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NEW LIFE-SPAN RESULTS FOR THE NONLINEAR HEAT EQUATION

SLIM TAYACHI AND FRED B. WEISSLER

ABSTRACT. We obtain new estimates for the existence time of the maximal solutions to the nonlinear heat equation $\partial_t u - \Delta u = |u|^\alpha u$, $\alpha > 0$ with initial values in Lebesgue, weighted Lebesgue spaces or measures. Non-regular, sign-changing, as well as non polynomial decaying initial data are considered. The proofs of the lower-bound estimates of life-span are based on the local construction of solutions. The proofs of the upper-bounds exploit a well-known necessary condition for the existence of nonnegative solutions. In addition, we establish new results for life-span using dilation methods and we give new life-span estimates for Hardy-Hénon parabolic equations.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this paper, we consider the nonlinear heat equation

$$\partial_t u = \Delta u + |u|^\alpha u, \tag{1.1}$$

where $u = u(t, x) \in \mathbb{R}$, $t > 0$, $x \in \Omega$, a domain of \mathbb{R}^N not necessarily bounded, $N \geq 1$ and $\alpha > 0$. In the case where the boundary $\partial\Omega \neq \emptyset$, we suppose $\partial\Omega$ sufficiently smooth and we impose Dirichlet conditions on the boundary:

$$u(t, x) = 0, \quad t > 0, \quad x \in \partial\Omega.$$

If Ω is not bounded, we impose Dirichlet conditions at infinity:

$$\lim_{|x| \rightarrow \infty, x \in \Omega} u(t, x) = 0, \quad t > 0,$$

or perhaps other convenient formulation (see for example [42, Definition 15.1, p. 75]). We usually consider the equation (1.1) with the initial value

$$u(0, \cdot) = u_0. \tag{1.2}$$

The Cauchy problem (1.1)-(1.2) is locally well-posed in various Banach spaces. In other words, each element or initial value u_0 in that space gives rise to a trajectory $u(t) = u(t, \cdot)$ which is a solution in some appropriate sense to the given equation, here equation (1.1), and such that $u(0) = u_0$. In many cases, this trajectory cannot exist for all time t , and we denote by $T_{\max}(u_0)$ the maximal possible existence time of such a trajectory. The term life-span refers to the study of the maximal existence time of solutions with initial data of the form $u_0 = \lambda\varphi$ for some fixed element φ in the considered Banach space and all $\lambda > 0$. Our aim is to establish lower and upper

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bounds of the life-span for a large class of initial data φ in terms of λ and study the asymptotic behavior of $T_{\max}(\lambda\varphi)$, either as $\lambda \rightarrow \infty$ or as $\lambda \rightarrow 0$.

It is well known that if $u_0 \in C_b(\mathbb{R}^N)$, the Banach space of continuous bounded functions on \mathbb{R}^N , there exists $T_{\max}(u_0) > 0$ such that (1.1)-(1.2) has a unique classical solution $u \in C^{1,2}((0, T_{\max}(u_0)) \times \mathbb{R}^N) \cap C([0, T_{\max}(u_0)] \times \mathbb{R}^N)$ which is bounded in $[0, T] \times \mathbb{R}^N$ for all $T < T_{\max}(u_0)$, and $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \rightarrow \infty$ as $t \rightarrow T_{\max}(u_0)$, if $T_{\max}(u_0) < \infty$. It is proved in [15] that if $\alpha < 2/N$ and $\varphi \in C_b(\mathbb{R}^N)$ with $\varphi \geq 0$, $\varphi \not\equiv 0$, then $T_{\max}(\lambda\varphi) < \infty$ for any $\lambda > 0$. For all $\alpha > 0$, if $\varphi \in C_b(\mathbb{R}^N)$, $\varphi \geq 0$ and $\liminf_{|x| \rightarrow \infty} |x|^\gamma \varphi(x) > 0$ with $\gamma < 2/\alpha$, then $T_{\max}(\lambda\varphi) < \infty$, for all $\lambda > 0$, as shown in [27]. This last result has been improved in many papers, see [50] for instance and some references therein. If we do not impose the positivity of the initial data, it has been proven in [29] that for a given φ sufficiently regular (i.e. with finite energy), $\varphi \not\equiv 0$, and $\lambda > 0$ is sufficiently large then $T_{\max}(\lambda\varphi) < \infty$. If $\alpha < 2/N$ and φ not necessarily positive but $\varphi \in L^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \varphi \neq 0$ then it is proved, in [11], that $T_{\max}(\lambda\varphi) < \infty$ for $\lambda > 0$ sufficiently small. Other blow-up results for λ small are proved in [16, 49, 50]. The above mentioned results show in particular the interest of studying the behavior of $T_{\max}(\lambda\varphi)$ for any value of λ and with or without any sign restriction on the initial data.

For example, it is proved in [27] that given any nontrivial nonnegative initial data $\varphi \in C_b(\mathbb{R}^N)$ then $T_{\max}(\lambda\varphi) \sim \lambda^{-\alpha}$, as $\lambda \rightarrow \infty$ and $T_{\max}(\lambda\varphi) \sim \lambda^{-\alpha}$, as $\lambda \rightarrow 0$ provided that $\varphi_\infty = \liminf_{|x| \rightarrow \infty} \varphi(x) > 0$. Shortly thereafter the exact limits were given in [18], that is $\lim_{\lambda \rightarrow \infty} \lambda^\alpha T_{\max}(\lambda\varphi) = \frac{1}{\alpha} \|\varphi\|_\infty^{-\alpha}$ and $\lim_{\lambda \rightarrow 0} \lambda^\alpha T_{\max}(\lambda\varphi) = \frac{1}{\alpha} \varphi_\infty^{-\alpha}$.

An other example is the study of the asymptotic behavior of the life-span $T_{\max}(\lambda\varphi)$ when $\varphi \in C_b(\mathbb{R}^N)$ is nonnegative nontrivial and having also a polynomial decay at infinity, that is

$$0 < \liminf_{|x| \rightarrow \infty} |x|^\gamma \varphi(x) \leq \limsup_{|x| \rightarrow \infty} |x|^\gamma \varphi(x) < \infty, \quad (1.3)$$

$0 < \gamma < N$. This is studied in [27] for small $\lambda > 0$. It is shown in [27, Theorem 3.15 (ii), p. 375] and [27, Theorem 3.21 (ii), p. 376] that if $\alpha < 2/N$ then

$$0 < \liminf_{\lambda \rightarrow 0} \lambda^{[(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}]} T_{\max}(\lambda\varphi) \leq \limsup_{\lambda \rightarrow 0} \lambda^{[(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}]} T_{\max}(\lambda\varphi) < \infty.$$

These results have been generalized recently in [23] replacing $\varphi \in C_b(\mathbb{R}^N)$ by $\varphi \in L^\infty(\mathbb{R}^N)$. See [23, Theorem 5.1, p. 128] and [23, Theorem 5.2, p. 130]. We notice that refined asymptotic is given in [33] for large λ and for φ not necessarily positive but still continuous and bounded. Other estimates are obtained for the life-span for regular and slowly decaying initial data in [32, 11, 50].

A class of initial data φ where φ is not necessarily in $C_b(\mathbb{R}^N)$ or in $L^\infty(\mathbb{R}^N)$ and satisfying either (1.3) or

$$0 < \liminf_{|x| \rightarrow 0} |x|^\gamma \varphi(x) \leq \limsup_{|x| \rightarrow 0} |x|^\gamma \varphi(x) < \infty, \quad (1.4)$$

has been considered in [50]. In fact, the asymptotic behavior of the life-span for initial data $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$, $|\varphi(x)| \leq c|x|^{-\gamma}$, where $0 < \gamma < N$ and $c > 0$ a constant, is studied in [50]. It is shown there that for some initial data φ singular at the origin or satisfying $\liminf_{|x| \rightarrow \infty} \varphi(x) := \varphi_\infty = 0$ the situation is quite different from previously known life-span results. In particular,

unlike [18], the limits of $\lambda^{(\frac{1}{\alpha}-\frac{\gamma}{2})^{-1}}T_{\max}(\lambda\varphi)$ as $\lambda \rightarrow 0$ or as $\lambda \rightarrow \infty$ may not exist. Also, if φ satisfies (1.4), so that φ is singular at the origin, then $T_{\max}(\lambda\varphi) \sim \lambda^{-(\frac{1}{\alpha}-\frac{\gamma}{2})^{-1}}$, as $\lambda \rightarrow \infty$ instead of $\lambda^{-\alpha}$ if φ is regular. See [50, Corollary 1.13 and Proposition 4.6]. It is also proved that if $\varphi(x) = |x|^{-\gamma}$, $|x| \leq 1$, $\varphi(x) = 0$, $|x| > 1$, $0 < \gamma < N$, $0 < \alpha < 2/\gamma$, $(N-2)\alpha < 4$ then $\lim_{\lambda \rightarrow \infty} \lambda^{(\frac{1}{\alpha}-\frac{\gamma}{2})^{-1}}T_{\max}(\lambda\varphi) = C > 0$. This last behavior shows the impact of the singularity of the initial data on the behavior of the life-span for large λ .

The goal of this paper is to improve and extend the above mentioned results by considering a large class of initial data, including singular, sign changing, not necessarily polynomially decaying initial data. To carry out this goal, we use three different methods. The first is based on the contraction mapping argument used to prove local existence. We recently introduced and used this method in [50]. Here, we apply it to the nonlinear heat equation and nonlinear Hardy-Hénon parabolic equations. In a forthcoming paper, it will be applied to a variety of evolution equations in order to exhibit the generality of this method ([48]). We know of some cases where the idea behind this method was previously used in other papers (see for example [24]) but to our knowledge, this method has never been presented as such or exploited in a systematic way. The second method is based on a necessary condition for local existence of non-negative solutions. The third method is based on scaling properties of the equation. Details are given below later in the introduction.

We begin with the first method and we consider the case where u_0 belongs to a Lebesgue space, where we can use the contraction mapping argument done in [53, 54]. It is well known that the problem (1.1)-(1.2) is locally well-posed in $L^q(\Omega)$ whenever $q \geq 1$, $q > q_c$ where

$$q_c = \frac{N\alpha}{2}. \quad (1.5)$$

See [53, 54, 42] and references therein. For any $u_0 \in L^q(\Omega)$, we denote by $T_{\max}(u_0)$ the existence time of the maximal (regular) solution to (1.1)-(1.2) in $L^q(\Omega)$. Our first result on lower bound of the life-span is derived from [53, 54] using an argument from [50].

Theorem 1.1 (Initial data in Lebesgue spaces). *Let $N \geq 1$, $\alpha > 0$ and q_c be given by (1.5). Let $\varphi \in L^q(\Omega)$ with $1 \leq q \leq \infty$, $q > q_c$ or $\varphi \in C_0(\Omega)$. Let $u \in C([0, T_{\max}(\lambda\varphi)); L^q(\Omega))$ be the maximal classical solution of (1.1)-(1.2) with initial data $u_0 = \lambda\varphi$, $\lambda > 0$ (we replace $[0, T_{\max}(\lambda\varphi))$ by $(0, T_{\max}(\lambda\varphi))$ if $q = \infty$). Then there exists a constant $C = C(\alpha, q) > 0$ such that*

$$T_{\max}(\lambda\varphi) \geq \frac{C}{(\lambda\|\varphi\|_q)^{(\frac{1}{\alpha}-\frac{N}{2q})^{-1}}}, \quad (1.6)$$

for all $\lambda > 0$.

Hereafter, $\|\cdot\|_q$ denotes the norm in the Lebesgue space $L^q(\Omega)$.

Remark 1.

- 1) If $T_{\max}(\varphi, L^q)$ denotes the existence time of the maximal (regular) solution to (1.1)-(1.2) for $\varphi \in L^q(\Omega)$, it is known (see for example [53] and Proposition 2.2 below) that if $\varphi \in L^p(\Omega) \cap L^q(\Omega)$ we have $T_{\max}(\varphi, L^q) = T_{\max}(\varphi, L^p)$. It follows that if $\varphi \in L^{\underline{q}}(\Omega) \cap L^{\bar{q}}(\Omega)$, $1 \leq$

$\underline{q} \leq \bar{q} \leq \infty$, $q_c < \underline{q}$, then the estimate (1.6) reads

$$T_{\max}(\lambda\varphi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2q}\right)^{-1}}, \quad \text{if } 0 < \lambda < 1,$$

$$T_{\max}(\lambda\varphi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2q}\right)^{-1}}, \quad \text{if } \lambda > 1.$$

- 2) Theorem 1.1 includes many known results on lower bound of the life-span. For example, it is shown in [27, Theorem 3.21 (ii), p. 376] that if $\varphi \in C_b(\mathbb{R}^N)$, $\varphi \geq 0$ and $\limsup_{|x| \rightarrow \infty} |x|^\gamma \varphi(x) < \infty$, then for $N < \gamma < 2/\alpha$, (hence $q_c < 1$) we have $T_{\max}(\lambda\varphi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2}\right)^{-1}}$, as $\lambda \rightarrow 0$. This result is a special case of Theorem 1.1. Indeed, it follows that $\varphi \in C_0(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ with $q \geq 1 > N/\gamma$. Thus, the estimate of [27] follows by taking $q = 1$ in (1.6). Other examples will be given throughout the paper.
- 3) Theorem 1.1 is valid for the equation $\partial_t u = \Delta u + a(x)|u|^\alpha u$, with $a \in L^\infty(\Omega)$. Under the additional assumption $a > 0$, it is shown in [40, Theorem 3 (i), p. 35] that, for $\varphi \in C_b(\mathbb{R}^N)$, $T_{\max}(\lambda\varphi) \geq C\lambda^{-\alpha}$, as $\lambda \rightarrow \infty$. For such initial data we may take $q = \infty$ and (1.6) recover the last estimate.
- 4) For positive initial data and bounded domain, estimate (1.6) is established in [46, Theorem 3.1, p. 2526] where it is also assumed that $N \geq 3$, $q > \max(1, q_c)$. See also [39] for other estimates with $N = 3$.
- 5) Let us consider the nonlinear heat equation with diffusivity:

$$\partial_t u = \mu \Delta u + |u|^\alpha u, \quad u(0) = \varphi \in L^q(\Omega), \quad q \geq 1, \quad q > q_c, \quad (1.7)$$

on $(0, T_{\max}^\mu(\varphi)) \times \Omega$, where $\alpha > 0$, $\mu > 0$ and $T_{\max}^\mu(\varphi)$ denotes the existence time of the maximal solution of (1.7) with initial data φ . We want to find a lower estimate of $T_{\max}^\mu(\varphi)$ with respect to μ . Let

$$v(t, x) = \mu^{-1/\alpha} u(t/\mu, x).$$

Then v satisfies the equation

$$\partial_t v = \Delta v + |v|^\alpha v, \quad v(0) = \mu^{-1/\alpha} \varphi,$$

on $(0, \mu T_{\max}^\mu(\varphi)) \times \Omega = (0, T_{\max}(\mu^{-1/\alpha} \varphi)) \times \Omega$, where $T_{\max}(\mu^{-1/\alpha} \varphi)$ is the existence time of the maximal solution v . Using (1.6) we get,

$$T_{\max}^\mu(\varphi) = \mu^{-1} T_{\max}(\mu^{-1/\alpha} \varphi) \geq C\mu^{-1} \left(\mu^{-1/\alpha} \|\varphi\|_q \right)^{-\left(\frac{1}{\alpha} - \frac{N}{2q}\right)^{-1}}.$$

That is,

$$T_{\max}^\mu(\varphi) \geq C\mu^{\left(\frac{2q}{N\alpha} - 1\right)^{-1}} \|\varphi\|_q^{-\left(\frac{1}{\alpha} - \frac{N}{2q}\right)^{-1}}.$$

For $q = \infty$, the right-hand side term does not depend on μ and we have

$$T_{\max}^\mu(\varphi) \geq C\|\varphi\|_\infty^{-\alpha}.$$

See [34, 13, 14] for related estimates. Note that if $q < \infty$ then $\lim_{\mu \nearrow \infty} T_{\max}^\mu(\varphi) = \infty$.

Using the same method based on the contraction mapping argument as in [50], we also derive from [54] the following lower estimate for the life-span for the case of finite Borel measure. We denote by \mathcal{M} the set of finite Borel measures on Ω .

Theorem 1.2 (Initial data measure). *Let $N \geq 1$, $\alpha > 0$ and q_c be given by (1.5). If \mathbf{m} is a finite Borel measure on Ω , i.e. $\mathbf{m} \in \mathcal{M}$, and if $q_c < 1$, i.e. $\alpha < \frac{2}{N}$, then the existence time of the maximal solution for (1.1)-(1.2) with initial data $u_0 = \lambda \mathbf{m}$ satisfies*

$$T_{\max}(\lambda \mathbf{m}) \geq \frac{C}{(\lambda \|\mathbf{m}\|_{\mathcal{M}})^{\left(\frac{1}{\alpha} - \frac{N}{2}\right)^{-1}}} \quad (1.8)$$

for all $\lambda > 0$, where $C = C(\alpha) > 0$ is a constant.

We now estimate from below the life-span of solutions for the nonlinear heat equation (1.1) in \mathbb{R}^N with decaying initial data $u_0 = \lambda \varphi$, which may be singular, without sign restriction and for any $\lambda > 0$. For $\gamma > 0$, $1 \leq q \leq \infty$, we consider the weighted Lebesgue space

$$L_{\gamma}^q(\mathbb{R}^N) = \{f : \mathbb{R}^N \rightarrow \mathbb{R}, \text{ measurable, } |\cdot|^{\gamma} f \in L^q(\mathbb{R}^N)\},$$

endowed with the norm

$$\|f\|_{L_{\gamma}^q(\mathbb{R}^N)} := \| |\cdot|^{\gamma} f \|_q.$$

In Theorem 4.1 below, we give a well-posedness result in weighted Lebesgue spaces for the nonlinear heat equation. As a consequence, we obtain the following lower bound estimates of the life-span.

Theorem 1.3 (Initial data in weighted Lebesgue spaces). *Let $N \geq 1$ and $\alpha > 0$. If $\varphi \in L_{\gamma}^q(\mathbb{R}^N)$, where $0 < \gamma < N$, $\gamma < 2/\alpha$, $q \in (1, \infty]$ and*

$$\frac{1}{q} + \frac{\gamma}{N} < 1, \quad \frac{N\alpha}{2q} + \frac{\alpha\gamma}{2} < 1, \quad (1.9)$$

then the existence time of the maximal solution of (1.1)-(1.2) in $L_{\gamma}^q(\mathbb{R}^N)$ with $u_0 = \lambda \varphi$ satisfies

$$T_{\max}(\lambda \varphi) \geq \frac{C}{\left(\lambda \|\varphi\|_{L_{\gamma}^q}\right)^{\left(\frac{1}{\alpha} - \frac{N}{2q} - \frac{\gamma}{2}\right)^{-1}}}, \quad (1.10)$$

for all $\lambda > 0$, where $C = C(\alpha, q, \gamma, N) > 0$ is a constant.

Remark 2. Under the conditions $0 < \gamma < N$, $\gamma < 2/\alpha$, $1 < q \leq \infty$, (1.9) is equivalent to

$$q > \frac{N}{N - \gamma} \quad \text{and} \quad q > \frac{N\alpha}{2 - \gamma\alpha}.$$

Combining the results of Theorems 1.3 and 1.1, we get the following estimates of the existence time of the maximal solution to (1.1)-(1.2).

Corollary 1.4. *Let $N \geq 1$, $\alpha > 0$, $\gamma > 0$, $\gamma \neq N$ and q_c be given by (1.5). Assume that*

$$\min \left[\frac{N\alpha}{2}, \frac{\gamma\alpha}{2} \right] < 1.$$

Let $\varphi \in L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)$, where $q \in (1, \infty]$, $q > q_c$. If $\gamma < N$ we assume further (1.9). Then the existence time of the maximal solution of (1.1)-(1.2) in $L^q(\mathbb{R}^N)$ with $u_0 = \lambda\varphi$ satisfies

$$T_{\max}(\lambda\varphi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{1}{2} \min\left(\frac{N}{q} + \gamma, N\right)\right)^{-1}}, \quad (1.11)$$

for all $0 < \lambda \leq 1$, where C is a positive constant, $C = C(\alpha, q, \gamma, N, \|\varphi\|_{L_\gamma^q \cap L^q})$ if $\gamma < N$ and $C = C(\alpha, q, \gamma, N, \|\varphi\|_1)$ if $\gamma > N$.

Remark 3.

- 1) For the particular case $q = \infty$, Corollary 1.4 includes that of [27, Theorem 3.21, p. 376] and [23, Theorem 5.1, p. 128], where φ is continuous, $\varphi \geq 0$, $\varphi \in L^\infty(\mathbb{R}^N) \cap L_\gamma^\infty(\mathbb{R}^N)$, $\gamma > 0$. The novelty of our estimate is that it holds without any condition on the sign of the initial data. Unlike [27, 23], the case $\gamma = N$ is not considered here.
- 2) Corollary 1.4 is totally new if $q < \infty$.
- 3) Obviously, if $\lambda > 1$, (1.6) is better than (1.11), which itself holds for all $\lambda > 0$, as shown in the proof.

The solution of (1.1)-(1.2) constructed with initial data in $L_\gamma^q(\mathbb{R}^N)$ is in $C_0(\mathbb{R}^N)$ for $t > 0$, by Proposition 4.2 below. This is well-known to hold also for the solution with initial data in $L^p(\mathbb{R}^N)$, $p < \infty$. So the constructed solution for initial data in $L^p(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)$ can be extended to a maximal solution of (1.1)-(1.2), $u : (0, T_{\max}) \rightarrow C_0(\mathbb{R}^N)$. This maximal existence time is equal to that in $L^p(\mathbb{R}^N)$ or in $L_\gamma^q(\mathbb{R}^N)$, as shown in Proposition 4.2 below. In the following result, which extends Corollary 1.4 for $0 < \gamma < N$, we give a lower bound estimate of the life-span for initial data in $L^p(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)$.

Corollary 1.5. *Let $N \geq 1$, $\alpha > 0$ and q_c be given by (1.5). If $\varphi \in L^p(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)$, where $p > q_c$, $1 \leq p \leq \infty$, $0 < \gamma < N$, $\gamma < 2/\alpha$, $q \in (1, \infty]$ and satisfies (1.9), then the existence time of the maximal solution of (1.1)-(1.2) with $u_0 = \lambda\varphi$ satisfies*

$$T_{\max}(\lambda\varphi) \geq C \begin{cases} \lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2} \max\left(\frac{1}{q} + \frac{\gamma}{N}, \frac{1}{p}\right)\right)^{-1}}, & \text{if } 0 < \lambda \leq 1, \\ \lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2} \min\left(\frac{1}{q} + \frac{\gamma}{N}, \frac{1}{p}\right)\right)^{-1}}, & \text{if } \lambda > 1, \end{cases} \quad (1.12)$$

where $C = C(\alpha, p, q, \gamma, N, \|\varphi\|_{L^p \cap L_\gamma^q}) > 0$ is a constant.

We now turn to results based on the second method, which gives upper-bounds on the life-span and which requires positivity. We distinguish the cases when $\lambda > 0$ is large or $\lambda > 0$ is small and begin with the case λ is large. By [29], $T_{\max}(\lambda\varphi) < \infty$ in this case, if φ is sufficiently regular. By [56, Theorem 1], it follows that if $\varphi \geq 0$ is either a locally integrable function or a positive Borel measure on Ω , $\varphi \not\equiv 0$, then $T_{\max}(\lambda\varphi) < \infty$ for all sufficiently large $\lambda > 0$. See section 5. See also [36, Theorem 2, p. 882] for $\varphi \in C_0(\Omega)$. Our first life-span upper bound is as follows.

Theorem 1.6. *Let $N \geq 1$ and $\alpha > 0$. Let $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \not\equiv 0$. It follows that the existence time for the maximal solution of (1.1)-(1.2) with $u_0 = \lambda\varphi$ satisfies $T_{\max}(\lambda\varphi) < \infty$ for*

$\lambda > 0$ sufficiently large and

$$\limsup_{\lambda \rightarrow \infty} \lambda^\alpha T_{\max}(\lambda\varphi) \leq \frac{1}{\alpha \|\varphi\|_\infty^\alpha}. \quad (1.13)$$

Remark 4.

1) Theorem 1.1 and Theorem 1.6 together show that

$$T_{\max}(\lambda\varphi) \sim (\lambda \|\varphi\|_\infty)^{-\alpha}, \quad \lambda \rightarrow \infty \quad (1.14)$$

whenever $\varphi \geq 0$, $\varphi \not\equiv 0$, $\varphi \in L^\infty(\Omega)$. This extends the result of [27, Theorem 3.2 (ii), p. 372] to L^∞ initial data. The lower estimate is valid even if φ is not necessarily positive.

2) With the notation of Part 5) of Remark 1 and using Theorem 1.6, we have that for $\varphi \geq 0$, $\varphi \not\equiv 0$, $\varphi \in L^\infty(\Omega)$, the maximal existence time of (1.7) satisfies

$$\limsup_{\mu \searrow 0} T_{\max}^\mu(\varphi) \leq \frac{1}{\alpha} \|\varphi\|_\infty^{-\alpha}$$

and hence combined with Part 5) of Remark 1 we have $T_{\max}^\mu(\varphi) \sim \frac{1}{\alpha} \|\varphi\|_\infty^{-\alpha}$, as $\mu \rightarrow 0$. It is shown in [34, Theorem 1, p. 351] that $\lim_{\mu \searrow 0} T_{\max}^\mu(\varphi) = \frac{1}{\alpha} \|\varphi\|_\infty^{-\alpha}$, without sign restriction on φ but, unlike our case, only for Ω a bounded domain and assuming also φ a continuous function on $\bar{\Omega}$.

3) Theorem 1.6 is known for bounded domain and regular initial data, see [45, 43]. See also [42, Remark 17.2(i), p. 92] for other estimates in bounded domain.

We now consider positive initial data which are singular near the origin, where we restrict ourselves to the case $\Omega = \mathbb{R}^N$. We have obtained the following.

