(\sigma)\|_\infty^{\alpha+1} \leq C^{\alpha+1}\sigma^{-\frac{2-\gamma}{2\alpha_1}-\delta+\frac{\gamma}{2}-1}$ , we get

$$t^{\frac{2-\gamma}{2\alpha_1}+\delta}\|u(t) - w(t)\|_\infty \leq C(\delta) + C^{\alpha+1}|a|C \int_{\frac{1}{2}}^1 (1-\sigma)^{-\frac{\gamma}{2}} \sigma^{-\frac{2-\gamma}{2\alpha_1}-\delta+\frac{\gamma}{2}-1} d\sigma.$$

Thus, it follows that for all  $\delta > 0$  sufficiently small there exists a constant  $C'(\delta)$  such that

$$\|u(t) - w(t)\|_\infty \leq C'(\delta)t^{-\frac{2-\gamma}{2\alpha_1}-\delta},$$

for all  $t > 0$ . This prove (6.23) for  $q = \infty$ . The general result (6.23) follows now by the Hölder inequality. The estimate (6.24) follows by a simple dilation argument. The proof of the Theorem is now complete.  $\square$

We now give the proof of Theorem 1.5 for the linear case.

*Proof of Theorem 1.5 part (ii).* Let  $\alpha_1$  be as in Theorem 6.2. Put  $\sigma = (2-\gamma)/\alpha_1$ . Since  $\alpha > \alpha_1 > (2-\gamma)/N$  then  $(2-\gamma)/\alpha < \sigma < N$ . Then (ii) follows by Theorem 6.2, precisely by (6.23) with  $q = \infty$ .  $\square$

## 7. GENERAL SINGULAR PROBLEM

In this section we study the general singular nonlinear parabolic equation

$$u(t) = e^{t\Delta}\varphi + \int_0^t e^{(t-\sigma)\Delta} [V(\cdot)|u(\sigma)|^\alpha u(\sigma)] d\sigma, \quad (7.1)$$

$u = u(t, x) \in \mathbb{R}$ ,  $t > 0$ ,  $x \in \mathbb{R}^N$ ,  $\alpha > 0$  and  $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ . The potential  $V$  verifies

$$|V(x)| \leq C|x|^{-\gamma}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \quad (7.2)$$

and satisfies one of the hypotheses:

$$(H_1) \quad V(x) = a(1 - f(x))|x|^{-\gamma},$$

$$(H_2) \quad V(x) = af(x)|x|^{-\gamma},$$

where  $a \in \mathbb{R}$  and  $f$  is such that

$$f(x)|x|^{-\gamma} \in L^s(\mathbb{R}^N), \quad \frac{\gamma}{N} < \frac{1}{s} < \frac{2\gamma}{N} + \frac{\alpha+1}{r} - \frac{1}{q_c}, \quad \frac{1}{s} < \frac{2}{N} - \frac{\alpha}{r}, \quad (7.3)$$

with  $r$  satisfies (4.1). As an example for such function  $f$  we may take a cut-off function compactly supported and  $f \equiv 1$  near the origin.

It is clear that Proposition 2.1, the  $C_0$ -well-posedness,  $L^q$ -well-posedness and the global existence results hold as for the case  $V(x) = a|x|^{-\gamma}$ . In particular, we can prove similar results of global existence as in Theorem 1.3. Here, we are mainly concerned by the asymptotic behavior of global solutions with initial values  $\varphi(x) \sim \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$  as  $|x| \rightarrow \infty$ , where  $\omega \in L^\infty(\mathbb{R}^N)$  is homogeneous of degree 0 and  $\|\omega\|_\infty$  is sufficiently small. We show that for the case  $(H_1)$ , we have the same asymptotic behavior as for the equation (1.1). While for the case  $(H_2)$  the behavior is linear, in particular it is different from the case of equation (1.1). Precisely, we have the following result.

**Theorem 7.1** (General Potential). *Let  $0 < \gamma < \min(2, N)$ . Suppose that*

$$\alpha > \frac{2 - \gamma}{N}. \quad (7.4)$$

Let  $\varphi \in C_0(\mathbb{R}^N)$  be such that

$$|\varphi(x)| \leq \frac{c}{(1 + |x|^2)^{\frac{2-\gamma}{2\alpha}}}, \quad \forall x \in \mathbb{R}^N,$$

for  $c > 0$  sufficiently small, and

$$\varphi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}, \quad |x| \geq A,$$

for some constant  $A > 0$  and some  $\omega \in L^\infty(\mathbb{R}^N)$ , homogeneous of degree 0, with  $\|\omega\|_\infty$  sufficiently small. Assume that  $V$  satisfies (7.2).