Theorem 1.7. *Let $N \geq 1$ and $\alpha > 0$. Let $0 < \gamma < N$, $\gamma < \frac{2}{\alpha}$ and let $\omega \in L^\infty(\mathbb{R}^N)$ be homogeneous of degree 0, $\omega \geq 0$ and $\omega \not\equiv 0$. Suppose that $\varphi \in L_{loc}^1(\mathbb{R}^N)$, $\varphi \geq 0$ is such that $\varphi(x) \geq \omega(x)|x|^{-\gamma}$ for $|x| \leq \epsilon$, and some $\epsilon > 0$. It follows that $T_{\max}(\lambda\varphi) < \infty$ for $\lambda > 0$ sufficiently large and*

$$\limsup_{\lambda \rightarrow \infty} \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda\varphi) \leq \frac{1}{(\alpha^{1/\alpha} \|e^\Delta(\omega \cdot |\cdot|^{-\gamma})\|_\infty)^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}}}. \quad (1.15)$$

Remark 5.

1) If φ is as in Theorem 1.7 such that $\varphi \in L^\infty(\mathbb{R}^N)$, $0 < \gamma < N$, $\gamma < \frac{2}{\alpha}$ then Theorem 1.3 and Theorem 1.7 together show that $T_{\max}(\lambda\varphi) \sim \lambda^{-(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}}$, as $\lambda \rightarrow \infty$. This extends the result of [50, Proposition 4.5] by removing the condition $(N - 2)\alpha < 4$, as well as the condition $\liminf_{|x| \rightarrow 0} |x|^\gamma \varphi(x) > 0$.

2) If $N < \gamma < \frac{2}{\alpha}$, then there is no local nonnegative solution to (1.1) with initial value $\lambda\tilde{\varphi}$ for all $\lambda > 0$, where

$$\tilde{\varphi}(x) = \begin{cases} \omega(x)|x|^{-\gamma}, & |x| \leq \epsilon \\ 0, & |x| > \epsilon. \end{cases} \quad (1.16)$$

See [56, 6].

We now turn to upper estimates on $T_{\max}(\lambda\varphi)$ as $\lambda \rightarrow 0$. For this we need to assume that Ω is not bounded, and for simplicity we consider $\Omega = \mathbb{R}^N$. Our first result of this type is for measures. Consider $u_0 = \lambda\mathbf{m}$, where $\lambda > 0$ and $\mathbf{m} \in \mathcal{M}$, the set of finite Borel measures on \mathbb{R}^N . We suppose that \mathbf{m} is a positive measure. To insure that (1.1) is locally well-posed on \mathcal{M} we assume $\alpha < \frac{2}{N}$, and this implies (by Fujita's result) that $T_{\max}(\lambda\mathbf{m}) < \infty$ for all $\lambda > 0$.

Theorem 1.8. *Let $N \geq 1$ and $\alpha > 0$. Suppose $\alpha < \frac{2}{N}$ and let $\mathbf{m} \in \mathcal{M}$ be a positive finite Borel measure on \mathbb{R}^N . It follows that $T_{\max}(\lambda\mathbf{m}) < \infty$ for all $\lambda > 0$ and*

$$\limsup_{\lambda \rightarrow 0} \lambda^{(\frac{1}{\alpha} - \frac{N}{2})^{-1}} T_{\max}(\lambda\mathbf{m}) \leq \frac{1}{(\alpha^{1/\alpha} (4\pi)^{-N/2} \|\mathbf{m}\|_{\mathcal{M}})^{(\frac{1}{\alpha} - \frac{N}{2})^{-1}}}. \quad (1.17)$$

Remark 6. Theorem 1.8 includes the case $u_0 = \lambda\varphi$ where $\varphi \in L^1(\mathbb{R}^N)$, $\varphi \geq 0$, $\varphi \not\equiv 0$ and $\alpha < \frac{2}{N}$. Indeed, consider the measure $d\mathbf{m} = \varphi dx$ where dx denotes Lebesgue measure. It follows then that Theorem 1.1 and Theorem 1.8 together show that

$$T_{\max}(\lambda\varphi) \sim (\lambda \|\varphi\|_1)^{-(\frac{1}{\alpha} - \frac{N}{2})^{-1}}, \quad \lambda \rightarrow 0, \quad (1.18)$$

whenever $\varphi \geq 0$, $\varphi \not\equiv 0$, $\varphi \in L^1(\mathbb{R}^N)$. The lower estimate is valid even if φ is not necessarily positive.

We have obtained the following for positive initial data having some decay at infinity.

Theorem 1.9. *Let $N \geq 1$ and $\alpha > 0$. Let $\varphi \in L^1_{loc}(\mathbb{R}^N)$, $\varphi \geq 0$ and suppose that $\varphi(x) \geq \omega(x)|x|^{-\gamma}$ for $|x| \geq R$, for some $R > 0$, where $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0, $\omega \geq 0$ and $\omega \not\equiv 0$. If $0 < \gamma < N$ and $\gamma < \frac{2}{\alpha}$, then $T_{\max}(\lambda\varphi) < \infty$ for all $\lambda > 0$ and*

$$\limsup_{\lambda \rightarrow 0} \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda\varphi) \leq \frac{1}{(\alpha^{1/\alpha} \|e^{\Delta}(\omega \cdot |\cdot|^{-\gamma})\|_\infty)^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}}}. \quad (1.19)$$

Remark 7.

- 1) If φ is as in Theorem 1.9 such that $\varphi \in L^\infty(\mathbb{R}^N)$, $0 < \gamma < N$, $\gamma < \frac{2}{\alpha}$ then Theorem 1.3 and Theorem 1.9 together show that $T_{\max}(\lambda\varphi) \sim \lambda^{-(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}}$, as $\lambda \rightarrow 0$. This extends the result of [50, Proposition 4.5] by removing the condition $(N - 2)\alpha < 4$, as well as the condition $\liminf_{|x| \rightarrow \infty} |x|^\gamma \varphi(x) > 0$.
- 2) If $N < \gamma < \frac{2}{\alpha}$, then $\tilde{\varphi} \in L^1(\mathbb{R}^N)$, where

$$\tilde{\varphi}(x) = \begin{cases} 0, & |x| < R \\ \omega(x)|x|^{-\gamma}, & |x| \geq R, \end{cases} \quad (1.20)$$

for some $R > 0$, and so Theorem 1.8 gives an upper life-span bound as $\lambda \rightarrow 0$. So for φ as in Theorem 1.9 with $\gamma > 0$, $\gamma \neq N$, and by comparison argument, Remark 6 and the above one together show that $T_{\max}(\lambda\varphi) \sim \lambda^{-(\frac{1}{\alpha} - \frac{1}{2} \min(\gamma, N))^{-1}}$, as $\lambda \rightarrow 0$.

- 3) In the particular case where φ is continuous and bounded such that $\liminf_{|x| \rightarrow \infty} |x|^\gamma \varphi(x) > 0$, a similar result is obtained in [27, Theorem 3.15 (ii)]. If $\varphi \in L^\infty(\mathbb{R}^N)$ and is nonnegative satisfying $\varphi(x) \geq (1 + |x|)^{-\gamma}$ for almost all $x \in \mathbb{R}^N$ a similar result is also obtained in [23, Theorem 5.2 (ii)]. Here φ is only $L^1_{loc}(\mathbb{R}^N)$, and so the condition on lower bound on

φ is imposed only near infinity and we do not require $\liminf_{|x| \rightarrow \infty} |x|^\gamma \varphi(x) > 0$. In fact, by taking for example $\omega(x) = |x_1|/|x|$, we have $\liminf_{|x| \rightarrow \infty} |x|^\gamma \varphi(x) = 0$. We also give an explicit upper bound.

We now consider upper-bounds of the life-span for sign changing initial data. We define the sector

$$\Omega_m = \left\{ x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N; x_1 > 0, x_2 > 0, \dots, x_m > 0 \right\}, \quad (1.21)$$

where $1 \leq m \leq N$ is an integer. For $0 < \gamma < N$ and integer $1 \leq m \leq N$, we let $\psi_0 : \Omega_m \rightarrow \mathbb{R}$ be given by

$$\psi_0(x) = c_{m,\gamma} x_1 \cdots x_m |x|^{-\gamma-2m}, \quad x \in \Omega_m, \quad (1.22)$$

where

$$c_{m,\gamma} = \gamma(\gamma+2) \cdots (\gamma+2m-2). \quad (1.23)$$

In [50] local well-posedness for $\varphi \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$, anti-symmetric with respect to x_1, x_2, \dots, x_m , and $\varphi|_{\Omega_m}$ is in the Banach space

$$\mathcal{X} = \left\{ \psi \in L^1_{loc}(\Omega_m); \frac{\psi}{\psi_0} \in L^\infty(\Omega_m) \right\}, \quad (1.24)$$

have been shown for $0 < \alpha < 2/(\gamma+m)$. The solution can be extended to maximal solution $u : (0, T_{\max}(\varphi)) \rightarrow C_0(\mathbb{R}^N)$. Furthermore, there exists a constant $C > 0$, such that

$$\lambda^{[(\frac{1}{\alpha} - \frac{\gamma+m}{2})^{-1}]} T_{\max}(\lambda\varphi) \geq C, \quad (1.25)$$

for all $\lambda > 0$. We denote by $e^{t\Delta_m}$ the heat semigroup on Ω_m . We have obtained the following for large λ .

Theorem 1.10. *Let the positive integer m and the real numbers α, γ be such that*

$$1 \leq m \leq N, \quad 0 < \gamma < N, \quad 0 < \alpha < \frac{2}{\gamma+m}.$$

Suppose that $\varphi \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$, anti-symmetric with respect to x_1, x_2, \dots, x_m , $\varphi|_{\Omega_m} \in \mathcal{X}$, $\varphi \geq 0$ in Ω_m is such that $\varphi(x) \geq \omega(x)\psi_0(x)$ for $x \in \Omega_m \cap \{|x| \leq \epsilon\}$, for some $\epsilon > 0$, where $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0, anti-symmetric with respect to x_1, x_2, \dots, x_m , $\omega \geq 0$ on Ω_m and $\omega \not\equiv 0$. It follows that $T_{\max}(\lambda\varphi) < \infty$ for $\lambda > 0$ sufficiently large and

$$\limsup_{\lambda \rightarrow \infty} \lambda^{(\frac{1}{\alpha} - \frac{\gamma+m}{2})^{-1}} T_{\max}(\lambda\varphi) \leq \frac{1}{(\alpha^{1/\alpha} \|e^{\Delta_m}(\omega\psi_0)\|_\infty)^{(\frac{1}{\alpha} - \frac{\gamma+m}{2})^{-1}}}.$$

We have obtained the following for small λ .

Theorem 1.11. *Let the positive integer m and the real numbers α, γ be such that*

$$1 \leq m \leq N, \quad 0 < \gamma < N, \quad 0 < \alpha < \frac{2}{\gamma+m}.$$

Suppose that $\varphi \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$, anti-symmetric with respect to x_1, x_2, \dots, x_m , $\varphi|_{\Omega_m} \in \mathcal{X}$, $\varphi \geq 0$ in Ω_m is such that $\varphi(x) \geq \omega(x)\psi_0(x)$ for $x \in \Omega_m \cap \{|x| \geq R\}$, for some $R > 0$, where $\omega \in L^\infty(\mathbb{R}^N)$

is homogeneous of degree 0, anti-symmetric with respect to x_1, x_2, \dots, x_m , $\omega \geq 0$ on Ω_m and $\omega \not\equiv 0$. It follows that $T_{\max}(\lambda\varphi) < \infty$ for all $\lambda > 0$ and

$$\limsup_{\lambda \rightarrow 0} \lambda^{(\frac{1}{\alpha} - \frac{\gamma+m}{2})^{-1}} T_{\max}(\lambda\varphi) \leq \frac{1}{(\alpha^{1/\alpha} \|e^{\Delta_m}(\omega\psi_0)\|_{\infty})^{(\frac{1}{\alpha} - \frac{\gamma+m}{2})^{-1}}}.$$

Remark 8.

- 1) Theorems 1.10 and 1.11 improve the results of [50, Theorem 1.10, Proposition 4.5] by removing the condition $(N-2)\alpha < 4$. Also the conditions on the $\liminf_{|x| \rightarrow \infty} \frac{|x|^{\gamma+m}}{x_1 x_2 \dots x_m} \varphi(x) > 0$ or on the $\liminf_{|x| \rightarrow 0} \frac{|x|^{\gamma+m}}{x_1 x_2 \dots x_m} \varphi(x) > 0$ are not required here.
- 2) Theorem 1.10 (respectively Theorem 1.11) together with (1.25) show that

$$T_{\max}(\lambda\varphi) \sim \lambda^{-(\frac{1}{\alpha} - \frac{\gamma+m}{2})^{-1}},$$

as $\lambda \rightarrow \infty$ (respectively as $\lambda \rightarrow 0$).

The proofs of the known results cited above are based on careful constructions of super and sub-solutions, comparison and Kaplan's arguments. See, for example, [43, 30, 57, 58, 38] and some references therein. In the case of decaying initial data, the results are derived via a careful analysis of the asymptotic in the L^∞ -norm of the solutions to the linear heat equation on \mathbb{R}^N with initial data having specific orders of decay at space infinity as well as Kaplan's arguments and comparison principles, see [27]. This method, [27], has been used in many papers in the last three decades, see for example [30, 60, 61, 5, 59] and references therein. Most of the results require that λ be either sufficiently large or sufficiently small and initial data are positive and regular. Also, some scaling arguments are applied to derive life-span estimates, such as in [18, 11].

It is interesting to compare the two methods used to prove our results above. The proof of lower bounds as already mentioned, is based on the contraction mapping argument which gives local well-posedness of solutions (as in [50]). Consequently, it does not require any positivity condition or maximum principle. To prove the upper estimates, we use a necessary condition for local existence of non-negative solutions established in [56] (see Proposition 5.1 below), combined with the maximum principle, continuity properties of the heat semigroup and scaling argument. For these estimates, positivity is required. There is a certain unity in these two methods. On the one hand, the contraction mapping argument gives a sufficient condition on $T > 0$ for the existence of a solution on the interval $[0, T]$ for some initial value u_0 . This condition takes the form of an inequality involving both T and u_0 . This condition must fail for $T = T_{\max}$, which implies that the opposite inequality must hold. When this inequality is applied to initial values of the form $u_0 = \lambda\varphi$, this results in a lower life-span estimate. On the other hand, inequality in [56, Theorem 1] gives a necessary condition on $T > 0$ for the existence of a (positive) solution on the interval $[0, T]$, for some initial value $u_0 \geq 0$. This condition must hold for all $T < T_{\max}$. Moreover, this condition is stable under limits, and so must hold in the case $T = T_{\max}$. When the resulting inequality is applied to initial values of the form $u_0 = \lambda\varphi$, an upper life-span estimate is obtained. We note that the lower estimates for $T_{\max}(\lambda\varphi)$ do not in and of themselves prove finite time blowup, while the upper estimates do so.

Our results based on scaling, the third approach in this paper, on the one hand use ideas introduced in [11], and on the other hand comparison arguments. In particular, we give life-span estimates for an initial value of the form

$$\Phi(x) = \begin{cases} \omega(x)|x|^{-\gamma_1}, & |x| \leq 1 \\ \omega(x)|x|^{-\gamma_2}, & |x| > 1. \end{cases} \quad (1.26)$$

where $0 < \gamma_1, \gamma_2 < N$ and $\gamma_1, \gamma_2 < \frac{2}{\alpha}$ ($\gamma_1 \neq \gamma_2$) and $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0, $\omega \geq 0$, $\omega \not\equiv 0$. See Corollary 6.6 below. We show, in particular, the impact of the singularity on the life-span for λ large and the impact of the decay at infinity on the life-span for λ small.

The rest of this paper is organized as follows. In Section 2, we consider the standard nonlinear heat equation and prove Theorems 1.1 and 1.2. In Section 3, we prove new estimates for the heat kernel in weighted Lebesgue spaces, see Proposition 3.1 below. Section 4 is devoted to the case of slowly decaying initial data and the proofs of Theorem 1.3 and Corollaries 1.4 and 1.5. The upper estimates, Theorems 1.6–1.11, are proved in Section 5. In Section 6, we establish life-span estimates via nonlinear scaling. In the appendix, we give some estimates of the life-span for Hardy-Hénon parabolic equations.

Throughout the paper, C will be a positive constant which may vary from line to line. For positive functions f and g , we say that $f(x) \sim g(x)$ as $x \rightarrow x_0$ if there exists two positive constants C_1 and C_2 such that $C_1g(x) \leq f(x) \leq C_2g(x)$ in a neighborhood of x_0 .

2. LOWER BOUNDS FOR INITIAL DATA IN LEBESGUE SPACES

We consider the integral equation corresponding to the problem (1.1)-(1.2)

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\sigma)\Delta} [|u(\sigma)|^\alpha u(\sigma)] d\sigma, \quad (2.1)$$

where $e^{t\Delta}$ is the heat semigroup on Ω . It is known that the integral kernel corresponding to $e^{t\Delta}$ is bounded by the Gauss kernel for the heat semigroup on \mathbb{R}^N . Hence the $L^q - L^r$ smoothing inequalities are independent of Ω , i.e.

$$\|e^{t\Delta}u_0\|_{L^r(\Omega)} \leq (4\pi t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})} \|u_0\|_{L^q(\Omega)} \quad (2.2)$$

whenever $1 \leq q \leq r \leq \infty$.

We recall for future use that in the case of $\Omega = \mathbb{R}^N$

$$D_\tau e^{t\Delta} = e^{(t/\tau^2)\Delta} D_\tau \quad (2.3)$$

where D_τ is the dilation operator $D_\tau f(x) = f(\tau x)$. In particular

$$D_{\sqrt{t}} e^{t\Delta} = e^\Delta D_{\sqrt{t}}. \quad (2.4)$$

In this section the goal is to establish lower bounds for the life-span of solutions as an immediate consequence of the fixed point argument used to prove well-posedness of (2.1) in certain Banach spaces. While this argument is well-known, in order to show the applications to life-span, it is more convenient to recall some of the details.

To this end we recall the value

$$q_c = \frac{N\alpha}{2},$$

and we require that q and r satisfy the following conditions:

$$q > q_c \tag{2.5}$$

and

$$1 \leq \frac{r}{\alpha + 1} \leq q \leq r. \tag{2.6}$$

Note that given $q > q_c$ and $q \geq 1$, one can always choose $r = q(\alpha + 1)$. Also, if $q > q_c$ and $q \geq \alpha + 1$, one can choose $r = q$. Furthermore, in all cases above, we have $r \geq \alpha + 1$. Finally, if $q = \infty$, then necessarily $r = \infty$. We set

$$\beta = \frac{N}{2} \left(\frac{1}{q} - \frac{1}{r} \right). \tag{2.7}$$

We next define the space of curves in which we seek a solution to (2.1), i.e. the space in which we carry out the contraction mapping argument. For a fixed $M > 0$ and $T > 0$, (and q , r and β as above), we set

$$Y_{M,T}^{q,r} = \{u \in C((0, T]; L^r(\Omega)) : \sup_{t \in (0, T]} t^\beta \|u(t)\|_r \leq M\}. \tag{2.8}$$

With the distance

$$d(u, v) = d_{M,T}^{q,r}(u, v) = \sup_{t \in (0, T]} t^\beta \|u(t) - v(t)\|_r \tag{2.9}$$

the space $Y_{M,T}^{q,r}$ is a complete metric space.

To carry out the fixed point argument, we let $u_0 \in \mathcal{D}'(\Omega)$ and suppose that there exists $K > 0$ such that

$$\sup_{t \in (0, T]} t^\beta \|e^{t\Delta} u_0\|_r \leq K. \tag{2.10}$$

This condition includes implicitly the condition that $e^{t\Delta} u_0$ be well-defined and in $L^r(\Omega)$ for $t > 0$. Recall that if $u_0 \geq 0$, then $e^{t\Delta} u_0$ indeed is well-defined, but perhaps infinite. In order for (2.10) to hold, it suffices for example that $\sup_{t \in (0, T]} t^\beta \|e^{t\Delta} |u_0|\|_r \leq K$, since $|e^{t\Delta} u_0| \leq e^{t\Delta} |u_0|$. We define the iterative operator by

$$\mathcal{F}_{u_0} u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\sigma)\Delta} [|u(\sigma)|^\alpha u(\sigma)] d\sigma. \tag{2.11}$$

The following theorem is well-known. Since we are particularly interested here in the contraction mapping property, we sketch that part of the proof.

Theorem 2.1. *Let $N \geq 1$, $\alpha > 0$, $1 \leq q \leq \infty$ and $q > q_c$. There is a constant $C = C(\alpha, q) > 0$ such that if $K > 0$, $M > 0$ and $T > 0$ satisfy*

$$K + CT^{1-\frac{N\alpha}{2q}} M^{\alpha+1} \leq M, \tag{2.12}$$

and if $u_0 \in \mathcal{D}'(\Omega)$ satisfies (2.10) for some $r \geq q$ with $1 \leq \frac{r}{\alpha+1} \leq q \leq r$, then \mathcal{F}_{u_0} is a strict contraction on $Y_{M,T}^{q,r}$ and so has a unique fixed point $u = \mathcal{F}_{u_0} u \in Y_{M,T}^{q,r}$. This solution u of (2.1) is a classical solution of (1.1) on $(0, T]$.

Furthermore, if $u_0 \in L^q(\Omega)$ and $q < \infty$, then this fixed point has the property that $u \in C([0, T]; L^q(\Omega))$ with $u(0) = u_0$.

Remark 9. Of course the sufficient condition (2.12) can be taken as

$$\sup_{t \in (0, T]} t^\beta \|e^{t\Delta} u_0\|_r + CT^{1-\frac{N\alpha}{2q}} M^{\alpha+1} \leq M, \quad (2.13)$$

i.e. taking equality in (2.10)

Proof. We first consider when the space $Y_{M, T}^{q, r}$ is preserved by the iterative operator \mathcal{F}_{u_0} . Thus we suppose $u \in Y_{M, T}^{q, r}$, and we estimate $\mathcal{F}_{u_0} u(t)$ as follows.

$$\begin{aligned} t^\beta \|\mathcal{F}_{u_0} u(t)\|_r &\leq t^\beta \|e^{t\Delta} u_0\|_r + t^\beta \int_0^t \|e^{(t-\sigma)\Delta} [|u(\sigma)|^\alpha u(\sigma)]\|_r d\sigma \\ &\leq K + t^\beta \int_0^t [4\pi(t-\sigma)]^{-\frac{N\alpha}{2r}} \| |u(\sigma)|^\alpha u(\sigma) \|_{\frac{r}{\alpha+1}} d\sigma \\ &= K + (4\pi)^{-\frac{N\alpha}{2r}} t^\beta \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}} \|u(\sigma)\|_r^{\alpha+1} d\sigma \\ &\leq K + (4\pi)^{-\frac{N\alpha}{2r}} t^\beta \left(\int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) M^{\alpha+1} \\ &\leq K + (4\pi)^{-\frac{N\alpha}{2r}} t^{1-\frac{N\alpha}{2q}} \left(\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) M^{\alpha+1} \\ &\leq K + (4\pi)^{-\frac{N\alpha}{2r}} \left(\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) T^{1-\frac{N\alpha}{2q}} M^{\alpha+1} \\ &\leq K + 2(\alpha+1)(4\pi)^{-\frac{N\alpha}{2r}} \left(\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) T^{1-\frac{N\alpha}{2q}} M^{\alpha+1}. \end{aligned}$$

It follows that if (2.12) holds, where

$$C = C(\alpha, q, r) = 2(\alpha+1)(4\pi)^{-\frac{N\alpha}{2r}} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma,$$

then $Y_{M, T}^{q, r}$ is stable by \mathcal{F}_{u_0} .

Next we show that \mathcal{F}_{u_0} is a strict contraction on $Y_{M,T}^{q,r}$. We estimate as follows.

$$\begin{aligned}
t^\beta \|\mathcal{F}_{u_0}u(t) - \mathcal{F}_{u_0}v(t)\|_r &\leq t^\beta \int_0^t \|e^{(t-\sigma)\Delta} [|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma)]\|_r d\sigma \\
&\leq (4\pi)^{-\frac{N\alpha}{2r}} t^\beta \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}} \| [|u(\sigma)|^\alpha u(\sigma) - |v(\sigma)|^\alpha v(\sigma)] \|_{\frac{r}{\alpha+1}} d\sigma \\
&\leq (\alpha+1)(4\pi)^{-\frac{N\alpha}{2r}} t^\beta \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}} \| [u(\sigma) - v(\sigma)] [|u(\sigma)|^\alpha + |v(\sigma)|^\alpha] \|_{\frac{r}{\alpha+1}} d\sigma \\
&\leq (\alpha+1)(4\pi)^{-\frac{N\alpha}{2r}} t^\beta \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}} \|u(\sigma) - v(\sigma)\|_r [\|u(\sigma)\|_r^\alpha + \|v(\sigma)\|_r^\alpha] d\sigma \\
&\leq 2(\alpha+1)(4\pi)^{-\frac{N\alpha}{2r}} t^\beta \left(\int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) M^\alpha d_{M,T}^{q,r}(u,v) \\
&\leq 2(\alpha+1)(4\pi)^{-\frac{N\alpha}{2r}} t^{1-\frac{N\alpha}{2q}} \left(\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) M^\alpha d_{M,T}^{q,r}(u,v) \\
&\leq 2(\alpha+1)(4\pi)^{-\frac{N\alpha}{2r}} \left(\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) T^{1-\frac{N\alpha}{2q}} M^\alpha d_{M,T}^{q,r}(u,v).
\end{aligned}$$

Thus

$$d_{M,T}^{q,r}(\mathcal{F}_{u_0}u, \mathcal{F}_{u_0}v) \leq 2(\alpha+1)(4\pi)^{-\frac{N\alpha}{2r}} \left(\int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) T^{1-\frac{N\alpha}{2q}} M^\alpha d_{M,T}^{q,r}(u,v).$$

It follows that if (2.12) holds then \mathcal{F}_{u_0} is a strict contraction on $Y_{M,T}^{q,r}$.

The only difficulty is that C potentially depends on r as well as q and α . To rectify this, one can replace $C(\alpha, q, r)$ by $C(\alpha, q) = \max_{q \leq r \leq q(\alpha+1)} C(\alpha, q, r)$ and the result holds with $C = C(\alpha, q)$. \square

As is well-known, Theorem 2.1 is used to show that the integral equation (2.1) is locally well-posed on $L^q(\Omega)$. In particular, if $u_0 \in L^q(\Omega)$ the resulting solution given by the fixed point argument can be extended to a unique maximal solution on an interval $[0, T_{\max}(u_0))$. We will not belabor this point further.

We have also the following.

Proposition 2.2. *Let $N \geq 1$, $\alpha > 0$, $1 \leq q < \infty$ and $q > q_c$. Let $T_{\max}(\varphi, L^q)$ denotes the existence time of the maximal solution of (2.1) with initial data $\varphi \in L^q(\Omega)$. Then the following hold.*

- (i) $u(t) \in C_0(\Omega)$ for $t \in (0, T_{\max}(\varphi, L^q))$.
- (ii) If $\varphi \in L^q(\Omega) \cap C_0(\Omega)$ then $T_{\max}(\varphi, L^q) = T_{\max}(\varphi, C_0(\Omega))$, the existence time of the maximal solution of (2.1) with initial data $\varphi \in C_0(\Omega)$.
- (iii) If $\varphi \in L^q(\Omega) \cap L^p(\Omega)$ with $q_c < p \leq \infty$ then $T_{\max}(\varphi, L^q) = T_{\max}(\varphi, L^p)$, the existence time of the maximal solution of (2.1) with initial data $\varphi \in L^p(\Omega)$.

Proof. (i) By iterative argument, as in [2], $u(t) \in L^r(\Omega)$ for $q \leq r \leq \infty$. It is known that $e^{t\Delta} : L^q(\Omega) \rightarrow C_0(\Omega)$, is bounded for $t > 0$ and $1 \leq q < \infty$. See [42, 9]. Hence, by (2.1), $u(t) \in C_0(\Omega)$.

(ii) By (i) we have $T_{\max}(\varphi, L^q) \leq T_{\max}(\varphi, C_0(\Omega))$. Using (2.1) and (2.2), we have

$$\begin{aligned} \|u(t)\|_q &\leq \|e^{t\Delta}\varphi\|_q + \int_0^t \| |u(\sigma)|^\alpha u(\sigma) \|_q d\sigma \\ &\leq \|\varphi\|_q + \int_0^t \|u(\sigma)\|_\infty^\alpha \|u(\sigma)\|_q d\sigma. \end{aligned}$$

By Gronwall's inequality, we get

$$\|u(t)\|_q \leq \|\varphi\|_q e^{\int_0^t \|u(\sigma)\|_\infty^\alpha d\sigma}.$$

Hence u can not blow up in $L^q(\Omega)$ before it blows up in $C_0(\Omega)$. That is $T_{\max}(\varphi, C_0(\Omega)) \leq T_{\max}(\varphi, L^q)$.

(iii) Let $\varepsilon \in (0, \min(T_{\max}(\varphi, L^q), T_{\max}(\varphi, L^p)))$. By (i) we have $u(\varepsilon) \in C_0(\Omega)$. Using (ii) we have if $p < \infty$,

$$T_{\max}(u(\varepsilon), L^q) = T_{\max}(u(\varepsilon), C_0(\Omega)) = T_{\max}(u(\varepsilon), L^p).$$

That is $T_{\max}(\varphi, L^q) - \varepsilon = T_{\max}(\varphi, L^p) - \varepsilon$. If $p = \infty$, then $q < p$ hence (i)-(ii) hold and $T_{\max}(\varphi, L^q) - \varepsilon = T_{\max}(u(\varepsilon), C_0(\Omega)) = T_{\max}(u(\varepsilon), L^\infty) = T_{\max}(\varphi, L^\infty) - \varepsilon$. Hence we get the result. \square

As a first application of Theorem 2.1 to life-span estimates, we prove Theorem 1.1.

Proof of Theorem 1.1. Consider $u_0 = \lambda\varphi$, where $\lambda > 0$ and $\varphi \in L^q(\Omega)$. The key observation is that if $T_{\max}(\lambda\varphi) < \infty$, it is impossible to carry out the fixed point argument on the interval $[0, T_{\max}(\lambda\varphi)]$ with initial value $u_0 = \lambda\varphi$. Hence, by (2.13)

$$\sup_{t \in (0, T_{\max}(\lambda\varphi))} t^\beta \|e^{t\Delta} u_0\|_r + CT_{\max}(\lambda\varphi)^{1 - \frac{N\alpha}{2q}} M^{\alpha+1} > M,$$

for all $M > 0$. Recall that $(4\pi t)^\beta \|e^{t\Delta} u_0\|_r \leq \|u_0\|_q$ by the $L^q - L^r$ smoothing properties of the heat semigroup (2.2), so that

$$(4\pi)^\beta \lambda \|\varphi\|_q + CT_{\max}(\lambda\varphi)^{1 - \frac{N\alpha}{2q}} M^{\alpha+1} > M,$$

for all $M > 0$. In particular, if we set $M = 2(4\pi)^\beta \lambda \|\varphi\|_q$, this gives

$$CT_{\max}(\lambda\varphi)^{1 - \frac{N\alpha}{2q}} [\lambda \|\varphi\|_q]^\alpha > 1.$$

Thus we have proved Theorem 1.1. \square

As a second application, consider $u_0 = \lambda\mathbf{m}$, where $\lambda > 0$ and $\mathbf{m} \in \mathcal{M}$, the set of finite Borel measures on Ω . For example, \mathbf{m} could be a point mass. In order to apply Theorem 2.1, we observe first that

$$|e^{t\Delta}\mathbf{m}| \leq (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} d|\mathbf{m}|(y),$$

where, by abuse of notation, \mathbf{m} denotes both the measure on Ω and its natural extension to \mathbb{R}^N , and $|\mathbf{m}|$ is the total variation of \mathbf{m} . Hence

$$\begin{aligned} \|e^{t\Delta}\mathbf{m}\|_\infty &\leq (4\pi t)^{-\frac{N}{2}}\|\mathbf{m}\|_{\mathcal{M}}, \\ \|e^{t\Delta}\mathbf{m}\|_1 &\leq \|\mathbf{m}\|_{\mathcal{M}}; \end{aligned}$$

and so by interpolation

$$\|e^{t\Delta}\mathbf{m}\|_r \leq (4\pi t)^{-\frac{N}{2}(1-\frac{1}{r})}\|\mathbf{m}\|_{\mathcal{M}}, \quad (2.14)$$

for all $1 \leq r \leq \infty$.

Theorem 2.1 thus implies that (2.1) is locally well-posed on \mathcal{M} if $q_c < 1$. Simply take $q = 1$ and $r = \alpha + 1$. (This of course is well-known.)

Proof of Theorem 1.2. To obtain a life-span estimate, we again note that if the maximal existence time is finite, i.e. $T_{\max}(\lambda\mathbf{m}) < \infty$, then (2.13) can not hold with $u_0 = \lambda\mathbf{m}$ and $T = T_{\max}(\lambda\mathbf{m})$. Hence, also using (2.14), for $\beta = \frac{N}{2}(1 - \frac{1}{\alpha+1})$, we must have

$$(4\pi)^\beta \lambda \|\mathbf{m}\|_{\mathcal{M}} + CT_{\max}(\lambda\mathbf{m})^{1-\frac{N\alpha}{2}} M^{\alpha+1} > M,$$

for all $M > 0$. As above, we take $M = 2(4\pi)^\beta \lambda \|\mathbf{m}\|_{\mathcal{M}}$, which gives the lower estimate

$$CT_{\max}(\lambda\mathbf{m})^{\frac{1}{\alpha}-\frac{N}{2}} \lambda \|\mathbf{m}\|_{\mathcal{M}} > 1.$$

This completes the proof of Theorem 1.2. \square

As a third application of Theorem 2.1 to life-span estimates we consider $u_0 = \lambda\varphi$ where $\lambda > 0$, $\varphi \in L^1_{loc}(\mathbb{R}^N)$ and $|\varphi| \leq |\cdot|^{-\gamma}$ for some $0 < \gamma < N$. We recall that if $\frac{N}{\gamma} < r$, then

$$\|e^{t\Delta}|\cdot|^{-\gamma}\|_r = t^{-\frac{\gamma}{2}+\frac{N}{2r}}\|e^\Delta|\cdot|^{-\gamma}\|_r \quad (2.15)$$

for all $t > 0$. This follows from a scaling argument. For convenience, we set

$$L = \|e^\Delta|\cdot|^{-\gamma}\|_r. \quad (2.16)$$

Hence if $|\varphi| \leq |\cdot|^{-\gamma}$, then

$$\|e^{t\Delta}(\lambda\varphi)\|_r \leq \lambda\|e^{t\Delta}|\cdot|^{-\gamma}\|_r = L\lambda t^{-\frac{\gamma}{2}+\frac{N}{2r}}. \quad (2.17)$$

We next set $q = \frac{N}{\gamma}$, so that $1 < q < r$, and we may choose r so that (2.6) holds. Also, $\beta = \frac{N}{2}(\frac{1}{q} - \frac{1}{r}) = \frac{\gamma}{2} - \frac{N}{2r}$. Theorem 2.1 clearly shows that (2.1) is locally well-posed for initial values bounded by a multiple of $|x|^{-\gamma}$ with $0 < \gamma < N$ and $\frac{N}{\gamma} > q_c$, i.e. $\gamma < \frac{2}{\alpha}$. This of course is known. See [11, Theorem 2.8], and also [50, Theorem 2.3].

As for a life-span estimate, if $u_0 = \lambda\varphi$ where $|\varphi(x)| \leq |x|^{-\gamma}$, then the existence time of the solution, $T_{\max}(\lambda\varphi)$, if it is finite, must verify

$$\lambda L + CT_{\max}(\lambda\varphi)^{1-\frac{\gamma\alpha}{2}} M^{\alpha+1} > M$$

for all $M > 0$, where L is given by (2.16). For $M = 2\lambda L$, this gives

$$CT_{\max}(\lambda\varphi)^{1-\frac{\gamma\alpha}{2}} (\lambda L)^\alpha > 1.$$

In other words, we have the following result.

Corollary 2.3. *Let $0 < \gamma < N$ and $\gamma < \frac{2}{\alpha}$. Suppose $\varphi \in L^1_{loc}(\mathbb{R}^N)$ is such that $|\varphi(x)| \leq |x|^{-\gamma}$. It follows that*

$$T_{\max}(\lambda\varphi) \geq \frac{C}{(\lambda L)^{\left(\frac{1}{\alpha} - \frac{\gamma}{2}\right)^{-1}}, \quad \lambda > 0, \quad (2.18)$$

where L is given by (2.16) and C depends only on α and γ .

This last result recovers [50, Theorem 2.6(ii)] in the case $m = 0$, by a different but related method: the contraction mapping argument is formulated differently. It does not seem possible that the contraction mapping argument used in the proof of Theorem 2.1 can be used to recover [50, Theorems 2.3 and 2.6] in the case $1 \leq m \leq N$. Indeed, that is the point of the paper [50]. Note also that Theorem 1.3 gives also the result but here the constant at the right-hand side is explicit.

3. ESTIMATES FOR THE HEAT KERNEL IN WEIGHTED SPACES

In this section we prove the following heat kernel estimates. For simplicity, the space $L^p(\mathbb{R}^N)$, will be denoted by L^p . We recall that the norm in $L^p(\mathbb{R}^N)$, $\|\cdot\|_{L^p(\mathbb{R}^N)}$ is denoted by $\|\cdot\|_p$.

Proposition 3.1. *Let $N \geq 1$, $0 \leq \gamma \leq \mu < N$, $q_1 \in (1, \infty]$ and $q_2 \in (1, \infty]$ satisfy*

$$0 \leq \frac{1}{q_2} < \frac{\mu - \gamma}{N} + \frac{1}{q_1} \leq \frac{\mu}{N} + \frac{1}{q_1} < 1.$$

Then there exists a constant $C > 0$ depending on N, γ, μ, q_1 and q_2 such that

$$\left\| |\cdot|^\gamma e^{t\Delta} u \right\|_{q_2} \leq C t^{-\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) - \frac{\mu - \gamma}{2}} \left\| |\cdot|^\mu u \right\|_{q_1}, \quad t > 0, \quad |\cdot|^\mu u \in L^{q_1}. \quad (3.1)$$

Remark 10.

- 1) The estimate (3.1) is well-known for $\mu = \gamma = 0$, that is (2.2) (see for example [53]). For the case $\gamma = 0$, $0 < \mu < N$, (3.1) is established in [2]. Estimate (3.1) is known for $0 < \gamma \leq \mu < N$, $0 \leq \frac{1}{q_2} \leq \frac{1}{q_1} < \frac{1}{q_1} + \frac{\mu}{N} < 1$ in [51, 8]. See also [25] for the case $q_1 = q_2 = \infty$. It follows by [51, 8] that (3.1) holds for $0 < \gamma = \mu < N$, $q_1 = q_2 = q \in (1, \infty]$, $\frac{1}{q} + \frac{\mu}{N} < 1$.
- 2) The power of t in (3.1) is optimal. This can be shown by scaling argument as in [2]. In fact, for $t > 0$, we have

$$e^{\Delta} u = D_{\sqrt{t}} e^{t\Delta} D_{\frac{1}{\sqrt{t}}} u$$

for all $u \in \mathcal{S}'(\mathbb{R}^N)$. Also

$$\left\| |\cdot|^\gamma D_{\sqrt{t}} f \right\|_r = t^{-\frac{N}{2r} - \frac{\gamma}{2}} \left\| |\cdot|^\gamma f \right\|_r$$

for all $|\cdot|^\gamma f \in L^r$, $r \geq 1$. Writing (3.1) for $t = 1$ as follows

$$\left\| |\cdot|^\gamma e^{\Delta} u \right\|_{q_2} = \left\| |\cdot|^\gamma D_{\sqrt{t}} e^{t\Delta} D_{1/\sqrt{t}} u \right\|_{q_2} \leq C \left\| |\cdot|^\mu u \right\|_{q_1}.$$

Setting $D_{1/\sqrt{t}} u = v$ that is $u = D_{\sqrt{t}} v$, we get

$$\left\| |\cdot|^\gamma D_{\sqrt{t}} e^{t\Delta} v \right\|_{q_2} \leq C \left\| |\cdot|^\mu D_{\sqrt{t}} v \right\|_{q_1}.$$

That is

$$t^{-\frac{N}{2q_2}-\frac{\gamma}{2}} \left\| |\cdot|^\gamma e^{t\Delta} v \right\|_{q_2} \leq C t^{-\frac{N}{2q_1}-\frac{\mu}{2}} \left\| |\cdot|^\mu v \right\|_{q_1}.$$

This gives (3.1) for all $t > 0$.

- 3) The fact that $\gamma \leq \mu$ is necessary. This follows by translation argument. See also [10]. We take, in (3.1), $t = 1$, $u = G_1(\cdot - \tau x_0)$, $\tau > 0$, $x_0 \in \mathbb{R}^N$, $|x_0| = 1$. In fact, we have that

$$e^\Delta G_1(\cdot - \tau x_0) = G_1 \star G_1(\cdot - \tau x_0) = G_2(\cdot - \tau x_0).$$

On the other hand,

$$\left\| |\cdot|^\gamma G_2(\cdot - \tau x_0) \right\|_{q_2} = \left\| |\cdot + \tau x_0|^\gamma G_2 \right\|_{q_2} = \tau^\gamma \left\| \left| \frac{\cdot}{\tau} + x_0 \right|^\gamma G_2 \right\|_{q_2}$$

and

$$\left\| |\cdot|^\mu G_1(\cdot - \tau x_0) \right\|_{q_1} = \tau^\mu \left\| \left| \frac{\cdot}{\tau} + x_0 \right|^\mu G_1 \right\|_{q_1}.$$

Hence, (3.1) reads,

$$\tau^{-(\mu-\gamma)} \left\| \left| \frac{\cdot}{\tau} + x_0 \right|^\gamma G_2 \right\|_{q_2} \leq C \left\| \left| \frac{\cdot}{\tau} + x_0 \right|^\mu G_1 \right\|_{q_1}.$$

Then we let $\tau \rightarrow \infty$, since $\left\| \left| \frac{\cdot}{\tau} + x_0 \right|^\mu G_1 \right\|_{q_1} \rightarrow \|G_1\|_{q_1} < \infty$ and $\left\| \left| \frac{\cdot}{\tau} + x_0 \right|^\gamma G_2 \right\|_{q_2} \rightarrow \|G_2\|_{q_2} > 0$, to deduce that $\gamma \leq \mu$ if $q_2, q_1 < \infty$.

- 4) Our estimate is different from that of [51, 8] since we do not require $q_1 \leq q_2$ if $\gamma < \mu$. In fact, since the condition $\gamma \leq \mu$, is necessary by the above remark, all that we require is that the power of t in (3.1) is negative.

To prove Proposition 3.1, we establish the following estimates for the heat kernel in weighted Lorentz spaces. Since the cases $0 = \gamma < \mu < N$, ([2]) and $0 < \gamma = \mu < N$ ([51, 8]) are known, we only give the proof for $0 < \gamma < \mu < N$.

Proposition 3.2. *Let $N \geq 1$, $0 < \gamma < \mu < N$, $1 \leq q \leq \infty$, $q_1 \in (1, \infty]$ and $q_2 \in (1, \infty]$ satisfy*

$$0 \leq \frac{1}{q_2} < \frac{\mu - \gamma}{N} + \frac{1}{q_1} < \frac{\mu}{N} + \frac{1}{q_1} < 1.$$

Then there exists a constant $C > 0$ depending on N, γ, μ, q, q_1 and q_2 such that

$$\left\| |\cdot|^\gamma e^{t\Delta} u \right\|_{L^{q_2, q}} \leq C t^{-\frac{N}{2} \left(\frac{1}{q_1} - \frac{1}{q_2} \right) - \frac{\mu - \gamma}{2}} \left\| |\cdot|^\mu u \right\|_{L^{q_1, \infty}}, \quad t > 0, \quad |\cdot|^\mu u \in L^{q_1, \infty}, \quad (3.2)$$

with if $q_2 = \infty$ then $q = \infty$.

Remark 11. A Young's inequality is proved in [26, Theorem 3.1, p. 201] for weighted Lebesgue spaces where it is assumed also $q_1, q_2 < \infty$. We do not use this here and we provide a simpler proof for our case as a convolution with a Gaussian. See [47] for (3.2) with $\gamma = 0 < \mu < N$, $0 \leq \frac{1}{q_2} < \frac{\mu}{N} + \frac{1}{q_1} < 1$.

Proof of Proposition 3.2. From the embedding $L^{q_2, 1} \hookrightarrow L^{q_2, q}$, $q \geq 1$, $q_2 < \infty$, it is sufficient to give the proof for $q = 1$. Since $\gamma > 0$, then by the inequality $|x|^\gamma \leq C(|y|^\gamma + |x - y|^\gamma)$, we write

$$\left\| |\cdot|^\gamma e^{t\Delta} u \right\| = \left\| |\cdot|^\gamma (G_t \star u) \right\| \leq C G_t \star (|\cdot|^\gamma |u|) + C (|\cdot|^\gamma G_t) \star |u|, \quad t > 0. \quad (3.3)$$

Let $\gamma < \mu < N$, $q_1 \in (1, \infty)$ and $q_2 \in (1, \infty)$ be such that

$$\frac{1}{q_2} < \frac{\mu - \gamma}{N} + \frac{1}{q_1} < \frac{\mu}{N} + \frac{1}{q_1} < 1.$$

Set

$$\frac{1}{p_1} = 1 + \frac{1}{q_2} - \frac{1}{p_2},$$

with

$$\frac{1}{p_2} = \frac{\mu - \gamma}{N} + \frac{1}{q_1}.$$

Since $\gamma < N$, then $p_1 \in (1, \infty)$ and satisfies

$$\frac{\gamma}{N} < 1 - \left(\frac{\mu - \gamma}{N} + \frac{1}{q_1} \right) < \frac{1}{p_1} < 1.$$

Let us introduce the numbers \tilde{p}_1, \tilde{p}_2 defined by

$$\frac{1}{\tilde{p}_1} = \frac{1}{p_1} - \frac{\gamma}{N}, \quad \frac{1}{\tilde{p}_2} = \frac{\mu}{N} + \frac{1}{q_1}.$$

We have

$$\frac{1}{\tilde{p}_1} = 1 + \frac{1}{q_2} - \frac{1}{\tilde{p}_2},$$

Since $0 < \gamma < \mu$, and by the conditions on q_1, q_2 , we have that

$$0 < \frac{1}{p_2} < \frac{1}{\tilde{p}_2} < 1, \quad 0 < \frac{1}{\tilde{p}_1} < 1, \quad 0 < \frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p_1} - 1 + \frac{1}{p_2} < \frac{1}{p_2} < 1.$$

Using generalized Young's inequality, see [37, Theorem 2.6, p. 137] or [17, 28], we deduce that

$$\begin{aligned} \|\cdot\|^{\gamma} e^{t\Delta} u \|_{L^{q_2,1}} &\leq C \|G_t \star |\cdot|^{\gamma} u\|_{L^{q_2,1}} + C \|\cdot\|^{\gamma} G_t \star |u\|_{L^{q_2,1}} \\ &\leq C \|G_t\|_{L^{p_1,1}} \|\cdot\|^{\gamma} u\|_{L^{p_2,\infty}} + C \|\cdot\|^{\gamma} G_t \|_{L^{\tilde{p}_1,1}} \|u\|_{L^{\tilde{p}_2,\infty}} \\ &:= CI_1 + CI_2, \end{aligned} \tag{3.4}$$

where

$$1 + \frac{1}{q_2} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2}, \quad 1 < q_2, p_2, \tilde{p}_2 < \infty, \quad 1 < p_1, \tilde{p}_1 < \infty.$$

We first estimate I_1 . Using the generalized Hölder inequality, see [37, Theorem 3.4, p. 141] or [28, 17]), we get

$$I_1 \leq C \|G_t\|_{L^{p_1,1}} \|\cdot\|^{-(\mu-\gamma)} \|_{L^{\frac{N}{\mu-\gamma},\infty}} \|\cdot\|^{\mu} u\|_{L^{q_1,\infty}}, \quad 0 < \mu - \gamma < N, \quad \frac{1}{p_2} = \frac{\mu - \gamma}{N} + \frac{1}{q_1} < 1.$$

Since $G_t(x) = t^{-\frac{N}{2}} G_1(x/\sqrt{t}) = t^{-\frac{N}{2}} (4\pi)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} \in L^{p_1,1}$, we deduce from [17] that

$$\|G_t\|_{L^{p_1,1}} = t^{-\frac{N}{2}} \|D_{1/\sqrt{t}} G_1\|_{L^{p_1,1}} = t^{-\frac{N}{2}} t^{\frac{N}{2p_1}} \|G_1\|_{L^{p_1,1}} = Ct^{-\frac{N}{2}} \left(1 - \frac{1}{p_1}\right),$$

with

$$1 - \frac{1}{p_1} = \frac{1}{p_2} - \frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{q_2} + \frac{\mu - \gamma}{N} > 0.$$

Then, we deduce that

$$I_1 \leq Ct^{-\frac{N}{2}} \left(\frac{1}{q_1} - \frac{1}{q_2}\right)^{-\frac{\mu-\gamma}{2}} \|\cdot\|^{\mu} u\|_{L^{q_1,\infty}}. \tag{3.5}$$

We now estimate I_2 . Using the generalized Hölder inequality, we get

$$I_2 \leq C \|\cdot\|^{\gamma} G_t \|_{L^{\tilde{p}_1, 1}} \|\cdot\|^{-\mu} \|_{L^{\frac{N}{\mu}, \infty}} \|\cdot\|^{\mu} u \|_{L^{q_1, \infty}},$$

$$0 < \mu < N, \quad \frac{1}{\tilde{p}_2} = \frac{\mu}{N} + \frac{1}{q_1} < 1.$$

Since $|x|^{\gamma} G_t(x) = t^{-\frac{N}{2} + \frac{\gamma}{2}} |x/\sqrt{t}|^{\gamma} G_1(x/\sqrt{t}) = t^{-\frac{N}{2}} (4\pi)^{-\frac{N}{2}} |x|^{\gamma} e^{-\frac{|x|^2}{4t}} \in L^{\tilde{p}_1, 1}$, we deduce from [17] that

$$\|\cdot\|^{\gamma} G_t \|_{L^{\tilde{p}_1, 1}} = t^{-\frac{N}{2} + \frac{\gamma}{2}} \|D_{1/\sqrt{t}}(\cdot\|^{\gamma} G_1)\|_{L^{\tilde{p}_1, 1}} = t^{-\frac{N}{2} + \frac{\gamma}{2}} t^{\frac{N}{2\tilde{p}_1}} \|\cdot\|^{\gamma} G_1 \|_{L^{\tilde{p}_1, 1}} = Ct^{-\frac{N}{2}} \left(1 - \frac{1}{\tilde{p}_1}\right).$$

Then, we deduce that

$$I_2 \leq Ct^{-\frac{N}{2}} \left(\frac{1}{q_1} - \frac{1}{q_2}\right)^{-\frac{\mu-\gamma}{2}} \|\cdot\|^{\mu} u \|_{L^{q_1, \infty}}. \quad (3.6)$$

Plugging (3.5) and (3.6) in (3.4) we get (3.2).

If $q_2 \in (1, \infty)$ and $q_1 = \infty$, hence $q_2 > N/(\mu - \gamma)$, the above calculations for estimating I_1, I_2 hold using the generalized Hölder inequality in [37, Theorem 3.4, p. 141] (see also [28, Proposition 2.3 a), p. 19]).