Let  $u$  be the global solution of (7.1) with initial data  $\varphi$ . Let  $u_S$  be the global mild self-similar solution of (1.1)-(1.2) with initial data  $\Phi(x) = \omega(x)|x|^{-\frac{2-\gamma}{\alpha}}$ , given by Theorem 1.4. Then we have the following.

- (i) *Nonlinear behavior: If  $V(x) = a(1 - f(x))|x|^{-\gamma}$ , where  $f$  satisfies (7.3), then there exists  $\delta > 0$  such that*

$$\|u(t) - u_S(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha} - \delta}, \quad \forall t > 0.$$

- (ii) *Linear behavior: If  $V(x) = af(x)|x|^{-\gamma}$ , where  $f$  satisfies (7.3), then there exists  $\delta > 0$  such that*

$$\left\| u(t) - e^{t\Delta} \left( \omega(\cdot) |\cdot|^{-\frac{2-\gamma}{\alpha}} \right) \right\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha} - \delta}, \quad \forall t > 0,$$

where  $C$  is a positive constant.

One should emphasize that the equation (7.1) has no self-similar structure in general, but the previous result shows that global solutions are asymptotically self-similar. In [8], Pinsky consider all positive solutions to the equation (7.1) where  $0 \leq V \in C^\delta(\mathbb{R}^N)$  and that for large  $|x|$  and constants  $c_1, c_2 > 0$ ,

$$c_1|x|^{-\gamma} \leq V(x) \leq c_2|x|^{-\gamma}, \quad 0 < \gamma < \min(N, 2).$$

He shows that if  $0 < \alpha < (2 - \gamma)/N$ , then (7.1) does not have global solutions, for any choice of initial data  $u_0(x) \geq 0$ . This shows that the condition (7.4) is optimal.

*Proof of Theorem 7.1.* Let  $r$  and  $\beta$  verify (4.1) and (4.2). Define  $\beta(p)$  by

$$\beta(p) = \frac{2 - \gamma}{2\alpha} - \frac{N}{2p}, \quad \forall p > 1.$$

Let  $q \geq 1$  and  $\delta' > 0$  be such that

$$\frac{\gamma}{N} + \frac{\alpha + 1}{r} < \frac{1}{q} < \frac{2\gamma}{N} + \frac{2(\alpha + 1)}{r} - \frac{2 - \gamma}{N\alpha}, \quad (7.5)$$

$$\delta' = \frac{N}{2} \left( \frac{1}{q} - \frac{\alpha + 1}{r} \right) - \frac{\gamma}{2}, \quad \frac{N}{2} \left( \frac{1}{q} - \frac{1}{r} \right) < 1. \quad (7.6)$$

Proof of Part (i). We will show the following:

$$\|u(t) - u_S(t)\|_p \leq Ct^{-\beta(p) - \delta'}, \quad \forall p \in [r, \infty]. \quad (7.7)$$

We have

$$\begin{aligned}
u(t) - u_{\mathcal{S}}(t) &= e^{t\Delta}(\varphi - \Phi) + \int_0^t e^{(t-\sigma)\Delta} [V(\cdot)|u(\sigma)|^\alpha u(\sigma) - a|\cdot|^{-\gamma}|u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma)] d\sigma \\
&= e^{t\Delta}(\varphi - \Phi) + a \int_0^t e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))] d\sigma \\
&\quad - a \int_0^t e^{(t-\sigma)\Delta} [f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)] d\sigma.
\end{aligned}$$

Then using Proposition 2.1 with  $(q_1, q_2) = (\frac{r}{\alpha+1}, r)$  we obtain

$$\begin{aligned}
t^{\beta+\delta'} \|u(t) - u_{\mathcal{S}}(t)\|_r &\leq t^{\beta+\delta'} \|e^{t\Delta}(\varphi - \Phi)\|_r + \\
&\quad |a| t^{\beta+\delta'} \int_0^t \left\| e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_{\mathcal{S}}(\sigma)|^\alpha u_{\mathcal{S}}(\sigma))] \right\|_r d\sigma \\
&\quad + |a| t^{\beta+\delta'} \int_0^t \left\| e^{(t-\sigma)\Delta} [f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)] \right\|_r d\sigma \\
&\leq t^{\beta+\delta'} \|e^{t\Delta}(\varphi - \Phi)\|_r + C t^{\beta+\delta'} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} (\|u(\sigma)\|_r^\alpha + \|u_{\mathcal{S}}(\sigma)\|_r^\alpha) \|u(\sigma) - u_{\mathcal{S}}(\sigma)\|_r d\sigma \\
&\quad + |a| t^{\beta+\delta'} \int_0^t \left\| e^{(t-\sigma)\Delta} [f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)] \right\|_r d\sigma.
\end{aligned}$$