If $q_2 = q = \infty$, the proof follows by using the generalized Young inequality, [37, Theorem 3.6, p. 141] (see also [28, Proposition 2.4 b), p. 20]) as follows

$$\|\cdot\|^{\gamma} e^{t\Delta} u \|_{\infty} \leq C \|G_t\|_{L^{p_1, 1}} \|\cdot\|^{\gamma} |u| \|_{L^{p_2, \infty}} + C \|\cdot\|^{\gamma} G_t \|_{L^{\tilde{p}_1, 1}} \|u\|_{L^{\tilde{p}_2, \infty}},$$

with

$$1 - \left(\frac{\mu - \gamma}{N} + \frac{1}{q_1}\right) = \frac{1}{p_1} \in (\gamma/N, 1)$$

and by similar calculations as above. This completes the proof. \square

We now give the proof of Proposition 3.1.

Proof of Proposition 3.1. The proof follows by taking $q = q_2$ in (3.2) and using the fact that

$$\|\cdot\|^{\mu} u \|_{L^{q_1, \infty}} \leq C \|\cdot\|^{\mu} u \|_{L^{q_1, q_1}} = C \|\cdot\|^{\mu} u \|_{L^{q_1}}.$$

\square

4. LOWER BOUNDS FOR SLOWLY DECAYING INITIAL DATA

In this section we apply Proposition 3.1 in order to show local well-posedness in weighted Lebesgue spaces for the nonlinear heat equation (2.1). This allows us to obtain more precise estimates for the lower bound of the life-span in relation with the weight. For $\gamma \geq 0, 1 \leq q \leq \infty$, we consider the weighted Lebesgue space

$$L_{\gamma}^q(\mathbb{R}^N) = \{f : \mathbb{R}^N \rightarrow \mathbb{R}, \text{mesurable}, |\cdot|^{\gamma} f \in L^q(\mathbb{R}^N)\}.$$

Endowed with the norm

$$\|f\|_{L_{\gamma}^q} := \|\cdot\|^{\gamma} f \|_{L^q},$$

$L_\gamma^q(\mathbb{R}^N)$ is a Banach space. Clearly, if $0 < \gamma < N$, $1 \leq q \leq \infty$, and $\frac{1}{q} + \frac{\gamma}{N} < 1$, using the Hölder inequality, we have $L_\gamma^q(\mathbb{R}^N) \subset \mathcal{S}'(\mathbb{R}^N)$. Also, for $u_0 \in L_\gamma^q(\mathbb{R}^N)$, $0 < \gamma < N$, $\frac{N}{N-\gamma} < q < \infty$, we have $\lim_{t \rightarrow 0} \|e^{t\Delta}u_0 - u_0\|_{L_\gamma^q(\mathbb{R}^N)} = 0$. This follows as for the standard $L^q(\mathbb{R}^N)$ case, that is $\gamma = 0$.

We are interested in the local well-posedness for the nonlinear heat equation (2.1) in $L_\gamma^q(\mathbb{R}^N)$. We consider initial data $u_0 \in L_\gamma^q(\mathbb{R}^N)$ where q, γ satisfy

$$0 < \gamma < N, \quad \gamma < \frac{2}{\alpha}.$$

$$\frac{1}{q} + \frac{\gamma}{N} < 1, \quad \frac{N\alpha}{2q} + \frac{\alpha\gamma}{2} < 1.$$

The critical exponent in the weighted Lebesgue space $L_\gamma^q(\mathbb{R}^N)$ is given by

$$q_c(\gamma) = \frac{N\alpha}{2 - \gamma\alpha}. \quad (4.1)$$

The value of the critical exponent $q_c(\gamma)$ can be explained by scaling argument. In fact, if u is a solution of the equation (1.1), with $\Omega = \mathbb{R}^N$, then for any $\mu > 0$, u_μ is also a solution of (1.1), where

$$u_\mu(t, x) = \mu^{\frac{2}{\alpha}} u(\mu^2 t, \mu x).$$

We have $\|u_\mu(t)\|_{L_\gamma^q} = \mu^{\frac{2}{\alpha} - \gamma - \frac{N}{q}} \|u(t)\|_{L_\gamma^q}$, and on initial data $u(0) = u_0$ we have

$$\|\mu^{\frac{2}{\alpha}} u_0(\mu \cdot)\|_{L_\gamma^q} = \mu^{\frac{2}{\alpha} - \gamma - \frac{N}{q}} \|u_0\|_{L_\gamma^q}.$$

The only weighted Lebesgue exponent (obviously if its exponent is greater than 1) for which the norm is invariant under these dilations is

$$\frac{N}{q_c(\gamma)} = \frac{2}{\alpha} - \gamma.$$

Hence $q_c(\gamma)$ is given by (4.1). We have the following local well-posedness result.

Theorem 4.1 (Local well-posedness in L_γ^q). *Let $N \geq 1$ be an integer, $\alpha > 0$ and γ such that*

$$0 < \gamma < N, \quad \gamma < 2/\alpha. \quad (4.2)$$

Let $q_c(\gamma)$ be given by (4.1). Then we have the following.

(i) *If $\gamma(\alpha + 1) < N$ and q is such that*

$$q > \frac{N(\alpha + 1)}{N - \gamma(\alpha + 1)}, \quad q > q_c(\gamma) \quad \text{and} \quad q \leq \infty,$$

then equation (2.1) is locally well-posed in $L_\gamma^q(\mathbb{R}^N)$. More precisely, given $u_0 \in L_\gamma^q(\mathbb{R}^N)$, then there exist $T > 0$ and a unique solution $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N))$ of (2.1) (we replace $[0, T]$ by $(0, T]$ if $q = \infty$ and u satisfies $\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta}u_0\|_{L_\gamma^\infty(\mathbb{R}^N)} = 0$). Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} \|u(t)\|_{L_\gamma^q} = \infty$.

(ii) Assume that $q > q_c(\gamma)$ with $\frac{N}{N-\gamma} < q \leq \infty$. It follows that equation (2.1) is locally well-posed in $L_\gamma^q(\mathbb{R}^N)$ as in part (i) except that uniqueness is guaranteed only among functions $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N))$ which also verify $t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})+\frac{\nu\alpha}{2}} \|u(t)\|_{L_\nu^r}$ is bounded on $(0, T]$, where $r = (\alpha+1)q$, $\nu(\alpha+1) = \gamma$, (we replace $[0, T]$ by $(0, T]$ if $q = \infty$ and u satisfies $\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta} u_0\|_{L_\gamma^\infty(\mathbb{R}^N)} = 0$). Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} \|u(t)\|_{L_\gamma^q} = \infty$. Furthermore,

$$\|u(t)\|_{L_\gamma^q} \geq C(T_{\max} - t)^{\frac{N}{2q} - \frac{2-\gamma\alpha}{2\alpha}}, \quad \forall t \in [0, T_{\max}), \quad (4.3)$$

where C is a positive constant.

Proof. (i) Let us define the maps

$$K_t(u) = e^{t\Delta} (|u|^\alpha u), \quad t > 0.$$

Using the following inequality, which follows by the Hölder inequality,

$$\left\| |\cdot|^{\nu(\alpha+1)} (|u|^\alpha u - |v|^\alpha v) \right\|_{\frac{p}{\alpha+1}} \leq C \left(\|u\|_{L_\nu^p}^\alpha + \|v\|_{L_\nu^p}^\alpha \right) \|u - v\|_{L_\nu^p}, \quad p \geq \alpha + 1, \quad \nu \geq 0, \quad (4.4)$$

and Proposition 3.1 that is $e^{t\Delta} : L_{(\alpha+1)\gamma}^{\frac{q}{\alpha+1}} \rightarrow L_\gamma^q$ is bounded for each $t > 0$, we have that $K_t : L_\gamma^q \rightarrow L_\gamma^q$ is locally Lipschitz with

$$\begin{aligned} \|K_t(u) - K_t(v)\|_{L_\gamma^q} &\leq C t^{-\frac{N}{2}(\frac{\alpha+1}{q}-\frac{1}{q})-\frac{\alpha\gamma}{2}} \| |u|^\alpha u - |v|^\alpha v \|_{L_{(\alpha+1)\gamma}^{\frac{q}{\alpha+1}}} \\ &\leq C t^{-\frac{N\alpha}{2q}-\frac{\alpha\gamma}{2}} (\|u\|_{L_\gamma^q}^\alpha + \|v\|_{L_\gamma^q}^\alpha) \|u - v\|_{L_\gamma^q} \\ &\leq 2CM^\alpha t^{-\frac{N\alpha}{2q}-\frac{\alpha\gamma}{2}} \|u - v\|_{L_\gamma^q}, \end{aligned}$$

for $\|u\|_{L_\gamma^q} \leq M$ and $\|v\|_{L_\gamma^q} \leq M$. We have also, that $t^{-\frac{N\alpha}{2q}-\frac{\alpha\gamma}{2}} \in L_{loc}^1(0, \infty)$, since $q > q_c(\gamma)$. Obviously $t \mapsto \|K_t(0)\|_\infty = 0 \in L_{loc}^1(0, \infty)$, also $e^{s\Delta} K_t = K_{t+s}$ for $s, t > 0$. Then the proof follows by [53, Theorem 1, p. 279].

(ii) We consider r and $\nu > 0$ such that $\nu < \gamma$, $\nu(\alpha+1) < N$, $r > q$. Hence we have

$$\frac{1}{r} < \frac{\alpha+1}{r} + \frac{\nu\alpha}{N} < \frac{\alpha+1}{r} + \frac{\nu(\alpha+1)}{N} < 1, \quad \frac{1}{r} < \frac{1}{q} + \frac{\gamma-\nu}{N} < \frac{1}{q} + \frac{\gamma}{N} < 1.$$

The choice of r, ν is to guaranties that the maps $e^{t\Delta} : L_{(\alpha+1)\nu}^{\frac{r}{\alpha+1}} \rightarrow L_\nu^r$ and $e^{t\Delta} : L_\gamma^q \rightarrow L_\nu^r$ are bounded so that we may apply Proposition 3.1. In order that $e^{t\Delta} : L_{(\alpha+1)\nu}^{\frac{r}{\alpha+1}} \rightarrow L_\gamma^q$ is bounded, we choose for simplicity,

$$r = (\alpha+1)q, \quad \nu(\alpha+1) = \gamma,$$

(If $q = \infty$ we have $r = \infty$), and we may apply [8, Lemma 2.1] to get that $e^{t\Delta} : L_{(\alpha+1)\nu}^{\frac{r}{\alpha+1}} = L_\gamma^q \rightarrow L_\gamma^q$ is bounded. With this choice, the conditions on r and ν are satisfied, since

$$\frac{1}{q} + \frac{\gamma}{N} < 1.$$

Define

$$\beta(\nu) = \frac{N}{2q} - \frac{N}{2r} + \frac{\alpha\nu}{2}. \quad (4.5)$$

We choose $K > 0$, $T > 0$, $M > 0$ such that

$$K + CM^{\alpha+1}T^{1-\frac{N\alpha}{2q}-\frac{\gamma\alpha}{2}} \leq M, \quad (4.6)$$

where C is a positive constant. We will show that there exists a unique solution u of (2.1) such that $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N)) \cap C((0, T]; L_\nu^r(\mathbb{R}^N))$ with

$$\|u\| = \max \left[\sup_{t \in [0, T]} \|u(t)\|_{L_\gamma^q}, \sup_{t \in (0, T]} t^{\beta(\nu)} \|u(t)\|_{L_\nu^r} \right] \leq M.$$

The proof is based on a contraction mapping argument in the set

$$Y_{M,T}^{q,\gamma} = \{u \in C([0, T]; L_\gamma^q(\mathbb{R}^N)) \cap C((0, T]; L_\nu^r) : \|u\| \leq M\}.$$

Endowed with the metric $d(u, v) = \|u - v\|$, $Y_{M,T}^{q,\gamma}$ is a nonempty complete metric space. We note that for $u_0 \in L_\gamma^q$ we have

$$\|e^{t\Delta}u_0\|_{L_\nu^r} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{\gamma-\nu}{2}} \|u_0\|_{L_\gamma^q} = Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{\nu\alpha}{2}} \|u_0\|_{L_\gamma^q} = Ct^{-\beta(\nu)} \|u_0\|_{L_\gamma^q}.$$

We will show that \mathcal{F}_{u_0} defined in (2.11) is a strict contraction on $Y_{M,T}^{q,\gamma}$. The condition on the initial data $\|u_0\|_{L_\gamma^q} \leq K$ will implies that $t^\beta \|e^{t\Delta}u_0\|_{L_\nu^r} \leq K$. We have

$$\begin{aligned} t^{\beta(\nu)} \|\mathcal{F}_{u_0}u(t)\|_{L_\nu^r} &\leq t^{\beta(\nu)} \|e^{t\Delta}u_0\|_{L_\nu^r} + t^{\beta(\nu)} \int_0^t \|e^{(t-\sigma)\Delta} [|u(\sigma)|^\alpha u(\sigma)]\|_{L_\nu^r} d\sigma \\ &\leq K + Ct^{\beta(\nu)} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\nu(\alpha+1)-\nu}{2}} \| |\cdot|^{\nu(\alpha+1)} |u(\sigma)|^\alpha u(\sigma) \|_{r/(\alpha+1)} d\sigma \\ &= K + Ct^{\beta(\nu)} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\nu\alpha}{2}} \|u(\sigma)\|_{L_\nu^r}^{\alpha+1} d\sigma \\ &\leq K + CM^{\alpha+1} t^{\beta(\nu)} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r}-\frac{\nu\alpha}{2}} \sigma^{-\beta(\nu)(\alpha+1)} d\sigma \\ &\leq K + CM^{\alpha+1} t^{1-\frac{N\alpha}{2q}-\frac{\gamma\alpha}{2}} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\nu\alpha}{2}} \sigma^{-\beta(\nu)(\alpha+1)} d\sigma \\ &\leq K + CM^{\alpha+1} T^{1-\frac{N\alpha}{2q}-\frac{\gamma\alpha}{2}} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}-\frac{\nu\alpha}{2}} \sigma^{-\beta(\nu)(\alpha+1)} d\sigma. \end{aligned}$$

By the hypotheses and the fact that $q < r$ and $\nu < \gamma$ we have

$$\frac{N\alpha}{2r} + \frac{\nu\alpha}{2} < \frac{N\alpha}{2q} + \frac{\gamma\alpha}{2} < 1, \quad \beta(\nu)(\alpha+1) = \frac{N\alpha}{2q} + \frac{\alpha\gamma}{2} < 1.$$

We estimate in L_γ^q as follows,

$$\begin{aligned}
\|\mathcal{F}_{u_0}u(t)\|_{L_\gamma^q} &\leq \|e^{t\Delta}u_0\|_{L_\gamma^q} + \int_0^t \|e^{(t-\sigma)\Delta}[|u(\sigma)|^\alpha u(\sigma)]\|_{L_\gamma^q} d\sigma \\
&\leq K + C \int_0^t \|\cdot\|^{\nu(\alpha+1)} |u(\sigma)|^\alpha u(\sigma)\|_{r/(\alpha+1)} d\sigma \\
&= K + C \int_0^t \|u(\sigma)\|_{L_\nu^r}^{\alpha+1} d\sigma \\
&\leq K + CM^{\alpha+1} \int_0^t \sigma^{-\beta(\nu)(\alpha+1)} d\sigma \\
&\leq K + CM^{\alpha+1} t^{1-\frac{N\alpha}{2q}-\frac{\gamma\alpha}{2}} \int_0^1 \sigma^{-\beta(\nu)(\alpha+1)} d\sigma \\
&\leq K + CM^{\alpha+1} T^{1-\frac{N\alpha}{2q}-\frac{\gamma\alpha}{2}} \int_0^1 \sigma^{-\beta(\nu)(\alpha+1)} d\sigma.
\end{aligned}$$

The condition (4.6) implies that the space $Y_{M,T}^{q,r}$ is preserved by the iterative operator \mathcal{F}_{u_0} . We show similarly the contraction. The proof of the other parts follows as in [2]. So we omit the details. This completes the proof of the theorem. \square

We note that uniqueness in Part (ii) of Theorem 4.1 holds in $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N)) \cap C((0, T]; L_\nu^r(\mathbb{R}^N))$. This follows by similar argument as in [3]. We will not belabor this point further.

Proof of Theorem 1.3. Consider $u_0 = \lambda\varphi$, where $\lambda > 0$ and $\varphi \in L_\gamma^q$. If $T_{\max}(\lambda\varphi) < \infty$, it is impossible to carry out the fixed point argument on the interval $[0, T_{\max}(\lambda\varphi)]$ with initial value $u_0 = \lambda\varphi$. Hence, by (4.6)

$$K + CT_{\max}(\lambda\varphi)^{1-\frac{N\alpha}{2q}-\frac{\gamma\alpha}{2}} M^{\alpha+1} > M,$$

for all $M > K$. Letting $K = \|u_0\|_{L_\gamma^q} = \lambda\|\varphi\|_{L_\gamma^q}$, so that

$$\lambda\|\varphi\|_{L_\gamma^q} + CT_{\max}(\lambda\varphi)^{1-\frac{N\alpha}{2q}-\frac{\gamma\alpha}{2}} M^{\alpha+1} > M,$$

for all $M > \lambda\|\varphi\|_{L_\gamma^q}$. In particular, if we set $M = 2\lambda\|\varphi\|_{L_\gamma^q}$, this gives

$$CT_{\max}(\lambda\varphi)^{1-\frac{N\alpha}{2q}-\frac{\gamma\alpha}{2}} [\lambda\|\varphi\|_{L_\gamma^q}]^\alpha > 1.$$

Thus we have proved Theorem 1.3. \square

Remark 12. If $u_0 \in L_\gamma^q$ with $\gamma > 0$, $q \leq \infty$ are as in Theorem 4.1 then writing

$$\begin{aligned}
|u_0| &= |u_0 \mathbf{1}_{\{|x| \leq 1\}} + u_0 \mathbf{1}_{\{|x| > 1\}}| \\
&\leq |x|^{-\gamma} \mathbf{1}_{\{|x| \leq 1\}} (|x|^\gamma |u_0| \mathbf{1}_{\{|x| \leq 1\}}) + |x|^{-\gamma} \mathbf{1}_{\{|x| > 1\}} (|x|^\gamma |u_0|) \\
&\leq |x|^{-\gamma} \mathbf{1}_{\{|x| \leq 1\}} (|x|^\gamma |u_0| \mathbf{1}_{\{|x| \leq 1\}}) + |x|^\gamma |u_0|,
\end{aligned}$$

where $\mathbf{1}_A$ is the indicator function of a subset A of \mathbb{R}^N , we see by the Hölder inequality that $u_0 \in L^r + L^s$ with $\frac{1}{r} = \frac{1}{q} + \frac{\gamma}{N}$, $s = q$ if $q < \infty$ and $r \geq 1$, $\frac{N\alpha}{2} < r < \frac{N}{\gamma} < s < \infty$ if $q = \infty$. Then the local well-posedness is proved in [11, Theorem 2.8]. The fixed point argument used in [11] seems

not to give an explicit lower bound estimate of life span, as $T < 1$, the minimal local existence time, is required in the proof there and the constants in particular in [11, Inequality (2.10)], seems to depend on T .

The construction of solutions to (2.1) with initial data in the intersection of two metric spaces follows by well-known argument. See also the proof of [2, Proposition 3.2, p. 126]. We have the following result for the existence time of the maximal solution.

Proposition 4.2. *Let $N \geq 1$ be an integer, $\alpha > 0$ and $0 < \gamma < N$, $\gamma < 2/\alpha$. Let $q_c(\gamma)$ be given by (4.1). Let $q > q_c(\gamma)$, $\frac{N}{N-\gamma} < q < \infty$ and $T_{\max}(\varphi, L_\gamma^q)$ denotes the existence time of the maximal solution of (2.1) with initial data $\varphi \in L_\gamma^q$. Then the following hold.*

- (i) $u(t) \in C_0(\mathbb{R}^N) \cap L^\infty$ for $t \in (0, T_{\max}(\varphi, L_\gamma^q))$.
- (ii) If $\varphi \in L_\gamma^q \cap C_0(\mathbb{R}^N)$ then $T_{\max}(\varphi, L_\gamma^q) = T_{\max}(\varphi, C_0(\mathbb{R}^N))$, the existence time of the maximal solution of (2.1) with initial data $\varphi \in C_0(\mathbb{R}^N)$.
- (iii) If $\varphi \in L_\gamma^q \cap L^p$ with $p > q_c(\gamma)$, $\frac{N}{N-\gamma} < p < \infty$ then $T_{\max}(\varphi, L_\gamma^q) = T_{\max}(\varphi, L^p)$, the existence time of the maximal solution of (2.1) with initial data $\varphi \in L^p$.
- (iv) If $\varphi \in L_\gamma^q \cap L^p$ with $q_c < p \leq \infty$, then $T_{\max}(\varphi, L_\gamma^q) = T_{\max}(\varphi, L^p)$, the existence time of the maximal solution of (2.1) with initial data $\varphi \in L^p$.

Proof. (i) Let $\varphi \in L_\gamma^q(\mathbb{R}^N)$, $q > q_c(\gamma)$ and $q > \frac{N}{N-\gamma}$. Let $r = (\alpha + 1)q$, $\nu(\alpha + 1) = \gamma$ and $\beta(\nu)$ be given by (4.5). Let p be such that $r < p \leq \infty$. Hence $p > q$ and

$$0 \leq \frac{1}{p} < \frac{\alpha + 1}{r} < \frac{\gamma}{N} + \frac{\alpha + 1}{r} < 1, \quad \frac{1}{p} < \frac{1}{q} < \frac{\gamma}{N} + \frac{1}{q} < 1.$$

For $0 < T < T_{\max}(\varphi, L_\gamma^q)$, we have

$$\begin{aligned} \|u(t)\|_{L_\gamma^p} &\leq \|e^{t\Delta}\varphi\|_{L_\gamma^p} + C \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})} \|u(\sigma)\|_{L_\gamma^r}^{\alpha+1} d\sigma \\ &\leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \|\varphi\|_{L_\gamma^q} + Ct^{1-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\beta(\nu)(\alpha+1)} \sup_{s \in (0, T]} \left(s^{\beta(\nu)(\alpha+1)} \|u(s)\|_{L_\gamma^r}^{\alpha+1} \right) \times \\ &\quad \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})} \sigma^{-\beta(\nu)(\alpha+1)} d\sigma \\ &\leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \|\varphi\|_{L_\gamma^q} + M^{\alpha+1} Ct^{1-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\frac{\gamma\alpha}{2}} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})} \sigma^{-\beta(\nu)(\alpha+1)} d\sigma. \end{aligned}$$

Since $r > q > q_c(\gamma)$, it follows that if

$$\frac{\alpha + 1}{r} - \frac{2}{N} < \frac{1}{p} < \frac{1}{r},$$

then $u(t)$ is in L_γ^p for all $t \in (0, T_{\max}(\varphi, L_\gamma^q))$. The result for general $p > q$ follows by iteration. Hence $u(t)$ is in L_γ^∞ , for $t \in (0, T_{\max}(\varphi, L_\gamma^q))$. Then $u(t) \in L^r + L^s$ for $r \geq 1$, $\frac{N\alpha}{2} < r < \frac{N}{\gamma} < s < \infty$. Hence by [11, Theorem 2.8] $u(t) \in L^p$ for $s \leq p \leq \infty$. Then it follows that $u(t) \in C_0(\mathbb{R}^N)$, for $t \in (0, T_{\max}(\varphi, L_\gamma^q))$.

(ii) By (i) we have $T_{\max}(\varphi, L_\gamma^q) \leq T_{\max}(\varphi, C_0(\mathbb{R}^N))$. Using (2.1), we have

$$\begin{aligned} \|u(t)\|_{L_\gamma^q} &\leq \|e^{t\Delta}\varphi\|_{L_\gamma^q} + C \int_0^t \| |u(\sigma)|^\alpha u(\sigma) \|_{L_\gamma^q} d\sigma \\ &\leq C\|\varphi\|_{L_\gamma^q} + C \int_0^t \|u(\sigma)\|_\infty^\alpha \|u(\sigma)\|_{L_\gamma^q} d\sigma \end{aligned}$$

By Gronwall's inequality, we get

$$\|u(t)\|_{L_\gamma^q} \leq C\|\varphi\|_{L_\gamma^q} e^{C \int_0^t \|u(\sigma)\|_\infty^\alpha d\sigma}.$$

Hence u can not blow up in L_γ^q before it blows up in $C_0(\mathbb{R}^N)$. That is $T_{\max}(\varphi, C_0(\mathbb{R}^N)) \leq T_{\max}(\varphi, L_\gamma^q)$.

(iii) Let $\varepsilon \in (0, \min(T_{\max}(\varphi, L_\gamma^q), T_{\max}(\varphi, L_\gamma^p)))$. By (i) we have $u(\varepsilon) \in C_0(\mathbb{R}^N)$. Using (ii) we have

$$T_{\max}(u(\varepsilon), L_\gamma^q) = T_{\max}(u(\varepsilon), C_0(\mathbb{R}^N)) = T_{\max}(u(\varepsilon), L_\gamma^p).$$

That is $T_{\max}(\varphi, L_\gamma^q) - \varepsilon = T_{\max}(\varphi, L_\gamma^p) - \varepsilon$, hence we get the result.

(iv) Follows similarly as (iii). This completes the proof of Proposition 4.2. \square

Proof of Corollary 1.4. Since $T_{\max}(\varphi, L^q)$, the maximal existence time in L^q , is equal to $T_{\max}(\varphi, L^q \cap L_\gamma^q)$ the maximal existence time in $L^q \cap L_\gamma^q$, we deduce that

1) If $N\alpha < 2$ we discuss the two cases

(i) $\gamma < N$ hence $\gamma < 2/\alpha$ and we have

$$\frac{1}{q} + \frac{\gamma}{N} < 1, \quad \frac{N\alpha}{2q} + \frac{\gamma\alpha}{2} = \frac{N\alpha}{2} \left(\frac{1}{q} + \frac{\gamma}{N} \right) < \frac{N\alpha}{2} < 1,$$

hence, we apply Theorem 1.3 to get

$$T_{\max}(\lambda\varphi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2q} - \frac{\gamma}{2}\right)^{-1}}.$$

(ii) $\gamma > N$ then $\varphi \in L^1$. In fact, we write $\varphi = \varphi\mathbf{1}_{\{|x| \leq 1\}} + \varphi\mathbf{1}_{\{|x| > 1\}}$. On one hand, since $\varphi \in L^q$, $q > 1$, hence $\varphi\mathbf{1}_{\{|x| \leq 1\}} \in L^1$. On the other hand, by the Hölder inequality,

$$\begin{aligned} \|\varphi\mathbf{1}_{\{|x| > 1\}}\|_1 &= \|\varphi|x|^\gamma|x|^{-\gamma}\mathbf{1}_{\{|x| > 1\}}\|_1 \\ &\leq \|\varphi|x|^\gamma\|_q \|\ |x|^{-\gamma}\mathbf{1}_{\{|x| > 1\}}\|_{q'} \\ &= \|\varphi\|_{L_\gamma^q} \| |x|^{-\gamma}\mathbf{1}_{\{|x| > 1\}} \|_{q'} < \infty, \end{aligned}$$

since $\gamma q' \geq \gamma > N$ and since $\varphi \in L_\gamma^q$, that is $\varphi\mathbf{1}_{\{|x| > 1\}} \in L^1$. Hence both results give that $\varphi \in L^1$. We may then apply Theorem 1.1 in L^1 , using $1 > \frac{N\alpha}{2}$ to get $T_{\max}(\lambda\varphi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2}\right)^{-1}}$.

2) If $N\alpha > 2$ then we assume $\gamma < 2/\alpha$ and we have only one case, $\gamma < N$. Hence since $q > q_c(\gamma)$

we apply Theorem 1.3 to get $T_{\max}(\lambda\varphi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2q} - \frac{\gamma}{2}\right)^{-1}}$.

In the all cases we have $T_{\max}(\lambda\varphi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{1}{2} \min\left(\frac{N}{q} + \gamma, N\right)\right)^{-1}}$. This completes the proof of Corollary 1.4. \square

Proof of Corollary 1.5. Since $\varphi \in L^p \cap L^q_\gamma$, then the existence time of the maximal solution is the same to that in L^p and to that in L^q_γ . By the Hölder inequality $\varphi \in L^r_\tau$ for $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$, $\tau = (1-\theta)\gamma$, $\theta \in [0, 1]$. The existence time of the maximal solution is also the same in L^r_τ . The function $x \in (0, 2/(N\alpha)) \rightarrow \lambda^{-(\frac{1}{\alpha} - \frac{N}{2}x)^{-1}}$ is increasing for $\lambda \in (0, 1)$ and decreasing for $\lambda \in (1, \infty)$. Letting $x = x_\theta = \frac{1}{r} + \frac{\tau}{N} = \theta \left(\frac{1}{p} - \frac{1}{q} - \gamma \right) + \frac{1}{q} + \frac{\gamma}{N}$, we have that $\max_{\theta \in [0, 1]} x_\theta = \max\left(\frac{1}{q} + \frac{\gamma}{N}, \frac{1}{p}\right) = \max(x_0, x_1)$ and $\min_{\theta \in [0, 1]} x_\theta = \min\left(\frac{1}{q} + \frac{\gamma}{N}, \frac{1}{p}\right) = \min(x_0, x_1)$.

Using Theorem 1.3 if $\theta = 0$ or Theorem 1.1 if $\theta = 1$, we have that

$$T_{\max}(\lambda\varphi) \geq C (\lambda \|\varphi\|_{L^r_\tau})^{-(\frac{1}{\alpha} - \frac{N}{2}[\frac{1}{r} + \frac{\tau}{N}])^{-1}} \geq C \left(\lambda \|\varphi\|_{L^p \cap L^q_\gamma} \right)^{-(\frac{1}{\alpha} - \frac{N}{2}[\frac{1}{r} + \frac{\tau}{N}])^{-1}}.$$

The result follows then by taking in the last inequality $\max_{\theta \in [0, 1]} x_\theta$, for $\lambda \in (0, 1)$ and $\min_{\theta \in [0, 1]} x_\theta$ for $\lambda \in (1, \infty)$. This completes the proof of the Corollary. \square

Example 4.3. Let $0 < \gamma < N$ and $\gamma < 2/\alpha$. Let $\tilde{\varphi}$ be given by (1.16). Then $\tilde{\varphi} \in L^p \cap L^q_\gamma$ with $\frac{N\alpha}{2} < p < \frac{N}{\gamma}$. By Corollary 1.5 we have

$$T_{\max}(\lambda\varphi) \geq C \begin{cases} \lambda^{-(\frac{1}{\alpha} - \frac{N}{2p})^{-1}}, & \text{if } 0 < \lambda \leq 1, \\ \lambda^{-(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}}, & \text{if } \lambda > 1. \end{cases}$$

5. UPPER BOUNDS FOR NONNEGATIVE SOLUTIONS

In this section we exploit a well-known necessary condition for the existence of a nonnegative solution to (2.1). More precisely, if u is a nonnegative solution of the integral equation (2.1) on $(0, T) \times \Omega$ then

$$\alpha t (e^{t\Delta} u_0)^\alpha \leq 1, \tag{5.1}$$

for all $t \in (0, T]$, where $u_0 \geq 0$ can be either a locally integrable function or a positive Borel measure on Ω . See [56, Theorem 1].

We let $T_{\max}(u_0)$ denote the maximal existence time of a nonnegative solution of (2.1), and so $0 \leq T_{\max}(u_0) \leq \infty$. Indeed, there are three possibilities, all of which can be realized: there is no local nonnegative solution with initial value u_0 , there is at least one local solution on some interval $(0, T)$, but no global solution, i.e. on $(0, \infty)$, or there is indeed a global solution. In the case $u_0 = \lambda\varphi$, then (5.1) becomes

$$\alpha \lambda^\alpha t (e^{t\Delta} \varphi)^\alpha \leq 1, \tag{5.2}$$

for all $t \in (0, T]$. If $\varphi \geq 0$, $\varphi \not\equiv 0$, this implies that $T_{\max}(\lambda\varphi) < \infty$ for all sufficiently large $\lambda > 0$ and that

$$\lim_{\lambda \rightarrow \infty} T_{\max}(\lambda\varphi) = 0. \tag{5.3}$$

Indeed, given any $t > 0$, (5.2) can not be true for sufficiently large $\lambda > 0$, and so $t \geq T_{\max}(\lambda\varphi)$ for sufficiently large $\lambda > 0$. This shows the first statements of Theorems 1.6, 1.7 and 1.10.

It is important to realize that $T_{\max}(u_0)$ as just defined, i.e. the maximal existence time of a nonnegative solution, is not necessarily the same as the maximal existence time of a regular nonnegative solution. Indeed, in some cases, a nonnegative solution can be continued after blowup.

See [1] for example. However, any upper bound on $T_{\max}(u_0)$ is also an upper bound on the maximal existence time of a regular nonnegative solution.

Proposition 5.1. *Let $u_0 \geq 0$ be either a locally integrable function or a positive Borel measure on Ω , and let $T_{\max}(u_0)$ denote the maximal existence time of a nonnegative solution of (2.1). If $0 < T_{\max}(u_0) < \infty$, then*

$$\alpha T_{\max}(u_0) (e^{T_{\max}(u_0)\Delta} u_0)^\alpha \leq 1. \quad (5.4)$$

In particular, if $u_0 = \lambda\varphi$, then

$$\alpha \lambda^\alpha T_{\max}(\lambda\varphi) \|e^{T_{\max}(\lambda\varphi)\Delta} \varphi\|_\infty^\alpha \leq 1. \quad (5.5)$$

Proof. Inequality (5.1) is true for all $0 < t < T_{\max}(u_0)$. Hence it is true for $t = T_{\max}(u_0)$. \square

We now give the proofs of the upper bounds.

Proof of Theorem 1.6. Since $T_{\max}(\lambda\varphi) \rightarrow 0$ as $\lambda \rightarrow \infty$, it suffices by (5.5) to observe that if $\varphi \in L^\infty(\Omega)$, then $\|e^{t\Delta}\varphi\|_\infty \rightarrow \|\varphi\|_\infty$ as $t \rightarrow 0$. Indeed, $\|e^{t\Delta}\varphi\|_\infty \leq \|\varphi\|_\infty$ so $\limsup_{t \rightarrow 0} \|e^{t\Delta}\varphi\|_\infty \leq \|\varphi\|_\infty$. On the other hand, $e^{t\Delta}\varphi \rightarrow \varphi$ weak* as $t \rightarrow 0$, so $\|\varphi\|_\infty \leq \liminf_{t \rightarrow 0} \|e^{t\Delta}\varphi\|_\infty$. \square

Proof of Theorem 1.7. In order to estimate $T_{\max}(\lambda\varphi)$ from above, it suffices to estimate $T_{\max}(\lambda\tilde{\varphi})$ from above, where $\tilde{\varphi}$ is defined in (1.16). Indeed, since $0 \leq \tilde{\varphi}(x) \leq \varphi(x)$ it follows that $T_{\max}(\lambda\varphi) \leq T_{\max}(\lambda\tilde{\varphi})$.

To find an upper estimate on $T_{\max}(\lambda\tilde{\varphi})$ as $\lambda \rightarrow \infty$, it suffices by Proposition 5.1 to determine the behavior of $\|e^{t\Delta}\tilde{\varphi}\|_\infty$ as $t \rightarrow 0$. Let D_τ be the dilation operator $D_\tau f(x) = f(\tau x)$. We have

$$\|e^{t\Delta}\tilde{\varphi}\|_\infty = \|D_{\sqrt{t}} e^{t\Delta}\tilde{\varphi}\|_\infty = \|e^\Delta D_{\sqrt{t}}\tilde{\varphi}\|_\infty = t^{-\frac{\gamma}{2}} \|e^\Delta [t^{\frac{\gamma}{2}} D_{\sqrt{t}}\tilde{\varphi}]\|_\infty. \quad (5.6)$$

Since $t^{\frac{\gamma}{2}} D_{\sqrt{t}}\tilde{\varphi} \rightarrow \omega |\cdot|^{-\gamma}$ in $\mathcal{D}'(\mathbb{R}^N)$ as $t \rightarrow 0$, it follows by [7, Proposition 3.8 (i), page 1123] that

$$t^{\frac{\gamma}{2}} \|e^{t\Delta}\tilde{\varphi}\|_\infty \rightarrow \|e^\Delta(\omega |\cdot|^{-\gamma})\|_\infty,$$

as $t \rightarrow 0$. Since by (5.3), $T_{\max}(\lambda\varphi) \rightarrow 0$ as $\lambda \rightarrow \infty$, this along with (5.5) implies

$$\limsup_{\lambda \rightarrow \infty} \lambda^\alpha T_{\max}(\lambda\varphi)^{1-\frac{\alpha\gamma}{2}} \leq \frac{1}{\alpha \|e^\Delta(\omega |\cdot|^{-\gamma})\|_\infty^\alpha},$$

which is the desired result. \square

Proof of Theorem 1.8. Applying (5.5) we see that

$$\alpha \lambda^\alpha T_{\max}(\lambda\mathbf{m}) \|e^{T_{\max}(\lambda\mathbf{m})\Delta} \mathbf{m}\|_\infty^\alpha \leq 1.$$

Furthermore,

$$\|e^{t\Delta}\mathbf{m}\|_\infty = \|D_{\sqrt{t}} e^{t\Delta}\mathbf{m}\|_\infty = \|e^\Delta D_{\sqrt{t}}\mathbf{m}\|_\infty = t^{-\frac{N}{2}} \|e^\Delta [t^{\frac{N}{2}} D_{\sqrt{t}}\mathbf{m}]\|_\infty$$

so that

$$\alpha \lambda^\alpha T_{\max}(\lambda\mathbf{m})^{1-\frac{N\alpha}{2}} \|e^\Delta [T_{\max}(\lambda\mathbf{m})^{\frac{N}{2}} D_{\sqrt{T_{\max}(\lambda\mathbf{m})}}\mathbf{m}]\|_\infty^\alpha \leq 1. \quad (5.7)$$

The result follows since $T_{\max}(\lambda \mathbf{m}) \rightarrow \infty$ as $\lambda \rightarrow 0$ (by continuous dependence or Theorem 1.2) and $t^{\frac{N}{2}} D_{\sqrt{t}} \mathbf{m} \rightarrow \|\mathbf{m}\|_{\mathcal{M}} \delta$ as $t \rightarrow \infty$. In fact, $\mu_t = t^{N/2} D_{\sqrt{t}} \mathbf{m}$ is the measure defined by

$$\int_{\mathbb{R}^N} f(x) d\mu_t(x) dx = \int_{\mathbb{R}^N} D_{1/\sqrt{t}} f(x) d\mathbf{m}(x), \quad f \in C_0(\mathbb{R}^N).$$

We have $D_{1/\sqrt{t}} f \rightarrow f(0)$ as $t \rightarrow \infty$ \mathbf{m} a.e. Since \mathbf{m} is finite, then by the dominated convergence theorem $\int_{\mathbb{R}^N} f(x) d\mu_t(x) dx \rightarrow \|\mathbf{m}\|_{\mathcal{M}} \delta$ as $t \rightarrow \infty$ for every f in $C_0(\mathbb{R}^N)$. Then $t^{\frac{N}{2}} D_{\sqrt{t}} \mathbf{m} \rightarrow \|\mathbf{m}\|_{\mathcal{M}} \delta$ as $t \rightarrow \infty$ in the dual space $(C_0(\mathbb{R}^N))'$. We know that $e^\Delta : L^1(\mathbb{R}^N) \rightarrow C_0(\mathbb{R}^N)$ is a continuous operator then, by duality, $e^\Delta : (C_0(\mathbb{R}^N))' \rightarrow (L^1(\mathbb{R}^N))' = L^\infty(\mathbb{R}^N)$, is a continuous operator. Hence $\|e^\Delta [T_{\max}(\lambda \mathbf{m})^{\frac{N}{2}} D_{\sqrt{T_{\max}(\lambda \mathbf{m})}} \mathbf{m}]\|_\infty$ converges to $\|\mathbf{m}\|_{\mathcal{M}} \|e^\Delta \delta\|_\infty = \|\mathbf{m}\|_{\mathcal{M}} (4\pi)^{-N/2}$ as $\lambda \rightarrow 0$. This along with (5.7) implies

$$\limsup_{\lambda \rightarrow 0} \lambda^\alpha T_{\max}(\lambda \mathbf{m})^{1 - \frac{N\alpha}{2}} \leq \frac{1}{(\alpha^{1/\alpha} (4\pi)^{-N/2} \|\mathbf{m}\|_{\mathcal{M}})^\alpha}.$$

This gives the desired result. \square

Proof of Theorem 1.9. If φ is too singular, it may happen that $T_{\max}(\lambda \varphi) = 0$, i.e. there is no local nonnegative solution with initial value $\lambda \varphi$. This is not a problem, since we will be obtaining upper bounds.

Since $\varphi \geq \tilde{\varphi}$, where $\tilde{\varphi}$ is defined in (1.20), it suffices to estimate $T_{\max}(\lambda \tilde{\varphi})$. The calculation in (5.6) gives

$$\|e^{t\Delta} \tilde{\varphi}\|_\infty = t^{-\frac{\gamma}{2}} \|e^\Delta [t^{\frac{\gamma}{2}} D_{\sqrt{t}} \tilde{\varphi}]\|_\infty.$$

Moreover, $t^{\frac{\gamma}{2}} D_{\sqrt{t}} \tilde{\varphi} \rightarrow \omega \cdot |\cdot|^{-\gamma}$ as $t \rightarrow \infty$ in $\mathcal{D}'(\mathbb{R}^N)$. It follows, by [7, Proposition 3.8 (i), page 1123] that

$$t^{\frac{\gamma}{2}} \|e^{t\Delta} \tilde{\varphi}\|_\infty = \|e^\Delta [t^{\frac{\gamma}{2}} D_{\sqrt{t}} \tilde{\varphi}]\|_\infty \rightarrow \|e^\Delta [\omega \cdot |\cdot|^{-\gamma}]\|_\infty,$$

as $t \rightarrow \infty$. If $0 < t < T_{\max}(\lambda \tilde{\varphi})$, then by (5.2) we must have

$$\alpha \lambda^\alpha t^{1 - \frac{\alpha\gamma}{2}} [t^{\frac{\gamma}{2}} \|e^{t\Delta} \tilde{\varphi}\|_\infty]^\alpha \leq 1. \quad (5.8)$$

It follows that if $\gamma < \frac{2}{\alpha}$, then $T_{\max}(\lambda \tilde{\varphi}) < \infty$ for all $\lambda > 0$. This is of course a consequence of Fujita's result (including the limiting case) if $\alpha \leq \frac{2}{N}$. (See for example [55].) If $\alpha > \frac{2}{N}$, i.e. $q_c > 1$, this is a consequence of the more general result [50, Theorem 1.7] in the case $m = 0$. (See also [27, Theorem 3.2(i)] and [44, Theorem 2].) In these cases, putting $t = T_{\max}(\lambda \tilde{\varphi})$ in (5.8) and letting $\lambda \rightarrow 0$, we obtain

$$\limsup_{\lambda \rightarrow 0} \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \leq \frac{1}{(\alpha^{1/\alpha} \|e^\Delta (\omega \cdot |\cdot|^{-\gamma})\|_\infty)^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}}}.$$

\square

Proof of Theorem 1.10. The proof follows similarly as that of Theorem 1.7, replacing \mathbb{R}^N by Ω_m , $|x|^{-\gamma}$ by ψ_0 hence γ by $\gamma + m$, and using [35, Proposition 4.1 (ii), p. 359], for the convergence. \square

Proof of Theorem 1.11. The proof follows similarly as that of Theorem 1.9 by replacing \mathbb{R}^N by Ω_m , $|x|^{-\gamma}$ by ψ_0 , hence γ by $\gamma + m$, using [50, Theorem 1.7] for the blow up of the solution and [35, Proposition 4.1 (ii), p. 359], for the convergence. \square

6. LIFE-SPAN ESTIMATES VIA NONLINEAR SCALING

In this section we show how certain scaling arguments can give upper (and lower) life-span bounds for solutions of (2.1) on \mathbb{R}^N . Similar arguments can be used on sectors of \mathbb{R}^N . The previous section likewise used scaling arguments, but only in regard to properties of $e^{t\Delta}\varphi$. In this section, we use nonlinear scaling arguments, which can then be adapted to other equations which are scale invariant. Some of the results in this section are the same as in the previous section, but obtained by a different method.

We begin with some observations in a general context. We consider an evolution partial differential equation defined either on \mathbb{R}^N or on some domain Ω which is a cone, i.e. if $x \in \Omega$ then $\mu x \in \Omega$ for all $\mu > 0$. We also suppose that the set of solutions of the evolution equation is invariant under the transformation

$$u_\mu(t, x) = \mu^\sigma u(\mu^2 t, \mu x). \quad (6.1)$$

In other words, u is a solution if and only if u_μ is a solution for all $\mu > 0$. If u has initial value $u(0, \cdot) = u_0$, then u_μ has initial value $u_\mu(0, \cdot) = \mu^\sigma u_0(\mu \cdot) = \mu^\sigma D_\mu u_0 \equiv u_{\mu,0}$. It is clear that

$$T_{\max}(u_{\mu,0}) = T_{\max}(\mu^\sigma D_\mu u_0) = \frac{1}{\mu^2} T_{\max}(u_0)$$

If $u_0 = \lambda\varphi$, it follows that $u_{\mu,0} = \lambda\mu^\sigma D_\mu\varphi$, so that

$$\mu^{-2} T_{\max}(\lambda\varphi) = T_{\max}(\lambda\mu^\sigma D_\mu\varphi).$$

Now let us suppose that φ has certain properties with respect to a scaling different from that of the equation, for example $\mu^\gamma D_\mu\varphi$ where $\gamma \neq \sigma$. If so, we may set

$$\lambda = \mu^{\gamma-\sigma}, \quad (6.2)$$

hence $\mu = \lambda^{\frac{1}{\gamma-\sigma}}$, $\mu^{-2} = \lambda^{\frac{2}{\sigma-\gamma}}$, so that

$$\lambda^{\frac{2}{\sigma-\gamma}} T_{\max}(\lambda\varphi) = T_{\max}(\mu^\gamma D_\mu\varphi). \quad (6.3)$$

In the simplest case, $\mu^\gamma D_\mu\varphi \equiv \varphi$, i.e. φ is homogeneous of degree $-\gamma$, we have therefore the following formal proposition.

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain which is also a cone. Suppose that the solutions of an evolution equation (the set of trajectories of a dynamical system over Ω) are invariant under the transformation (6.1). If $\varphi \in L^1_{loc}(\Omega)$ or $\varphi \in \mathcal{M}(\Omega)$ is homogeneous of degree $-\gamma$, where $\gamma \neq \sigma$, then*

$$\lambda^{\frac{2}{\sigma-\gamma}} T_{\max}(\lambda\varphi) = T_{\max}(\varphi)$$

for all $\lambda > 0$.

In the case of the nonlinear heat equation, $\sigma = \frac{2}{\alpha}$ and so (6.2) and (6.3) become

$$\lambda = \mu^{\gamma-\frac{2}{\alpha}}, \quad \lambda^{\left(\frac{1}{\alpha}-\frac{\gamma}{2}\right)^{-1}} T_{\max}(\lambda\varphi) = T_{\max}(\mu^\gamma D_\mu\varphi). \quad (6.4)$$

We immediately deduce the following.

Corollary 6.2. *Let $T_{\max}(u_0)$ denote the maximal solution to (2.1) on \mathbb{R}^N with initial value u_0 .*

(i) If $\alpha < \frac{2}{N}$, then

$$\lambda^{(\frac{1}{\alpha} - \frac{N}{2})^{-1}} T_{\max}(\lambda\delta) = T_{\max}(\delta)$$

for all $\lambda > 0$.

(ii) If $0 < \gamma < N$ and $\gamma < \frac{2}{\alpha}$, and if $\psi(x) = \omega(x)|x|^{-\gamma}$ where $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0, then

$$\lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda\psi) = T_{\max}(\psi)$$

for all $\lambda > 0$.

There are two other possibilities which allow us to obtain life-span estimates. On the one hand, it could be that $\mu^\gamma D_\mu \varphi$ has a limit as $\mu \rightarrow 0$ or as $\mu \rightarrow \infty$, possibly along a subsequence. If one can control $T_{\max}(\mu^\gamma D_\mu \varphi)$ as this limit is attained, one obtains a corresponding life-span estimate from (6.3). This procedure was introduced in the paper [11]. For results of this type, we refer the reader to [11, Theorems 1.3, 1.4, 1.5] and [50, Theorems 1.9, 1.10, 1.12, Corollary 1.13, Propositions 4.5, 4.6]. As these latter results show, one can have different life-span behaviors along different subsequences, either as $\lambda \rightarrow 0$ or as $\lambda \rightarrow \infty$. We recall that all of these results depend on delicate continuity properties of the blowup time.

The other approach uses comparison. As a first, and simple, example, we have the following immediate consequence of Corollary 6.2.

Corollary 6.3. *If $0 < \gamma < N$ and $\gamma < \frac{2}{\alpha}$, and if $\varphi \in L^1_{loc}(\mathbb{R}^N)$, $|\varphi(x)| \leq \omega(x)|x|^{-\gamma}$ where $\omega \in L^\infty(\mathbb{R}^N)$, $\omega \geq 0$ is homogeneous of degree 0, then*

$$\lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda\varphi) \geq T_{\max}(\omega|\cdot|^{-\gamma})$$

for all $\lambda > 0$.

This is essentially the same as Corollary 2.3. Theorem 1.3 gives also the result but here the constant at the right-hand side is explicit.

Proof. The absolute value of the solution with initial value $\lambda\varphi$ is bounded above by the solution with initial value $\lambda\omega|\cdot|^{-\gamma}$. We then apply the second assertion of Corollary 6.2. \square

We have the following for the function $\tilde{\varphi}$ given by (1.16).

Corollary 6.4. *Let $N \geq 1$, $\alpha > 0$, $0 < \gamma < N$, $\gamma < \frac{2}{\alpha}$, $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0, $\omega \geq 0$, $\omega \not\equiv 0$ and $\tilde{\varphi}$ be given by (1.16). Then the following hold.*

(i) *There exists $T_1 \in [T_{\max}(\omega|\cdot|^{-\gamma}), T_{\max}(\tilde{\varphi})]$ such that*

$$\lim_{\lambda \rightarrow \infty} \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda\tilde{\varphi}) = T_1.$$

(ii) $\lim_{\lambda \rightarrow 0} \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda\tilde{\varphi}) = \infty$.

Proof. (i) The function $\mu \rightarrow \mu^\gamma D_\mu \tilde{\varphi}$ is decreasing on $(0, \infty)$, and

$$\lim_{\mu \rightarrow 0} \mu^\gamma D_\mu \tilde{\varphi} = \omega|\cdot|^{-\gamma}, \quad \lim_{\mu \rightarrow \infty} \mu^\gamma D_\mu \tilde{\varphi} = 0,$$

where the first limit is realized in $L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$ and the second in $L^{q_1}(\mathbb{R}^N)$ whenever $0 \leq \frac{N}{q_2} < \gamma < \frac{N}{q_1} \leq N$. Consequently

$$\tilde{\varphi} \leq \mu^\gamma D_\mu \tilde{\varphi} \leq \omega |\cdot|^{-\gamma}, \quad \forall \mu \leq 1.$$

Applying (6.4), and since $\gamma < 2/\alpha$, we conclude that $\lambda \rightarrow \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi})$ is decreasing on $(0, \infty)$ and

$$T_{\max}(\omega |\cdot|^{-\gamma}) \leq \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \leq T_{\max}(\tilde{\varphi}), \quad \forall \lambda \geq 1.$$

The existence of the limit T_1 follows by monotonicity.