We begin by estimating the first term in the left hand side of the previous inequality. Since  $q$  satisfies (7.5) it follows that  $0 < \delta' < \frac{N}{2} - \frac{2-\gamma}{2\alpha}$  and so as in the proof of Theorem 5.1 we obtain the estimate of the first term

$$t^{\beta+\delta'} \|e^{t\Delta}(\varphi - \Phi)\|_r \leq C.$$

We now estimates of the second term. Let  $T > 0$  be an arbitrary real number. Since  $\sup_{t>0} \|u(t)\|_r \leq M$  and  $\sup_{t>0} \|u_{\mathcal{S}}(t)\|_r \leq M$  and by using the expression of  $\beta$ , we have

$$\begin{aligned}
C t^{\beta+\delta'} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} (\|u(\sigma)\|_r^\alpha + \|u_{\mathcal{S}}(\sigma)\|_r^\alpha) \|u(\sigma) - u_{\mathcal{S}}(\sigma)\|_r d\sigma &\leq \\
2CM^\alpha \left( t^{\beta+\delta'} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)-\delta'} d\sigma \right) \left( \sup_{t \in (0, T]} t^{\beta+\delta'} \|u(t) - u_{\mathcal{S}}(t)\|_r \right) & \\
\leq 2CM^\alpha \left( t^{\beta+\delta'-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\beta(\alpha+1)-\delta'+1} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)-\delta'} d\sigma \right) \times & \\
\left( \sup_{t \in (0, T]} t^{\beta+\delta'} \|u(t) - u_{\mathcal{S}}(t)\|_r \right) & \\
\leq 2CM^\alpha \left( \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)-\delta'} d\sigma \right) \left( \sup_{t \in (0, T]} t^{\beta+\delta'} \|u(t) - u_{\mathcal{S}}(t)\|_r \right) & \\
\leq C' \left( \sup_{t \in (0, T]} t^{\beta+\delta'} \|u(t) - u_{\mathcal{S}}(t)\|_r \right), &
\end{aligned}$$

where  $C' = 2CM^\alpha \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}} \sigma^{-\beta(\alpha+1)-\delta'} d\sigma$ . By the hypotheses on  $r$ ,  $q$  and the expressions of  $\beta$  and  $\delta'$ ,  $C'$  is a finite positive constant.

We turn now to estimate the third term. Let  $r'$  be such that

$$\frac{1}{q} = \frac{1}{r'} + \frac{\alpha + 1}{r}. \quad (7.8)$$

By the assumptions on  $f$  we have that  $f(\cdot)|\cdot|^{-\gamma}$  is in  $L^{r'}(\mathbb{R}^N)$ . Using the Hölder inequality

$$\|f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_q \leq \|f(\cdot)|\cdot|^{-\gamma}\|_{r'} \|u(\sigma)\|_r^{\alpha+1}.$$

By the smoothing effect (2.1) with  $(s_1, s_2) = (q, r)$ ,

$$\begin{aligned} \|e^{(t-\sigma)\Delta} f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_r &\leq C(t-\sigma)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})} \|f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_q \\ &\leq C(t-\sigma)^{-(\frac{N\alpha}{2r}+\frac{\gamma}{2})-\delta'} \|u(\sigma)\|_r^{\alpha+1}. \end{aligned}$$

Then, by (4.3) and (7.6), we obtain

$$\begin{aligned} |a|t^{\beta+\delta'} \int_0^t \|e^{(t-\sigma)\Delta} f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_r d\sigma &\leq Ct^{\beta+\delta'} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\delta'} \|u(\sigma)\|_r^{\alpha+1} d\sigma \\ &\leq CM^{\alpha+1} t^{\beta+\delta'-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\delta'-\beta(\alpha+1)+1} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\delta'} \sigma^{-\beta(\alpha+1)} d\sigma \\ &\leq CM^{\alpha+1} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\gamma}{2}-\delta'} \sigma^{-\beta(\alpha+1)} d\sigma \leq C. \end{aligned}$$

Since the constants in the estimate of the three terms do not depend on  $T$ , we obtain the asymptotic behavior (7.7) for  $p = r$ .

We now turn to the asymptotic result in the  $L^\infty$ -norm. Write

$$\begin{aligned} u(t) - u_S(t) &= e^{\frac{t}{2}\Delta} (u(t/2) - u_S(t/2)) + \int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta} [V(\cdot)|u(\sigma)|^\alpha u(\sigma) - a|\cdot|^{-\gamma}(|u_S(\sigma)|^\alpha u_S(\sigma))] d\sigma \\ &= e^{\frac{t}{2}\Delta} (u(t/2) - u_S(t/2)) + a \int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta} [|\cdot|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_S(\sigma)|^\alpha u_S(\sigma))] d\sigma \\ &\quad - a \int_{\frac{t}{2}}^t e^{(t-\sigma)\Delta} f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma) d\sigma. \end{aligned}$$

By using the smoothing properties of the heat semi-group, we have that

$$\begin{aligned} t^{\frac{2-\gamma}{2\alpha}+\delta'} \|u(t) - u_S(t)\|_\infty &\leq Ct^{\beta+\delta'} \|u(t/2) - u_S(t/2)\|_r + \\ &\quad |a|t^{\frac{2-\gamma}{2\alpha}+\delta'} \int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_S(\sigma)|^\alpha u_S(\sigma)))\|_\infty d\sigma + \\ &\quad |a|t^{\frac{2-\gamma}{2\alpha}+\delta'} \int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma}(f(\cdot)|u(\sigma)|^\alpha u(\sigma)))\|_\infty d\sigma. \end{aligned}$$

Using the fact that  $\|u_S(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}}$ ,  $\|u(t)\|_\infty \leq Ct^{-\frac{2-\gamma}{2\alpha}}$  we get

$$\begin{aligned} t^{\frac{2-\gamma}{2\alpha}+\delta'} \int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta} (|\cdot|^{-\gamma}(|u(\sigma)|^\alpha u(\sigma) - |u_S(\sigma)|^\alpha u_S(\sigma)))\|_\infty d\sigma &\leq C \times \\ &\quad \left[ \int_{\frac{1}{2}}^1 (1-\sigma)^{-\frac{\gamma}{2}} \sigma^{-(\alpha+1)\frac{2-\gamma}{2\alpha}-\delta'} d\sigma \right] \sup_{t \in (0, T]} \left( t^{\frac{2-\gamma}{2\alpha}+\delta'} \|u(t) - u_S(t)\|_\infty \right). \end{aligned}$$

We turn now to estimate the last term. Let  $r'$  be given by (7.8). By the smoothing effect

$$\begin{aligned} \|e^{(t-\sigma)\Delta}(f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma))\|_\infty &\leq C(t-\sigma)^{-\frac{N}{2r'}} \|f(\cdot)|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)\|_{r'} \\ &\leq C(t-\sigma)^{-\frac{N}{2r'}} \|f(\cdot)|\cdot|^{-\gamma}\|_{r'} \|u(\sigma)\|_\infty^{\alpha+1} \\ &\leq C' M^{\alpha+1} (t-\sigma)^{-\frac{N}{2r'}} \sigma^{-\frac{2-\gamma}{2\alpha}(\alpha+1)}. \end{aligned}$$

Since  $r'$  satisfies (7.8) and  $\delta'$  satisfies (7.6) we have,

$$t^{\frac{2-\gamma}{2\alpha}+\delta'} \int_{\frac{t}{2}}^t \|e^{(t-\sigma)\Delta}(|\cdot|^{-\gamma}(f(\cdot)|u(\sigma)|^\alpha u(\sigma))\|_\infty d\sigma \leq C.$$

Finally we can conclude that,  $\sup_{t \in (0, T]} t^{\frac{2-\gamma}{2\alpha}+\delta'} \|u(t) - u_S(t)\|_\infty \leq C'(\delta')$ . Since the constant  $C'(\delta')$  does not depend on  $T > 0$ , one can take the supremum over  $(0, \infty)$ . It follows that there exists a constant  $C'(\delta')$  such that

$$\|u(t) - u_S(t)\|_\infty \leq C'(\delta') t^{-\frac{2-\gamma}{2\alpha}-\delta'},$$

for all  $t > 0$ . This prove (7.7) for  $r = \infty$ . The general result (7.7) follows now by the Hölder inequality.