(ii) We have that $\tilde{\varphi} \in L^q$ for $q \geq 1$, $\frac{N\alpha}{2} < q < \frac{N}{\gamma}$. Hence, by Theorem 1.1

$$T_{\max}(\lambda \tilde{\varphi}) \geq C \lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2q}\right)^{-1}}.$$

Then

$$\lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \geq C \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1} - \left(\frac{1}{\alpha} - \frac{N}{2q}\right)^{-1}},$$

that is

$$\lim_{\lambda \rightarrow 0} \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) = \infty.$$

For the second assertion, we may also use the continuous dependence in $L^q(\mathbb{R}^N)$, where $\gamma < \frac{N}{q} < \frac{2}{\alpha}$. In fact, since $\lim_{\mu \rightarrow \infty} \mu^\gamma D_\mu \tilde{\varphi} = 0$, we know that $T_{\max}(\mu^\gamma D_\mu \tilde{\varphi}) \rightarrow \infty$ as $\mu \rightarrow \infty$; so that by (6.4) we have

$$\lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \rightarrow \infty, \quad \text{as } \lambda \rightarrow 0.$$

□

Remark 13.

- 1) It is natural to conjecture that $\lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \rightarrow T_{\max}(\omega |\cdot|^{-\gamma})$, as $\lambda \rightarrow \infty$. This holds in particular for $(N-2)\alpha < 4$, by continuous dependence of the maximal time of existence.
- 2) We remark that the upper bound on $\lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi})$ for large $\lambda > 0$ is of the same order as given in Theorem 1.7. As for small $\lambda > 0$, if $\alpha < \frac{2}{N}$, then Theorem 1.2 gives the stronger estimate $\lambda^{(\frac{1}{\alpha} - \frac{N}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \geq c > 0$ for all $\lambda > 0$, and is of the same order as given in Theorem 1.8 and Remark 6. However, Part (ii) improves the estimate of Corollary 2.3 in the case $\alpha = \frac{2}{N}$, where $T_{\max}(\lambda \tilde{\varphi}) < \infty$ for all $\lambda > 0$. If $\alpha > \frac{2}{N}$, then $T_{\max}(\lambda \tilde{\varphi}) = \infty$ for sufficiently small $\lambda > 0$, since $\tilde{\varphi} \in L^{q_c}(\mathbb{R}^N)$, see [55].

For the function $\tilde{\tilde{\varphi}}$, we have the following.

Corollary 6.5. *Let $N \geq 1$, $\alpha > 0$, $0 < \gamma < N$, $\gamma < \frac{2}{\alpha}$, $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0, $\omega \geq 0$, $\omega \not\equiv 0$ and $\tilde{\tilde{\varphi}}$ be given by (1.20). Then the following hold.*

(i) *There exists $T_2 \in [T_{\max}(\omega |\cdot|^{-\gamma}), T_{\max}(\tilde{\tilde{\varphi}})]$ such that*

$$\lim_{\lambda \rightarrow 0} \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\tilde{\varphi}}) = T_2.$$

(ii) $\lim_{\lambda \rightarrow \infty} \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\tilde{\varphi}}) = \infty$.

Proof. (i) The function $\mu \rightarrow \mu^\gamma D_\mu \tilde{\varphi}$ is increasing on $(0, \infty)$, and

$$\lim_{\mu \rightarrow 0} \mu^\gamma D_\mu \tilde{\varphi} = 0, \quad \lim_{\mu \rightarrow \infty} \mu^\gamma D_\mu \tilde{\varphi} = \omega \cdot |\cdot|^{-\gamma},$$

where the first limit is realized in $L^{q_2}(\mathbb{R}^N)$ and the second in $L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$ whenever $0 \leq \frac{N}{q_2} < \gamma < \frac{N}{q_1} \leq N$. Consequently

$$\tilde{\varphi} \leq \mu^\gamma D_\mu \tilde{\varphi} \leq \omega \cdot |\cdot|^{-\gamma}, \quad \forall \mu \geq 1.$$

Applying (6.4) we conclude that $\lambda \rightarrow \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi})$ is increasing on $(0, \infty)$ and

$$T_{\max}(\omega \cdot |\cdot|^{-\gamma}) \leq \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \leq T_{\max}(\tilde{\varphi}), \quad \forall \lambda \leq 1.$$

(ii) By continuous dependence in $L^\infty(\mathbb{R}^N)$, we know that $T_{\max}(\mu^\gamma D_\mu \tilde{\varphi}) \rightarrow \infty$ as $\mu \rightarrow 0$; so that by (6.4) we have

$$\lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \rightarrow \infty, \quad \text{as } \lambda \rightarrow \infty.$$

For the second assertion, we may also use the following argument. We have that $\tilde{\varphi} \in L_\gamma^\infty \cap L^\infty$. That is, by Corollary 1.4

$$T_{\max}(\lambda \tilde{\varphi}) \geq C \lambda^{-(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}}$$

and by Theorem 1.1

$$T_{\max}(\lambda \tilde{\varphi}) \geq C \lambda^{-(\frac{1}{\alpha})^{-1}}.$$

Then

$$\lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \geq \max \left(C, C \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1} - (\frac{1}{\alpha})^{-1}} \right).$$

Hence

$$\lim_{\lambda \rightarrow \infty} \lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) = \infty.$$

□

Remark 14.

- 1) It is natural to conjecture that $\lambda^{(\frac{1}{\alpha} - \frac{\gamma}{2})^{-1}} T_{\max}(\lambda \tilde{\varphi}) \rightarrow T_{\max}(\omega \cdot |\cdot|^{-\gamma})$, as $\lambda \rightarrow 0$.
- 2) We remark that $\tilde{\varphi} \in L^\infty(\mathbb{R}^N)$, and Theorem 1.1 and Theorem 1.6 give the precise order of magnitude of $T_{\max}(\lambda \tilde{\varphi})$ as $\lambda \rightarrow \infty$.

Next, we consider the function Φ given by (1.26). We have the following.

Corollary 6.6. *Let $N \geq 1$, $\alpha > 0$, $0 < \gamma_1, \gamma_2 < N$ and $\gamma_1, \gamma_2 < \frac{2}{\alpha}$ ($\gamma_1 \neq \gamma_2$). Let $\omega \in L^\infty(\mathbb{R}^N)$ is homogeneous of degree 0, $\omega \geq 0$, $\omega \not\equiv 0$ and let Φ be defined by (1.26). Then we have the following.*

- (i) *There exists $T_3 \in (T_{\max}(\Phi), T_{\max}(\omega \cdot |\cdot|^{-\gamma_1})]$ such that*

$$\lim_{\lambda \rightarrow \infty} \lambda^{(\frac{1}{\alpha} - \frac{\gamma_1}{2})^{-1}} T_{\max}(\lambda \Phi) = T_3$$

and if $\alpha < \frac{4}{N-2}$, or $\gamma_1 > \gamma_2$ then

$$T_3 = T_{\max}(\omega \cdot |\cdot|^{-\gamma_1}).$$

(ii) There exists $\tilde{T}_3 \in (T_{\max}(\Phi), T_{\max}(\omega|\cdot|^{-\gamma_2})]$ such that

$$\lim_{\lambda \rightarrow 0} \lambda^{(\frac{1}{\alpha} - \frac{\gamma_2}{2})^{-1}} T_{\max}(\lambda\Phi) = \tilde{T}_3$$

and if $\alpha < \frac{4}{N-2}$, or $\gamma_1 > \gamma_2$ then

$$\tilde{T}_3 = T_{\max}(\omega|\cdot|^{-\gamma_2}).$$

(iii) If $\gamma_1 < \gamma_2$, then $\lim_{\lambda \rightarrow 0} \lambda^{(\frac{1}{\alpha} - \frac{\gamma_1}{2})^{-1}} T_{\max}(\lambda\Phi) = \lim_{\lambda \rightarrow \infty} \lambda^{(\frac{1}{\alpha} - \frac{\gamma_2}{2})^{-1}} T_{\max}(\lambda\Phi) = \infty$.

Remark 15. Corollary 6.6 shows that the asymptotic behavior of the life-span as $\lambda \rightarrow \infty$ is determined by the singularity of the initial data and when $\lambda \rightarrow 0$ it is determined by the decay rate at infinity of the initial value.

Proof of Corollary 6.6.

1) Analysis of $T_{\max}(\lambda\Phi)$ in the case $\gamma_1 < \gamma_2$. In this case $\Phi = \omega \min[|\cdot|^{-\gamma_1}, |\cdot|^{-\gamma_2}]$, so $\Phi \leq \omega|\cdot|^{-\gamma_1}$ and $\Phi \leq \omega|\cdot|^{-\gamma_2}$. Hence

$$\mu^{\gamma_1} D_\mu \Phi \leq \omega|\cdot|^{-\gamma_1} \quad (6.5)$$

and

$$\mu^{\gamma_2} D_\mu \Phi \leq \omega|\cdot|^{-\gamma_2} \quad (6.6)$$

for all $\mu > 0$.

We claim that

- The function $\mu \rightarrow \mu^{\gamma_1} D_\mu \Phi$ is decreasing on $(0, \infty)$, and

$$\lim_{\mu \rightarrow 0} \mu^{\gamma_1} D_\mu \Phi = \omega|\cdot|^{-\gamma_1}, \quad \lim_{\mu \rightarrow \infty} \mu^{\gamma_1} D_\mu \Phi = 0,$$

where the limits are in $L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$ whenever $0 \leq \frac{N}{q_2} < \gamma_1 < \frac{N}{q_1} \leq N$, by (6.5).

- The function $\mu \rightarrow \mu^{\gamma_2} D_\mu \Phi$ is increasing on $(0, \infty)$, and

$$\lim_{\mu \rightarrow 0} \mu^{\gamma_2} D_\mu \Phi = 0, \quad \lim_{\mu \rightarrow \infty} \mu^{\gamma_2} D_\mu \Phi = \omega|\cdot|^{-\gamma_2},$$

where the limits are in $L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$ whenever $0 \leq \frac{N}{q_2} < \gamma_2 < \frac{N}{q_1} \leq N$, by (6.6).

Proof of the claim. Let $0 < \mu < \nu < \infty$. In particular, $\mu^{\gamma_1 - \gamma_2} > \nu^{\gamma_1 - \gamma_2}$. We have

$$\begin{aligned} \mu^{\gamma_1} \Phi(\mu x) &= \begin{cases} \omega(x)|x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \omega(x)|x|^{-\gamma_1}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \mu^{\gamma_1 - \gamma_2} \omega(x)|x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} = \begin{cases} \omega(x)|x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ (\frac{1}{|x|})^{\gamma_1 - \gamma_2} \omega(x)|x|^{-\gamma_2}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \mu^{\gamma_1 - \gamma_2} \omega(x)|x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} \\ &\geq \begin{cases} \omega(x)|x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \nu^{\gamma_1 - \gamma_2} \omega(x)|x|^{-\gamma_2}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \nu^{\gamma_1 - \gamma_2} \omega(x)|x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} = \nu^{\gamma_1} \Phi(\nu x). \end{aligned}$$

Also,

$$\begin{aligned} \mu^{\gamma_2} \Phi(\mu x) &= \begin{cases} \mu^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \mu^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_1}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \omega(x) |x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} = \begin{cases} \mu^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \mu^{\gamma_2 - \gamma_1} |x|^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_2}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \omega(x) |x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} \\ &\leq \begin{cases} \nu^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \omega(x) |x|^{-\gamma_2}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \omega(x) |x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} = \nu^{\gamma_1} \Phi(\nu x). \end{aligned}$$

It follows that

- The function $\mu \rightarrow T_{\max}(\mu^{\gamma_1} D_{\mu} \Phi)$ is increasing on $(0, \infty)$, and

$$T_{\max}(\Phi) \geq \lim_{\mu \rightarrow 0} T_{\max}(\mu^{\gamma_1} D_{\mu} \Phi) \geq T_{\max}(\omega | \cdot |^{-\gamma_1}),$$

and if $\alpha < \frac{4}{N-2}$, then

$$\lim_{\mu \rightarrow 0} T_{\max}(\mu^{\gamma_1} D_{\mu} \Phi) = T_{\max}(\omega | \cdot |^{-\gamma_1}).$$

Also

$$\lim_{\mu \rightarrow \infty} T_{\max}(\mu^{\gamma_1} D_{\mu} \Phi) = \infty.$$

- The function $\mu \rightarrow T_{\max}(\mu^{\gamma_2} D_{\mu} \Phi)$ is decreasing on $(0, \infty)$, and

$$T_{\max}(\Phi) \geq \lim_{\mu \rightarrow \infty} T_{\max}(\mu^{\gamma_2} D_{\mu} \Phi) \geq T_{\max}(\omega | \cdot |^{-\gamma_2}),$$

and if $\alpha < \frac{4}{N-2}$, then

$$\lim_{\mu \rightarrow \infty} T_{\max}(\mu^{\gamma_2} D_{\mu} \Phi) = T_{\max}(\omega | \cdot |^{-\gamma_2}).$$

Also

$$\lim_{\mu \rightarrow 0} T_{\max}(\mu^{\gamma_2} D_{\mu} \Phi) = \infty.$$

Next, applying (6.4), we have

$$\lambda^{\left(\frac{1}{\alpha} - \frac{\gamma_1}{2}\right)^{-1}} T_{\max}(\lambda \Phi) = T_{\max}(\mu^{\gamma_1} D_{\mu} \Phi), \quad \lambda = \mu^{\gamma_1 - \frac{2}{\alpha}}, \quad (6.7)$$

$$\lambda^{\left(\frac{1}{\alpha} - \frac{\gamma_2}{2}\right)^{-1}} T_{\max}(\lambda \Phi) = T_{\max}(\mu^{\gamma_2} D_{\mu} \Phi), \quad \lambda = \mu^{\gamma_2 - \frac{2}{\alpha}}. \quad (6.8)$$

This completes the proof of (i)-(ii) if $\gamma_1 < \gamma_2$ and (iii).

2) Analysis of $T_{\max}(\lambda \Phi)$ in the case $\gamma_1 > \gamma_2$.

In this case $\Phi = \omega \max[|\cdot|^{-\gamma_1}, |\cdot|^{-\gamma_2}]$, so $\Phi \geq \omega |\cdot|^{-\gamma_1}$, $\Phi \geq \omega |\cdot|^{-\gamma_2}$ and $\Phi \leq \omega(|\cdot|^{-\gamma_1} + |\cdot|^{-\gamma_2})$.

Hence

$$\omega |\cdot|^{-\gamma_1} \leq \mu^{\gamma_1} D_{\mu} \Phi \leq \omega(|\cdot|^{-\gamma_1} + \mu^{\gamma_1 - \gamma_2} |\cdot|^{-\gamma_2}) \quad (6.9)$$

and

$$\omega |\cdot|^{-\gamma_2} \leq \mu^{\gamma_2} D_{\mu} \Phi \leq \omega(\mu^{\gamma_2 - \gamma_1} |\cdot|^{-\gamma_1} + |\cdot|^{-\gamma_2}) \quad (6.10)$$

for all $\mu > 0$.

We claim that

- The function $\mu \rightarrow \mu^{\gamma_1} D_\mu \Phi$ is increasing on $(0, \infty)$, and

$$\lim_{\mu \rightarrow 0} \mu^{\gamma_1} D_\mu \Phi = \omega | \cdot |^{-\gamma_1}, \quad \lim_{\mu \rightarrow \infty} \mu^{\gamma_1} D_\mu \Phi = \infty,$$

where the limits are in $L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$ whenever $0 \leq \frac{N}{q_2} < \gamma_2 < \gamma_1 < \frac{N}{q_1} \leq N$, by (6.9).

- The function $\mu \rightarrow \mu^{\gamma_2} D_\mu \Phi$ is decreasing on $(0, \infty)$, and

$$\lim_{\mu \rightarrow 0} \mu^{\gamma_2} D_\mu \Phi = \infty, \quad \lim_{\mu \rightarrow \infty} \mu^{\gamma_2} D_\mu \Phi = \omega | \cdot |^{-\gamma_2},$$

where the limits are in $L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$ whenever $0 \leq \frac{N}{q_2} < \gamma_2 < \gamma_1 < \frac{N}{q_1} \leq N$, by (6.10).

Proof of the claim. Let $0 < \mu < \nu < \infty$, so that $\mu^{\gamma_2 - \gamma_1} > \nu^{\gamma_2 - \gamma_1}$. We have

$$\begin{aligned} \mu^{\gamma_1} \Phi(\mu x) &= \begin{cases} \omega(x) |x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \omega(x) |x|^{-\gamma_1}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \mu^{\gamma_1 - \gamma_2} \omega(x) |x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} = \begin{cases} \omega(x) |x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ (\frac{1}{|x|})^{\gamma_1 - \gamma_2} \omega(x) |x|^{-\gamma_2}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \mu^{\gamma_1 - \gamma_2} \omega(x) |x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} \\ &\leq \begin{cases} \omega(x) |x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \nu^{\gamma_1 - \gamma_2} \omega(x) |x|^{-\gamma_2}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \nu^{\gamma_1 - \gamma_2} \omega(x) |x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} = \nu^{\gamma_1} \Phi(\nu x). \end{aligned}$$

Also,

$$\begin{aligned} \mu^{\gamma_2} \Phi(\mu x) &= \begin{cases} \mu^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \mu^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_1}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \omega(x) |x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} = \begin{cases} \mu^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \mu^{\gamma_2 - \gamma_1} |x|^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_2}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \omega(x) |x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} \\ &\geq \begin{cases} \nu^{\gamma_2 - \gamma_1} \omega(x) |x|^{-\gamma_1}, & |x| \leq \frac{1}{\nu} \\ \omega(x) |x|^{-\gamma_2}, & \frac{1}{\nu} \leq |x| \leq \frac{1}{\mu} \\ \omega(x) |x|^{-\gamma_2}, & |x| \geq \frac{1}{\mu}. \end{cases} = \nu^{\gamma_2} \Phi(\nu x). \end{aligned}$$

It follows that

- The function $\mu \rightarrow T_{\max}(\mu^{\gamma_1} D_\mu \Phi)$ is decreasing on $(0, \infty)$, and

$$\lim_{\mu \rightarrow 0} T_{\max}(\mu^{\gamma_1} D_\mu \Phi) = T_{\max}(\omega | \cdot |^{-\gamma_1}).$$

- The function $\mu \rightarrow T_{\max}(\mu^{\gamma_2} D_\mu \Phi)$ is increasing on $(0, \infty)$, and

$$\lim_{\mu \rightarrow \infty} T_{\max}(\mu^{\gamma_2} D_\mu \Phi) = T_{\max}(\omega | \cdot |^{-\gamma_2}).$$

Next, applying (6.4), we have

$$\begin{aligned} \lambda^{\left(\frac{1}{\alpha} - \frac{\gamma_1}{2}\right)^{-1}} T_{\max}(\lambda \Phi) &= T_{\max}(\mu^{\gamma_1} D_\mu \Phi), \quad \lambda = \mu^{\gamma_1 - \frac{2}{\alpha}}, \\ \lambda^{\left(\frac{1}{\alpha} - \frac{\gamma_2}{2}\right)^{-1}} T_{\max}(\lambda \Phi) &= T_{\max}(\mu^{\gamma_2} D_\mu \Phi), \quad \lambda = \mu^{\gamma_2 - \frac{2}{\alpha}}. \end{aligned}$$

To show that we reach the above exact limits we use the following.

Observation. If $\phi_k \rightarrow \phi$ and assume we are in a situation of continuous dependence, then we know $\liminf_{k \rightarrow \infty} T_{\max}(\phi_k) \geq T_{\max}(\phi)$. Suppose also that $\phi \leq \phi_k$ for all k , hence $T_{\max}(\phi_k) \leq T_{\max}(\phi)$ for all k , so $\limsup_{k \rightarrow \infty} T_{\max}(\phi_k) \leq T_{\max}(\phi)$. Hence $\lim_{k \rightarrow \infty} T_{\max}(\phi_k) = T_{\max}(\phi)$.

This allows us to obtain the above exact limits for the case $\gamma_1 > \gamma_2$. This completes the proof of (i)-(ii) if $\gamma_1 > \gamma_2$.

We may also show (iii) as follows. The function Φ verifies: if $\gamma_1 < \gamma_2$ then $\Phi \in L^q$,

$$\frac{\gamma_1}{N} < \frac{1}{q} < \frac{\gamma_2}{N} < \min(1, \frac{2}{N\alpha}).$$

Hence, by Theorem 1.1

$$T_{\max}(\lambda\Phi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2q}\right)^{-1}}.$$

Then

$$\lim_{\lambda \rightarrow 0} \lambda^{\left(\frac{1}{\alpha} - \frac{N}{2}\right)^{-1}} T_{\max}(\lambda\Phi) = \infty = \lim_{\lambda \rightarrow \infty} \lambda^{\left(\frac{1}{\alpha} - \frac{N}{2}\right)^{-1}} T_{\max}(\lambda\Phi).$$

This completes the proof of Corollary 6.6. \square

APPENDIX A. NONLINEAR HARDY PARABOLIC EQUATIONS

Our purpose in the appendix is to estimate the life-span of solutions for the nonlinear Hardy-Hénon parabolic equations

$$\partial_t u = \Delta u + |\cdot|^l |u|^\alpha u, \quad (\text{A.1})$$

$u = u(t, x) \in \mathbb{R}$, $t > 0$, $x \in \mathbb{R}^N$, $N \geq 1$, $\alpha > 0$, $-\min(2, N) < l$ and with initial value

$$u(0) = u_0. \quad (\text{A.2})$$

A mild solution of the problem (A.1)-(A.2) is a solution of the integral equation

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\sigma)\Delta} \left(|\cdot|^l |u(\sigma)|^\alpha u(\sigma) \right) d\sigma, \quad (\text{A.3})$$

and it is in this form that we consider problem (A.1)-(A.2).

In this first part of the appendix we consider the case $l < 0$, that is the Hardy case. The problem (A.3) is well-posed in $C([0, T]; L^q(\mathbb{R}^N)) \cap C((0, T]; L^r(\mathbb{R}^N))$, $T > 0$, for $u_0 \in L^q(\mathbb{R}^N)$, $1 < q < \infty$, $q > q_c(l)$ or $u_0 \in C_0(\mathbb{R}^N)$, where

$$q_c(l) = \frac{N\alpha}{2+l}, \quad (\text{A.4})$$

and $r > q$ satisfies

$$\frac{1}{q(\alpha+1)} + \frac{l}{N(\alpha+1)} < \frac{1}{r} < \frac{N+l}{N(\alpha+1)}. \quad (\text{A.5})$$

See [2, Theorem 1.1, p. 117] and [3]. This solution can be extended to a maximal solution defined on $[0, T_{\max}(u_0))$. We have obtained the following.

Theorem A.1 (The nonlinear Hardy parabolic equations). *Let $N \geq 1$, $-\min(2, N) < l < 0$, $\alpha > 0$, and $q_c(l)$ be given by (A.4). Let $\varphi \in L^q(\mathbb{R}^N)$ with $1 < q < \infty$, $q > q_c(l)$ or $\varphi \in C_0(\mathbb{R}^N)$ and $r > q$ satisfies (A.5). Let $u \in C([0, T_{\max}(\lambda\varphi)]; L^q(\mathbb{R}^N)) \cap C((0, T_{\max}(\lambda\varphi)); L^r(\mathbb{R}^N))$ be the*

maximal solution of (A.3) with initial data $u_0 = \lambda\varphi$, $\lambda > 0$. Then there exists a constant $C = C(N, l, \alpha, q) > 0$ such that

$$T_{\max}(\lambda\varphi) \geq C (\lambda \|\varphi\|_q)^{-\left(\frac{2+l}{2\alpha} - \frac{N}{2q}\right)^{-1}}, \quad (\text{A.6})$$

for all $\lambda > 0$.

Proof of Theorem A.1. For $q > q_c(l)$, let $r > q$ satisfying (A.5). Let us define

$$\beta(l) = \frac{N}{2} \left(\frac{1}{q} - \frac{1}{r} \right).$$

We note that r depends on l , hence β also. The well-posedness results for (A.1) has been obtained in [2, 3]. We now give the proof of (A.6). Let $-\min(2, N) < l < 0$, $\alpha > 0$, $\lambda > 0$, $K > 0$ and $\varphi \in C_0(\mathbb{R}^N)$, $\|\varphi\|_\infty \leq K$ or $\varphi \in L^q$, $q > 1$, $q > q_c(l)$ such that $\|\varphi\|_q \leq K$. Let $u \in C([0, T_{\max}(\lambda\varphi)]; L^q(\mathbb{R}^N)) \cap C((0, T_{\max}(\lambda\varphi)); L^r(\mathbb{R}^N))$ with $r > q$ satisfying (A.5), be the maximal solution of (A.1) on $[0, T_{\max}(\lambda\varphi))$. It is proved in [2, Inequalities (3.5), (3.6), p. 124] that for $K, T, M > 0$ such that

$$K + CT^{1-\frac{N\alpha}{2q}+\frac{l}{2}}M^{\alpha+1} \leq M$$

the solution u of (A.1) is defined on $[0, T]$ and verifies $\max \left[\sup_{t \in (0, T]} t^{\beta(l)} \|u(t)\|_r, \sup_{[0, T]} \|u(t)\|_q \right] \leq M$. Here C is a positive constant. Then for $T_{\max}(\varphi)$ we should have

$$K + C \left(T_{\max}(\varphi) \right)^{1-\frac{N\alpha}{2q}+\frac{l}{2}} M^{\alpha+1} > M,$$

for all $M > K$. That is it must be

$$\lambda K + C \left(T_{\max}(\lambda\varphi) \right)^{1-\frac{N\alpha}{2q}+\frac{l}{2}} M^{\alpha+1} > M,$$

for all $M > \lambda K$. If we set $M = 2\lambda K$, we get

$$2^{\alpha+1} K^\alpha C T_{\max}(\lambda\varphi)^{1-\frac{N\alpha}{2q}+\frac{l}{2}} \lambda^\alpha > 1.$$

Then taking $K = \|\varphi\|_q$, we get that there exists $C = C(N, \alpha, l, q) > 0$ (since r itself depends on q) such that (A.6) holds. This completes the proof of the Theorem. \square

Using similar argument developed to prove Theorem A.1, we derive the same result for the equation

$$\partial_t u = \Delta u + a(x)|u|^\alpha u, \quad (\text{A.7})$$

where $a(x)|\cdot|^{-l}$ is in $L^\infty(\mathbb{R}^N)$. In particular, we may take a regular near the origin. Then we have the following.

Corollary A.2. *Let $N \geq 1$, $-\min(2, N) < l < 0$, $\alpha > 0$, and $q_c(l)$ be given by (A.4). Let $\varphi \in L^q(\mathbb{R}^N)$ with $1 < q \leq \infty$, $q > q_c(l)$ or $\varphi \in C_0(\mathbb{R}^N)$ and $r > q$ satisfies (A.5). Let $u \in C([0, T_{\max}(\lambda\varphi)]; L^q(\mathbb{R}^N)) \cap C((0, T_{\max}(\lambda\varphi)); L^r(\mathbb{R}^N))$ be the maximal mild solution of (A.7) such that $a(x)|\cdot|^{-l} \in L^\infty(\mathbb{R}^N)$ and with initial data $u_0 = \lambda\varphi$, $\lambda > 0$, constructed by [2, Theorem 1.1, p. 117] and [2, p. 142] (we replace $[0, T_{\max}(\lambda\varphi))$ by $(0, T_{\max}(\lambda\varphi))$ if $q = \infty$). Then (A.6) holds for all $\lambda > 0$.*

Corollary A.2 includes many known results. We will compare our results with those of [40]. For this we restrict ourselves to the case where a is positive and Hölder continuous as assumed in [40]. Also, it is supposed in [40] that $\varphi \in C_b(\mathbb{R}^N)$, $\varphi \geq 0$. For $\lambda > 0$ small, two classes of initial data are considered in [40].

The first class is for φ dominated by a Gaussian. It is shown in [40, Theorem 1 (i), p. 33] that if $0 < \alpha < (2 + l)/N$, then

$$T_{\max}(\lambda\varphi) \geq C\lambda^{-\left(\frac{1}{\alpha} - \frac{N}{2}\right)^{-1}}, \quad \text{as } \lambda \rightarrow 0.$$

For this class $\varphi \in L^q(\mathbb{R}^N)$ for all $q \geq 1$. Since $a \in L^\infty(\mathbb{R}^N)$ because $l < 0$ and a is regular, then we may use Theorem 1.1 (which is valid for such a as noted before) and apply (1.6) with $q = 1$ since $q_c < 1$, and then recover the result of [40].

The second class considered in [40] is for φ such that there exist constants $c_1, c_2 > 0$ and $c_1 \leq \varphi \leq c_2$. The estimates, as stated in [40, Theorem 2 (i)-(a), (ii)-(a), (iii)-(a), pp. 33-34], reads

$$T_{\max}(\lambda\varphi) \geq C\lambda^{-\frac{2\alpha}{2+l}}, \quad \text{as } \lambda \rightarrow 0.$$

Here $\varphi \in L^\infty(\mathbb{R}^N)$, the previous estimate is the same as (A.6) with $q = \infty$. We then recover the results of [40].

APPENDIX B. NONLINEAR HÉNON PARABOLIC EQUATIONS

In this part of the appendix we study a nonlinear heat equation with a spatially growing variable coefficient. We consider the equation (A.1) for $l > 0$ and with the initial condition (A.2). Local well-posedness is known in $C(\mathbb{R}^N) \cap L^\infty \cap L_{l/\alpha}^\infty$, (see [52, 31]). Recently local well-posedness is established in L_s^q for some $q \geq \alpha + 1$ and s satisfying some conditions (see [8]). Not much is known about this equation in comparison with the standard nonlinear heat equation, that is the case $l = 0$. In particular, the life-span is only known for small lambda and rapidly decaying positive initial data, see [40]. Note that the blowup may hold at the origin as it may also not hold at the origin. See [22, 19, 20, 21, 12]. To show lower-bound estimates of the life-span, we establish local well-posedness results. Using Proposition 3.1, we prove local existence for (A.1) in L_γ^q for

$$\gamma = \frac{l}{\alpha} < N$$

and q is such that

$$q > q_c = \frac{N\alpha}{2}, \quad \frac{N}{N-\gamma} < q \leq \infty. \quad (\text{B.1})$$

This value of γ is inspired by [52].

We note that for $0 < \gamma = l/\alpha < N$,

$$q_c > \frac{N}{N-\gamma} \Leftrightarrow q_F := \frac{N\alpha}{2+l} > 1,$$

where q_F is the Fujita exponent for the equation (A.1) (see [41]). We have the following local well-posedness result.

Theorem B.1 (Local well-posedness in L_γ^q). *Let $N \geq 1$ be an integer, $\alpha > 0$ and $l > 0$ be such that*

$$\gamma := \frac{l}{\alpha} < N. \quad (\text{B.2})$$

Let q_c be given by (B.1). Then we have the following.

(i) *If q is such that*

$$q > \frac{N(\alpha + 1)}{N - \gamma}, \quad q > q_c \quad \text{and} \quad q \leq \infty,$$

then equation (A.3) is locally well-posed in $L_\gamma^q(\mathbb{R}^N)$. More precisely, given $u_0 \in L_\gamma^q(\mathbb{R}^N)$, then there exist $T > 0$ and a unique solution of (A.3) $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N))$ if $q < \infty$ and $u \in C((0, T]; L_\gamma^\infty(\mathbb{R}^N))$, $\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta} u_0\|_{L_\gamma^\infty(\mathbb{R}^N)} = 0$ if $q = \infty$. Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} \|u(t)\|_{L_\gamma^q} = \infty$.

(ii) *Assume that $q > q_c$ with $\frac{N}{N-\gamma} < q \leq \infty$. It follows that equation (A.3) is locally well-posed in $L_\gamma^q(\mathbb{R}^N)$ as in part (i) except that uniqueness is guaranteed only among functions $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N))$ which also verify $t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{r})} \|u(t)\|_{L_\gamma^r}$, is bounded on $(0, T]$, where r is given below, (we replace $[0, T]$ by $(0, T]$ if $q = \infty$ and u satisfies $\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta} u_0\|_{L_\gamma^\infty(\mathbb{R}^N)} = 0$). Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} \|u(t)\|_{L_\gamma^q} = \infty$. Furthermore,*

$$\|u(t)\|_{L_\gamma^q} \geq C (T_{\max} - t)^{\frac{N}{2q} - \frac{1}{\alpha}}, \quad \forall t \in [0, T_{\max}), \quad (\text{B.3})$$

where C is a positive constant.

Remark 16. Unlike in [8], here we do not impose $q \geq \alpha + 1$. Also, our strategy is different from that of [8]. In fact, we use a method of [53, 2]. Precisely, to prove the local well-posedness in L_γ^q , we use an auxiliary space L_γ^r for some r as an auxiliary parameter, while in [8] the weight $\gamma = l/\alpha$ is replaced by a real number s that is considered as an auxiliary parameter.

Proof of Theorem B.1. (i) Let us define the maps

$$K_{t,l}(u) = e^{t\Delta} \left(|\cdot|^l |u|^\alpha u \right), \quad t > 0.$$

By the Hölder inequality and Proposition 3.1 with $\gamma = \frac{l}{\alpha} = \mu$, $q_1 = q/(\alpha + 1)$, $q_2 = q$, for each $t > 0$ and if $q > \frac{N(\alpha+1)}{N-\gamma}$, $q \leq \infty$, $K_{t,l} : L_\gamma^q \rightarrow L_\gamma^q$ is locally Lipschitz with

$$\begin{aligned} \|K_{t,l}(u) - K_{t,l}(v)\|_{L_\gamma^q} &\leq C t^{-\frac{N}{2}(\frac{\alpha+1}{q} - \frac{1}{q})} \left\| |\cdot|^l (|u|^\alpha u - |v|^\alpha v) \right\|_{L_\gamma^{\frac{q}{\alpha+1}}} \\ &= C t^{-\frac{N\alpha}{2q}} \left\| |\cdot|^{\alpha\gamma} (|u|^\alpha u - |v|^\alpha v) \right\|_{L_\gamma^{\frac{q}{\alpha+1}}} \\ &\leq C t^{-\frac{N\alpha}{2q}} (\|u\|_{L_\gamma^q}^\alpha + \|v\|_{L_\gamma^q}^\alpha) \|u - v\|_{L_\gamma^q} \\ &\leq 2CM^\alpha t^{-\frac{N\alpha}{2q}} \|u - v\|_{L_\gamma^q}, \end{aligned}$$

for $\|u\|_{L_\gamma^q} \leq M$ and $\|v\|_{L_\gamma^q} \leq M$. We have also, that $t^{-\frac{N\alpha}{2q}} \in L_{loc}^1(0, \infty)$, since $q > q_c = \frac{N\alpha}{2}$. Obviously $t \mapsto \|K_{t,l}(0)\|_\infty = 0 \in L_{loc}^1(0, \infty)$, also $e^{s\Delta}K_{t,l} = K_{t+s,l}$ for $s, t > 0$. Then the proof follows by [53, Theorem 1, p. 279].

(ii) We begin with the observation that, since $q > \frac{N}{N-\gamma}$, there exists $r > q$ satisfying

$$\frac{1}{q(\alpha+1)} < \frac{1}{r} < \frac{N-\gamma}{N(\alpha+1)}. \quad (\text{B.4})$$

We then observe that, since $q > q_c$, we have

$$\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{1}{q(\alpha+1)}.$$

Hence any $r > q$ satisfying (B.4) verifies

$$\frac{1}{q} - \frac{2}{N(\alpha+1)} < \frac{1}{r}.$$

This last inequality implies that

$$\beta(\alpha+1) < 1,$$

where

$$\beta = \frac{N}{2q} - \frac{N}{2r}. \quad (\text{B.5})$$

We choose $K > 0$, $T > 0$, $M > 0$ such that

$$K + \mathcal{C}M^{\alpha+1}T^{1-\frac{N\alpha}{2q}} \leq M, \quad (\text{B.6})$$

where \mathcal{C} is a positive constant given below. We will show that there exists a unique solution u of (A.3) such that $u \in C([0, T], L_\gamma^q(\mathbb{R}^N))$ and $u \in C((0, T], L_\gamma^r(\mathbb{R}^N))$ with

$$\|u\| = \max \left[\sup_{t \in [0, T]} \|u(t)\|_{L_\gamma^q}, \sup_{t \in (0, T]} t^\beta \|u(t)\|_{L_\gamma^r} \right] \leq M.$$

The proof is based on a contraction mapping argument in the set

$$Y_{M,T}^{q,\gamma} = \{u \in C([0, T], L_\gamma^q(\mathbb{R}^N)) \cap C((0, T], L_\gamma^r) : \|u\| \leq M\}.$$

Endowed with the metric $d(u, v) = \|u - v\|$, $Y_{M,T}^{q,\gamma}$ is a nonempty complete metric space. We note that for $u_0 \in L_\gamma^q$ we have

$$\|e^{t\Delta}u_0\|_{L_\gamma^r} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}\|u_0\|_{L_\gamma^q} = Ct^{-\beta}\|u_0\|_{L_\gamma^q}.$$

The condition on initial data $\|u_0\|_{L_\gamma^q} \leq K$ implies that $t^\beta \|e^{t\Delta}u_0\|_{L_\gamma^r} \leq K$. We will show that

$$\mathcal{F}_{u_0}u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\sigma)\Delta} [|\cdot|^l |u(\sigma)|^\alpha u(\sigma)] d\sigma. \quad (\text{B.7})$$

is a strict contraction on $Y_{M,T}^{q,\gamma}$.

Using Proposition 3.1, that is the boundedness of the map $e^{t\Delta} : L_\gamma^q \rightarrow L_\gamma^r$, for the first term and the boundedness of the map $e^{t\Delta} : L_\gamma^{\frac{r}{\alpha+1}} \rightarrow L_\gamma^r$, for the second term, we have

$$\begin{aligned}
t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{L_\gamma^r} &\leq t^\beta \|e^{t\Delta}(\varphi - \psi)\|_{L_\gamma^r} + t^\beta \int_0^t \|e^{(t-s)\Delta} [|\cdot|^l (|u|^\alpha u(s) - |v|^\alpha v(s))]\|_{L_\gamma^r} ds \\
&\leq \|\varphi - \psi\|_{L_\gamma^q} + Ct^\beta \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{r})} \| |\cdot|^{\gamma\alpha} (|u|^\alpha u(s) - |v|^\alpha v(s)) \|_{L_\gamma^{\frac{r}{\alpha+1}}} ds \\
&\leq \|\varphi - \psi\|_{L_\gamma^q} + \left(2(\alpha+1)CM^\alpha t^\beta \int_0^t (t-s)^{-\frac{N\alpha}{2r}} s^{-\beta(\alpha+1)} ds \right) d(u, v) \\
&\leq \|\varphi - \psi\|_{L_\gamma^q} + \left(2(\alpha+1)CM^\alpha t^{1-\frac{N\alpha}{2q}} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) d(u, v) \\
&\leq \|\varphi - \psi\|_{L_\gamma^q} + C_1 M^\alpha T^{1-\frac{N\alpha}{2q}} d(u, v),
\end{aligned}$$

where $C_1 = 2(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma < \infty$.

Using [8, Lemma 2.1] that is the boundedness of the map $e^{t\Delta} : L_\gamma^q \rightarrow L_\gamma^q$, for the first term and Proposition 3.1, the boundedness of the map $e^{t\Delta} : L_\gamma^{\frac{r}{\alpha+1}} \rightarrow L_\gamma^q$, for the second term, we have

$$\begin{aligned}
\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_{L_\gamma^q} &\leq \|e^{t\Delta}(\varphi - \psi)\|_{L_\gamma^q} + \int_0^t \|e^{(t-s)\Delta} [|\cdot|^l (|u|^\alpha u(s) - |v|^\alpha v(s))]\|_{L_\gamma^q} ds \\
&\leq \|\varphi - \psi\|_{L_\gamma^q} + C \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})} \| |\cdot|^{\gamma\alpha} (|u|^\alpha u(s) - |v|^\alpha v(s)) \|_{L_\gamma^{\frac{r}{\alpha+1}}} ds \\
&\leq \|\varphi - \psi\|_{L_\gamma^q} + \left(2(\alpha+1)CM^\alpha \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})} s^{-\beta(\alpha+1)} ds \right) d(u, v) \\
&\leq \|\varphi - \psi\|_q + \left(2(\alpha+1)CM^\alpha t^{1-\frac{N\alpha}{2q}} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})} \sigma^{-\beta(\alpha+1)} d\sigma \right) d(u, v) \\
&\leq \|\varphi - \psi\|_{L_\gamma^q} + C_2 M^\alpha T^{1-\frac{N\alpha}{2q}} d(u, v).
\end{aligned}$$

where $C_2 = 2(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{q})} \sigma^{-\beta(\alpha+1)} d\sigma < \infty$.

From the above estimates, it follows that

$$d(\mathcal{F}_\varphi(u), \mathcal{F}_\psi(v)) \leq \|\varphi - \psi\|_{L_\gamma^q} + \mathcal{C} M^\alpha T^{1-\frac{N\alpha}{2q}} d(u, v), \quad (\text{B.8})$$

where $\mathcal{C} = \max(C_1, C_2)$. The rest of the proof follows similarly as that of Theorem 4.1 and as in [2]. This completes the proof. \square

Theorem B.1 allows us to obtain the following.

Corollary B.2 (Hénon parabolic equations). *Let $N \geq 1$, $\alpha > 0$, $0 < l < N\alpha$. If $\varphi \in L_\gamma^q(\mathbb{R}^N)$, where*

$$\gamma = l/\alpha < N, \quad q > \frac{N\alpha}{2} \quad \text{and} \quad \frac{N}{N-\gamma} < q \leq \infty,$$

then the life-span of (A.3) with initial data $\lambda\varphi$ satisfies

$$T_{\max}(\lambda\varphi) \geq C(\lambda\|\varphi\|_{L_\gamma^q})^{-\left(\frac{1}{\alpha}-\frac{N}{2q}\right)^{-1}}, \quad (\text{B.9})$$

for all $\lambda > 0$, where $C = C(\alpha, q, l, N)$.

Remark 17.

- 1) We see that l has no effect on the lower bound of the life span. This is because blow up may not occur at the origin nor at $|x|$ infinite.
- 2) Corollary B.2 is totally new for $q < \infty$.

Proof of Corollary B.2. The proof follows using (B.6) and is similar to that of Theorem 1.1. \square

In the case of initial data in $L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)$ we have the following result which generalizes that of [52] known for $q = \infty$.

Theorem B.3 (Local well-posedness in $L^q \cap L_\gamma^q$). *Let $N \geq 1$ be an integer, $\alpha > 0$ and $l > 0$ be such that*

$$0 < \gamma := \frac{l}{\alpha} < N.$$

Let q_c be given by (B.1). Then we have the following.

- (i) Equation (A.3) is locally well-posed in $L^\infty(\mathbb{R}^N) \cap L_\gamma^\infty(\mathbb{R}^N)$. More precisely, given $u_0 \in L^\infty(\mathbb{R}^N) \cap L_\gamma^\infty(\mathbb{R}^N)$, then there exist $T > 0$ and a unique solution $u \in C((0, T]; L^\infty(\mathbb{R}^N) \cap L_\gamma^\infty(\mathbb{R}^N))$ of (A.3) and u satisfies $\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta} u_0\|_{L^\infty \cap L_\gamma^\infty(\mathbb{R}^N)} = 0$. Moreover, u can be extended to a maximal interval $(0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} (\|u(t)\|_\infty + \|u(t)\|_{L_\gamma^\infty}) = \infty$.
- (ii) If q is such that

$$q > \frac{N(\alpha + 1)}{N - \gamma}, \quad q > q_c \quad \text{and} \quad q < \infty,$$

then equation (A.3) is locally well-posed in $L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)$. More precisely, given $u_0 \in L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)$, then there exist $T > 0$ and a unique solution $u \in C([0, T]; L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N))$ of (A.3). Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} (\|u(t)\|_q + \|u(t)\|_{L_\gamma^q}) = \infty$.

- (iii) Assume that $q > q_c$ with $\frac{N}{N-\gamma} < q \leq \infty$. It follows that equation (A.3) is locally well-posed in $L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)$ as in part (ii) except that uniqueness is guaranteed only among functions $u \in C([0, T]; L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N))$ which also verify $t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{r})} \|u(t)\|_{L_r^q}$, $t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{r})} \|u(t)\|_r$ are bounded on $(0, T]$, where r is as above (we replace $[0, T]$ by $(0, T]$ if $q = \infty$ and u satisfies $\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta} u_0\|_{L^\infty \cap L_\gamma^\infty(\mathbb{R}^N)} = 0$). Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} (\|u(t)\|_q + \|u(t)\|_{L_\gamma^q}) = \infty$. Furthermore,

$$\|u(t)\|_{L^q \cap L_\gamma^q} \geq C (T_{\max} - t)^{\frac{N}{2q} - \frac{1}{\alpha}}, \quad \forall t \in [0, T_{\max}), \quad (\text{B.10})$$

where C is a positive constant.

Proof of Theorem B.3. We will just give the new elements of the proof.

(i)-(ii) By the Hölder inequality and Proposition 3.1 with $\gamma = \frac{l}{\alpha} = \mu$, $q_1 = q/(\alpha + 1)$, $q_2 = q$, for each $t > 0$ and if $q > \frac{N(\alpha+1)}{N-\gamma}$, $q \leq \infty$, $K_{t,t} : L^q \cap L_\gamma^q \rightarrow L^q \cap L_\gamma^q$ is locally Lipschitz and, since

$$q > \frac{N(\alpha+1)}{N-\gamma} > \alpha + 1,$$

$$\begin{aligned} \|K_{t,l}(u) - K_{t,l}(v)\|_q &\leq C t^{-\frac{N}{2}(\frac{\alpha+1}{q}-\frac{1}{q})} \left\| |\cdot|^l (|u|^\alpha u - |v|^\alpha v) \right\|_{q/(\alpha+1)} \\ &\leq C t^{-\frac{N\alpha}{2q}} (\|u\|_{L_\gamma^q}^\alpha + \|v\|_{L_\gamma^q}^\alpha) \|u - v\|_q \\ &\leq 2CM^\alpha t^{-\frac{N\alpha}{2q}} \|u - v\|_q, \end{aligned}$$

for $\max(\|u\|_q, \|u\|_{L_\gamma^q}) \leq M$ and $\max(\|v\|_q, \|v\|_{L_\gamma^q}) \leq M$. We have also, that $t^{-\frac{N\alpha}{2q}} \in L_{loc}^1(0, \infty)$, since $q > q_c = \frac{N\alpha}{2}$. Then the proof follows by [53, Theorem 1, p. 279].

(iii) We choose $r > q$ satisfying (B.4). We consider $K > 0$, $T > 0$, $M > 0$ such that

$$K + \mathcal{C}M^{\alpha+1}T^{1-\frac{N\alpha}{2q}} \leq M, \quad (\text{B.11})$$

where \mathcal{C} is a positive constant given below. We will show that there exists a unique solution u of (A.3) such that $u \in C([0, T]; L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N))$ and $u \in C((0, T]; L^r(\mathbb{R}^N) \cap L_\gamma^r(\mathbb{R}^N))$ with

$$\|u\| = \max \left[\sup_{t \in [0, T]} \|u(t)\|_q, \sup_{t \in [0, T]} \|u(t)\|_{L_\gamma^q}, \sup_{t \in (0, T]} t^\beta \|u(t)\|_{L_\gamma^r}, \sup_{t \in (0, T]} t^\beta \|u(t)\|_r \right] \leq M.$$

The proof is based on a contraction mapping argument in the set

$$Y_{M, T}^{q, \gamma} = \{u \in C([0, T]; L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)) \cap C((0, T]; L^r \cap L_\gamma^r) : \|u\| \leq M\}.$$

Endowed with the metric $d(u, v) = \|u - v\|$, $Y_{M, T}^{q, \gamma}$ is a nonempty complete metric space. We note that for $u_0 \in L^q$,

$$\|e^{t\Delta} u_0\|_r \leq C t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})} \|u_0\|_q = C t^{-\beta} \|u_0\|_q.$$

The condition on initial data $\max(\|u_0\|_q, \|u_0\|_{L_\gamma^q}) \leq K$ implies that $t^\beta \|e^{t\Delta} u_0\|_{L_\gamma^r} \leq K$, $t^\beta \|e^{t\Delta} u_0\|_{L^r} \leq K$. We will show that \mathcal{F}_{u_0} defined in (2.11) is a strict contraction on $Y_{M, T}^{q, \gamma}$. Using Proposition 3.1, that is $e^{t\Delta} : L^q \rightarrow L^r$, for the first term and $e^{t\Delta} : L^{\frac{r}{\alpha+1}} \rightarrow L^r$, for the second term, we have

$$\begin{aligned} t^\beta \|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_r &\leq t^\beta \|e^{t\Delta}(\varphi - \psi)\|_r + t^\beta \int_0^t \|e^{(t-s)\Delta} [|\cdot|^l (|u|^\alpha u(s) - |v|^\alpha v(s))]\|_r ds \\ &\leq \|\varphi - \psi\|_q + C t^\beta \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{r})} \left\| |\cdot|^{\alpha\gamma} (|u|^\alpha u(s) - |v|^\alpha v(s)) \right\|_{\frac{r}{\alpha+1}} ds \\ &\leq \|\varphi - \psi\|_q + \left(2(\alpha+1)CM^\alpha t^\beta \int_0^t (t-s)^{-\frac{N\alpha}{2r}} s^{-\beta(\alpha+1)} ds \right) d(u, v) \\ &\leq \|\varphi - \psi\|_q + \left(2(\alpha+1)CM^\alpha t^{1-\frac{N\alpha}{2q}} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma \right) d(u, v) \\ &\leq \|\varphi - \psi\|_{L^q} + C_3 M^\alpha T^{1-\frac{N\alpha}{2q}} d(u, v), \end{aligned}$$

where $C_3 = 2(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r}} \sigma^{-\beta(\alpha+1)} d\sigma < \infty$.

Using Proposition 3.1, that is $e^{t\Delta} : L^q \rightarrow L^q$, for the first term and $e^{t\Delta} : L^{\frac{r}{\alpha+1}} \rightarrow L^q$, for the second term, we have

$$\begin{aligned}
\|\mathcal{F}_\varphi(u)(t) - \mathcal{F}_\psi(v)(t)\|_q &\leq \|e^{t\Delta}(\varphi - \psi)\|_q + \int_0^t \|e^{(t-s)\Delta} [|\cdot|^l (|u|^\alpha u(s) - |v|^\alpha v(s))]\|_q ds \\
&\leq \|\varphi - \psi\|_q + C \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r} - \frac{1}{q})} \| |\cdot|^{\alpha\gamma} (|u|^\alpha u(s) - |v|^\alpha v(s)) \|_{\frac{r}{\alpha+1}} ds \\
&\leq \|\varphi - \psi\|_q + \left(2(\alpha+1)CM^\alpha \int_0^t (t-s)^{-\frac{N}{2}(\frac{\alpha+1}{r} - \frac{1}{q})} s^{-\beta(\alpha+1)} ds \right) d(u, v) \\
&\leq \|\varphi - \psi\|_q + \left(2(\alpha+1)CM^\alpha t^{1-\frac{N\alpha}{2q}} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r} - \frac{1}{q})} \sigma^{-\beta(\alpha+1)} d\sigma \right) d(u, v) \\
&\leq \|\varphi - \psi\|_{L^q} + C_4 M^\alpha T^{1-\frac{N\alpha}{2q}} d(u, v).
\end{aligned}$$

where $C_4 = 2(\alpha+1)C \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r} - \frac{1}{q})} \sigma^{-\beta(\alpha+1)} d\sigma < \infty$. From the above estimates, it follows that

$$d(\mathcal{F}_\varphi(u), \mathcal{F}_\psi(v)) \leq \|\varphi - \psi\|_{L^q \cap L_\gamma^q} + CM^\alpha T^{1-\frac{N\alpha}{2q}} d(u, v), \quad (\text{B.12})$$

where $C = \max(C_1, C_2, C_3, C_4)$. The rest of the proof follows similarly as above and as in [2]. \square

We have also the following result.