Proof of Part (ii). Let  $v(t) = e^{t\Delta}(\omega(\cdot)|\cdot|^{-\frac{2-\gamma}{\alpha}})$ . We will show the following:

$$\|u(t) - v(t)\|_p \leq C t^{-\beta(p)-\delta'}, \quad \forall p \in [r, \infty].$$

We have

$$\begin{aligned} u(t) - v(t) &= e^{t\Delta}(\varphi - \Phi) + \int_0^t e^{(t-\sigma)\Delta} [V(\cdot)|u(\sigma)|^\alpha u(\sigma)] d\sigma \\ &= e^{t\Delta}(\varphi - \Phi) + \int_0^t e^{(t-\sigma)\Delta} (af|\cdot|^{-\gamma}|u(\sigma)|^\alpha u(\sigma)) d\sigma. \end{aligned}$$

Then we obtain

$$\|u(t) - v(t)\|_r \leq \|e^{t\Delta}(\varphi - \Phi)\|_r + |a| \int_0^t \|e^{(t-\sigma)\Delta} (f(\cdot)|\cdot|^{-\gamma}) |u(\sigma)|^\alpha u(\sigma)\|_r d\sigma.$$

The rest of the proof is similar to that of Part (i), so we omit the details.  $\square$

#### APPENDIX A. AUXILIARY LEMMAS

The proof of Lemma 6.1 follows by the following two lemmas.

**Lemma A.1.** *Let  $N \geq 1$  be an integer. Let  $\gamma$  be a real number such that  $0 < \gamma < \min(2, N)$ . Let the real number  $\alpha_1$  be such that*

$$\alpha_1 > \frac{2-\gamma}{N}.$$

*Given such  $\gamma$  and  $\alpha_1$  one can always choose  $r_1$  such that*

- (i)  $\frac{\gamma}{N} + \frac{\alpha_1+1}{r_1} < 1$ ,
- (ii)  $\frac{N\alpha_1}{2r_1} + \frac{\gamma}{2} < 1$ ,
- (iii)  $(\frac{2-\gamma}{2\alpha_1} - \frac{N}{2r_1})(\alpha_1 + 1) < 1$ .

*Proof of Lemma A.1.* Note that (i)-(iii) are equivalent to

- (i)  $\frac{N(\alpha_1+1)}{N-\gamma} < r_1$ ,
- (ii)  $\frac{N\alpha_1}{2-\gamma} < r_1$ ,



(iii)  $r_1 < \frac{N\alpha_1(\alpha_1+1)}{2-\gamma(\alpha_1+1)}$  if  $2 - \gamma(\alpha_1 + 1) > 0$ .

Since  $\alpha_1$  and  $\gamma$  satisfy the conditions of Lemma A.1 we can easily show that if  $2 - \gamma(\alpha_1 + 1) > 0$ , that is  $\alpha_1 < \frac{2-\gamma}{\gamma}$ , then

$$\frac{N(\alpha_1 + 1)}{N - \gamma} < \frac{N\alpha_1(\alpha_1 + 1)}{2 - \gamma(\alpha_1 + 1)},$$

$$\frac{N\alpha_1}{2 - \gamma} < \frac{N\alpha_1(\alpha_1 + 1)}{2 - \gamma(\alpha_1 + 1)}.$$

And so we can choose  $r_1$  satisfying Lemma A.1.  $\square$

Next, we set

**Lemma A.2.** *Let  $N \geq 1$  be an integer. Let  $\gamma$  be a real number such that  $0 < \gamma < \min(2, N)$ . Let the real number  $\alpha$  be such that  $\alpha > \frac{2-\gamma}{N}$ . Let the real number  $\alpha_1$  such that*

$$\alpha > \alpha_1 > \frac{2-\gamma}{N}.$$

*Choose  $r_1$  satisfying (6.1) and  $r_2$  satisfying (6.2). It follows that*

- (i)  $r_1 < r_2$ ,
- (ii)  $\frac{\gamma}{N} + \frac{\alpha_1+1}{r_1} < 1$ ,  $\frac{\gamma}{N} + \frac{\alpha+1}{r_2} < 1$ ,
- (iii)  $\frac{N\alpha_1}{2r_1} + \frac{\gamma}{2} < 1$ ,  $\frac{N\alpha}{2r_2} + \frac{\gamma}{2} < 1$ ,
- (iv)  $(\frac{2-\gamma}{2\alpha_1} - \frac{N}{2r_1})(\alpha_1 + 1) < 1$ ,  $(\frac{2-\gamma}{2\alpha} - \frac{N}{2r_2})(\alpha + 1) < 1$ ,
- (v)  $\frac{2-\gamma}{N\alpha_1} < \frac{\gamma}{N} + \frac{1}{r_1} + \frac{\alpha}{r_2} < 1$ ,
- (vi)  $\frac{2-\gamma}{N\alpha} < \frac{\gamma}{N} + \frac{\alpha_1}{r_1} + \frac{1}{r_2} < 1$ .

*Proof.* The proof of Lemma A.2 is obvious and can be omitted.  $\square$

We now give the proof of Lemma 6.2.

*Proof of Lemma 6.2.* One verifies, using the expression for  $r'$ ,  $\beta'$  and  $\beta_1$ , that condition (b) is equivalent to (6.21). Since  $\delta > 0$ , one must have  $\theta_\delta > \frac{1}{\alpha+1}$ . Write now

$$\theta_\delta = \frac{1}{\alpha + 1} + \varepsilon.$$

In the limiting case  $\varepsilon = 0$ , we see that  $\theta_\delta = \frac{1}{\alpha+1}$ ,  $r' = r_{12}$ ,  $\beta' = \beta_{12}$  and  $\delta = 0$ ; and so conditions (a), (b) and (c) are consequence of Lemma 6.1. Since conditions (a) and (c) are open, it is clear that they still hold for small  $\varepsilon > 0$ , assuming (6.21).

In order to get a specific bound on allowable  $\varepsilon > 0$ , note that (a)-(c) are equivalent to

- (a')  $\frac{1}{r_1} < \frac{\gamma}{N} + \frac{\alpha+1}{r_{12}} + \varepsilon(\alpha + 1)(\frac{1}{r_1} - \frac{1}{r_2}) < 1$ ,
- (b')  $\delta = \varepsilon \frac{(2-\gamma)(\alpha+1)(\alpha-\alpha_1)}{2\alpha_1\alpha}$ ,
- (c')  $\frac{N\alpha}{2r_2} + \varepsilon \frac{N(\alpha+1)}{2}(\frac{1}{r_1} - \frac{1}{r_2}) + \frac{\gamma}{2} < 1$ ,  $\beta_{12}(\alpha + 1) + \varepsilon(\alpha + 1)(\beta_1 - \beta_2) < 1$ ,

where  $r_{12}$  and  $\beta_{12}$  are as in Lemma 6.1. We clearly have  $\theta_\delta < 1$ , for small  $\varepsilon$ , and thanks to (ii)-(iv) of Lemma 6.1, (a')-(c') are satisfied for

$$0 < \varepsilon < \varepsilon_{\min} = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}, \tag{A.1}$$

where

$$\begin{aligned}\varepsilon_1 &= \frac{\alpha}{\alpha+1}; \\ \varepsilon_2 &= \frac{1}{\alpha+1} \left(1 - \frac{\gamma}{N} - \frac{\alpha+1}{r_{12}}\right) \left(\frac{1}{r_1} - \frac{1}{r_2}\right)^{-1}; \\ \varepsilon_3 &= \frac{2}{N(\alpha+1)} \left(\frac{2-\gamma}{2} - \frac{N\alpha}{2r_2}\right) \left(\frac{1}{r_1} - \frac{1}{r_2}\right)^{-1}; \\ \varepsilon_4 &= \frac{1}{\alpha+1} (1 - \beta_{12}(\alpha+1)) |\beta_1 - \beta_2|^{-1}.\end{aligned}$$

That is (6.19), (a)-(c) are satisfied for  $0 < \delta < \delta_0$ , where

$$\delta_0 = \varepsilon_{\min} \frac{(2-\gamma)(\alpha+1)(\alpha-\alpha_1)}{2\alpha_1\alpha}, \quad (\text{A.2})$$

and  $\varepsilon_{\min}$  is given by (A.1).  $\square$

**Remark A.3.** One can verify easily that  $\frac{1}{r_{12}} = \frac{1}{\alpha+1} \frac{1}{r_1} + \frac{\alpha}{\alpha+1} \frac{1}{r_2}$ ,  $\beta_{12} = \frac{1}{\alpha+1} \beta_1 + \frac{\alpha}{\alpha+1} \beta_2 = \frac{2-\gamma}{2\alpha_1} \frac{\alpha_1+1}{\alpha+1} - \frac{N}{2r_{12}}$ , and  $\beta_2 = \frac{\alpha_1}{\alpha} \beta_1$ .

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