Proposition B.4. *Let $\alpha > 0$ and let $0 < \gamma := l/\alpha < N$. Assume the hypotheses of Theorem B.3. Let $T_{\max}(\varphi, L^q \cap L_\gamma^q)$ denotes the existence time of the maximal solution of (A.3) with initial data $\varphi \in L^q \cap L_\gamma^q$. Then we have the following.*

- (i) *If $\varphi \in L^q \cap L_\gamma^q$, then for $t \in (0, T_{\max}(\varphi, q))$, $u(t) \in L^\infty \cap L_\gamma^\infty$.*
- (ii) *If $\varphi \in L^p \cap L_\gamma^p \cap L^q \cap L_\gamma^q$, $\frac{N}{N-\gamma} < q < p \leq \infty$ and $q > q_c$. Then $T_{\max}(\varphi, L^p \cap L_\gamma^p) = T_{\max}(\varphi, L^q \cap L_\gamma^q)$.*

Proof. (i) Let $\varphi \in L^q(\mathbb{R}^N)$, $q > q_c$ and $q > \frac{N}{N-\gamma}$. Let r and β be as above and (B.5). Let p be such that $r < p \leq \infty$. Hence $p > q$,

$$0 \leq \frac{1}{p} < \frac{\gamma}{N} + \frac{\alpha+1}{r} < 1, \quad \frac{1}{p} < \frac{\gamma}{N} + \frac{1}{q} < 1,$$

and for $0 < T < T_{\max}(\varphi, q)$, we have

$$\begin{aligned}
\|u(t)\|_p &\leq \|e^{t\Delta}\varphi\|_p + C \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r} - \frac{1}{p})} \|u(\sigma)\|_{L_\gamma^q}^\alpha \|u(\sigma)\|_r d\sigma \\
&\leq (4\pi t)^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p})} \|\varphi\|_q + Ct^{1-\frac{N}{2}(\frac{\alpha+1}{r} - \frac{1}{p})-\beta(\alpha+1)} \sup_{s \in (0, T]} \left(s^{\beta\alpha} \|u(s)\|_{L_\gamma^q}^\alpha \right) \times \\
&\quad \sup_{s \in (0, T]} \left(s^\beta \|u(s)\|_r \right) \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r} - \frac{1}{p})} \sigma^{-\beta(\alpha+1)} d\sigma \\
&\leq (4\pi t)^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p})} \|\varphi\|_q + M^{\alpha+1} Ct^{1-\frac{N}{2}(\frac{\alpha+1}{r} - \frac{1}{p})} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r} - \frac{1}{p})} \sigma^{-\beta(\alpha+1)} d\sigma.
\end{aligned}$$

Also,

$$\begin{aligned}
\|u(t)\|_{L_\gamma^p} &\leq \|e^{t\Delta}\varphi\|_{L_\gamma^p} + C \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})} \|u(\sigma)\|_{L_\gamma^{\alpha+1}} d\sigma \\
&\leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \|\varphi\|_{L_\gamma^q} + Ct^{1-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})-\beta(\alpha+1)} \sup_{s \in (0, T]} \left(s^{\beta(\alpha+1)} \|u(s)\|_{L_\gamma^{\alpha+1}} \right) \times \\
&\quad \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})} \sigma^{-\beta(\alpha+1)} d\sigma \\
&\leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \|\varphi\|_{L_\gamma^q} + M^{\alpha+1} Ct^{1-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})} \int_0^1 (1-\sigma)^{-\frac{N}{2}(\frac{\alpha+1}{r}-\frac{1}{p})} \sigma^{-\beta(\alpha+1)} d\sigma.
\end{aligned}$$

Since $r > q > q_c$, it follows that if

$$\frac{\alpha+1}{r} - \frac{2}{N} < \frac{1}{p} < \frac{1}{r},$$

then $u(t)$ is in $L^p \cap L_\gamma^p$ for all $t \in (0, T_{\max}(\varphi, q))$. The result for general $p > q$ follows by iteration. Hence $u(t)$ is in $L^\infty \cap L_\gamma^\infty$, for $t \in (0, T_{\max}(\varphi, q))$.

(ii) Follows as in Proposition 4.2. This finishes the proof of the proposition. \square

Theorem B.3 and inequality (B.11) allow us to obtain that under the same hypotheses of Corollary B.2 if $\varphi \in L^q(\mathbb{R}^N) \cap L_\gamma^q(\mathbb{R}^N)$ then the life-span of (A.1) with initial data $\lambda\varphi$ satisfies

$$T_{\max}(\lambda\varphi) \geq C \left(\lambda \|\varphi\|_{L_\gamma^q \cap L^q} \right)^{-\left(\frac{1}{\alpha} - \frac{N}{2q}\right)^{-1}}, \quad (\text{B.13})$$

for all $\lambda > 0$, which gives a power of λ not depending on l , unlike the case $l < 0$.

APPENDIX C. THE HARDY-HÉNON EQUATIONS WITH DECAYING INITIAL DATA

In this part of the appendix, we investigate lower bound estimates for life-span for the solutions of the equation (A.1) with initial data having some decay. As in Section 4, we work in L_γ^q with $\gamma > 0$, $\gamma > l/\alpha$. This allows us to obtain a lower bound of the life span for initial data having more decay than l/α , if $l > 0$. We consider the Duhamel formulation of (A.1)-(A.2), that is the equation (A.3) and suppose that

$$N \geq 1, \quad \alpha > 0, \quad -\min(2, N) < l < N\alpha. \quad (\text{C.1})$$

Let γ be such that

$$0 < \gamma < N, \quad \frac{l}{\alpha} < \gamma < \frac{2+l}{\alpha} \quad (\text{C.2})$$

and q satisfying

$$\frac{N}{N-\gamma} < q \leq \infty, \quad q > \frac{N\alpha}{2+l-\gamma\alpha} =: q_c(\gamma, l). \quad (\text{C.3})$$

$q_c(\gamma, l)$ is the critical exponent of (A.1) for initial data in L_γ^q . The condition (C.3) can be reformulated as follows:

$$\frac{\gamma}{N} \leq \frac{1}{q} + \frac{\gamma}{N} < 1, \quad \frac{N\alpha}{2q} + \frac{\gamma\alpha}{2} - \frac{l}{2} < 1.$$

Let $0 < \nu < \gamma$ be such that

$$\frac{\gamma+l}{\alpha+1} < \nu < \frac{N+l}{\alpha+1}.$$

Hence, using (C.1) and (C.2), we have

$$\frac{l}{\alpha} < \nu < \gamma, \quad 0 < \nu < \nu(\alpha + 1) - l < N, \quad 0 < \gamma < \nu(\alpha + 1) - l < N.$$

Let now $r > q$ be such that

$$\frac{1}{q(\alpha + 1)} - \frac{\nu(\alpha + 1) - l - \gamma}{N(\alpha + 1)} < \frac{1}{r} < \frac{N - \nu(\alpha + 1) + l}{N(\alpha + 1)}.$$

This is possible by (C.3). Hence, we have

$$\frac{1}{r} < \frac{\alpha + 1}{r} + \frac{\nu(\alpha + 1) - l - \nu}{N} < \frac{\alpha + 1}{r} + \frac{\nu(\alpha + 1) - l}{N} < 1,$$

$$\frac{1}{q} < \frac{\alpha + 1}{r} + \frac{\nu(\alpha + 1) - l - \gamma}{N} < \frac{\alpha + 1}{r} + \frac{\nu(\alpha + 1) - l}{N} < 1.$$

That is, by [8, Lemma 2.1] $e^{t\Delta} : L_\gamma^q \rightarrow L_\gamma^q$, is bounded and we may apply Proposition 3.1, so that the maps $e^{t\Delta} : L_\gamma^q \rightarrow L_\nu^r$, $e^{t\Delta} : L_{\nu(\alpha+1)-l}^{r/(\alpha+1)} \rightarrow L_\gamma^q$ and $e^{t\Delta} : L_{\nu(\alpha+1)-l}^{r/(\alpha+1)} \rightarrow L_\nu^r$ are bounded.

Let us introduce

$$\tilde{\beta}_l = \frac{N}{2} \left(\frac{1}{q} - \frac{1}{r} \right) + \frac{\gamma - \nu}{2}.$$

Hence

$$\tilde{\beta}_l > \frac{\gamma - \nu}{2} > 0.$$

We have,

$$\begin{aligned} \tilde{\beta}_l(\alpha + 1) &= \frac{N}{2} \left(\frac{\alpha + 1}{q} - \frac{\alpha + 1}{r} \right) + (\alpha + 1) \frac{\gamma - \nu}{2} \\ &\leq \frac{N}{2} \left(\frac{\alpha + 1}{q} - \frac{1}{q} + \frac{\nu(\alpha + 1) - l - \gamma}{N} \right) + (\alpha + 1) \frac{\gamma - \nu}{2} \\ &= \frac{N\alpha}{2q} + \frac{\alpha\gamma}{2} - \frac{l}{2} < 1. \end{aligned}$$

We also have

$$\begin{aligned} \frac{N\alpha}{2r} + \frac{\nu\alpha}{2} - \frac{l}{2} &< \frac{N\alpha}{2q} + \frac{\gamma\alpha}{2} - \frac{l}{2} < 1, \\ \frac{N}{2} \left(\frac{\alpha + 1}{r} - \frac{1}{q} \right) + \frac{\nu(\alpha + 1) - l - \gamma}{2} &< \frac{N\alpha}{2r} + \frac{\nu\alpha}{2} - \frac{l}{2} < 1. \end{aligned}$$

These last three estimates are crucial to the local existence argument below. We note that if $l > 0$, we may take $\nu(\alpha + 1) - l = \gamma$, $r = (\alpha + 1)q$. With the above choice of the parameters, we can show the following local well-posedness result.

Theorem C.1. *Let $N \geq 1$ be an integer, $\alpha > 0$, $-\min(2, N) < l$ and*

$$\frac{l}{N} < \alpha. \tag{C.4}$$

Let γ be satisfying (C.2) and $q_c(\gamma, l)$ be given by (C.3). Then we have the following.

(i) If $\gamma(\alpha + 1) < N + l$ and q is such that

$$q > \frac{N(\alpha + 1)}{N + l - \gamma(\alpha + 1)}, \quad q > q_c(\gamma, l) \quad \text{and} \quad q \leq \infty,$$

then equation (A.3) is locally well-posed in $L_\gamma^q(\mathbb{R}^N)$. More precisely, given $u_0 \in L_\gamma^q(\mathbb{R}^N)$, then there exist $T > 0$ and a unique solution $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N))$ of (A.3) (we replace $[0, T]$ by $(0, T]$ if $q = \infty$ and u satisfies $\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta} u_0\|_{L_\gamma^\infty(\mathbb{R}^N)} = 0$). Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} \|u(t)\|_{L_\gamma^q} = \infty$.

(ii) Assume that $q > q_c(\gamma, l)$ with $\frac{N}{N-\gamma} < q \leq \infty$. It follows that equation (A.3) is locally well-posed in $L_\gamma^q(\mathbb{R}^N)$ as in part (i) except that uniqueness is guaranteed only among functions $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N))$ which also verify $t^{\tilde{\beta}l} \|u(t)\|_{L_\nu^r}$ is bounded on $(0, T]$, where r and ν are as above (we replace $[0, T]$ by $(0, T]$ if $q = \infty$ and u satisfies $\lim_{t \rightarrow 0} \|u(t) - e^{t\Delta} u_0\|_{L_\gamma^\infty(\mathbb{R}^N)} = 0$). Moreover, u can be extended to a maximal interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} \|u(t)\|_{L_\gamma^q} = \infty$. Furthermore,

$$\|u(t)\|_{L_\gamma^q} \geq C (T_{\max} - t)^{\frac{N}{2q} + \frac{\gamma}{2} - \frac{2+l}{2\alpha}}, \quad \forall t \in [0, T_{\max}), \quad (\text{C.5})$$

where C is a positive constant.

Proof. (i) Using the inequality (4.4), and Proposition 3.1 that is $e^{t\Delta} : L_{(\alpha+1)\gamma-l}^{\frac{q}{\alpha+1}} \rightarrow L_\gamma^q$ is bounded, for each $t > 0$ we have that $K_{t,l} : L_\gamma^q \rightarrow L_\gamma^q$ is locally Lipschitz with

$$\begin{aligned} \|K_{t,l}(u) - K_{t,l}(v)\|_{L_\gamma^q} &\leq C t^{-\frac{N}{2}(\frac{\alpha+1}{q} - \frac{1}{q}) - \frac{\alpha\gamma-l}{2}} \| |u|^\alpha u - |v|^\alpha v \|_{L_{(\alpha+1)\gamma}^{\frac{q}{\alpha+1}}} \\ &\leq C t^{-\frac{N\alpha}{2q} - \frac{\alpha\gamma}{2} + \frac{l}{2}} (\|u\|_{L_\gamma^q}^\alpha + \|v\|_{L_\gamma^q}^\alpha) \|u - v\|_{L_\gamma^q} \\ &\leq 2CM^\alpha t^{-\frac{N\alpha}{2q} - \frac{\alpha\gamma}{2} + \frac{l}{2}} \|u - v\|_{L_\gamma^q}, \end{aligned}$$

for $\|u\|_{L_\gamma^q} \leq M$ and $\|v\|_{L_\gamma^q} \leq M$. The rest of the proof is similar to that of Theorems 4.1 and B.1.

(ii) For $u_0 \in L_\gamma^q$ we have $\|e^{t\Delta} u_0\|_{L_\nu^r} \leq C t^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{\gamma-\nu}{2}} \|u_0\|_{L_\gamma^q} = C t^{-\tilde{\beta}l} \|u_0\|_{L_\gamma^q}$. We choose $K > 0$, $T > 0$, $M > 0$ such that

$$K + CM^{\alpha+1} T^{1 - \frac{N\alpha}{2q} - \frac{\gamma\alpha}{2} + \frac{l}{2}} \leq M, \quad (\text{C.6})$$

where C is a positive constant. We will show that there exists a unique solution u of (A.3) such that $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N))$ and $u \in C((0, T]; L_\nu^r(\mathbb{R}^N))$ with

$$\|u\| = \max \left[\sup_{t \in [0, T]} \|u(t)\|_{L_\gamma^q}, \sup_{t \in (0, T]} t^{\tilde{\beta}l} \|u(t)\|_{L_\nu^r} \right] \leq M.$$

The proof is based on a contraction mapping argument in the set

$$Y_{M,T}^{q,\gamma} = \{u \in C([0, T]; L_\gamma^q(\mathbb{R}^N)) \cap C((0, T]; L_\nu^r) : \|u\| \leq M\}.$$

Endowed with the metric $d(u, v) = \|u - v\|$, $Y_{M,T}^{q,\gamma}$ is a nonempty complete metric space. We consider u_0 such that $\|u_0\|_{L_\gamma^q} \leq K$ and we estimate as follows:

$$\begin{aligned}
t^{\tilde{\beta}_l} \|\mathcal{F}_{u_0} u(t)\|_{L_\nu^r} &\leq t^{\tilde{\beta}_l} \|e^{t\Delta} u_0\|_{L_\nu^r} + t^{\tilde{\beta}_l} \int_0^t \|e^{(t-\sigma)\Delta} [|\cdot|^l |u(\sigma)|^\alpha u(\sigma)]\|_{L_\nu^r} d\sigma \\
&\leq K + Ct^{\tilde{\beta}_l} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r} - \frac{(\nu(\alpha+1)-l)-\nu}{2}} \| |\cdot|^{\nu(\alpha+1)} |u(\sigma)|^\alpha u(\sigma) \|_{r/(\alpha+1)} d\sigma \\
&= K + Ct^{\tilde{\beta}_l} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r} - \frac{\nu\alpha}{2} + \frac{l}{2}} \|u(\sigma)\|_{L_\nu^r}^{\alpha+1} d\sigma \\
&\leq K + CM^{\alpha+1} t^{\tilde{\beta}_l} \int_0^t (t-\sigma)^{-\frac{N\alpha}{2r} - \frac{\nu\alpha}{2} + \frac{l}{2}} \sigma^{-\tilde{\beta}_l(\alpha+1)} d\sigma \\
&\leq K + CM^{\alpha+1} t^{1 - \frac{N\alpha}{2q} - \frac{\gamma\alpha}{2} + \frac{l}{2}} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r} - \frac{\nu\alpha}{2} + \frac{l}{2}} \sigma^{-\tilde{\beta}_l(\alpha+1)} d\sigma \\
&\leq K + CM^{\alpha+1} T^{1 - \frac{N\alpha}{2q} - \frac{\gamma\alpha}{2} + \frac{l}{2}} \int_0^1 (1-\sigma)^{-\frac{N\alpha}{2r} - \frac{\nu\alpha}{2} + \frac{l}{2}} \sigma^{-\tilde{\beta}_l(\alpha+1)} d\sigma,
\end{aligned}$$

and similarly for the contraction. The other estimates can be handled similarly as above, see also [2]. So we omit the details. This completes the proof of Theorem C.1. \square

Remark 18.

- 1) We can take $\gamma = \max(0, \frac{l}{\alpha})$ in Theorem C.1 as well as $l = 0$, it is then a generalization of Theorems 4.1 and B.1.
- 2) See [8, Theorem 1.13] for related results. The range of the values of q in (ii) are larger than in [8], while (i) is essentially contained in [8] which we give for completeness. Also the methods are different. In fact, we work in an auxiliary space L_ν^r , for some r and ν while in [8] some auxiliary spaces L_ν^q for some $\tilde{\nu}$ but q is fixed are considered.
- 3) If $q = \infty$ we may replace L_γ^∞ by the space obtained by the closure, with respect to the L_γ^∞ -topology, of $\mathcal{D}(\mathbb{R}^N)$, the space of compactly supported $C^\infty(\mathbb{R}^N)$ functions. For initial data in this sub-space of L_γ^∞ the result holds on $[0, T]$ instead of $(0, T]$.
- 4) Using argument of [3], we can show that uniqueness in the part (ii) of Theorem C.1 holds in $u \in C([0, T]; L_\gamma^q(\mathbb{R}^N)) \cap C((0, T]; L_\nu^r(\mathbb{R}^N))$.

Theorem C.1 gives the following.

Corollary C.2 (Hénon parabolic equations with decaying initial data). *Let $N \geq 1$ be an integer, $\alpha > 0$ and $-\min(2, N) < l < N\alpha$. If $\varphi \in L_\gamma^q(\mathbb{R}^N)$, where*

$$\begin{aligned}
0 < \gamma < N, \quad \frac{l}{\alpha} < \gamma < \frac{2+l}{\alpha}, \\
\frac{\gamma}{N} \leq \frac{1}{q} + \frac{\gamma}{N} < 1, \quad \frac{N\alpha}{2q} + \frac{\gamma\alpha}{2} - \frac{l}{2} < 1.
\end{aligned}$$

Then the life-span of (A.3) with initial data $\lambda\varphi$ satisfies

$$T_{\max}(\lambda\varphi) \geq C \left(\lambda \|\varphi\|_{L_\gamma^q} \right)^{-\left(\frac{2+l}{2\alpha} - \frac{N}{2q} - \frac{\gamma}{2}\right)^{-1}}, \quad (\text{C.7})$$

for all $\lambda > 0$, where $C = C(\alpha, q, l, \gamma, N) > 0$ is a constant.

Remark 19.

- 1) Corollary C.2 answers a problem left open in [40]. In fact, when $l > 0$ only exponentially decaying initial data are considered in [40].
- 2) Similar results, using scaling argument, seems to be proved in [60, 61] for related equations, but only for small λ , $q = \infty$ and positive initial data.
- 3) If $\varphi \in L^q \cap L_\gamma^q$, or $\varphi \in L_{l_+/\alpha}^q \cap L_\gamma^q$, where $l_+ = \max(l, 0)$ then (C.7) is better than (A.6) and (B.9) for $0 < \lambda < 1$.

Proof of Corollary C.2. The proof follows using (C.6) and is similar to that of Theorem 1.1, so we omit the details. \square

We complement Corollary C.2 by the following upper bound estimates.

Proposition C.3 (Upper bounds of life-span for Hardy-Hénon equations). *Let $N \geq 1$ be an integer $\alpha > 0$ and $-\min(2, N) < l$. Assume that*

$$\frac{l}{N} < \alpha < \frac{2+l}{N}. \quad (\text{C.8})$$

Let $\omega \in L^\infty(\mathbb{R}^N)$ be homogeneous of degree 0, $\omega \geq 0$ and $\omega \not\equiv 0$, $\tilde{\varphi}$ be given by (1.16) and $\tilde{\tilde{\varphi}}$ be given by (1.20). Let $0 < \gamma < N$ be such that

$$\frac{l}{\alpha} < \gamma,$$

and $\varphi \in L_\gamma^q(\mathbb{R}^N)$, where $\frac{N}{N-\gamma} < q \leq \infty$. Then we have the following.

- (i) If $\varphi \geq \tilde{\varphi}$ then $T_{\max}(\lambda\varphi) \leq C\lambda^{-\left(\frac{2+l}{2\alpha}-\frac{\gamma}{2}\right)^{-1}}$, $\lambda > 1$.
- (ii) If $\varphi \geq \tilde{\tilde{\varphi}}$ then $T_{\max}(\lambda\varphi) \leq C\lambda^{-\left(\frac{2+l}{2\alpha}-\frac{\gamma}{2}\right)^{-1}}$, $0 < \lambda < 1$.

To prove Proposition C.3, we use a scaling argument. We recall the definition of the dilation operators $D_\mu\varphi = \varphi(\mu \cdot)$, $\mu > 0$. It is clear that if u is a solution of the equation (A.1) then for any $\mu > 0$, u_μ is also a solution of (A.1), where $u_\mu(t, x) = \mu^{\frac{2+l}{\alpha}} u(\mu^2 t, \mu x)$. Hence, σ in (6.1) is given by

$$\sigma = \frac{2+l}{\alpha}.$$

So that, for $\lambda = \mu^{\gamma - \frac{2+l}{\alpha}}$, (6.3) reads

$$\lambda^{-\left[\left(\frac{2+l}{2\alpha}-\frac{\gamma}{2}\right)^{-1}\right]} T_{\max}(\lambda\varphi) = T_{\max}(\lambda\mu^{\frac{2+l}{\alpha}} D_\mu\varphi) = T_{\max}(\mu^\gamma D_\mu\varphi). \quad (\text{C.9})$$

Let $0 < \gamma < (2+l)/\alpha$. Let φ be a nonnegative function, satisfying $\mu^\gamma D_\mu\varphi \leq \varphi$, for some $\mu > 0$. Then, since $\lambda = \mu^{\gamma - \frac{2+l}{\alpha}}$, we have by comparison argument (see [52, Theorem 2.4, p. 564]) and (C.9) that

$$T_{\max}(\lambda\varphi) \geq \lambda^{-\left(\frac{2+l}{2\alpha}-\frac{\gamma}{2}\right)^{-1}} T_{\max}(\varphi).$$

Similarly, if $\mu^\gamma D_\mu\varphi \geq \varphi$, for some $\mu > 0$, and $T_{\max}(\varphi) < \infty$, we have that

$$T_{\max}(\lambda\varphi) \leq \lambda^{-\left(\frac{2+l}{2\alpha}-\frac{\gamma}{2}\right)^{-1}} T_{\max}(\varphi).$$

Proof of Proposition C.3. The condition (C.8) implies that $T_{\max}(\lambda\varphi) < \infty$ as well as $T_{\max}(\lambda\tilde{\varphi}) < \infty$, and $T_{\max}(\lambda\tilde{\tilde{\varphi}}) < \infty$, for any $\lambda > 0$. See [41].

(i) By comparison argument it suffices to give the proof for $T_{\max}(\lambda\tilde{\varphi})$. We have that

$$\mu^\gamma D_\mu \tilde{\varphi} \geq \tilde{\varphi}, \quad \mu < 1.$$

Since $\gamma < (2+l)/\alpha$ and $\lambda = \mu^{\gamma - \frac{2+l}{\alpha}}$ then $\mu < 1$ is equivalent to $\lambda > 1$. By the above calculations,

$$T_{\max}(\lambda\tilde{\varphi}) \leq C\lambda^{-\left(\frac{2+l}{2\alpha} - \frac{\gamma}{2}\right)^{-1}}, \quad \lambda > 1.$$

(ii) By comparison argument it suffices to give the proof for $T_{\max}(\lambda\tilde{\tilde{\varphi}})$. We have that

$$\mu^\gamma D_\mu \tilde{\tilde{\varphi}} \geq \tilde{\tilde{\varphi}}, \quad \mu > 1.$$

Since $\gamma < (2+l)/\alpha$ and $\lambda = \mu^{\gamma - \frac{2+l}{\alpha}}$ then $\mu > 1$ is equivalent to $\lambda < 1$. Then by the above calculations,

$$T_{\max}(\lambda\tilde{\tilde{\varphi}}) \leq C\lambda^{-\left(\frac{2+l}{2\alpha} - \frac{\gamma}{2}\right)^{-1}}, \quad \lambda < 1.$$

This completes the proof of the proposition. \square

Remark 20. We may take $q = \infty$ in Proposition C.3. In particular, combining Corollary C.2 and Proposition C.8, we have $T_{\max}(\lambda\tilde{\varphi}) \sim \lambda^{-\left(\frac{2+l}{2\alpha} - \frac{\gamma}{2}\right)^{-1}}$, as $\lambda \rightarrow \infty$, and $T_{\max}(\lambda\tilde{\tilde{\varphi}}) \sim \lambda^{-\left(\frac{2+l}{2\alpha} - \frac{\gamma}{2}\right)^{-1}}$, as $\lambda \rightarrow 0$. This shows that, for large initial data the life-span increases as the power l increases, while, for small initial data the life-span decreases as the power l increases.

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